

Joint Realizability of Monotone Boolean Functions

Peter Crawford-Kahrl, Bree Cummins, and Tomáš Gedeon*

Department of Mathematical Sciences, Montana State University, Bozeman, MT 59715

Abstract

The study of monotone Boolean functions (MBFs) has a long history. We explore a connection between MBFs and ordinary differential equation (ODE) models of gene regulation, and, in particular, a problem of the *realization* of an MBF as a function describing the state transition graph of an ODE. We formulate a problem of *joint realizability* of finite collections of MBFs by establishing a connection between the parameterized dynamics of a class of ODEs and a collection of MBFs. We pose a question of what collections of MBFs can be realized by ODEs that belong to nested classes defined by increased algebraic complexity of their right-hand sides. As we progressively restrict the algebraic form of the ODE, we show by a combination of theory and explicit examples that the class of jointly realizable functions strictly decreases. Our results impact the study of regulatory network dynamics, as well as the classical area of MBFs. We conclude with a series of potential extensions and conjectures.

Keywords: Monotone Boolean functions,, gene regulatory networks, switching systems,

2010 MSC: 92C42, 34C23, 94C11

1. Introduction

The study of Boolean functions in general and monotone Boolean functions in particular has a long history [1, 2, 3, 4, 5, 6, 7]. One area in which monotone Boolean functions (MBFs) have been used is in modeling the dynamics of gene

*Corresponding author. Email: tgedeon@montana.edu

5 regulatory networks. In these models the (Boolean) state of each node i in the regulatory network is updated based on the (Boolean) states of the nodes j that are connected by an edge from j to i . The monotonicity requirement on a Boolean function (Definition 2.2) reflects the fact that the edges in gene regulatory networks are signed and thus the effect of one gene on another is always
10 either monotonically increasing (activating edge) or monotonically decreasing (repressing edge).

An alternative class of network models uses continuous time dynamics of ordinary differential equations. We are interested in a particular class of such models with piecewise linear right-hand sides [8, 9, 10, 11, 12, 13, 14]. For the
15 most general of these models, which we call *K-systems*, the right-hand side is fully determined by a finite collection of constants $K = (K_1, \dots, K_n)$, where K_i is a collection of constants that describes the activity of node i in the regulatory network. Each collection K_i within K also satisfies a monotonicity condition that reflects the monotone effect of the edges in the network.

20 The main goal of this paper is to show that there is a close relationship between *K*-systems and collections of monotone Boolean functions. In order to show this connection, we first show that to each *K*-system one can associate a state transition graph (STG), which is a finite directed graph that coarsely captures the progression of the trajectories of the *K*-system. There are a finite
25 number of STGs, which permits an imposition of an equivalence relation on the (infinite) set of *K*-systems, with an equivalence class denoted $[K]$.

Our first major result is the correspondence between the equivalence classes $[K]$ and collections of MBFs. For a fixed regulatory network with n nodes, each equivalence class $[K]$ has the form $[K] = ([K_1], \dots, [K_n])$. Then each $[K_i]$ for a
30 node i with m_i input edges and b_i output edges corresponds to a collection of b_i monotone Boolean functions with m_i inputs. Moreover, each such collection of b_i MBFs, satisfying an additional condition that the truth sets are linearly ordered by inclusion, is associated to an equivalence class $[K_i]$. Using this result, the equivalence classes $[K]$ are arranged into a parameter graph (PG) specific to the
35 regulatory network under consideration. The edges of the PG are determined

by the adjacency of the collections of MBFs associated to each $[K]$.

Our next set of results examines the effect of imposing algebraic restrictions on the form of the right-hand side of the differential equations of the network model, which results in additional structure on the set K . These additional
40 algebraic restrictions decrease the size of the parameter graph. We ask which MBFs, and which collections of MBFs, are realizable as parameter nodes of the corresponding restricted parameter graphs.

The classes of algebraic functions that we chose to examine are nested; the most restricted and smallest class consists of linear functions, Σ , which is a sub-
45 set of functions that can be obtained as products of sums of individual variables, $\Pi\Sigma$, and lastly sums of products of sums, $\Sigma\Pi\Sigma$. These classes are all special cases of K -systems and therefore admit STGs and PGs. These algebraic restrictions are motivated by the software DSGRN [15, 16, 17, 18], which calculates the PGs and STGs for network models with the class of $\Pi\Sigma$ functions, and in
50 principle can be extended to other classes of algebraic expressions.

We show that the classes Σ , $\Pi\Sigma$, and $\Sigma\Pi\Sigma$ do impose constraints on pairs of MBFs that can be realized as parameter nodes of a PG. In fact, we show that the classes of pairs of MBFs with three inputs that are realizable as linear functions are a strict subset of $\Pi\Sigma$ -jointly realizable pairs, which is in turn a
55 strict subset of $\Sigma\Pi\Sigma$ -jointly realizable pairs of MBFs. We also show that there are pairs of MBFs for any $n \geq 4$ inputs that are K -jointly realizable but are not $\Sigma\Pi\Sigma$ realizable.

These results show that the increased complexity of the algebraic expression provides a richer class of models as measured by the set of MBFs that can be
60 realized in a PG. At the same time, the connection between differential equation models and collections of MBFs allows for the formulation of a host of interesting questions (see the Discussion) about what k -tuples of MBFs can be realized as nodes of parameter graphs of differential equation models as the complexity of the right-hand side varies.

65 2. K -systems and Monotone Boolean Functions

A regulatory network is a useful abstraction for organizing information about interacting units. Nodes represent units and (directed) edges interaction between the nodes.

Definition 2.1. A *regulatory network* $\mathbf{RN}$ is a triple $\mathbf{RN} := \{V, E, s\}$ where

- 70 • V is the set of *vertices*;
- $E \subset V \times V$ is a finite set of oriented *edges*, where (i, j) denotes the edge from i to j ;
- $s : E \rightarrow \{+, -\}$ is the *sign* of the edge.

We will generally use $n = |V|$. We denote $S(i)$ to be the set of *sources* of node i and $T(i)$ the set of *targets* of node i :

$$S(i) = \{j \in V \mid (j, i) \in E\} \text{ and } T(i) = \{j \in V \mid (i, j) \in E\}.$$

We split the set of sources into activating and repressing inputs as $S(i) = S(i)^+ \cup S(i)^-$ where

$$j \in S^+(i) \text{ iff } e = (j, i) \in E \text{ and } s(e) = +$$

and

$$j \in S^-(i) \text{ iff } e = (j, i) \in E \text{ and } s(e) = -$$

The interpretation of the signed edges comes from biology; a positive edge
75 signifies up-regulation, where the rate of change of the target node concentration increases as the concentration of the source node increases. A negative edge signifies down-regulation, where the rate of change of the target node concentration decreases as the concentration of the source node increases. Inherent in this description is *monotonicity* of the rate of change of the target node with
80 respect to changes in each of the source nodes [19, 18, 20, 21, 22, 23].

One of the natural ways to associate dynamics to a network is using Boolean functions. Every node is assumed to be either OFF, corresponding to low concentration, represented by the state 0, or ON, corresponding to high concentration, represented by the state 1. A Boolean function is given by $f : \mathbb{B}^n \rightarrow \mathbb{B}$ with $\mathbb{B} := \{0, 1\}$. In examples, we will often write elements of $\mathbb{B}^n$ as strings (e.g. 10010 $\in \mathbb{B}^5$).

Definition 2.2. [24] A Boolean function $f : \mathbb{B}^n \rightarrow \mathbb{B}$ is *positive* (resp. *negative*) in y_i if $f|_{y_i=0} \leq f|_{y_i=1}$ (resp. $f|_{y_i=0} \geq f|_{y_i=1}$), where $f|_{y_i=\beta}$ is the function with y_i fixed at $\beta \in \mathbb{B}$. We say that f is *monotone in y_i* if it is either positive or negative in y_i and is *monotone* if it is monotone in y_i for all i .

Positive and negative monotone Boolean functions (MBF) capture the effect of positive and negative edges in the network **RN**, respectively. We will use the notation

$$\mathbf{MBF}(n) := \{f : \mathbb{B}^n \rightarrow \mathbb{B} \mid f \text{ is monotone}\}$$

$$\mathbf{MBF}^+(n) := \{f : \mathbb{B}^n \rightarrow \mathbb{B} \mid f \text{ is positive in } x_i \text{ for all } i \in \{1, \dots, n\}\}$$

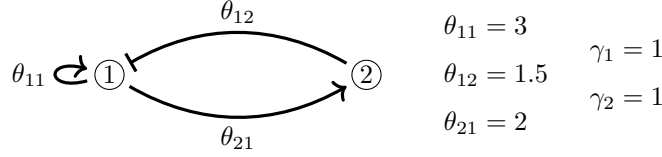
The dynamics of the network with n nodes is described by iteration of $f : \mathbb{B}^n \rightarrow \mathbb{B}^n$, where $f := (f_1, f_2, \dots, f_n)$ is a collection of MBFs. Monotone Boolean function models are widely used due to their simplicity, but matching their predictions to experimental values of continuous variables like concentration always poses a challenge. An effort to combine the simplicity of Boolean maps with a continuous time description was initiated by [9, 10, 25]. To explain this approach we extend the definition of regulatory network given in Definition 2.1.

Definition 2.3. A *weighted regulatory network* is a regulatory network **RN** with positive, real-valued weights assigned to each node,

$$\gamma = (\gamma_1, \dots, \gamma_n)$$

and each edge

$$\theta_i = \{\theta_{s_1 i}, \theta_{s_2 i}, \dots, \theta_{s_{b_i} i}\} \text{ where } \{s_1, s_2, \dots, s_{b_i}\} = T(i). \quad (1)$$



$$\begin{aligned}
S^+(1) &= \{1\} & S^-(1) &= \{2\} \\
S^+(2) &= \{1\} & S^-(2) &= \emptyset
\end{aligned}$$

Figure 1: An example weighted regulatory network. Here $V = \{1, 2\}$. We use $\rightarrow$ to denote a positive (activating) edge, and $\dashv$ to denote a negative (inhibiting) edge. We also illustrate the sets of sources for each node.

We want to bring attention to the indexing we use: θ_{ji} is associated to the edge from i to j , in the tradition of [15]. The node weights are called *decay rates* and the edge weights are called *thresholds*. We make a generic assumption that for each node i , the b_i thresholds in the collection θ_i are distinct.

The idea of decay rates comes from biology and indicates how quickly a gene product will break down under natural cellular processes. A common model of enzymatic gene regulation is the sigmoidal Hill function model, which has a half-saturation value. These half-saturation values are sometimes treated as thresholds, here represented as weights on edges. An example weighted regulatory network is shown in Figure 1.

2.1. K -systems

The most general attempt to combine the simplicity of Boolean maps with a continuous time description resulted in *switching K -systems*, consisting of a system of differential equations on $\mathbb{R}_+^n$. The “ K ” in K -system denotes a finite collection of real values that satisfy a monotonicity assumption (Definition 2.4) and are used to parameterize a system of ordinary differential equations (ODEs) with discontinuous right-hand sides.

Given a weighted regulatory network, we associate to each node i a continuous non-negative variable $x_i \in \mathbb{R}_+$, usually representing the concentration of

a gene product. The phase space of the gene concentrations $x = (x_1, \dots, x_n)$ is $\mathbb{R}_+^n$. The thresholds θ_i divide the x_i axis into $b_i + 1$ intervals, where $b_i = |\theta_i|$ is the number of targets of node i . We enumerate these by the integers $0, \dots, b_i$ in ascending order. Then

$$X = \mathbb{R}_+^n \setminus \{x_i = \theta_{ji} \mid i \in 1, \dots, n, \quad \theta_{ji} \in \theta_i\}$$

115 is an open rectangular grid where each component of X is an open domain. As we will see next, the collection of real numbers K determines an ODE system defined on X whose solutions are consistent with a discrete mapping between open domains in X and this discrete map can be interpreted as a collection of **MBFs**. The following definition of the K -system goes back to at least Thieffry
120 and Romero [26]; we follow the exposition in [13].

Definition 2.4. Recall the definition of a regulatory network **RN** in Definition 2.1, particularly the nodes V and the sources $S^\pm(i)$. Let

$$K := \{K_{i,A,B} \in \mathbb{R}_+ \mid i \in V, A \subset S^+(i), B \subset S^-(i)\} \quad (2)$$

be a collection of non-negative numbers that satisfies the *monotonicity assumption*:

- For each $i \in V$, if $A \subsetneq A' \subset S^+(i)$ then

$$K_{i,A,B} \leq K_{i,A',B} \quad \text{for all } B \subset S^-(i).$$

- For each $i \in V$, if $B \subsetneq B' \subset S^-(i)$ then

$$K_{i,A,B} \geq K_{i,A,B'} \quad \text{for all } A \subset S^+(i).$$

For convenience, we associate the real-valued collection K to a collection of *parameter assignment functions*

$$k_i : 2^{S^+(i)} \times 2^{S^-(i)} \rightarrow \mathbb{R}_+, \quad k_i(A, B) := K_{i,A,B}$$

denoting $k = (k_1, \dots, k_n)$.

Note that the class of K-systems is very large since the number of constants
 125 K_i associated to a node $i \in V$ with a_i source nodes is proportional to the size
 of the power set of $\{1, \dots, a_i\}$

We continue the example from Figure 1 by listing an example assignment of
 numbers K that satisfy the monotonicity assumption.

$$\begin{aligned} K_{1,\emptyset,2} &= 0.1 & K_{2,\emptyset,\emptyset} &= 0.2 \\ K_{1,\emptyset,\emptyset} &= 0.5 & K_{2,1,\emptyset} &= 0.4 \\ K_{1,1,2} &= 5 \\ K_{1,1,\emptyset} &= 6 \end{aligned} \tag{3}$$

Up to now, the construction of K has depended only on the structure of an
 130 unweighted regulatory network $\mathbf{RN}$. We now take into account the weights
 associated to $\mathbf{RN}$ as in Definition 2.3. The K-system ODE described next
 depends on the these weights.

The dynamics of variables x_i are affected by the incoming edges to node
 i in the regulatory network $\mathbf{RN}$. For each $x_j \in S(i)$, the value of x_j is either
 above or below the threshold θ_{ij} assigned to the edge from j to i in the weighted
 regulatory network. If $x_j \in S^+(i)$ has an activating effect, then $x_j > \theta_{ij}$ implies
 that x_i will be produced at a greater rate than when $x_j < \theta_{ij}$. The inequalities
 are swapped for a repressing effect, $x_j \in S^-(i)$. With this in mind, we define
 the *activity function* for a node i and $x \in X$, as follows:

$$\zeta_i : X \rightarrow 2^{S^+(i)} \times 2^{S^-(i)}, \quad \zeta_i(x) = (A_i, B_i) \tag{4}$$

$$\begin{aligned} A_i &= \{j \in S^+(i) \mid x_j > \theta_{ij}\} \\ B_i &= \{j \in S^-(i) \mid x_j > \theta_{ij}\}. \end{aligned}$$

The map $\zeta := (\zeta_1, \dots, \zeta_n)$, defined on X , is constant on each open domain of
 X .

135 The composition of the parameter assignment function with the activity
 function, $k_i \circ \zeta_i$, assigns to a vector $x \in X$ a scalar parameter $k_i(A, B) = k_i(\zeta_i(x))$

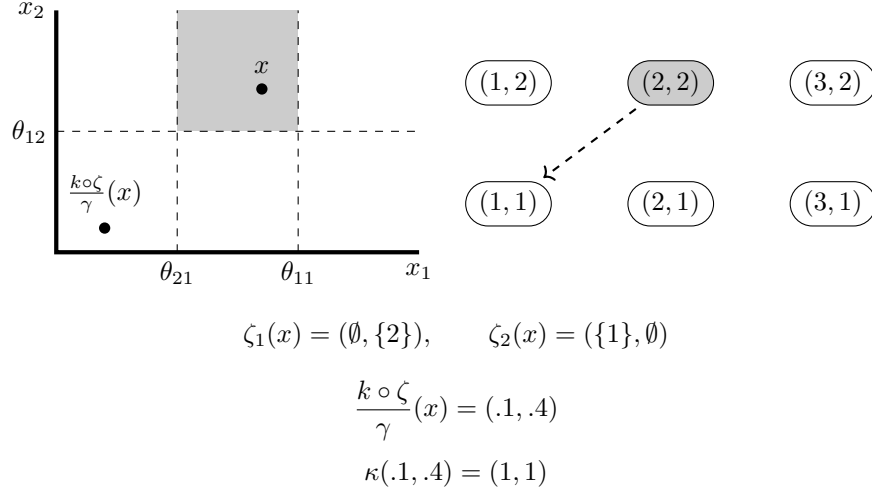


Figure 2: Continuing the example from Figure 1 we construct the state transition graph. Above left is X and right is D . For any x in the shaded domain, the value of $\frac{k \circ \zeta}{\gamma}(x)$ is constant and located in the lower left domain. This determines the value $\Phi^K(2, 2) = (1, 1)$ denoted with a dashed arrow; see (8).

in the set K . Recalling the decay rates γ from Definition 2.3, we are now in a position to define a differential equation parameterized by K and defined on X .

Definition 2.5. The system

$$\dot{x}_i = -\gamma_i x_i + k_i(\zeta_i(x)) \quad (5)$$

is called the K -system on X associated to the weighted regulatory network $\mathbf{RN}$.

140 Note that since ζ is constant on the open domains of X , the differential equation is linear in each such domain. On the boundaries of the domains, the system is undefined due to the discontinuity in $k_i(\zeta_i(x))$. However, we extend the system by continuity, whenever possible, from X to $\mathbb{R}_+^n$. The assumption of the non-negativity of k guarantees that the non-negative orthant $\mathbb{R}_+^n$ is positively
145 invariant and the concentrations x_i remain non-negative for all $t \geq 0$.

2.1.1. State transition graph

Let

$$D := \prod_{i=1}^n \{0, 1, \dots, b_i\}$$

be a set of n integer sequences that will be referred to as *states*. Recall that b_i is the number of targets of the node i in $\mathbf{RN}$, $b_i = |\theta_i|$. We construct a function between the points of X and the states in D . Let $\kappa_i : X \rightarrow \{0, 1, \dots, b_i\}$ be a map

$$\kappa_i(x) = \ell \quad \text{when } x_i \in (\theta_{*i}^\ell, \theta_{*i}^{\ell+1}) \quad (6)$$

where superscript ℓ indicates the ℓ -th domain of the x_i axis. Collecting the maps κ_i in a single map we define $\kappa : X \rightarrow D$ to be the *index assignment map* $\kappa = (\kappa_1, \dots, \kappa_n)$. We say that D *indexes* the open domains of X . By
150 construction, the index assignment map is constant on each domain in X . The map κ takes a vector $x \in X$ and assigns it to the state representing the domain of X in which x lies.

The function

$$\frac{k \circ \zeta}{\gamma} = \left(\frac{k_1 \circ \zeta_1}{\gamma_1}, \dots, \frac{k_n \circ \zeta_n}{\gamma_n} \right)$$

is a map $\frac{k \circ \zeta}{\gamma} : X \rightarrow X$. We define a discrete map $\Phi^K : D \rightarrow D$ on the set of states D by requiring that

$$\Phi^K \circ \kappa = \kappa \circ \frac{k \circ \zeta}{\gamma}, \quad (7)$$

i.e. that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\frac{k \circ \zeta}{\gamma}} & X \\ \downarrow \kappa & & \downarrow \kappa \\ D & \xrightarrow{\Phi^K} & D \end{array} \quad (8)$$

Note that the solution of (5) with initial condition x_0 converges to the target point $\frac{k \circ \zeta}{\gamma}(x_0)$. The map Φ^K takes the state $d = \kappa(x_0)$ and assigns it to the
155 state $\Phi^K(d)$ which contains the target point $\frac{k \circ \zeta}{\gamma}(x_0)$. In this way, the map Φ^K captures the behavior of solutions of (5). It is important to note that the convergence of the solution starting at x_0 toward its target point $\frac{k \circ \zeta}{\gamma}(x_0)$ is

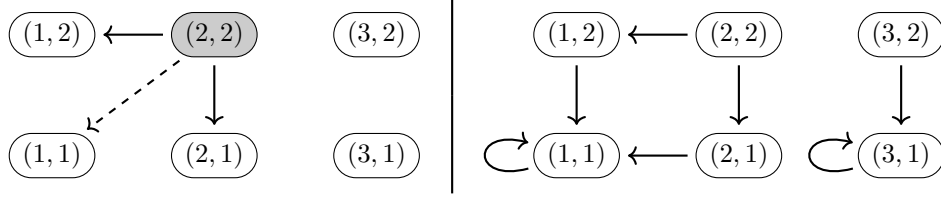


Figure 3: Left: The state transition graph is the asynchronous update dynamics, and so does not allow the diagonal transition; instead, we replace the dashed arrow with arrows capturing one-step adjacency. Right: the completed STG, where the process that was illustrated for state (2,2) is repeated for each state.

only valid while the solutions remain in the component of X containing x_0 ; when they enter a neighboring domain, the target point will change.

160 To capture this behavior, we define a *state transition graph* on states $d \in D$ that coarsely describes solutions of (5). It represents the *asynchronous* update dynamics for the discrete valued function Φ^K .

Definition 2.6 (State transition graph). The *state transition graph* (STG) is a directed graph with nodes D , where two nodes $d, d' \in D$ are connected by a
165 directed edge $d \rightarrow d'$, if and only if

1. either $d = d'$ and $\Phi^K(d) = d$; or
2. d and d' differ in exactly one component, say i , and

$$d'_i = d_i + 1 \text{ and } \Phi_i^K(d) > d_i, \text{ or}$$

$$d'_i = d_i - 1 \text{ and } \Phi_i^K(d) < d_i$$

We construct the state transition graph of our example network in Figures 2 and 3.

The number of maps Φ^K for a given **RN** is finite. This induces an equivalence relation over all collections K satisfying the monotonicity condition in
170 Definition 2.4 that are consistent with the structure of **RN**.

Definition 2.7. For a given weighted regulatory network, we define an equiv-

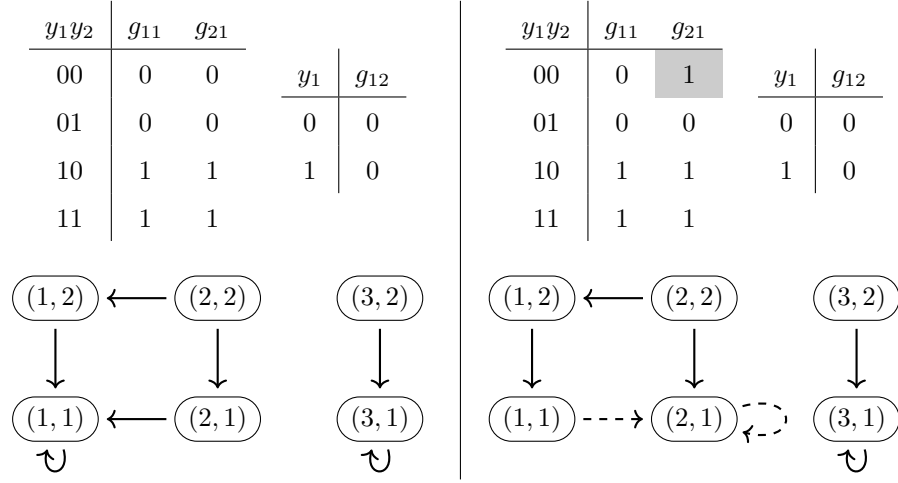


Figure 4: Left: the collection of MBFs corresponding to our example network and choice of K in (3), and the associated state transition graph. Right: A collection of MBFs adjacent (in the parameter graph) to the collection on the left. The single change is highlighted in gray. The corresponding state transition graph is also pictured; differences caused by the shaded entry are shown as dashed edges

alence relation on the collection of all parameter sets K . We set

$$K \cong K' \iff \Phi^K(d) = \Phi^{K'}(d) \quad \text{for all } d \in D.$$

Notice that each equivalence class $[K]$ has a component structure composed of n independent equivalence classes, $[K] = ([K_1], \dots, [K_n])$, one for each node $i \in V$ in the regulatory network. This is because the monotonicity assumption in Definition 2.4 applies independently to each node.

2.2. Equivalence classes $[K]$ are collections of MBFs

We now discuss the connection between equivalence classes $[K]$ and monotone Boolean functions. Each equivalence class $[K]$ is uniquely associated to a collection of $\prod_{i=1}^n b_i$ MBFs, where $b_i = |T(i)|$ is the number of targets of node i in $\mathbf{RN}$. We label these MBFs g_{si} , one for each threshold θ_{si} in the weighted regulatory network, and construct them below.

Let $[K] = ([K_1], \dots, [K_n])$ be an equivalence class, and consider an element $K_i \in [K_i]$, where $K_i = \{K_{i,A,B}\}_{A \subset S^+(i), B \subset S^-(i)}$ is a collection of constants for node i in **RN**. Define a function

$$\alpha_i : \mathbb{B}^{S(i)} \rightarrow 2^{S^+(i)} \times 2^{S^-(i)}$$

where $\alpha_i(\vec{y}) = (A_i, B_i)$ for

$$A_i = \{j \in S(i) \mid y_j = 1\} \cap S^+(i), \quad B_i = \{j \in S(i) \mid y_j = 1\} \cap S^-(i).$$

Here we use the standard multi-index notation $\mathbb{B}^{S(i)} = \{y_{j_1}y_{j_2} \dots y_{j_{m_i}} \mid j_k \in S(i), y_{j_k} \in \mathbb{B}\}$, i.e. elements of $\mathbb{B}^{S(i)}$ are Boolean strings of length $|S(i)|$ indexed by elements of $S(i)$ in order. As an example, if $S(i) = \{2, 4, 5, 7\}$, then $\vec{y} =$

185 $y_2y_4y_5y_7$, where $y_i \in \mathbb{B}$.

Let $\{\theta_{s_1 i}, \theta_{s_2 i}, \dots, \theta_{s_{b_i} i}\}$ be the b_i thresholds associated to the b_i targets of node i in **RN**. With this assignment, we define positive Boolean functions $g_{si} : \mathbb{B}^{S(i)} \rightarrow \mathbb{B}$ for all $s \in T(i)$, as

$$g_{si}(\vec{y}) = \begin{cases} 1 & \text{when } K_{i, \alpha_i(\vec{y})} > \theta_{si} \gamma_s \\ 0 & \text{when } K_{i, \alpha_i(\vec{y})} < \theta_{si} \gamma_s \end{cases}$$

and negative Boolean functions as

$$g_{si}(\vec{y}) = \begin{cases} 0 & \text{when } K_{i, \alpha_i(\vec{y})} > \theta_{si} \gamma_s \\ 1 & \text{when } K_{i, \alpha_i(\vec{y})} < \theta_{si} \gamma_s \end{cases}$$

Note that the strict inequality can be changed to non-strict inequality in one of the two equations and this will not affect our results. Since none of these possibilities is *a priori* better, we chose to require that both inequalities are strict.

We observe that if $j \in S^+(i)$, then g_{si} will be positive in x_j , and if $j \in S^-(i)$, then g_{si} will be negative in x_j . Therefore, any g_{si} constructed in this way will be a monotone Boolean function $g_{si} \in \mathbf{MBF}(|S_i|)$. As we show next, the collection

$$G = \{g_{si} \mid i = 1, \dots, n, s \in T(i)\}$$

190 is an equivalent representation of the equivalence class $[K]$.

Proposition 2.8. *Fix a weighted regulatory network $\mathbf{RN}$ and thus sets $S^+(i)$, $S^-(i)$, and $T(i)$, as well as the weights γ_i and θ_{ji} for all $i = 1, \dots, n$, $j \in T(i)$. Then*

$$\Phi^K = \Phi^{K'} \iff G = G'.$$

Proof. Consider two different collections K, K' associated to parameter assignment functions k, k' respectively. Note that the set of states D , the function $\kappa : X \rightarrow D$ and the function $\zeta : X \rightarrow 2^{S^+(i)} \times 2^{S^-(i)}$ are uniquely determined by the weighted regulatory network $\mathbf{RN}$. Therefore, Φ^K and $\Phi^{K'}$ differ only in the functions k and k' :

$$\Phi^K \circ \kappa = \kappa \circ \frac{k \circ \zeta}{\gamma} \quad \Phi^{K'} \circ \kappa = \kappa \circ \frac{k' \circ \zeta}{\gamma}.$$

Therefore it follows that

$$\Phi^K = \Phi^{K'} \iff \kappa \circ \frac{k}{\gamma} = \kappa \circ \frac{k'}{\gamma} \iff \kappa \circ \frac{k}{\gamma} \circ \alpha = \kappa \circ \frac{k'}{\gamma} \circ \alpha.$$

This in turn leads to the equivalencies:

$$\begin{aligned} &\iff \kappa(K_{i, \alpha_i(\vec{y})}/\gamma_i) = \kappa(K'_{i, \alpha_i(\vec{y})}/\gamma_i) \quad \forall i = 1, \dots, n \text{ and } \forall \vec{y} \in \mathbb{B}^{S(i)} \\ &\iff K_{i, \alpha_i(\vec{y})}/\gamma_i, K'_{i, \alpha_i(\vec{y})}/\gamma_i \in (\theta_{*i}^\ell, \theta_{*i}^{\ell+1}) \text{ for some } \ell \\ &\iff K_{i, \alpha_i(\vec{y})}, K'_{i, \alpha_i(\vec{y})} \in (\theta_{*i}^\ell \gamma_i, \theta_{*i}^{\ell+1} \gamma_i) \text{ for some } \ell \\ &\iff (K_{i, \alpha_i(\vec{y})} > \theta_{ji} \gamma_i \iff K'_{i, \alpha_i(\vec{y})} > \theta_{ji} \gamma_i) \quad \forall j \in T(i) \\ &\iff (g_{ji} \in G \iff g_{ji} \in G'). \end{aligned}$$

□

For an example, compare the collection K in (3) for the weighted regulatory network in Figure 1 to the equivalent collection of three monotone Boolean functions in the left panel of Figure 4.

195 2.3. Parameter Graph

The fact that we can view equivalence classes $[K]$ as collections of monotone Boolean functions allows us to organize the equivalence classes $[K]$ into a graph,

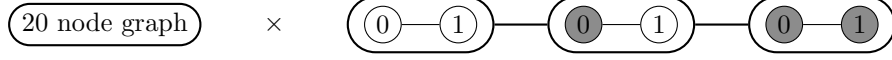


Figure 5: Continuing the example, we show the parameter graph for the network in Figure 1. The parameter graph takes the form of a product graph, with one factor for each node in $\mathbf{RN}$. The factor associated to node 1, which has two inputs and two outputs in $\mathbf{RN}$, is the 20 node graph on the left. It is isomorphic to the graph shown in Appendix B, Figure B.9. The factor associated to node 2, which has one input and one output in $\mathbf{RN}$, is shown on right. Each parameter node in the factor on the right contains the associated monotone Boolean function, where gray shading means $g_{12} = 0$, and similarly white shading implies $g_{12} = 1$.

called the *parameter graph* (PG), where each node is associated to an equivalence class. Let $[K]$ and $[K']$ be different equivalence classes, with associated collections of MBFs $G = \{g_{si}\}$ and $G' = \{g'_{si}\}$. The nodes $[K]$ and $[K']$ will be connected by an edge, if, and only if, there is $i \in \{1, \dots, n\}$, $s \in T(i)$, and $\vec{y} \in \mathbb{B}^{S(i)}$ such that

$$\begin{aligned} g_{si}(\vec{y}) &\neq g'_{si}(\vec{y}) \\ g_{si}(\vec{z}) &= g'_{si}(\vec{z}) \quad \forall \vec{z} \neq \vec{y} \\ g_{\ell j} &= g'_{\ell j} \text{ whenever } j \neq i \text{ or } \ell \neq s. \end{aligned}$$

In other words, there is exactly one MBF whose value differs on one input. An example of a single adjacency is shown in Figure 4 and the parameter graph for our running example is shown in Figure 5.

2.4. Representative networks

200 It will be convenient to consider a subset of weighted regulatory networks $\mathbf{RN}$ with $\gamma_i = 1$ for all i . It turns out that the class of weighted regulatory networks with this property exhibits the same parameter graphs with the same collection of state transition graphs as the collection of weighted graphs with general positive decay rates $\gamma = (\gamma_1, \dots, \gamma_n)$.

205 To see this, fix a set K and its parameter assignment function k from Definition 2.4. Consider a weighted regulatory network $\mathbf{RN}$ with the collections of sources $S^+(i)$, $S^-(i)$, targets $T(i)$, decay rates $\gamma = (\gamma_1, \dots, \gamma_n)$, and thresholds $\{\theta_{ji}\}$. Consider a network $\widehat{\mathbf{RN}}$ with the same collection of sources $S^+(i)$, $S^-(i)$

and targets $T(i)$, but with all decay rates set to $\hat{\gamma}_i = 1$ and the thresholds set
 210 to $\{\hat{\theta}_{ji} = \gamma_i \theta_{ji}\}$.

The threshold assignment induces a bijection $x \mapsto \hat{x}$ with $\hat{x} = \gamma x$, from X to $\hat{X}$:

$$\hat{X} = \mathbb{R}_+^n \setminus \{\hat{x}_i = \hat{\theta}_{ji} \mid i = 1, \dots, n, \quad j \in T(i)\}.$$

The key observation is that

$$\hat{x}_i \in (\gamma_i \theta_{*i}^\ell, \gamma_i \theta_{*i}^{\ell+1}) \Leftrightarrow x_i = \frac{\hat{x}_i}{\gamma_i} \in (\theta_{*i}^\ell, \theta_{*i}^{\ell+1}). \quad (9)$$

This allows us to conclude that the activity functions ζ and $\hat{\zeta}$ (defined in (4)) satisfy $\hat{\zeta}(\hat{x}) = \zeta(\hat{x}/\gamma) = \zeta(x)$, which leads to

$$\theta_{ji} < \frac{k_i \circ \zeta}{\gamma_i}(x) < \theta_{si}$$

for some $j, s \in T(i)$, if and only if

$$\hat{\theta}_{ji} < k_i \circ \hat{\zeta}(\hat{x}) < \hat{\theta}_{si}.$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\frac{k \circ \zeta}{\gamma}} & X \\ \downarrow \gamma x & & \downarrow \gamma x \\ \hat{X} & \xrightarrow{k \circ \hat{\zeta}} & \hat{X} \end{array}$$

Since the underlying network topology is the same between two weighted networks $\mathbf{RN}$ and $\widehat{\mathbf{RN}}$, the discrete states of the state transition graph are the same $D = \hat{D}$. Using (9) again, we conclude that the index assignment functions κ and $\hat{\kappa}$ from (6) satisfy the following:

$$\hat{\kappa}(\hat{x}) = \ell \Leftrightarrow \kappa(x) = \ell.$$

Recalling that $\Phi^K \circ \kappa = \kappa \circ (k \circ \zeta)/\gamma$ from (7), we see that

$$\hat{\Phi}^K \circ \hat{\kappa}(\hat{x}) = \Phi^K \circ \kappa(x).$$

This means that the state transition graphs are identical under K applied to $\mathbf{RN}$ and $\widehat{\mathbf{RN}}$.

We conclude that by considering the restricted class of weighted regulatory networks with $\gamma_i = 1$ for all i we will recover the same set of state transition graphs as the general system. Therefore we will assume $\gamma_i = 1$ from now on, and we will write g_{si} as

$$g_{si}(\vec{y}) = \begin{cases} 1 & \text{when } K_{i,\alpha_i}(\vec{y}) > \theta_{si} \\ 0 & \text{when } K_{i,\alpha_i}(\vec{y}) < \theta_{si} \end{cases}$$

or

$$g_{si}(\vec{y}) = \begin{cases} 0 & \text{when } K_{i,\alpha_i}(\vec{y}) > \theta_{si} \\ 1 & \text{when } K_{i,\alpha_i}(\vec{y}) < \theta_{si} \end{cases}. \quad (10)$$

2.5. Differential equations from monotone Boolean functions

In Definition 2.5 we associated an ordinary differential equation to a weighted regulatory network **RN**. A more explicit way to do so is due to [9, 10, 25]. Again consider a weighted regulatory network **RN** with nodes $i \in V$ summarizing regulatory activity for continuous variables x_i . Assume that regulation of x_i by its regulatory input x_j switches abruptly at the real-valued threshold θ_{ij} from **RN**, written as one of two maps

$$\sigma_{ij}^+(x_j) = \begin{cases} 1 & x_j > \theta_{ij} \\ 0 & x_j < \theta_{ij} \end{cases} \quad \sigma_{ij}^-(x_j) = \begin{cases} 0 & x_j > \theta_{ij} \\ 1 & x_j < \theta_{ij} \end{cases}$$

whenever $j \in S(i)$ is a source of node i . We write $\sigma_i = (\sigma_{is_1}, \dots, \sigma_{is_{m_i}})$, where
215 $s_j \in S(i)$, $m_i = |S(i)|$, $\sigma_{is} = \sigma_{is}^+$ whenever $s \in S^+(i)$, and $\sigma_{is} = \sigma_{is}^-$ whenever $s \in S^-(i)$. In other words, σ^+ models an activating input and σ^- a repressing input.

We shall again assume that any two thresholds θ_{ji} and θ_{ki} are distinct for variable x_i . Also as before, these thresholds $\{\theta_{ij} \mid i = 1, \dots, n, j \in S(i)\}$ divide $\mathbb{R}_+^n$ into an open rectangular grid X . In addition, assume that for every node $i \in V$ there is an associated Boolean function $f_i : \mathbb{B}^{S(i)} \rightarrow \mathbb{B}$ that converts inputs of the node i to the new state of node i . Then we consider the following system of ODEs on X :

$$\dot{x}_i = -x_i + f_i(\sigma_i(x)) \quad (11)$$

3. Algebraic switching systems

The system of ODEs (11) has no continuous parameters. In order to introduce such parameters and allow comparison with K-systems we parameterize both the domain and the range of each function $f_i : \mathbb{B}^n \rightarrow \mathbb{B}$. To parameterize the domain we replace in the definition of σ_{ij} the Boolean values $0 < 1$ by continuous, non-negative, real parameter values $L_{ij} < U_{ij}$. To capture the sign along the network edges, we again consider two types of σ_{ij} functions

$$\sigma_{ij}^+(x_j) = \begin{cases} U_{ij} & x_j > \theta_{ij} \\ L_{ij} & x_j < \theta_{ij} \end{cases} \quad \sigma_{ij}^-(x_j) = \begin{cases} L_{ij} & x_j > \theta_{ij} \\ U_{ij} & x_j < \theta_{ij} \end{cases}.$$

We introduce the *interaction function* Λ_i as a real-valued replacement for the function f_i . All interaction functions will be algebraic expressions over the real numbers using addition and multiplication. In this notation, Equation (11) reads

$$\dot{x}_i = -x_i + \Lambda_i(\sigma_i(x)),$$

and we refer to it as a *switching system*, as in [15, 17].

220 For every $x \in X$, the composition $\Lambda_i(\sigma_i(x))$ assigns a real number that is a combination of numbers $\{L_{ij} \text{ or } U_{ij} \mid j \in S(i)\}$, where for each j only one of L_{ij} or U_{ij} enters the function Λ_i . This combination is constant on each domain in X . For monotone functions Λ_i , the image of $\Lambda_i(\sigma_i(x))$ is a set K that satisfies the monotonicity assumption in Definition 2.4. Therefore the switching
225 system with monotone Λ_i for all i is a K-system (5) and therefore gives rise to a parameter graph and to a state transition graph for each parameter node.

In this paper we consider three basic algebraic forms of functions Λ_i . The set of linear (Σ) Λ functions is given by

$$\Sigma(n) := \left\{ \Lambda_i : \mathbb{R}_+^n \rightarrow \mathbb{R}_+ \mid V \subseteq \{1, \dots, n\}, \Lambda_i(z_1, \dots, z_n) = \sum_{i \in V} z_i \right\}.$$

The products of sums ($\Pi\Sigma$) Λ functions are

$$\Pi\Sigma(n) := \left\{ \Lambda_i : \mathbb{R}_+^n \rightarrow \mathbb{R}_+ \mid \Lambda_i(z_1, \dots, z_n) = \prod_{W_k} \left(\sum_{i \in W_k} z_i \right) \right\},$$

where the sets W_k partition $S(i)$. The sums of products of sums ($\Sigma\Pi\Sigma$) Λ functions are

$$\Sigma\Pi\Sigma(n) := \left\{ \Lambda_i : \mathbb{R}_+^n \rightarrow \mathbb{R}_+ \mid \Lambda_i(z_1, \dots, z_n) = \sum_{W_k} \left(\prod_{V_{k,j}} \left(\sum_{i \in V_{k,j}} z_i \right) \right) \right\}, \quad (12)$$

where the disjoint union of $V_{k,j}$ is W_k . As before, the sets W_k partition $S(i)$, and the sets $V_{k,j}$ partition the set W_k . Observe that these classes of functions contain progressively more functions, i.e. for $n \geq 3$,

$$\Sigma(n) \subsetneq \Pi\Sigma(n) \subsetneq \Sigma\Pi\Sigma(n).$$

The restriction of the class of functions Λ to a product of sums ($\Pi\Sigma$) goes back to at least Snoussi [14], and was used extensively in the development of DSGRN (Dynamic Signatures Generated by Regulatory Networks) [15, 17, 16, 27, 28]. The main contribution of the DSGRN approach is the definition and explicit construction of a parameter graph (Section 2.3) in terms of inequalities in the input combinations of $\{L_{ij}, U_{ij} \mid j \in S(i)\}$ and thresholds $\{\theta_{ki} \mid k \in T(i)\}$ given a collection of nonlinearities in $\Pi\Sigma$.

As we have shown in Section 2.2, each component $[K_i]$ of the parameter node $[K] = ([K_1], \dots, [K_n])$ is equivalent to a collection of monotone Boolean functions, one for each edge in the regulatory network. When a node i has a single target, then there is only one MBF associated to node i , namely g_{ji} , where j is the sole target of i . In the case of multiple targets, $|T(i)| > 1$, there is a collection of $|T(i)|$ MBFs for node i . We consider a single component $[K_i]$, first where node i has a single target and second where node i has more than one target. We ask which such singletons or collections of monotone Boolean functions can be associated to a parameter node in the parameter graphs of K -, Σ -, $\Pi\Sigma$ - and $\Sigma\Pi\Sigma$ -systems.

Definition 3.1. We say that a monotone Boolean function $h : \mathbb{B}^n \rightarrow \mathbb{B}$ is **-realizable*, where $*$ can stand for K , Σ , $\Pi\Sigma$ or $\Sigma\Pi\Sigma$, if there exists a regulatory network $\mathbf{RN}$ with a node j with a single target ℓ and weight $\theta_{\ell j}$ and a parameter

node $[K] = ([K_1], \dots, [K_n])$ for the $*$ -system, such that the Boolean map g_{ℓ_j} that corresponds to $[K_j]$ is h .

We say that a k -tuple of monotone Boolean functions $h_1, h_2, \dots, h_k : \mathbb{B}^n \rightarrow \mathbb{B}$ is $*$ -jointly realizable if there exists a regulatory network **RN** with a node j with k targets $\ell_1, \dots, \ell_k$ and weights $\{\theta_{\ell_i j}\}$ and a parameter node $[K] = ([K_1], \dots, [K_n])$ for the $*$ -system such that the collection of Boolean maps $g_{\ell_1 j}, g_{\ell_2 j}, \dots, g_{\ell_k j}$ that corresponds to $[K_j]$ are the maps $h_1, h_2, \dots, h_k$ respectively.

Remark 3.2. We note that Σ -realizability is a special case of $\Pi\Sigma$ -realizability, which is in turn a special case of $\Sigma\Pi\Sigma$ -realizability, which is in turn a special case of K -realizability. These observations rely on the fact that $\Sigma \subset \Pi\Sigma \subset \Sigma\Pi\Sigma$ and that the images of any monotone Λ functions give rise to K -systems.

The definition of realizability 3.1 uses arbitrary functions $h \in \mathbf{MBF}$. In some cases, it will be convenient to assume $g_{ki} \in \mathbf{MBF}^+(m_i)$, instead of the weaker condition $g_{ki} \in \mathbf{MBF}(m_i)$, where recall $m_i = |S(i)|$ is the number of sources of i . This is achieved via a coordinate change. For each $j \in S(i)$, define the function $\beta_j^{(i)} : \mathbb{B} \rightarrow \mathbb{B}$ as

$$\beta_j^{(i)}(b) = \begin{cases} b & \text{if } j \in S^+(i) \\ 1 - b & \text{if } j \in S^-(i) \end{cases} \quad (13)$$

Then define

$$\beta^{(i)} : \mathbb{B}^{m_i} \rightarrow \mathbb{B}^{m_i}, \quad \beta^{(i)} = (\beta_1^{(i)}, \dots, \beta_{m_i}^{(i)})$$

component-wise. We observe that $g_{ki} \circ \beta^{(i)} \in \mathbf{MBF}^+(m_i)$ and that $\beta^{(i)}$ is an involution i.e. $\beta^{(i)} \circ \beta^{(i)} = \text{Id}$. We will use the notation β for a function where we do not specify the network node identity i .

Using the coordinate change β , we can see that $h : \mathbb{B}^{m_i} \rightarrow \mathbb{B}$ is $*$ -realizable if and only if, $f := h \circ \beta \in \mathbf{MBF}^+(m_i)$ is a positive Boolean function and is also $*$ -realizable, via the collection K' defined as

$$K'_{i, A', \emptyset} = K_{i, A, B}$$

where

$$A' = A \cup B,$$

and network $\mathbf{RN}'$ which is the same as $\mathbf{RN}$ except that all edges are now activating. Likewise, if $h_1, h_2, \dots, h_k : \mathbb{B}^n \rightarrow \mathbb{B}$ is $*$ -jointly realizable if, and only if, $f_1 = h_1 \circ \beta^{(1)}, \dots, f_k := h_k \circ \beta^{(k)} \in \mathbf{MBF}^+(n)$ are positive Boolean functions and are also $*$ -jointly realizable. Therefore it is sufficient to consider
265 functions and are also $*$ -jointly realizable. Therefore it is sufficient to consider in proofs only positive Boolean functions.

The central question that we pose in this paper is to ask how much restriction the algebraic forms Σ , $\Pi\Sigma$, and $\Sigma\Pi\Sigma$ impose on the richness of the potential dynamics of the switching system. We will interpret the number of
270 k -tuples of MBFs that can be represented in the parameter graph as the richness of that particular class of switching systems. This question generalizes and extends a classical problem of determining when a monotone Boolean function is a threshold function.

Definition 3.3. [24] A Boolean function $f : \mathbb{B}^n \rightarrow \mathbb{B}$, is called a *threshold function* (or a *linearly separable function*) if there exist real numbers $a_1, \dots, a_n \in \mathbb{R}$ and a threshold $\theta \in \mathbb{R}$ such that for all $\vec{y} = (y_1, \dots, y_n) \in \mathbb{B}^n$,

$$f(\vec{x}) = \begin{cases} 1 & \text{if } \sum_{j=1}^n a_j y_j > \theta \\ 0 & \text{otherwise} \end{cases}. \quad (14)$$

The $(n+1)$ -tuple $(a_1, a_2, \dots, a_n, \theta)$ is called a (*separating*) *structure* of f .

275 As we will see later in Lemma 4.21, any monotone Boolean function f is a threshold function if and only if f is Σ -realizable, i.e. representable in the parameter graph of a Σ system. Framed in terms of threshold functions, determining which f functions are Σ -realizable is then equivalent to determining which MBFs are indeed threshold functions.

280 This is a classical problem in the Boolean literature. Paull [2] showed that monotonicity is a necessary condition for a Boolean function to be a threshold function. As shown by Chow [6] and Elgot [3], a Boolean function is a threshold function if, and only if, it is *assumable*, where assumable was defined by Winder

[5]. An algorithm for determining whether a Boolean function is a threshold
 285 function was given by Peled and Simeone [29]. Their algorithm will produce
 $a_1, \dots, a_n$ in the sense of Definition 3.3 if the Boolean function is indeed a
 threshold function. An algorithm for producing additional linearly separable
 Boolean functions and further characterization of threshold functions was given
 in Rao and Zhang [30]. The number of threshold functions for $n \leq 8$ was found
 290 in [4], and extended to $n = 9$ by Gruzling [31].

Note that f is a threshold function if the set of points in $\mathbb{B}^n \subset \mathbb{R}^n$ at which
 f attains value 1 is linearly separable from those points where f attains value
 0. Following this connection, Pantovic et al. [32], Zunic [33], and Wang and
 Williams [34] all examined partitions of sets of points with surfaces that are not
 295 necessarily hyperplanes. This is intimately related to the questions of $\Pi\Sigma$ - and
 $\Sigma\Pi\Sigma$ -realizability, i.e. which monotone Boolean functions are representable in
 a parameter graph of a $\Pi\Sigma$ - vs. $\Sigma\Pi\Sigma$ - systems.

The parameter graph node $[K] = ([K_1], \dots, [K_n])$ represents n collections
 of monotone Boolean functions, where each $[K_j]$ corresponds to $b_j = |T(j)|$
 300 MBFs, the number of targets of j in $\mathbf{RN}$. Not every collection of b_j monotone
 Boolean functions is $*$ -jointly realizable for the algebraic classes we consider.
 We introduce the idea of considering multiple Boolean functions simultaneously
 and asking which of them are $*$ -jointly realizable, i.e. realizable in a node in the
 parameter graph of a K , Σ , $\Pi\Sigma$ or $\Sigma\Pi\Sigma$ system.

305 4. Realizability Results

The main results are summarized in Table 1, where we consider pairs of $*$ -
 jointly realizable MBFs. The equality sign between two categories indicates that
 whenever a pair of MBFs with given set of inputs n (row index) is realizable
 in one category, it is also realizable in the other category. The strict subset
 310 relationship shows that any pair of functions realizable in the smaller category
 is also realizable in the larger category, and, furthermore, there is a pair of
 Boolean functions $f \prec g$ that is realizable in the larger category that is not

n	*-Joint Realizability of $f \prec g$				
1	Σ	$=$	$\Pi\Sigma$	$=$	$\Sigma\Pi\Sigma = K$
2	Σ	$=$	$\Pi\Sigma$	$=$	$\Sigma\Pi\Sigma = K$
3	Σ	$\subsetneq$	$\Pi\Sigma$	$\subsetneq$	$\Sigma\Pi\Sigma = K$
≥ 4	Σ	$\subsetneq$	$\Pi\Sigma$	$\subsetneq$	$\Sigma\Pi\Sigma \subsetneq K$

Table 1: Summary of the results. The n is the number of inputs for each of the pair of MBF, where $f \prec g$ means that the truth set of f is a subset of truth set of g (Definition 4.5). The (in)equalities express realizability relations among categories of functions (see text). Row 1 and 2 are a result of Proposition 4.22. Row 3 is a result of Subsections 4.4.1, 4.4.2, and 4.5. Row 4 is a result of Subsection 4.4.3.

realizable in the smaller category.

4.1. K -realizability

315 The main goal of this section is to prove the following two results: (1) any monotone Boolean function is K -realizable, and (2) any k -tuple $f_1 \prec \dots \prec f_k$ of MBFs is K -jointly realizable. Therefore K -systems are general enough to represent any collection of monotone Boolean functions.

Definition 4.1. For a Boolean function $f : \mathbb{B}^n \rightarrow \mathbb{B}$, we define the *truth set* of f as

$$\text{True}(f) := \{\vec{y} \in \mathbb{B}^n \mid f(\vec{y}) = 1\}.$$

Similarly we define the *false set* of f as

$$\text{False}(f) := \{\vec{y} \in \mathbb{B}^n \mid f(\vec{y}) = 0\}.$$

For $U \subseteq \mathbb{B}^n$ any subset, we will also use the notation

$$\text{True}(f)|_U := \{\vec{y} \in U \mid f(\vec{y}) = 1\}, \quad \text{False}(f)|_U := \{\vec{y} \in U \mid f(\vec{y}) = 0\}.$$

320 We start our discussion of K -realizability and K -joint realizability by proving two results relating K -realizability to the existence of what we call a *realizing function*.

Definition 4.2. The *positive monotonicity assumption* for a function $\mathcal{R} : \mathbb{B}^n \rightarrow \mathbb{R}_+$ is the following: for all $j \in \{1, \dots, n\}$, for all $\vec{y} \in \mathbb{B}^n$ with $y_j = 0$

$$\mathcal{R}(\vec{y}) \leq \mathcal{R}(\vec{y} + \hat{e}_j).$$

Next Theorem provides characterization of K -realizability and K -joint realizability of monotone Boolean functions.

Theorem 4.3.

325

1. $f \in \mathbf{MBF}^+(n)$ is K -realizable if, and only if, there exist

- a weighted regulatory network $\mathbf{RN}$ with a node i with only one target j and a weight θ_{ji} and
- a function $\mathcal{R}^{(i)}$ such that $\mathcal{R}^{(i)} : \mathbb{B}^n \rightarrow \mathbb{R}_+$ satisfies the positive monotonicity assumption and

$$f(\vec{y}) = \begin{cases} 1 & \text{if } \mathcal{R}^{(i)}(\vec{y}) > \theta_{ji} \\ 0 & \text{if } \mathcal{R}^{(i)}(\vec{y}) < \theta_{ji} \end{cases}.$$

2. A k -tuple of MBFs $f_1, f_2, \dots, f_k \in \mathbf{MBF}^+(n)$ is K -jointly realizable if, and only if, there exist

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- a weighted regulatory network $\mathbf{RN}$ with a node i with k targets $\ell_1, \dots, \ell_k$ and weights $\{\theta_{\ell_j i}\}$ and
- a function $\mathcal{R}^{(i)}$ such that $\mathcal{R}^{(i)} : \mathbb{B}^n \rightarrow \mathbb{R}_+$ satisfies the positive monotonicity assumption and for all $j \in \{1, \dots, k\}$, f_j can be expressed as

$$f_j(\vec{y}) = \begin{cases} 1 & \text{if } \mathcal{R}^{(i)}(\vec{y}) > \theta_{\ell_j i} \\ 0 & \text{if } \mathcal{R}^{(i)}(\vec{y}) < \theta_{\ell_j i} \end{cases}. \quad (15)$$

Proof. Using Definition 3.1, Equation (10), and setting

$$\mathcal{R}^{(i)}(\vec{y}) \equiv K_{i, \alpha_i}(\vec{y}), \quad \vec{y} \in \mathbb{B}^n,$$

the theorem follows. It remains to note that the positive monotonicity assumption on $\mathcal{R}^{(i)}$ induces the (positive) monotonicity condition on K_i from Definition 2.4. Likewise, when $f_1, \dots, f_k \in \mathbf{MBF}^+(n)$, then $S(i) = S^+(i)$ and K_i satisfying the monotonicity condition in Definition 2.4 implies that $\mathcal{R}^{(i)}$ must satisfy the positive monotonicity condition. $\square$

Definition 4.4. If Theorem 4.3 is satisfied, then the pair $(\mathcal{R}^{(i)}, \mathbf{RN})$ is called a *realizing function* and *realizing network* for $f_1, \dots, f_k$, respectively.

Theorem 4.3 shows that K -systems can be thought of as arising from monotone Boolean functions via realizing functions $\mathcal{R}^{(i)}$, one for each node i in a realizing network $\mathbf{RN}$. In the following, we will restrict our focus to a single node in $\mathbf{RN}$ and drop the superscript.

Definition 4.5. For two Boolean functions $f, g : \mathbb{B}^n \rightarrow \mathbb{B}$, we say f *implies* g and write $f \prec g$ if, and only if, $\text{True}(f) \subseteq \text{True}(g)$.

Now we prove the main result of this section, namely that all k -tuples of MBFs that are linearly ordered $f_1 \prec f_2 \prec \dots \prec f_k$ are K -(jointly) realizable for all $k \geq 1$.

Theorem 4.6.

1. $f \in \mathbf{MBF}^+(n)$ if and only if f is K -realizable.
2. A collection $f_1, \dots, f_b \in \mathbf{MBF}^+(n)$ of monotone Boolean functions has a linear order $f_1 \prec f_2 \prec \dots \prec f_b$ if and only if it is K -jointly realizable.

Proof. Since realizability is a special case of joint realizability and since a single MBF trivially has an order, it is sufficient to prove the second point.

($\Rightarrow$) Let $\mathcal{R}(\vec{y}) := \sum_{j=1}^b f_j(\vec{y})$. $\mathcal{R}$ satisfies the positive monotonicity assumption of Definition 4.2, since if $y_i = 0$ for some $\vec{y}$, then $\mathcal{R}(\vec{y}) \leq \mathcal{R}(\vec{y} + \hat{e}_i)$ by summation and the positivity of f_j . Now for each $j \in \{1, \dots, b\}$, let $\theta_j = b - j + \frac{1}{2}$. Suppose $\vec{y} \in \text{True}(f_j)$. Then $\vec{y} \in \text{True}(f_k)$ for $k = j, \dots, b$, since $\text{True}(f_j) \subseteq \text{True}(f_k)$ by the $\prec$ relationship. So $\mathcal{R}(\vec{y}) = b - j + 1$ and we

360 have $\theta_j = b - j + 1/2 < \mathcal{R}(\vec{y})$ as desired. Then the function $\mathcal{R}$ and thresholds $\theta_1, \dots, \theta_b$ satisfy the assumptions of Theorem 4.3.

($\Leftarrow$) Given the b thresholds, establish the indexing $\theta_b < \dots < \theta_1$ using the order of $\mathbb{R}$. Then for any $i < j$, we have $\theta_i > \theta_j$ and $(\theta_i, \infty) \subseteq (\theta_j, \infty)$. Given the realizing function $\mathcal{R}$, define a collection of b positive monotone Boolean functions by

$$\text{True}(f_i) = \mathcal{R}^{-1}(\theta_i, \infty).$$

Then by construction

$$f_i(\vec{y}) = \begin{cases} 1 & \text{if } \mathcal{R}(\vec{y}) > \theta_i \\ 0 & \text{if } \mathcal{R}(\vec{y}) < \theta_i \end{cases}.$$

Moreover, if $i < j$, we have $\text{True}(f_i) \subset \text{True}(f_j)$, implying $f_i \prec f_j$. $\square$

4.2. *-Realizability

In this section we discuss technical points needed later for Σ , $\Pi\Sigma$, and $\Sigma\Pi\Sigma$ realizability.

Since Σ , $\Pi\Sigma$, and $\Sigma\Pi\Sigma$ realizability are based on Λ functions, $\Lambda : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$, we need to consider a restricted class of realizing functions of the form

$$\mathcal{R} := \Lambda \circ \phi,$$

where $\phi : \mathbb{B}^n \rightarrow \mathbb{R}_+^n$ component-wise monotonically encodes a Boolean vector into a real valued vector, i.e. $\phi = (\phi_1, \dots, \phi_n)$, where $\phi_i : \mathbb{B} \rightarrow \mathbb{R}_+$, $\phi_i(0) < \phi_i(1)$ and Λ is an algebraic function that belongs to one of the classes $\Sigma(n) \subsetneq \Pi\Sigma(n) \subsetneq \Sigma\Pi\Sigma(n)$.

370 The following Lemma is a direct consequence of Theorem 4.3, the definition of the classes of algebraic functions $\Sigma(n)$, $\Pi\Sigma(n)$ and $\Sigma\Pi\Sigma(n)$, and the previously made observation that all switching systems are K -systems.

Lemma 4.7. *In the following, $*$ could be Σ , $\Pi\Sigma$, or $\Sigma\Pi\Sigma$. A function $f \in \mathbf{MBF}^+(n)$ is $*$ -realizable if, and only if, there exist a realizing network $\mathbf{RN}$ and realizing function $\mathcal{R} = \Lambda \circ \phi$, where*

1. the $*$ -interaction function $\Lambda : \mathbb{R}^n \rightarrow \mathbb{R}$ belongs to the class $\Lambda \in \Sigma\Pi\Sigma(n)$, $\Lambda \in \Pi\Sigma(n)$, or $\Lambda \in \Sigma(n)$ if $*$ = $\Sigma\Pi\Sigma$, $*$ = $\Pi\Sigma$, or $*$ = Σ respectively;
and
2. for each $i \in \{1, \dots, n\}$, the function $\phi_i : \mathbb{B} \rightarrow \mathbb{R}_+$ satisfies $\phi_i(0) < \phi_i(1)$.

380 Similarly, a k -tuple of MBFs $f_1, f_2, \dots, f_k \in \mathbf{MBF}^+(n)$ is $*$ -jointly realizable if and only if there exist a realizing network $\mathbf{RN}$ and a realizing function $\mathcal{R} = \Lambda \circ \phi$ for Λ a $*$ -interaction function from (1) and ϕ a map satisfying (2).

The general question of which k -tuples of Boolean functions are $*$ -jointly realizable seems very difficult and is likely connected to fundamental problems
385 in algebraic geometry. We focus here on some initial results for $k = 2$ and will consider pairs of Boolean functions with different numbers of inputs. We start with definitions and results that enumerate consequences of joint realizability of $f \prec g$ on relationships between **True** and **False** sets on subsets of the space of Boolean inputs.

390 In the following, and many times throughout the rest of the manuscript, it will be useful to view $\mathbb{B}^n$ as a hypercube embedded in $\mathbb{R}_+^n$ with side lengths of 1. This gives rise to a geometrical structure of $\mathbb{B}^n$, where if $\vec{y} = (y_1, \dots, y_{i-1}, 0, y_{i+1}, \dots, y_n)$, then $\vec{y} + \hat{e}_i = (y_1, \dots, y_{i-1}, 1, y_{i+1}, \dots, y_n)$, where $\hat{e}_i$ is the standard i -th basis vector in $\mathbb{R}^n$. This defines the geometrical idea of *floor* and *ceiling* in the i -th
395 direction of the hypercube $\mathbb{B}^n$. When $f \prec g$, there are relationships between the **True** and **False** sets on the floors and ceilings in all directions.

Definition 4.8. We define the *ceiling (of $\mathbb{B}^n$) in the i -th normal direction* as the set

$$\mathbf{C}_i := \{(y_1, y_2, \dots, y_n) \in \mathbb{B}^n \mid y_i = 1\}$$

and similarly we define the *floor (of $\mathbb{B}^n$) in the i -th normal direction* as the set

$$\mathbf{F}_i := \{(y_1, y_2, \dots, y_n) \in \mathbb{B}^n \mid y_i = 0\}.$$

Notice that $\mathbf{C}_i$ and $\mathbf{F}_i$ are both hypercubes of dimension $(n - 1)$, that $\mathbb{B}^n = \mathbf{C}_i \cup \mathbf{F}_i$, and that $\mathbf{C}_i = \mathbf{F}_i + \hat{e}_i$.

Next we define the idea of a *collapse*, in which a floor and ceiling of $\mathbb{B}^n$ are
 400 considered objects embedded in the hypercube $\mathbb{B}^{n-1}$.

Definition 4.9. For a given $i \in \{1, \dots, n\}$, we define the i -th *collapse* as the
 function $\text{Col}_i : \mathbb{B}^n \rightarrow \mathbb{B}^{n-1}$ which removes the i -th coordinate, defined as

$$\text{Col}_i((y_1, \dots, y_n)) := (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$$

Then for any subset $U \subset \mathbb{B}^n$ we have

$$\text{Col}_i(U) = \{(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n) \in \mathbb{B}^{n-1} \mid (y_1, \dots, y_n) \in U\}.$$

Using the notions of floor, ceiling, and collapse, we move through a series
 of results that are critical to future proofs involving $\ast$ -joint realizability for Λ
 function classes. The following Lemma shows the relationship between truth
 sets of any monotone Boolean function on the floor and ceiling in any direction
 405 of the Boolean cube.

Lemma 4.10. *If $f \in \mathbf{MBF}^+(n)$, then for all $i \in \{1, \dots, n\}$,*

$$\text{Col}_i(\text{True}(f)|_{\mathbf{F}_i}) \subseteq \text{Col}_i(\text{True}(f)|_{\mathbf{C}_i})$$

Proof. Observe that the hypercube can be viewed as a distributive lattice via
 the relation $\leq$ on the corners of the hypercube by

$$\vec{y} \leq \vec{z} \in \mathbb{B}^n \iff (y_i = 1 \Rightarrow z_i = 1) \text{ for all } i = 1, \dots, n.$$

Notice that since any $f \in \mathbf{MBF}^+(n)$ is positive monotone, $\text{True}(f)$ is an upper-
 set of $\mathbb{B}^n$ viewed as a lattice. Therefore, for any $\vec{y} \in \text{True}(f)|_{\mathbf{F}_i}$, we have $\vec{y} + \hat{e}_i \in$
 $\text{True}(f)|_{\mathbf{C}_i}$. Under the collapse operation, we have $\text{Col}_i(\vec{y}) = \text{Col}_i(\vec{y} + \hat{e}_i)$, com-
 pleting the proof. $\square$

410 The next Lemma compares truth sets of a pair of monotone Boolean func-
 tions where both are evaluated on the same subset of the Boolean hypercube.

Proposition 4.11. *If $f, g \in \mathbf{MBF}^+(n)$ and $f \prec g$, then for all $i \in \{1, \dots, n\}$,*

$$\text{Col}_i(\text{True}(f)|_{\mathbf{F}_i}) \subseteq \text{Col}_i(\text{True}(g)|_{\mathbf{F}_i}) \text{ and } \text{Col}_i(\text{True}(f)|_{\mathbf{C}_i}) \subseteq \text{Col}_i(\text{True}(g)|_{\mathbf{C}_i}). \quad (16)$$

Proof. Recall that $f \prec g$ implies $\text{True}(f) \subseteq \text{True}(g)$, which implies that $\text{True}(f)|_{\mathbf{F}_i} \subseteq \text{True}(g)|_{\mathbf{F}_i}$ and that $\text{True}(f)|_{\mathbf{C}_i} \subseteq \text{True}(g)|_{\mathbf{C}_i}$. Since in both cases, the collapse operation occurs over the same set, we have (16) as desired. $\square$

415 Finally, we arrive at a comparison of truth sets of a pair of compatible monotone Boolean functions on different subset of the hypercube.

Proposition 4.12. *If $f, g \in \mathbf{MBF}^+(n)$ and $f \prec g$, then for all $i \in \{1, \dots, n\}$,*

$$\text{Col}_i(\text{True}(f)|_{\mathbf{F}_i}) \subseteq \text{Col}_i(\text{True}(g)|_{\mathbf{C}_i})$$

Proof. By Lemma 4.10 we see have $\text{Col}_i(\text{True}(f)|_{\mathbf{F}_i}) \subseteq \text{Col}_i(\text{True}(f)|_{\mathbf{C}_i})$, and by Proposition 4.11 we have $\text{Col}_i(\text{True}(f)|_{\mathbf{C}_i}) \subseteq \text{Col}_i(\text{True}(g)|_{\mathbf{C}_i})$, which completes the proof. $\square$

420 The following two technical results for special forms of Λ functions are used extensively in the coming sections to derive the results in Table 1. Proofs for Theorem 4.14 and Theorem 4.15 are found in Appendix A.

Definition 4.13. Let Λ be a $*$ -interaction function. We say z_i is a *factor* (of Λ) if there is another map Λ' that does not depend on z_i such that $\Lambda = z_i \Lambda'$.
425 Similarly we say z_i is a *simple term* (of Λ) if we can write Λ as $\Lambda = z_i + \Lambda'$. Notice if Λ is a Σ -interaction function, for all $i \in \{1, \dots, n\}$, z_i is a simple term.

The following theorem shows that the truth sets of jointly realizable monotone Boolean functions evaluated along the directions that correspond to factors or simple terms have special properties.

Theorem 4.14. *Let $*$ be Σ , $\Pi\Sigma$, or $\Sigma\Pi\Sigma$. Let $f, g \in \mathbf{MBF}^+(n)$, with $f \prec g$, be $*$ -jointly realizable. Let $(\Lambda \circ \phi, \mathbf{RN})$ $*$ -jointly realize (f, g) . For each $\ell \in \{1, \dots, n\}$, if z_ℓ is a factor or a simple term of Λ , then*

$$\text{Col}_\ell(\text{True}(f)|_{\mathbf{C}_\ell}) \subseteq \text{Col}_\ell(\text{True}(g)|_{\mathbf{F}_\ell}), \text{ or} \quad (17)$$

$$\text{Col}_\ell(\text{True}(f)|_{\mathbf{C}_\ell}) \supseteq \text{Col}_\ell(\text{True}(g)|_{\mathbf{F}_\ell}) \quad (18)$$

430 The final theorem of this Section shows that each pair of jointly realizable monotone Boolean functions with n inputs gives rise to a pair of jointly realizable monotone Boolean function with $n-1$ inputs associated to the floor and ceiling.

Theorem 4.15. *Let $*$ be Σ , $\Pi\Sigma$, or $\Sigma\Pi\Sigma$. Let $n > 1$. Let $f, g \in \mathbf{MBF}^+(n)$ be $*$ -jointly realizable MBFs on $\mathbb{B}^n$ with $f \prec g$. Let $\ell \in \{1, \dots, n\}$ and $U \in \{\mathbf{F}_\ell, \mathbf{C}_\ell\}$. Then*

a) the functions $f'_U, g'_U \in \mathbf{MBF}^+(n-1)$ defined by

$$\text{True}(f'_U) = \text{Co1}_\ell(\text{True}(f)|_U) \text{ and } \text{True}(g'_U) = \text{Co1}_\ell(\text{True}(g)|_U)$$

are $*$ -jointly realizable MBFs on $\mathbb{B}^{n-1}$, and

b) there is a single $*$ -interaction function Λ' along with maps $\phi_{\mathbf{C}_\ell}, \phi_{\mathbf{F}_\ell}$ and weighted regulatory networks $\mathbf{RN}_{\mathbf{C}_\ell}$ and $\mathbf{RN}_{\mathbf{F}_\ell}$ such that $(\Lambda' \circ \phi_{\mathbf{F}_\ell}, \mathbf{RN}_{\mathbf{F}_\ell})$ $*$ -jointly realizes $(f'_{\mathbf{F}_\ell}, g'_{\mathbf{F}_\ell})$ and $(\Lambda' \circ \phi_{\mathbf{C}_\ell}, \mathbf{RN}_{\mathbf{C}_\ell})$ $*$ -jointly realizes $(f'_{\mathbf{C}_\ell}, g'_{\mathbf{C}_\ell})$.

440 4.3. Joint realizability in $\mathbb{B}^n$ and realizability in $\mathbb{B}^{n+1}$

We will show that there is a bijection η between pairs $(f, g) \in \mathbf{MBF}^+(n) \times \mathbf{MBF}^+(n)$ satisfying $f \prec g$ and $h \in \mathbf{MBF}^+(n+1)$. We use η to relate the $*$ -joint realizability of a pair $f \prec g$ and the $*$ -realizability of single function $\eta(f, g) \in \mathbf{MBF}^+(n+1)$. We will use this fact at the end of the section to prove

445 rows 1 and 2 of Table 1.

Definition 4.16. Define the map

$$\eta : \{(f, g) \mid f, g \in \mathbf{MBF}^+(n) \text{ and } f \prec g\} \rightarrow \mathbf{MBF}^+(n+1)$$

by $h = \eta(f, g)$, where for $\vec{y} \in \mathbb{B}^{n+1}$,

$$h(\vec{y}) = \begin{cases} f(y_1, \dots, y_n) & \text{if } y_{n+1} = 0 \\ g(y_1, \dots, y_n) & \text{if } y_{n+1} = 1 \end{cases}.$$

Lemma 4.17. *The map η is a bijection onto $\mathbf{MBF}^+(n+1)$.*

Proof. First we describe the range of η . Let $f, g \in \mathbf{MBF}^+(n)$ with $f \prec g$. By definition f and g describe the floor and ceiling of $\eta(f, g) = h$ in the $(n+1)$ -th direction,

$$\text{True}(f) = \text{Col}_{n+1} \left(\text{True}(h)|_{\mathbf{F}_{n+1}} \right) \text{ and } \text{True}(g) = \text{Col}_{n+1} \left(\text{True}(h)|_{\mathbf{C}_{n+1}} \right). \quad (19)$$

Then the positive monotonicity of f and g induce positive monotonicity on the floor and ceiling of h , and $f \prec g$ gives positive monotonicity in the $(n+1)$ -th direction. So the range of η is contained in $\mathbf{MBF}^+(n+1)$.

450 It is clear that η is injective. Indeed, if $\eta(f, g) = \eta(f', g')$ then it follows immediately from the definition that $f = f'$ and $g = g'$.

To show that η is surjective, consider $h \in \mathbf{MBF}^+(n+1)$ and define f and g by setting (19) to be true. Since h satisfies positive monotonicity on its floor and ceiling, $f, g \in \mathbf{MBF}^+(n)$. Also, since

$$\text{Col}_{n+1} \left(\text{True}(h)|_{\mathbf{F}_{n+1}} \right) \subseteq \text{Col}_{n+1} \left(\text{True}(h)|_{\mathbf{C}_{n+1}} \right),$$

by Lemma 4.10, we have that $f \prec g$. We have then constructed the desired pair (f, g) with $f \prec g$ such that $\eta(f, g) = h$ is well-defined. $\square$

As a consequence of this result, note that $\eta^{-1}(h) = (f, g)$ is well defined.

455 **Theorem 4.18.** *Let $n \geq 1$. Suppose $f \prec g$ is $*$ -jointly realizable, where $f, g \in \mathbf{MBF}^+(n)$. Then $h = \eta(f, g)$ is $*$ -realizable.*

Proof. Suppose $f \prec g$ is $*$ -jointly realizable. Then by Lemma 4.7 there exists $(\Lambda \circ \phi, \mathbf{RN})$ that $*$ -jointly realizes f and g . Moreover, the proof of Theorem 4.6 tells us that the thresholds in $\mathbf{RN}$ associated to these functions satisfy $\theta_g < \theta_f$.

460 We seek to construct $(\Lambda' \circ \phi', \mathbf{RN}')$ that $*$ -realizes h . To build $\mathbf{RN}'$, we take $\mathbf{RN}$ and add a source to the node under consideration from any other node in the network. It remains to discover the weight, θ' of that edge.

Case 1: ($*$ = Σ or $*$ = $\Sigma\Pi\Sigma$). By the assumption since $f \prec g$ is $*$ -jointly

realizable, we have

$$f(y_1, \dots, y_n) = \begin{cases} 1 & \text{if } \Lambda(\phi_1(y_1), \dots, \phi_n(y_n)) > \theta_f \\ 0 & \text{if } \Lambda(\phi_1(y_1), \dots, \phi_n(y_n)) < \theta_f \end{cases}$$

$$g(y_1, \dots, y_n) = \begin{cases} 1 & \text{if } \Lambda(\phi_1(y_1), \dots, \phi_n(y_n)) > \theta_g \\ 0 & \text{if } \Lambda(\phi_1(y_1), \dots, \phi_n(y_n)) < \theta_g \end{cases}$$

For $\vec{y} \in \mathbb{B}^{n+1}$, we assign $\Lambda' \circ \phi' = \Lambda \circ \phi + \phi'_{n+1}$ for $\phi' = (\phi_1, \dots, \phi_n, \phi'_{n+1})$ with some choice of ϕ'_{n+1} . We know $f(y_1, \dots, y_n) = h(y_1, \dots, y_n, 0)$ and $g(y_1, \dots, y_n) = h(y_1, \dots, y_n, 1)$, so whatever ϕ'_{n+1} and θ' we choose must satisfy

$$f(y_1, \dots, y_n) = h(y_1, \dots, y_n, 0) = \begin{cases} 1 & \text{if } \Lambda(\phi_1(y_1), \dots, \phi_n(y_n)) + \phi'_{n+1}(0) > \theta' \\ 0 & \text{if } \Lambda(\phi_1(y_1), \dots, \phi_n(y_n)) + \phi'_{n+1}(0) < \theta' \end{cases}$$

and

$$g(y_1, \dots, y_n) = h(y_1, \dots, y_n, 1) = \begin{cases} 1 & \text{if } \Lambda(\phi_1(y_1), \dots, \phi_n(y_n)) + \phi'_{n+1}(1) > \theta' \\ 0 & \text{if } \Lambda(\phi_1(y_1), \dots, \phi_n(y_n)) + \phi'_{n+1}(1) < \theta' \end{cases}$$

Consider the assignment $\phi'_{n+1}(0) = \epsilon$, $\phi'_{n+1}(1) = \theta_f + \epsilon - \theta_g$, and $\theta' := \theta_f + \epsilon$, where ϵ is any sufficiently small real number $0 < \epsilon < \theta_f - \theta_g$. It is easy to check
465 that with this assignment, $h(y_1, \dots, y_n, 0) = 1$ if and only if $f(y_1, \dots, y_n) = 1$ and that $h(y_1, \dots, y_n, 1) = 1$ if and only if $g(y_1, \dots, y_n) = 1$. This completes the construction of $(\Lambda' \circ \phi', \mathbf{RN}')$ that $*$ -realizes h .

Case 2: ($*$ = $\Pi\Sigma$) The proof proceeds analogously with Case 1, where the only difference is replacement of a simple term by a factor in Λ' . It is easy to verify that the following assignments $*$ -realize h : $\phi' = (\phi_1, \dots, \phi_n, \phi'_{n+1})$,

$$\Lambda' \circ \phi'(y_1, \dots, y_n, y_{n+1}) = \phi'_{n+1}(y_{n+1}) \cdot (\Lambda \circ \phi(y_1, \dots, y_n)) ,$$

$$\theta' = \theta_f, \phi'_{n+1}(0) = 1, \phi_{n+1}(1) = \theta_f / \theta_g. \quad \square$$

We do not know if the converse of Theorem 4.18 is true in general. However,
470 with an additional constraint we obtain the following partial converse.

Theorem 4.19. Let $n \geq 1$. Suppose $(\Lambda \circ \phi, \mathbf{RN})$ $*$ -realizes $h \in \mathbf{MBF}(n+1)$. If z_i is a factor or simple term of Λ , then $f, g \in \mathbf{MBF}^+(n)$ defined by

$$\text{True}(f) = \text{Col}_i(\text{True}(h)|_{\mathbf{F}_i}), \quad \text{True}(g) = \text{Col}_i(\text{True}(h)|_{\mathbf{C}_i})$$

are $*$ -jointly realizable.

Proof. Without loss of generality assume $i = n+1$. Let θ be the threshold associated to the realization of h .

Case 1: (z_{n+1} is a factor) Then $\Lambda = z_{n+1}\Lambda'$ and from Lemma 4.7

$$h(y_1, \dots, y_{n+1}) = \begin{cases} 1 & \text{if } \phi_{n+1}(y_{n+1})\Lambda'(\phi_1(y_1), \dots, \phi_n(y_n)) > \theta \\ 0 & \text{if } \phi_{n+1}(y_{n+1})\Lambda'(\phi_1(y_1), \dots, \phi_n(y_n)) < \theta \end{cases}$$

If we restrict our attention to f , dividing by $\phi_{n+1}(0)$, the above equation gives us

$$f(y_1, \dots, y_n) = h(y_1, \dots, y_n, 0) = \begin{cases} 1 & \text{if } \Lambda'(\phi_1(y_1), \dots, \phi_n(y_n)) > \theta/\phi_{n+1}(0) \\ 0 & \text{if } \Lambda'(\phi_1(y_1), \dots, \phi_n(y_n)) < \theta/\phi_{n+1}(0) \end{cases}.$$

Restricting our attention to g we see that

$$g(x_1, \dots, x_n) = h(x_1, \dots, x_n, 1) = \begin{cases} 1 & \text{if } \Lambda'(\phi_1(x_1), \dots, \phi_n(x_n)) > \theta/\phi_{n+1}(1) \\ 0 & \text{if } \Lambda'(\phi_1(x_1), \dots, \phi_n(x_n)) < \theta/\phi_{n+1}(1). \end{cases}$$

Construct $\mathbf{RN}'$ by removing the source edge associated to $n+1$ and adding
 475 one target edge to the node under consideration. Assign to one target edge the weight $\theta_f = \theta/\phi_{n+1}(0)$ and assign $\theta_g = \theta/\phi_{n+1}(1)$ to the other. After setting $\phi' = (\phi_1, \dots, \phi_n)$, we obtain $(\Lambda' \circ \phi', \mathbf{RN}')$ that $*$ -jointly realizes (f, g) .

Case 2: (z_{n+1} is a simple term) The argument for this case is similar but instead of dividing by $\phi_{n+1}(y_{n+1})$, we will subtract. Specifically, since y_{n+1} is a simple term,

$$h(y_1, \dots, y_{n+1}) = \begin{cases} 1 & \text{if } \phi_{n+1}(y_{n+1}) + \Lambda'(\phi_1(y_1), \dots, \phi_n(y_n)) > \theta \\ 0 & \text{if } \phi_{n+1}(y_{n+1}) + \Lambda'(\phi_1(y_1), \dots, \phi_n(y_n)) < \theta \end{cases}$$

and so

$$f(y_1, \dots, y_n) = h(y_1, \dots, y_n, 0) = \begin{cases} 1 & \text{if } \Lambda'(\phi_1(y_1), \dots, \phi_n(y_n)) > \theta - \phi_{n+1}(0) \\ 0 & \text{if } \Lambda'(\phi_1(y_1), \dots, \phi_n(y_n)) < \theta - \phi_{n+1}(0) \end{cases}$$

$$g(y_1, \dots, y_n) = h(y_1, \dots, y_n, 1) = \begin{cases} 1 & \text{if } \Lambda'(\phi_1(y_1), \dots, \phi_n(y_n)) > \theta - \phi_{n+1}(1) \\ 0 & \text{if } \Lambda'(\phi_1(y_1), \dots, \phi_n(y_n)) < \theta - \phi_{n+1}(1) \end{cases}$$

Construct $\mathbf{RN}'$ as before with threshold assignments $\theta_f = \max\{0, \theta - \phi_{n+1}(0)\}$, $\theta_g = \max\{0, \theta - \phi_{n+1}(1)\}$, and a further perturbation by small enough $\epsilon > 0$ if
480 $\theta_f = \theta_g$. Then tuple $(\Lambda' \circ \phi', \mathbf{RN}')$ $*$ -jointly realizes (f, g) . $\square$

The following Corollary is an immediate result of Theorems 4.18 and 4.19. It states that, in the Σ class of functions, joint realizability of a pair (f, g) in dimension n is equivalent to the realizability of $\eta(f, g)$ in dimension $n + 1$, since every term in Λ is simple.

485 **Corollary 4.20.** *Let $n \geq 1$. Suppose $f \prec g$ and let $h = \eta(f, g)$. Then (f, g) is Σ -jointly realizable if and only if h is Σ -realizable.*

As promised, we now show the equivalence of threshold (linearly separable) functions and Σ -realizability, see Definition 3.3.

Lemma 4.21. *Let $f \in \mathbf{MBF}^+(n)$.*

- 490 1. *If f is Σ -realizable then f is a threshold function.*
2. *If f is a threshold function with separating structure $(a_1, \dots, a_n, \theta')$ such that $a_1, \dots, a_n \geq 0$ and $\theta' > -n$, then f is Σ -realizable.*

Proof. (1) Suppose f is Σ -realizable, and let $(\Lambda \circ \phi, \mathbf{RN})$ Σ -realize f . We construct $a_1, \dots, a_n$ and θ' as in the sense of Definition 3.3 as follows: set
495 $a_i = \phi_i(1) - \phi_i(0)$, and let $\theta' = \max\{0, \theta - (\phi_1(0) + \dots + \phi_n(0))\}$.

(2) Now suppose f is a threshold function with separating structure $(a_1, \dots, a_n, \theta')$ such that $a_1, \dots, a_n \geq 0$ and $\theta' > -n$. Set $\Lambda = z_1 + \dots + z_n$. Set $\phi_i(0) = 1$, $\phi_i(1) = 1 + a_i$, and $\theta = \theta' + n$, to obtain the desired $(\Lambda \circ \phi, \theta)$. $\square$

The following proposition, together with Remark 3.2, proves the first two
500 rows of Table 1.

Proposition 4.22. *Assume $f \prec g \in \mathbf{MBF}^+(n)$ with $n = 1$ or $n = 2$. Then the pair (f, g) is Σ -realizable.*

Proof. By simple enumeration, one can check that, for $n = 1, 2, 3$, all functions in $\mathbf{MBF}^+(n)$ are threshold functions, and admit separating structures with
505 $a_1, \dots, a_n, \theta > 0$. Via Lemma 4.21, these functions are Σ -realizable. This fact, when combined with Corollary 4.20, proves the first two rows of Table 1. $\square$

4.4. Strict subset relations in Table 1.

This section contains a series of examples illustrating the differences between Σ , $\Pi\Sigma$, and $\Sigma\Pi\Sigma$ realizability, proving some of the strict subset results
510 in Table 1. We will use Theorems 4.14 and 4.15 extensively.

The idea behind all of the examples is to show that there exists a pair of $*$ -jointly realizable functions (f, g) that are not $*'$ -jointly realizable, where $*'$ is a more restrictive class than $*$. The proofs are inductive, with different base case constructions and very similar inductive steps. The methodology for the induction is to take an (f, g) $*$ -jointly realizable, but not $*'$ -jointly realizable, pair in $\mathbb{B}^n$ and to set (f, g) to be the floors of new $(\tilde{f}, \tilde{g})$ functions in $\mathbb{B}^{n+1}$. It then remains to construct ceilings that ensure $\tilde{f}, \tilde{g} \in \mathbf{MBF}^+(n+1)$. For any $\vec{y} \in \mathbf{C}_{n+1}$, we choose to set $\tilde{f}(\vec{y}) = \tilde{g}(\vec{y}) = 1$. By this choice,

$$\text{True}(\tilde{f}) \supseteq \mathbf{C}_{n+1}, \quad \text{True}(\tilde{g}) \supseteq \mathbf{C}_{n+1},$$

which ensures that $\tilde{f}, \tilde{g} \in \mathbf{MBF}^+(n+1)$. By the contrapositive of Theorem 4.15 (a), this construction ensures that $(\tilde{f}, \tilde{g})$ are not $*'$ -jointly realizable. We then show that $(\tilde{f}, \tilde{g})$ are $*$ -jointly realizable.

4.4.1. Σ -jointly realizable $\subsetneq \Pi\Sigma$ -jointly realizable for $n \geq 3$.

515 In this section we prove the first strict inclusion in the third and fourth rows of Table 1.

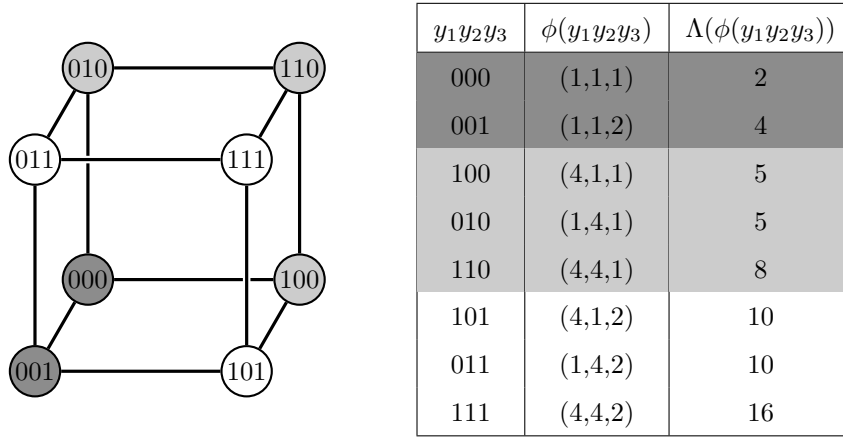


Figure 6: Left: An example pair $f, g : \mathbb{B}^3 \rightarrow \mathbb{B}$ with $f \prec g$. Dark grey is $\text{False}(f) \cap \text{False}(g)$, light grey is $\text{False}(f) \cap \text{True}(g)$, and white is $\text{True}(f) \cap \text{True}(g)$. Nodes are labels with $y_1y_2y_3$. The pair (f, g) are $\Pi\Sigma$ -jointly realizable, but not Σ -jointly realizable. Right: A table of values proving (f, g) are $\Pi\Sigma$ -jointly realizable. Here $\Lambda = (z_1 + z_2)z_3$ and $\theta_1 = 4.5$, $\theta_2 = 9$. The coloring in the table column is consistent with vertex coloring on the left.

Lemma 4.23. *Let $n \geq 3$. There exists a pair $f \prec g \in \mathbf{MBF}^+(n)$ such that (f, g) is not Σ -jointly realizable, but is $\Pi\Sigma$ -jointly realizable.*

Proof. We first construct an explicit pair (f, g) for $n = 3$. Consider the pair
520 $f \prec g$ of MBFs depicted on the left of Figure 6. We use Theorem 4.14 to show that (f, g) is not Σ -jointly realizable, and provide an explicit $(\Lambda \circ \phi, \mathbf{RN})$ that $\Pi\Sigma$ -jointly realizes (f, g) . Choose any $\mathbf{RN}$ with a node with three sources and two targets, with threshold values to be determined.

First, we illustrate the use of Theorem 4.14. Observe that $\text{True}(f)|_{\mathbf{C}_1} = \{101, 111\}$, and so $\text{Col}_1(\text{True}(f)|_{\mathbf{C}_1}) = \{01, 11\}$. Similarly $\text{True}(g)|_{\mathbf{F}_1} = \{010, 011\}$, so $\text{Col}_1(\text{True}(g)|_{\mathbf{F}_1}) = \{10, 11\}$. Therefore, we can see that

$$\text{Col}_1(\text{True}(f)|_{\mathbf{C}_1}) \not\subseteq \text{Col}_1(\text{True}(g)|_{\mathbf{F}_1}) \text{ and } \text{Col}_1(\text{True}(f)|_{\mathbf{C}_1}) \not\supseteq \text{Col}_1(\text{True}(g)|_{\mathbf{F}_1}).$$

By the contrapositive of Theorem 4.14, we see that if $(\Lambda \circ \phi, \mathbf{RN})$ *-jointly
525 realizes (f, g) , then Λ cannot have a simple term or factor z_1 . Therefore, (f, g) is not Σ -jointly realizable, as any Σ -interaction function Λ has every variable as a simple term.

However, (f, g) is $\Pi\Sigma$ -jointly realizable. To see this, set $\Lambda = (z_1 + z_2)z_3$,
 $\phi_1(0) = \phi_2(0) = \phi_3(0) = 1$, $\phi_1(1) = 4$, $\phi_2(1) = 4.1$, $\phi_3(1) = 2$, and $\theta_1 = 4.5$,
530 $\theta_2 = 9$. The results of such an assignment are displayed in the table in Figure 6.

We now prove the inductive step. Let $n \geq 3$. Assume there exists $f, g : \mathbb{B}^n \rightarrow \mathbb{B}$ such that (f, g) is not Σ -jointly realizable, but is $\Pi\Sigma$ -jointly realizable. Let $(\Lambda \circ \phi, \mathbf{RN})$ with thresholds θ_1 and θ_2 $\Pi\Sigma$ -jointly realize (f, g) . Now over $\mathbb{B}^{n+1}$ define

$$\tilde{f}(y_1 \dots y_{n+1}) := \begin{cases} f(y_1 \dots y_n) & y_{n+1} = 0 \\ 1 & y_{n+1} = 1 \end{cases}$$

$$\tilde{g}(y_1 \dots y_{n+1}) := \begin{cases} g(y_1 \dots y_n) & y_{n+1} = 0 \\ 1 & y_{n+1} = 1 \end{cases}$$

Observe that

$$\text{Col}_{n+1}(\text{True}(\tilde{f})|_{\mathbf{F}_{n+1}}) = \text{True}(f), \quad \text{Col}_{n+1}(\text{True}(\tilde{g})|_{\mathbf{F}_{n+1}}) = \text{True}(g),$$

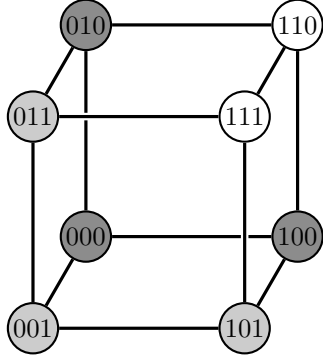
in other words the floor of $\tilde{f}$ has the same truth set as f and the floor of $\tilde{g}$ has the same truth set as g . Since (f, g) are not Σ -jointly realizable, the contrapositive of Theorem 4.15 (a) tells us that $(\tilde{f}, \tilde{g})$ are not Σ -jointly realizable.
535 Let $m = \min\{\Lambda(\phi(\mathbb{B}^n))\}$ and define $\tilde{\phi}_{n+1}(0) = 1$ and $\tilde{\phi}_{n+1}(1) = \max\{2, C\}$, where C is large enough such that $mC > \max\{\theta_1, \theta_2\}$. Define $\tilde{\Lambda} = z_{n+1}\Lambda$ and $\tilde{\phi} := (\phi_1, \dots, \phi_n, \tilde{\phi}_{n+1})$. Then $\tilde{\Lambda}$ is a valid $\Pi\Sigma$ -interaction function, and $(\tilde{\Lambda} \circ \tilde{\phi}, \mathbf{RN})$ $\Pi\Sigma$ -jointly realizes $(\tilde{f}, \tilde{g})$, completing the proof. $\square$

4.4.2. $\Pi\Sigma$ -jointly realizable $\subsetneq \Sigma\Pi\Sigma$ -jointly realizable for $n \geq 3$

540 In this section we prove the second strict inclusion in the third and fourth rows of Table 1.

Lemma 4.24. *Let $n \geq 3$. There exists a pair $f \prec g \in \mathbf{MBF}^+(n)$ such that (f, g) is not $\Pi\Sigma$ -jointly realizable, but is $\Sigma\Pi\Sigma$ -jointly realizable.*

Proof. Again, we construct an explicit pair (f, g) for $n = 3$. Consider the pair
545 $f \prec g$ shown in Figure 7. We will show that (f, g) is not $\Pi\Sigma$ -jointly realizable.



$y_1y_2y_3$	$\phi(y_1y_2y_3)$	$\Lambda(\phi(y_1y_2y_3))$
000	(1,1,1)	2
100	(3,1,1)	4
010	(1,3,1,1)	4.1
001	(1,1,4)	5
101	(3,1,4)	7
011	(1,3,1,4)	7.1
110	(3,3,1,1)	10.3
111	(3,3,1,4)	13.3

Figure 7: An example pair $f, g : \mathbb{B}^3 \rightarrow \mathbb{B}$ with $f \prec g$. Dark grey is $\text{False}(f) \cap \text{False}(g)$, light grey is $\text{False}(f) \cap \text{True}(g)$, and white is $\text{True}(f) \cap \text{True}(g)$. Nodes are labels with $y_1y_2y_3$. The pair (f, g) is $\Sigma\Pi\Sigma$ -jointly realizable, but not $\Pi\Sigma$ -jointly realizable. Right: A table of values proving (f, g) are $\Sigma\Pi\Sigma$ -jointly realizable. Here $\Lambda = z_1z_2 + z_3$ and $\theta_1 = 4.5$, $\theta_2 = 9$. The coloring in the rightmost column is consistent with vertex coloring on the left.

Suppose, by way of contradiction, that $(\Lambda \circ \phi, \mathbf{RN})$ $\Pi\Sigma$ -jointly realizes (f, g) for some $\mathbf{RN}$ with a node with three sources and two targets.

From Lemma 4.7, we can see that the only allowable $\Pi\Sigma$ -interaction function for $n = 3$ are

$$z_1 + z_2 + z_3, \quad (z_1 + z_2)z_3, \quad (z_1 + z_3)z_2, \quad (z_2 + z_3)z_1, \quad z_1z_2z_3$$

For $f \prec g$ in Figure 7, observe that

$$\text{Col}_1(\text{True}(f)|_{\mathbf{C}_1}) \not\subseteq \text{Col}_1(\text{True}(g)|_{\mathbf{F}_1}) \text{ and } \text{Col}_1(\text{True}(f)|_{\mathbf{C}_1}) \not\supseteq \text{Col}_1(\text{True}(g)|_{\mathbf{F}_1}), \quad (20)$$

and

$$\text{Col}_2(\text{True}(f)|_{\mathbf{C}_2}) \not\subseteq \text{Col}_2(\text{True}(g)|_{\mathbf{F}_2}) \text{ and } \text{Col}_2(\text{True}(f)|_{\mathbf{C}_2}) \not\supseteq \text{Col}_2(\text{True}(g)|_{\mathbf{F}_2}). \quad (21)$$

By Theorem 4.14, we see that z_1 and z_2 cannot be simple terms or factors of Λ . This constraint implies that $\Lambda = (z_1 + z_2)z_3$. The following inequality argument will show that this choice of Λ is also impossible. To reduce notation, we will write $\phi_i(0) = \ell_i$ and $\phi_i(1) = u_i$.

550

We have the following relations from Figure 7:

$$\Lambda(\phi(100)) < \Lambda(\phi(001)) \quad (\text{dark grey} < \text{light grey})$$

$$\Lambda(\phi(010)) < \Lambda(\phi(001)) \quad (\text{dark grey} < \text{light grey})$$

--

$$\Lambda(\phi(101)) < \Lambda(\phi(110)) \quad (\text{light grey} < \text{white})$$

$$\Lambda(\phi(011)) < \Lambda(\phi(110)) \quad (\text{light grey} < \text{white})$$

Written in the language of ℓ and u this means

$$(u_1 + \ell_2)\ell_3 < (\ell_1 + \ell_2)u_3$$

$$(\ell_1 + u_2)\ell_3 < (\ell_1 + \ell_2)u_3$$

$$(u_1 + \ell_2)u_3 < (u_1 + u_2)\ell_3$$

$$(\ell_1 + u_2)u_3 < (u_1 + u_2)\ell_3.$$

We consider first and fourth equation; the second and third together lead to similar contradiction. First equation:

$$(u_1 + \ell_2)\ell_3 < (\ell_1 + \ell_2)u_3$$

$$u_1\ell_3 + \ell_2\ell_3 < \ell_1u_3 + \ell_2u_3$$

$$u_1\ell_3 - \ell_1u_3 < \ell_2(u_3 - \ell_3)$$

Fourth equation:

$$(\ell_1 + u_2)u_3 < (u_1 + u_2)\ell_3$$

$$\ell_1u_3 + u_2u_3 < u_1\ell_3 + u_2\ell_3$$

$$u_2(u_3 - \ell_3) < u_1\ell_3 - \ell_1u_3$$

Comparing the last line in each equation block we get

$$u_2(u_3 - \ell_3) < u_1\ell_3 - \ell_1u_3 < \ell_2(u_3 - \ell_3),$$

which, after cancellation, gives

$$u_2 < \ell_2.$$

Therefore $\phi_2(0) > \phi_2(1)$, contradicting Lemma 4.7, so $\Lambda = (z_1 + z_2)z_3$ is also impossible. Therefore, (f, g) is not $\Pi\Sigma$ -jointly realizable.

However, the pair (f, g) is $\Sigma\Pi\Sigma$ -jointly realizable. To see this, let $\Lambda =$
555 $z_1z_2 + z_3$, let $\phi_1(0) = \phi_2(0) = \phi_3(0) = 1$, $\phi_1(1) = 3$, $\phi_2(1) = 3.1$, and $\phi_3(1) = 4$,
and let $\theta_1 = 4.5$, and $\theta_2 = 9$. Such an assignment is displayed in the table in
Figure 7.

We now prove the inductive step. Let $n \geq 3$. Assume there exists $f, g : \mathbb{B}^n \rightarrow$
 $\mathbb{B}$ such that (f, g) is not $\Pi\Sigma$ -jointly realizable, but is $\Sigma\Pi\Sigma$ -jointly realizable. Let
 $(\Lambda \circ \phi, \mathbf{RN})$ with thresholds θ_1 and θ_2 $\Sigma\Pi\Sigma$ -jointly realize (f, g) . Define

$$\tilde{f}(y_1 \dots y_{n+1}) := \begin{cases} f(y_1 \dots y_n) & y_{n+1} = 0 \\ 1 & y_{n+1} = 1 \end{cases}$$

$$\tilde{g}(y_1 \dots y_{n+1}) := \begin{cases} g(y_1 \dots y_n) & y_{n+1} = 0 \\ 1 & y_{n+1} = 1 \end{cases}$$

As in the proof of Lemma 4.23, observe that

$$\text{Col}_{n+1}(\text{True}(\tilde{f})|_{\mathbf{F}_{n+1}}) = \text{True}(f), \quad \text{Col}_{n+1}(\text{True}(\tilde{g})|_{\mathbf{F}_{n+1}}) = \text{True}(g),$$

in other words the floor of $\tilde{f}$ has the same truth set as f and the floor of $\tilde{g}$ has the
same truth set as g . Since (f, g) are not $\Pi\Sigma$ -jointly realizable, the contrapositive
560 of Theorem 4.15 (a) tells us that $(\tilde{f}, \tilde{g})$ are not $\Pi\Sigma$ -jointly realizable. Let $m =$
 $\min\{\Lambda(\phi(\mathbb{B}^n))\}$. Define $\tilde{\phi}_{n+1}(0) := 1$ and $\tilde{\phi}_{n+1}(1) := \max\{2, C\}$, where C
is large enough such that $m + C > \max\{\theta_1, \theta_2\}$. Define $\tilde{\Lambda} := z_{n+1} + \Lambda$ and
 $\tilde{\phi} := (\phi_1, \dots, \phi_n, \tilde{\phi}_{n+1})$. Then $\tilde{\Lambda}$ is a valid $\Sigma\Pi\Sigma$ -interaction function, and $(\tilde{\Lambda} \circ \tilde{\phi},$
 $\mathbf{RN})$ $\Sigma\Pi\Sigma$ -jointly realizes $(\tilde{f}, \tilde{g})$, completing the proof. $\square$

565 4.4.3. $\Sigma\Pi\Sigma$ -jointly realizable $\subsetneq$ K -jointly realizable for $n \geq 4$

In this section we prove the last strict inclusion in the fourth row of Table 1.

Lemma 4.25. *Let $n \geq 4$. There exists a pair $f \prec g \in \mathbf{MBF}^+(n)$ such that
 (f, g) is not $\Sigma\Pi\Sigma$ -jointly realizable, but (f, g) is K -jointly realizable.*

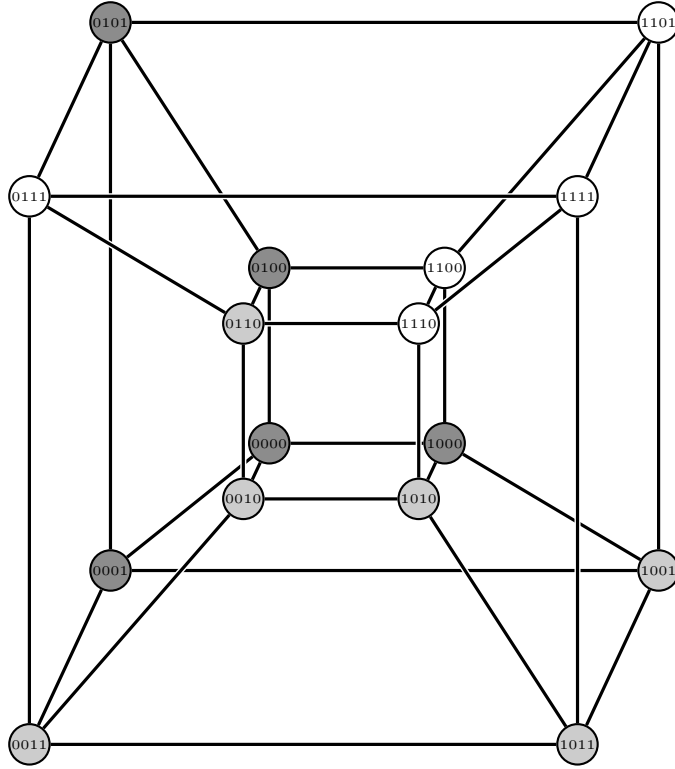


Figure 8: An example pair $f, g : \mathbb{B}^4 \rightarrow \mathbb{B}$ with $f \prec g$. Dark grey is $\text{False}(f) \cap \text{False}(g)$, light grey is $\text{False}(f) \cap \text{True}(g)$, and white is $\text{True}(f) \cap \text{True}(g)$. Nodes are labels with $y_1 y_2 y_3 y_4$. The pair (f, g) is K -jointly realizable, but not $\Sigma\Pi\Sigma$ -jointly realizable. Visually, the inner cube is the floor in the fourth direction, and the outer cube is the ceiling in the fourth direction.

Proof. Recall that any pair $f \prec g \in \mathbf{MBF}^+(n)$ is K -jointly realizable by Theorem 4.6. It remains to construct an example that is not $\Sigma\Pi\Sigma$ -jointly realizable and apply induction. We construct an explicit pair for $n = 4$. Consider the pair (f, g) in Figure 8. Observe that the floor (inner cube) has the same truth set as the cube in Figure 7. In Lemma 4.24 we showed that this pair can be realized by the $\Sigma\Pi\Sigma$ -interaction function $\Lambda = z_1 z_2 + z_3$. It turns out this is the only valid $\Sigma\Pi\Sigma$ -interaction function that can realize the pair. To see this, we first list all possible $\Sigma\Pi\Sigma$ -interaction functions for $n = 3$, which are

$$\begin{array}{cccc} z_1 + z_2 + z_3 & (z_1 + z_2)z_3 & (z_1 + z_3)z_2 & (z_2 + z_3)z_1 \\ z_1 z_2 z_3 & z_1 z_2 + z_3 & z_1 z_3 + z_2 & z_2 z_3 + z_1 \end{array}$$

Via Equations (20) and (21) we can rule out all but $z_1 z_2 + z_3$ and $(z_1 + z_2)z_3$.

570 In addition, Lemma 4.24 showed that $\Lambda = (z_1 + z_2)z_3$ also does not work. Therefore, we can make the following claim: if $(\Lambda \circ \phi, \mathbf{RN})$ $\Sigma\Pi\Sigma$ -jointly realizes the pair of MBFs from Figure 8, then $\Lambda = z_1 z_2 + z_3$.

Suppose that $f \prec g \in \mathbf{MBF}^+(4)$ in Figure 8 are $\Sigma\Pi\Sigma$ -jointly realizable. Then by Theorem 4.15 the floor (inner cube) and ceiling (outer cube) pairs $(f'_{\mathbf{F}_4}, g'_{\mathbf{F}_4})$ and $(f'_{\mathbf{C}_4}, g'_{\mathbf{C}_4})$ defined rigorously in Theorem 4.15 are $\Sigma\Pi\Sigma$ -jointly realizable, and there is a single $\Sigma\Pi\Sigma$ -interaction function Λ along with maps $\phi_{\mathbf{C}_4}, \phi_{\mathbf{F}_4}$, realizing networks $\mathbf{RN}'_{\mathbf{C}_4}, \mathbf{RN}'_{\mathbf{F}_4}$, and thresholds $\theta_{\mathbf{C}_4,1}, \theta_{\mathbf{C}_4,2}, \theta_{\mathbf{F}_4,1}, \theta_{\mathbf{F}_4,2}$ such that $(\Lambda \circ \phi_{\mathbf{F}_4}, \mathbf{RN}'_{\mathbf{F}_4})$ $\Sigma\Pi\Sigma$ -jointly realizes $(f'_{\mathbf{F}_4}, g'_{\mathbf{F}_4})$ and $(\Lambda \circ \phi_{\mathbf{C}_4}, \mathbf{RN}'_{\mathbf{C}_4})$ $\Sigma\Pi\Sigma$ -jointly realizes $(f'_{\mathbf{C}_4}, g'_{\mathbf{C}_4})$. By our above claim, we know $\Lambda = z_1 z_2 + z_3$. However, consider $(f'_{\mathbf{C}_4}, g'_{\mathbf{C}_4})$. By inspection, we see that

$$\text{Col}_3(\text{True}(f'_{\mathbf{C}_4})|_{\mathbf{C}_3}) \not\subseteq \text{Col}_3(\text{True}(g'_{\mathbf{C}_4})|_{\mathbf{F}_3})$$

$$\text{Col}_3(\text{True}(f'_{\mathbf{C}_4})|_{\mathbf{C}_3}) \not\supseteq \text{Col}_3 \text{True}(g'_{\mathbf{C}_4})|_{\mathbf{F}_3})$$

By Theorem 4.14 we know that if $(\Lambda \circ \phi_{\mathbf{C}_4}, \mathbf{RN}'_{\mathbf{C}_4})$ $\Sigma\Pi\Sigma$ -jointly realizes $(f'_{\mathbf{C}_4}, g'_{\mathbf{C}_4})$, then z_3 cannot be a factor or simple term of Λ . This contradicts our claim that

575 $\Lambda = z_1 z_2 + z_3$, and so (f, g) are not $\Sigma\Pi\Sigma$ -jointly realizable.

We now prove the inductive step. Let $n \geq 4$. Assume there exists $f, g : \mathbb{B}^n \rightarrow \mathbb{B}$ such that (f, g) is not $\Sigma\Pi\Sigma$ -jointly realizable, but is K -jointly realizable.

Define

$$\begin{aligned}\tilde{f}(y_1 \dots y_{n+1}) &:= \begin{cases} f(y_1 \dots y_n) & y_{n+1} = 0 \\ 1 & y_{n+1} = 1 \end{cases} \\ \tilde{g}(y_1 \dots y_{n+1}) &:= \begin{cases} g(y_1 \dots y_n) & y_{n+1} = 0 \\ 1 & y_{n+1} = 1 \end{cases}\end{aligned}$$

Observe that $\tilde{f}', \tilde{g}' \in \mathbf{MBF}^+(n+1)$ such that $\tilde{f}' \prec \tilde{g}'$. Since $\tilde{f}'_{\mathbf{F}_{n+1}} = f$ and $\tilde{g}'_{\mathbf{F}_{n+1}} = g$, by Theorem 4.15 we know $(\tilde{f}, \tilde{g})$ is not $\Sigma\Pi\Sigma$ -jointly realizable. By Theorem 4.6, we know all pairs $f \prec g \in \mathbf{MBF}^+(n+1)$ are always K -jointly realizable. $\square$

580 4.5. $\Sigma\Pi\Sigma$ -joint realizability = K -joint realizability for $n = 3$

This is the final result remaining to be proven in Table 1. To find the total number of pairs (f, g) such that $f, g \in \mathbf{MBF}^+(3)$ and $f \prec g$, we use the bijection given in Definition 4.16. The number of K -realizable pairs (f, g) where $f \prec g \in \mathbf{MBF}^+(3)$ is the same as $|\mathbf{MBF}^+(4)|$. In [1] it was found
585 that $|\mathbf{MBF}^+(4)| = 168$. We used the software DSGRN to find that 150 MBFs in $\mathbf{MBF}^+(4)$ are Σ -realizable. The software was also used to computationally check that there are exactly 150 pairs $f \prec g \in \mathbf{MBF}^+(3)$ that are Σ -jointly realizable. We explicitly constructed the remaining 18 pairs $(f_i \prec g_i), i = 1, \dots, 18$ that are provably not Σ -joint realizable by applying Theorem 4.14 to
590 at least one direction in each case. These pairs are all presented in Appendix Table B.2 with the direction that allows application of Theorem 4.14 indicated. Finally, in Appendix Table B.3, we provide specific realizing functions $\Lambda \circ \phi$ and realizing network thresholds θ_1, θ_2 that $\Sigma\Pi\Sigma$ -jointly realize all 18 pairs.

5. Discussion

595 In this work we linked two classes of dynamical systems, one a continuous time ordinary differential equation (ODE) model and the other a discrete time monotone Boolean function (MBF) model. Both of these classes have been used

to model dynamics of gene regulatory networks. We show that a very general class of ODE models with a discontinuous right hand side admits an equivalence relation, such that all ODEs in an equivalence class share the approximate description of dynamics in terms of the identical state transition graph STG. We then showed that each equivalence class corresponds to a collection of MBFs. The collections of MBFs can be arranged into a parameter graph where edges between the collections indicate a one-step change in one of the MBFs.

After establishing the equivalence between collections of MBFs and equivalence classes of K systems of ODEs, we pose the question about what restrictions, if any, the algebraic form of the right-hand side of the ODE imposes on k -tuples of MBFs that correspond to realizable equivalence classes of ODEs.

We show that the classes of pairs of MBFs with three inputs that are realizable as linear functions are a strict subset of $\Pi\Sigma$ -jointly realizable pairs, which is in turn a strict subset of $\Sigma\Pi\Sigma$ -jointly realizable pairs of MBFs. We also show that there are pairs of MBFs with any $n \geq 4$ inputs that are K -jointly realizable, but are not $\Sigma\Pi\Sigma$ realizable. To summarize, the increased complexity of the algebraic expression provides a richer class of models as measured by the set of MBFs that can be realized in a parameter graph.

Our work opens up many interesting questions about the joint realizability of collections of MBFs. We will briefly discuss two potential sets of questions. First, we defined an infinite nested set of classes of nonlinearities. While we only discussed the first three $\Sigma \subsetneq \Pi\Sigma \subsetneq \Sigma\Pi\Sigma$, adding alternating products and sums creates larger and larger classes of functions. Do our results extend in this direction? In other words, are there pairs of monotone Boolean functions that are realizable in a parameter graph via class $s + 1$, but are not realizable in class s ? Furthermore, is it possible that there are pairs of MBFs that are not realizable in any of the infinite progression of algebraic classes with alternating products and sums, but are K -jointly realizable?

The second class of questions generalizes pairs of monotone Boolean functions to k -tuples. Our results derive some constraints for pairs of MBFs, which are then used to prove the main results about differences in $*$ -joint realizability.

While these results apply pairwise to any k -tuple of MBFs with $f_1 \prec f_2 \prec \dots \prec f_k$, to rule out realizability of some tuples, we do not know if there any additional constraints that arise from considering, say, triples of functions $f \prec g \prec h$, or k -tuples of MBFs.

By providing a link between a class of discontinuous differential equations and the collection of k -tuples of MBFs, this paper provides an opening to a class of new problems in the field of monotone Boolean functions.

Appendix A. Proofs of Theorem 4.14 and 4.15.

Proof of Theorem 4.14. We will proceed by contradiction. Suppose z_ℓ is a factor or simple term and suppose the negation of Equations (17) and (18). The negation of (17) is there exists some point $\vec{y} \in \mathbf{F}_\ell$ such that $g(\vec{y}) = 0$ and $f(\vec{y} + \hat{e}_\ell) = 1$. The negation of (18) implies there exists some point $\vec{w} \in \mathbf{F}_\ell$ such that $g(\vec{w}) = 1$ and $f(\vec{w} + \hat{e}_\ell) = 0$. Let θ_f and θ_g be the thresholds from $\mathbf{RN}$ associated to the two functions respectively. From the definition of $*$ -jointly realizable, we know

$$g(\vec{y}) = 0 \iff \Lambda(\phi(\vec{y})) < \theta_g \quad (\text{A.1})$$

$$f(\vec{y} + \hat{e}_\ell) = 1 \iff \Lambda(\phi(\vec{y} + \hat{e}_\ell)) > \theta_f \quad (\text{A.2})$$

$$g(\vec{w}) = 1 \iff \Lambda(\phi(\vec{w})) > \theta_g \quad (\text{A.3})$$

$$f(\vec{w} + \hat{e}_\ell) = 0 \iff \Lambda(\phi(\vec{w} + \hat{e}_\ell)) < \theta_f \quad (\text{A.4})$$

We now proceed by cases.

Case 1: (z_ℓ is a factor) Recall from the definition of factor that there exists a function Λ' that does not depend on z_ℓ such that $\Lambda = z_\ell \Lambda'$. We have $z_\ell = \phi_\ell(v_\ell)$ for any $\vec{v} \in \mathbb{B}^n$. Taking $\vec{v} \in \mathbf{F}_\ell$ we have

$$\Lambda(\phi(\vec{v})) = \phi_\ell(0) \Lambda'(\phi'(\mathbf{Co1}_\ell(\vec{v})))$$

$$\Lambda(\phi(\vec{v} + \hat{e}_\ell)) = \phi_\ell(1) \Lambda'(\phi'(\mathbf{Co1}_\ell(\vec{v}))).$$

We used the collapse operation because Λ' is independent of z_ℓ and we took $\phi'(\mathbf{Co1}_\ell(\vec{v})) = (\phi_1(v_1), \dots, \phi_{\ell-1}(v_{\ell-1}), \phi_{\ell+1}(v_{\ell+1}), \dots, \phi_n(v_n))$. Lastly, we used

$\text{Col}_\ell(\vec{v}) = \text{Col}_\ell(\vec{v} + \hat{e}_\ell)$. We conclude that for any $\vec{v} \in \mathbf{F}_\ell$

$$\frac{\phi_\ell(1)}{\phi_\ell(0)} \Lambda(\phi(\vec{v})) = \Lambda(\phi(\vec{v} + \hat{e}_\ell)).$$

From (A.2) and (A.4), we may write

$$\begin{aligned} \frac{\phi_\ell(1)}{\phi_\ell(0)} \Lambda(\phi(\vec{y})) > \theta_f &\iff \Lambda(\phi(\vec{y})) > \frac{\phi_\ell(0)}{\phi_\ell(1)} \theta_f \\ \frac{\phi_\ell(1)}{\phi_\ell(0)} \Lambda(\phi(\vec{w})) < \theta_f &\iff \Lambda(\phi(\vec{w})) < \frac{\phi_\ell(0)}{\phi_\ell(1)} \theta_f \end{aligned}$$

and combining with (A.1) and (A.3) we obtain

$$\theta_g > \frac{\phi_\ell(0)}{\phi_\ell(1)} \theta_f \text{ and } \theta_g < \frac{\phi_\ell(0)}{\phi_\ell(1)} \theta_f$$

a clear contradiction.

Case 2: (z_ℓ is a simple term) Similar to Case 1, the key fact is, if z_ℓ is a simple term, we know for any $\vec{v} \in \mathbf{F}_\ell$

$$\Lambda(\phi(\vec{v})) + (\phi_\ell(1) - \phi_\ell(0)) = \Lambda(\phi(\vec{v} + \hat{e}_\ell)),$$

We then make a similar argument as before. Equations (A.2) and (A.4) give

$$\begin{aligned} \Lambda(\phi(\vec{y})) + (\phi_\ell(1) - \phi_\ell(0)) > \theta_1 &\iff \Lambda(\phi(\vec{y})) > \theta_1 - (\phi_\ell(1) - \phi_\ell(0)) \\ \Lambda(\phi(\vec{w})) + (\phi_\ell(1) - \phi_\ell(0)) < \theta_1 &\iff \Lambda(\phi(\vec{w})) < \theta_1 - (\phi_\ell(1) - \phi_\ell(0)) \end{aligned}$$

and combining with (A.1) and (A.3) we obtain

$$\theta_g > \theta_f - (\phi_\ell(1) - \phi_\ell(0)) \text{ and } \theta_g < \theta_f - (\phi_\ell(1) - \phi_\ell(0))$$

which is our desired contradiction. $\square$

640 *Proof of Theorem 4.15.* Let $f, g : \mathbb{B}^n \rightarrow \mathbb{B}$, with $f \prec g$, be $*$ -jointly realizable MBFs, and let $(\Lambda \circ \phi, \mathbf{RN})$ jointly $*$ -realize (f, g) . Let θ_f and θ_g be the thresholds associated to the two realizations of these functions.

We seek to construct a $*$ -interaction function $\Lambda' : \mathbb{R}_+^{n-1} \rightarrow \mathbb{R}_+$, a map $\phi' : \mathbb{B}^{n-1} \rightarrow \mathbb{R}_+^{n-1}$, and a weighted regulatory network $\mathbf{RN}'_U$ with thresholds θ'_f and θ'_g so that $(\Lambda' \circ \phi', \mathbf{RN}'_U)$ $*$ -jointly realizes $f'_U, g'_U : \mathbb{B}^{n-1} \rightarrow \mathbb{B}$. In doing so

645

we will prove a). In each of the following cases, the construction of Λ' does not depend on whether $U = \mathbf{F}_\ell$ or $U = \mathbf{C}_\ell$, and so b) will follow immediately.

Recall that we are going to collapse over the ℓ^{th} dimension. In the following proof, we will be considering a node in $\mathbf{RN}$ that has an incoming edge from node ℓ and two targets, one of which is associated to the MBF f and the other to g . It will be useful to define the graph $\mathbf{RN}'$ to be the network $\mathbf{RN}$ without the edge from ℓ to the node under consideration. $\mathbf{RN}'$ is an intermediate step to the construction of $\mathbf{RN}'_U$.

Using the observations in Remark 3.2, we note that regardless of the specific value of $*$, we can always assume that $\Lambda \in \Sigma\Pi\Sigma$ defined in (12). Therefore there exist sets $W_1, \dots, W_L$, where $W_1, \dots, W_L$ partitions the set $\{1, \dots, n\}$, and for each $k \in \{1, \dots, L\}$, there exists $M_k \in \mathbb{N}$ so that the sets $V_{k,1}, \dots, V_{k,M_k}$ partition the set W_k , such that

$$\Lambda = \sum_{W_k} \left(\prod_{V_{k,j}} \left(\sum_{i \in V_{k,j}} z_i \right) \right)$$

There is exactly one set in the partition, call it W_p , such that $\ell \in W_p$. Furthermore, there is exactly one $V_{p,*}$ that contains ℓ , call it $V_{p,q}$. We now proceed by cases. Define the map $\delta : \{1, \dots, n\} \setminus \{\ell\} \rightarrow \{1, \dots, n-1\}$ as

$$\delta(j) = \begin{cases} j & \text{if } j < \ell \\ j-1 & \text{if } j > \ell \end{cases}$$

We will use the map δ to construct the interaction function Λ' . If $*$ is $\Sigma\Pi\Sigma$, we need to consider Cases 1, 2, and 3. However, if $*$ is $\Pi\Sigma$, then $W_1 = \{1, \dots, n\}$, and we only need to consider Cases 2 and 3, and if $*$ is Σ , then $W_1 = V_{1,1} = \{1, \dots, n\}$, and we only need to consider Case 3.

Case 1: ($W_p = \{\ell\}$) In this case the map Λ has the form

$$\Lambda = z_\ell + \sum_{\substack{W_k \\ k \neq p}} \left(\prod_{V_{k,j}} \left(\sum_{i \in V_{k,j}} z_i \right) \right)$$

where $z_i = \phi_i(y_i)$ for $\vec{y} \in \mathbb{B}^n$. We construct the interaction function Λ' as

$$\Lambda' = \sum_{\substack{W_k \\ k \neq p}} \left(\prod_{V_{k,j}} \left(\sum_{i \in V_{k,j}} z_{\delta(i)} \right) \right)$$

and define $z_{\delta(i)} = \phi'_i(y_i) = \phi_{\delta^{-1}(i)}(y_i)$. We then set $\phi' = (\phi'_1, \dots, \phi'_{n-1})$. This construction ensures, for any $\vec{y} = (y_1, \dots, y_n) \in \mathbb{B}^n$,

$$\Lambda(\phi(\vec{y})) = \Lambda'(\phi'(y_{\delta^{-1}(1)}, \dots, y_{\delta^{-1}(n-1)})) + \phi_\ell(y_\ell)$$

If $U = \mathbf{F}_\ell$, then for all $\vec{y} = (y_1, \dots, y_n) \in U$, we know $y_\ell = 0$. Therefore,

$$\Lambda(\phi(\vec{y})) = \Lambda'(\phi'(y_{\delta^{-1}(1)}, \dots, y_{\delta^{-1}(n-1)})) + \phi_\ell(0).$$

Set $\theta'_f = \max\{0, \theta_f - \phi_\ell(0)\}$ and $\theta'_g = \max\{0, \theta_g - \phi_\ell(0)\}$. It is possible that $\theta'_f = \theta'_g$ at this point. However, since $\Lambda \circ \phi$ takes on finitely many values, we can always perturb one threshold by a small enough ϵ to guarantee our inequalities still hold. After this potential perturbation, replace θ_f and θ_g in $\mathbf{RN}'$ with θ'_f and θ'_g to complete the construction of $\mathbf{RN}'_{\mathbf{F}_\ell}$. This construction means that

$$\Lambda(\phi(y_1, \dots, y_n)) \leq \theta_f \iff \Lambda'(\phi'(y_{\delta^{-1}(1)}, \dots, y_{\delta^{-1}(n-1)})) \leq \theta'_f$$

and likewise for θ_g and θ'_g . Therefore $(\Lambda' \circ \phi', \mathbf{RN}'_{\mathbf{F}_\ell})$ *-jointly realizes $f'_{\mathbf{F}_\ell}$ and $g'_{\mathbf{F}_\ell}$.

Similarly, if $U = \mathbf{C}_\ell$, then for all $\vec{y} = (y_1, \dots, y_n) \in U$, we have

$$\Lambda(\phi(\vec{y})) = \Lambda'(\phi'(y_{\delta^{-1}(1)}, \dots, y_{\delta^{-1}(n-1)})) + \phi_\ell(1).$$

660 We set $\theta'_f = \max\{0, \theta_f - \phi_\ell(1)\}$ and $\theta'_g = \max\{0, \theta_g - \phi_\ell(1)\}$, perturbed by small enough $\epsilon > 0$, if necessary, to replace θ_f and θ_g in $\mathbf{RN}'$ and complete the construction of $\mathbf{RN}'_{\mathbf{C}_\ell}$. Then $(\Lambda' \circ \phi', \mathbf{RN}'_{\mathbf{C}_\ell})$ *-jointly realizes $f'_{\mathbf{C}_\ell}$ and $g'_{\mathbf{C}_\ell}$.

Case 2: ($W_p \setminus \{\ell\} \neq \emptyset$ and $V_{p,q} = \{\ell\}$) The interaction function Λ' without the ℓ^{th} element is

$$\Lambda' = \sum_{W_k} \left(\prod_{\substack{V_{k,j} \\ j \neq q}} \left(\sum_{i \in V_{k,j}} z_{\delta(i)} \right) \right)$$

In this case we know that W_p is a partition of at minimum size two. Pick exactly one $t \in W_p \setminus \{q\}$. Construct $z_{\delta(j)} = \phi'_j(y_j)$ as follows: if $U = \mathbf{F}_i$, then for $\vec{y} \in \mathbb{B}^n$

$$\phi'_j(y_j) = \begin{cases} \phi_{\delta^{-1}(j)}(y_j)\phi_\ell(0) & \text{if } j \in V_{p,t} \\ \phi_{\delta^{-1}(j)}(y_j) & \text{otherwise} \end{cases}.$$

However, if $U = \mathbf{C}_i$, then

$$\phi'_j(y_j) = \begin{cases} \phi_{\delta^{-1}(j)}(y_j)\phi_\ell(1) & \text{if } j \in V_{p,t} \\ \phi_{\delta^{-1}(j)}(y_j) & \text{otherwise} \end{cases}.$$

We have ensured for any $\vec{y} = (y_1, \dots, y_n) \in U$,

$$\Lambda(\phi(\vec{y})) = \Lambda'(\phi'(y_{\delta^{-1}(1)}, \dots, y_{\delta^{-1}(n-1)}))$$

Setting $\theta'_f = \theta_f$ and $\theta'_g = \theta_g$ obtains the desired result; i.e., $\mathbf{RN}'_U = \mathbf{RN}'$ and $(\Lambda' \circ \phi', \mathbf{RN}') *$ -jointly realizes f'_U, g'_U .

Case 3: ($W_p \setminus \{\ell\} \neq \emptyset$ and $V_{p,q} \setminus \{\ell\} \neq \emptyset$) In this case the original interaction function Λ takes the form

$$\Lambda = \sum_{W_k} \left(\prod_{\substack{V_{k,j} \\ (k,j) \neq (p,q)}} \left(\sum_{i \in V_{k,j}} z_i \right) \right) + \left(\prod_{V_{p,q}} \left(\sum_{i \in V_{p,q}} z_i \right) \right).$$

The interaction function Λ' is constructed as

$$\Lambda' = \sum_{W_k} \left(\prod_{\substack{V_{k,j} \\ (k,j) \neq (p,q)}} \left(\sum_{i \in V_{k,j}} z_{\delta(i)} \right) \right) + \left(\prod_{V_{p,q}} \left(\sum_{\substack{i \in V_{p,q} \\ i \neq \ell}} z_{\delta(i)} \right) \right).$$

Pick exactly one element $t \in V_{p,q} \setminus \{\ell\}$, and construct ϕ'_j as follows: if $U = \mathbf{F}_\ell$, then

$$\phi'_j(y_j) := \begin{cases} \phi_{\delta^{-1}(j)}(y_j) + \phi_\ell(0) & \text{if } j = \delta(t) \\ \phi_{\delta^{-1}(j)}(y_j) & \text{otherwise} \end{cases}.$$

However, if $U = \mathbf{C}_\ell$, then

$$\phi'_j(y_j) := \begin{cases} \phi_{\delta^{-1}(j)}(y_j) + \phi_\ell(1) & \text{if } j = \delta(t) \\ \phi_{\delta^{-1}(j)}(y_j) & \text{otherwise} \end{cases}.$$

We then set $\phi' = (\phi'_1, \dots, \phi'_{n-1})$. This construction ensures, for any $\vec{y} = (y_1, \dots, y_n) \in U$,

$$\Lambda(\phi(\vec{y})) = \Lambda'(\phi'(y_{\delta^{-1}(1)}, \dots, y_{\delta^{-1}(n-1)}))$$

665 and so by setting $\mathbf{RN}'_U = \mathbf{RN}'$ as in Case 2, we obtain the desired result. $\square$

Appendix B. Supporting tables and figures

Table B.2 lists explicitly all pairs $f \prec g \in \mathbf{MBF}^+(\mathbf{3})$ of Boolean functions with three inputs that are not Σ -jointly realizable. These correspond to non-threshold monotone Boolean functions in $\mathbf{MBF}^+(4)$ with 4 inputs. In each case
670 the direction that allows us to use Theorem 4.14 to rule out Σ -joint realizability is indicated in the last column.

In Table B.3 we show explicitly the form of the interaction function $\Lambda \in \mathbf{\Sigma\Pi\Sigma}$, the values $\phi(1) = (\phi_1(1), \phi_2(1), \phi_3(1))$, and the value of the thresholds θ_g, θ_f that $\Sigma\Pi\Sigma$ -jointly realize all pairs $f_i \prec g_i$ given in Table B.2. In all cases,
675 we set $\phi_1(0) = \phi_2(0) = \phi_3(0) = 1$. This list, together with 150 pairs that are Σ -jointly realizable, exhausts all pairs $f \prec g$ of functions in $\mathbf{MBF}^+(\mathbf{3})$ and proves that for $n = 3$ every such pair is $\Sigma\Pi\Sigma$ -joint realizable.

Figure B.9 shows all pairs $f \prec g \in \mathbf{MBF}^+(\mathbf{2})$. It is also the factor of the parameter graph associated to node 1 in the network in Figure 1, after
680 transforming the Boolean functions to be positive under the map β given in (13).

Acknowledgments This research was supported by NSF grant DMS-1839299 (B.C, T.G), DARPA FA8750-17-C-0054 (B.C, T.G) and NIH 5R01GM126555-01 (P.CK, B.C, T.G).

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	Input								Direction(s)
	000	001	010	100	110	101	011	111	
$f_1 \prec g_1$	00	01	00	01	11	01	01	11	y_1
$f_2 \prec g_2$	00	00	01	01	01	01	11	11	y_2
$f_3 \prec g_3$	00	01	01	00	01	11	01	11	y_3
$f_4 \prec g_4$	00	01	01	00	11	01	01	11	y_2
$f_5 \prec g_5$	00	01	00	01	01	01	11	11	y_3
$f_6 \prec g_6$	00	00	01	01	01	11	01	11	y_1
$f_7 \prec g_7$	00	01	00	00	11	01	01	11	y_1, y_2
$f_8 \prec g_8$	00	00	00	01	01	01	11	11	y_2, y_3
$f_9 \prec g_9$	00	00	01	00	01	11	01	11	y_1, y_3
$f_{10} \prec g_{10}$	00	01	00	01	11	01	11	11	y_1, y_3
$f_{11} \prec g_{11}$	00	00	01	01	01	11	11	11	y_1, y_2
$f_{12} \prec g_{12}$	00	01	01	00	11	11	01	11	y_2, y_3
$f_{13} \prec g_{13}$	00	01	00	00	11	01	11	11	y_1
$f_{14} \prec g_{14}$	00	00	00	01	01	11	11	11	y_2
$f_{15} \prec g_{15}$	00	00	01	00	11	11	01	11	y_3
$f_{16} \prec g_{16}$	00	00	00	01	11	01	11	11	y_1
$f_{17} \prec g_{17}$	00	00	01	00	01	11	11	11	y_2
$f_{18} \prec g_{18}$	00	01	00	00	11	11	01	11	y_3

Table B.2: All 18 non Σ -jointly realizable pairs (f, g) for $n = 3$ such that $f \prec g$. The column for input $\vec{y}$ shows the pair of values of $f_i(y)g_i(y)$. For example, column 001 has 01 in the first row. This means that $f_1(001) = 0$ and $g_1(001) = 1$. The direction(s) that allow the use of Theorem 4.14 to rule out Σ -joint realizability are indicated in the last column.

i	Λ	$\phi(111)$	θ_g	θ_f
1	$z_1 z_2 + z_3$	(3, 2, 3)	3.5	6.5
2	$z_1 + z_2 z_3$	(3, 3, 2)	3.5	6.5
3	$z_2 + z_1 z_3$	(2, 3, 3)	3.5	6.5
4	$z_1 z_2 + z_3$	(2, 3, 3)	3.5	6.5
5	$z_1 + z_2 z_3$	(3, 2, 3)	3.5	6.5
6	$z_2 + z_1 z_3$	(3, 3, 2)	3.5	6.5
7	$z_1 z_2 + z_3$	(3, 3, 4)	4.5	8
8	$z_1 + z_2 z_3$	(4, 3, 3)	4.5	8
9	$z_2 + z_1 z_3$	(3, 4, 3)	4.5	8
10	$z_2(z_1 + z_3)$	(4, 2, 4)	4.5	9
11	$z_3(z_1 + z_2)$	(4, 4, 2)	4.5	9
12	$z_1(z_2 + z_3)$	(2, 4, 4)	4.5	9
13	$z_1 z_2 + z_3$	(2, 3, 4)	4.5	6.5
14	$z_1 + z_2 z_3$	(4, 2, 3)	4.5	6.5
15	$z_2 + z_1 z_3$	(3, 4, 2)	4.5	6.5
16	$z_2(z_1 + z_3)$	(4, 2, 3)	4.5	7.5
17	$z_3(z_1 + z_2)$	(3, 4, 2)	4.5	7.5
18	$z_1(z_2 + z_3)$	(2, 3, 4)	4.5	7.5

Table B.3: Example $(\Lambda \circ \phi, \theta_1, \theta_2)$ that $\Sigma\Pi\Sigma$ -jointly realize all pairs $f_i \prec g_i$ given in Table B.2. For all rows, $\phi_1(0) = \phi_2(0) = \phi_3(0) = 1$.

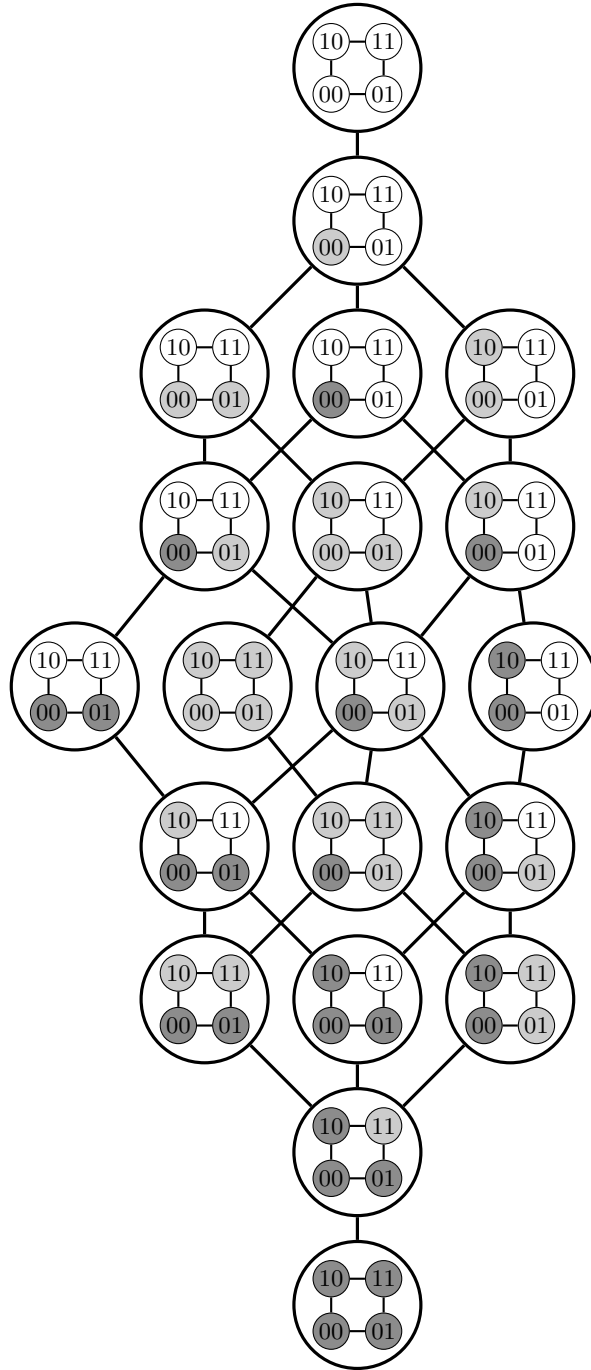


Figure B.9: All 20 pairs of functions $f \prec g \in \mathbf{MBF}^+(2)$. Each node in the above graph represents a pair of functions, where dark grey is $\text{False}(f) \cap \text{False}(g)$, light grey is $\text{False}(f) \cap \text{True}(g)$, and white is $\text{True}(f) \cap \text{True}(g)$.

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