Conceptualizing Mathematical Transformation as Substitution Equivalence:
The Critical Role of Student Definitions

Claire Wladis
BMCC/CUNY Graduate Center

Benjamin Sencindiver
CUNY Graduate Center

Kathleen Offenholley
BMCC/CUNY

Elisabeth Jaffe
BMCC/CUNY

Joshua Taton
CUNY Graduate Center

This theoretical paper explores student conceptions of transformation as substitution equivalence by linking it to their definitions of substitution and equivalence. This work draws on the work of Sfard (1995) to conceptualize substitution equivalence and its components, equivalence and substitution, as a spectrum from computational to structural. We provide examples of students' work to illustrate how student notions of substitution, equivalence, and substitution equivalence as an approach to justifying transformation may related to one another.

Keywords: Equivalence, Substitution, Substitution Equivalence, Structural Thinking, Definitions

Transformation has often been framed as a core mathematical activity (Kieran, 2004), and all mathematical calculation, whether arithmetic, simplifying expressions, or finding the solution sets of equations, can be viewed as a process of transformation. Thus, with the goal of exploring the core mathematical ideas that justify why particular transformations are mathematically valid, we view mathematical transformation through the lens of substitution equivalence, conceptualizing it as a process of replacing one symbolic object with an equivalent one, and naming this process substitution (Wladis et al., 2020). This also includes the process of identifying sub-objects and replacing them with equivalent ones in order to generate a new equivalent object. This process is non-trivial for many students, and we hypothesize that substitution equivalence may be intimately connected to many of the struggles that students have with symbolic mathematics at various levels and domains. Little attention has been paid formally to students' notions of substitution equivalence, even though these notions may be intricately linked to the ways in which students think about and execute various types of mathematical transformation. In this paper we attempt to address that gap, by providing a model of student thinking around substitution equivalence. First we describe the model, including the theories and body of research literature which have informed its creation, and then we proceed to use the model to analyze a few vignettes of student work, in order to illustrate its potential affordances.

Substitution Equivalence as a Lens for Mathematical Transformation

In this paper, we focus specifically on student thinking around *substitution equivalence*, or the notion that two expressions, equations, or other mathematical objects are equivalent if one can be generated from the other through a sequence of substitutions carried out through a combination of correct interpretation of syntactic structure and appropriate use of mathematical properties (Wladis et al., 2020).

Definition of Substitution: In order to see clearly how mathematical activity could be viewed through the lens of substitution equivalence, we define substitution more broadly than has been done explicitly in much existing research and curricula, as the process of replacing any mathematical object (or any unified subpart of an object) with an equivalent object, regardless of complexity. This includes the replacement of x in $2x^2 - 2x + 1$ with -3, but also, e.g., the replacement of $x^2 - 6x = 1$ with the equivalent equation $x^2 - 6x - 1 = 0$ during solving.

Definition of Equivalence: We note that the idea of substitution equivalence is wholly dependent upon an underlying equivalence relation of some kind and depends upon a specific stipulated definition of equivalence. This may be a particular context-specific definition of equivalence (e.g., two equations are equivalent if they have the same solution set), or a more generalized concept of equivalence (e.g., an equivalence relation); however, any definition of equivalence that satisfies the definition of an equivalence relation could be used.

Definition of Substitution Equivalence: We define the domain of substitution equivalence as composed of two main ideas, which we illustrate in more detail in subsequent sections. According to our model, students who have a notion of substitution equivalence recognize:

1. **The *general* notion of substitution equivalence:** They understand that we can replace an object with *any* other equivalent object when problem-solving.
2. **That substitution of unified sub-objects can be used to generate equivalent objects:** They understand that objects can be broken into unified sub-objects, and that we can replace any unified sub-object with any equivalent unified sub-object (and the process of substitution leaves the rest of the structure of that object unchanged).

The second notion leads us to another core definition: We use the term *subexpression* (or sub-object, more generally) to denote a substring of an expression (or other object) that can be treated as a unified object without changing the syntactic meaning of the original expression (or object). E.g., $a - b$ is a subexpression of $a - b - c$, but $b - c$ is not (because putting parentheses around $b - c$ would change the syntactic meaning of the expression).

Model of Operational and Structural Thinking about Substitution Equivalence

Wladis et al (2020) described key features of student thinking around substitution equivalence on a spectrum from structural versus operational approaches. This paper aims to take this further by describing explicitly how student conceptions of substitution equivalence may be dependent upon student definitions of substitution and equivalence (see Figure 1).

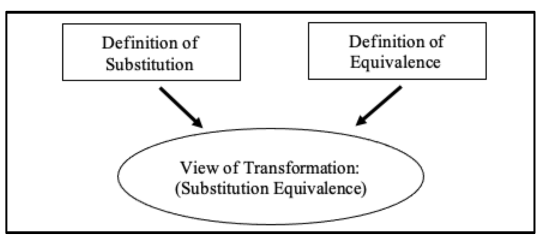


Figure 1: *Model of Student Thinking about Substitution Equivalence*

In the model in Figure 1, holding well-defined and standard definitions of both substitution and equivalence are necessary but not sufficient conditions for students to develop a view of transformation justified by substitution equivalence. A student may have trouble thinking of transformation as substitution equivalence because (a) their definitions of substitution are too narrow; (b) their definitions of equivalence are ill-defined, unstable, or invalid; (c) they do not draw on their knowledge of substitution and/or equivalence when performing transformation; or a combination of all of these. We conceptualize student views of substitution, equivalence, and transformation as being on a continuum from operational to structural (Table 1). This model is based on the notion that the *ability* to conceptualize transformation as a process of substitution equivalence may be useful for students in developing deeper understanding of the justification behind their transformation work (and a way of checking the validity of that work).

Table 1: Components of substitution equivalence model

	Operational Thinking	Structural Thinking
View of Transformation	Students see transformations of expressions and equations (or other objects) as a process of “operating on” the original object itself. They may or may not see this as linked to any notion of equivalence.	Students see each step in a transformation as the process of replacing one object with an equivalent one through substitution, using properties and existing syntactic structure. They appear to have some notion of an equivalence class as an object (which need not be formally defined).
Definition of Equivalence	Students either ignore the notion of equivalence entirely, or appear to have only vague, ill-defined, or unstable notions of equivalence, or try to apply one definition of equivalence that works only in one context to another context.	Students have a well-defined and relatively stable definition of equivalence, and recognize that it is context-dependent. They recognize that equivalence is a fixed trait (two objects are either equivalent under a particular definition or not—they do not “become” equivalent).
Definition of Substitution	Students see substitution only as plugging a number in for a variable (and then computing the result). They see variables as representing only numbers.	Students see replacement of any object (or sub-object) with an equivalent one as substitution. They see variables as representing any valid mathematical object, including numbers or (potentially complex) expressions.

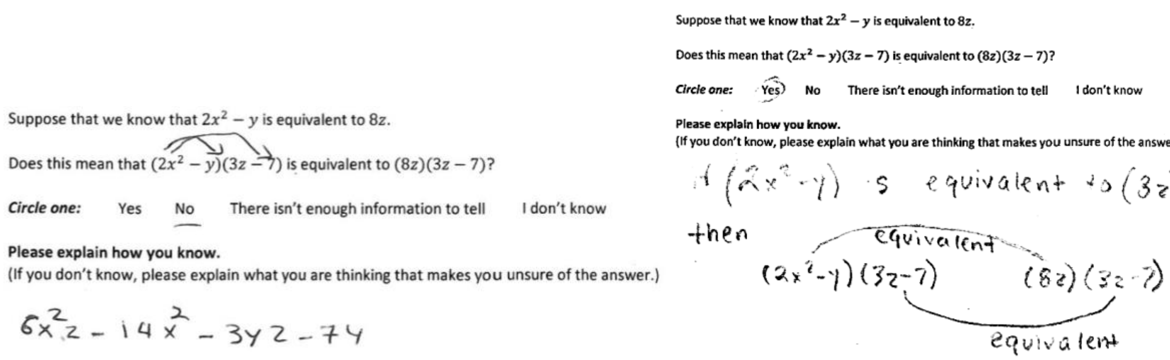
Development of the Model

This work draws on data collected from multiple classes across six years at a northeastern community college, including classroom observations, cognitive interviews, and open-ended questionnaires. These data were analyzed using conceptual analysis (Thompson, 2008) to generate and refine models of students’ thinking to explain their written work and utterances. We note that these models of students’ thinking are based on what the students communicate in the moment and are situated within the given task. Further, their strategies and responses may be impacted a myriad of factors, including but not limited to the wording of the question, the environment they responded in, or the established sociomathematical norms of the classrooms they participate (Yackel et al., 2000).

This analysis was heavily influenced by the work of Sfard (1995), and existing literature about the students’ definitions of mathematical concepts (Edwards & Ward, 2004) and their understanding of equals sign (e.g., Knuth et al., 2006). Sfard (1995) describes that students can conceive of a mathematical concept as a combination of two ways: operationally (as a process, often of computation) or structurally (abstract entities in and of themselves; Sfard, 1995). In terms of equality, similar language and ideas are used in the literature to describe the students’ conceptions of the equals sign, often either operationally (as a ‘do something symbol’; Kieran, 1981), or relationally (as a relationship between two entities; Knuth et al., 2006), though further refining these categories (Rittle-Johnson et al., 2011; Stephens et al., 2013) has been the focus of other research. Though research on equality is plentiful, research on substitution and substitution equivalence as a broader concept is comparatively minimal. For example, substitutive aspects of equivalence have been investigated in the context of arithmetic (Jones & Pratt, 2012), and Musgrave, Hatfield, and Thompson (2015) have found that secondary teachers had particular difficulty correctly applying a given substitution property to expressions when they found operations to be unfamiliar or had difficulty thinking of symbols simultaneously as both a process and an object. They argue that if teachers are having difficulties with these ideas, then these are likely stumbling blocks for students as well.

Vignettes: A Model in Action

We now provide examples of students' written work to illustrate how one might use the model we present here. These are intended to highlight the continuum of the operational and structural views. To see how students' views of transformation as substitution equivalence can vary along this spectrum, we present two developmental elementary algebra students' responses about assessing whether or not two expressions are equivalent (Figure 2), where the first response (Figure 2a) exemplifies an operational view and the second response (Figure 2b) exemplifies a structural view. The first student's response (Figure 2a) appears to foreground computation and symbolic manipulation. In cognitive interviews (not included here because of lack of space), students on similar problems have explained similar work by stating that they can only tell if two expressions are equivalent if they both simplify to the same final "answer", so this approach may happen when students have an internal computational definition of equivalence as "expressions that simplify to the same thing". Regardless, this student's response foregrounds computation, and hence would be considered as an operational view of transformation. In contrast, the response in Figure 2b illustrates exactly how the two equivalent subexpressions are substituted into the larger expressions using arrows to indicate the relationship between each piece and to highlight the structure of the two expressions. They map each unified subexpression in the first expression to an equivalent unified-subexpression in the same place in the second expression, in order to illustrate how they know that the two expressions are equivalent. Though the student doesn't explicitly use the substitution, we do see evidence that they are looking at the underlying structure and visualizing a replacement or exchange of one equivalent sub-part with another.



(a) (b)

Figure 2: *Examples of students' responses rooted in an operational view (a) and structural view (b) of equivalence*

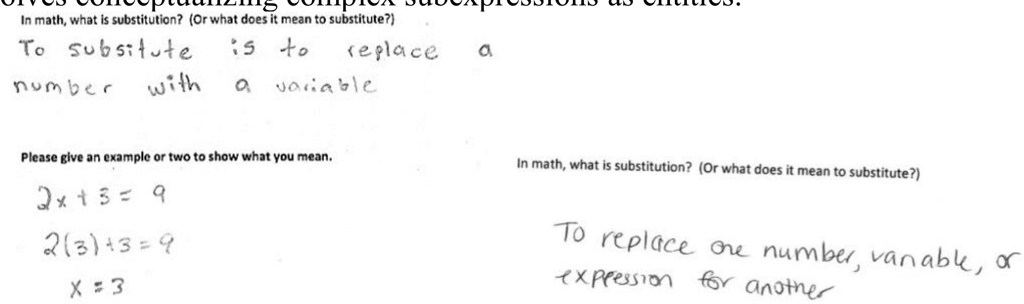
Students' Definitions of Equivalence

To see how students' definitions of equivalence can vary along this spectrum, we refer to the previous two examples and consider the definitions of equivalence the students seem to be evoking. These responses exemplify operational and structural definitions of equivalence, respectively. In the first response (Figure 2a), the student attempted to simplify the expressions to determine whether they are equivalent, and then appeared to decide that they are not equivalent after they could not immediately simplify them both to the same expression. This definition of equivalence appeared to be computational (e.g., "two expressions are equivalent only if they simplify to the same thing"), and their work doesn't seem acknowledge the equivalence within their work. Because the student abandoned the attempt after this did not work, this suggests that they did not see a way to use the structure of the given expressions to determine equivalence beyond simplifying both sides to see if the results are the same.

In contrast, the response in Figure 2b that the student may have a structural definition of equivalence. In this example, they are drawing on the structure of two complex expressions to show how they map to one another in such a way that each subexpression is either the same or equivalent, and leverage that equivalence to show that the final result will be equivalent. This apparent definition of equivalence appears to be well-defined and potentially could be a fixed trait of a set of objects.

Students' Definitions of Substitution

To exemplify the differences along this spectrum, we look at two students' definitions of substitution (Figure 3). Throughout data collection, the response in Figure 3a ("putting a number in for a letter") is one of the most common given by students at all levels, from elementary algebra through linear algebra. This narrow definition of substitution would be considered operational, while the response in Figure 3b would be considered structural view of substitution. This is because their definition affords a greater variety of terms to be replaced for one another, which involves conceptualizing complex subexpressions as entities.



(a) *(b)*

Figure 3: *Examples of an operational (a) and structural (b) definition of substitution.*

In order to see how student views of substitution may impact their view of transformation of expressions, we further examined students' responses to a task to identify instances of substitution, and found that their responses were typically consistent with their definitions (e.g., only recognizing transformation as substitution when it involved a number being substituted in for a letter if that was their stated definition); we include one example of this in the next section.

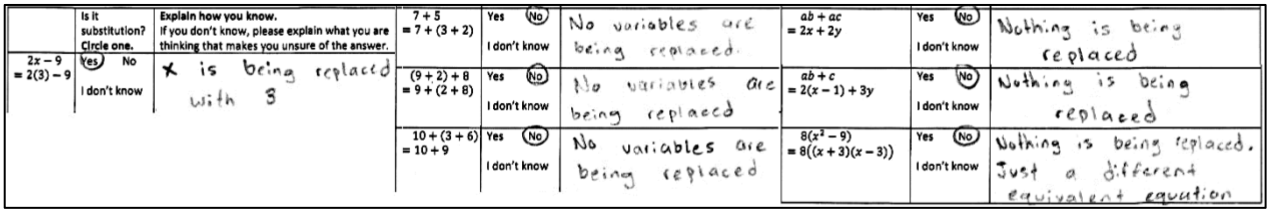
Using the Framework to Analyze Student work longitudinally

In order to illustrate the potential of this model for deeper analysis, we consider responses from an Algebra I student (whom we call Epsilon, like ε) across multiple tasks and points in time.

Substitution: We first consider Epsilon's definition of substitution (Figure 3a), where they have given an operational definition, rather than a structural one. This correlates with the extent to which they identify different computations as substitution in the following work (Figure 4).

We can see in Figure 4 that Epsilon rarely identified computation as substitution when it was more complex or generalized. They notice, for example, that the expressions in the last example in Figure 4 are equivalent, but they do not see replacement of the subexpression $x^2 - 9$ with

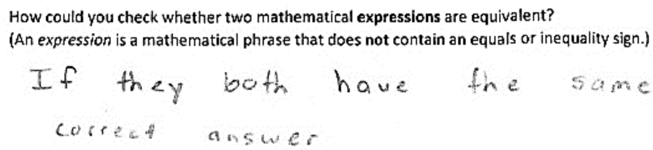
$(x+3)(x-3)$ as an instance of substitution ("nothing is being replaced"), which is consistent with the more limited operational definition of substitution that they gave in Figure 3a.



	Is it substitution? Circle one.	Explain how you know. If you don't know, please explain what you are thinking that makes you unsure of the answer.
$2x-9 = 2(3)-9$	(Yes) No I don't know	x is being replaced with 3
$7+5 = 7+(3+2)$	Yes (No) I don't know	No variables are being replaced.
$(9+2)+8 = 9+(2+8)$	Yes (No) I don't know	No variables are being replaced
$10+(3+6) = 10+9$	Yes (No) I don't know	No variables are being replaced
$ab+ac = 2x+2y$	Yes (No) I don't know	Nothing is being replaced
$ab+c = 2(x-1)+3y$	Yes (No) I don't know	Nothing is being replaced
$8(x^2-9) = 8((x+3)(x-3))$	Yes (No) I don't know	Nothing is being replaced. Just a different equivalent equation

Figure 4: *Epsilon's interpretations of substitution in specific contexts*

Equivalence: Now we consider Epsilon's definition of equivalent expressions (Figure 5).

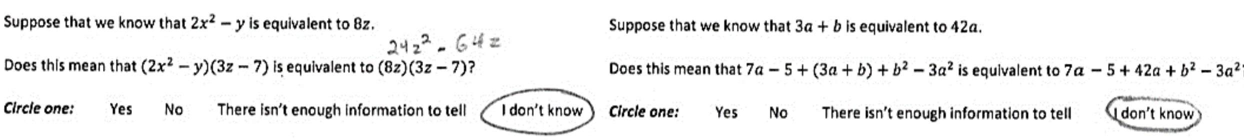


How could you check whether two mathematical **expressions** are equivalent?
(An *expression* is a mathematical phrase that does **not** contain an equals or inequality sign.)
If they both have the same correct answer

Figure 5: *Epsilon's definitions of equivalent expressions*

Epsilon provided a seemingly correct (if perhaps incomplete or ill-defined) definition of equivalent expressions. We cannot be sure the extent to which they understand that expressions have to have the same value for every possible combination of variable values or that this applies to algebraic and not just arithmetic expressions, and the word "answer" is also ill-defined; however, this definition is in line with the standard definition used in algebra, and they have been able to correctly identify equivalent algebraic expressions in last example in Figure 4 (as well as other questions not shown here), suggesting that their definition is at least somewhat standard. Their definition also appears to be operational, as it is rooted in computations with expressions.

Substitution equivalence: Now we consider the extent to which Epsilon recognizes instances of substitution equivalence in certain algebra examples (see Figure 6).



Suppose that we know that $2x^2-y$ is equivalent to $8z$.
$24z^2 - 64z$
Does this mean that $(2x^2-y)(3z-7)$ is equivalent to $(8z)(3z-7)$?
Circle one: Yes No There isn't enough information to tell (I don't know)

Suppose that we know that $3a+b$ is equivalent to $42a$.
Does this mean that $7a-5+(3a+b)+b^2-3a^2$ is equivalent to $7a-5+42a+b^2-3a^2$?
Circle one: Yes No There isn't enough information to tell (I don't know)

Figure 6. *Epsilon's recognition of substitution equivalence in some examples*

In Figure 6, Epsilon does not recognize either example as substitution equivalence. On the left in Figure 6, they attempt to simplify one of the expressions, but this does not help them to identify whether the two expressions are equivalent. They do not appear to draw on the given fact that $2x^2-y$ is equivalent to $8z$ when attempting to determine if the two expressions are equivalent. This suggests that they may not have a notion of substitution equivalence or are unable to draw on it in this problem context. Epsilon's operational approach to determining if the two expressions are equivalent suggests that their operational conception of equivalence may be limiting Epsilon's ability to recognize and use substitution equivalence when performing mathematical transformations. Another barrier to Epsilon developing a robust notion of substitution equivalence and linking this to their transformation work may be their narrow notion of substitution itself. Just as they do not recognize most of the transformations in Figure 4 as substitution, they likely do not recognize the transformations in Figure 6 as substitution either.

Potential impacts of instruction: Epsilon was actually part of a cohort that took part in a semester-long classroom intervention in which students were taught broader structural definitions of substitution, equivalence, and how to view transformation as substitution equivalence explicitly (as well as other concepts). One sample of Epsilon's work after the intervention can be seen in Figure 7.

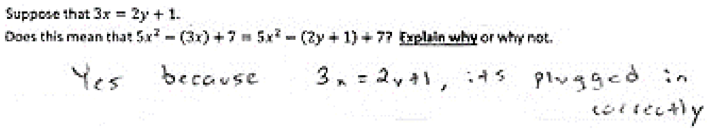


Figure 7: *Epsilon's identification of substitution equivalence after an intervention that addressed it explicitly*

After the intervention, Epsilon was not able to identify substitution equivalence in all cases, but they were able to recognize it in cases similar to questions where they had not recognized it at the start of the term. In Figure 7 we see how they are able to see a complex equation as an equivalence relationship between two structurally identical expressions where one equivalent subexpression could be conceptualized as having been substituted for another. Epsilon's use of the words "plugged in" are a common phrase often used by students to indicate substitute. We do note, however, that this language still suggests a computational approach. However, Epsilon is drawing on structural features of equivalent algebraic expressions through the lens of substitution equivalence, even if their approach still contains some computational elements. We have insufficient space to discuss the intervention at length here—we simply include this short example as a demonstration that more structural and well-defined definitions of substitution, equivalence, and substitution equivalence approaches to transformation can all be learned, even by students in developmental mathematics courses in college, given the right supports.

Conclusion

We have presented a model which describes how student definitions of substitution and equivalence may related to their ability to justify computational work through the lens of substitution equivalence. Using student examples, we have illustrated some of the affordances of this lens. We have demonstrated how students may struggle with substitution equivalence for different reasons, which may then require different instructional approaches. For example, if a student's definition of equivalence is ill-defined, it mays be important to find ways for them to correct their internal definition; whereas if a student has broad and well-defined definitions of substitution and equivalence, a more effective intervention may be one which helps them to see the connections between this existing knowledge and the work that they do when they perform transformations. These are very different approaches to solving what might on the surface look like similar errors, but which actually stem from very different underlying patterns of student thinking about the mathematics. Thus, we hope that this model may aid us to better tailor instruction to respond to student thinking, and to better think about how definitions of substitution and equivalence are presented in instruction. We have also shown through one particular student example that students are able to learn to think about transformation through a substitution equivalence lens with the right kind of instructional approaches, even when they are in developmental math courses. Further research is needed to better understand what approaches may be most effective, as well as to investigate which ways of thinking may be most productive for students in different contexts.

Acknowledgments

This work has been supported in part by NSF grant #1760491. The opinions expressed here are those of the authors and do not represent those of the granting agency.

References

Edwards, B. S., & Ward, M. B. (2004). Surprises from Mathematics Education Research: Student (Mis)use of Mathematical Definitions. *The American Mathematical Monthly*, *111*(5), 411–424. https://doi.org/10.1080/00029890.2004.11920092

Jones, I., & Pratt, D. (2012). A substituting meaning for the equals sign in arithmetic notating tasks. *Journal for Research in Mathematics Education*, *43*(1), 2–33. https://doi.org/10.5951/jresematheduc.43.1.0002

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*(3), 317–326. https://doi.org/10.1007/BF00311062

Kieran, C. (2004). The core of algebra: Reflections on its main activities. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The Future of the Teaching and Learning of Algebra. The 12th ICMI Study* (pp. 21–33). Springer. https://doi.org/10.1007/1-4020-8131-6_2

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, *37*(4), 297–312.

Rittle-Johnson, B., Matthews, P. G., Taylor, R. S., & McEldoon, K. L. (2011). Assessing Knowledge of Mathematical Equivalence: A Construct-Modeling Approach. *Journal of Educational Psychology*, *103*(1), 85–104. https://doi.org/10.1037/a0021334

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behavior*, *14*(1), 15–39. https://doi.org/10.1016/0732-3123(95)90022-5

Stephens, A. C., Knuth, E. J., Blanton, M. L., Isler, I., Gardiner, A. M., & Marum, T. (2013). Equation structure and the meaning of the equal sign: The impact of task selection in eliciting elementary students' understandings. *Journal of Mathematical Behavior*, *32*(2), 173–182. https://doi.org/10.1016/j.jmathb.2013.02.001

Thompson, P. W. (2008). Conceptual Analysis of Mathematical Ideas: Some Spadework at the Foundation of Mathematics Education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. SÈpulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education* (pp. 45–64). PME.

Wladis, C., Offenholley, K., Beiting-Parish, M., Griffith, S., Jaffe, E., Thakkar, N., & Dawes, D. (2020). A Proposed Framework of Student Thinking around Substitution Equivalence: Structural versus Operational Views. *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education*, 762–768.

Yackel, E., Rasmussen, C., & King, K. (2000). Social and sociomathematical norms in an advanced undergraduate mathematics course. *Journal of Mathematical Behavior*, *19*(3), 275–287. https://doi.org/10.1016/s0732-3123(00)00051-1