Modeling Student Definitions of Equivalence: Operational vs. Structural Views and Extracted vs. Stipulated Definitions

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This paper describes a model of student thinking around equivalence (conceptualized as any type of equivalence relation), presenting vignettes from student conceptions from various college courses ranging from developmental to linear algebra, and courses in between (e.g., calculus). In this model, we conceptualize student definitions along a continuous plane with two-dimensions: the extent to which definitions are extracted vs. stipulated; and the extent to which conceptions of equivalence are operational or structural. We present examples to illustrate how this model may help us to recognize ill-defined or limited thinking on the part of students even when they appear to be able to provide "standard" definitions of equivalence, as well as to highlight cases in which students are providing mathematically valid, if non-standard, definitions of equivalence. We hope that this framework will serve as a useful tool for analyzing student work, as well as exploring instructional and curricular handling of equivalence.

Keywords: Equivalence, Equation, Solution Set, Operational Thinking, Structural Thinking, Definitions

Equivalence is central to mathematics at all levels, and across all domains. In mathematics education, much research has focused on studying how students think about the equals sign in primary school (Knuth et al., 2006) through post-secondary (Fyfe et al., 2020), because students' conceptions of the equals sign have been shown to be related to their ability to perform arithmetic and algebraic calculations. However, equality is just one example of the larger concept of equivalence—other types of equivalence occur extensively throughout the K-16 curriculum, but are rarely, if ever, taught under one unifying idea called equivalence (Wladis et al., 2020). On the other hand, multiple types of equivalence (e.g., similar/congruent figures, function types, expressions or equations with the "same form") are contained in the *Common Core Mathematics Standards* but are never explicitly labeled as a type of equivalence.

When equivalence is not explicitly defined, students may extract their own non-standard, ill-defined, or unstable definitions, or they may inappropriately use the definition of equivalence from one area (e.g., expressions) in another area where it cannot be directly applied to obtain the "standard" definition expected of them (e.g., equations). In this paper we will illustrate this problem by presenting examples of students' definitions around equivalence and a model for analyzing student definitions, focusing on college students' definitions of equivalent equations. Student examples will be used as vignettes to illustrate the model. Our aim in presenting this model is to start a conversation about student definitions of equivalence and to present an initial framework that can then be further tested, refined, and revised by future empirical work.

Theoretical Framework

Formally, we define equivalence through the notion of an equivalence relation. The formal definition of an equivalence relation most often given in advanced mathematics classes is that of a binary relation that follows the identity, symmetry and transitive properties. However, another equivalent but more accessible definition of an ***equivalence relation*** is that of a partition on a set, or more informally: If we have a set of objects, and a rule for sorting objects into sets so that each goes into one and only one set (and this rule is mathematically well-defined), then this "sorting" is an equivalence relation, and two objects are equivalent if they belong to the same set.

We do not advocate at this time for teaching any particular group of students this generalized definition of an equivalence relation; we simply note that if we did want to discuss this more generalized definition with students, that the definition of a partition on a set is accessible to students at many different developmental levels (in fact, it bears a striking similarity to preschool sorting tasks in the mathematics curriculum). Our primary motivation for introducing this definition is to define equivalence rigorously—this includes not just definitions of equality, or insertionally equivalent equations (i.e., equations that have the same solution set), but also anything in the curriculum which meets the definition of an equivalence relation.

This will also help us to more precisely discuss student definitions. "Experts" often point out when students use "incorrect" definitions, but we note that in existing curricula and classroom practice the word equivalence is often ill-defined (or never explicitly defined), even though it takes on different definitions in different contexts. When students have no explicit definitions of equivalence, this presents several potential problems: students may incorrectly apply one definition to another context where it fails to produce the standard definition (e.g., definition of equivalent expressions to equations); they may have only ill-defined or operational definitions of equivalence which inhibit their ability to reason through problems; or they may use valid but non-standard definitions of equivalence, in which case they are being penalized for not knowing certain socio-mathematical norms even when they are reasoning correctly. We hope that the model presented here will allow us to better understand student thinking about equivalence, and to better recognize when these three situations (as well as others) might be occurring.

Model of Equivalence

Our model of student thinking about equivalence conceptualizes student definitions as existing on a two-dimensional plane with two axes: operational vs. structural conceptions of equivalence (Sfard, 1991, 1992, 1995), and extracted vs. stipulated definitions of equivalence (Edwards & Ward, 2004, 2008). In operational thinking, a student thinks of mathematical entities as a process of computation; in structural thinking, they think of them as abstract objects in and of themselves which can then been seen as objects for even higher-order processes; objects are seen as reified processes (e.g., $6x$ is seen as an object itself, and not just as the process of multiplying x by 6), however when students view something as an object which is not the reification of any process, this is called a *pseudostructural* conception (p.75, Sfard, 1992)[1].

Extracted definitions are created to describe actual observed usage (e.g., a student may extract a meaning for equivalence their instructional experiences, whether or not they have encountered an explicit definition). In contrast, *stipulated* definitions are those definitions that are stated explicitly—to determine if something fits the definition one must consult the definition directly (Edwards & Ward, 2008)[2]. We note that in our model, a stipulated definition may be

[1] We note that process and object dichotomy is also related to other theories such as APOS theory (Arnon et al., 2014) and the notion of a procept (Gray & Tall, 2011), but we have insufficient space to discuss these distinctions.
[2] Mathematical definitions are typically seen as stipulated rather than extracted, although there may be many (both correct and incorrect) features of students' concept images that stem from extracted rather than stipulated knowledge around the concept definition (see e.g., Edwards and Ward, 2004 for examples).

stipulated by the student or an authority—the key features we use to determine if a definition is stipulated in our framework is whether or it appears to be explicit, well-defined, and stable. We note that while we have displayed our model in Table 1 as a two-by-two grid for the sake of simplicity, these categories are not necessarily binary, but conceptualized as more of a spectrum. In that sense, Table 1 could perhaps better be represented by a 2D coordinate plane.

Table 1: Model of Student Thinking About Equivalence.

	Extracted Definition	Stipulated Definition
Operational Conception of Equivalence	**Pseudo-Process View**: Students see equivalence as a computational process, and their approaches to those processes are dictated by prior experience in ways that are extracted rather than stipulated. Definitions of equivalence are typically non-standard, ill-defined, and/or unstable.	**Process-View**: Students see equivalence as a process, but do process computations by referring to stipulated rules or properties. Students with this view may be able to perform calculations correctly but this does not necessarily translate to being able to use stipulated definitions to recognize equivalent objects.
Structural Conception of Equivalence	**Pseudo-object view:** The student is able to consider whether two objects are equivalent without reverting to an explicit computation, perhaps by considering the structure of the objects; but definitions of equivalence are typically extracted in some way from experience rather than based on stipulated definitions of equivalence, and as a result are typically non-standard, ill-defined, and/or unstable	**Object view:** The student is able to consider whether two objects are equivalent without reverting to an explicit computation, perhaps by considering the structure of the objects; definitions of equivalence used to determine equivalence are stipulated. The student conceptualizes equivalence classes (or solution sets) as objects, although they need not do this formally.

Method

Data for this study were collected from 124 students at an urban community college through open-ended questions in 18 different courses, from developmental elementary algebra (similar to Algebra I in high school) to linear algebra. Student responses were analyzed using thematic analysis (Braun & Clarke, 2006). Responses coded as indicative of an operational-view of equivalence provided evidence of thinking of equivalence as an algorithm; those coded as indicative of a structural-view of equivalence provided evidence of thinking of equivalence as a fixed trait of an object, or reasoning about equivalence via its general properties.

In coding student work, students often struggled provide definitions of equivalent equations for several different reasons. One issue appeared to be that students attempted to apply the definition of equivalent expressions to equivalent equations. For example, in Figure 1, we see the work of two students-- one in elementary algebra and one in linear algebra-- both of whom give somewhat similar definitions of equivalent equations. The elementary algebra student gives a more ill-defined definition ("same answer") but we see from the examples that they provide that they appear to be thinking about equivalent arithmetic expressions. We would classify this response as a pseudo-process view, as the definition is not well-defined, and because it appears to center around arithmetic calculation.

We see similar work by the linear algebra student in Figure 1, with some differences; they give broader examples of equivalence (describing also when two vectors are equivalent) and their definition is a bit more detailed ("when two quantities are the same on both sides of an equation"). But like the elementary algebra student in Figure 1, they conflate the definition of equivalent expressions with equations (they include an algebra example, but only show identical expressions as equal). Their definition of equivalent equations is also not fully well-defined ("check if both sides are the same"), because the word "same" here is not well-defined. While their answer does show signs of having been exposed to more examples of mathematical equivalence, this does not appear to have positively impacted their definition of equivalent equations; we would still classify their definitions as extracted, because they are ill-defined.

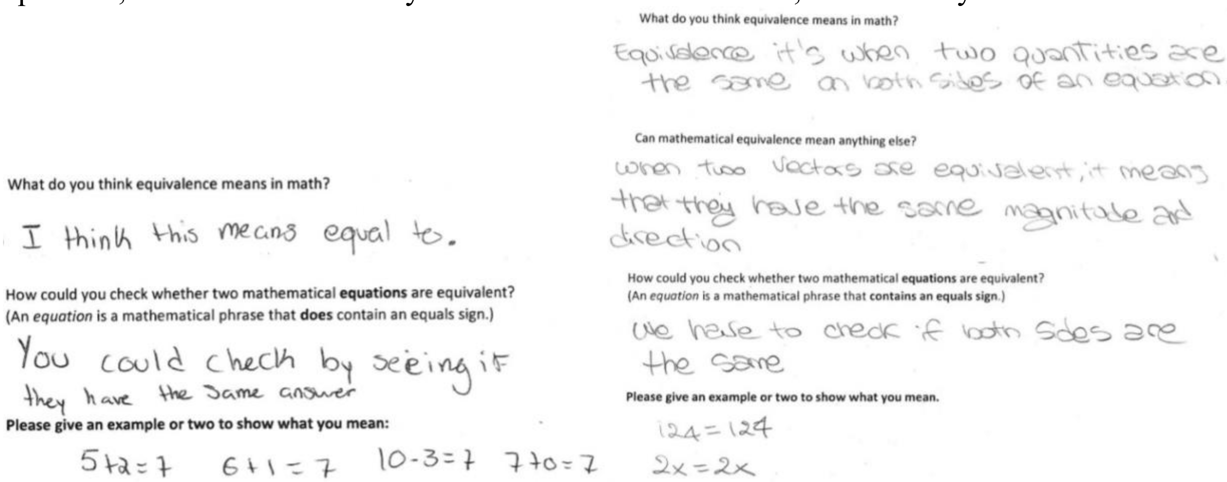


Figure 1: *Definitions from an elementary algebra student (on left) and a linear algebra student (on right), conflating the definition of equivalent expressions with equivalent equations*

Students who applied the definition of equivalent expressions to equations may even do this in a way that is mathematically valid (i.e., fits the definition of an equivalence relation), even though it is not one of the "standard" definitions of equivalent equations (e.g., same solution set).

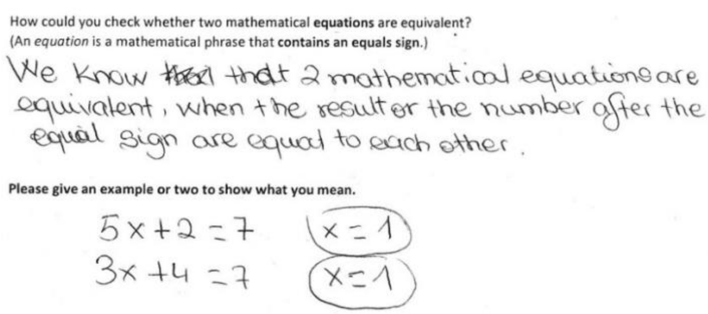
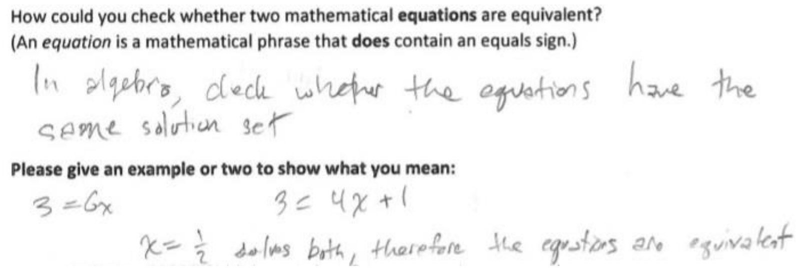
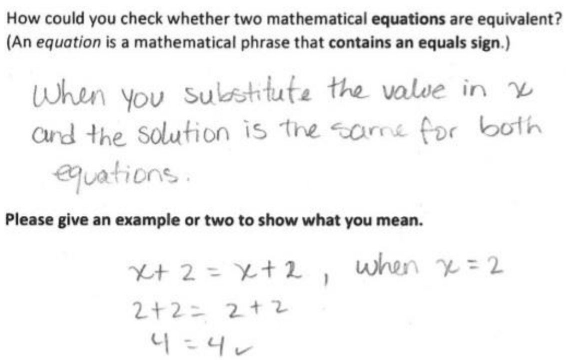


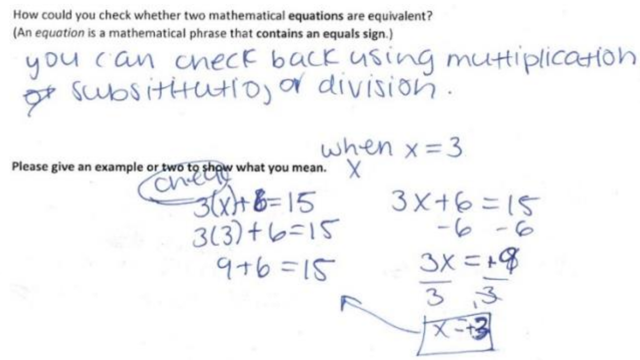
Figure 2. *Precalculus student's non-standard structural definition of equivalent equations*

Consider Figure 2, where a precalculus student has defined equivalent equations as two equations where "the result or the number after the equal sign are equivalent", and based on their examples, this seems to suggest that any equations of the form $expression = n$ for fixed n would be equivalent to one another. This is similar to definitions given by other students in other research (Wladis et al., 2020). This student is particularly interesting, because the two equations that they have given also happen to have the same solution set, so it is unclear if this is an implied part of their definition as well. Whether it includes this feature or not, we would classify this definition as a structural view even though it is a "non-standard" definition, because the student has given what could be a well-defined but alternate definition of equivalence (whether or not their definition is fully well-defined is unclear, as they haven't filled out all the details)[3].



(a) Calculus III student





(b) introductory stats student (c) intermediate algebra/precalculus student

Figure 3*: Examples of different ways that students used notion of "solving" in defining equivalent equations*

In contrast to the previous examples, some students did draw in some way on the notion of "solving" equations or the solution sets of equations when defining equivalence. However, the ways in which students drew on notions of "solving" also fell into different areas of our framework. Simply talking about the "solution" of an equation was not sufficient to classify work as either stipulated or structural even though it sounds like it is related to the standard insertional equivalence definition of equations (i.e., same solution set). In Figure 3(a), we see the work of a Calculus III student, who appears to have a well-defined and structural view of equivalent equations: they define equivalent equations as having the same solution set (seeming to conceptualize the solution set as a fixed object); and their definition appears to be well-defined, not just because of their definitions, but also because they have provided an example which shows that their interpretation of "same solution" appears to be the "standard" one. We note that this is critical, as many students used the language of "same solution" but actually meant it to describe equivalent sides of an equation (equivalent expressions) rather than equivalent equations. See, for example, the work of an introductory statistics student in Figure 3(b). This student wrote that two equations are equivalent if you "substitute the value in for x and the solution is the same for both equations": this sounds like the standard definition of equivalent equations (if an incomplete one that does not account for the possibility that x may have more than one value), however, looking at the example this student has provided, we see that to them "solution" actually denotes the quantity which results from simplifying each side of an equation (not the solution set of two different equations). In this sense, the students' definition is ill-defined, because the vocabulary that they are using appears to be ill-defined and has

[3] This student may be drawing on notions of equations with the "same form" (e.g., $y = mx + b$, $ax^2 + bx + c = 0$) which is another type of equivalence that is commonly used in the algebra curriculum, even if it is not called equivalence in the curriculum (however, "same form" could in fact be codified as a formal equivalence relation, and students may be noticing this when they draw on it in their equivalence definitions (Wladis et al., 2020).

multiple, perhaps vague, meanings. For these reasons, we would classify this work in (b) as a pseudo-process view, even though on the surface the definition initially looked very similar to the one given in (a). The third example of student work in Figure 3(c) shows another common approach that students used, in which they drew on notions of solving when asked about equivalent equations, but struggled to relate these notions to any well-defined definition of equivalence. This student has solved an equation and checked the solution by substituting it back into the original equation; however, it is unclear what the definition of equivalent equations is, or even which two objects the student is claiming are equivalent (perhaps equivalence for them is not about the relationship between two objects, but is instead names a process of checking the solution of an equation). Because of this, we classify this as a pseudo-process view—there is no well-defined stated definition, and the student's focus is on computation.

Students also gave a variety of other non-standard definitions of equivalence that might possibly have been well-defined definitions of equivalence relations (e.g., equivalent arithmetic equations as ones that express the same additive relationship; equivalent algebraic equations which express the same relationship between the variables), which for the sake of space we do not share here. However, we note that by de-coupling our categorization of student definitions of equivalence from notions of what is "standard" and thinking more carefully about the extent to which student definitions of equivalence are stipulated definitions which meet the criteria of an equivalence relation; and the extent to which student conceptions of equivalence are structural or operational, we may be able to achieve two critical goals more effectively: 1) we may be able to better identify student thinking which "sounds right", but is actually ill-defined; and 2) we may be able to identify valid student thinking that simply does not adhere to "standard" definitions. Both of these goals may better help us to tailor instruction to students.

We now briefly describe some overall trends we found in coding open-ended questions on definitions of equivalence (Table 2). Students primarily associated equivalence with equality, and rarely cited other forms (e.g., equivalent equations), although the incidence of non-equality examples rose somewhat with course level. Similarly, students at all levels were extremely likely to give ill-defined or vague definitions of equivalence when asked. In terms of student definitions of equivalent equations, most students conflated this with the definition of equivalent expressions; this did not appear to improve with course level, suggesting that the lack of explicit definitions of equivalent equations in textbooks and curricula (Wladis et al., 2020) may well be contributing to student difficulty in understanding the how definitions of equivalence vary in different contexts. Some of these definitions, while non-standard, may have qualified as formal equivalence relations, and therefore mathematically valid reasoning—the prevalence of this was not correlated with course level, suggesting that students at all levels may sometimes be generating valid but non-standard definitions. Many students associated equivalent equations with solving, but this was rarely done in a well-defined way: roughly one quarter of all students at all course levels solved an equation but did not relate this in any well-defined way to the definition of equivalent equations (most commonly this involved solving a single equation, and then checking the answer, with no clear mention of which two things were actually equivalent); fewer students did this at levels of precalculus and above, but the differences by course level were not large. Small numbers of students did interpret equivalent equations to mean equations which have the same solution set, and did so in a well-defined way; this was slightly more common as course levels went up; however, the vast majority of these students did so in a operational way (i.e., solved two equations and said they were equivalent, without discussing the solution set in a more general or structural way); this is perhaps to be expected, given the operational way in which the question itself was phrased, however, this does follow patterns observed in questions without this more operational wording, such as the more general question about the definition of equivalence given on this set of questions (although student tendencies to use structural rather than operational definitions did increase with course level). However, we note that overall, structural and well-defined definitions were rare among all students, suggesting that instruction which specifically includes explicit stipulated definitions, and which encourages structural reasoning is needed at all levels.

Table 2. Summary of student definitions of equivalence

	elem. alg. or below	inter. alg. or 100-level	200-level or above
general definition of equivalence			
ill-defined or vague	67%	71%	60%
cited equality	94%	87%	80%
other valid definition	0%	3%	16%
operational definition	41%	18%	17%
structural definition	0%	2%	17%
how to tell if two equations are equivalent			
conflated w/ equiv. expressions	44%	48%	44%
of these, possible WD defn.	19%	6%	16%
finding solution set, operational	0%	3%	8%
related to "solving" but ill-defined	22%	29%	16%
solution set, structural	0%	2%	4%
total *n*	36	62	25

Discussion and Conclusion

The model of student thinking around definitions of equivalence that is presented here aims to refocus our attention from whether definitions look like a "standard" definition so that we consider more carefully the extent to which student definitions are explicit and well-defined as well as the extent to which students are able to think structurally rather than just operationally. Using this lens allows us to pinpoint places where students appear to understand a standard definition but upon further reflection we find that this definition is not well-defined or is wholly operational, limiting the student's ability to use it. On the other hand, it also allows us to recognize when students' reasoning is mathematically valid, and when students are recognizing more generalized instances of equivalence relations, even when they are not able to define them fully. Evidence from student examples here suggests that students do notice many kinds of "sameness", yet struggle to articulate this in mathematically well-defined ways, just as they struggle to articulate "standard" definitions of equivalence in well-defined ways. This suggests that students are capable of noticing and assimilating more generalized notions of equivalence, but need more explicit definitions and language in order to be able to do this rigorously. Future research is necessary to better understand what kinds of explicit definitions of equivalence work best for students in different contexts, and the extent to which discussions of the more general notion of an equivalence relation might be helpful in instruction. This framework may also be able to serve as a measure of instruction and curricula, to assess how the concept of equivalence is presented to students as they are learning at various levels in the curriculum.

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