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Entanglement spread area law in gapped ground states

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Ground-state entanglement governs various properties of quantum many-body systems at low temperatures and is the key to understanding gapped quantum phases of matter. Here we identify a structural property of entanglement in the ground state of gapped local Hamiltonians. This property is captured using a quantum information quantity known as the entanglement spread, which measures the difference between Rényi entanglement entropies. Our main result shows that gapped ground states possess limited entanglement spread across any partition of the system, exhibiting an area-law scaling. Our result applies to systems with interactions described by any graph, but we obtain an improved bound for the special case of lattices. These interaction graphs include cases where entanglement entropy is known not to satisfy an area law. We achieve our results first by connecting the ground-state entanglement to the communication complexity of testing bipartite entangled states and then devising a communication scheme for testing ground states using recently developed quantum algorithms for Hamiltonian simulation.

The ground states of local Hamiltonians are quantum many-body states with central importance in condensed-matter physics, quantum chemistry and quantum complexity theory. A unique property of these states is the presence of multipartite entanglement, which makes them suitable for quantum computation[1] and leads to novel phenomena such as exotic phases of matter[2–4] and quantum phase transitions[5]. However, the existence of entanglement also complicates the theoretical and numerical study of these states, since an increase in entanglement is often tied with an increase in complexity of their representation.

There have been extensive efforts in the past to characterize and classify various features of ground-state entanglement. An important problem in this direction is the 'area-law conjecture' for entanglement entropy in the ground state of gapped local Hamiltonians[6–8]. This conjecture states that in a gapped ground state $|\Omega\rangle$, the amount of entanglement between a subset of particles A and its complement B is at most proportional to the number of interaction terms that cross the boundary ∂A (Fig. 1). Here the entanglement is measured by the von Neumann entropy of the eigenvalues of the reduced state $\Omega_A = \mathrm{tr}_B|\Omega\rangle\langle\Omega|$ in subset A. This behaviour is drastically different from the generic situation where the entanglement across the cut ∂A scales with the size of the smaller partition $|A|$ rather than $|\partial A|$. The entanglement entropy area law has been rigorously proven for one-dimensional (1D) gapped systems, with some remarkable implications that justify the success of numerical methods like the density matrix renormalization group algorithm[9,10] and the validity of the matrix-product state representation of ground states[6,7].

Unfortunately, despite research efforts spanning more than a decade[11–16], the entanglement entropy area law has remained a conjecture for lattices of higher dimensions, non-Euclidean geometries or more general interaction graphs. In fact, a 'generalized area law' is known to be false when the systems are not placed on a lattice[17]. Such systems have become increasingly relevant in physics in connection to quantum chaotic dynamics, holographic duality and disordered systems[18], and have also been engineered in various experimental setups[19–21].

In this Article, we reveal a new structural property for ground-state entanglement that applies to all gapped local Hamiltonians with arbitrary interaction graphs. The entanglement entropy area-law conjecture concerns the von Neumann entropy of Ω_A, which could be seen as an average of the log of the eigenvalues of the state. We consider, instead, a variance-like quantity of the log of the eigenvalues, which is a well-known quantum information quantity called the entanglement spread. This quantity gives an estimate of the spread of the eigenvalues of state Ω_A (Fig. 1). We prove that as long as the Hamiltonian is gapped, its ground state possesses limited entanglement spread on general interaction graphs, exhibiting an 'area-law' scaling for this quantity. We further show that on lattices, this can be improved to sub-area scaling. The strength of our results is that it relies on no assumption other than the presence of a gap and uniqueness of the ground state, which are the same assumptions made in the 1D entanglement entropy area laws.

Our findings relate to a series of previous works[22–27] that suggest in two-dimensional (2D) gapped systems, the ground state in region A resembles the Gibbs state of a local Hamiltonian (known as the modular or entanglement Hamiltonian) acting only on boundary ∂A. We show how the sub-area-law scaling of the entanglement spread proved here for lattices matches the predicted locality of the modular Hamiltonian. We also reveal a connection between the entanglement spread and efficient algorithms for contracting the tensor network representations of 2D gapped ground states. Building on another study[28], we show that if a 2D gapped ground state is represented by a projected entangled-pair state (PEPS)[29] that satisfies a variant of our sub-area entanglement spread without any smoothing, then the information from this PEPS can be used to efficiently compute the expectation value of any local observable in the system.

Statement of main result

Consider a many-body quantum system comprising many qudits (d-dimensional systems) interacting according to Hamiltonian H. This can define an interaction graph (Fig. 1) where vertices

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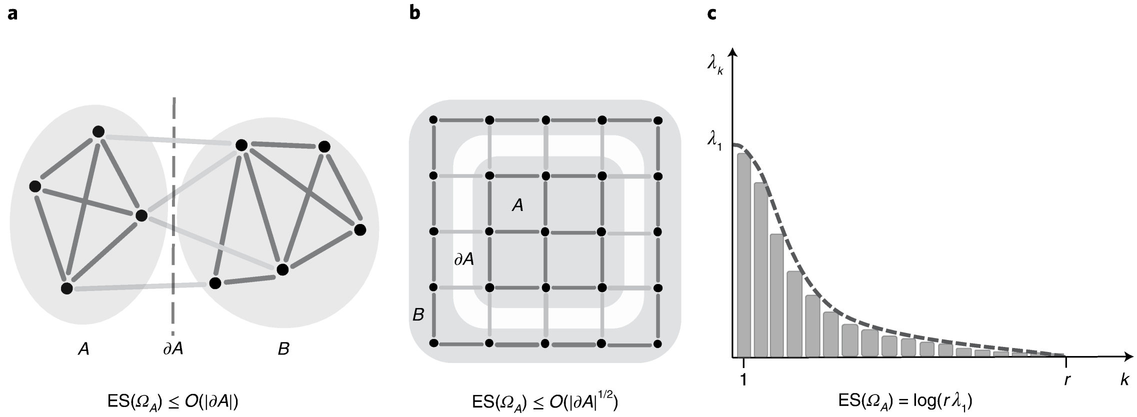

Fig. 1 | Entanglement spectrum across a general partition. a, Interacting quantum systems with a general interaction graph partitioned into two parts A and B (shaded regions) with boundary ∂A. **b**, A similar partition for a system on a lattice. **c**, Profile of the eigenvalues (also known as Schmidt coefficients) $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_r$ of the reduced ground state in region A. This distribution fully encodes the information about the bipartite entanglement between particles in regions A and B. Up to a smoothing step (explained in the main text) and the entanglement spread across the cut ∂A is defined as $\log(r\boldsymbol{\lambda}_1) \approx \log(\boldsymbol{\lambda}_1/\boldsymbol{\lambda}_r)$. We show that the entanglement spread scales as $O(|\partial A|)$. We improve this to $O(\sqrt{|\partial A|})$ for lattice Hamiltonians. In comparison, the area-law conjecture for entanglement entropy asserts that the entropy of the distribution of Schmidt coefficients is bounded by the size of the cut $|\partial A|$, that is, $S(\Omega_A) = O(|\partial A|)$.

correspond to qudits and edges to two-body interaction terms; our results apply equally well to k-local Hamiltonians for $k>2$, but the interactions are easier to visualize for $k=2$.

Suppose we fix a partition of qudits into two parts A and B. We denote the ground state by $|\Omega\rangle_{AB}$ and the non-zero eigenvalues of the reduced state $\Omega_A = \mathrm{tr}_B|\Omega\rangle\langle\Omega|$ (or equivalently Ω_B) by $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_r$. These eigenvalues, which form a probability distribution, are often called the Schmidt coefficients. We assume these coefficients are arranged in the descending order, that is, $\boldsymbol{\lambda}_1 \geq \boldsymbol{\lambda}_2 \geq \ldots \geq \boldsymbol{\lambda}_r > 0$, and r is referred to as the Schmidt rank of state $|\Omega\rangle_{AB}$.

We are interested in obtaining a bound on the spread of the distribution of these Schmidt coefficients $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_r$. A natural quantity for measuring this spread is the entanglement spread, denoted by $\mathrm{ES}(\Omega_A)$, which in its simplest form is defined by $\mathrm{ES}(\Omega_A) = \log(r\boldsymbol{\lambda}_1)$ (ref. [30]). The connection between this quantity and the spread of the distribution of Schmidt coefficients can be better seen by noticing that $\mathrm{ES}(\Omega_A) \approx \log(\boldsymbol{\lambda}_1/\boldsymbol{\lambda}_r)$, which follows from $r \approx 1/\boldsymbol{\lambda}_r$. Hence, if all the Schmidt coefficients are concentrated around the same value, this quantity is small, and as these coefficients spread out, it grows larger.

This definition of entanglement spread is, however, not robust to small perturbation of the spectrum of eigenvalues. For instance, the addition of a series of small eigenvalues to the tail of the distribution significantly affects the Schmidt rank r, but alters the state negligibly in trace distance. To this end, we allow for some small $\delta \in [0,1]$ fraction of Schmidt coefficients to be removed from $|\Omega\rangle_{AB}$ (Fig. 2) before finding their entanglement spread. Let $\boldsymbol{\lambda}_1'$ be the largest Schmidt coefficient in the remainder and r' be the remaining Schmidt rank. The robust or smooth entanglement spread, denoted by $\mathrm{ES}_\delta(\Omega_A)$, is defined by $\log(r'\boldsymbol{\lambda}_1')$ (ref. [30]). This can be alternatively written in an entropic version as

$$\mathrm{ES}_\delta(\Omega_A) = S^\delta_{\max}(\Omega_A) - S^\delta_{\min}(\Omega_A), \quad (1)$$

where $S^\delta_{\max}(\Omega_A) = \log r'$ is the smooth max-entropy and $S^\delta_{\min}(\Omega_A) = -\log \boldsymbol{\lambda}_1'$ is the smooth min-entropy, both of which are examples of Rényi entanglement entropies. Although it is known that von Neumann entropy captures the performance of certain information-processing tasks involving an asymptotic number of copies of a quantum state, the smooth min-entropy and max-entropy have operational meanings in the 'single-shot' regime where only a single copy of a quantum state is considered[31–33]. This single-shot property has been used before to derive an entanglement entropy area law for 1D systems[8]. As discussed later, the entanglement spread $\mathrm{ES}_\delta(\Omega_A)$ also has a single-shot operational interpretation in terms of the communication complexity of testing $|\Omega\rangle_{AB}$, which is the basis of our analysis. Having set the definition of entanglement spread, we are now ready to state our main result.

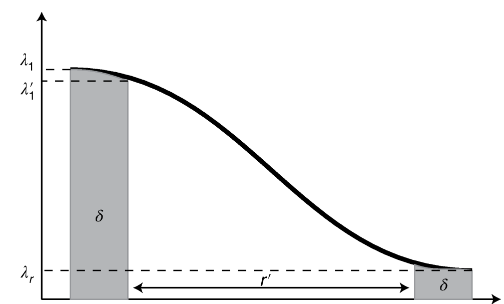

Fig. 2 | Smooth entanglement spread. To define a robust version of entanglement spread, we remove the δ fraction of mass from both 'ends' of a given distribution. The smooth entanglement spread is then defined as $\mathrm{ES}_\delta(\Omega_A) = \log(r'\boldsymbol{\lambda}')$.

Main result. Given a local Hamiltonian over an arbitrary interaction graph, the entanglement spread of a subset of qudits A in the ground state $|\Omega\rangle_{AB}$ is bounded by

$$\mathrm{ES}_\delta(\Omega_A) \leq O\left(\frac{|\partial A|}{\gamma} \times \log\left[\frac{1}{\delta}\right]\right), \quad (2)$$

where γ is the spectral gap above the ground state, $|\partial A|$ is the size of the boundary of A and δ is the smoothness parameter in equation (1). This establishes an area-law scaling for the entanglement spread. Some other factors with logarithmic dependence on $|\partial A|$, γ and δ are hidden in $O(\cdot)$.

This bound can be improved to a sub-area-law scaling if the interaction graph is a lattice of a constant dimension. That is, qudits are located at vertices of the cubic lattice $\mathbb{Z}^D$ with $D = O(1)$,

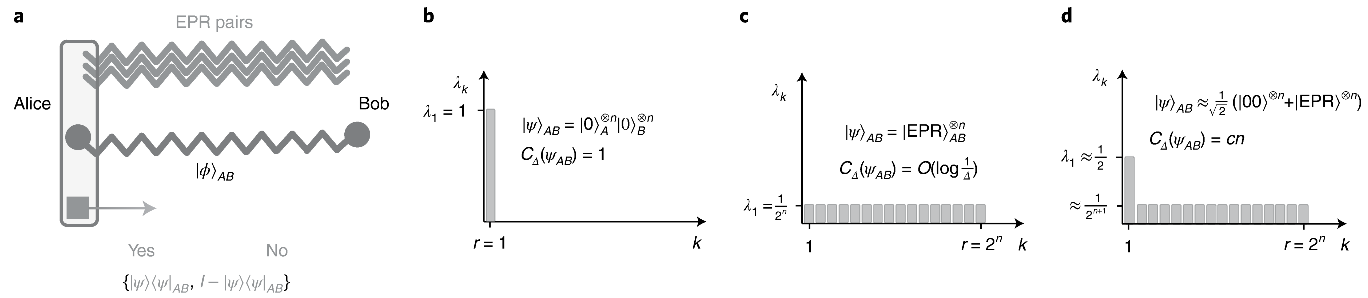

Fig. 3 | Communication complexity of identity testing. a, Suppose a bipartite state $|\phi\rangle_{AB}$ is shared between Alice and Bob who also have access to unlimited shared EPR pairs. The parties want to determine if their shared state is the state $|\psi\rangle_{AB}$ or not by performing the two-outcome measurement $\{|\psi\rangle\langle\psi|_{AB}, \mathbb{1} - |\psi\rangle\langle\psi|_{AB}\}$. In one round of the communication protocol, Alice may perform a joint unitary operation on her share of $|\phi\rangle_{AB}$, the EPR pairs and an ancillary register, which is then sent to Bob who proceeds in the same way. **b**, Testing a product state simply requires one qubit of communication since Alice can coherently test if her registers are in the desired state. She then sends the answer to Bob who compares that with the result of testing his registers by performing a projective measurement. **c**, One might initially suspect that testing the maximally entangled state requires exchanging a large number of qubits. However, it turns out that by using quantum expanders[17], one can perform this up to an error Δ by exchanging $C_\Delta(\psi) = O(\log[1/\Delta])$ qubits independent of size n. **d**, When the target state is a superposition of the last two cases, it holds that $C_\Delta(\psi) = cn$ for some constant c that depends on Δ. This is because a coherent measurement on this state can be also used to reflect about it, which, in turn, allows one to create n EPR pairs from the product states. However, it is known that this requires at least $O(n)$ qubits of communication[39]. This motivates the bound in equation (1). Indeed, in the first two examples, $\mathrm{ES}(\psi) = 0$, whereas in the third case, $\mathrm{ES}(\psi) = O(n)$.

and they only interact if they are close in Euclidean distance. In this case, we have

$$\mathrm{ES}_\delta(\Omega_A) \le O\left(\sqrt{\frac{|\partial A|}{\gamma}} \times \log\left[\frac{1}{\delta}\right]\right). \qquad (3)$$

The intuition behind the quadratically better bound in equation (3) comes from the exponential decay of correlations for gapped ground states on finite-dimensional lattices[34–36], which implies that the distant qudits along boundary ∂A are almost uncorrelated. This suggests that the ground state across the boundary is roughly in a product form $|\phi\rangle_{AB}^{\otimes|\partial A|}$ composed of $O(|\partial A|)$ partially entangled states $|\phi\rangle_{AB}$. By using conventional concentration bounds[37], it can be shown that the smooth entanglement spread obeys $\mathrm{ES}_\delta(\phi_A^{\otimes k}) = O(\sqrt{k})$ matching the scaling of the bound in equation (3). In Supplementary Information, we show that the bound in equation (3) for lattices cannot be, in general, improved.

Entanglement spread and test of entanglement

The key to our findings is a connection to the field of quantum communication complexity. A basic question here concerns the scenario where two parties (Alice and Bob) want to test whether they share a specific entangled state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, with Alice and Bob owning registers $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. The testing is achieved by performing the measurement $\{|\psi\rangle\langle\psi|_{AB}, \mathbb{1} - |\psi\rangle\langle\psi|_{AB}\}$ and accepting if the first outcome is obtained. Since, in general, the state $|\psi\rangle_{AB}$ is an entangled state, Alice and Bob need to exchange qubits to perform this operation. For instance, Alice can send her register $\mathcal{H}_A$ to Bob who then performs the joint operation on $\mathcal{H}_A \otimes \mathcal{H}_B$ and sends back Alice's register. But an improved performance can be obtained by adapting our test to the state $|\psi\rangle_{AB}$ and also when the parties are only required to implement an approximate testing measurement $\{K, \mathbb{1} - K\}$ such that $||K - |\psi\rangle\langle\psi|_{AB}|| \le \Delta$. The communication complexity, $C_\Delta(\psi_{AB})$, is defined as the minimum number of qubits that the parties need to exchange to perform the test with an error of at most Δ.

Since by applying local unitaries, a bipartite state $|\psi\rangle_{AB}$ can always be transformed to the Schmidt form $\sum_{\ell=1}^{r} \sqrt{\lambda_\ell}|\ell\rangle_A|\ell\rangle_B$, the communication complexity $C_\Delta(\psi_{AB})$ depends only on the Schmidt coefficients $\lambda_1,\ldots,\lambda_r$. Figure 3 presents the communication setup in more detail and demonstrates the dependency of $C_\Delta(\psi_{AB})$ on the Schmidt coefficients for various examples. These examples reveal that it is not the amount of entanglement that determines the cost of testing the state but rather the spread of the distribution of Schmidt coefficients. This motivates a general connection between the communication complexity and entanglement spread $\mathrm{ES}_\Delta(\psi_{AB})$ of the state $|\psi\rangle_{AB}$. To this end, we show that

$$C_\Delta(\psi_{AB}) \ge \mathrm{ES}_{\Delta'}(\psi_A) - 1, \qquad (4)$$

where the smoothness parameter $\Delta' = \Delta^{2/3}$. Different variants of this bound have been proven before, first in ref. [30] and also in refs. [38,39]. In Supplementary Section 3.1, we prove a version of this bound tailored for our own application. An interesting aspect of the bound in equation (4) is that it continues to hold even if we allow Alice and Bob to use unlimited EPR pairs during their testing protocol (Supplementary Section 3.1 discusses such EPR-assisted testing protocols). In Methods, we explain how this can be exploited to improve our results.

With these tools in hand, our strategy is to devise an EPR-assisted testing protocol for the gapped ground state $|\Omega\rangle_{AB}$ and use its communication complexity as in equation (4) to obtain the upper bound in equation (2) on the entanglement spread. Following other work[40,41], we will refer to the operator K in the testing measurement $\{K, \mathbb{1} - K\}$ that the parties implement as an EPR-assisted approximate ground-state projector (AGSP). In Methods, we overview our testing protocol that involves a novel construction of an AGSP based on the quantum phase estimation (QPE) algorithm. We show that Alice and Bob can implement a distributed version of QPE with communication complexity $O(|\partial A|/\gamma)$ and hence achieve the upper bound in equation (2).

Discussion

How do our bounds on entanglement spread relate to entanglement entropy area laws? One implication of our results is that if one can prove an area law for $S_{\min}^\delta(\Omega_A)$, then this yields an area law for $S_{\max}^\delta(\Omega_A)$ and hence for entanglement entropy. The utility of this is that smooth min-entropy area law may be easier to prove in comparison to the entanglement entropy area law. For instance, for specific models such as the sign-free local Hamiltonians, proving a min-entropy area law can be reduced to a classical problem[42].

The area-law scaling for entanglement spread stated in equation (2) provably applies even to interaction graphs in which the entanglement entropy is known not to satisfy an area law. An example of such a local Hamiltonian is constructed in another work[17] whose interaction graph can be partitioned into sets A and B such that the size of the boundary $|\partial A|=1$, but the entanglement entropy of A, in violation of the area law, scales as $S(\Omega_A)\geq|A|^c$ for some positive constant $c<1$. Nevertheless, it is shown elsewhere[17] that the ground state $|\Omega\rangle_{AB}$ across this cut is a maximally entangled state, which has a constant (in fact, vanishing) entanglement spread according to equation (1) (Fig. 3c), as predicted by our area-law bound on $\mathrm{ES}_\delta(\Omega_A)$ in equation (2). When combined with our previous discussion on area law, this suggests the following: the ground state of a gapped Hamiltonian always exhibits a small entanglement spread. However, it either has a large min-entropy (such as maximally entangled states in the counter-example Hamiltonian[17]) and hence does not obey an entropy area law or it possesses limited min-entropy (such as 1D ground states) and thus obeys an entropy area law.

Our results connect to a body of work that aims to understand the properties of gapped phases of matter by studying not only the entanglement entropy but also more broadly the features of the entanglement spectrum of ground states. Given the reduced state Ω_A, the modular (or entanglement) Hamiltonian H_{mod} is defined such that $\Omega_A=\mathrm{e}^{-H_{\mathrm{mod}}}$. In other words, Ω_A corresponds to the Gibbs (thermal) state of H_{mod}. The eigenvalues of H_{mod} are known as the entanglement spectrum[22]. Various works, initiated elsewhere[22], relate the entanglement spectrum to different characteristics of a given phase[23–27]. These studies suggest that in 2D gapped systems, the entanglement spectrum often has features of the spectrum of a 1D local Hamiltonian, although its features are not always isomorphic to the original Hamiltonian[43]. In particular, the entanglement entropy area law predicts an $O(|\partial A|)$ scaling for the entropy of Ω_A, which matches the entropy of the Gibbs state of a 1D Hamiltonian. A more general question is which aspects of the modular Hamiltonian H_{mod} are similar to that of a 1D local Hamiltonian beyond simply the $O(|\partial A|)$ scaling of entropy. We contribute to this by showing one further feature, namely, the $O(\sqrt{|\partial A|})$ scaling of the entanglement spread of Ω_A.

To see why $O(\sqrt{|\partial A|})$ scaling predicted by our bound in equation (3) for the gapped ground states is in agreement with H_{mod} being a 1D local Hamiltonian, we use the well-known fact that at thermal equilibrium, the energy distribution of a many-body quantum system is concentrated around the average energy.

Indeed, if H_{mod} is a sum of $O(|\partial A|)$ local terms, each of norm $O(1)$, then the energy variance $\sigma^2=\mathrm{Tr}[H_{\mathrm{mod}}^2\,\Omega_A]-\mathrm{Tr}[H_{\mathrm{mod}}\,\Omega_A]^2\leq O(|\partial A|)$, which can be shown using the exponential decay of correlations in the Gibbs state of such Hamiltonians[44,45]. It follows from Chebyshev's inequality on the concentration of probability distributions that the δ-smooth entanglement spread of the spectrum of the Gibbs state satisfies $\mathrm{ES}_\delta(\Omega_A)\leq O(\sqrt{|\partial A|/\delta})$, yielding the same $O(\sqrt{|\partial A|})$ dependency as in the bound in equation (3).

Finally, there is a close link between the features of entanglement and developing efficient algorithms for simulating gapped ground states. The tensor network states such as PEPS[29] are used to efficiently represent 2D ground states. Although contracting these tensor networks to estimate the expectation of local observables is believed to be computationally intractable in general[46], finding physically motivated conditions on gapped ground states that imply efficient contraction methods has been the subject of many past studies. This includes exploiting the locality of entanglement spectrum[25], area laws for entanglement entropy[10,47,48] and uniformity conditions on boundary eigenvalues[28]. In this work, we further show that there is an efficient algorithm for computing the expectation value $\langle O\rangle$ of any local observable O if the ground state is given by a PEPS of a small bond dimension that satisfies the exponential decay of correlations and for which a variant of our entanglement spread bound without any smoothing (that is, $\delta=0$) holds. This algorithm simply estimates the local expectation $\langle O\rangle$ by $\mathrm{tr}[\Pi O]/\mathrm{tr}[\Pi]$, where Π is the projector onto a disk of sufficiently large, but still constant, radius in the ground state that encloses the support of the local observable O (Supplementary Sections 2.3 and 3.4 provide the precise statements and proof). Whether this result can be extended to the case of smooth entanglement spread—for which we establish a sub-area scaling—is an interesting open problem.

In conclusion, this work connects two previously unrelated topics: ground-state entanglement and communication complexity of testing bipartite entangled states. We uncover a new property of entanglement in the ground state of any gapped local Hamiltonian, namely, an area law for the entanglement spread. We discuss the importance of our results as follows: (1) showing how entanglement spread can capture the unique features of gapped ground states beyond what is evident from entanglement entropy, (2) connecting the improved sub-area law for entanglement spread on lattices to the conjectures regarding the locality of modular or entanglement Hamiltonian and (3) demonstrating the relevance of entanglement spread in devising efficient algorithms for gapped ground states.

Online content

Any methods, additional references, Nature Research reporting summaries, source data, extended data, supplementary information, acknowledgements, peer review information; details of author contributions and competing interests; and statements of data and code availability are available at https://doi.org/10.1038/s41567-022-01740-7.

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Methods

Testing protocol for gapped ground states. Here we detail the construction of AGSP used in our testing protocol to prove the entanglement spread area law in equation (2). This AGSP is based on the QPE algorithm. One advantage of using QPE compared with conventional Chebyshev polynomials (used in earlier works[7,41]) is that it applies not only to geometrically local Hamiltonians on lattices but also to any local Hamiltonian on arbitrary interaction graphs. One can view QPE as a procedure that conditions on a time register $|t\rangle$, which is in a uniform superposition of $t\in[0,O(1/\gamma)]$; applies the 'Hamiltonian simulation' operator e^{-itH} on an input state; and determines the energy of the state. Setting γ to equal the gap of the Hamiltonian and repeating QPE in parallel $O(\log\left[\frac{1}{\Delta}\right])$ times, we can use this algorithm to approximately distinguish the ground state $|\Omega\rangle$ from other eigenstates. That is, we can perform the two-outcome measurement $\{K,\mathbb{1}-K\}$ on any input state, where $||K-|\Omega\rangle\langle\Omega|||\le\Delta$ (Supplementary Section 3.2.3 provides a more precise derivation).

To implement this algorithm in a distributed fashion involving two parties, Alice and Bob need to share two copies of the time register $|t\rangle$ and work together to apply the operator e^{-itH} conditioned on the register $|t\rangle$. For a given partition of qudits between Alice A and Bob B, we can write the Hamiltonian as $H=H_A+H_{\partial A}+H_B$, where $[H_A,H_B]=0$. One of our main technical contributions is designing a communication protocol for performing the Hamiltonian simulation operator e^{-itH} with a communication cost that scales as $O(t||H_{\partial A}||)$ instead of the conventional $O(t||H||)$. It is not hard to see how one can achieve this if the boundary term $H_{\partial A}$ also commutes with H_A and H_B. In that case, we have $e^{-itH}=e^{-itH_A}e^{-itH_{\partial A}}e^{-itH_B}$ and the parties can implement e^{-itH} if one of them sends the boundary qudits that are in support of $H_{\partial A}$ to the other. This yields a communication cost that scales as $O(|\partial A|)$. In general, however, $H_{\partial A}$ does not commute with H_A and H_B and finding a non-trivial protocol for the Hamiltonian simulation becomes challenging.

One attempt to remedy this might be to use the Trotterization technique[49]. This divides the simulation into η segments and implements $e^{-itH/\eta}$ for η consecutive times. If η is large enough, $[H_{\partial A}/\eta, H_{A\text{ or }B}/\eta]\approx 0$, and we again recover the commuting case. That is, the parties collaboratively implement $e^{-itH_{\partial A}/\eta}$. Unfortunately, for this to work, we need η (and therefore, the communication cost) to be $O(t||H||)$, which is far from the bound $O(t||H_{\partial A}||)$ that we are aiming for here.

We, instead, use a recent framework for Hamiltonian simulation developed in another work[50] known as the 'interaction-picture' Hamiltonian simulation. Intuitively, one can view this as a sophisticated but widely used change of variables that allows us to separate the contribution of the boundary term from H_A and H_B. Suppose we want to prepare the state $|\psi(t)\rangle=e^{-itH}|\psi(0)\rangle$. For any $|\psi(t)\rangle$, we define its counterpart in the interaction picture by $|\psi_I(t)\rangle=e^{it(H_A+H_B)}|\psi(t)\rangle$. Since the operator $e^{it(H_A+H_B)}$ can be locally applied by the parties, the states $|\psi_I(t)\rangle$ and $|\psi(t)\rangle$ can be switched with each other with no extra communication. The point of this transformation is that the state $|\psi_I(t)\rangle$ can be prepared starting from $|\psi(0)\rangle$ by applying unitary $U(t)$, which is the Hamiltonian simulation operator associated with a time-dependent Hamiltonian $H_I(t)=e^{it(H_A+H_B)}H_{\partial A}e^{-it(H_A+H_B)}$. Putting the time dependence of $H_I(t)$ aside, the main advantage is that $||H_I(t)||=||H_{\partial A}||$. This solves the earlier issue as the number of segments η used when implementing $U(t)$ can be taken as small as $O(t||H_{\partial A}||)$ instead of the original $O(t||H||)$. The remaining task is to find a communication protocol for performing $U(t/\eta)$, which now is a more complicated operator than the previous one, that is, $e^{-itH_{\partial A}/\eta}$. This is done in other work[50,51] using the linear combination of unitaries method. In Supplementary Section 3.2.2, we present a modification of this algorithm based on the following idea.

As explained earlier, Alice and Bob may use unlimited EPR pairs in their protocol without affecting the bound in equation (4) on the entanglement spread. Moreover, as shown elsewhere[17] and discussed in Fig. 3, testing or performing reflections around maximally entangled states can be done with a cost independent of the dimension of this state. Together, these imply that we can replace the ancillary registers that need to be shared during the linear-combination-of-unitaries-based Hamiltonian simulation with free EPR pairs. This allows us to improve the cost of Hamiltonian simulation and obtain tighter bounds on the entanglement spread. Although we use the interaction-picture Hamiltonian simulation to implement this AGSP, achieving the same results with the Trotter methods—using ideas on information complexity[52]—would be valuable.

AGSP for lattices. Our improved bound in equation (3) for lattice Hamiltonians is obtained using the AGSPs based on Chebyshev polynomials. These were first developed in the context of the entanglement entropy area law in 1D systems[7,41,48]. The AGSP framework[40,41] already provides a framework to connect the min-entropy and entanglement entropy[40,41]. However, this connection does not give us the desired bound on entanglement spread, as it relates entanglement entropy and min-entropy by a certain multiplicative factor, which may be large. For instance, ref. [41] implies that by choosing the Chebyshev-based AGSP that has a shrinking of $O(1)$ and Schmidt rank of $2^{O(\sqrt{|\partial A|})}$, we get $S(\Omega_A)=O(\sqrt{|\partial A|}S_{\min}(\Omega_A))$, where $S(\Omega_A)$ is the von Neumann entropy of Ω_A.

We show that a simple adaptation of Chebyshev-based AGSP, along with appropriate smoothing, leads to a qualitatively stronger theorem for lattices, which shows that entanglement spread scales as $O(\sqrt{|\partial A|})$. We utilize the 'truncation step[7]', which is used to lower the norm of the Hamiltonian away from a cut and maintaining its gap and ground state. We apply truncation to both frustration-free and frustrated cases. In the former, we use the detectability lemma operator[40], whereas in the latter, we rely on recent techniques[53] to perform the truncation.

Data availability

No data or code have been generated in this work.

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All the authors contributed equally to all aspects of this work. The authors are arranged alphabetically by last name.

Competing interests

The authors declare no competing interests.

Additional information

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