Student definitions of equivalence: structural vs. operational conceptions, and extracted vs. stipulated definition construction

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In this study, we describe a model of student thinking around equivalence (conceptualized as any type of equivalence relation), presenting vignettes from student conceptions from various college courses ranging from developmental to linear algebra. In this model, we conceptualize student definitions along a continuous plane with two-dimensions: the extent to which definitions are extracted vs. stipulated; and the extent to which conceptions of equivalence are operational or structural. We present examples to illustrate how this model may help us to recognize ill-defined or operational thinking on the part of students even when they appear to be able to provide "standard" definitions of equivalence, as well as to highlight cases in which students are providing mathematically valid, if non-standard, definitions of equivalence. We hope that this framework will serve as a useful tool for analyzing student work and exploring instructional and curricular handling of equivalence.

Keywords: Equivalence, solution set, operational-views, structural-views, definitions.

Equivalence is central to mathematics at all levels, and across all domains. In mathematics education, much research has focused on studying how students think about the equals sign in primary (Knuth et al., 2006) through post-secondary (Fyfe et al., 2020) school, because student conceptions of the equals sign are related to their arithmetic and algebraic calculations. However, equality is just one example of the larger concept of equivalence—other types of equivalence occur extensively throughout the K-16 curriculum, but are rarely, if ever, taught under one unifying idea called equivalence (Wladis et al., 2020). On the other hand, multiple types of equivalence (e.g., similar/congruent figures, function types, equations with the "same form") are contained in the *U.S. Common Core Mathematics Standards* but are never explicitly labeled as a type of equivalence. When equivalence is not explicitly defined, students may extract their own non-standard, ill-defined, or unstable definitions, or they may inappropriately use the definition of equivalence from one area (e.g., expressions) in another area where it cannot be directly applied to obtain the "standard" definition expected of them (e.g., equations). In this paper we will illustrate this problem by presenting examples of student definitions around equivalence and a model for analyzing student definitions, focusing on college students' definitions of equivalent equations. Examples of student work will be used as vignettes to illustrate the model. Our aim in presenting this model is to start a conversation about student definitions of equivalence and to present an initial framework that can then be further tested, refined, and revised by future empirical work.

Theoretical framework

We frame the analysis of student definitions of concepts in terms of Tall and Vinner's (1981) concept image and concept definition constructs. A person's *concept image* describes the "total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p.152). Their *concept definition* describes a "form of words that a learner uses for his own explanation of his (evoked) concept image" (p.152). Hence an individual's *personal* concept definition is idiosyncratic to the individual, may vary based on the context it is evoked, and may deviate from the larger mathematical community (the *formal* concept definition). When we refer to a student's 'definition', we are referring to their personal concept definition.

In this paper, we define the *formal* concept definition of equivalence as an equivalence relation. The formal definition of an equivalence relation most often given in advanced mathematics classes is that of a binary relation that follows the identity, symmetry and transitive properties. However, another equivalent but more accessible definition of an *equivalence relation* is that of a partition on a set, or more informally: If we have a set of objects and a mathematically well-defined rule for sorting objects into sets so that each goes into one and only one set, then this "sorting" is an equivalence relation, and two objects are equivalent if they belong to the same set. Using this definition as an analysis tool allows us to account for many types of equivalence, with many different mathematical objects (e.g., numbers, algebraic expressions, algebraic equations) and equivalence relations (e.g., equality of expressions; insertion equivalence of equations; Wladis et al., 2020; Zwetzschler & Prediger, 2013). (We note that this definition is an analysis tool that is not necessarily intended to be given to students.)

From this perspective, a student's personal definition of equivalence in a given context is valid in so far as it is an equivalence relation and can be expressed in a mathematically well-defined way by the individual. When students have no explicit definitions of equivalence, this presents several potential problems: (1) students may incorrectly apply one definition to another context where it fails to produce the "standard" definition (e.g., definition of equivalent expressions to equations); (2) they may have only ill-defined or operational definitions of equivalence which inhibit their ability to reason through problems; or (3) they may use valid but non-standard definitions of equivalence, in which case they are being penalized for not knowing certain socio-mathematical norms (Yackel & Cobb, 1996) even when they are reasoning correctly. We argue that the model presented here allows us to better recognize when these three situations (as well as others) might be occurring with students.

Model of equivalence

Our model of student thinking about equivalence conceptualizes student definitions as existing on a two-dimensional plane with two axes: operational vs. structural conceptions of equivalence (Sfard, 1992), and extracted vs. stipulated definitions of equivalence (Edwards & Ward, 2004). A student with an *operational* conception thinks of mathematical entities as a process of computation, while a student with a *structural* conception thinks of them as abstract objects which can then be acted on by even higher-order processes. A student with a structural conception sees objects as reified processes (e.g., $6x$ is seen as an object itself, and not just as the process of multiplying x by 6), however when students view something as an object which is not the reification of any process, this is called a *pseudostructural* conception (p.75, Sfard, 1992). We see Sfard's constructs as related to the computational/relational distinction made in research on the equals sign, where the computational view is a cue to calculate, and the relational view focuses on equality as a relationship (e.g., Knuth et al., 2006). Our model can be seen as a generalization of the computational/relational distinction made in research on the equals sign, where equivalence structures (the equivalence relationships) are core objects that justify computation. This is in contrast to Sfard's description of structures (e.g., algebraic expressions) being viewed as a normative process which is reified into an object.

Extracted definitions are created to describe actual observed usage of a term (e.g., a student may extract a meaning for equivalence from their instructional experiences, whether or not they have encountered an explicit definition). In contrast, *stipulated* definitions are those definitions that are stated explicitly—to determine if something fits the definition one must consult the definition directly (Edwards & Ward, 2004). We note that in our model, a stipulated definition may be stipulated by the student or an authority—the key features we use to determine if a definition is stipulated in our framework is whether it appears to be explicit, well-defined, and stable across contexts. We note that while we have displayed our model in Table 1 as a two-by-two grid for the sake of simplicity, but we conceptualize these categories as a spectrum (thus, Table 1 is actually a continuous 2D plane).

Table 1: Model of student thinking about equivalence

	Extracted Definition	**Stipulated Definition**
Operational Conception of Equivalence	**Pseudo-process view:** Students see equivalence as a computational process, and their approaches to those processes are dictated by prior experience in ways that are extracted rather than stipulated. Definitions of equivalence are typically non-standard, ill-defined, and/or unstable.	**Process view:** Students see equivalence as a process, but do process computations by referring to stipulated rules or properties. Students with this view may be able to perform calculations correctly but this does not necessarily translate to being able to use stipulated definitions to recognize equivalent objects.
Structural Conception of Equivalence	**Pseudo-object view:** The student is able to consider whether two objects are equivalent without reverting to an explicit computation, perhaps by considering the structure of the objects; but definitions of equivalence are typically extracted in some way from experience rather than based on stipulated definitions of equivalence, and as a result are typically non-standard, ill-defined, and/or unstable	**Object view:** The student is able to consider whether two objects are equivalent without reverting to an explicit computation, perhaps by considering the structure of the objects; definitions of equivalence used to determine equivalence are stipulated. The student conceptualizes equivalence classes (or solution sets) as objects, although they need not do this formally.

Methods

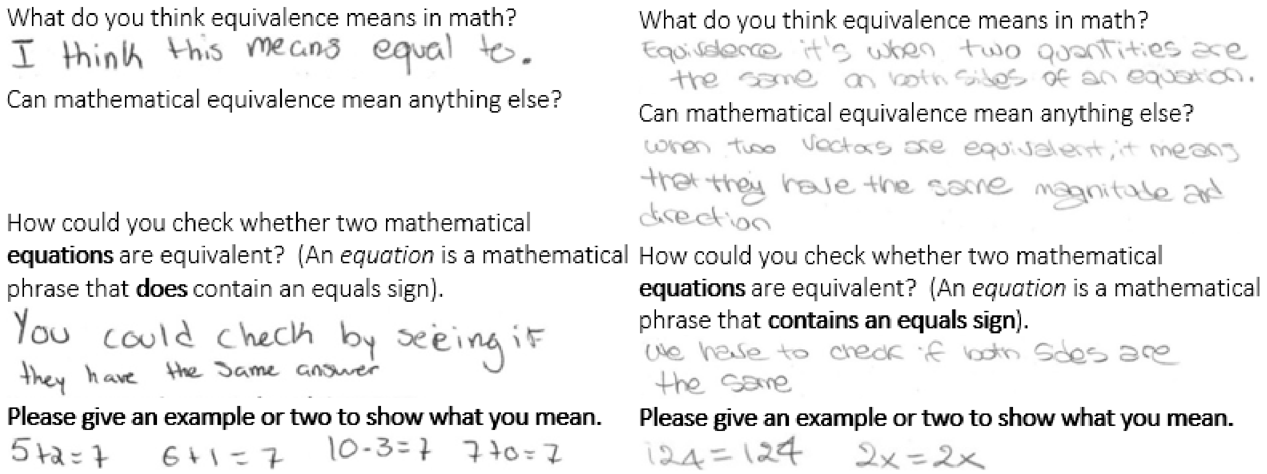
Data for this study were collected from 124 students at an urban community college in the US through open-ended questions in 18 different courses, from developmental elementary algebra (similar to Algebra I in secondary school) to linear algebra. Student responses were analyzed using thematic analysis (Braun & Clarke, 2006), combining codes from the model above with an emergent coding scheme. Multiple coders participated in several rounds of coding until consensus was reached. Responses coded as indicative of an operational view of equivalence provided evidence of thinking of equivalence as an algorithm; those coded as indicative of a structural view of equivalence provided evidence of thinking of equivalence as a fixed trait of an object, or reasoning about equivalence via its general properties. Further coding details are described below.

Results

Students often struggled to provide definitions of equivalent equations for several different reasons.

One issue appears to be that students attempted to apply the definition of equivalent expressions to that of equivalent equations.

Pseudo-process view (extracted and operational): For example, in Figure 1, we see the work of two students, one in elementary algebra, and one in linear algebra, both of whom give somewhat similar definitions of equivalent equations. The elementary algebra student gives a more ill-defined definition ("same answer") but we see from the examples that they provide that they appear to be thinking about equivalent arithmetic expressions. We would classify this response as a pseudo-process view, as the definition is not well-defined and appears to center around arithmetic calculation.


What do you think equivalence means in math?
I think this means equal to.
Can mathematical equivalence mean anything else?
How could you check whether two mathematical **equations** are equivalent? (An *equation* is a mathematical phrase that **does** contain an equals sign).
You could check by seeing if they have the same answer
Please give an example or two to show what you mean.
5+2=7 6+1=7 10-3=7 7+0=7

What do you think equivalence means in math?
Equivalence it's when two quantities are the same on both sides of an equation.
Can mathematical equivalence mean anything else?
when two Vectors are equivalent, it means that they have the same magnitude and direction
How could you check whether two mathematical **equations** are equivalent? (An *equation* is a mathematical phrase that **contains an equals sign**).
We have to check if both sides are the same
Please give an example or two to show what you mean.
124 = 124 2x = 2x

Figure 1: Definitions from an elementary algebra student, pseudo-process view (on left) and a linear algebra student, extracted view (on right)

Operational view: We see similar work by the linear algebra student in Figure 1, with some differences; they give broader examples of equivalence (describing also vectors) and their definition is more detailed ("when two quantities are the same on both sides of an equation"). But like the elementary algebra student in Figure 1, they conflate the definition of equivalent expressions with equations (they include an algebra example, but only show identical expressions as equal). Their definition of equivalent equations is also not fully well-defined ("check if both sides are the same"), because the word "same" is not well-defined. While their answer shows signs of having seen more examples of mathematical equivalence, this does not appear to have positively impacted their definition of equivalent equations; we classify their definition as extracted, because it is ill-defined.

Structural view: Students who apply the definition of equivalent expressions to equations may even do this in a way that is mathematically valid (i.e., fits the definition of an equivalence relation), even though it is not a "standard" definitions of equivalent equations (e.g., same solution set). Consider Figure 2, where a precalculus student has defined equivalent equations as two equations where "the result or the number after the equal sign are equivalent". Based on their examples, they seem to be suggesting that any equations of the form $expression = n$ for fixed n would be equivalent. This is similar to definitions given by other students in other research (Wladis et al., 2020). This example is particularly interesting, because the two equations given here also happen to have the same solution set, so it is unclear if this is also an implied part of the student's definition. Whether the definition includes this feature or not, we would classify it as structural even though it is a "non-standard" definition, because the student has given what could be a well-defined but alternate definition of equivalence (whether or not their definition is fully well-defined is unclear)[1].

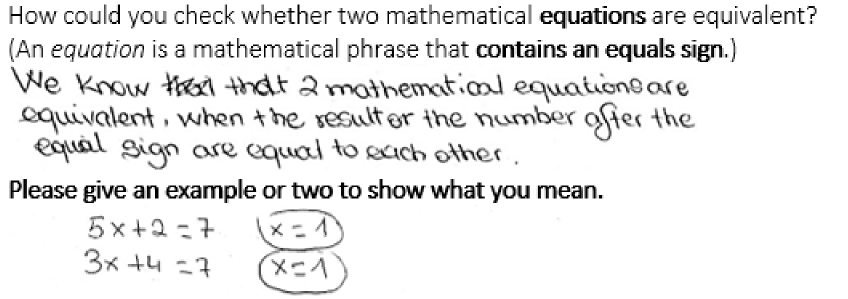


Figure 2. Precalculus student's non-standard structural definition of equivalent equations

In contrast to the previous examples, some students did draw in some way on the notion of "solving" equations or the solution sets of equations when defining equivalence. However, the ways in which students drew on notions of "solving" also fell into different areas of our framework. Simply talking about the "solution" of an equation was not sufficient to classify work as either stipulated or structural even though it sounds like it is related to the standard insertional equivalence definition of equations (i.e., same solution set).

Structural view: In Figure 3(a), we see the work of a Calculus III student, who appears to have a well-defined and structural view of equivalent equations: they define equivalent equations as having the same solution set (seeming to conceptualize the solution set as a fixed object); and their definition appears to be well-defined, not just because of their stated definition, but also because the example they give which shows that their interpretation of "same solution" appears to be the "standard" one. We note that this is critical, as many students used the language of "same solution" but actually meant it to describe equivalent sides of an equation (equivalent expressions) rather than solution set.

Pseudo-process views (operational and extracted): See, for example, the work of an introductory statistics student in Figure 3(b). This student wrote that two equations are equivalent if you "substitute the value in for x and the solution is the same for both equations": this sounds like the standard definition of equivalent equations (if an incomplete one that does not account for the possibility that x may have more than one value), however, looking at the example they provided, we see that to them "solution" denotes the quantity resulting from simplifying one side of an equation (not the solution set of an equation). In this sense, the student's definition is ill-defined, because the vocabulary that they are using appears to be ill-defined and has multiple, perhaps vague, meanings. For these reasons, we would classify this work in (b) as a pseudo-process view, even though on the surface the definition initially looked similar to the one in (a). The student work in Figure 3(c) shows another common approach that students used, in which they drew on notions of solving when asked about equivalent equations, but struggled to relate these notions to any well-defined definition of equivalence. This

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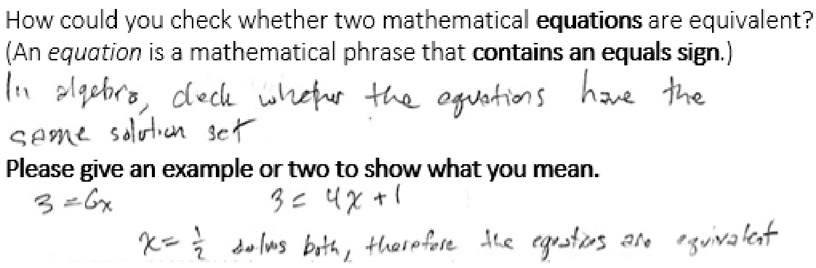
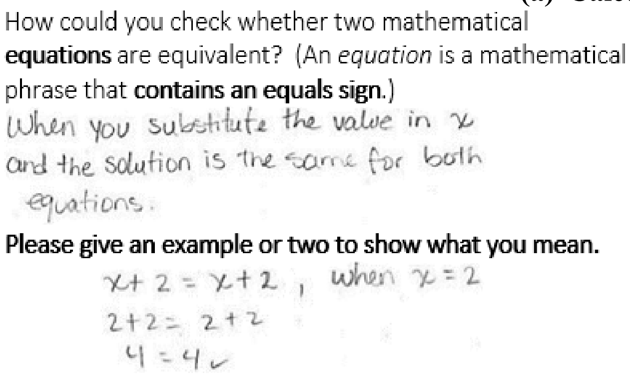
[1] This student may be drawing on notions of equations with the "same form" (e.g., $y = mx + b$, $ax^2 + bx + c = 0$) which is another type of equivalence that is commonly used in the algebra curriculum, even if it is not called equivalence in the curriculum (however, "same form" could in fact be codified as a formal equivalence relation, and students may be noticing this when they draw on it in their equivalence definitions (Wladis et al., 2020)).

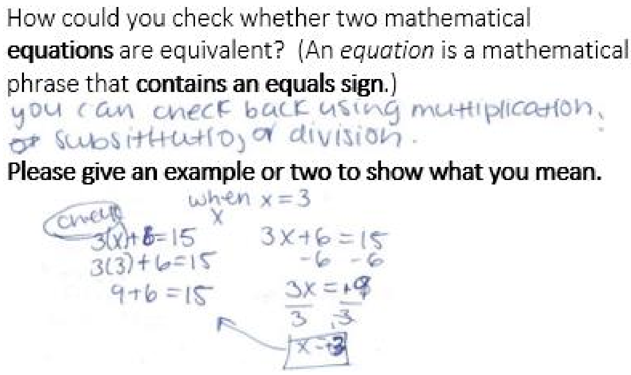
student has solved an equation and checked the solution by substituting it back into the original equation; however, it is unclear what the definition of equivalent equations is, or even which two objects the student is claiming are equivalent (perhaps equivalence for them is not about the relationship between two objects, but instead names a process of checking the solution of an equation). Thus, we classify this as a pseudo-process view—there is no well-defined stated definition, and the student's focus is on computation.



(a) Calculus III student



(b) introductory stats student



(c) intermediate algebra/precalculus student

Figure 3: Examples of different ways that students used "solving" in defining equivalent equations

Students also gave other non-standard definitions of equivalence that might have been well-defined equivalence relations (e.g., equivalent arithmetic equations as ones that express the same additive relationship; equivalent algebraic equations as ones that express the same relationship between the variables). However, we note that by de-coupling our categorization of student definitions of equivalence from what is "standard" and thinking more carefully about the extent to which student definitions of equivalence are stipulated (and an equivalence relation); and the extent to which student conceptions of equivalence are structural or operational, we may be able to achieve two critical goals more effectively: (1) we may be able to better identify student thinking which "sounds right", but is actually ill-defined; and (2) we may be able to identify valid student thinking that simply does not adhere to "standard" definitions. Both of these goals may better help us to tailor instruction to students.

We now briefly describe some overall trends we found in coding responses to open-ended questions on definitions of equivalence (Table 2). Students primarily associated equivalence with equality, and rarely cited other forms (e.g., equivalent equations), although the incidence of non-equality examples rose somewhat with course level. Similarly, students at all levels were extremely likely to give ill-defined or vague definitions of equivalence when asked. When asked about their definitions of equivalent equations, most students conflated this with the definition of equivalent expressions; this did not appear to improve with course level, suggesting that the lack of explicit definitions of equivalent equations in textbooks and curricula (Wladis et al., 2020) may well be contributing to student difficulties in understanding how definitions of equivalence vary in different contexts. Some of these definitions, while non-standard, may have been equivalence relations, and therefore reflect mathematically valid reasoning—the prevalence of this was not correlated with course level, suggesting that students at all levels may sometimes be generating valid but non-standard definitions. Many students associated equivalent equations with solving, but this was rarely done in a well-defined way: roughly one quarter of all students at all course levels solved an equation but did not relate this in any well-defined way to the definition of equivalent equations (most commonly they solved a single equation and then checked the answer, with no mention of which two things were equivalent); a smaller percentage of students did this at levels of precalculus and above, but the differences by course level were small. Some students interpreted equivalent equations as equations with the same solution set, and did so in a well-defined way; this was slightly more common as course levels went up; however, the vast majority of these students did so in an operational way (i.e., solved two equations and said they were equivalent, without discussing the solution set in a more general or structural way). This is perhaps to be expected, given the operational way in which the question itself was phrased, however, this does follow patterns observed in questions without this more operational wording, such as the more general question about the definition of equivalence given on this set of questions (although the tendency to use structural rather than operational definitions did increase with course level). However, we note that overall, structural and well-defined definitions were rare among all students, suggesting that instruction which specifically includes explicit stipulated definitions, and which encourages structural reasoning, is needed at all levels.

Table 2. Summary of student definitions of equivalence

	elementary alg. or below	intermediate alg. or 100-level	200-level or above
General definition of equivalence			
ill-defined or vague	67%	71%	60%
cited equality	94%	87%	80%
other valid definition	0%	3%	16%
operational definition	41%	18%	17%
structural definition	0%	2%	17%
How to tell if two equations are equivalent			
conflated w/ equiv. expressions	44%	48%	44%
of these, possible well-defined defn.	19%	6%	16%
finding solution set, operational	0%	3%	8%
related to "solving" but ill-defined	22%	29%	16%
solution set, structural	0%	2%	4%
total *n*	36	62	25

Discussion and conclusion

Our model of student definitions of equivalence aims to refocus our attention from whether definitions look "standard" to whether student definitions are well-defined equivalence relations, and whether their definitions are structural vs. operational. Using this lens allows us to pinpoint when students appear to understand a standard definition, but upon deeper analysis we find that their definition is ill-defined or wholly operational, limiting their ability to use it. On the other hand, this model also allows us to recognize students' mathematically valid definitions even when they are nonstandard or students are not able to explain them fully formally. Evidence from examples of student work suggests that students do notice many kinds of "sameness", yet struggle to articulate this in mathematically well-defined ways, just as they struggle to articulate "standard" definitions of equivalence in well-defined ways. This suggests that students are capable of noticing and using more generalized notions of equivalence, but need more explicit definitions and language in order to be able to do this rigorously. Future research is necessary to better understand what kinds of explicit definitions of equivalence work best for students in different contexts, and the extent to which discussions of the more general notion of an equivalence relation might be helpful in instruction. This framework may also be able to serve as a framework for instruction and curricula, to assess how the concept of equivalence is presented to students as they are learning at various levels in the curriculum.

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