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Egalitarian Steiner triple systems for data popularity

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Abstract
For an ordering of the blocks of a design, the point sum of an element is the sum of the indices of blocks containing that element. Block labelling for popularity asks for the point sums to be as equal as possible. For Steiner systems of order v and strength t in general, the average point sum is $O(v^{2t-1})$; under various restrictions on block partitions of the Steiner system, the difference between the largest and smallest point sums is shown to be $O(v^{(t+1)/2} \log v)$. Indeed for Steiner triple systems, direct and recursive constructions are given to establish that systems exist with all point sums equal for more than two thirds of the admissible orders.

Keywords Steiner system · Steiner triple system · Transversal design · Group-divisible design · Hill-climbing algorithm

Mathematics Subject Classification 05B07 · 05B05 · 51E10 · 68M07 · 68P20

1 Introduction

A *set system* $D = (X, \mathcal{B})$ consists of X, a set of elements (*points*), and $\mathcal{B}$, a multiset of subsets of X (*blocks*). We examine block and point labellings (orderings) for a number of classes of set systems. Let $k \geq t \geq 1$. A t-(v, k, λ) *packing* is a set system $(X, \mathcal{B})$ with $|X| = v$ and $|B| = k$ for each $B \in \mathcal{B}$ for which every t-subset of X is contained in at most λ blocks. The packing is a t-(v, k, λ) *design* when every t-subset of X is a subset of exactly λ blocks. The *replication number* of the design is $r = \frac{\lambda\binom{v-1}{t-1}}{\binom{k-1}{t-1}}$; every element appears in exactly r blocks. A t-$(v, k, 1)$ design is a *Steiner system* or *Steiner t-design*, denoted by $S(t, k, v)$. A 2-$(v, 3, 1)$ design is a *Steiner triple system of order v*, denoted by STS(v). A 3-$(v, 4, 1)$

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design is a *Steiner quadruple system of order* v, denoted by $\mathrm{SQS}(v)$. A t-$(v, k, 1)$ packing is also referred to as a *partial* $S(t, k, v)$ or *partial Steiner system*.

A t-(v, k, λ) packing, $(X, \mathcal{B})$, is *s-partitionable of type* $(\tau_1, \ldots, \tau_\ell)$ when $\mathcal{B}$ admits a partition into ℓ classes $\{\mathcal{B}_i : 1 \leq i \leq \ell\}$ so that $(X, \mathcal{B}_i)$ is an s-$(|X|, k, \tau_i)$ design for each $1 \leq i \leq \ell$. (Often we write types in exponential notation.) When $s = 1$, the class $\mathcal{B}_i$ is an *α-parallel class* when $\tau_i = \alpha$. A set system that admits a 1-partitioning with $\tau_1 = \cdots = \tau_\ell = 1$ is *resolvable*, the 1-partitioning is a *resolution*, and each class in it is a *parallel class*. A Steiner triple system $\mathrm{STS}(v)$, together with a resolution of it, is a *Kirkman triple system*, $\mathrm{KTS}(v)$. For relevant background in design theory, see the standard texts [7,48], and for triple systems in particular, see [22].

Steiner systems arise in numerous ways in assigning data items to storage units in storage systems; see [12,23] and references therein, as well as [6,50]. Dau and Milenkovic [23] identify the need to consider the long-term popularities of the data items stored when selecting a particular design for such applications. They propose a combinatorial model that ranks data items by popularity. We introduce their model next using an equivalent matrix formulation.

Let $A = (a_{ij}) \in \{0, 1\}^{n \times m}$, in which rows are indexed by $\{0, \ldots, n-1\}$ and columns by $\{0, \ldots, m-1\}$. In the intended applications, each row corresponds to a data item, and each column corresponds to a storage location; an entry is 1 if and only if the item is stored at that location. The data items are totally ordered by popularity (long-term frequency of access). A data item d is associated with row i when there are exactly i data items that are less popular than d; this is the *popularity labelling* of the rows. The order of the columns is (for the moment) inconsequential.

The column sum $\sum_{i=0}^{n-1} a_{ij}$ counts the data items placed at location j. In data placement, typically one wants these counts to be all equal so that each storage location maintains the same number of data items. As a surrogate for the frequency of access to this location, Dau and Milenkovic [23] suggest also considering the weighted column sum $\sigma_j = \sum_{i=0}^{n-1} i \cdot a_{ij}$; this is a *block sum*. For a specific $n \times m$ matrix A, define $\mathsf{MinSum}(A) = \min(\sigma_j : 0 \leq j < m\}$ and $\mathsf{MaxSum}(A) = \max(\sigma_j : 0 \leq j < m\}$. Then 'large' MinSum ensures that no location contains only unpopular items, while 'small' MaxSum ensures that no location contains only very popular items. To treat both simultaneously, define the *difference sum* $\mathsf{DiffSum}(A) = \mathsf{MaxSum}(A) - \mathsf{MinSum}(A)$; when the $\mathsf{DiffSum}$ is 'small', accesses to each storage location have been, to an extent, balanced.

In [12,23], the *ratio sum* $\mathsf{MaxSum}(A)/\mathsf{MinSum}(A)$ is also proposed as a measure of access balance. Because the ratio sum is closely related to the difference sum, we do not pursue ratio sums here. In [51], minimizing the variance of the block sums is proposed. Because we are concerned with the worst case rather than the average, we do not pursue variance sums here.

Once the association of data items with storage units is fixed by matrix A, the sum metrics are determined. Nevertheless, we can relabel or permute the rows of A to alter, and perhaps improve, these metrics. We pursue this next. Let $D = (X, \mathcal{B})$ be a set system with $|X| = n$ in which the multiset $\mathcal{B}$ consists of m blocks on X. Given a labelling $(x_0, \ldots x_{n-1})$ of X and a labelling $(B_0, \ldots B_{m-1})$ of $\mathcal{B}$, an *incidence matrix* for D is the $n \times m$ matrix $A = (a_{ij})$ in which $a_{ij} = 1$ when $x_i \in B_j$, and 0 otherwise. Let $\mathsf{Inc}(D)$ be the set of all incidence matrices for D. We are free to choose any incidence matrix from $\mathsf{Inc}(D)$. We extend the sum metrics to set systems by defining $\mathsf{MinSum}(D) = \max(\mathsf{MinSum}(A) : A \in \mathsf{Inc}(D))$, $\mathsf{MaxSum}(D) = \min(\mathsf{MaxSum}(A) : A \in \mathsf{Inc}(D))$, and $\mathsf{DiffSum}(D) = \min(\mathsf{DiffSum}(A) : A \in \mathsf{Inc}(D))$. (Although $\mathsf{DiffSum}(D) \geq \mathsf{MaxSum}(D) - \mathsf{MinSum}(D)$, equality is not ensured.) The *reverse* of incidence matrix A is the matrix R obtained by interchanging row i and $n-1-i$ for

$0 \le i \le \lfloor \frac{n-1}{2} \rfloor$; when column j has k '1' entries, its weighted column sums in A and in R total $k(n-1)$. Hence when each block has size k in D, $\mathsf{MaxSum}(D) = k(n-1) - \mathsf{MinSum}(D)$.

The *dual* D^{T} of a set system $D = (X, \mathcal{B})$ is the set system whose incidence matrix is the transpose of an incidence matrix of D. When there are no repeated blocks, the dual is obtained by interchanging blocks and elements. Applying the framework to the dual set system treats (weighted) *row* sums (i.e., *point sums*) in an incidence matrix, rather than column sums (block sums).

One may be free to choose not only point and block labellings, but also the set system itself. When $\mathcal{D}_{n,m}$ is a set of set systems, each having n points and m blocks, define $\mathsf{MinSum}(\mathcal{D}_{n,m}) = \max(\mathsf{MinSum}(D) : D \in \mathcal{D}_{n,m})$, $\mathsf{MaxSum}(\mathcal{D}_{n,m}) = \min(\mathsf{MaxSum}(D) : D \in \mathcal{D}_{n,m})$, and $\mathsf{DiffSum}(\mathcal{D}_{n,m}) = \min(\mathsf{DiffSum}(D) : D \in \mathcal{D}_{n,m})$. In order to make these metrics meaningful in an application, numerous criteria dictate requirements on the set systems to select (and label). Because of the many applications in which Steiner systems are used for data placement, Dau and Milenkovic [23] focus on set systems that are Steiner systems, and duals of Steiner systems. In particular, they establish bounds on the sum metrics for $S(t, k, v)$ designs:

Theorem 1 [23] *Let $\mathcal{S}_{t,k,v}$ be the set of all $S(t, k, v)$ designs.*

$$\begin{aligned}
\mathsf{MinSum}(\mathcal{S}_{t,k,v}) &\le \frac{1}{2}(v(k-t+1) + k(t-2));\\
\mathsf{MaxSum}(\mathcal{S}_{t,k,v}) &\ge \frac{1}{2}(v(k+t-1) - kt);\\
\mathsf{DiffSum}(\mathcal{S}_{t,k,v}) &\ge (v-k)(t-1).
\end{aligned}$$

Moreover, $\mathsf{MinSum}(\mathcal{S}_{t,t+1,v}) \le (v-1) + \binom{t}{2}$, $\mathsf{MaxSum}(\mathcal{S}_{t,t+1,v}) \ge t(v-1) - \binom{t}{2}$, *and* $\mathsf{DiffSum}(\mathcal{S}_{t,t+1,v}) \ge (t-1)(v-t-1)$. *Finally,* $\mathsf{DiffSum}(\mathcal{S}_{2,3,v}) \ge v$.

They establish the existence of an STS(v) D with $\mathsf{MinSum}(D) = v$, the largest possible by Theorem 1. Chee et al. [12] construct dense t-$(v, t+1, 1)$ packings that meet these bounds; they also improve on the $\mathsf{DiffSum}$ lower and upper bounds for Steiner triple systems to establish that $\mathsf{DiffSum}(\mathcal{S}_{2,3,v}) \ge v + 1$ when $v \ge 13$, and that for every admissible v, an $S(2, 3, v)$ D exists having $\mathsf{DiffSum}(D) \le v + 7$.

Our focus in this paper is instead with the duals of Steiner systems. Throughout, let $\mathcal{D}_{t,k,v}$ be the set of duals of $S(t, k, v)$ designs. Applying the general matrix framework to the dual has a natural interpretation for the $S(t, k, v)$ design itself. Labelling the b blocks of an $S(t, k, v)$ design $(X, \mathcal{B})$ as $B_0, \ldots, B_{b-1}$, the (weighted) sum for a point $x \in X$ is $\sum_{j: x \in B_j} j$. The sum metrics for point sums of any set system are just the sum metrics for the block sums of its dual set system. We typically treat block labellings of a set system and examine point sums, rather than point labellings of its dual and associated block sums; naturally, either perspective can be used.

Let $\mathcal{M}_{v,b,k}$ be the set of all set systems on v elements having b blocks, each of size k, in which each element occurs in exactly $r = bk/v$ blocks. Let $M \in \mathcal{M}_{v,b,k}$. A block labelling of M is *split* if the number b of blocks is even, and every point appears equally often in the first $b/2$ blocks and the last $b/2$ blocks. A block labelling of M is *egalitarian* if the corresponding incidence matrix $X \in \mathsf{Inc}(M)$ has $\mathsf{DiffSum}(X^{\mathsf{T}}) = 0$, so that $\mathsf{DiffSum}(M^{\mathsf{T}}) = 0$. In this case $\mathsf{MinSum}(M^{\mathsf{T}}) = \mathsf{MaxSum}(M^{\mathsf{T}}) = \frac{b(b-1)k}{2v}$. A block labelling that is both split and egalitarian is *split egalitarian*.

We give an example to make the many definitions more clear. There is exactly one SQS(10), D, and it admits a cyclic automorphism [29]; its blocks are $\{\{i, i+1, i+3, i+4\}, \{i, i+1, i+$

111111110000011110000000000000	000111010010000011010111000100
111000000111111000000000111000000	010100010011101000100100110010
100110000100101001110001011000	110000110001000111001000001110
100001010011000101010110001010	010010001110000101111000100001
001011001000100010011110010001	101100000000101101001100110000 1
001000110110001000111000100011	111000000110000100001011001110 0
010100010001100010101100101100	100101101000001000110001001001 1
010010101010000101100101000101	001011001000110110000100101010
000101101101010001001000010110	000010110100111000101011001000
000000000000001111000000111111 11	001001100111100100000001010101

Fig. 1 Incidence matrices A_1 and A_2 for the SQS(10)

$2, i+6\}, \{i, i+2, i+4, i+7\} : 0 \leq i < 10\}$. Two incidence matrices, $A_1, A_2 \in \mathsf{Inc}(D)$, are given in Fig. 1.

The average weighted column sum (block sum) in each is 18, but the two incidence matrices differ in the sum metrics. In particular, $\mathsf{MinSum}(A_1) = 6$, $\mathsf{MaxSum}(A_1) = 30$, and $\mathsf{DiffSum}(A_1) = 24$; in contrast, $\mathsf{MinSum}(A_2) = 11$, $\mathsf{MaxSum}(A_2) = 25$, and $\mathsf{DiffSum}(A_2) = 14$. There is a point labelling for D yielding an incidence matrix A_3 with $\mathsf{MinSum}(A_3) = 12$. An exhaustive treatment of incidence matrices establishes that $\mathsf{MinSum}(D) = 12$, $\mathsf{MaxSum}(D) = 24$, and $\mathsf{DiffSum}(D) = 14$, each equality being realized a different incidence matrix. Because there is only one SQS(10) up to isomorphism, the same equalities hold for $\mathcal{S}_{3,4,10}$.

Considering instead the weighted row sums (point sums) of A_1 and A_2, the average for each incidence matrix is $174 = \frac{1}{2}12 \cdot 29$. We find that $\mathsf{MinSum}(A_1^{\mathsf{T}}) = 86$, $\mathsf{MaxSum}(A_1^{\mathsf{T}}) = 262$, and $\mathsf{DiffSum}(A_1^{\mathsf{T}}) = 176$. The metrics for A_2^{T} are dramatically different: $\mathsf{MinSum}(A_2^{\mathsf{T}}) = \mathsf{MaxSum}(A_2^{\mathsf{T}}) = 174$, and $\mathsf{DiffSum}(A_2^{\mathsf{T}}) = 0$. So A_2 gives an egalitarian block labelling. Hence $\mathsf{DiffSum}(D^{\mathsf{T}}) = \mathsf{DiffSum}(\mathcal{D}_{3,4,10}) = 0$, the best one could hope for. Although $b = 30$ is even and each point occurs in 12 blocks, neither A_1 nor A_2 is a split block labelling.

In [18], a complete characterization is given of orderings for which an $S(2, 2, v)$, a trivial design equivalent to a complete graph, admits an egalitarian edge ordering; see also [47] The Steiner triple systems form the next (and more interesting) case. In Sect. 2 we establish lower and upper bounds on the sum metrics for duals of Steiner systems using the probabilistic method. In Sect. 3, we establish that certain transversal designs can be block-labelled to have all point sums equal. In Sect. 4, we apply an inflation method to these egalitarian labellings to produce egalitarian labellings of more general group-divisible designs. In Sect. 5, we treat duals of Steiner triple systems, first specializing the general bounds. We then establish the existence of an egalitarian STS(v) whenever $v \geq 13$, and $v \equiv 1, 9 \pmod{12}$ or $v \equiv 3, 27 \pmod{36}$; existence is also established for infinitely many orders in each of the congruence classes $v \equiv 7, 15, 19, 31 \pmod{36}$.

2 Steiner systems in general

In the first part of this paper, we employ probabilistic methods; see [2] for the important background, and [14] for relevant work on concentration inequalities.

An $S(t, k, v)$ has $b = \frac{\binom{v}{t}}{\binom{k}{t}}$ blocks, and every point appears in $r = \frac{\binom{v-1}{t-1}}{\binom{k-1}{t-1}}$ blocks.

2.1 Elementary necessary conditions

Dau and Milenkovic [23] present bounds for duals of $S(2, 3, v)$s, which we state more generally next.

Lemma 1 *For every* $M \in \mathcal{M}_{v,b,k}$, $\mathsf{MinSum}(M^{\mathsf{T}}) \leq \frac{k}{v}\binom{b}{2} \leq \mathsf{MaxSum}(M^{\mathsf{T}})$.

Proof Label the blocks of M arbitrarily as $B_0, \ldots, B_{b-1}$. The sum of the v point sums is $\sum_{j=0}^{b-1} j|B_j| = k\sum_{j=0}^{b-1} j = k\binom{b}{2}$, so the average point sum is $\frac{k}{v}\binom{b}{2}$. □

Naturally, an egalitarian system could exist only when $\frac{k}{v}\binom{b}{2}$ is an integer.

Corollary 1 $\mathsf{MinSum}(\mathcal{D}_{t,k,v}) \leq \frac{1}{2}r(b-1) = \frac{1}{2}\frac{\binom{v-1}{t-1}}{\binom{k-1}{t-1}}\left[\frac{\binom{v}{t}}{\binom{k}{t}} - 1\right] \leq \mathsf{MaxSum}(\mathcal{D}_{t,k,v})$.

Proof An $S(t, k, v)$, D, has $b = \frac{\binom{v}{t}}{\binom{k}{t}}$ and $r = \frac{\binom{v-1}{t-1}}{\binom{k-1}{t-1}}$, and $D \in \mathcal{M}_{v,b,k}$. Because $vr = bk$, the average point sum is $\frac{r(b-1)}{2} = \frac{1}{2}\frac{\binom{v-1}{t-1}}{\binom{k-1}{t-1}}\left[\frac{\binom{v}{t}}{\binom{k}{t}} - 1\right]$. □

Because b and r are integral for any $S(t, k, v)$, the bound of Corollary 1 is always either integral or *half-integral* (i.e., twice the bound is an integer). Specializing to $S(2, k, v)$s, we obtain:

Lemma 2 $\mathsf{MinSum}(\mathcal{D}_{2,k,v}) \leq \frac{(v+k-1)(v-1)(v-k)}{2k(k-1)^2} = \frac{(v+k-1)(r(r-1))}{2k} \leq \mathsf{MaxSum}(\mathcal{D}_{2,k,v})$. *This bound is integral except possibly when* $k = 2^s k'$ *for* k' *odd with* $s \geq 1$, *and* $r \equiv 2^s + 1 \pmod{2^{s+1}}$.

Proof Algebraic simplification of the average given in Corollary 1 when $t = 2$ yields the average stated. Because $v(v-1) \equiv 0 \pmod{k(k-1)}$, we have $(r(k-1)+1)(r(k-1)) \equiv 0 \pmod{k(k-1)}$, so $(-r+1)(r) \equiv 0 \pmod{k}$. When k is odd, $r(r-1) \equiv 0 \pmod{2k}$, so the average is an integer. When k and r are both even, $v+k-1 = (r+1)(k-1)+1 \equiv 0 \pmod{2}$, so the average is an integer. Now $r \equiv 0, 1 \pmod{2^s}$ because $r(r-1) \equiv 0 \pmod{2^s k'}$; when $r \equiv 1 \pmod{2^{s+1}}$, $r(r-1) \equiv 0 \pmod{2k}$, so the average is an integer. □

There are cases in which the bound is not integral: An $S(2, 4, v)$ has integral average point sum when $v \equiv 1, 4, 13 \pmod{24}$, half-integral when $v \equiv 16 \pmod{24}$. An $S(2, 4, 16)$ has average point sum $\frac{19\cdot 15\cdot 12}{2\cdot 4\cdot 3\cdot 3} = \frac{19\cdot 5}{2}$, for example. Similar results can be established for larger values of t. For example, a Steiner quadruple system ($S(3, 4, v)$) has integral average point sum when $v \equiv 2, 4, 10, 14, 20, 22 \pmod{24}$, half-integral when $v \equiv 8, 16 \pmod{24}$.

2.2 General sufficient conditions

There is a large literature on ordering the blocks of various classes of designs [24] to address practical concerns. None appears to address the sum metrics studied here; "pessimal orderings" [15,20] ask for block labellings in which, for every point, every two blocks B_i and B_j containing that point have $|j - i|$ exceeding a stated threshold. Although these labellings certainly determine bounds on the point sums, we obtain better bounds for our sum metrics directly.

A straightforward probabilistic argument establishes the following.

Theorem 2 [17] *Fix k and t. When D is an $S(t,k,v)$,* $\mathsf{DiffSum}(D^\mathsf{T})$ *is $O(v^{3t/2})$.*

Theorem 2 demonstrates that, for k and t fixed, $k > t \geq 3$, and v sufficiently large, *all* Steiner systems admit a block labelling far better than the worst labelling. To obtain better bounds for specific Steiner systems, one can employ structural restrictions. By labelling all blocks in each class of a block partition consecutively, one obtains:

Theorem 3 [17] *Let $D = (X, \mathcal{B})$ be an $S(t,k,v)$ that is s-partitionable of type $(\tau_1, \ldots, \tau_\ell)$, and let D^T be its dual. Then*

$$\mathsf{MinSum}(D^\mathsf{T}) \geq \frac{r(b-1)}{2} + \frac{r}{2} - \frac{b}{2}\left[\prod_{i=1}^{s-1}\frac{v-i}{k-i} + \frac{1}{r}\left[\prod_{i=1}^{s-1}\frac{v-i}{k-i}\right]^2 \sum_{i=1}^{\ell}\tau_i(\tau_i-1)\right],$$

$$\mathsf{MaxSum}(D^\mathsf{T}) \leq \frac{r(b-1)}{2} - \frac{r}{2} + \frac{b}{2}\left[\prod_{i=1}^{s-1}\frac{v-i}{k-i} + \frac{1}{r}\left[\prod_{i=1}^{s-1}\frac{v-i}{k-i}\right]^2 \sum_{i=1}^{\ell}\tau_i(\tau_i-1)\right],$$

$$\mathsf{DiffSum}(D^\mathsf{T}) \leq b\left[\prod_{i=1}^{s-1}\frac{v-i}{k-i} + \frac{1}{r}\left[\prod_{i=1}^{s-1}\frac{v-i}{k-i}\right]^2 \sum_{i=1}^{\ell}\tau_i(\tau_i-1)\right] - r, .$$

Corollary 2 [17] *Let $D = (X, \mathcal{B})$ be an $S(t,k,v)$ that is 1-partitionable of type $(\tau_1, \ldots, \tau_\ell)$, and let D^T be its dual. Then*

$$\mathsf{MinSum}(D^\mathsf{T}) \geq \frac{1}{2}\left[r(b-1) - (b-r) - \frac{b}{r}\sum_{i=1}^{\ell}\tau_i(\tau_i-1)\right],$$

$$\mathsf{MaxSum}(D^\mathsf{T}) \leq \frac{1}{2}\left[r(b-1) + (b-r) + \frac{b}{r}\sum_{i=1}^{\ell}\tau_i(\tau_i-1)\right],$$

$$\mathsf{DiffSum}(D^\mathsf{T}) \leq b - r + \frac{b}{r}\sum_{i=1}^{\ell}\tau_i(\tau_i-1), .$$

When in addition $\alpha = \max\{\tau_i\}$,

$$\mathsf{MinSum}(D^\mathsf{T}) \geq \frac{1}{2}\left[r(b-1) - (b\alpha - r)\right],$$
$$\mathsf{MaxSum}(D^\mathsf{T}) \leq \frac{1}{2}\left[r(b-1) + (b\alpha - r)\right],$$
$$\mathsf{DiffSum}(D^\mathsf{T}) \leq b\alpha - r.$$

The best bounds from Theorem 3 appear to arise from 1-partitionable systems, as in Corollary 2. Among the 1-partitionings, the best bounds arise from partitions with $\ell = r$ and $\tau_1 = \cdots = \tau_r = 1$ (i.e., from resolvable designs).

Corollary 3 [17] *Let $D = (X, \mathcal{B})$ be a resolvable $S(t,k,v)$, and let D^T be its dual. Then*

$$\mathsf{MinSum}(D^\mathsf{T}) \geq \frac{1}{2}\left[r(b-1) - (b-r)\right],$$
$$\mathsf{MaxSum}(D^\mathsf{T}) \leq \frac{1}{2}\left[r(b-1) + (b-r)\right],$$
$$\mathsf{DiffSum}(D^\mathsf{T}) \leq b - r.$$

A resolvable $S(t,k,v)$ can exist only when $v \equiv 0 \pmod{k}$. To treat cases when no resolvable $S(t,k,v)$ exists, or when none is known, we consider other partitions. Suppose that the automorphism group of an $S(t,k,v)$, D, contains a v-cycle, so that the Steiner system is *cyclic*. The blocks of D are partitioned into *orbits* $\mathcal{B}_1,\ldots,\mathcal{B}_\ell$ by the cyclic automorphism. When $|\mathcal{B}_i| = v$, the orbit is *full*. Otherwise $v/(|\mathcal{B}_i|) \geq 2$ is a divisor of k, and the orbit is *short*. It follows that the action of the cyclic automorphism yields a 1-partitioning of type $(\tau_1,\ldots,\tau_\ell)$ in which τ_i is a divisor of k for $1 \leq i \leq \ell$. Hence Corollary 2 yields

Corollary 4 *Let $D = (X, \mathcal{B})$ be a cyclic $S(t,k,v)$, and let D^{T} be its dual. Then*

$$\begin{aligned}\mathsf{MinSum}(D^{\mathsf{T}}) &\geq \frac{1}{2}\left[r(b-1)-(bk-r)\right],\\ \mathsf{MaxSum}(D^{\mathsf{T}}) &\leq \frac{1}{2}\left[r(b-1)+(bk-r)\right],\\ \mathsf{DiffSum}(D^{\mathsf{T}}) &\leq bk-r.\end{aligned}$$

We expect that better probabilistic methods can improve upon these results. To illustrate this, we refine Corollary 3.

Theorem 4 *Let $D = (X,\mathcal{B})$ be a resolvable $S(t,k,v)$, and let D^{T} be its dual. Then $\mathsf{DiffSum}(D^{\mathsf{T}}) \leq \frac{4\ln(2v)\sqrt{r}(v-k)}{k}$, which is $O(v^{\frac{t+1}{2}}\log v)$ for fixed t and k.*

Proof Let $r = \frac{\binom{v-1}{t-1}}{\binom{k-1}{t-1}}$. Let $\{\mathcal{B}_0,\ldots,\mathcal{B}_{r-1}\}$ be the partition of $\mathcal{B}$ into r parallel classes. We label the blocks so that the blocks of $\mathcal{B}_i$ are labelled with a random permutation of the integers $\{\frac{iv}{k},\ldots,\frac{(i+1)v}{k}-1\}$ for $0 \leq i < r$. For each $x \in X$ and each $0 \leq i < r$, denote by $\ell_{x,i}$ the integer in $\{0,\ldots,\frac{v}{k}-1\}$ for which $\frac{iv}{k}+\ell_{x,i}$ is the label of the unique block containing x in $\mathcal{B}_i$.

We define a number of random variables. Because the permutation of $\mathcal{B}_i$ is chosen randomly, $\ell_{x,i}$ is the value of a random variable $R_{x,i}$, with values $\{0,\ldots,\frac{v}{k}-1\}$, each equally likely. Let random variable $R_x = \sum_{i=0}^{r-1} R_{x,i}$. Define the random variable $T_{x,i} = \frac{k}{v-k}R_{x,i}$, so that $T_{x,i}$ takes on values between 0 and 1 (inclusive), and define random variable $T_x = \sum_{i=0}^{r-1} T_{x,i}$. The point sum of $x \in X$ is the random variable $P_x = \sum_{i=0}^{r-1}\left(\frac{iv}{k}+R_{x,i}\right) = \frac{vr(r-1)}{2k}+R_x$. We tabulate the expected value (mean) and variance for each. Using linearity of expectations and the mutual independence of $\{R_{x,i} : 0 \leq i < r\}$, the calculations are routine (see, e.g., [40]).

	$T_{x,i}$	T_x	$R_{x,i}$	R_x	P_x
Expected Value	$\frac{1}{2}$	$\frac{r}{2}$	$\frac{(v-k)}{2k}$	$\frac{r(v-k)}{2k}$	$\frac{1}{2}r(b-1)$
Variance	$\frac{(v+k)}{12(v-k)}$	$\frac{r(v+k)}{12(v-k)}$	$\frac{(v-k)(v+k)}{12k^2}$	$\frac{r(v-k)(v+k)}{12k^2}$	$\frac{r(v-k)(v+k)}{12k^2}$

Now we employ a concentration inequality (see [14] for relevant background). Apply the two-sided Chernoff bound [13,14] $Pr[|X-\mu| \geq \varepsilon\mu] \leq 2\exp(-\frac{\varepsilon^2\mu}{2+\varepsilon})$ (to $X = T_x$ with $\mu = \frac{r}{2}$) to get $Pr[|R_x - \frac{r(v-k)}{2k}| \geq \varepsilon\frac{r(v-k)}{2k}] \leq 2\exp(-\frac{\varepsilon^2}{2+\varepsilon}\frac{r}{2})$. Choose $\varepsilon = \frac{4\ln(2v)}{\sqrt{r}}$. Then $\frac{r\varepsilon^2}{4+2\varepsilon} = \frac{16\sqrt{r}\ln^2(2v)}{4\sqrt{r}+8\ln(2v)} > \ln(2v)$. So $Pr[|R_x - \frac{r(v-k)}{2k}| \geq \frac{4\ln(2v)\sqrt{r}(v-k)}{2k}] < \frac{1}{v}$.

The union bound (or Boole's inequality) states the elementary observation that the probability that at least one of a set of events occurs does not exceed the sum of the probabilities of occurrence for each event. Then applying the union bound, the probability that $|R_x - \frac{r(v-k)}{2k}| \geq \frac{4\ln(2v)\sqrt{r}(v-k)}{2k}$ for *at least one* $x \in X$ is strictly less than $v\cdot(\frac{1}{v}) = 1$. Hence the probability that $|R_x - \frac{r(v-k)}{2k}| < \frac{4\ln(2v)\sqrt{r}(v-k)}{2k}$ for *all* $x \in X$ is strictly greater than 0, Therefore a block ordering exists that meets the stated bound on the DiffSum. □

When $s = 1$ and α is fixed, the corollaries of Theorem 3 yield a DiffSum (for fixed t and k) that is $O(v^t)$, while the average point sum is $\Theta(v^{2t-1})$. Theorem 4 demonstrates that a reduction is possible for resolvable designs, and this can be generalized to the cyclic case and other partitionings. However, we pursue a different direction, next exploring certain partial Steiner systems in which the DiffSum bound can be further reduced.

3 Transversal designs

A *transversal design* of *order* v, *blocksize* k, and *strength* t, denoted by $TD(t, k, v)$, is a triple $(X, \mathcal{G}, \mathcal{B})$ so that

1. $|X| = kv$;
2. $\mathcal{G} = \{G_1, \ldots, G_k\}$ is a partition of X into *groups*, where each group G_i contains v elements of X;
3. $\mathcal{B}$ is a set of k-subsets (*blocks*) of X, for which $|B \cap G_i| = 1$ whenever B is a block and G_i is a group; and
4. every t-subset of X either appears in exactly one block, or contains two or more elements from a group (but not both).

Whenever T is a $TD(t, k, v)$, $T \in \mathcal{M}_{vk,v^t,k}$. By Lemma 1, the average point sum of T is $\frac{1}{2}v^{t-1}(v^t - 1)$, which is always integral when $t \geq 2$.

Theorem 5 *Whenever a $TD(t, k+t, v)$ exists with $k \geq t \geq 2$, there exists an egalitarian resolvable $TD(t, k, v)$. It is split egalitarian when v is even.*

Proof Let $T' = (X', \mathcal{G}', \mathcal{B}')$ be a $TD(t, k+t, v)$ with $X' = \{0, \ldots, v-1\} \times \{1, \ldots, k+t\}$, and $\mathcal{G}' = \{G_1, \ldots, G_{k+t}\}$ with $G_j = \{0, \ldots, v-1\} \times \{j\}$. We form the $TD(t, k, v)$, $T = (X, \mathcal{G}, \mathcal{B})$, with $X = \{0, \ldots, v-1\} \times \{1, \ldots, k\}$, and $\mathcal{G} = \{G_1, \ldots, G_k\}$. To define *and label* the blocks $B_0, \ldots, B_{v^t-1}$ of $\mathcal{B}$, proceed as follows. Whenever $B = \{x_1, \ldots, x_{k+t}\} \in \mathcal{B}'$ so that $\{x_j\} = B \cap G_j$ for $1 \leq j \leq k+t$, write $x_j = (a_j, j)$ and compute $\beta = \sum_{i=0}^{t-1} a_{k+1+i} v^i$ and set $B_\beta = \{x_1, \ldots, x_k\}$. Because we obtain a $TD(t, t, v)$ from T' by considering only the last t groups, each block B_β with $0 \leq \beta < v^t$ is defined exactly once by this procedure. Moreover, partitioning blocks according to the v^{t-1} tuples using the last $t-1$ groups of T gives the parallel classes of a resolution of $\mathcal{B}$.

Consider a point $x \in X$. In order to calculate the point sum of x, one can calculate the contribution to the point sum from each of the groups $G_{k+1}, \ldots, G_{k+t}$, as follows. When a block containing x contains $y \in G_{k+1+i}$, writing $y = (a, k+1+i)$, the contribution is av^i. For every choice of x and $y \in (G_{k+1} \cup \cdots \cup G_{k+t})$, $\{x, y\}$ appears in exactly v^{t-2} blocks of $\mathcal{B}'$. When x is fixed, the total contribution from the group G_{k+1+i} is $v^{t-2}\binom{v}{2}v^i$ because there are v^{t-2} occurrences of each of the v coordinates in G_{k+1+i} and $\sum_{i=0}^{v-1} i = \binom{v}{2}$. Summing the contributions over all groups $G_{k+1}, \ldots, G_{k+t}$, the point sum of x is $\sum_{i=0}^{t-1} v^{t-2}\binom{v}{2}v^i = \frac{v^{t-1}}{2}\sum_{i=0}^{t-1}(v^{i+1} - v^i) = \frac{1}{2}v^{t-1}(v^t - 1)$. Because every point of X has the same point sum, $\mathsf{DiffSum}(T^\mathsf{T}) = 0$. Every point of X appears equally often in blocks with elements of the $(k+t)$th group, so the labelling is split when v is even. □

Theorem 5 exploits not just one partition (or resolution) of the $TD(t, k, v)$ into parallel classes, but rather employs t orthogonal resolutions. Ingredients for the theorem arise from the well-known equivalence of transversal designs and orthogonal arrays of index one [32]. Perhaps unfortunately, while a $TD(t, k, v)$ is a partial $S(t, k, vk)$ with v^t blocks, an $S(t, k, vk)$ would have $\frac{(vk)!(k-t)!}{(vk-t)!k!}$ blocks, a larger number.

4 GDDs

A *group divisible design* (GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ which satisfies the following properties:

(1) $\mathcal{G}$ is a partition of a set X (of *points*) into subsets (*groups*),
(2) $\mathcal{B}$ is a set of subsets of X (*blocks*) such that a group and a block contain at most one common point,
(3) every pair of points from distinct groups occurs in a unique block.

The *group-type (type)* of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. We usually use exponential notation for the group-type. The group-type is *uniform* when all groups have the same size, in which case it is of the form g^u. If K is a set of positive integers, none less than 2, then a GDD $(X, \mathcal{G}, \mathcal{B})$ is a K-GDD if $|B| \in K$ for every block $B \in \mathcal{B}$. When $K = \{k\}$, we simply write k for K. In this notation, a transversal design $TD(2, k, v)$ is a k-GDD of group-type v^k.

We extend certain definitions given earlier. Let D be a k-GDD of group-type g^u. An m-suitable set for D is a set of incidence matrices $A_1, \ldots, A_m \in \mathsf{Inc}(D)$ for which each is obtained from each other by permuting columns (and hence each has the same point labelling) . If $A_1, \ldots, A_m$ is an m-suitable set for D, the $gu \times bm$ array $A = (A_1 \cdots A_m)$, and $\mathsf{DiffSum}(A^\mathsf{T}) = 0$, then D is *m-egalitarian* and $A_1, \ldots, A_m$ is an *m-egalitarian set* for D. When $A_1, \ldots, A_m$ is an m-suitable set, the set is *split* if m is even, or if m is odd and $A_{(m+1)/2}$ is split.

We collect some easy observations:

Lemma 3 *Let D be a k-GDD of group-type g^u, which has $b = \frac{g^2u(u-1)}{k(k-1)}$ blocks.*

1. *D is split m-egalitarian for every even m.*
2. *If D is both m_1-egalitarian and m_2-egalitarian, it is $(m_1 + m_2)$-egalitarian.*
3. *D is b-egalitarian.*

Proof For the first statement, let A be any incidence matrix of D. Let $A_1 = \cdots = A_{m/2} = A$. Let A' be the result of reversing the block order for A, and set $A_{m/2+1} = \cdots = A_m = A'$. Then $A_1, \ldots, A_m$ is a split m-egalitarian set. For the second statement, if $A_1, \ldots, A_{m_1}$ is an m_1-egalitarian set and $B_1, \ldots, B_{m_2}$ is an m_2-egalitarian set, then $A_1, \ldots, A_{m_1}, B_1, \ldots, B_{m_2}$ is $(m_1 + m_2)$-egalitarian. For the third statement, let A be any incidence matrix of a k-GDD of group-type g^u. For $0 \le j < b$ form incidence matrix A_j from A by letting each column c of A_j be column $(c + j) \bmod b$ of A. Then $\{A_j : 0 \le j < b\}$ is a b-egalitarian set. □

Next we describe a method for inflation.

Theorem 6 *Suppose that there exist*

1. *an m-egalitarian k-GDD of group-type g^u having a 1-partition of type $x_1^{r_1} \cdots x_\ell^{r_\ell}$, and*
2. *an egalitarian resolvable $TD(2, k, m)$ from Theorem 5.*

Then an egalitarian k-GDD of group-type $(gm)^u$ exists having a 1-partition of type $x_1^{mr_1} \cdots x_\ell^{mr_\ell}$. If either ingredient is split, the resulting GDD is also split.

Proof Let $(X, \mathcal{G}, \mathcal{D})$ be the m-egalitarian k-GDD of group-type g^u, which is split when possible. This k-GDD has $d = \frac{g^2u(u-1)}{k(k-1)}$ blocks; denote its replication number by r. Let

$$(\{0, \dots, m-1\} \times \{1, \dots, k\}, \{\{0, \dots, m-1\} \times \{j\} : 1 \leq j \leq k\}, \mathcal{T})$$

be an egalitarian resolvable $TD(2, k, m)$ from Theorem 5, which is split when possible. Although blocks in $\mathcal{D}$ are unordered sets, we treat each as a k-tuple by selecting an arbitrary but fixed order for the elements in each; this is used throughout. We form the k-GDD of group-type $(gm)^u$ as $(X \times \{0, \dots, m-1\}, \mathcal{H} = \{G \times \{0, \dots, m-1\} : G \in \mathcal{G}\}, \mathcal{B})$. When $D = (z_1, \dots, z_k)$ and $T = \{(y_1, 1) \dots, (y_k, k)\}$, we denote $\{(z_1, y_1), \dots, (z_k, y_k)\}$ by $D \odot T$. Then set $\mathcal{B} = \{D \odot T : D \in \mathcal{D}, T \in \mathcal{T}\}$. The verification that $(X \times \{0, \dots, m-1\}, \mathcal{H}, \mathcal{B})$ is a k-GDD of group-type $(gm)^u$ is straightforward. Moreover, when $\mathcal{D}'$ forms an x_j-parallel class of the GDD of type g^u and $\mathcal{T}'$ forms a parallel class of the TD, $\{\{(z_1, y_1), \dots, (z_k, y_k)\} : (z_1, \dots, z_k) \in \mathcal{D}', ((y_1, 1) \dots, (y_k, k)) \in \mathcal{T}'\}$ forms an x_j-parallel class of the GDD of type $(gm)^u$, yielding the stated 1-partitioning.

Let $T_{0,0}, \dots, T_{0,m-1}, T_{1,0}, \dots, T_{1,m-1}, \dots, T_{m-1,0}, \dots T_{m-1,m-1}$ be the blocks of the TD in an egalitarian labelling (split when possible), so that $\{T_{j,0}, \dots, T_{j,m-1}\}$ is a parallel class for $0 \leq j < m$. Consider an m-egalitarian set $(A_0, \dots A_{m-1})$ for $(X, \mathcal{G}, \mathcal{D})$ (split when possible); for $0 \leq j < m$, let $D_{j,0}, \dots D_{j,d-1}$ be the block labelling of $\mathcal{D}$ specified by the column ordering of A_j.

Now label the blocks of the k-GDD of group-type $(gm)^u$ as $(B_i : 0 \leq i < m^2 d)$, defined by $B_{\beta md+\alpha m+\gamma} = D_{\beta,\alpha} \odot T_{\beta,\gamma}$ for $0 \leq \beta, \gamma < m$ and $0 \leq \alpha < d$. Partition the labelling into m consecutive intervals of md blocks each; each point appears r times in each interval because each interval is isomorphic to m disjoint copies of the k-GDD of group-type g^u, which together span all points of the k-GDD of group-type $(gm)^u$. Hence when the egalitarian $TD(2, k, m)$ is split, because m is even, the labelling is split. When m is odd but the k-GDD of group-type g^u is split, recall that $A_{\frac{m-1}{2}}$ is split. Consider the middle interval, with $\beta = \frac{m-1}{2}$. (In the preceding intervals, each point appears in the same number of blocks as in the following intervals, so we focus on this middle interval.) Now because $A_{\frac{m-1}{2}}$ is split, each point appears in the first half of the middle interval the same number of times as in the second half, and hence the labelling is split.

Consider a point (x, y) with $x \in X$ and $0 \leq y < m$. Form m intervals as before. Let f_β denote the point sum of (x, y) within the interval β for $0 \leq \beta < m$ (to be specific, index the blocks within each interval from 0 to $md - 1$). The point sum of (x, y) can be written as $\sum_{\beta=0}^{m-1} r \cdot \beta \cdot md + \sum_{\beta=0}^{m-1} f_\beta$. We focus on $\sum_{\beta=0}^{m-1} f_\beta$, because the first sum depends only on the parameters but neither on x nor on y. Within the interval β, let $\{a_{\beta,\ell}m + b_{\beta,\ell} : 0 \leq \ell < r, 0 \leq b_{\beta,\ell} < m\}$ be the indices of blocks containing (x, y), noting that $f_\beta = \sum_{\ell=0}^{r-1} a_{\beta,\ell}m + b_{\beta,\ell}$. Let d_β be the point sum of x in $D_{\beta,0}, \dots D_{\beta,d-1}$, for $0 \leq \beta < m$. Then $\sum_{\beta=0}^{m-1} \sum_{\ell=0}^{r-1} a_{\beta,\ell} = \sum_{\beta=0}^{m-1} d_\beta$, which is the same for every choice of x and y because the k-GDD of group-type g^u is m-egalitarian. Finally consider $\sum_{\beta=0}^{m-1} \sum_{\ell=0}^{r-1} b_{\beta,\ell}$. Because blocks of D have been treated as k-tuples, each point x is associated with each of the m parallel classes of the transversal design, r times each. Because the TD is egalitarian, $\sum_{\beta=0}^{m-1} \sum_{\ell=0}^{r-1} b_{\beta,\ell}$ is the same for every choice of x and y, Hence the point sums do not depend on which point is considered, and the GDD produced is egalitarian. □

5 Steiner triple systems

We refer the reader to [22] for an in-depth discussion of Steiner triple systems. Dau and Milenkovic [23] use the Bose [8] and Skolem [30,46] constructions to establish that there exists a block labelled $\mathrm{STS}(v)$ whose dual has MinSum at least $\frac{55}{1728}v^3 + O(v^2)$, achieving a MinSum for the dual that approaches $\frac{55}{72}$ of the average point sum. First we improve upon these to obtain bounds on the MinSum of the dual approaching $\frac{1}{24}v^3 + O(v^2)$, agreeing in the dominant term with the average point sum.

One might hope that *every* Steiner triple system admits a 1-partitioning of type $(\tau_1, \ldots, \tau_\ell)$ in which the largest of the $\{\tau_i\}$ is 'small', in order to apply Theorem 3 effectively to all Steiner triple systems. However, little is known about 1-partitionings for arbitrary STSs; see [16]. We therefore focus on specific classes of Steiner triple systems.

By Corollary 4, for a cyclic $\mathrm{STS}(v)$, D, we have $\mathsf{MinSum}(D^{\mathsf{T}}) \geq \frac{v(v-1)(v-7)}{24}$, $\mathsf{MaxSum}(D^{\mathsf{T}}) \leq \frac{(v-1)(v^2+5v-12)}{24}$, and $\mathsf{DiffSum}(D^{\mathsf{T}}) \leq \frac{(v-1)^2}{2}$. A cyclic $\mathrm{STS}(v)$ exists whenever $v \equiv 1, 3 \pmod 6$ and $v \neq 9$ [42]. An improvement results from considering cyclic STSs with disjoint starter blocks. Such $\mathrm{STS}(v)$s are studied in [25,34] when $v \equiv 1 \pmod 6$, and in [9,26] when $v \equiv 3 \pmod 6$. Consider a cyclic $\mathrm{STS}(v = 6t + 1)$; suppose that it has starter blocks (orbit representatives) $\{\{a_i, b_i, c_i\} : 0 \leq i < t\}$ so that $|\cup_{i=0}^{t-1}\{a_i, b_i, c_i\}| = 3t$. Labelling the triples within the orbit of $\{a_i, b_i, c_i\}$ so that $\{a_i + j, b_i + j, c_i + j\}$ (reducing modulo v) is placed in position j, we find that the $3t$ positions in which a specific element σ appears are all distinct. This increases the bound on the MinSum by at least $\binom{(v-1)/2}{2}$, to $\frac{(v-1)(v^2-4v-9)}{24}$, and reduces the $\mathsf{DiffSum}$ to $\frac{(v+1)(v-1)}{4}$. Novák [41] advances a strong conjecture, which would imply that every cyclic $\mathrm{STS}(6t + 1)$ can have starter blocks chosen to be disjoint, and this holds at least for all $v \leq 61$ [22]. Recently Novák's conjecture has been shown to hold for all admissible orders, with finitely many possible exceptions [27].

By Corollary 2, whenever a Kirkman triple system $\mathrm{KTS}(v)$, D, exists, we have $\mathsf{MinSum}(D^{\mathsf{T}}) \geq \frac{v(v-1)(v-3)}{24}$, $\mathsf{MaxSum}(D^{\mathsf{T}}) \leq \frac{(v+4)(v-1)(v-3)}{24}$, and $\mathsf{DiffSum}(D^{\mathsf{T}}) \leq \frac{(v-1)(v-3)}{6}$. A $\mathrm{KTS}(v)$ exists if and only if $v \equiv 3 \pmod 6$ [35,44].

Parallelling Theorem 5, one can also employ multiple resolutions of an STS. When an $\mathrm{STS}(v)$, $(X, \mathcal{B})$, has a resolution into parallel classes $\{\mathcal{P}_1, \ldots, \mathcal{P}_{(v-1)/2}\}$, and a resolution into parallel classes $\{\mathcal{Q}_1, \ldots, \mathcal{Q}_{(v-1)/2}\}$ so that $|\mathcal{P}_i \cap \mathcal{Q}_j| \leq 1$, it is a *doubly resolvable Kirkman triple system*, $\mathrm{DRKTS}(v)$. From a $\mathrm{DRKTS}(v)$ one can form a certain type of *Kirkman square*, a $(v-1)/2 \times (v-1)/2$ array in which the cell in row i and column j contains the triple in $\mathcal{P}_i \cap \mathcal{Q}_j$, if one exists; otherwise the cell is empty. The existence of $\mathrm{DRKTS}(v)$ is settled with few possible exceptions [1,21]. Applying Corollary 2 using the resolution $\{\mathcal{P}_1, \ldots, \mathcal{P}_{(v-1)/2}\}$, labelling the triples within each so that a triple from $\mathcal{Q}_j$ is placed in position $j - 1$, for each $1 \leq j \leq v/3$, we obtain:

Lemma 4 *Whenever a doubly resolvable Kirkman triple system DRKTS(v), D, exists,* $\mathsf{MinSum}(D^{\mathsf{T}}) \geq \frac{v(3v+1)(v-3)}{72}$, $\mathsf{MaxSum}(D^{\mathsf{T}}) \leq \frac{(v+3)(3v-4)(v-3)}{72}$, *and* $\mathsf{DiffSum}(D^{\mathsf{T}}) \leq \frac{(v-3)^2}{18}$.

Each restriction mentioned reduces the $\mathsf{DiffSum}$, but although the coefficient of v^2 has been reduced, the rate of growth has not. Theorem 4 reduces the rate of growth to $O(v^{3/2} \log v)$ when $v \equiv 3 \pmod 6$ using Kirkman triple systems. Next we outline an analogue for cases when $v \equiv 1 \pmod 6$. Write $v = 6s + 1$. An $\mathrm{STS}(6s + 1)$, $(V, \mathcal{B})$, together with a partition of $\mathcal{B}$ into sets $\{\mathcal{B}_0, \ldots, \mathcal{B}_{3s}\}$ so that $\mathcal{B}_j$ is a set of $2s$ disjoint blocks for $0 \leq j < 3s$, and $\mathcal{B}_{3s}$ is a set of s disjoint blocks, is a *Hanani triple system*, $\mathrm{HATS}(6s + 1)$. A $\mathrm{HATS}(v)$ exists if and only if $v \equiv 1 \pmod 6$ and $v \geq 19$ [49]. Without loss of generality, we write a HATS$(6s+1)$ on symbols $\{0, \ldots, 6s\}$ so that (1) for $0 \leq j < 3s$, $\bigcup_{B \in \mathcal{B}_j} B = \{0, \ldots, 6s\} \setminus \{j\}$, and (2) $\mathcal{B}_{3s} = \{\{i, s+i, 2s+i\} : 0 \leq i < s\}$. Letting $\mathcal{C}_i = \mathcal{B}_i \cup \mathcal{B}_{s+i} \cup \mathcal{B}_{2s+i} \cup \{\{i, s+i, 2s+i\}\}$ for $0 \leq i < s$, we have that $\mathcal{C}_0, \ldots, \mathcal{C}_{s-1}$ is a 1-partitioning of type 3^s.

In order to label the blocks, we label blocks in $\mathcal{C}_{i-1}$ before blocks in $\mathcal{C}_i$ for $0 < i < s$. To label the blocks of $\mathcal{C}_i$, we label the blocks in $\mathcal{B}_i$ with a random permutation of $\{i(6s+1), \ldots, i(6s+1)+2s-1\}$; those in $\mathcal{B}_{2s+i}$ with a random permutation of $\{i(6s+1)+2s, \ldots, i(6s+1)+4s\} \setminus \{i(6s+1)+3s\}$; and those in $\mathcal{B}_{s+i}$ with a random permutation of $\{i(6s+1)+4s+1, \ldots, i(6s+1)+6s\}$. Finally label $\{i, s+i, 2s+i\}$ with $i(6s+1)+3s$. Then within the blocks of $\mathcal{C}_i$, the expected values of the point sums are $3i(6s+1)+9s$, *except that* point i has expected point sum $3i(6s+1)+11s+\frac{1}{2}$ and point $s+i$ has expected point sum $3i(6s+1)+7s-\frac{1}{2}$. Then computing variances and applying Chernoff bounds, an analogue of Theorem 4 can be established when $v \equiv 1 \pmod 6$. However, we pursue a different direction next.

For Steiner triple systems, we can do much better. In the remainder of the paper we show that, in most cases, the DiffSum can be 0.

5.1 Small orders

When D is the STS(7), $D = D^{\mathsf{T}}$, and $\mathsf{DiffSum}(D^{\mathsf{T}}) = 7$ [23]. Hence it is not egalitarian, but it is m-egalitarian except possibly when $m \in \{1, 3, 5\}$ by Lemma 3.

When D is the STS(9), D^{T} is a $TD(2, 4, 3)$. When the point labelling of the $TD(2, 4, 3)$ has groups $\{\{3i, 3i+1, 3i+2\} : 0 \leq i < 4\}$, define an array A that contains row (a, b, c, d) whenever $\{a, b+3, c+6, d+9\}$ is a block of the $TD(2, 4, 3)$. Then A is an orthogonal array, an $OA(4, 3)$ on symbols $\{0, 1, 2\}$. The block sums of the $TD(2, 4, 3)$ (equivalently, the point sums of the STS(9)) are obtained from the row sums of A by adding 18 to each.

Lemma 5 *There is an* $OA(4, 3)$ *with minimum row sum 2, but none with 3. There is an* $OA(4, 3)$ *with maximum row sum 6, but none with 5. There is an* $OA(4, 3)$ *with the difference between the maximum and minimum row sums equal to 5, but none with 4.*

Proof The $OA(4, 3)$ with rows 0011, 0120, 0202, 1022, 1101, 1210, 2000, 2112, and 2221 has smallest row sum 2 and largest row sum 7. Interchanging symbols 0 and 2 throughout gives an $OA(4, 3)$ with maximum row sum 6.

Now suppose to the contrary that some $OA(4, 3)$ has minimum row sum at least 3. Because the three rows containing 0 in a specified column must have total row sum 9, each row containing a 0 must have row sum 3. Then no row can contain three 0s, and there must be six containing two 0s, and three containing no 0s. Without loss of generality, three of the six containing two 0s are 0012, 0201, and 0120. But for some x, y, row $x00y$ must be present, but $y \notin \{1, 2\}$, a contradiction. Hence the minimum row sum is at most 2, and the maximum is at least 6 (by reflecting symbols).

Next suppose to the contrary that there is an $OA(4, 3)$ with the difference between the maximum and minimum row sums equal to 4, so that the minimum row sum is 2 and the maximum is 6. No row can contain four 0s. If a row contains three 0s, the fourth entry is a 2. The sum of the three rows containing a 2 in the same coordinate is 15, so one of these rows has row sum at least 7, which cannot be. Symmetrically, no row can contain more than two 2s. Because in each of the $\binom{4}{2} = 6$ pairs of columns, a row contains 00 in the specified columns, the same holds for 22, and there are nine rows in total, we must have at least three rows each containing two 0s and two 2s. Without loss of generality, one is 0022. But the only row consistent with this is 2200, and hence three such rows cannot arise, which is a contradiction. □

Lemma 6 *The STS*(9)*, D, has* $\mathsf{MinSum}(D^{\mathsf{T}}) = 20$*,* $\mathsf{MaxSum}(D^{\mathsf{T}}) = 24$*, and* $\mathsf{DiffSum}(D^{\mathsf{T}}) = 5$.

Proof Dau and Milenkovic [23] establish the bounds for MinSum and MaxSum by examining all 12! block labellings; we give a computer-free proof. We consider point labellings of a $TD(2, 4, 3)$. When $\{a, b, c\}$ is a group, the three blocks containing a have total block sum $66+2a-b-c$. If the MinSum is greater than 20, the groups must be $\{\{3i, 3i+1, 3i+2\} : 0 \leq i < 4\}$. So apply Lemma 5 to the corresponding $OA(4, 3)$ to establish that $\mathsf{MinSum}(D^{\mathsf{T}}) = 20$, $\mathsf{MaxSum}(D^{\mathsf{T}}) = 24$ by reversal, and $\mathsf{DiffSum}(D^{\mathsf{T}}) \leq 5$. (Unfortunately, this does not establish that $\mathsf{DiffSum}(D^{\mathsf{T}}) \neq 4$, because the application of Lemma 5 is predicated on the labelling of groups.)

Suppose to the contrary that $\mathsf{DiffSum}(D^{\mathsf{T}}) = 4$, and consider point labellings with MinSum 20 and MaxSum 24. Whenever $\{x, x + a, x + b\}$ with $0 \leq x < x + a < x + b \leq 11$ is a group, the blocks containing x have total block sum $66 - a - b$, and those containing $x + b$ have total block sum $66 + 2b - a$. Because each block sum must be between 20 and 24, $a + b \leq 6$ and $2b - a \leq 6$. Hence $3b \leq 12$, so $b \leq 4$. If $b = 4$, then $a = 2$.
Case 1. The group containing 0 is $\{0, 2, 4\}$. The three blocks containing 0 have total block sum 60, so each has block sum 20. But the three block sums containing 11 have total block sum at least 69, so none has block sum 20, a contradiction because some block contains $\{0, 11\}$.
Case 2. The group containing 0 is $\{0, 2, 3\}$. No group $\{1, 1 + a, 1 + b\}$ disjoint from the group containing 0 with $b \leq 4$ can be chosen except when $b = 4$ and $a = 3$, a contradiction.
Case 3. The group containing 0 is $\{0, 1, 3\}$. Some block contains $\{2, 11\}$. Blocks containing 2 must have total block sum at most 61, while those containing 11 have total block sum at least 69, so the block containing $\{2, 11\}$ has block sum 21. For a block containing 2 to have block sum at least 21, the group containing 2 is $\{2, 4, 5\}$. For a block containing 11 to have block sum at most 21, the group containing 11 is $\{9, 10, 11\}$. Hence the final group is $\{6, 7, 8\}$. Two possibilities arise; we treat them with a single argument. For $a \in \{0, 1\}$, if $\{\{0, 2, 8, 10 + a\}, \{1, 2, 7, 11 - a\}, \{2, 3, 6, 9\}\}$ are blocks, no block containing $\{a, 11\}$ can have block sum 24 (the two remaining elements are each at most 6).
Case 4. The group containing 0 is $\{0, 1, 2\}$. If the group containing 11 is not $\{9, 10, 11\}$ then reverse the labelling to obtain an earlier case. If the group containing 3 is $\{3, 4, 6\}$, then the final group is $\{5, 7, 8\}$. But every block containing 5 has block sum at most 21, and every block containing 6 has block sum at least 21, and $\{5, 6\}$ must be in a block of sum 21. The other two blocks containing 5 have block sum 20, and the other two containing 6 have block sum 24. It follows that 0 must be in the block containing $\{5, 6\}$ (for otherwise 0 must be in blocks with sums 20 and 24), so it must be $\{0, 5, 6, 10\}$. The other two blocks containing 5 must be $\{2, 4, 5, 9\}$ and $\{1, 3, 5, 11\}$. But then no block can contain $\{2, 6\}$, because it cannot contain 5 or 9, but must have block sum 24. So the remaining two groups must be $\{3, 4, 5\}$ and $\{6, 7, 8\}$, and Lemma 5 completes the proof. □

The STS(7) and STS(9) are not good indicators of the best DiffSum for larger orders. Indeed, for certain $v \geq 13$, we find an STS(v), D, that is egalitarian, i.e., has $\mathsf{DiffSum}(D^{\mathsf{T}}) = 0$.

To construct small egalitarian STSs, we employ a local optimization (hill-climbing) method, described next. For a fixed STS(v), $(V, \mathcal{B})$, the target point sum is computed as $\rho = \frac{(v+2)(v-1)(v-3)}{24}$. Denote the point sum of $x \in V$ as $ps(x)$, once a block labelling is chosen. The *score* of the block labelling is $\sum_{x \in V} |ps(x) - \rho|$. Initially select a random block labelling and compute its score. Repeatedly choose two blocks whose indices in the labelling do not differ by more than the current score, and interchange them in the block labelling if the score is not increased by the exchange. If the score becomes 0, an egalitarian labelling has been found. If a threshold number of block exchanges is examined without reducing the score, the search is restarted from scratch. If a threshold number of restarts complete without success, the search is abandoned in failure. We set both thresholds at 10000, so that a search typically completes in minutes of computation time.

We present STS(v)s using lower- and upper-case letters as elements, with the blocks in an egalitarian labelling. An STS(13) is given next; such an STS is necessarily partitionable into two 3-parallel classes [22]. Every point sum equals 75. For example, point b appears in blocks 0, 6, 10, 18, 20, and 21.

```
bij adg fil hkm elm cfh bgh ceg cik djk abm aek dfm afj gjl acl dei ehj
bcd dhl bef bkl cjm fgk gim ahi
```

The STS(15) is isomorphic to the first of eighty STS(15)s from [37], and hence is resolvable into seven parallel classes.

```
hlm cik dno gln afk bef abo ejn gio cfh cjl fjm deh agj eko bij dil adm
bcd bgh gkm hkn cmo eim ahi dfg bmn ael acn djk bkl ceg fin flo hjo
```

The STS(19) is cyclic (it is isomorphic to $\{\{i, i+1, i+5\}, \{i, i+2, i+8\}, \{i, i+3, i+10\} : 0 \leq i < 19\}$), and hence is partitionable into three 3-parallel classes.

```
gps jlr adk cnp gio bir him enq aln fqs efj ahq hkr fip cjs jko ajm bmo
ghl dei bpq dmp dfl bkn clo doq egm cek cqr bhs cfm bel aes agr bcg fhn
for kms cdh nos drs klp bdj dgn mnr aop fgk eho gjq ikq ijn abf ils epr
aci lmq hjp
```

The STS(21) is cyclic and resolvable. Every point has sum $345 = \frac{10 \cdot 69}{2}$. Moreover, it is split egalitarian, because every point appears five times in the first 35 blocks and five times in the last 35.

```
jkm bfg bce qrt eno dmu fnr gkt ost pqs fhp dhq kln ait cjq bip cdf rsu
alu bmo els ghs aef imn cko ajr ijl ptu hlm giq bju dkr cnp deg aho
chu anq ams gjo glr dlp fiu eku dio clt emq hjn bhr hik abd cis akp opr
djt gmp ejp flo eir fjs bnt acg bks blq gnu dns eht fmt oqu cmr fkq
```

The STS(25) is cyclic (it is isomorphic to $\{\{i, i+1, i+6\}, \{i, i+2, i+11\}, \{i, i+3, i+10\}, \{i, i+4, i+17\} : 0 \leq i < 25\}$), and hence is partitionable into four 3-parallel classes. The ordering shown is split egalitarian.

```
ips dhu cfu cjm bow bet ksw aim bjn cko muy hly anv hpt gnq gjy ehw fnr
ovy gos bqx aer dlp ads cry hor dkn ltx emq ahk dgv fix cgt fjw apw kru
cpx bil iqu elo jqt bfs mtw dqy gkx nux jrv eiv lsv fmp
pqv ijo klq cdi dwx ajx ftv efk kmv fhq egp npy gir rsx exy jlu lmr fgl
lnw sty abg buv bdm esu atu not hvx bpr qrw mox drt ikt cvw hjs cqs mns
guw cen aoq afy bch iwy acl dfo opu ghm hin dej jkp bky
```

The STS(31) is cyclic (it is isomorphic to $\{\{i, i+1, i+7\}, \{i, i+2, i+11\}, \{i, i+3, i+20\}, \{i, i+4, i+23\}, \{i, i+5, i+21\} : 0 \leq i < 31\}$), and hence is partitionable into five 3-parallel classes.

```
jmD glB jlu qsB abh bfy npy iCE hsv alo ksw ejz fhq nxC dwE owA jkq eCD
aqA prA cko mnt dtD tvE bdm pAD agE bvx ilC hrw aex rtC gos bzA cvD uvB
fqt lwz pxB tuA hio ghn cwy isx rsy suD kuz fDE dek fnr arC hkB xyE cfw
gqv aim cdj buC opv cAB oqz gir efl muy bjn cmr cgz ikt acl diy cen bev
gkD fzB ijp qyC dlp emq hpt bgw fiz pzE dBC euE joE mxA fpu ehy dxz eps
mox mwB jux dor nyB stz dgx atB adu rzD knE hmC gjA jrv dfo lvA wxD eyA
chx nou inD klr nvz ctE lms lnw jty kmv akp kvy cnq eiB gru sAE bmp bkE
ayz iqu fgm bsD qBE dns itw fjC brB fkA gAC vwC blq egp csC bci ltx eot
pqw qrx hBD hlE ozC hjs oyD auw afv ajD dhA
```

The STS(33) is cyclic and can be partitioned into four 3-parallel classes and four parallel classes. The ordering given is split egalitarian.

```
hmG dmt ktA bjo oAG cix ams duD glF ckp ajq myE afz bsB mvC lty bqC gnE
eqw rzE luB gpw diC cwE ahy frx qzG emr ioD nwD flA dxF jsz bhw bnt hqx
lxD dpv nzF etF dkB iuA hpu ksx elC agv pyF nvA owB hoF jpE iry fuG fns
kwC cls hnC tBG djy apB cjA htz dsE bgA biz iqv oxE ekz kqF jvB crD auC
fmD fwF ipG enu gxG lrG gsy ejD cou ctC bkr qyD arA fov evE gmB
rsu fjt cvz alw kvG bvD rvF grC efh juF bfp gku DEG quE eis nrB eyG jnx
koy lmo cgq cny epA lpz fyC got itE iBF fqB zAC ABD abd deg vwy noq BCE
exB cmF hsD hik uvx ijl akD acG dnG hAE gzD sAF jrw bmx opr wxz ain chB
pxC pqs dwA osC ptD dhr imw CDF doz atx fkE bce swG mqA aeo xyA buy jCG
cdf ghj bFG jkm qrt stv kln yzB tuw hlv dlq aEF blE muz mnp fgi
```

We have established

Lemma 7 *An egalitarian STS*(v) *exists for* $v \in \{13, 15, 19, 21, 25, 31, 33\}$. *It is split egalitarian when* $v \in \{21, 25, 33\}$.

5.2 A doubling construction

To obtain infinite families of egalitarian Steiner triple systems, we adapt some standard direct and recursive constructions (see [22]). First we consider a direct construction.

Theorem 7 *Let* $\Gamma = (X, \oplus)$ *be an abelian group of order* $v = 6t + 1$ *with* $v > 7$. *Suppose that* $D = ((X \times \{1, 2\}) \cup \{\infty\}, \mathcal{B})$ *is a KTS*$(2v + 1)$ *with a parallel class* $\mathcal{P}$ *whose orbits under the action of* Γ *form the partition of* D *into parallel classes, and so that* $|\{B \in \mathcal{P} : B \subset X \times \{1\}\}| \neq 1$ *or* $t - 1$. *Then* D *is egalitarian.*

Proof For a triple B and any $z \in X$, let $B + z$ be the triple obtained by replacing (x, i) by $(x \oplus z, i)$ for $x \in X$, leaving ∞ fixed. For each $z \in X$, define $\mathcal{P}_z = \{B + z : B \in \mathcal{P}\}$. Then $\{\mathcal{P}_z : z \in X\}$ forms the v parallel classes of the KTS. Arbitrarily label the nonzero elements of X as $e_1, \ldots, e_{v-1}$.

First we label the blocks of $\mathcal{P}$ as $\{B_i : 0 \leq i \leq 4t\}$. In the process, we define $\mu(\infty) = i$ when $\infty \in B_i$ and $0 \leq i \leq 4t$, and $\mu(j) = \frac{1}{v} \sum\{i : (x, j) \in B_i, 0 \leq i \leq 4t, x \in X\}$ for $j \in \{1, 2\}$. Because $\mu(\infty) + (6t + 1)\mu(1) + (6t + 1)\mu(2) = 3\binom{4t+1}{2}$, we have $\mu(2) = 2t$ whenever $\mu(1) = \mu(\infty) = 2t$.

There is exactly one block of $\mathcal{P}$ that contains ∞. Label this block as B_{2t} so that $\mu(\infty) = 2t$. For $0 \leq \ell \leq 3$, let $\mathcal{P}_\ell$ be the set $\{B \in \mathcal{P} \setminus \{B_{2t}\} : |B \cap (X \times \{1\})| = \ell$, and let $\rho_\ell = |\mathcal{P}_\ell|$. (These counts satisfy $3\rho_0 + \rho_1 = \rho_2 + 3\rho_3 = 3t$ and $\rho_1 + \rho_2 = 3t$, so $\rho_0 + \rho_3 = t$. By hypothesis, $\{\rho_0, \rho_3\} \cap \{1, t - 1\} = \emptyset$.)

Now we treat cases, ensuring that $\mu(1) = 2t$ for each:

Case 1. t **is even and** ρ_0 **is even.** Then ρ_1, ρ_2, and ρ_3 are all even. Label the remaining $4t$ blocks so that $B_{2t-j} \in \mathcal{P}_i$ if and only if $B_{2t+j} \in \mathcal{P}_i$ for all $1 \leq j \leq 2t$.
Case 2. t **is even and** ρ_0 **is odd.** Then ρ_1, ρ_2, and ρ_3 are all odd. Choose one block in $\mathcal{P}_0$ to be B_{2t-1}, one block in $\mathcal{P}_1$ to be B_{2t+3}, one block in $\mathcal{P}_2$ to be B_{2t-3}, and one block in $\mathcal{P}_3$ to be B_{2t+1}. Label the remaining $4t-4$ blocks so that $B_{2t-j} \in \mathcal{P}_i$ if and only if $B_{2t+j} \in \mathcal{P}_i$ for $j = 2$ and all $4 \leq j \leq 2t$.
Case 3. t **is odd and** ρ_0 **is even.** Then ρ_1 and ρ_3 are odd, but ρ_2 is even. Choose three blocks in $\mathcal{P}_1$ to be B_{2t-2}, B_{2t-1},and B_{2t+3}, and three blocks in $\mathcal{P}_3$ to be B_{2t+2}, B_{2t+1},and B_{2t-3}. Label the remaining $4t-6$ blocks so that $B_{2t-j} \in \mathcal{P}_i$ if and only if $B_{2t+j} \in \mathcal{P}_i$ for all $4 \leq j \leq 2t$.
Case 4. t **is odd and** ρ_0 **is odd.** Then ρ_1 and ρ_3 are even, but ρ_2 is odd. This is symmetric to Case 3, by swapping the second coordinate of the points.

Extend the labelling of $\mathcal{P} = \mathcal{P}_0$ as $B_0, \ldots, B_{4t}$ to all triples by setting $B_{i(4t+1)+j} = B_j + e_i$ for all $1 \leq i \leq v-1$ and $0 \leq j \leq 4t$. The point sum of ∞ is $(4t+1)\binom{v}{2} + (4t+1)\mu(\infty)$. For $x \in X$ and $j \in \{1, 2\}$, the point sum of (x, j) is $(4t+1)\binom{v}{2} + (4t+1)\mu(j)$. Hence all points have the same point sum, and the labelling is egalitarian. □

Corollary 5 *An egalitarian resolvable STS*(v) *exists when* $v = 12t + 3$, $t \geq 2$, *and either every prime divisor of* $6t + 1$ *is congruent to* $1 \bmod 6$, *or* $6t + 1$ *is a prime power.*

Proof A KTS$(2t+1)$ is constructed in [4,5] when $6t+1$ is a prime, satisfying the conditions of the lemma and having $\rho_3 = t$, $\rho_0 = \rho_2 = 0$, and $\rho_1 = 3t$. Anderson [3] generalizes this to handle cases when every prime divisor of $6t+1$ is congruent to 1 mod 6. The same pattern is obtained when $6t+1$ is a prime power in [44]; also see [22, Theorem 19.10]. See also [10,11] for related work. □

Corollary 5 establishes that an egalitarian resolvable STS(v) exists when $v \in \{27, 39, 51, 63, 75, 87, 99\}$, for example. An STS$(v)$ having an automorphism whose orbits consist of one fixed point and two $(v-1)/2$-cycles is *2-rotational*. A 2-rotational STS(v) exists if and only if $v \equiv 3, 7 \pmod{12}$ [43]; however the existence of a resolvable 2-rotational STS(v) when $v \equiv 3 \pmod{12}$ appears to be open in general. In light of Theorem 7, a complete solution for this problem would be useful.

5.3 Filling in holes

Next we fill holes in egalitarian 3-GDDs from Theorem 6. We need a technical lemma.

Lemma 8 *Let* $u \geq 3$ *be an integer. Let* $(V, \mathcal{B})$ *be an STS*(v) *whose triples admit a partition into* τ_s *s-parallel classes* ($1 \leq s \leq \frac{v-1}{2}$), *so that for every* s *either* τ_s *is even, or* $\tau_s \neq 1$ *and* u *is odd. There exist* u *(not necessarily proper) block colourings* $\gamma_0, \gamma_1, \ldots, \gamma_{u-1}$ *with* $\gamma_j : \mathcal{B} \mapsto \{0, 1, \ldots, u-1\}$ *for which*

1. *for* $B \in \mathcal{B}$, $\{\gamma_0(B), \gamma_1(B), \ldots, \gamma_{u-1}(B)\} = \{0, 1, \ldots, u-1\}$; *and*
2. *for* $x \in V$, $\sum_{j=0}^{u-1} \sum_{B:x\in B} \gamma_j(B) = \frac{v-1}{2}\binom{u}{2}$.

Proof For each $1 \leq s \leq \frac{v-1}{2}$, let $\mathcal{P}_{s,0}, \ldots, \mathcal{P}_{s,\tau_s-1}$ be the s-parallel classes in the partition. We colour blocks in the s-parallel classes as follows. Write $\tau_s = 2\ell$ when τ_s is even, $\tau_s = 2\ell + 3$ when τ_s is odd. For $0 \leq i < 2\ell$ and each $B \in \mathcal{P}_{s,i}$, for $0 \leq j < u$, set $\gamma_j(B) = j$ if i is even or $\gamma_j = u - 1 - j$ if i is odd. This provides the required colourings when each of $\{\tau_s\}$ is even. To complete the colourings in the remaining cases, we must provide the colourings for blocks in $\mathcal{P}_{s,\tau_s-3}, \mathcal{P}_{s,\tau_s-2}, \mathcal{P}_{s,\tau_s-1}$ when τ_s is odd. When $B \in \mathcal{P}_{s,\tau_s-3}$, set $\gamma_j(B) = j$ for $0 \le j < u$. When $B \in \mathcal{P}_{s,\tau_s-2}$, set $\gamma_j(B) = u-1-2j$ for $0 \le j \le \frac{u-1}{2}$, and $\gamma_j(B) = 2u-1-2j$ for $\frac{u-1}{2} < j \le u-1$. When $B \in \mathcal{P}_{s,\tau_s-1}$, set $\gamma_j(B) = \frac{u-1}{2} + j$ for $0 \le j \le \frac{u-1}{2}$, and $\gamma_j(B) = j - \frac{u+1}{2}$ for $\frac{u-1}{2} < j \le u-1$. Note that $j + u - 1 - 2j + \frac{u-1}{2} + j = j + 2u - 1 - 2j + j - \frac{u+1}{2} = 3\frac{u-1}{2}$.

The verification is routine. □

Lemma 9 *Let $u \ge 3$ be odd. Suppose that there exists an egalitarian STS(v) having a 1-partitioning of type $(s^{\tau_s} : 1 \le s \le \frac{v-1}{2})$ with $\tau_s \ne 1$, for $1 \le s \le \frac{v(u-1)}{2}$. Further suppose that an egalitarian 3-GDD of type v^u exists, having a 1-partitioning of type $(s^{\mu_s} : 1 \le s \le \frac{v(u-1)}{2})$. Then there exists an egalitarian STS(uv) having a 1-partitioning of type $(s^{\tau_s+\mu_s} : 1 \le s \le \frac{v(u-1)}{2})$.*

Proof Let $D = (X, \mathcal{D})$ be the STS(v) with $b = \frac{v(v-1)}{6}$ blocks having egalitarian labelling $D_0, \ldots, D_{b-1}$, so that every point sum is κ. Let $E = (\{0, \ldots, u-1\} \times X, \{\{j\} \times X : 0 \le j < u\}, \mathcal{E})$ be the egalitarian 3-GDD of type v^u, with d blocks, egalitarian labelling $E_0, \ldots, E_{d-1}$, and each point with sum σ. Apply Lemma 8 to produce u block colourings $\gamma_0, \gamma_1, \ldots \gamma_{u-1}$ for the blocks of $\mathcal{D}$.

To form the STS(uv) we form a set $\mathcal{B}$ of blocks, by starting with the blocks of E, and adjoining a copy of the blocks of D on $\{j\} \times X$ for $0 \le j < u$. We must order these $d + u\frac{v(v-1)}{6}$ blocks to produce an egalitarian labelling. To do this, define $B_i = E_i$ for $0 \le i < d$. Then set $B_{d+uj+\gamma_\ell(D_j)} = \{\ell\} \times D_j$ for $0 \le \ell < u$ and $0 \le j < b$. Then every point has point sum $\sigma + d\frac{v-1}{2} + u\kappa + \frac{(v-1)(u-1)}{4}$, and the STS$(uv)$ is egalitarian. □

Next we describe some (but by no means all) of the consequences.

Corollary 6 *Suppose that $v \ge 13$ and there exists an egalitarian STS(v) having a 1-partitioning of type $(s^{\tau_s} : 1 \le s \le \frac{v-1}{2})$ so that $\tau_s \ne 1$ when $1 \le s \le \frac{v-1}{2}$. Then there exist*

1. *an egalitarian STS$(3v)$ having a 1-partitioning of type $(1^{\tau_1+v})(s^{\tau_s} : 2 \le s \le \frac{v-1}{2})$.*
2. *an egalitarian STS$(7v)$ having a 1-partitioning of type $(1^{\tau_1})(2^{\tau_2})(3^{\tau_3+v})(s^{\tau_s} : 4 \le s \le \frac{v-1}{2})$.*
3. *an egalitarian STS$(13v)$ having a 1-partitioning of type $(1^{\tau_1})(2^{\tau_2})(3^{\tau_3+2v})(s^{\tau_s} : 4 \le s \le \frac{v-1}{2})$.*
4. *an egalitarian STS$(25v)$ having a 1-partitioning of type $(1^{\tau_1})(2^{\tau_2})(3^{\tau_3+4v})(s^{\tau_s} : 4 \le s \le \frac{v-1}{2})$.*

Proof We employ 3-GDDs of type 1^3, 1^7, 1^{13}, and 1^{25} with 1-partitioning of type 1^1, 3^1, 3^2, and 3^4, respectively, taking $m = v$; there is a $TD(2, 5, m)$ in each of these cases. By Lemma 3, the first 3-GDD is m-egalitarian for all $m \ge 1$ and the second is m-egalitarian for all $m \ge 7$. The rest are egalitarian by Lemma 7. Apply Theorem 6 to construct the egalitarian 3-GDD of type v^u. Now apply Lemma 9. □

Lemma 10 *Let $u \ge 3$ and $v \ge 13$. Suppose that there exists an egalitarian STS(v) having a 1-partitioning of type $(s^{\tau_s} : 1 \le s \le \frac{(v-1)(u-1)}{2})$, with u odd and $\tau_s \ne 1$, or u even and τ_s even, for $1 \le s \le \frac{(v-1)(u-1)}{2}$. Suppose that a split egalitarian 3-GDD of type $(v-1)^u$ having a 1-partitioning of type $(s^{\mu_s} : 1 \le s \le \frac{(v-1)(u-1)}{2})$ exists. Then an egalitarian STS$(u(v-1)+1)$ exists.*

Proof Let $D = (X \cup \{\infty\}, \mathcal{D})$ be the STS(v) with $b = \frac{v(v-1)}{6}$ blocks having egalitarian labelling $D_0, \ldots, D_{b-1}$, so that every point sum is κ. Apply Lemma 8 to produce u block colourings $\gamma_0, \gamma_1, \ldots \gamma_{u-1}$ for the blocks of $\mathcal{D}$. Let $E = (\{0, \ldots, u-1\} \times X, \{\{j\} \times X : 0 \leq j < u\}, \mathcal{E})$ be the split egalitarian 3-GDD of type v^u, with d blocks, split egalitarian labelling $E_0, \ldots, E_{d-1}$, and each point with sum σ. In E, each point appears in an even number $r = \frac{(v-1)(u-1)}{2}$ of blocks because the labelling is split. Hence d is also even.

We form the STS($u(v-1)+1$) on elements $(\{0, \ldots, u-1\} \times X) \cup \{\infty\}$, whose block set $\mathcal{B}$ is obtained by starting with the blocks of E, and adjoining a copy of the blocks of D on $(\{j\} \times X) \cup \{\infty\}$ for $0 \leq j < u$. We must order these $d + ub$ blocks to produce an egalitarian labelling. To do this, first define $B_i = E_i$ for $0 \leq i < \frac{d}{2}$, and $B_{ub+i} = E_i$ for $\frac{d}{2} \leq i < d$. (This orders the prefix of $d/2$ blocks and the suffix of $d/2$ blocks, leaving a consecutive set of ub blocks to be determined "in the middle".)

For $0 \leq \ell < u$, let π_ℓ be a function that maps x to (ℓ, x) for $x \in X$, and maps ∞ to itself. When $D \subseteq X \cup \{\infty\}$, let $\pi_\ell(D)$ denote $\{\pi_\ell(y) : y \in D\}$. Then set $B_{d/2+uj+\gamma_\ell(D_j)} = \pi_\ell(D_j)$ for $0 \leq \ell < u$ and $0 \leq j < b$.

Now following the proof of Lemma 9, every point of $\{0, \ldots, u-1\} \times X$ has the same point sum. Because this point sum equals the average point sum. the point sum of ∞ is the same and the block labelling is egalitarian. $\square$

We do not specify the type of a 1-partitioning of the egalitarian STS($u(v-1)+1$) from Lemma 10 in general, because we cannot simply append the classes of the ingredient designs as in Lemma 9. Nevertheless we determine these types for some specific applications.

Corollary 7 *Suppose that $v \geq 13$ and there exists an egalitarian STS(v) having a 1-partitioning of type $(s^{\tau_s} : 1 \leq s \leq \frac{v-1}{2})$, with $\tau_s \neq 1$ for $1 \leq s \leq \frac{v-1}{2}$. Then there exist*

1. *an egalitarian STS*$(3v-2)$ *having a 1-partitioning of type* $((3s)^{\tau_s} : 1 \leq s \leq \frac{v-1}{2})$.
2. *an egalitarian STS*$(5v-4)$ *having a 1-partitioning of type* $((5s)^{\tau_s} : 1 \leq s \leq \frac{v-1}{2})$ *provided that* $v \equiv 1 \pmod 6$ *and* $v \notin \{19, 31\}$.
3. *an egalitarian STS*$(7v-6)$ *having a 1-partitioning of type* $((7s)^{\tau_s} : 1 \leq s \leq \frac{v-1}{2})$.
4. *an egalitarian STS*$(11v-10)$ *having a 1-partitioning of type* $((11s)^{\tau_s} : 1 \leq s \leq \frac{v-1}{2})$ *provided that* $v \equiv 1 \pmod 6$, $v \notin \{19, 31\}$, *and* $\tau_s = 0$ *unless* $s \equiv 0 \pmod 3$.
5. *an egalitarian STS*$(19v-18)$ *having a 1-partitioning of type* $((19s)^{\tau_s} : 1 \leq s \leq \frac{v-1}{2})$.

If in addition the 1-partitioning satisfies τ_s even for $1 \leq s \leq \frac{v-1}{2}$ (and hence $v \equiv 1, 9 \pmod{12}$), then there exists

6. *an egalitarian STS*$(4v-3)$ *having a 1-partitioning of type* $((4s)^{\tau_s} : 1 \leq s \leq \frac{v-1}{2})$ *provided that* $v \geq 25$.

Proof In each case, we apply Theorem 6 to make an egalitarian 3-GDD of group-type $(v-1)^u$ and apply Lemma 10 to fill the holes. Next we give details.

For $u = 3$, use a 3-GDD of group-type 1^3 with weight $m = v-1$ to construct an egalitarian 3-GDD of group-type $(v-1)^3$ having a 1-partitioning of type 1^{v-1}. The 1-partitioning of the egalitarian STS($3v-2$) from Lemma 10 is obtained by first forming the union of corresponding s-parallel classes from the three STS(v)s to obtain a set of blocks in which ∞ appears $3s$ times and each other element appears s times. Adjoining $2s$ parallel classes of the egalitarian 3-GDD of group-type $(v-1)^3$ yields a $(3s)$-parallel class.

For $u \in \{5, 11\}$, form a 3-GDD of group-type 3^u by treating the blocks of one parallel class of a KTS($3u$) as groups; this has a 1-partitioning of type $1^{3(u-1)/2}$. Because $\frac{v-1}{3}$ is even, the GDD is $\frac{v-1}{3}$-egalitarian. Give weight $m = \frac{v-1}{3}$ (a $TD(2, 5, \frac{v-1}{3})$ exists because $v \notin \{19, 31\}$) to construct an egalitarian 3-GDD of group-type $(v-1)^u$ having a 1-partitioning of type $1^{(u-1)(v-1)/2}$. The 1-partitioning of the egalitarian STS$(uv-(u-1))$ from Lemma 10 is obtained by first forming the union of corresponding s-parallel classes from the STS(v)s to obtain a set of blocks in which ∞ appears us times and each other element appears s times. Adjoining $(u-1)s$ 1-parallel classes of the egalitarian 3-GDD of group-type $(v-1)^u$ yields a (us)-parallel class.

For $u \in \{7, 13, 19\}$, use a 3-GDD of group-type 1^u (having a 1-partitioning of type $3^{(u-1)/6}$) with weight $m = v-1$ to construct an egalitarian 3-GDD of group-type $(v-1)^u$ having a 1-partitioning of type $3^{(v-1)(u-1)/6}$. The 1-partitioning of the egalitarian STS$(uv-(u-1))$ from Lemma 10 is obtained by first forming the union of corresponding s-parallel classes from the STS(v)s to obtain a set of blocks in which ∞ appears us times and each other element appears s times. Adjoining $\frac{s(u-1)}{3}$ 3-parallel classes of the egalitarian 3-GDD of group-type $(v-1)^u$ yields a (us)-parallel class.

For $u = 4$, use a 3-GDD of group-type 2^4 (having a 1-partitioning of type 3^1) with weight $m = \frac{v-1}{2}$ (noting that a $TD(2, 5, \frac{v-1}{2})$ exists when $v \geq 25$) to construct an egalitarian 3-GDD of group-type $(v-1)^4$ having a 1-partitioning of type $3^{\frac{3(v-1)}{2}}$. The 1-partitioning of the egalitarian STS$(4v-3)$ from Lemma 10 is obtained by first forming the union of corresponding s-parallel classes from the STS(v)s to obtain a set of blocks in which ∞ appears $4s$ times and each other element appears s times. Adjoining s 3-parallel classes of the egalitarian 3-GDD of group-type $(v-1)^4$ yields a $(4s)$-parallel class. □

One application of Lemma 10 uses a special ingredient.

Lemma 11 *There exists a split egalitarian STS(49).*

Proof Using the hill-climbng method described earlier, a split egalitarian 3-GDD of type 4^4 follows:

```
flm cij den djk hno ajp ekl bhi agh iop efo gmn abk fgp bcl cdm
gil bmo acf bdg cnp fhk kmp ado bep ceh egj aln hjm ikn jlo dfi
```

Interpret each triple as an ordered set, as shown. Then within the first 16 triples and within the last 16, every element appears once each as the first, second, and third element of a triple. Form a 3-GDD of type 3^3 with ordered triples $\{B_\ell = (i, j, (i+j) \bmod 3) : 0 \leq i, j < 3, \ell = 3i + j\}$. We inflate the GDD of type 4^4 giving weight 3, treating the ordered triples in the order specified. In the first half, replace the triple (a, b, c) by the ordered list of nine triples $((a_i, b_j, c_m) : B_\ell = (i, j, m), 0 \leq \ell < 9)$. In the second half, replace the triple (a, b, c) by the ordered list of nine triples $((a_i, b_j, c_m) : B_{8-\ell} = (i, j, m), 0 \leq \ell < 9)$. The result is a split egalitarian 3-GDD of type 12^4. Now apply the strategy of the proof of Lemma 10 using four copies of an egalitarian STS(13) to fill the holes. □

We apply an extension of Lemma 10 next.

Lemma 12 *There exists an egalitarian STS(v) having a 1-partitioning into two $\frac{v-1}{4}$-parallel classes whenever $v \equiv 1, 9 \pmod{12}$ and $v \neq 9$.*

Proof When $v \in \{13, 21, 25, 33\}$, apply Lemma 7. When $v \in \{45, 57, 81\}$, apply Corollary 6(1). When $v = 49$, apply Lemma 11. Otherwise write

$$v = \begin{cases} 24m + 25 \; if \quad v \equiv 1 \pmod{24}, \text{ setting } s = m, g = 6 \text{ and } u = 6 \\ 24m + 33 \; if \quad v \equiv 9 \pmod{24}, \text{ setting } s = m, g = 6 \text{ and } u = 8 \\ 24m + 13 \; if \; v \equiv 13 \pmod{24}, \text{ setting } s = 2m, g = 3 \text{ and } u = 3 \\ 24m + 21 \; if \; v \equiv 21 \pmod{24}, \text{ setting } s = 2m, g = 3 \text{ and } u = 5 \end{cases}$$

Then a 3-GDD of type $g^s u^1$ exists [19]. Each of these 3-GDDs is 4-egalitarian (by Lemma 3) and has a 1-partitioning of type $((gs+u)/2)^1$. When $v \equiv 1 \pmod{24}$, apply Theorem 6 to get a split egalitarian 3-GDD of type 24^{m+1} having a 1-partitioning of type $(3m)^4$. Apply Theorem 10 with $v = 25$ (noting that the STS(25) from Lemma 7 has a 1-partitioning of type 3^4) to produce an egalitarian STS$(24m+1)$, When $v \equiv 13 \pmod{24}$, apply Theorem 6 to get a split egalitarian 3-GDD of type 12^{2m+1} having a 1-partitioning of type $(3m)^4$. Apply Theorem 10 with $v = 13$ (noting that the STS(13) from Lemma 7 has a 1-partitioning of type 3^2) to produce an egalitarian STS$(24m+13)$.

Now we treat cases with $v \equiv 9 \pmod{12}$, which are more involved. The STS(21) and STS(33) from Lemma 7 are split egalitarian. Inflate the non-uniform 3-GDD of type $g^s u^1$ to form a 3-GDD of type $(4g)^s(4u)^1$ using the method in the proof of Theorem 6. The ordering of the resulting 3-GDD is split and has $d = \frac{1}{6}[\binom{s}{2}16g^2 + 16gsu]$ blocks; most importantly, every point in a group of size g appears in $2g(s-1)+2u$ blocks and has point sum $(g(s-1)+u)(d-1)$, and every point in the group of size u appears in $2gs$ blocks and has point sum $gs(d-1)$.

To form the STS$(4gs+4u+1)$, add an infinite point to the 3-GDD of type $(4g)^s(4u)^1$, placing an STS$(4g+1)$ or STS$(4u+1)$ on each group along with the infinite point. Now we order its blocks. As in the proof of Lemma 10, the prefix of the ordering is the first half of the blocks of the 3-GDD of type $(4g)^s(4u)^1$, and the suffix is the last half. Following this prefix, place the first half of the blocks of the (split egalitarian) STS$(4u+1)$; immediately preceding the suffix, place the last half. In the 'middle', place the blocks of the s copies of the STS$(4g+1)$, ordered as in the proof of Lemma 10. The 1-partitioning is obtained by observing that the 3-GDD of type $(4g)^s(4u)^1$ is split, the STS$(4u+1)$ is split egalitarian, and the STS$(4g+1)$ partitions into two g-parallel classes. □

Lemma 13 *There exists an egalitarian STS(v) whenever* $v \equiv 3, 27 \pmod{36}$.

Proof For $v = 27$, apply Lemma 7. Otherwise apply Corollary 6(1) to an egalitarian STS$(v/3)$ from Lemma 12. □

5.4 Consequences

An egalitarian STS(v) exists whenever $v \equiv 1, 3, 9, 13, 21, 25, 27, 33 \pmod{36}$ except when $v = 9$ by Lemmas 12 and 13. It remains to treat cases with $v \equiv 7, 15, 19, 31 \pmod{36}$. Because $3v - 2 \equiv 7, 19 \pmod{36}$ when $v \equiv 3, 7 \pmod{12}$, it follows from Corollary 7(1) that if an egalitarian STS(v) exists whenever $v \equiv 15, 31 \pmod{36}$ and $v \geq 15$, then an egalitarian STS(v) exists for all admissible $v \geq 13$. Unfortunately the remaining constructions fail to treat all values of $v \equiv 15, 31 \pmod{36}$. Table 1 reports the current status for the existence of egalitarian STS(v)s with $v < 720$.

6 Concluding remarks

One can treat block labellings for other classes of Steiner systems starting with the techniques developed here. For example, a Steiner quadruple system, SQS(v), has average point sum $\frac{(v-1)(v-2)(v-4)(v^2+v+6)}{288}$. A resolvable SQS$(v)$ exists for all $v \equiv 4, 8 \pmod{12}$ [31,33]; doubly resolvable SQSs are studied in [38]. A Steiner quadruple system with a 1-partitioning of type $(\tau_1, \ldots, \tau_{r/2})$ with each $\tau_i = 2$ is *(1,2)-resolvable*; such an SQS(v) exists whenever $v \equiv 2, 10 \pmod{12}$ except when $v = 10$ [39]. Cyclic SQSs are also well-studied; see

Table 1 Existence of egalitarian STS(v) for $v \equiv 7, 15, 19, 31 \pmod{36}$ and $v < 720$

$v \pmod{72}$ 7	15	19	31	43	51	55	67
7 ×	15 ✓	19 ✓	31 ✓	43 C1	51 A	55 C1	67 ?
79 C1	87 A	91 C1	103 ?	115 C1	123 A	127 C1	139 ?
151 C1	159 A	163 C1	175 B2	187 C1	195 A	199 ?	211 C3
223 C1	231 B2	235 C1	247 B3	259 C1	267 A	271 C1	283 ?
295 C1	303 A	307 ?	319 ?	331 C1	339 A	343 C1	355 ?
367 C1	375 B4	379 C1	391 C2	403 C1	411 ?	415 ?	427 B2
439 C1	447 A	451 C1	463 C4	475 C1	483 A	487 C1	499 ?
511 C1	519 A	523 C1	535 ?	547 C1	555 A	559 C1	571 C5
583 C1	591 ?	595 B2	607 ?	619 C1	627 A	631 C1	643 ?
655 C1	663 A	667 C1	679 B2	691 C1	699 A	703 C1	715 B3

Entries provide a ✓ when it is produced by Lemma 7, A when it is an application of Corollary 5, Bj when it is produced by statement (j) of Corollary 6, Cj when it is produced by statement (j) of Corollary 7, or '?' when no construction for an egalitarian system is known

[28,36,45], for example. Each of these results can be employed in a variant of Theorem 4 to produce useful bounds on the DiffSum of their duals. However, as for STS(v)s, such general bounds are probably weak for SQS(v)s. Although $\mathsf{DiffSum}(\mathcal{D}_{3,4,8}) = 4$, the SQS(10) is egalitarian (see Fig. 1). Unlike STS(v)s, there cannot be an egalitarian SQS(v) for every admissible order v that is large enough, because $\mathsf{DiffSum}(\mathcal{D}_{3,4,v}) \geq 1$ when $v \equiv 8, 16 \pmod{24}$. Nevertheless, we expect that the DiffSum is, in some sense, small for all orders.

Returning to Steiner triple systems, although every STS(v) with $v \geq 13$ has DiffSum at least $v + 1$ [12], a reasonable conjecture is that there is an egalitarian STS(v) for every admissible $v \geq 13$. Moreover, for v sufficiently large, there exists an STS(v) whose DiffSum is at least $3v - o(v)$ [12]; but in contrast it may happen that the dual of *every* STS(v) has small DiffSum, say at most v. Indeed it is conceivable that, for v sufficiently large, every STS(v) is egalitarian. If so, much different techniques would be needed to establish this.

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