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Natural parametrization of percolation interface and pivotal points

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Abstract. We prove that the interface of critical site percolation on the triangular lattice converges to SLE_6 in its natural parametrization, where the discrete interface is parametrized such that each edge is crossed in one unit of time, while the limiting curve is parametrized by its $7/4$-dimensional Minkowski content. We also prove that the scaling limit of counting measure on the pivotal points, which was proved to exist by Garban, Pete, and Schramm (*J. Amer. Math. Soc.* **26** (2013) 939–1024), is its $3/4$-dimensional Minkowski content up to a deterministic multiplicative constant.

Résumé. Nous montrons que l'interface de la percolation du site à paramètre critique sur le réseau triangulaire converge vers la courbe SLE_6 dans sa parmétrisation naturelle, où l'interface discrète est paramétrisée de telle sorte que chaque arête est traversée en une unité de temps, tandis que la courbe limite est paramétrée par son contenu $7/4$-dimensionnel de Minkowski. Nous montrons également que la limite d'échelle de la mesure de comptage sur les points pivots, dont l'existence a été confirmée par Garban, Pete et Schramm (*J. Amer. Math. Soc.* **26** (2013) 939–1024), est son contenu $3/4$-dimensionnel de Minkowski jusqu'à une constante multiplicative déterministe.

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1. Introduction

Percolation is one of the most studied statistical mechanics models in probability. Since the breakthrough works of Smirnov [28], who proved the conformal invariance of critical site percolation on the triangular lattice, and of Schramm [25], who introduced the Schramm-Loewner evolution (SLE), the understanding of the scaling limit of percolation on planar lattices has greatly improved.

Garban, Pete, and Schramm [7–9] made important contributions in this direction. In [8] they proved scaling limit results for several important classes of points for critical percolation, including pivotal points and points on the percolation interface. They proved that the limiting measures are conformally covariant, and that they are measurable with respect to the scaling limit of percolation.

In the continuum, a substantial effort has been made to understand natural measures on special points of SLE_κ curves. For example, SLE_κ curves have non-trivial $2 \wedge (1 + \kappa/8)$-dimensional Minkowski content, which defines a parametrization of the curve called the *natural parametrization*. SLE with its natural parametrization is uniquely characterized by conformal invariance and domain Markov property, with the constraint that the parametrization is rescaled in a covariant way under the application of a conformal map. See [15,16,20]. SLE with its natural parametrization is believed to describe the scaling limit of curves in statistical physics models parametrized such that one edge/face/vertex is visited in one unit of time. This conjecture was proved for the case of the loop-erased random walk (LERW) and SLE_2 by Lawler and Viklund [17,18].

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In this paper, we link the limiting measures in [8] with the natural measures on special points of SLE_6. The purpose of building the link is two-fold:

1. It makes the limiting measures in [8] more intrinsic and concrete. In the case of percolation pivotal points, this link is important for the work of the first and third authors on the conformal embedding of uniform triangulations [12].
2. It allows us to prove that the percolation interface converges to SLE_6 in its natural parametrization.

1.1. *The scaling limit of the percolation interface under its natural parametrization*

Let us briefly recall the definition of SLE. Fix $\kappa > 0$, and let $(B_t)_{t\geq 0}$ be a standard linear Brownian motion. Consider the Loewner differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \qquad g_0(z) = z, \quad \forall z \in \overline{\mathbb{H}}.$$

Then for each $z \in \mathbb{H}$, $g_t(z)$ is well-defined up to some time $\tau_z \in [0, \infty]$. Let $K_t = \overline{\{z : \tau_z < t\}}$. Then a.s. there exists a unique continuous non-self-crossing curve γ such that K_t is the closure of points disconnected from ∞ on $\mathbb{H}$ by $\gamma([0,t])$. We call γ the chordal SLE_κ on $\mathbb{H}$ from 0 to ∞ (under the capacity parametrization). Let Ω be a simply connected domain whose set of prime ends $\partial\Omega$ is a continuous image of a circle.[4] Let a, b be two distinct points on $\partial\Omega$. Consider a conformal map $f : \mathbb{H} \to \Omega$ with $f(0) = a$ and $f(\infty) = b$. Although there is one degree of freedom when choosing f, the law of $f(\gamma)$ (viewed as a continuous curve modulo increasing reparametrizations) does not depend on this choice. We call this probability measure the *chordal* SLE_κ *on* Ω *from* a *to* b, or simply SLE_κ on (Ω, a, b).

Let $\mathbb{T}$ denote the regular triangular lattice where each face is an equilateral triangle. For $\eta > 0$, let $\eta\mathbb{T}$ be $\mathbb{T}$ rescaled by η. Each vertex on $\eta\mathbb{T}$ is called a site. Let $\eta\mathbb{T}^*$ denote the regular hexagonal lattice dual to $\eta\mathbb{T}$ such that each vertex on $\mathbb{T}$ corresponds to a hexagonal face on $\eta\mathbb{T}^*$. Given an edge e of $\eta\mathbb{T}$, let e^* be its dual edge in $\eta\mathbb{T}^*$. Recall that a Jordan domain is a bounded simply connected domain on $\mathbb{C}$ whose boundary is homeomorphic to a circle. A Jordan domain D is called a η-*polygon* if ∂D lies on the lattice $\eta\mathbb{T}$. A vertex v on $\eta\mathbb{T}$ is called an *inner vertex* (resp., *boundary vertex*) of D if $v \in D$ (resp., $v \in \partial D$). We similarly define *boundary/inner edges* of D.

Suppose Ω is a Jordan domain. Let Ω_η be the largest η-polygon whose set of inner vertices is contained in Ω and forms a connected set on $\eta\mathbb{T}$. (In case of a draw, choose Ω_η arbitrarily from the set of largest η-polygons, but note that Ω_η will be uniquely determined for all sufficiently small η.) Including all inner vertices and edges of Ω_η, we obtain a planar graph embedded in $\mathbb{C}$ which we call the η-*approximation* of Ω and still denote by Ω_η.[5] To distinguish with the continuum, we write the union of boundary vertices and edges of Ω_η as $\Delta\Omega_\eta$. A *path* on a graph is a sequence of vertices such that each vertex is adjacent to its successor. Given two distinct boundary edges of Ω_η, removing $\{e, e'\}$ from $\Delta\Omega_\eta$ gives two paths on the boundary. We let $\Delta_{e,e'}\Omega_\eta$ denote the one tracing $\Delta\Omega_\eta$ counterclockwise from e to e'. Given $x \in \partial\Omega$, let x_η be the edge on $\Delta\Omega_\eta$ closest to x (if there is a tie, choose one arbitrarily).

A *site percolation* on Ω_η is a black/white coloring of inner vertices of Ω_η. The *critical Bernoulli site percolation* on Ω_η, which we denote by $\mathrm{Ber}(\Omega_\eta)$, is the uniform measure on site percolations on Ω_η. A coloring of vertices on $\Delta\Omega_\eta$ is called a *boundary condition*. A site percolation on Ω_η together with a boundary condition determines a coloring of vertices on Ω_η. The (a,b)-*boundary condition* is the coloring where vertices on $\Delta_{a_\eta,b_\eta}\Omega_\eta$ (resp., $\Delta_{b_\eta,a_\eta}\Omega_\eta$) are black (resp., white). Note that this is well-defined since we required that Ω_η does not have any cut vertices. Given a site percolation ω_η on Ω_η with (a,b)-boundary condition, there is a unique path γ_η on $\eta\mathbb{T}^*$ from a_η^* to b_η^*, such that each edge on the path has a white vertex on its left side and a black vertex on its right side. We call γ_η the *percolation interface* of ω_η on $(\Omega_\eta, a_\eta, b_\eta)$.

Let $(\mathcal{U}, d_{\mathcal{U}})$ denote the separable metric space of continuous curves modulo reparametrization, with the distance $d_{\mathcal{U}}$ between curves $\gamma^1 : [0, T_1] \to \mathbb{C}$ and $\gamma^2 : [0, T_2] \to \mathbb{C}$ defined by

$$d_{\mathcal{U}}(\gamma_1, \gamma_2) = \inf_{\alpha,\beta} \Big[\sup_{0\leq t\leq 1} \big|\gamma^1\big(\alpha(t)\big) - \gamma^2\big(\beta(t)\big)\big| \Big], \tag{1}$$

where the infimum is taken over all choices of increasing bijections $\alpha : [0,1] \to [0, T_1]$ and $\beta : [0,1] \to [0, T_2]$. It is proved in [4,28] that γ_η converges to an SLE_6 on (Ω, a, b) for the $d_{\mathcal{U}}$-metric (see Theorem 2.1). Although this convergence result gives a powerful tool for analyzing large scale properties of percolation (e.g. arm exponents [29]), a more natural notion of convergence would be under the parametrization where γ_η traverses each edge in the same amount of

[4]This is the necessary and sufficient boundary condition for the Riemann mapping from the unit disk to Ω to continuously extend to the boundary. (See e.g. [24].)

[5]A notion of η-approximation of the Jordan domain Ω is also introduced in Definition 4.1 of [3], which is denoted by D^η in their notation. One can check that D_η equals the union of D^η and its so-called external boundary defined in [3, Section 4].

time. We prove this result in Theorem 1.4 below. Before stating this result, we need the notions of Minkowski content and occupation measure.

Definition 1.1. Given a set $\mathcal{A} \subset \mathbb{C}$, for $r > 0$, let $\mathcal{A}^r = \{z \in \mathbb{C} : B(z,r) \cap \mathcal{A} \neq \varnothing\}$. For $d \in [0,2]$ we define the *d-dimensional Minkowski content* of $\mathcal{A}$ to be the following limit, provided it exists

$$\mathrm{Mink}_d(\mathcal{A}) := \lim_{r\to 0} r^{d-2}\,\mathrm{Area}\big(\mathcal{A}^r\big). \tag{2}$$

If the limit does not exist, then the d-dimensional Minkowski content of $\mathcal{A}$ is not defined.

Definition 1.2. Fix $d \in [0,2]$. Let $\mathcal{A} \subset \mathbb{C}$ be a random closed set and let μ be a random Borel measure on $\mathbb{C}$. Suppose $\mathbb{P}[\mu(U) = \mathrm{Mink}_d(\mathcal{A} \cap U)] = 1$ for each Jordan domain U with piecewise smooth boundary. We call $\mu_{\mathcal{A}}$ the *occupation measure* of $\mathcal{A}$ and say that it is (a.s.) defined by the d-dimensional Minkowski content of $\mathcal{A}$.

Let γ be an SLE_6 on (Ω, a, b), where Ω is a Jordan domain with smooth boundary and a, b are two distinct boundary points. Assume the parametrization of γ comes from the image of a capacity-parametrized SLE_6 on $(\mathbb{H}, 0, \infty)$ under a conformal map $f : \mathbb{H} \to \Omega$ with $f(0) = a$, $f(\infty) = b$. By [15], we know the following.

1. For each $t \in (0,\infty]$, a.s. the $7/4$-dimensional Minkowski content of $\gamma([0,t])$ exists and defines the occupation measure of $\gamma([0,t])$ as in Definition 1.2. We denote the occupation measure of $\gamma((0,\infty))$ by $\mathfrak{m}_\gamma$.
2. The function $t \mapsto \mathrm{Mink}_{7/4}(\gamma([0,t]))$ is a.s. strictly increasing and Hölder continuous.

Definition 1.3. Suppose Ω is a Jordan domain with smooth boundary. Let γ be an SLE_6 on (Ω, a, b), where a, b are two distinct boundary points. Let $\widehat{\gamma} : [0, \mathfrak{m}_\gamma(\Omega)] \to \Omega$ be the parametrization of γ such that $\mathrm{Mink}_{7/4}(\gamma([0,t])) = t$ for any $t \in [0, \mathfrak{m}_\gamma(\Omega)]$. Then $\widehat{\gamma}$ is called the *natural parametrization* of γ.

In fact [15] mainly focuses on the upper half plane. However, as explained below Theorem 1.1 there, the case of Jordan domains with smooth boundary can be easily obtained by the covariance of Minkowski content under conformal mappings.

Define the following distance ρ between two parametrized curves $\gamma^1 : [0, T_1] \to \mathbb{C}$ and $\gamma^2 : [0, T_2] \to \mathbb{C}$.

$$\rho\big(\gamma^1, \gamma^2\big) = \Big[|T_2 - T_1| + \sup_{0\le s\le 1} \big|\gamma^2(sT_1) - \gamma^1(sT_2)\big|\Big]. \tag{3}$$

As mentioned above, for statistical mechanical models where SLE is the scaling limit in the $d_{\mathcal{U}}$-metric, it is believed that the convergence should also hold in the ρ-metric under the natural parametrization. In this paper, we prove this for the percolation interface.

Theorem 1.4. *Let (Ω, a, b), γ, and $\widehat{\gamma}$ be as in Definition 1.3. For $\eta > 0$, sample ω_η from $\mathrm{Ber}(\Omega_\eta)$ and let γ_η be the interface of ω_η from a_η to b_η. Pick $c_1 > 0$, write $\xi_\eta = c_1\eta^2/\alpha_2^\eta(\eta, 1)$, and let $\widehat{\gamma}_\eta$ be the parametrization of γ_η with constant speed (with respect to the Euclidean metric) such that each edge is crossed in ξ_η units of time. Then with an appropriate choice of c_1, the curve $\widehat{\gamma}_\eta$ converges weakly to $\widehat{\gamma}$ in the ρ-metric.*

Fix $c_1 > 0$, and let the (normalized) interface measure τ_η on γ_η be defined by

$$\tau_\eta := c_1 \sum_{e\in\gamma_\eta} \delta_e \frac{\eta^2}{\alpha_2^\eta(\eta, 1)}, \tag{4}$$

where $\alpha_2^\eta(\eta, 1)$ is a normalizing constant depending on η that we will specify in Section 2.4, and δ_e is the measure assigning unit mass uniformly along e and 0 elsewhere. The following is proved in [8].

Theorem 1.5 ([8]). *In the setting of Theorem 1.4, there is a coupling of $(\omega_\eta)_{\eta>0}$ and γ such that as $\eta \to 0$, it holds a.s. that γ_η converges to γ in the $d_{\mathcal{U}}$-metric, and τ_η in (4) converges to a random Borel measure τ supported on the range of γ in the weak topology. Moreover, τ is measurable with respect to γ.*

It was not proved in [8] that the measure τ defines a parametrization of γ. Some of the challenges in proving this are discussed in [8, Sections 1.2 and 5.3].

As a first step towards proving Theorem 1.4, we prove the following in Section 3.

Theorem 1.6. *In Theorem 1.4, one can choose $\mathfrak{c}_1$ in (4) such that $\tau = \mathfrak{m}_\gamma$ a.s.*

We end this subsection by commenting on our proof ideas for Theorems 1.6 and 1.4. On the one hand, the proof of Theorem 1.6 closely follows [8, Section 4] with the simplification that we work directly in the continuum, hence our one-point and two-point estimates are power laws with no sub-polynomial corrections, in contrast to the arm exponent estimates for percolation. On the other hand, an additional technicality arises when we try to implement the continuum analog of a strong coupling result from [8, Section 4]. See the beginning of Section 3 for more discussion. Our proof of Theorem 1.9 below uses the same idea as in Theorem 1.6. However, Theorem 1.6 itself is not sufficient for proving Theorem 1.4 due to the presence of double points in SLE_6. To deal with this issue, we prove that the occupation measure of the frontier of SLE_6 is 0 and use it to conclude the proof of Theorem 1.4 in Section 5.

1.2. *The natural measure on pivotal points*

Let Ω be a Jordan domain with smooth boundary and sample ω_η from $\mathrm{Ber}(\Omega_\eta)$. Let $a, b, c, d \in \partial\Omega$ be four distinct points ordered counterclockwise. For η small enough such that a_η, b_η, c_η, and d_η are distinct. the following three sentences describe the same event.

- There is a path $\{v_i\}_{1\le i\le n}$ such that v_1 and v_n are on $\Delta_{b_\eta,c_\eta}\Omega_\eta$ and $\Delta_{d_\eta,a_\eta}\Omega_\eta$ respectively, while v_i is a white inner vertex for all $1 < i < n$.
- Let e_η be the first edge crossed by the percolation interface on $(\Omega_\eta, a_\eta, c_\eta)$ with one endpoint lying on $\Delta_{b_\eta,d_\eta}\Omega_\eta$. Then e_η has an endpoint on $\Delta_{b_\eta,c_\eta}\Omega_\eta$.
- Let $\overline{e}_\eta$ be the first edge crossed by the percolation interface on $(\Omega_\eta, c_\eta, a_\eta)$ with one endpoint lying on $\Delta_{d_\eta,b_\eta}\Omega_\eta$. Then $\overline{e}_\eta$ has an endpoint on $\Delta_{d_\eta,a_\eta}\Omega_\eta$.

Denote this event by E_η. Consider the pair of curves $(\gamma_\eta^1, \gamma_\eta^2)$ defined as follows. When E_η occurs, let γ_η^1 and γ_η^2 be the percolation interfaces on (Ω, a_η, b_η) and $(\Omega_\eta, c_\eta, d_\eta)$, respectively. Otherwise, let γ_η^1 and γ_η^2 be the percolation interfaces on (Ω, a_η, d_η) and $(\Omega_\eta, c_\eta, b_\eta)$, respectively. Given an event defined in terms of ω_η, a site in Ω_η is called a *pivotal point* for this event if flipping the color of the site changes the outcome of the event. Let $\mathcal{P}_\eta$ be the set of pivotal points for E_η. Then a site of Ω_η belongs to $\mathcal{P}_\eta$ if and only if it is the endpoint of one edge crossed by γ_η^1 and one edge crossed by γ_η^2.

The picture above has a natural scaling limit. Let $\partial_{a,b}\Omega$ be the counterclockwise arc on $\partial\Omega$ between a and b. By locality, we can couple the chordal SLE_6 on (Ω, a, b) to the chordal SLE_6 on (Ω, a, d) such that the two curves agree until hitting the arc $\partial_{b,d}\Omega$, after which they evolve independently. Let E be the event that the hitting location on $\partial_{b,d}\Omega$ lies on $\partial_{b,c}\Omega$. If E occurs (resp., does not occur), let γ^1 be the SLE_6 from a to b (resp., d) so that there exists a unique connected component of $\Omega \setminus \gamma^1$ whose boundary contains c and d (resp. b). Conditioning on γ^1, let γ^2 be a chordal SLE_6 on this component from c to d (resp. b).

A point is called a *pivotal point* for E if and only if it is on the range of both γ^1 and γ^2. Let $\mathcal{P}$ denote the set of pivotal points of E. Fix $c_{\mathrm{p}} > 0$ and define

$$\mu_\eta := c_{\mathrm{p}} \sum_{z\in\mathcal{P}_\eta} \delta_z \frac{\eta^2}{\alpha_4^\eta(\eta, 1)}, \tag{5}$$

where $\alpha_4^\eta(\eta, 1)$ is a normalizing constant which will be specified in Section 2.4.

The following theorem follows from [8].

Theorem 1.7. *There is a coupling of $(\gamma_\eta^1, \gamma_\eta^2)$ and (γ^1, γ^2) such that a.s.,*

(1) *$\mathbf{1}_{\mathbb{E}_\eta}$ converges to $\mathbf{1}_E$,*
(2) *$(\gamma_\eta^1, \gamma_\eta^2)$ converges to (γ^1, γ^2) in the $d_{\mathcal{U}}$-metric, and*
(3) *μ_η converges to a measure μ supported on $\mathcal{P}$.*

Certain basic properties of μ were also proved in [8], for example that μ is measurable with respect to the scaling limit of percolation in quad-crossing space (see Section 2 for definitions) and the conformal covariance of μ.

As we will explain in more detail in Section 4, (see the discussion above Lemmas 4.1 and 4.3) the set $\mathcal{P}$ is locally absolutely continuous with respect to the set of cut points of two-dimensional Brownian motion, whose occupation measure is the subject of [11]. Using the relationship between $\mathcal{P}$ and Brownian cut points we will prove the following result in Section 4.

Proposition 1.8. *The occupation measure $\mathfrak{m}_{\mathcal{P}}$ of $\mathcal{P}$ a.s. exists and is defined by its $3/4$-dimensional Minkowski content in the sense of Definition 1.2.*

In Section 4, we use the same arguments as for Theorem 1.6 to conclude the following.

Theorem 1.9. *In Theorem 1.7, one can choose c_{p} in (5) such that $\mu = \mathfrak{m}_{\mathcal{P}}$ a.s.*

Theorem 1.9 confirms that the scaling limit of the pivotal measure in [8] is in fact induced by the $3/4$-dimensional Minkowski content of the continuum pivotal points. We can also consider double points of SLE_6 and the points of intersection of CLE_6 loops, which describe the full scaling limit of the interfaces between black and white clusters in critical percolation [3]. In these cases, the analog of Theorem 1.9 holds since their local pictures are absolutely continuous with respect to each other and the Minkowski content is defined locally. We restrict to the formulation in Theorem 1.9 for concreteness.

Theorem 1.9 is an important ingredient of the first and third authors' proof [12] of the convergence of uniform triangulations to Liouville quantum gravity (LQG) with parameter $\sqrt{8/3}$ under the so-called Cardy embedding, which is a discrete conformal embedding based on percolation. A key tool in the proof is the *Liouville dynamical percolation* (LDP) introduced in [6], which is a variant of the ordinary dynamical percolation considered in [9]. The discrete (ordinary) dynamical percolation in [9] is defined as follows. We start from a sample of critical Bernoulli site percolation and then use i.i.d. exponential clocks at each site to update the color. The discrete LDP is defined in the same way except that the rates of the exponential clocks are not identical but depend on a background LQG surface. The continuous LDP is the continuum limit of the discrete LDP as the lattice size and the clock rates are rescaled appropriately. By the existence of the scaling limit of the pivotal measure from [8], the existence of the continuous LDP was proved in [6] in the so-called quad-crossing topology (see Section 2.2), similarly as in [9].

The key idea of [12] is to consider ordinary dynamical percolation (namely, with i.i.d. clocks) on a uniform triangulation and realize that under the conformal embedding the scaling limit of this dynamic is the continuous LDP. Once this is proved, ergodicity of continuous LDP proved in [6] implies that uniform triangulations under the Cardy embedding converge to LQG. The quad-crossing topology allows [6] to apply the powerful machinery of noise sensitivity developed in [7] to prove the desired ergodicity of continuous LDP. However, it is not the natural topology to describe the scaling limit of the ordinary dynamical percolation on uniform triangulations. The natural topology is given by the *mating-of-trees* framework of Duplantier, Miller, and Sheffield [5]; also see [10].

The technical bulk of [12] is to show that the quad-crossing and mating-of-trees descriptions of continuous LDP are equivalent. In both descriptions, the dynamic is determined by its initial configuration and a Poisson point process whose intensity measure is supported on the set of pivotal points, and the equivalence of the LDP descriptions can therefore be reduced to the equivalence of two notions of pivotal measure. The notion of pivotal measure coming from [6] is given by the ordinary pivotal measure in [8] weighted by the exponential of a Gaussian free field. The notion coming from mating-of-trees, which is introduced in [2], is defined using Brownian motion and involves neither the ordinary pivotal measure from [8] nor Gaussian free field. To show the equivalence of these two notions of pivotal measure, the description of ordinary pivotal measure in terms of Minkowski content as in Theorem 1.9 plays an important role. In particular, with this definition the equivalence of the two pivotal measures becomes a natural and concrete statement for CLE_6. See [12, Section 5] for the detail of this argument and see [12, Section 1.4] for an overview of the entire program.

2. Preliminaries

In this section we review some basic facts about percolation which are used in later proofs. Most facts are either known or easy consequences of known results. Therefore we will be brief and refer to [8,26,29,30] for more details.

2.1. *Basic notations*

Throughout the paper, we use γ and γ_η to represent SLE_6 and the percolation interface, respectively. Both γ and γ_η are understood as continuous curves modulo reparametrization unless otherwise specified. When there is no risk of confusion, we also use γ, γ_η to denote the range of the curves.

For all $R > 0$ and $z \in \mathbb{C} = \mathbb{R}^2$, we let $\mathcal{B}_R(z) = z + [-R, R]^2$ denote the square of side length $2R$ centered at z. We call a set a *box* if it can be written on this form. We write $\mathcal{B}_R$ for $\mathcal{B}_R(0)$ and $c\mathcal{B}_R(z)$ for $z + [-Rc, Rc]^2$ (instead of $cz + [-Rc, Rc]^2$) for all $c > 0$. For $0 < r < R$, let $A(r, R) = \mathcal{B}_R \setminus \mathcal{B}_r$. We call a domain A an *annulus* if A is topologically equivalent to $A(1, 2)$, and we use $\partial_1 A$ and $\partial_2 A$ to denote its inner and outer boundaries, respectively.

Given any two sets $X, Y \subset \mathbb{R}^2$, we write $\operatorname{dist}(X, Y) := \inf\{|x - y| : x \in X, y \in Y\}$. Let $\overline{X}$ denote the closure of X. If $\overline{X} \subset Y$, we write $X \Subset Y$.

We use classical asymptotic notations. Given two non-negative functions f and g, we write $f \lesssim g$ (resp., $f \gtrsim g$) if there is a constant $C > 0$ such that $f(x) \le Cg(x)$ (resp., $f(x) \ge Cg(x)$) for all x. We also write $f = O(g)$ when $f \lesssim g$. We write $f \asymp g$ if $f \lesssim g$ and $g \lesssim f$. We say $f(x) = o_x(1)$ as $x \to a$ if $\lim_{x\to a} f(x) = 0$.

2.2. *Quad-crossing representations of percolation*

There are various ways to represent the scaling limit of critical planar percolation (see e.g. the introduction of [26]). One way is to use its crossing information, as we review now.

A *quad* in $\mathbb{C}$ is a homeomorphism $Q : [0, 1]^2 \to \mathbb{C}$. Let

$$\partial_1 Q := Q\big(\{0\} \times [0, 1]\big), \qquad \partial_2 Q := Q\big([0, 1] \times \{0\}\big),$$
$$\partial_3 Q := Q\big(\{1\} \times [0, 1]\big), \qquad \partial_4 Q := Q\big([0, 1] \times \{1\}\big).$$

We will identify a quad Q with $(Q[0, 1]^2, Q(0, 0), Q(1, 0), Q(1, 1), (0, 1))$, so quads giving the same such tuple are identified. Let $\mathcal{Q}$ be the space of quads in $\mathbb{C}$, equipped with the uniform topology. A *crossing* of a quad Q is a closed set in $\mathbb{C}$ containing a connected closed subset of $Q([0, 1]^2)$ that intersects both $\partial_1 Q$ and $\partial_3 Q$. Given Q_1, Q_2 in $\mathbb{C}$, we say $Q_1 \le Q_2$ if every crossing of Q_2 contains a crossing of Q_1. We say $Q_1 < Q_2$ if there exists a neighborhood of $\mathcal{N}_i$ $(i = 1, 2)$ of Q_i in $\mathcal{Q}$ such that $N_1 \le N_2$ for any $N_i \in \mathcal{N}_i$. A *quad-crossing configuration* on $\mathbb{C}$ is a function $\omega : \mathcal{Q} \to \{0, 1\}$ such that the set $\omega^{-1}(1)$ is closed in $\mathcal{Q}$ and for any Q_1, Q_2 with $Q_1 < Q_2$, we have $\omega(Q_2) \le \omega(Q_1)$. We denote the space of quad-crossing configurations on $\mathbb{C}$ by $\mathcal{H}$. The set $\mathcal{H}$ can be endowed with a metric $d_{\mathcal{H}}$ such that $(\mathcal{H}, d_{\mathcal{H}})$ is compact and separable.

Let $\Omega \subsetneq \mathbb{C}$ be an open set and let $\mathcal{Q}_\Omega$ be the space of quads with image in Ω. By restricting to $\mathcal{Q}_\Omega$, each element in $\mathcal{H}$ induces a quad-crossing configuration on Ω. Let $\mathcal{H}_\Omega$ be the space of such configurations, endowed with the metric induced by $d_{\mathcal{H}}$, which we still denote by $d_{\mathcal{H}}$. We refer to [8,26] for more details on $(\mathcal{H}_\Omega, d_{\mathcal{H}})$. Here we only record the following facts. Suppose Ω is a Jordan domain and that ω_η is sampled from $\mathrm{Ber}(\Omega_\eta)$. We identify ω_η with an element in $\mathcal{H}_\Omega$ by setting $\omega_\eta(Q) = 1$ if and only if the white sites of ω_η form a crossing of Q. Then ω_η weakly converges to a random variable ω in $\mathcal{H}_\Omega$ under the $d_{\mathcal{H}}$-metric. Moreover,

1. for each deterministic quad $Q \in \mathcal{Q}_\Omega$, in any coupling where $\omega_\eta \to \omega$ a.s., we have $\omega_\eta(Q) \to \omega(Q)$ in probability;
2. there exists a countable collection $\{Q_n\}_{n\in\mathbb{N}} \subset \mathcal{Q}_\Omega$ such that Q_n has piecewise smooth boundary and $\{\omega(Q_n)\}_{n\in\mathbb{N}}$ generates the Borel σ-field of $(\mathcal{H}_\Omega, d_{\mathcal{H}})$.

2.3. *Some scaling limit results*

The following scaling limit result is from [3] and [8].

Theorem 2.1. *Suppose Ω is a Jordan domain. Sample ω_η from* $\mathrm{Ber}(\Omega_\eta)$. *Then there is a coupling of $(\omega_\eta)_{\eta>0}$ such that the following hold.*

1. *For any fixed $x, y \in \partial\Omega$ with $x \neq y$, the interface γ_η^{xy} on $(\Omega_\eta, x_\eta, y_\eta)$ converges in probability to an* SLE_6 *curve γ^{xy} on (Ω, x, y) under the $d_{\mathcal{U}}$-metric.*
2. *The quad-crossing configuration ω_η converges to ω in probability under the $d_{\mathcal{H}}$-metric.*

In particular, this provides a coupling of ω and $\{\gamma^{xy} : x \neq y, x, y \in \partial\Omega\}$.

Theorem 2.1 is obtained by considering the collection of disjoint loops Γ_η which are interfaces between black and white clusters of ω_η. They converge to a random collection of loops Γ called the conformal loop ensemble with $\kappa = 6$ (CLE_6) on Ω. Moreover, both $\{\gamma^{xy}\}_{x,y\in\partial\Omega}$ and ω are measurable with respect to the CLE_6 (see [3] and [8, Section 2.3]). We will not give more detail on CLE_6 as it is not needed, but refer to [3,27] for further details.

We have seen several classes of domains so far. For the definition of SLE_6, we assumed that the boundary is a continuous image of a circle. For quad-crossing space, we considered general domains. For Theorem 2.1, we considered Jordan domains. In Theorems 1.4 and 1.9, we assumed that $\partial\Omega$ is a smooth Jordan curve. We will carefully organize the argument so that Theorem 2.1 does not have to be extended to domains with rougher boundary. See Remark 3.7.

The following gives the convergence of the interface at the hitting time of certain domains. The lemma will be used to prove Lemmas 2.9, 2.10, and 5.1.

Lemma 2.2. *In the setting of Theorem 1.4, view $\gamma, \gamma_\eta : [0,1] \to \Omega$ as parametrized curves coupled together such that $\lim_{\eta\to 0}\sup_{0\le t\le 1}\{|\gamma_\eta(t)-\gamma(t)|\} = 0$ a.s. (The existence of such parametrizations and couplings is guaranteed by $d_{\mathcal{U}}$-convergence of γ_η to γ.) Let σ_η, σ be stopping times for γ_η and γ, respectively, such that $\sigma_\eta \to \sigma$ a.s. Fix a piecewise smooth simple curve $\ell \Subset \Omega$ such that $\mathbb{P}[\gamma(\sigma) \in \ell] = 0$. Let $\lambda = \inf\{t \ge \sigma : \gamma(t) \in \ell\}$ and $\lambda_\eta = \inf\{t \ge \sigma_\eta : \gamma_\eta(t) \in \ell\}$. Then $\lambda_\eta \to \lambda$ a.s.*

Proof. With probability 1, there exist sequences of rational times $t_\eta \downarrow \lambda$ and $s_\eta \uparrow \lambda$ for $\eta \to 0$ in a countable set such that $\gamma([s_\eta, t_\eta]) \cap \ell \neq \varnothing$. This can be easily proved by way of contradiction, by using that an SLE_6 curve will a.s. cross a deterministic smooth curve upon hitting it. Now the lemma follows from the continuity of γ. $\square$

2.4. *Arm events*

Given a percolation configuration ω_η and an annulus A, we say that an *alternating 4-arm event* occurs for A if and only if there are four disjoint monochromatic paths connecting $\partial_1 A$ and $\partial_2 A$ such that the color sequence of the four paths is alternating between black and white. There is an ambiguity in the definition due to the lattice effect at the boundary. However the precise convention does not matter as $\eta \to 0$ so we ignore it. In the continuum, suppose A is an annulus such that $\partial_1 A$ and $\partial_2 A$ are piecewise smooth. For $A \Subset \Omega$, a quad-crossing configuration $\omega \in \mathcal{H}_\Omega$ is said to belong to the alternating 4-arm event of A if there exist quads $Q_i \subset \mathcal{Q}_\Omega$, $i = 1,2,3,4$, with the following properties:

(i) Q_1 and Q_3 are disjoint and at positive distance from each other, and the same hold for Q_2 and Q_4.
(ii) For $i \in \{1,3\}$, the side $\partial_1 Q_i$ lies inside $\partial_1 A$ and the side $\partial_3 Q_i$ lies outside $\partial_2 A$; for $i \in \{2,4\}$, the side $\partial_2 Q_i$ lies inside $\partial_1 A$ and the side $\partial_4 Q_i$ lies outside $\partial_2 A$; all these sides are of positive distance away from A and from the other Q_j's.
(iii) The four quads are ordered cyclically around A according to their indices.
(iv) $\omega(Q_1) = \omega(Q_3) = 1$ and $\omega(Q_2) = \omega(Q_4) = 0$.

In both the discrete and the continuum, the general k-arm event in A given any prescribed color pattern can be defined similarly. For ω_η coming from a percolation configuration, the two definitions of arm events agree.

Convention 2.3. In the rest of the paper, for each $k = 2,3,4,5$, we focus on arm events with particular color conditions. For $k = 4$, it is the alternating 4-arm event. For $k = 2,3,5$, it is the k-arm event where not all arms have the same color. We will call these events the k-arm event without mentioning the color pattern. We will not need the case $k \neq 2,3,4,5$.

Now we are ready to describe the normalizing constants in (4) and (5).

Remark 2.4 (Normalizing constants). We use $\alpha_k^\eta(\eta,1)$ $(k = 2,4)$ to denote the probability of the k-arm event (under Convention 2.3) from the single site at the origin to $\partial\mathcal{B}_1$. Then $\alpha_2^\eta(\eta,1)$ and $\alpha_4^\eta(\eta,1)$ are the normalizing constants in (4) and (5), respectively. It is known that $\alpha_k^\eta(\eta,1) = \eta^{(k^2-1)/12+o_\eta(1)}$ [29]. The up-to-constant asymptotics are open.

In the coupling of Theorem 2.1, for $k = 2,3,4,5$, let $\mathcal{A}_k$ be k-arm events for an annulus $A \subset \Omega$ as in Convention 2.3. Then the event $\mathcal{A}_k$ is a.s. measurable with respect to the Borel σ-algebra of $(\mathcal{H}_\Omega, d_{\mathcal{H}})$ [8, Section 2]. As explained in [29], the events $\mathcal{A}_k$ can be expressed in terms of percolation exploration to give

$$\lim_{\eta\to 0}\mathbb{P}[\omega_\eta \in \mathcal{A}_k] = \mathbb{P}[\omega \in \mathcal{A}_k] \quad \text{for } k = 2,3,4,5. \tag{6}$$

Lemma 2.9 in [8] gives the following stronger version of (6) when $k = 2,3,4$. (This is expected to be true also for $k = 5$, but this is not proved in [8] and is not needed.)

Lemma 2.5. $\lim_{\eta\to 0}\mathbb{P}[\{\omega \in \mathcal{A}_k\} \triangle \{\omega_\eta \in \mathcal{A}_k\}] = 0$ *for* $k = 2,3,4$.

For $R > r > 0$ and $A = A(r,R)$ write $\alpha_k(r,R) = \mathbb{P}[\omega \in \mathcal{A}_k]$. The up-to-constant asymptotic for $\alpha_k(r,R)$ is well-known [29, Equation (14)] (c.f. Remark 2.4):

$$\alpha_k(r,R) \asymp (r/R)^{(k^2-1)/12}, \quad \text{for } k = 2,3,4,5. \tag{7}$$

An important property of ω as an element in $\mathcal{H}$ is the monotonicity built in its definition. The following monotonicity results will be used repeatedly.

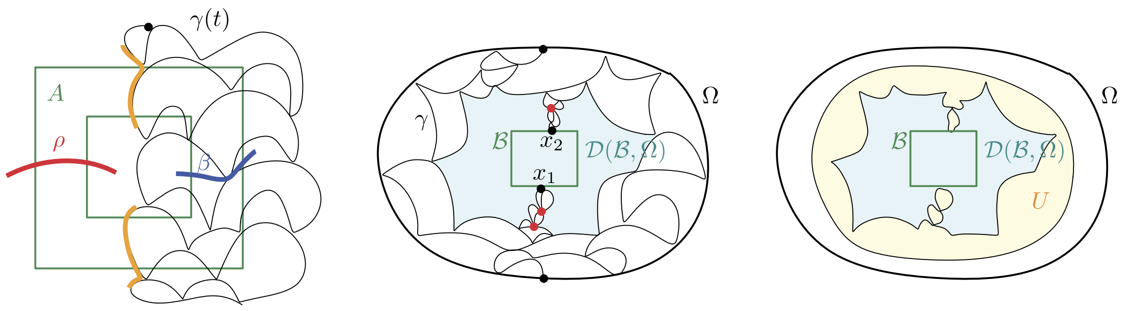

Fig. 1. Left: Illustration of Lemma 2.6, Assertion 2. The arms are shown in blue and orange instead of black and white, respectively. Middle: Illustration in light blue of the face $\mathcal{D}(\mathcal{B}, \Omega)$ at $\mathcal{B}$ induced by γ as defined in Section 2.5. Double points of $\partial\mathcal{D}(\mathcal{B}, \Omega)$ are marked in red and correspond to local cut points for γ (see Remark 3.7). Right: Illustration of the event $\mathcal{G}_\Omega(\mathcal{B}, U)$.

Lemma 2.6. *In the coupling of Theorem 2.1, let $\gamma := \gamma^{ab}$ and $\gamma_\eta := \gamma_\eta^{ab}$ for two given distinct points $a, b \in \partial\Omega$. View (γ_η, γ) as parametrized curves as in Lemma 2.2. For each fixed $t \in (0, 1)$, let K_t be the hull of $\gamma([0, t])$. Namely, K_t is the complement of the connected component of $\Omega \setminus \gamma([0, t])$ containing the target of γ. For any annulus $A \Subset \Omega$, let A_1 be the inside of $\partial_1 A$ and A_2 be the outside of $\partial_2 A$. Then for the quad-crossing configuration ω,*

1. *if $\partial_1 A \cap \gamma \neq \varnothing$, then the 2-arm event for A occurs a.s., and*
2. *if there exists $t \in [0, 1]$ such that $\partial K_t \cap A_1 \neq \varnothing$ and $\gamma(t) \in A_2$, then the 3-arm event for A occurs a.s.*

Proof. For Assertion 1, choose $t, \delta \in (0, 1)$ such that $\gamma(t)$ is inside $\partial_1 A$ and $\operatorname{dist}(\gamma(t), \partial_1 A) > \delta$. Since $\lim_{\eta\to 0}\sup_{0\le t\le 1}\{|\gamma_\eta(t) - \gamma(t)|\} = 0$ a.s., we have $\operatorname{dist}(\gamma_\eta(t), \partial_1 A) > 0.5\delta$ for small enough η. In this case the 2-arm event for A occurs for ω_η. Sending $\eta \to 0$ and applying Lemma 2.5, we get Assertion 1.

Assertion 2 can be proved similarly. See Figure 1. Since $\gamma(t)$ is a boundary point of K_t, there exists a $\delta > 0$ and a path ρ starting from some point in A_1 and ending at some point in A_2 such that $\operatorname{dist}(\rho, \gamma([0, t])) > \delta$. Without loss of generality assume the set $\partial K_t \cap A_1$ contains a point on left frontier of $\gamma([0, t])$. Now for small enough η, we have $\operatorname{dist}(\rho, \gamma_\eta([0, t])) > 0.5\delta$. On the other hand, by the argument for Assertion 1, for η small enough the there exists a black arm β of ω_η from $\partial_1 A$ to $\partial_2 A$. In this case, there must be a white arm of ω_η on each connected component of $A \setminus (\rho \cup \beta)$ from $\partial_1 A$ to $\partial_2 A$, hence the 3-arm event for A occurs for ω_η. Now Assertion 2 follows from Lemma 2.5. □

The following variant of Lemma 2.6 can be proved similarly. We omit the details.

Lemma 2.7. *Consider a coupling where both the conditions in Theorems 1.7 and 2.1 are satisfied so that $(\omega, \gamma^1, \gamma^2)$ are coupled. Let A, A_1, A_2 be defined as in Lemma 2.6. Then on the event $\mathcal{P} \cap A_1 \neq \varnothing$, the 4-arm event for A occurs a.s. for ω.*

The event $\gamma \cap A_1 \neq \varnothing$ in Lemma 2.6 is simply the 2-arm event with the further requirement that each of the two boundary arcs contain one endpoint of the arm. The similar statement holds for $\mathcal{P} \cap A_1 \neq \varnothing$ in Lemma 2.7. By the following lemma, these endpoint requirements only decrease the probability by a constant factor.

Lemma 2.8. *In the setting of Lemmas 2.6 and 2.7, let $\mathcal{B} \subset \Omega$ be a box of radius ε whose center is $r > 10\varepsilon$ away from $\partial\Omega$. Then $\mathbb{P}[\gamma \cap \mathcal{B} \neq \varnothing] \asymp \alpha_2(\varepsilon, r)$ and $\mathbb{P}[\mathcal{P} \cap \mathcal{B} \neq \varnothing] \asymp \alpha_4(\varepsilon, r)$, where the implicit constants in $\asymp$ only depend on Ω but not on other parameters.*

Proof. The 2-arm case follows from the classical one-point estimate of SLE_6. See e.g. [15]. The 4-arm case follows from [8, Proposition 4.9]. □

2.5. *Face induced by the percolation exploration*

Given a box $\mathcal{B}$ and two distinct points $x_1, x_2 \in \partial\mathcal{B}$, let θ_1 (resp., θ_2) be a simple path joining x_1 and x_2 (resp., x_2 and x_1). If the pair of paths $\Theta = \{\theta_1, \theta_2\}$ is such that there exists a domain $\mathcal{D}_\Theta$ with $\mathcal{B} \subset \mathcal{D}_\Theta$ and $\partial\mathcal{D}_\Theta = \theta_1 \cup \theta_2$, then we call Θ a *face* at $\mathcal{B}$ with endpoints x_1, x_2.

Let Ω be a simply connected domain whose boundary is a continuous curve and let $a, b \in \partial\Omega$ be such that $a \neq b$. Suppose γ is a SLE_6 on (Ω, a, b) parametrized in an arbitrary way and $\mathcal{B} \subset \Omega$ is a box. Throughout this subsection we write (Ω, a, b) as Ω whenever it simplifies the notation and cause no confusion. For example, we use $\mathcal{A}(\mathcal{B}, \Omega)$ to denote the event $\{\gamma \cap \mathcal{B} \neq \varnothing\}$ although this event depends on a, b. On $\mathcal{A}(\mathcal{B}, \Omega)$, let

$$\underline{\sigma} = \inf\{t : \gamma_t \in \mathcal{B}\}, \qquad \overline{\sigma} = \sup\{t : \gamma_t \in \mathcal{B}\},$$
$$x_1 = \gamma(\underline{\sigma}), \qquad x_2 = \gamma(\overline{\sigma}).$$

Let $\mathcal{D}(\mathcal{B}, \Omega)$ be the connected component of $\Omega \setminus (\gamma[0, \underline{\sigma}] \cup \gamma[\overline{\sigma}, \infty))$ containing $\mathcal{B}$. Then $\mathcal{D}(\mathcal{B}, \Omega)$ can be viewed as a (random) face at $\mathcal{B}$ with the arcs $\partial_{x_1,x_2}\mathcal{D}(\mathcal{B}, \Omega)$ and $\partial_{x_2,x_1}\mathcal{D}(\mathcal{B}, \Omega)$, which we call *the face at $\mathcal{B}$ induced by γ*. By setting $\mathcal{D}(\mathcal{B}, \Omega) = \varnothing$ when $\mathcal{A}(\mathcal{B}, \Omega)$ does not occur, we can view $\mathcal{D}(\mathcal{B}, \Omega)$ as a random domain with two ordered boundary marked points x_1, x_2 when it is nonempty. Given a simply connected domain U with piecewise smooth boundary such that $\mathcal{B} \Subset U \Subset \Omega$, set

$$\mathcal{G}_\Omega(\mathcal{B}, U) := \mathcal{A}(\mathcal{B}, \Omega) \cap \big\{\mathcal{D}(\mathcal{B}, \Omega) \subset U\big\}. \tag{8}$$

See the right part of Figure 1 for an illustration. The picture above has a discrete counterpart. Suppose Ω is a Jordan domain and ω_η is sampled from $\mathrm{Ber}(\Omega_\eta)$. Let γ_η denote the associated interface on $(\Omega_\eta, a_\eta, b_\eta)$ for some $a \neq b \in \partial\Omega$. Let $\mathcal{A}_\eta(\mathcal{B}, \Omega)$ be the event that there exists an edge on γ_η such that the two hexagons containing the edge are both in $\mathcal{B}$. Consider the first and last such edges on γ_η, whose visiting time are denoted by $\underline{\sigma}_\eta$ and $\overline{\sigma}_\eta$, respectively. Let Θ_η denote the face at $\mathcal{B}$ induced by γ_η, which forms the boundary of the domain $\mathcal{D}_\eta(\mathcal{B}, \Omega)$. Similarly as in (8), define $\mathcal{G}_{\Omega_\eta}(\mathcal{B}, U) := \mathcal{A}_\eta(\mathcal{B}, \Omega) \cap \{\mathcal{D}_\eta(\mathcal{B}, \Omega) \subset U\}$. We have the following two lemmas.

Lemma 2.9. *Suppose Ω is a Jordan domain and that $\mathcal{B}$ and U are defined as above. Suppose we are in the coupling of Theorem 2.1. We view γ_η and γ as parametrized curves as in Lemma 2.2. Then $\overline{\mathcal{D}_\eta(\mathcal{B}, \Omega)}$ converges to $\overline{\mathcal{D}(\mathcal{B}, \Omega)}$ in probability for the Hausdorff metric as closed sets with two ordered marked points.*

Proof. It suffices to show that

$$\lim_{\eta \to 0} \mathbb{P}\Big[\big\{\mathcal{B}' \cap \overline{\mathcal{D}_\eta(\mathcal{B}, \Omega)} = \varnothing\big\} \Delta \big\{\mathcal{B}' \cap \overline{\mathcal{D}(\mathcal{B}, \Omega)} = \varnothing\big\}\Big] = 0, \quad \text{for a fixed box } \mathcal{B}' \Subset \Omega. \tag{9}$$

Given a fixed piecewise smooth curve $p : [0, 1] \to \Omega$ with $p(0) \in \partial\Omega$, $p(1) \in \mathcal{B}$ and $p((0, 1)) \subset \Omega$. If $p \cap \partial\mathcal{D}(\mathcal{B}, \Omega) = \varnothing$, since γ_η converges to γ in the $d_{\mathcal{U}}$-metric, for small enough η we must have $p \cap \mathcal{D}_\eta(\mathcal{B}, \Omega) = \varnothing$. If $p \cap \partial\mathcal{D}(\mathcal{B}, \Omega) \neq \varnothing$, then by Lemma 2.2 for small enough η we must have $p \cap \mathcal{D}_\eta(\mathcal{B}, \Omega) \neq \varnothing$. This implies (9) by elementary topological consideration. □

Lemma 2.10. *In the setting of Lemma 2.9, let $\mathcal{A}_3$ represent the 3-arm event for $U \setminus \mathcal{B}$. Then $\mathbb{P}[\mathcal{A}(\mathcal{B}, \Omega) \setminus \mathcal{G}_\Omega(\mathcal{B}, U)] \leq \mathbb{P}[\omega \in \mathcal{A}_3]$.*

Proof. By Lemma 2.2, $\mathbb{P}[\overline{\mathcal{D}(\mathcal{B}, \Omega)} \subset U] = \mathbb{P}[\overline{\mathcal{D}(\mathcal{B}, \Omega)} \subset \overline{U}]$. By Lemma 2.9 it suffices to show that $\mathcal{A}_\eta(\mathcal{B}, \Omega) \cap \{\omega_\eta \notin \mathcal{A}_3\} \subset \mathcal{G}_{\Omega_\eta}(\mathcal{B}, U)$. To prove this, we see that if $\mathcal{A}_\eta(\mathcal{B}, \Omega) \cap \{\omega_\eta \notin \mathcal{A}_3\}$ occurs, the black sites adjacent to $\gamma_\eta([0, \underline{\sigma}_\eta])$ and $\gamma_\eta([0, \overline{\sigma}_\eta])$ must share a common hexagon within $U \setminus \mathcal{B}$. The similar statement holds for the white sites. This concludes the proof. □

In the setting of Theorem 2.1, it is clear that ω inherits the spatial independence property from ω_η. By Lemma 2.9, we get the following.

Lemma 2.11. *In the setting of Lemma 2.9, let Ω_1, Ω_2 be two disjoint open subsets in Ω. Then ω restricted to $\mathcal{Q}_{\Omega_1}$ and to $\mathcal{Q}_{\Omega_2}$ are independent as random variables in $\mathcal{H}_{\Omega_1}$ and $\mathcal{H}_{\Omega_2}$, respectively. Moreover, ω restricted to $\mathcal{Q}_{\mathcal{B}}$ is independent of $\mathcal{D}(\mathcal{B}, \Omega)$.*

3. Equivalence of the two measures on the interface

This section is devoted to proving Theorem 1.6 hence we retain the notations in the statement of the theorem. To prove Theorem 1.6, we use the L^2 framework as in [8], which is based on a strong coupling scheme and the spatial independence of percolation. Since we work in the continuum, some issues in [8, Section 4] can be simplified. In particular, the required one-point and two-point estimates that we will rely on are are power laws with no sub-polynomial corrections (see

Lemmas 3.3–3.5), while a major novelty of [8, Section 4] is obtaining scaling limit results despite the unknown sub-polynomial corrections in the percolation estimates. After we prepare the one-point and two-point estimates, we reduce Theorem 1.6 to a strong coupling estimate (21). This reduction is a straightforward adaptation of the L^2 argument in [8, Section 4], nevertheless we still include the full argument for completeness and hence follow closely both the method and the presentation in [8, Section 4]. To prove the strong coupling estimate (21), we would like to apply its discrete analog from [8] and then pass to the continuum. However, a straightforward implementation of this idea only gives Lemma 3.6, a weaker variant of (21). The reason is that when we pass from percolation to its continuum limit, we rely on Theorem 2.1, which is for Jordan domains. On the other hand, the domain boundary considered in (21) is the exterior boundary of SLE_6, which is not simple. Instead of trying to strengthen the convergence in Theorem 2.1 to include certain non-Jordan domains, we will use an argument directly in the continuum to go from Lemma 3.6 to the desired (21). We now carry out the plan above in detail.

Let $\mathcal{B} \Subset \Omega$ be a box whose four vertices are on $\bigcup_{k\in\mathbb{N}} 2^{-k}\mathbb{Z}^2$. Let $\varepsilon \in \{2^{-k} : k \in \mathbb{N}\}$. Assume ε is small enough such that $\varepsilon < \operatorname{dist}(\mathcal{B}, \partial\Omega)$. Then $\mathcal{B}$ is partitioned by certain boxes of radius ε centered at points on the lattice $2\varepsilon\mathbb{Z}^2$. Let $Q_1, \dots, Q_p$ be a list of these boxes in arbitrary order. For $i \ge 1$, let q_i denote the center of Q_i. Let

$$Y^\varepsilon = \#\{1 \le i \le p : \gamma \cap 2Q_i \ne \varnothing\}.$$

In [8, Section 5.3], the following is proved.[6]

Proposition 3.1. *There exists a deterministic constant $c > 0$ such that*

$$\tau(\mathcal{B}) = \lim_{\varepsilon\to 0} \frac{cY^\varepsilon}{\varepsilon^{-2}\alpha_2(\varepsilon, 1)} \quad \text{in } L^2.$$

Consider the square $Q_0 := \mathcal{B}_\varepsilon(0)$. Let γ^0 be a chordal SLE_6 on $(\mathcal{B}_1, -i, i)$ and $x_0 := \mathfrak{m}_{\gamma^0}(Q_0)$, where $\mathfrak{m}_{\gamma^0}$ is the occupation measure of γ^0. Let $\mathcal{A}_0(2\varepsilon, 1)$ be the event that $\gamma^0 \cap 2Q_0 \ne \varnothing$, and define

$$\beta_\varepsilon := \mathbb{E}\big[x_0 \,|\, \mathcal{A}_0(2\varepsilon, 1)\big]. \tag{10}$$

Theorem 1.6 is an immediate consequence of Proposition 3.1 and the following.

Proposition 3.2. *For each box $\mathcal{B} \Subset \Omega$ as above, we have that $\mathfrak{m}_\gamma(\mathcal{B}) = \lim_{\varepsilon\to 0} \beta_\varepsilon Y^\varepsilon$ in L^2.*

Before proving Proposition 3.2, we first record a few basic estimates in Lemmas 3.3–3.5.

Define y_i to be the indicator function of the event that $\gamma \cap 2Q_i \ne \varnothing$ so that $Y^\varepsilon = \sum_1^p y_i$. Similarly, for any $1 \le i \le p$, let $x_i = \mathfrak{m}_\gamma(Q_i)$ such that $\mathfrak{m}_\gamma(\mathcal{B}) = \sum_1^p x_i$. We first record some a priori estimates for the x_i's and the y_i's. These estimates would trivially follow from known Green function estimates for SLE_6 [19]. However, we instead present an argument that can be readily extended to the case of pivotal points in Section 4. The following result is classical, and we refer to [1] for a proof.

Lemma 3.3. *In the above setting, for all $1 \le i, j \le p$ with $i \ne j$,*

$$\mathbb{E}[y_i] \asymp \varepsilon^{1/4} \quad \text{and} \quad \mathbb{E}[y_i y_j] \lesssim \frac{\varepsilon^{1/2}}{|q_i - q_j|^{1/4}}. \tag{11}$$

where the constants in $\asymp$ and $\lesssim$ only depend on $\mathcal{B}$ and Ω.

A similar argument based on arm exponents gives the following.

Lemma 3.4. *For all $1 \le i, j \le p$ with $i \ne j$, we have*

$$\mathbb{E}[x_i] \lesssim \varepsilon^2, \qquad \mathbb{E}[x_i x_j] \lesssim \frac{\varepsilon^4}{|q_i - q_j|^{1/4}} \quad \text{and} \quad \mathbb{E}\big[x_i^2\big] \lesssim \varepsilon^{15/4}, \tag{12}$$

where the constants in $\lesssim$ only depend on B and Ω.

---

[6] To obtain Proposition 3.1 from [8, Section 5.3] we use that, in the notation of that paper, X appropriately renormalized converges to τ, $\mathbb{E}[(X - \beta_{\text{two-arm}} Y)^2] = o(\mathbb{E}[X^2])$, and $\beta_{\text{two-arm}} \asymp \varepsilon^2\eta^{-2}\alpha_4^\eta(\eta, \varepsilon)$.

Proof. For $r \in (0, 0.01\varepsilon)$ and $\bullet = i, j$, let $\mathcal{X}_\bullet = \gamma \cap 2Q_\bullet$ and $\mathcal{X}_\bullet^r = \{z : \mathrm{dist}(z, \mathcal{X}_\bullet) \le r\}$. It is clear that $\mathcal{X}_i^r \subset 4Q_i$. By Lemma 2.6 and (7), $\mathbb{P}[\mathrm{dist}(z, \mathcal{X}_i) \le r] \lesssim r^{1/4}$ for all $z \in 4Q_i$. Therefore, by Fubini's theorem, we have

$$\mathbb{E}\big[\mathrm{Area}\big(\mathcal{X}_i^r\big)\big] = \int_{4Q_i} \mathbb{P}\big[z \in \mathcal{X}_i^r\big]\, dz \lesssim \varepsilon^2 r^{1/4}.$$

Now Fatou's lemma and Definitions 1.1 and 1.2 yield $\mathbb{E}[x_i] \lesssim \varepsilon^2$.

For the second inequality, by Fubini's theorem, we have

$$\mathbb{E}\big[\mathrm{Area}\big(\mathcal{X}_i^r\big)\,\mathrm{Area}\big(\mathcal{X}_j^r\big)\big] = \int_{4Q_i \times 4Q_j} \mathbb{P}\big[z \in \mathcal{X}_i^r,\, w \in \mathcal{X}_j^r\big]\, dz\, dw.$$

By Lemma 3.3, we have $\mathbb{P}[z \in \mathcal{X}_i^r, w \in \mathcal{X}_j^r] \lesssim r^{1/2}/|z-w|^{1/4}$. Now the second inequality follows from Fatou's lemma and Definitions 1.1 and 1.2.

The third inequality follows from a similar argument as for the second one. □

By Lemmas 3.3 and 3.4, we have

$$\beta_\varepsilon \le \frac{\mathbb{E}[x_0]}{\mathbb{P}[\mathcal{A}_0(2\varepsilon, 1)]} \lesssim \frac{\varepsilon^2}{\varepsilon^{1/4}} = \varepsilon^{7/4}. \tag{13}$$

Lemma 3.5. *In the above setting, for all $1 \le i \le p$, let*

$$\widetilde{\mathcal{X}}_i = \bigcap_{\delta > 0} \left\{ z \in Q_i : \text{the 2-arm event occurs for the annulus } \mathcal{B}\left(q_i, \frac{3}{2}\varepsilon\right) \setminus \mathcal{B}(z, \delta) \right\}$$

and $\widetilde{\mathcal{X}}_i^r = \{z : \mathrm{dist}(z, \widetilde{\mathcal{X}}_i) \le r\}$. Let $\widetilde{x}_i = \liminf_{r \to 0} r^{-1/4}\,\mathrm{Area}(\widetilde{\mathcal{X}}_i^r)$. Then

$$x_i \le \widetilde{x}_i \quad \text{and} \quad \mathbb{E}[\widetilde{x}_i] \lesssim \varepsilon^{7/4} \quad \text{for all } 1 \le i \le p$$

where the constant in $\lesssim$ is independent ε, i, $\mathcal{B}$, Ω.

Proof. Lemma 2.6 and (2) imply that $x_i \le \widetilde{x}_i$. The bound $\mathbb{E}[\widetilde{x}_i] \lesssim \varepsilon^{7/4}$ follows from the same argument as for the first inequality in Lemma 3.4. Here the domain Ω is replaced by $\frac{3}{2}Q_i$. Therefore we get the upper bound $\varepsilon^{7/4}$ instead of ε^2. □

The advantage of considering $\widetilde{x}_i$ instead of x_i is that it is completely determined by ω restricted to $\mathcal{B}(q_i, \frac{3}{2}\varepsilon)$, hence is independent of what happen outside $2Q_i$.

Now we proceed to prove Proposition 3.2. Fix some $r > 0$ to be determined later. Write $\Delta_i = x_i - \beta_\varepsilon y_i$ for $1 \le i \le p$ and

$$\mathbb{E}\big[\big(\mathfrak{m}_\gamma(\mathcal{B}) - \beta_\varepsilon Y^\varepsilon\big)^2\big] = \sum_{i,j=1}^{p} \mathbb{E}[\Delta_i \Delta_j].$$

Split the summation into an "on-diagonal" term and an "off-diagonal" term:

$$\mathbb{E}\big[\big(\mathfrak{m}_\gamma(\mathcal{B}) - \beta_\varepsilon Y^\varepsilon\big)^2\big] = \sum_{|q_i - q_j| \le r} \mathbb{E}[\Delta_i \Delta_j] + \sum_{|q_i - q_j| > r} \mathbb{E}[\Delta_i \Delta_j]. \tag{14}$$

To estimate the on-diagonal term, take any i, j such that $|q_i - q_j| \le r$, and observe that since all variables and constants are positive, we have

$$\mathbb{E}[\Delta_i \Delta_j] \le \mathbb{E}\big[x_i x_j + \beta_\varepsilon^2 y_i y_j\big]. \tag{15}$$

There are $O(1)\varepsilon^{-2}$ choices for the box Q_i (where $O(1)$ depends on $\mathcal{B}$). For a fixed box Q_i and any $k \ge 0$ such that $2^k\varepsilon < r$, there are $O(1)2^{2k}$ boxes Q_j satisfying $2^k\varepsilon \le |q_i - q_j| < 2^{k+1}\varepsilon$. For any of these boxes, Lemma 3.4 gives $\mathbb{E}[x_i x_j] \lesssim \varepsilon^4/(2^k\varepsilon)^{1/4}$.

Therefore

$$\sum_{|q_i-q_j|\le r}\mathbb{E}[x_i x_j]\lesssim \varepsilon^{-2}\sum_{k\le \log_2(r/\varepsilon)} 2^{2k}\cdot\frac{\varepsilon^4}{(2^k\varepsilon)^{1/4}}. \tag{16}$$

By Lemma 3.3 and (13) we obtain the same bound on $\sum_{|q_i-q_j|\le r}\mathbb{E}[\beta_\varepsilon^2 y_i y_j]$. Therefore

$$\sum_{|q_i-q_j|\le r}\mathbb{E}[\Delta_i\Delta_j]\lesssim r^{7/4}. \tag{17}$$

Now consider the off-diagonal term in (14). We claim that for fixed δ, if ε is small enough, for any i, j such that $l:=|q_i-q_j|>r$ we have

$$\mathbb{E}[\Delta_i\Delta_j]\le\delta\cdot\frac{\varepsilon^4}{l^{1/4}}. \tag{18}$$

Let $\zeta\in(2\varepsilon,r/4)$ be some intermediate distance whose value will be fixed later. For $k=2,3$ and $\bullet=i,j$, let $\mathcal{A}_k^\bullet=\mathcal{A}_k^\bullet(\zeta,l/2)$ be the k-arm event for the annulus $\mathcal{B}(q_\bullet,l/2)\setminus\mathcal{B}(q_\bullet,\zeta)$. Following the notations of Section 2.5, let $\mathcal{D}_\bullet:=\mathcal{D}(\mathcal{B}(q_\bullet,\zeta),\Omega)$ and $\Theta_\bullet$ be the face at $\mathcal{B}(q_\bullet,\zeta)$ induced by γ. Let $\mathcal{G}_\bullet=\mathcal{G}(\mathcal{B}(q_\bullet,\zeta),\mathcal{B}(q_\bullet,l/2))$. Note that by Lemma 2.6, we have $\mathcal{G}_\bullet\subset\mathcal{A}_2^\bullet$.

Let $\mathcal{W}=\mathcal{G}_i\cap\mathcal{G}_j$ and $\mathcal{Z}=(\mathcal{A}_2^i\cap\mathcal{A}_2^j)\setminus\mathcal{W}$. By Lemma 2.6, if $\Delta_i\Delta_j\neq 0$, the event $\mathcal{A}_2^i\cap\mathcal{A}_2^j$ must occur. Therefore

$$\mathbb{E}[\Delta_i\Delta_j]=\mathbb{E}[\Delta_i\Delta_j\mathbf{1}_\mathcal{Z}]+\mathbb{E}[\Delta_i\Delta_j\mathbf{1}_\mathcal{W}]. \tag{19}$$

Let $\mathcal{A}_{i,j}$ be the event that two-arm events occur in the annuli $\mathcal{B}(q_j,l/2)\setminus 2Q_j$, $\mathcal{B}(q_i,l/2)\setminus 2Q_i$ and $\Omega\setminus\mathcal{B}(\frac{q_i+q_j}{2},l)$. Observe that if $(x_ix_j+\beta_\varepsilon^2 y_iy_j)\neq 0$ then $\mathcal{A}_{i,j}$ occurs. Recall $\widetilde{x}_i$ in Lemma 3.5. We have

$$\mathbb{E}\big[|\Delta_i\Delta_j|\mathbf{1}_{\mathcal{A}_2^i\setminus\mathcal{G}_i}\big]\le\mathbb{E}\big[\big(x_ix_j+\beta_\varepsilon^2y_iy_j\big)\mathbf{1}_{\mathcal{A}_2^i\setminus\mathcal{G}_i}\big]\le\mathbb{E}\big[\big(\widetilde{x}_i\widetilde{x}_j+\beta_\varepsilon^2\big)\cdot\mathbf{1}_{\mathcal{A}_2^i\setminus\mathcal{G}_i}\cdot\mathbf{1}_{\mathcal{A}_{i,j}}\big].$$

By Lemma 2.11, $\widetilde{x}_i$, $\widetilde{x}_j$ and $\mathbf{1}_{\mathcal{A}_2^i\setminus\mathcal{G}_i}\cdot\mathbf{1}_{\mathcal{A}_{i,j}}$ are independent. By Lemma 3.5 and (13), we have

$$\mathbb{E}\big[\big(\widetilde{x}_i\widetilde{x}_j+\beta_\varepsilon^2\big)\cdot\mathbf{1}_{\mathcal{A}_2^i\setminus\mathcal{G}_i}\cdot\mathbf{1}_{\mathcal{A}_{i,j}}\big]=\big(\mathbb{E}[\widetilde{x}_i]\mathbb{E}[\widetilde{x}_j]+\beta_\varepsilon^2\big)\mathbb{P}\big[\big(\mathcal{A}_2^i\setminus\mathcal{G}_i\big)\cap\mathcal{A}_{i,j}\big]\lesssim\varepsilon^{7/2}\mathbb{P}\big[\big(\mathcal{A}_2^i\setminus\mathcal{G}_i\big)\cap\mathcal{A}_{i,j}\big].$$

By the same argument as in Lemma 2.10, we have $\mathbb{P}[(\mathcal{A}_2^i\setminus\mathcal{G}_i)\cap\mathcal{A}_{i,j}]\le\mathbb{P}[\omega\in\mathcal{A}_3^i\cap\mathcal{A}_{i,j}]$. By Lemma 2.11 and (7), we have $\mathbb{P}[\omega\in\mathcal{A}_3^i\cap\mathcal{A}_{i,j}]=o_{\zeta/l}(1)\varepsilon^{1/2}/l^{1/4}$. Therefore

$$\mathbb{E}\big[|\Delta_i\Delta_j|\mathbf{1}_{\mathcal{A}_2^i\setminus\mathcal{G}_i}\big]=o_{\zeta/l}(1)\frac{\varepsilon^4}{l^{1/4}}.$$

Since $\mathcal{Z}\subset(\mathcal{A}_2^i\setminus\mathcal{G}_i)\cup(\mathcal{A}_2^j\setminus\mathcal{G}_j)$, we have

$$\big|\mathbb{E}[\Delta_i\Delta_j\mathbf{1}_\mathcal{Z}]\big|=o_{\zeta/l}(1)\frac{\varepsilon^4}{l^{1/4}}. \tag{20}$$

It remains to bound the second term on the right side of (19). Recall the notations introduced in Section 2.5. For $\bullet=i,j$, on the event $\mathcal{G}_\bullet$, let $\underline{\sigma}_\bullet=\inf\{t:\gamma_t\in B(q_\bullet,\zeta)\}$ and $\overline{\sigma}_\bullet=\sup\{t:\gamma_t\in B(q_\bullet,\zeta)\}$. Let $\gamma^\bullet$ be the curve $\gamma([\underline{\sigma}_\bullet,\overline{\sigma}_\bullet])$. Then by the reversibility of SLE_6, the curve $\gamma^\bullet$ conditioning on $\mathcal{D}_\bullet$ is a chordal SLE_6 inside $\mathcal{D}_\bullet$. We claim that

$$\mathbf{1}_\mathcal{W}\big|\mathbb{E}[x_i-\beta_\varepsilon y_i|\mathcal{D}_i]\big|=o_{\varepsilon/\zeta}(1)\frac{\varepsilon^2}{\zeta^{1/4}}\quad\text{and the same with } j \text{ in place of } i. \tag{21}$$

Let us first wrap up the proof of Proposition 3.2 given (21). On $\mathcal{W}$, the curves γ^i, γ^j are independent conditioned on $\mathcal{D}_i$, $\mathcal{D}_j$. Combining with (21), we get

$$\mathbb{E}\big[\mathbf{1}_\mathcal{W}|\Delta_i\Delta_j|\big]=o_{\varepsilon/\zeta}(1)\frac{\varepsilon^4}{\zeta^{1/2}}\mathbb{P}[\mathcal{W}].$$

On $\mathcal{W}$, the 2-arm event occurs in the disjoint annuli $\Omega \setminus \mathcal{B}(\frac{q_i+q_j}{2}, l)$, $\mathcal{B}(q_i, l/2) \setminus \mathcal{B}(q_i, \zeta)$ and $\mathcal{B}(q_j, l/2) \setminus \mathcal{B}(q_j, \zeta)$. By Lemma 2.11, we have $\mathbb{P}[\mathcal{W}] \lesssim \zeta^{1/2}/l^{1/4}$. Therefore,

$$\left|\mathbb{E}[\mathbf{1}_{\mathcal{W}} \Delta_i \Delta_j]\right| = o_{\varepsilon/\zeta}(1) \frac{\varepsilon^4}{l^{1/4}}. \tag{22}$$

Combining with (20) and setting $\zeta = r^2 = \varepsilon^{1/2}$, we get (18). Summing over $i,\ j$, we see that the off-diagonal term in (14) is less than δ for sufficiently small ε. Combining with (17), this concludes the proof of Proposition 3.2, and hence of Theorem 1.6.

Now we focus on the proof of (21), which crucially relies on the following lemma.

Lemma 3.6. *Let Ω' be a Jordan domain containing* 0. *Let $d = \operatorname{dist}(0, \partial\Omega')$ and $d' = d \wedge 1$. Let $a', b' \in \partial\Omega$ and γ' be a chordal* SLE_6 *on* (Ω, a', b'). *Let*

$$x' = \operatorname{Mink}_{7/4}\big(\gamma' \cap \mathcal{B}_{2\varepsilon}\big) \quad \textit{and} \quad y' = \mathbf{1}_{\gamma' \cap \mathcal{B}_{2\varepsilon} \neq \varnothing}.$$

Then there exist absolute constants $c, C > 0$ independent of Ω' such that for $0 < \varepsilon < d'/10$,

$$\left|\mathbb{E}\big[x' - \beta_\varepsilon y'\big]\right| \le C \left(\frac{2\varepsilon}{d'}\right)^c \cdot \varepsilon^{7/4} \cdot \alpha_2\big(2\varepsilon, d'\big). \tag{23}$$

Proof. Recall x_0 and $\mathcal{A}_0(2\varepsilon, 1)$ in the definition of β_ε. Also recall the notations in Section 2.5. Set $\mathcal{A} := \mathcal{A}(\mathcal{B}_{2\varepsilon}, \Omega') = \{\gamma' \cap \mathcal{B}_{2\varepsilon} \neq \varnothing\}$. We have

$$\mathbb{E}\big[x' - \beta_\varepsilon y' | \mathcal{A}\big] = \mathbb{E}\big[x' | \mathcal{A}\big] - \mathbb{E}\big[x_0 | \mathcal{A}_0(2\varepsilon, 1)\big].$$

Suppose ω'_η is a site percolation configuration on $\Omega'_\eta \setminus \mathcal{B}_{1.9\varepsilon}$. Then the discrete analog $\mathcal{A}_\eta(\mathcal{B}_{2\varepsilon}, \Omega)$ of $\mathcal{A}$ is an event measurable with respect to ω'_η. Moreover, the face $\mathcal{D}_\eta(\mathcal{B}_{2\varepsilon}, \Omega')$ induced by γ' at $\mathcal{B}_{2\varepsilon}$ is also measurable with respect to ω'_η. Now assume the law of ω'_η is the critical percolation conditioning on $\mathcal{A}_\eta$. Let ω^0_η be the random site percolation configuration defined in the same manner as ω'_η with $(\mathcal{B}_1, -i, i)$ in place of (Ω', a', b').

By [8, Proposition 3.6], there exist an absolute constant $c > 0$ independent of Ω' and a coupling $(\omega'_\eta, \omega^0_\eta)$ such that for $10\eta < \varepsilon < d'/10$, with probability at least $1 - (2\varepsilon/d')^c$, we have $\mathcal{D}_\eta(\mathcal{B}_{2\varepsilon}, \Omega') = \mathcal{D}_\eta(\mathcal{B}_{2\varepsilon}, \mathcal{B}_1)$. In fact, [8, Proposition 3.6] is stated for the 4-arm event but as explained in [8, Section 5.3], the result holds for the 2-arm case here with little adaption. In this coupling, we extend ω'_η and ω^0_η to $\mathcal{B}_{1.9\varepsilon}$ by coloring each vertex black with probability $1/2$ and white with probability $1/2$. Here we use the same randomness for ω'_η and ω^0_η on $\mathcal{B}_{1.9\varepsilon}$ while different vertices are colored independently. By Theorem 2.1 and Lemma 2.9, letting $\eta \to 0$, we have a continuum coupling $(\overline{\gamma}', \overline{\gamma}^0, \omega', \omega^0)$ such that

- $\overline{\gamma}'$ and $\overline{\gamma}^0$ are the scaling limits of the interfaces of ω'_η and ω^0_η, respectively;
- ω' and ω^0 are the scaling limits of ω'_η and ω^0_η, respectively, as quad-crossing configurations in the $d_{\mathcal{H}}$ metric;
- ω' has the law of ω as in Theorem 2.1 with Ω' in place of Ω, conditioning on $\mathcal{A}$;
- the law ω^0_η is the same as ω'_η with $(\mathcal{B}_1, -i, i)$ in place of (Ω', a', b');
- with probability at least $1 - (2\varepsilon/d')^c$, we have $\mathcal{D}(\mathcal{B}_{2\varepsilon}, \Omega') = \mathcal{D}(\mathcal{B}_{2\varepsilon}, \mathcal{B}_1)$;
- $\omega' = \omega^0$ inside $\mathcal{B}_{1.9\varepsilon}$, which is independent of $\mathcal{D}(\mathcal{B}_{2\varepsilon}, \Omega')$ and $\mathcal{D}(\mathcal{B}_{2\varepsilon}, \mathcal{B}_1)$.

Let F be the event that $\{\mathcal{D}(\mathcal{B}_{2\varepsilon}, \Omega') = \mathcal{D}(\mathcal{B}_{2\varepsilon}, \mathcal{B}_1)\}$. Let $\overline{x}', \overline{x}_0$ be defined in the same way as x', x_0 with (γ', γ^0) replaced by $(\overline{\gamma}', \overline{\gamma}^0)$. (Here the only difference between (γ', γ^0) and $(\overline{\gamma}', \overline{\gamma}^0)$ is that the former is unconditioned and the latter is conditioned.) Then $\overline{x}' = \overline{x}_0$ on F and $\mathbb{P}[F] \ge 1 - (2\varepsilon/d')^c$. Therefore

$$\left|\mathbb{E}\big[x' - \beta_\varepsilon y' | \mathcal{A}\big]\right| \le \left(\frac{2\varepsilon}{d'}\right)^c \big(\mathbb{E}\big[\overline{x}' | F^c\big] + \mathbb{E}\big[\overline{x}_0 | F^c\big]\big). \tag{24}$$

Let $\widetilde{x}'$ be defined as in Lemma 3.5 with $\overline{\gamma}'$ in place of γ. Then $\overline{x}' \le \widetilde{x}'$. By the nature of the coupling, $\widetilde{x}'$ is independent of F. Therefore,

$$\mathbb{E}\big[\overline{x}' | F^c\big] \le \mathbb{E}\big[\widetilde{x}'\big] \lesssim \varepsilon^{7/4}.$$

Similarly, we have $\mathbb{E}[\overline{x}_0 \mid F^c] \lesssim \varepsilon^{7/4}$. Combining with (24), and using that $x' - \beta_\varepsilon y' = 0$ unless $\mathcal{A}$ occurs, we see that there exists a constant $C > 0$

$$\left|\mathbb{E}\left[x' - \beta_\varepsilon y'\right]\right| \le C\left(\frac{\varepsilon}{d'}\right)^c \cdot \varepsilon^{7/4} \cdot \mathbb{P}[\mathcal{A}].$$

Now Lemma 2.6 yields (23). □

Remark 3.7. We assume that Ω' is a Jordan domain in Lemma 3.6 because our proof crucially relies on the coupling result of [8] in the discrete and the convergence result Theorem 2.1, which is only established for Jordan domains [3]. Lemma 3.6 is not directly applicable to $\mathcal{D}_i$, $\mathcal{D}_j$ in (21) since they are a.s. not Jordan (see Figure 1). To overcome this issue, we extend Lemma 3.6 to Lemma 3.8 below.

Lemma 3.8. *Suppose Ω' is a simply connected domain containing the origin whose boundary is a continuous curve. Let $\phi : I \to \mathbb{C}$ be a parametrization of $\partial\Omega'$ for $I \subset \mathbb{R}$ an interval, and let*

$$\mathrm{dbl} = \left\{z \in \partial\Omega' : \exists s \neq t \text{ such that } \phi(s) = \phi(t) = z\right\}.$$

Let $a', b' \in \partial\Omega' \setminus \mathrm{dbl}$ and $\phi : \overline{\mathbb{H}} \to \overline{\Omega'}$ be a conformal map with $\phi(0) = a'$, $\phi(\infty) = b'$. Let γ' be an SLE_6 on (Ω', a', b'). We say that (Ω', a', b') satisfies Property (S) *if $\mathbb{P}[\mathrm{dist}(\gamma', \mathrm{dbl}) > 0] = 1$. If (Ω', a', b') satisfies* (S) *then Lemma 3.6 holds for (Ω', a', b') with the same constants c, C.*

Proof. Suppose $\mathrm{dist}(\gamma', \mathrm{dbl}) > 0$ a.s. Then $\mathbb{P}[\mathrm{dist}(\phi^{-1}(\gamma'), \phi^{-1}(\mathrm{dbl})) < \delta] = o_\delta(1)$ for $\delta \in (0, 1)$. Let $\mathbb{H}^\delta = \{z \in \mathbb{H} : \mathrm{dist}(z, \phi^{-1}(\mathrm{dbl})) > \delta\}$ and $\Omega^\delta = \phi(\mathbb{H}^\delta)$. Then $\mathbb{P}[\gamma' \subset \Omega^\delta] = 1 - o_\delta(1)$. Since $\partial\mathbb{H}^\delta$ is a simple curve, we see that Ω^δ is a Jordan domain, thus satisfying Lemma 3.6. By the locality property of SLE_6, the total variation distance between the law of γ' and the SLE_6 on (Ω^δ, a', b') is $o_\delta(1)$. Since c, C in Lemma 3.6 are independent of δ, letting $\delta \to 0$, we prove Lemma 3.8. □

In the notation of Lemma 3.8, we say that (Ω', a', b') satisfies Property (W) if $\gamma' \cap \mathrm{dbl} = \varnothing$ a.s. The following lemma ensures that the complement of SLE_6 hulls satisfies Property (W).

Lemma 3.9. *Suppose γ is a chordal* SLE_6 *as in Theorem 1.4. Then a.s. there exists no point $p \in \gamma$ such that $\gamma \setminus \{p\}$ is disconnected and γ visits p at least twice.*

Proof. This is proved in [13, Remark 8.8]. □

Recall that $\mathcal{D}_\bullet$ is a domain induced by a face with two ordered marked points on its boundary. By Lemma 3.9, for $\bullet = i, j$, it is a.s. the case that $\mathcal{D}_\bullet$ (after recentering at 0) satisfies Property (W). However, $\mathcal{D}_\bullet$ does not satisfy Property (S) because the two boundary marked points could be accumulation points of dbl. (In fact, one can prove that the two boundary marked points a.s. are such accumulation points.) We overcome this issue by the following lemma.

Lemma 3.10. *For $\alpha \in (0, 1)$, let $\mathbb{H}_\alpha = (\mathbb{H} \setminus \alpha\mathbb{D}) \cap \alpha^{-1}\mathbb{D}$. Let γ^0 and γ^α be the chordal* SLE_6 *on $(\mathbb{H}, 0, \infty)$ and $(\mathbb{H}_\alpha, \alpha i, \alpha^{-1} i)$ respectively. Let $\mathcal{B} \subset \mathbb{H}$ be a box. Let σ and $\overline{\sigma}$ be the first and last, respectively, time that γ is contained in $\mathcal{B}$. Define σ_α and $\overline{\sigma}_\alpha$ for γ^α similarly. Then the total variation distance between $\gamma|_{[\sigma,\overline{\sigma}]}$ and $\gamma^\alpha|_{[\sigma_\alpha,\overline{\sigma}_\alpha]}$ as curves modulo monotone parametrizations is $o_\alpha(1)$ as $\alpha \to 0$.*

Proof. Let $\widetilde{\gamma}^\alpha$ be a chordal SLE_6 on $(\mathbb{H} \setminus \alpha\mathbb{D}, \alpha i, \infty)$. Let $\widetilde{\sigma}_\alpha$ and $\widetilde{\tau}^\alpha$ be the first and last, respectively, time that $\widetilde{\gamma}^\alpha$ is contained in $\mathcal{B}$. We couple $\widetilde{\gamma}^\alpha$ and γ^0 such that when running them backward, the two curves agree until hitting $\sqrt{\alpha}\mathbb{D}$; this is possible by reversibility of SLE_6. With probability $1 - o_\alpha(1)$, the remaining segments of the two curves will not touch $\mathcal{B}$. Then the total variation distance between $\gamma|_{[\sigma,\infty)}$ and $\widetilde{\gamma}^\alpha|_{[\widetilde{\sigma}_\alpha,\infty)}$ as curves modulo monotone parametrizations is $o_\alpha(1)$. Similarly, the total variation distance between $\widetilde{\gamma}^\alpha|_{[\widetilde{\sigma}_\alpha,\widetilde{\tau}_\alpha]}$ and $\gamma^\alpha|_{[\sigma_\alpha,\overline{\sigma}_\alpha]}$ as curves modulo monotone parametrizations is $o_\alpha(1)$. This concludes the proof. □

Now we are ready to prove (21). Let ϕ be defined as in Lemma 3.8 with $\mathcal{D}_\bullet$ and $x_\bullet$ in place of (Ω', a', b') and 0. Recall the notation in Lemma 3.10. We can define the analog of $x_\bullet$, $y_\bullet$ with $\gamma^\bullet$ replaced by the SLE_6 on $(\phi(\mathbb{H}_\alpha), \phi(\alpha i), \phi(\alpha^{-1} i))$ and denote these two quantities by x_α, y_α. By Lemma 3.10, the total variation distance between the laws of $(x_\bullet, y_\bullet)$ and (x_α, y_α) is $o_\alpha(1)$. On the other hand, $(\phi(\mathbb{H}_\alpha), \phi(\alpha i), \phi(\alpha^{-1} i))$ satisfies the stronger property (S) rather than just (W) because the boundary is simple and smooth near $\phi(\alpha i)$ and $\phi(\alpha^{-1}i)$. Since c, C in Lemma 3.8 are independent of α, letting $\alpha \to 0$, we arrive at

$$\mathbb{E}[x_\bullet - \beta_\varepsilon y_\bullet | \mathcal{D}_\bullet] \le C\left(\frac{2\varepsilon}{\zeta}\right)^c \cdot \varepsilon^{7/4} \cdot \alpha_2(2\varepsilon, \zeta) = o_{\varepsilon/\zeta}(1)\frac{\varepsilon^2}{\zeta^{1/4}}.$$

This concludes the proof of (21) and hence of Proposition 3.2.

4. Minkowski content for percolation pivotal points

This section is devoted to proving Proposition 1.8 and Theorem 1.9.

Recall that $\mathrm{SLE}_\kappa(\rho)$ and $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ processes are variants of SLE_κ whose driving functions have forcing terms prescribed by *force points* with certain *weights*. $\mathrm{SLE}_\kappa(\rho)$ has a single force point of weight ρ, while $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ has two force points of weight ρ_1 and ρ_2, respectively. We will not give the formal definition of these processes and refer instead to [21, Section 2.2], because we only use a few well-established facts about the processes developed in the framework of imaginary geometry [21].

We also recall the Brownian excursion on $\mathbb{H}$ from 0 to ∞. See [14, Chapter 2] for the precise definition. By the theory of conformal restriction [13] the left and right boundary of the Brownian excursion and those of $\mathrm{SLE}_6(2;2)$ have the same law. (In fact, the hull of both Brownian excursion and $\mathrm{SLE}_6(2,2)$ are the unique chordal restriction measure with exponent 1.) Let $\mathcal{C}$ denote the intersection of the left and right boundaries of the Brownian excursion, i.e. the set of cut points. By [11, Theorem 4.7], in the notation of Definition 1.2, we have the following.

Lemma 4.1. *The occupation measure of $\mathcal{C}$ a.s. exists and is defined by its $3/4$-dimensional Minkowski content.*

Remark 4.2. In [11, Theorem 4.7], the notion of a cut point is defined via cut times. Namely, given a Brownian excursion $(E(t))_{t\ge 0}$ on $\mathbb{H}$ from 0 to ∞, the set of cut points of E is defined by $\mathcal{C}' = \{\eta(t) : t \ge 0 \text{ and } E((0,t)) \cap E((t,\infty)) = \varnothing\}$. However, it can be checked that $\mathcal{C}' = \mathcal{C}$ a.s. The direction $\mathcal{C}' \subset \mathcal{C}$ a.s. is trivial. For the other direction, let $\mathcal{H}$ be the hull of E, which has the same law as the hull of $\mathrm{SLE}_6(2,2)$. By the SLE duality, the interior of $\mathcal{H}$ is a countable collection of simply connected open sets, ordered by the order in which they are first visited if we go from 0 to ∞ inside $\mathcal{H}$. In particular, for each $p \in \mathcal{C}$, $\mathcal{C} \setminus \{p\}$ has two components, one bounded and one unbounded, such that all the sets in the bounded component are ordered before the sets in the unbounded component. Suppose there exists $p \in \mathcal{C} \setminus \mathcal{C}'$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the bounded and unbounded component of $\mathcal{C} \setminus \{p\}$, respectively. Let t_1 be the first time $E(t) = p$ and $q \in E((0,t_1)) \cap E((t_1,\infty))$. Then $q \in \mathcal{C}_1$ and there exists $t' > t_1$ such that $E(t') = q$. For a rational $s \in (t_1, t')$, $E(s)$ a.s. is contained in a component B of the interior of $\mathcal{H}$. Moreover, the closure of B only has two points in $\mathcal{C}$, one of which must be visited by E twice. Since there are only countably many such points, this can be ruled out by the strong Markov property of E and the fact that a planar Brownian motion a.s. does not visit any fixed point. This gives $\mathcal{C} \subset \mathcal{C}'$ a.s.

Run an $\mathrm{SLE}_{8/3}(2;-4/3)$ on $(\mathbb{H}, 0, \infty)$ where the force points are at 0^- and 0^+. Conditioning on this curve, run an $\mathrm{SLE}_{8/3}(-4/3;4/3)$ on the domain to its left. Let $\mathbb{H}' \subset \mathbb{H}$ be the domain between these two curves. Conditioning on $\mathbb{H}'$, we run an $\mathrm{SLE}_6(1;1)$ on $(\mathbb{H}', 0, \infty)$. Then by the rule of interacting flow lines in [21], the marginal law of this curve is an $\mathrm{SLE}_6(2;2)$ on $(\mathbb{H}, 0, \infty)$ with force points at 0^+ and 0^-. See the left part of Figure 2 for an illustration.

Let Υ^1 be an $\mathrm{SLE}_6(2)$ on $(\mathbb{H}, 0, \infty)$ where the single force point is at 0^+. Then $\Upsilon^1 \cap \mathbb{R}_{>0} = \varnothing$. Now conditioning on Υ^1, let Υ^2 be a chordal SLE_6 from 0 to ∞ on the domain to the right of Υ^1. By SLE duality (see [31, Theorem 5.1] and [21, Theorem 1.4]) the right boundary of Υ^1 and the left boundary of Υ^2 have the same joint law as the left and right

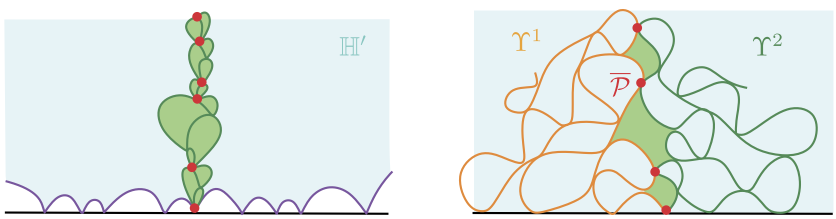

Fig. 2. Left: The green curve is an $\mathrm{SLE}_6(1,1)$ in $\mathbb{H}'$ (which is the domain in light blue) and has the law of an $\mathrm{SLE}_6(2,2)$ viewed as a curve in $\mathbb{H}$. The points of intersection of its left and right boundaries (red) have the law of the cut points $\mathcal{C}$ of a Brownian excursion in $\mathbb{H}$. Right: The region (light green) between the right boundary of Υ^1 and the left boundary of Υ^2 has the law of the region enclosed by an $\mathrm{SLE}_6(1,1)$ in $\mathbb{H}$.

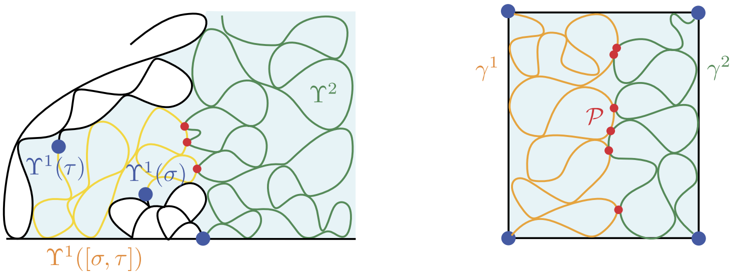

Fig. 3. Illustration of Lemma 4.4.

boundary of an $\mathrm{SLE}_6(1,1)$. See the right part of Figure 2 for an illustration. Denote their intersection by $\overline{\mathcal{P}}$. Combined with the paragraph above, we have the following.

Lemma 4.3. *There is a coupling of $\mathcal{C}$, $\mathbb{H}'$, and $\overline{\mathcal{P}}$ such that $\overline{\mathcal{P}}$ is independent of $\mathbb{H}'$, and $\mathcal{C}$ is the image of $\overline{\mathcal{P}}$ under a conformal map from $\mathbb{H}$ to $\mathbb{H}'$ fixing 0 and ∞.*

The next lemma links $\overline{\mathcal{P}}$ to the set $\mathcal{P}$ in Proposition 1.8 and Theorem 1.9.

Lemma 4.4. *Recall (Ω, a, b, c, d) and γ^1, γ^2 in Theorem 1.7. Let $\overline{\gamma}^2$ be the time-reversal of γ^2. There exist random times σ and τ for Υ^1 satisfying $0 < \sigma < \tau < \infty$ with positive probability, such that the following hold on the event that $0 < \sigma < \tau < \infty$.*

1. *The unbounded component of $\mathbb{H} \setminus (\Upsilon^1([0,\sigma]) \cup \Upsilon^1([\tau,\infty])$ can be conformally mapped to Ω with $(\Upsilon^1(\sigma), 0, \infty, \Upsilon^1(\tau))$ mapped to (a, b, c, d).*
2. *Conditioning on the realization of $\mathbb{H} \setminus (\Upsilon^1([0,\sigma]) \cup \Upsilon^1([\tau,\infty])$, the joint law of the conformal image of $\Upsilon^1([\sigma,\tau])$ and Υ^2 is the same as the conditional law of $(\gamma^1, \overline{\gamma}^2)$ conditioning on E^c.*

See Figure 3 for an illustration.

Proof. Let t be the last time where $\mathrm{Im}(\Upsilon^1) = 1$. By [23, Lemma 2.2], $\Upsilon^1[0,t]$ stays close to any deterministic smooth curve in $\mathbb{H}$ from the origin to a point on $\{z : \mathrm{Im}\, z = 1\}$ with positive probability. Therefore, with positive probability Υ^1 reaches a time $s < t$ when Condition 1 in Lemma 4.4 is achieved with s, t in place of σ, τ. On the event that there is such a time $s \in (0,t)$, let σ be the infimum of such times. Note that $\sigma > 0$ a.s. since as $s \to 0$ the extremal distance between the arc from $\Upsilon^1(s)$ to 0 and the arc from ∞ to $\Upsilon^1(\tau)$ in the unbounded component of $\mathbb{H} \setminus (\Upsilon^1([0,s]) \cup \Upsilon^1([\tau,\infty]))$ goes to zero. We set $\sigma = \infty$ if this event does not occur and let $\tau = t \vee \sigma$.

Viewing Υ^1, Υ^2 as two counterflow lines of different angles in the same imaginary geometry ([21]), for $i = 1, 2$, conditioning on Υ^i, the law of Υ^{3-i} is a chordal SLE_6. Therefore the domain Markov property and reversibility of SLE_6 yield that the same resampling property holds for the conformal image of $\Upsilon^1([\sigma,\tau])$ and Υ^2. By convergence of $(\gamma^1_\eta, \gamma^2_\eta)$ to (γ^1, γ^2), the same resampling property holds for $(\gamma^1, \overline{\gamma}^2)$ conditioning on E^c. As explained in [22, Appendix A], this resampling property uniquely determines the law of the pair of curves. Thus we conclude the proof. □

Combining Lemmas 4.1, 4.3, and 4.4, we see that on the event E^c the occupation measure of $\mathcal{P}$ exists and is defined by its $3/4$-dimensional Minkowski content. The same argument works when conditioning on E. This gives Proposition 1.8.

The proof of Theorem 1.9 follows from the exact same argument as in the proof of Theorem 1.6. We just need to replace one interface γ with the pair of interfaces γ^1, γ^2. Here we only point out the substitutes of the ingredients in the argument in Section 3.

Suppose we are in the coupling of Theorems 1.7 and 2.1. Then Lemma 2.7 and the 4-arm case of Lemma 2.8 and (7) give the analog of Lemmas 3.3, 3.4, and 3.5, in addition to (13). We can also adapt the concept of face in this setting, where the number of arcs becomes 4 instead of 2. We use the same notations as in Section 2.5. Given $\mathcal{B}$, let $\mathcal{A}(\mathcal{B},\Omega) = \{\mathcal{P} \cap \mathcal{B} \neq \varnothing\}$. On the event $\mathcal{A}(\mathcal{B},\Omega)$, we trace γ^1 and γ^2 and their time-reversals from a, b, c, d until first hitting $\mathcal{B}$. This defines a face at Θ at $\mathcal{B}$ induced by (γ^1, γ^2). Moreover, $\mathcal{D}(\mathcal{B},\Omega)$ and $\mathcal{G}_\Omega(\mathcal{B}, U)$ can be defined in the same way as in Section 2.5. Then Lemmas 2.9 and 2.10 still hold with $k = 2, 3$ replaced by $k = 4, 5$. Now if we carry out the argument in Section 3, Theorem 1.9 will be reduced to the analog of (21), which can still be proved by a coupling argument as in Lemma 3.6 combined with approximation arguments as in Lemmas 3.8, 3.9, and 3.10.

5. Convergence of the percolation interface under natural parametrization

In this section we prove Theorem 1.4. We pick the constant c_1 so that Theorem 1.6 holds.

Lemma 5.1. *The curve $\widehat{\gamma}_\eta$ is tight for the ρ-metric.*

Proof. We proceed by contradiction and suppose $\widehat{\gamma}_\eta$ is not tight. Then there exist $\delta_0 > 0$, $\eta_n \downarrow 0$ and $\varepsilon_n \downarrow 0$ such that

$$\mathbb{P}\big[\mathrm{osc}(\varepsilon_n; \widehat{\gamma}_{\eta_n}) > \delta_0\big] \geq \delta_0, \tag{25}$$

where $\mathrm{osc}(\varepsilon, f) = \sup_{|t-s|\leq\varepsilon} |f(t) - f(s)|$. Write $\widehat{\gamma}_{\eta_n}$ as $\widehat{\gamma}_n$ and γ_{η_n} as γ_n for simplicity. Given a realization of γ_n, let s_n, t_n be the smallest times such that $|\widehat{\gamma}_n(s_n) - \widehat{\gamma}_n(t_n)| = \mathrm{osc}(\varepsilon_n; \widehat{\gamma}_n)$. Then the random variables $(s_n, t_n, \widehat{\gamma}_n(s_n), \widehat{\gamma}_n(t_n))$ are tight. By the Skorokhod embedding theorem, we can couple $\{\gamma_n\}$ and γ such that (possibly) along a subsequence (which is still indexed by n for the sake of simplicity), the following occur a.s.

1. γ_n converge to γ in the topology of Theorem 1.5;
2. s_n and t_n converge to the same limit, which we denote by $\mathfrak{t}$;
3. there are $x, y \in \overline{\Omega}$ such that $\widehat{\gamma}_n(s_n) \to x$ and $\widehat{\gamma}_n(t_n) \to y$.

We first observe that both x and y are on the trace of γ a.s. In fact, consider an open ball $B(z, r)$ where z, r are both rational. (We call such balls *rational balls*.) Then on the event that $x \in B(z, r)$, it holds a.s. that for sufficiently large n, $\gamma_n \cap B(z, r) \neq \varnothing$. Given condition 1 in the coupling above, $\gamma \cap B(z, r) \neq \varnothing$ a.s. (see Lemma 2.2).

By (25), in the coupling above, $\mathbb{P}[|x - y| \geq \delta_0] \geq \delta_0$. Therefore we can find rational balls $B(z_1, r_1)$ and $B(z_2, r_2)$ such that the following event occurs with positive probability:

1. $x \in B(z_1, r_1)$ and $y \in B(z_2, r_2)$;
2. $\max\{r_1, r_2\} < 0.1\delta_0$ and $B(z_1, 2r_1) \cap B(z_2, r_2) = \varnothing$.

We work on this event hereafter. Let γ_n and γ be parametrized as in Lemma 2.2 and let

$$\rho_n^1 = \inf\big\{t : \gamma(t) \in B(z_1, r_1)\big\},$$
$$\sigma_n^1 = \inf\big\{t > \rho_n^1 : \gamma_n(t) \notin B(z_1, 2r_1)\big\},$$
$$\lambda_n^1 = \inf\big\{t > \sigma_n^1 : \gamma_n(t) \in B(z_2, r_2)\big\}.$$

Define ρ^1, σ^1, λ^1 similarly for γ. Note that $K_{\lambda^1} \setminus K_{\sigma^1}$ has non-empty interior a.s., where $K_\cdot$ is the hull process of γ. Therefore there exists a rational ball $B(z_3, r_3) \subset K_{\lambda^1} \setminus K_{\sigma^1}$ such that

$$\gamma \cap B(z_3, r_3) = \gamma\big(\big[\sigma^1, \lambda^1\big]\big) \cap B(z_3, r_3) \neq \varnothing.$$

By Lemma 2.2, we must have that for all sufficiently large n,

$$\gamma_n \cap B(z_3, r_3) = \gamma_n\big(\big[\sigma_n^1, \lambda_n^1\big]\big) \cap B(z_3, r_3) \neq \varnothing.$$

Since $\mathfrak{m}_\gamma(\partial B(z_3, r_3)) = 0$, by Theorem 1.6, $\lambda_n^1 - \sigma_n^1 \geq \tau_n(B(z_3, r_3)) \to c_* \mathfrak{m}_\gamma(B(z_3, r_3)) > 0$. Since $\gamma_n([0, \rho_n^1]) \subset \widehat{\gamma}_n([0, s_n])$ and $\widehat{\gamma}_n([0, \lambda_n^1]) \subset \gamma_n([0, t_n])$, while $t_n - s_n \to 0$, we must have $\gamma_n([0, \sigma_n^1]) \subset \widehat{\gamma}_n([0, s_n])$.

Now for $k \geq 2$ we define ρ_n^k and σ_n^k inductively by

$$\rho_n^k = \inf\big\{t > \sigma_n^{k-1} : \gamma_n(t) \in B(z_1, r_1)\big\},$$
$$\sigma_n^k = \inf\big\{t > \rho_n^k : \gamma_n(t) \notin B(z_1, 2r_1)\big\}.$$

Then $\rho_n^2 < \infty$. By the same argument as above, for any fixed k, for n sufficiently large, we have $\gamma_n([0, \sigma_n^k]) \subset \widehat{\gamma}_n([0, s_n])$ hence $\rho_n^{k+1} < \infty$. This yields that with positive probability, γ returns to $B(z_1, r_1)$ after hitting $\partial B(z_1, 2r_1)$ infinitely many times. This contradicts the fact that almost surely γ is a continuous curve and never returns to $B(z_1, 2r_1)$ after a certain finite time. □

Before proving Theorem 1.4 we prepare two lemmas.

Lemma 5.2. *Let γ, $\widehat{\gamma}$ be as in Theorem 1.4 and recall that $\mathfrak{m}_\gamma$ denotes the occupation measure of γ. For each fixed $t>0$, on the event $\mathfrak{m}_\gamma(\Omega)>t$ let K_t be the hull of $\widehat{\gamma}([0,t])$ (see Lemma 2.6). Then $\mathfrak{m}_\gamma(\partial K_t)=0$ a.s.*

Proof. By Assertion 2 above Definition 1.3, with probability 1, $\mathfrak{m}_\gamma(\widehat{\gamma}(t))=0$ for all $t\geq 0$. To prove Lemma 5.2, it suffices to show that for any fixed square $Q\subset\Omega$ and $\delta>0$, we have

$$\mathfrak{m}_\gamma\big(U^\delta\big)=0 \quad \text{a.s., where } U^\delta=Q\cap\partial K_t\setminus B\big(\widehat{\gamma}(t),\delta\big)$$

For $0<r<0.1\varepsilon<0.01\delta$, let $U^{\delta,\varepsilon}$ be the union of all boxes of side length ε on $\varepsilon\mathbb{Z}^2$ that has nonempty intersection with $U^\delta\neq\varnothing$. Let $U_r^{\delta,\varepsilon}$ be the r-neighborhood of $U^{\delta,\varepsilon}\cap\gamma$. Suppose we are in the coupling of Theorem 2.1 so that γ is coupled with $\omega\in\mathcal{H}_\Omega$. By Lemma 2.6, for each $x\in\Omega$, if $x\in U_r^{\delta,\varepsilon}$, then we have

1. the 2-arm event occurs for the annulus of $\mathcal{B}(x,\varepsilon)\setminus\mathcal{B}(x,r)$;
2. the 3-arm event occurs for the annulus of $\mathcal{B}(x,0.5\delta)\setminus\mathcal{B}(x,2\varepsilon)$.

By (7) and Lemma 2.11, there exists a constant $C=C(\delta)$ such that for any $x\in\Omega$,

$$\mathbb{P}\big[x\in U_r^{\delta,\varepsilon}\big]\leq C(\delta)(r/\varepsilon)^{1/4}\varepsilon^{2/3}=C(\delta)r^{1/4}\varepsilon^{5/12}.$$

By the definition of $\mathfrak{m}_\gamma$ in Definition 1.2, almost surely $\mathfrak{m}_\gamma(U^{\delta,\varepsilon})=\mathrm{Mink}_{7/4}(U^{\delta,\varepsilon}\cap\gamma)$. By Fatou's lemma and Definition 1.1,

$$\mathbb{E}\big[\mathfrak{m}_\gamma\big(U^\delta\big)\big]\leq\mathbb{E}\big[\mathfrak{m}_\gamma\big(U^{\delta,\varepsilon}\big)\big]\leq\liminf_{r\to 0}\mathbb{E}\big[r^{-1/4}\mathrm{Area}\big(U_r^{\delta,\varepsilon}\big)\big]\leq C(\delta)\varepsilon^{5/12}.$$

Sending $\varepsilon\to 0$ we have $\mathbb{E}[\mathfrak{m}_\gamma(U^\delta)]=0$ and we are done. □

Lemma 5.3. *In the setting of Theorem 1.4, conditioning on γ, sample U from $(0,\tau(\Omega))$ uniformly and sample $z\in\Omega$ according to $\tau(\cdot)/\tau(\Omega)$. Then $\widehat{\gamma}(U)$ has the same law as z.*

Proof. For $t>0$, let K_t be the hull of $\widehat{\gamma}([0,t])$. Namely A_t is the set A in Lemma 5.2. For $s\in[0,t)$, let $K_{s,t}=K_t\setminus K_s$. It is proved in [15, Equation (27)] that a.s.

$$\mathfrak{m}_\gamma\big(\widehat{\gamma}[s,t]\big)=t-s \quad \text{for all } t>s\geq 0. \tag{26}$$

By Theorem 1.6, Lemma 5.2, and (26), for a fixed pair of s,t we have $\tau(K_{s,t})=t-s$ a.s. For each $n\in\mathbb{N}$, we can couple z and U such that both $\widehat{\gamma}(U)$ and z fall in $K_{i/n,(i+1)/n}$ for some $i<n$. By the continuity of $\widehat{\gamma}$, we conclude the proof. □

Proof of Theorem 1.4. According to Lemma 5.1, by possibly restricting to a subsequence, we can assume that $\widehat{\gamma}_\eta$ converge a.s. to a curve γ' in the ρ-metric. Furthermore, we can assume that along this subsequence (τ_η,γ_η) converges to (τ,γ) a.s. in the topology of Theorem 1.5, where γ is a chordal SLE_6 (viewed as a curve modulo reparametrization of time). In this coupling, γ' is a parametrization of γ with total length $\tau(\Omega)$. Let $\widehat{\gamma}$ be γ with its natural parametrization. It suffices to show that $\gamma'=\widehat{\gamma}$ a.s.

Conditioning on all the data above, we can sample a random time U uniformly in $(0,\tau(\Omega))$ and a random edge e_η according to τ_η on γ_η such that $\mathbb{P}[\widehat{\gamma}_\eta(U)\in e_\eta]\geq 1-o_\eta(1)$. Notice that $\widehat{\gamma}_\eta(U)$ converges to $\gamma'(U)$ a.s. Therefore e_η converge to $\gamma'(U)$ a.s. Here we identify edges on the hexagonal lattice with their midpoints since the difference is negligible in the scaling limit. On the other hand, since τ_η converge to τ in the Prokhorov metric a.s., by Lemma 5.3 we have that e_η converges in law to $\widehat{\gamma}(U)$. This implies that $\gamma'(U)$ and $\widehat{\gamma}(U)$ are equal in law.

Given any fixed $t>0$, on the event that $t\in(0,\tau(\Omega))$, let A be the hull of $\widehat{\gamma}([0,t])$ and $A_\varepsilon=\{z\in\mathbb{C}:\mathrm{dist}(z,A))\leq\varepsilon\}$. Since $\tau(A\setminus\partial A)\leq t$ and (by Theorem 1.6 and Lemma 5.2) $\tau(\partial A)=0$, we have $\lim_{\varepsilon\to 0}\tau(A_\varepsilon)=t$. Thus for all $\delta>0$ with $t+2\delta<\tau(\Omega)$, we can find a (random) $\varepsilon>0$ a.s. such that $\tau(A_\varepsilon)\leq t+\delta$. Therefore $\limsup_{\eta\to 0}\tau_\eta(A_\varepsilon)\leq t+\delta$. Hence for sufficiently small η there exists $t_\eta^\delta\in[t,t+2\delta]$ such that $\widehat{\gamma}_\eta(t_\eta^\delta)\notin A_\varepsilon$. By possibly passing to a subsequence, we can assume $\lim_{\eta\to 0}t_\eta^\delta\to t^\delta$. Sending $\eta\to 0$, we have $\gamma'(t^\delta)\notin\widehat{\gamma}([0,t])$. Since γ' and $\widehat{\gamma}$ have the same range, we must have $\widehat{\gamma}(t)\in\gamma'([0,t^\delta])$. Letting $\delta\to 0$, we have $\widehat{\gamma}(t)\in\gamma'([0,t])$. By considering rational t's and then using the continuity of $\widehat{\gamma}$ and γ', we see that a.s. $\widehat{\gamma}([0,t])\subset\gamma'([0,t])$ for all $t\in(0,\tau(\Omega))$. Combined with $\gamma'(U)\overset{d}{=}\widehat{\gamma}(U)$, we have $\gamma'(U)=\widehat{\gamma}(U)$ hence $\gamma'=\widehat{\gamma}$ a.s. □

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