NEW CONSTRUCTIONS OF NEF CLASSES ON SELF-PRODUCTS OF CURVES

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Abstract. We study the nef cone of self-products of a curve. When the curve is very general of genus $g > 2$, we construct a nontrivial class of self-intersection 0 on the boundary of the nef cone. Up to symmetry, this is the only known nontrivial boundary example that exists for all $g > 2$. When the curve is general, we identify nef classes that improve on known examples for arbitrary curves. We also consider self-products of more than two copies of the curve.

1. Introduction

The closure of the ample cone of a projective variety X is the nef cone $\operatorname{Nef}(X)$. It is a fundamental invariant that controls morphisms from X to other projective varieties, in particular projective embeddings of X. It is important to compute this cone in specific examples; however, this is a difficult problem already on surfaces. A famous open question here is the following:

Conjecture 1.1 (Nagata; see [Nag59, Conjecture on p. 772])**.** *Let $\pi\colon X \to \mathbb{P}^2$ be the blow-up of $n \geq 10$ very general points in $\mathbb{P}^2$ with exceptional divisors $E_1, \ldots, E_n$. Let $H \subset \mathbb{P}^2$ be any line. Then*

$$\pi^*(\sqrt{n}H) - E_1 - \cdots - E_n \in \operatorname{Nef}(X).$$

Recall that a property is very general on a variety if it holds outside a countable union of Zariski closed proper subsets. Conjecture 1.1 is a particular case of the SHGH conjecture (see [CHMR13, Section 1.4] for an exposition). Note that the divisor in the Nagata conjecture has self-intersection 0, hence if it is nef, then it is on the boundary of the nef cone.

Another interesting class of surfaces is self-products of curves. Recall the following open problem:

Conjecture 1.2 (see [Laz04a, Remark 1.5.10])**.** *Let C be a smooth projective curve of genus g over $\mathbb{C}$. Denote by f_1 and f_2 (resp. δ) the classes of the fibers of the projections (resp. the class of the diagonal Δ) in $C \times C$. Then, we have*

$$(1 + \sqrt{g})(f_1 + f_2) - \delta \in \operatorname{Nef}(C \times C)$$

if g is sufficiently large and C has very general moduli.

The self-intersection of $(1 + \sqrt{g})(f_1 + f_2) - \delta$ is 0, just like in Conjecture 1.1. In fact, Ciliberto–Kouvidakis [CK99] and Ross [Ros07] prove that the Nagata conjecture implies Conjecture 1.2. In the direction of Conjecture 1.2, Kouvidakis [Kou93, Theorem 2] shows that

$$\left(1 + \frac{g}{\lfloor\sqrt{g}\rfloor}\right)(f_1 + f_2) - \delta \in \operatorname{Nef}(C \times C).$$

In particular, the conjecture holds when g is a perfect square. An improvement when g is not a perfect square is offered by [Ros07, (1.9)] who uses work of [SSS04] to prove that $\bigl(1 + \sqrt{g+1}\bigr)(f_1 + f_2) - \delta$ is nef.

It also makes sense to consider the non-symmetric divisors with zero self-intersection and ask:

2010 *Mathematics Subject Classification.* Primary 14C20, 14H10, 14H60, 14N05, 14D06.

The first author was partially supported by the Simons Foundation Collaboration Grant 579353.

The second author was partially supported by the National Science Foundation under Grant Nos. DMS-1701622 and DMS-1902616.

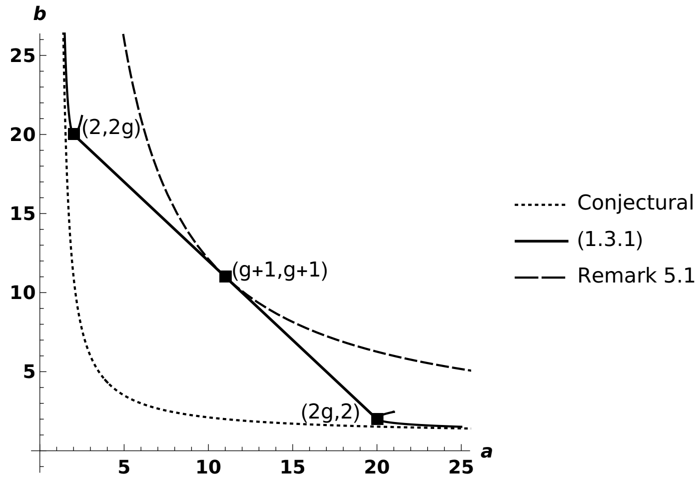


FIGURE 1. Nef classes on $C \times C$ for $g = 10$ and arbitrary C

The classes $af_1 + bf_2 - \delta$ are represented by the points (a, b). On the left, the outside curve is the conjectural nef boundary for very general curves in Question 1.3. Inside, on the upper left is the graph of $b = b_a + (a-1)(g-1)$ from (1.3.1). On the right is its reflection. The dotted curve in the middle is $b = 1 + \frac{g^2}{a-1}$ from Remark 5.1. The line segment bounds the convex hull. It is tangent to the curves at the specified points.

Question 1.3. Let C be a very general curve of (large) genus g. For any real $a > 1$, put

$$b_a \coloneqq 1 + \frac{g}{a-1}.$$

Is the class $af_1 + b_a f_2 - \delta$ nef on $C \times C$?

On arbitrary curves, the best known result is due to Rabindranath [Rab19, Proposition 3.2]. He adapts an idea of Vojta [Voj89] to prove that

$$af_1 + \big(b_a + (g-1)(a-1)\big)f_2 - \delta \in \mathrm{Nef}(C \times C). \tag{1.3.1}$$

See Figure 1. The line segment joining $(2, 2g)$ and $(2g, 2)$ is optimal for hyperelliptic curves.

Our sharpest result answers Question 1.3 in the affirmative for $a = 2$. Up to symmetry, this is the only settled case of Question 1.3 that we know of, other than Kouvidakis's when g is a perfect square.

Theorem (see Theorem 3.7)**.** *Let C be a* very general *smooth projective curve of genus $g \neq 2$. Then*

$$2f_1 + (1+g)f_2 - \delta \in \mathrm{Nef}(C \times C). \tag{1.3.2}$$

The idea is to degenerate C to a rational curve with g simple nodes in general position using a construction of [Ros07]. The nefness of the limit of the classes (1.3.2) follows from the elementary Proposition 3.8 concerning the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at g general symmetric pairs of points.

In Corollary 3.11 we apply the original techniques of Kouvidakis. We degenerate to simple covers of $\mathbb{P}^1$ to show that for all integers $2 \le d \le 1 + \sqrt{g}$ (so that $b_d \ge d$) and very general C:

$$df_1 + \big(2b_d - d\big)f_2 - \delta \in \mathrm{Nef}(C \times C). \tag{1.3.3}$$

See Figure 2.

Next, we propose an approach to Question 1.3 in terms of semistability of vector bundles and give partial results. We start with the simple observation that if $a > 1$, then $af_1 - \delta$ is ample on the fibers of the second projection $pr_2 \colon C \times C \to C$. If a is an integer and L_a is a divisor of degree a on

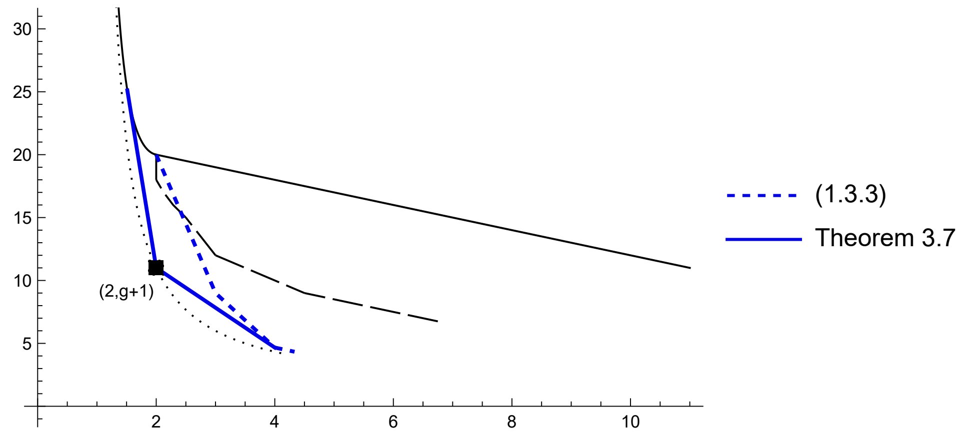


FIGURE 2. Nef classes on $C \times C$ for $g = 10$ and very general C

To improve visibility, we only show the picture above the diagonal $a = b$. Note the difference in scale. The point $(2, g+1)$ comes from Theorem 3.7. Keeping the contour from Figure 1 and from Figure 3 below, we have added two polygonal lines. The inner line represents classes coming from (1.3.3), and from convexity and symmetry. In the outer polygonal line we show classes induced by Theorem 3.7 and by previous results.

C, then we observe in Proposition 3.12.(ii) that the "positivity defect" of $pr_1^*L_a - \Delta$ (in this case the smallest b such that $af_1 + bf_2 - \delta$ is nef) is determined asymptotically by a similar measure of positivity defect of the sheaves $pr_{2*}\mathcal{O}\big(m(pr_1^*L_a - \Delta)\big)$. These are higher conormal bundles of C in the sense of [EL92]. The idea is an instance of a general "linearization" principle that we explain in Proposition 3.14.

Fix a curve C and a rational number $a > 1$. In Theorem 3.6, we use the above to prove that Question 1.3 is true for a and C if and only if the higher conormal bundles above are semistable in an asymptotic sense on C.

Even in the asymptotic sense, understanding the semistability of the terms in the sequence of higher conormal bundles seems very difficult. However, the first (or 0-th, depending on convention) term $pr_{2*}\mathcal{O}(pr_1^*L_a - \Delta)$ is well-understood. It is the syzygy bundle M_{L_a}, i.e., the kernel of the evaluation morphism $H^0(C, L_a) \otimes \mathcal{O}_C \to \mathcal{O}_C(L_a)$. Drawing on known results about its semistability, we obtain the following:

Theorem (see Theorem 3.4.(i))**.** *Let C be a* general *smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$. Denote by f_1 and f_2 (resp. δ) the classes of the fibers of the projections (resp. the class of the diagonal in $C \times C$). Then, we have*

$$df_1 + \left(b_d + \frac{g(g-1)}{(d-g)(d-1)}\right)f_2 - \delta \in \operatorname{Nef}(C \times C)$$

for every integer $d \geq \lfloor 3g/2 \rfloor + 1$.

When $d < 2g$ and g is large, we obtain examples outside the convex span of the known examples mentioned above due to Vojta and Rabindranath. Other examples are given in 3.4.(ii),(iii),(iv). In particular the latter is obtained from [CLV22], a semistability result for the normal bundle of general canonical curves. See also Figure 3.

Finally, we also consider self-products of more than two copies of C. Let f_i be the class of any fiber of the i-th projection $C^n \to C$. Let δ_{ij} be the class of the large diagonal $\{(x_1, \dots, x_n) \,|\, x_i = x_j\}$. With assumptions as in Theorem 3.4, it is immediate that

$$\sum_{i=2}^{n} \left(\left(1 + \frac{g}{d-g}\right)f_1 + df_i - \delta_{1i}\right) \in \operatorname{Nef}(C^n).$$

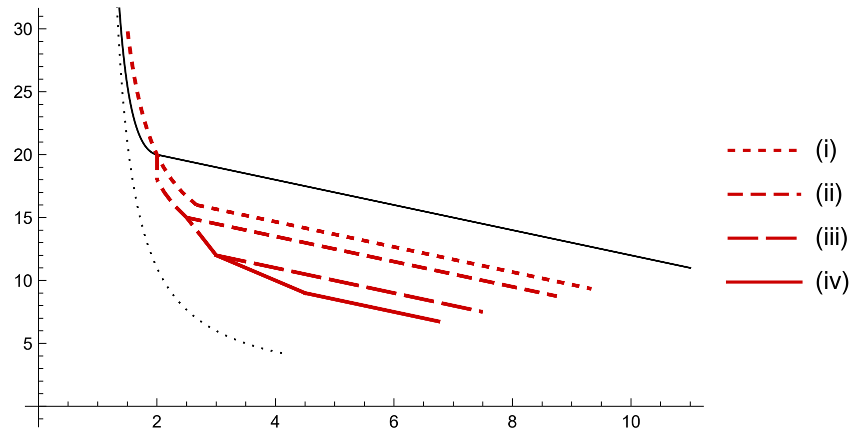


Figure 3. Nef classes on $C \times C$ for $g = 10$ and general C

We again focus above the diagonal. Keeping the contour from the arbitrary case, we have added classes coming from the corresponding four parts of Theorem 3.4 and from the convexity and symmetry of the nef cone.

We show furthermore in Theorem 4.3 that if C is an arbitrary smooth complex projective curve of positive genus and $d \in \mathbb{Z}$, then for certain values of n and d,

$$(n-1) \cdot \left(1 + \frac{g}{d-g}\right) f_1 + d \cdot \sum_{i=2}^{n} f_i - \sum_{1 \le i < j \le n} \delta_{ij} \in \operatorname{Nef}(C^n).$$

The proof makes use of the rich geometry of symmetric products of curves, and a result of Kempf on continuous global generation of vector bundles on abelian varieties.

Paper organization. In section 2 we review formal twists of sheaves by $\mathbb{R}$-Cartier $\mathbb{R}$-divisors and their positivity on curves. In section 3 we treat the nef cone of $C \times C$ starting from general considerations and known results for arbitrary curves, and then give our main results for general curves and for very general curves. In section 4 we treat higher products C^n. In section 5 we sketch what can be obtained by other natural approaches such as restricting from Jacobian varieties or from K3 surfaces.

Our proofs of the main results Theorems 3.7, 3.4, 3.6 use lemmas that are proved later in the paper in section 3.5. This is in an effort to present them as early as possible.

Acknowledgments. We thank Marian Aprodu, Thomas Bauer, Renzo Cavalieri, Alexandru Chirvasitu, Lawrence Ein, Mattias Jonsson, Alex Küronya, Robert Lazarsfeld, Emanuele Macrì, Eyal Markman, Mircea Mustaţă, Sönke Rollenske, Julius Ross, Praveen Kumar Roy, John Sheridan, and Brooke Ullery for useful discussions. We thank the referees for their comments and suggestions.

2. Background and notation

Let X be a projective scheme over an algebraically closed field. While our main results are over $\mathbb{C}$, some of our important tools (Proposition 3.12 and its generalization in Proposition 3.14) are valid in arbitrary characteristic.

2.1. **Formal twists of coherent sheaves.** Let $\mathcal{V}$ be a coherent sheaf on X, and let λ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Following the case of bundles in [Laz04b, Section 6.2], the *formal twist* of $\mathcal{V}$ by λ is the pair $(\mathcal{V}, \lambda)$, denoted by $\mathcal{V}\langle\lambda\rangle$. When D is an integral Cartier divisor, the formal twist $\mathcal{V}\langle D\rangle$ is identified with $\mathcal{V} \otimes \mathcal{O}_X(D)$.

The theory of twisted *vector bundles* has pullbacks. In particular, when $\mathcal{V}$ is a vector bundle and D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and $f\colon X' \to X$ is a finite morphism such that f^*D is actually Cartier, then $f^*\mathcal{V}\langle f^*D\rangle$ is $f^*\mathcal{V} \otimes \mathcal{O}_{X'}(f^*D)$.

If $\mathcal{V}$ is a vector bundle, we have Chern classes $c_1(\mathcal{V}\langle\lambda\rangle) \coloneqq c_1(\mathcal{V}) + \operatorname{rk}\mathcal{V}\cdot\lambda$. They are natural for pullbacks.

Tensor products are defined by $\mathcal{V}\langle\lambda\rangle \otimes \mathcal{V}'\langle\lambda'\rangle \coloneqq (\mathcal{V}\otimes\mathcal{V}')\langle\lambda+\lambda'\rangle$. Generally, when we talk about extensions, subsheaves, or quotients of twisted sheaves, or about morphisms between twisted sheaves, we understand that the twist λ is fixed.

Let $\mathbb{P}(\mathcal{V}) \coloneqq \operatorname{Proj}_{\mathcal{O}_X}\operatorname{Sym}^*\mathcal{V}$. Let $\rho\colon \mathbb{P}(\mathcal{V}) \to X$ denote the natural projection map, and let ξ denote the first Chern class of the relative $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ line bundle. If λ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, define $\mathbb{P}(\mathcal{V}\langle\lambda\rangle)$ as $\mathbb{P}(\mathcal{V})$, polarized with the ρ-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $\mathcal{O}_{\mathbb{P}(\mathcal{V}\langle\lambda\rangle)}(1) \coloneqq \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)\langle\rho^*\lambda\rangle$ whose first Chern class is $\xi + \rho^*\lambda$. This is in line with the classical formula $\mathcal{O}_{\mathbb{P}(\mathcal{V}\otimes\mathcal{O}_X(D))}(1) = \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)\otimes\rho^*\mathcal{O}_X(D)$ whenever D is a Cartier divisor.

The sheaf $\mathcal{V}$ is said to be *ample* (resp. *nef*) if the Cartier divisor class ξ has the same property. This extends formally to twists.

2.2. **Slopes and positivity.** Assume that C is a smooth projective curve. Let $\mathcal{V}$ be a coherent sheaf and let λ be an $\mathbb{R}$-divisor. The *degree* of $\mathcal{V}\langle\lambda\rangle$ is $\deg\mathcal{V} + \operatorname{rk}\mathcal{V}\cdot\deg\lambda$. The *slope* of the twisted coherent sheaf $\mathcal{V}\langle\lambda\rangle$ on X is

$$\mu(\mathcal{V}\langle\lambda\rangle) \coloneqq \frac{\deg\mathcal{V}\langle\lambda\rangle}{\operatorname{rk}\mathcal{V}}.$$

By convention, the slope of torsion sheaves is infinite. If $\mathcal{V}$ and $\mathcal{V}'$ are (twisted) coherent sheaves, then

$$\mu(\mathcal{V}\otimes\mathcal{V}') = \mu(\mathcal{V}) + \mu(\mathcal{V}'). \tag{2.0.1}$$

The smallest slope of any quotient of $\mathcal{V}$ is denoted by $\mu_{\min}(\mathcal{V})$. The equality $\mu(\mathcal{V}) = \mu_{\min}(\mathcal{V})$ is equivalent to the semistability of $\mathcal{V}$. Put $\mu_{\min}(\mathcal{V}\langle\lambda\rangle) \coloneqq \mu_{\min}(\mathcal{V}) + \deg\lambda$. A quotient of $\mathcal{V}$ with minimal slope exists, and is determined by the Harder–Narasimhan filtration of $\mathcal{V}$. In characteristic 0, set $\overline{\mu}_{\min}(\mathcal{V}) \coloneqq \mu_{\min}(\mathcal{V})$. In characteristic $p > 0$, let $F\colon C \to C$ be the absolute Frobenius morphism, and consider

$$\overline{\mu}_{\min}(\mathcal{V}) \coloneqq \lim_{n\to\infty}\frac{\mu_{\min}\big((F^n)^*\mathcal{V}\big)}{p^n}.$$

The sequence in the limit is weakly decreasing and eventually stationary. In fact, [Lan04, Theorem 2.7] proves that there exists $\delta = \delta_{\mathcal{V}} \geq 0$ such that the Harder–Narasimhan filtration of $(F^{\delta+n})^*\mathcal{V}$ is the pullback of the Harder–Narasimhan filtration of $(F^\delta)^*\mathcal{V}$ for all $n \geq 0$. In particular, the rational number $\overline{\mu}_{\min}(\mathcal{V}) = \frac{\mu_{\min}((F^\delta)^*\mathcal{V})}{p^\delta}$ is the smallest normalized slope of all quotients of all iterated Frobenius pullbacks $(F^n)^*\mathcal{V}$. For twisted sheaves, put $\overline{\mu}_{\min}(\mathcal{V}\langle\lambda\rangle) = \overline{\mu}_{\min}(\mathcal{V}) + \deg\lambda$.

Lemma 2.1 ([BP14, Theorem 1.1])**.** *Let C be a smooth projective curve over an algebraically closed field. Let $\mathcal{V}\langle\lambda\rangle$ be a twisted vector bundle on C. Denote $X \coloneqq \mathbb{P}(\mathcal{V}\langle\lambda\rangle)$ with bundle map $\rho\colon X \to C$. Denote by ξ the numerical first Chern class of the relative (twisted) $\mathcal{O}_{\mathbb{P}(\mathcal{V}\langle\lambda\rangle)}(1)$ sheaf, and by f the class of a fiber of ρ. Then, we have*

$$\operatorname{Nef}(X) = \big\langle \xi - \overline{\mu}_{\min}(\mathcal{V}\langle\lambda\rangle)f, f\big\rangle.$$

In particular, $\mathcal{V}\langle\lambda\rangle$ is nef if and only if $\overline{\mu}_{\min}(\mathcal{V}\langle\lambda\rangle) \geq 0$.

The version in [BP14] holds more generally for Grassmann bundles over curves. The result was seemingly first proved by Barton [Bar71, Theorem 2.1]. It is also stated explicitly by Brenner in [Bre04, Theorem 2.3] and [Bre06, p. 534], Biswas in [Bis05, Theorem 1.1], and Zhao in [Zha17, Theorem 4.3]. In characteristic zero it follows easily from Hartshorne's Theorem [Laz04b, Theorem 6.4.15], as observed by Miyaoka [Miy87]. A similar computation is carried out by the first author in [Ful11] to nef classes of arbitrary codimension.

Remark. In positive characteristic, it is necessary to work with $\overline{\mu}_{\min}(\mathcal{V})$ instead of $\mu_{\min}(\mathcal{V})$. See [Har71, Example 3.2] for a counterexample to the naïve positive characteristic analogue of Hartshorne's Theorem [Laz04b, Theorem 6.4.15].

3. Products of curves

Let C be a smooth projective curve of genus g over $\mathbb{C}$. Let p and q denote the projections onto each factor of $C \times C$. Let f_1 denote the class of the fiber of p and f_2 the class of a fiber of q. Denote by δ the class of the diagonal Δ.

For large genera, it is a tantalizing open problem to understand the nef cone of $C \times C$, even in the symmetric slice given by intersecting with the span of $f_1 + f_2$ and δ.

3.1. Elementary considerations.

Remark 3.1 (Necessary conditions for nefness)**.** Below, a, b, and c denote non-negative real numbers.

(1) The classes f_1 and f_2 are clearly on the boundary of $\mathrm{Nef}(C \times C)$.
(2) The class $af_1 + bf_2 + c\delta$ is nef if and only if $(af_1 + bf_2 + c\delta) \cdot \delta = a + b - c(2g - 2) \geq 0$. For example, $(g-1)f_1 + (g-1)f_2 + \delta$ is the pullback of the theta polarization on the Jacobian of C via the difference map

$$\begin{aligned} C \times C &\longrightarrow \mathrm{Jac}(C) \\ (x, y) &\longmapsto \mathcal{O}_C(x - y). \end{aligned}$$

(3) If b and c are not both zero, then the class $\pm af_1 - bf_2 - c\delta$ is not nef (or even pseudo-effective), because it has negative intersection with f_1. By symmetry, the analogous statement holds for $-af_1 \pm bf_2 - c\delta$ if a and c are not both zero.
(4) The class $-af_1 - bf_2 + c\delta$ is only pseudo-effective when $a = b = 0$, and only nef when $a = b = c = 0$.

Thus, the classes that are not well-understood (up to scaling and interchanging f_1 and f_2) are those of form

(1) $af_1 + bf_2 - \delta$. By intersecting with f_1 and f_2, we get $a \geq 1$ and $b \geq 1$ as necessary conditions for these classes to be nef. By considering their self-intersections, we also have $a > 1$ and

$$b \geq b_a := 1 + \frac{g}{a-1}$$

as necessary conditions.
(2) $-af_1 + bf_2 + \delta$. Here $0 \leq a < 1$ and $b \geq \frac{g}{1-a} - 1$ are necessary conditions for the class to be nef.
(3) $af_1 - bf_2 + \delta$ with $0 \leq b < 1$ and $a \geq \frac{g}{1-b} - 1$.

Remark 3.2 (Genus $g = 1$)**.** The conditions above are also sufficient when C is an elliptic curve. See [Laz04a, Lemma 1.5.4].

Question 1.3, which asks for the nefness of the class

$$af_1 + b_a f_2 - \delta, \tag{3.2.1}$$

predicts that the conditions above are sufficient for classes of the form $af_1 + bf_2 - \delta$ for very general curves of sufficiently large genus.

3.2. Nef classes for arbitrary curves.

Remark 3.3 (Rabindranath–Vojta divisors)**.** Let C be an *arbitrary* smooth projective curve of genus $g \geq 1$. Inspired by [Voj89], Rabindranath in [Rab19, Proposition 3.2] proves that if $r, s > 0$, then $(\sqrt{(g+s)r^{-1}}+1)f_1 + (\sqrt{(g+s)r}+1)f_2 - \delta$ is nef if $r \geq \frac{(g+s)(g-1)}{s}$.We thereby deduce the nefness of the divisor

$$af_1 + \big(b_a + (g-1)(a-1)\big)f_2 - \delta \tag{3.3.1}$$

for $a > 1$. These are close to the conjectural bound (3.2.1) for a close to 1. See Figure 1.

The original argument of Vojta [Voj89] applies to the classes $-af_1 + bf_2 + \delta$ with $0 \leq a < 1$ and proves that

$$-af_1 + \left(-1 + \frac{g}{1-a} + (g-1)(1-a)\right)f_2 + \delta \text{ is nef.} \tag{3.3.2}$$

3.3. Our main results for general curves.

We construct examples of nef classes for C general. They improve the examples in Remark 3.3 that were valid for arbitrary C.

Theorem 3.4. *Let C be a* general *smooth projective curve of genus g over $\mathbb{C}$. Then*

(i) *If $g \geq 2$, then for all integers $d \geq \lfloor 3g/2 \rfloor + 1$ the divisor class*

$$df_1 + \left(1 + \frac{g}{d-g}\right)f_2 - \delta \textit{ is nef.}$$

(ii) *If $g \geq 3$, then for all integers $2g-2 \geq d \geq \lfloor 3g/2 \rfloor$ the divisor class*

$$df_1 + \frac{d}{d-g+1}f_2 - \delta \textit{ is nef.}$$

(iii) *If $g \geq 7$, then the divisor class*

$$\big(\lfloor 3g/2 \rfloor - 3\big)f_1 + \frac{\lfloor 3g/2 \rfloor - 3}{\lfloor g/2 \rfloor - 1}f_2 - \delta \textit{ is nef.}$$

(iv) *If $g \geq 7$, then the divisor class*

$$(g-1)f_1 + \frac{3g-3}{g-4}f_2 - \delta \textit{ is nef.}$$

See Figure 3 for a representation in genus 10 for how Theorem 3.4 improves Remark 3.3 for general curves. When C is an arbitrary smooth projective curve of genus $g \in \{0, 1\}$, in fact $df_1 + \big(1 + \frac{g}{d-g}\big)f_2 - \delta$ is nef for all *real* $d > 1$. See Remark 3.2. The key ingredients of our proof of Theorem 3.4 are the simple nefness criterion in Proposition 3.12,(i), and the semistability of kernel bundles and of higher order versions of kernel bundles.

Definition 3.5. If L is a Cartier divisor on C and $i \geq 0$ is an integer, we denote

$$M^{(i-1)}(L) \coloneqq q_*\big(p^*\mathcal{O}_C(L) \otimes \mathcal{O}_{C\times C}(-i\Delta)\big)$$
$$T^{i-1}(L) \coloneqq q_*\big(p^*\mathcal{O}_C(L) \otimes \mathcal{O}_{C\times C}(i\Delta)\big).$$

Denote $M_L \coloneqq M^{(0)}(L)$. It sits in an exact sequence

$$0 \longrightarrow M_L \longrightarrow H^0(X, L) \otimes \mathcal{O}_C \xrightarrow{\text{ev}} L,$$

which is why it is called a *kernel bundle* (it is also called a *syzygy bundle*, or sometimes *Lazarsfeld–Mukai bundle*). When L is globally generated, then M_L is the pullback of the twisted cotangent space $\Omega_{\mathbb{P}^r}(1)$ via the morphism induced by $|L|$. If $|L|$ is an embedding, then $M^{(1)}(L) = N_C^\vee\mathbb{P}^r \otimes \mathcal{O}_C(L)$ is a twist of the conormal bundle. [EL92], Ein and Lazarsfeld use the notation $R^{i-1}(L)$ instead of $M^{(i-1)}(L)$, and call them *higher conormal bundles*.

Proof of Theorem 3.4. (i). On any smooth curve C, the bundle M_L is semistable if L is globally generated of degree d with $d-2(h^0(C,L)-1)\le \mathrm{Cliff}(C)$. This result appears in several references, e.g., [PR88], [But94], [Cam08], [BPO09], [MS12, Theorem 1.3], or [ES12, Proposition 3.1].

When C is general, $\mathrm{Cliff}(C)=\lfloor (g-1)/2\rfloor$ by [ACGH85]. If furthermore L is general of degree $d\ge \lfloor 3g/2\rfloor+1\ge g+2$, then L is globally generated, and so is $L(-x)$ for general $x\in C$ (e.g., by [KM21, §3]). Furthermore L is non-special (i.e., $h^1(C,L)=0$), therefore $h^0(C,L)-1=d-g$, and $d-2(h^0(C,L)-1)=2g-d\le \lfloor (g-1)/2\rfloor$.

We deduce that M_L is semistable of slope $-\frac{d}{d-g}=-\left(1+\frac{g}{d-g}\right)$. By Lemma 2.1, the twisted bundle $M_L\langle \frac{d}{d-g}x\rangle$ is nef. Furthermore the natural fiberwise evaluation map $\epsilon\colon q^*M_L\to p^*L(-\Delta)$ relative to q specializes over general x to $H^0(C,L(-x))\otimes \mathcal{O}_C\to L(-x)$, hence it is surjective on the general fiber. Note that $p^*L(-\Delta)$ has degree $d-1$ on the fibers of q, hence it is relatively ample. Proposition 3.12.(i) then proves the claim.

(ii). When L is globally generated of degree d and $h^1(C,L)=1$, then $\mu(M_L)=-\frac{d}{d-g+1}$. We look for L satisfying the following properties:

(a) L is globally generated of degree d with $2g-2\ge d\ge \lfloor 3g/2\rfloor$.
(b) $h^1(C,L)=1$.
(c) $L(-x)$ is globally generated for general $x\in C$.

Note that conditions (a) and (b) together imply $d-2(h^0(C,L)-1)\le \lfloor (g-1)/2\rfloor$, which gives the semistability of M_L. In fact the inequality is strict.

Condition (b) is equivalent by Serre duality to $L=\omega_C(-E)$ for some effective divisor E with $h^0(C,E)=1$. Let $e=\deg E\ge 0$. Condition (b) is then met if we pick E with $0\le e<\mathrm{gon}(C)$. For general C, we have $\mathrm{gon}(C)=\left\lfloor \frac{g+3}{2}\right\rfloor=\left\lceil \frac{g}{2}+1\right\rceil$.

We now focus on (c). Let $p\in C$ be an arbitrary point. The bundle $L(-x)$ is generated at p if $H^1(C,L(-x-p))\to H^1(C,L(-x))$ is an isomorphism. By Serre duality, this is equivalent to the natural map $H^0(C,E+x)\to H^0(C,E+x+p)$ being an isomorphism, where $L=\omega_C(-E)$ as above. The map is in any case injective, and both spaces are at least 1-dimensional. It is sufficient to ask $h^0(C,E+x+p)=1$. As above, for C general, this is implied by $2\le e+2<\left\lfloor \frac{g+3}{2}\right\rfloor$. In fact, under this assumption, $L(-x)$ is globally generated for all x.

Note that if $0\le e\le \left\lfloor \frac{g+3}{2}\right\rfloor-3$, then $d=2g-2-e$ is in the range $2g-2\ge d\ge \lfloor 3g/2\rfloor$.

Finally, to settle (a), we want to show that for all $0\le e\le \left\lfloor \frac{g+1}{2}\right\rfloor-3$ we can find E effective of degree e with $L=\omega_C(-E)$ globally generated. Arguing as in (c), we find that any effective E will do.

(iii). For $d=\lfloor \frac{3g}{2}\rfloor-3$, we have $e=2g-2-d=\lfloor \frac{g+3}{2}\rfloor$. This is the gonality of a general curve. On such a general curve C we can pick a divisor E of degree e that is globally generated and $h^0(C,E)\ge 2$. In fact $h^0(C,E+x+p)=2$ for all $x,p\in C$ because $e+2=\lfloor \frac{g+3}{2}\rfloor+2<\frac{2g}{3}+2$, the next Brill–Noether threshold, under the assumption $g\ge 10$. As in part (ii) we deduce that $L=\omega_C(-E)$ and $L(-x)$ are globally generated for all $x\in C$. In this case $h^1(C,L)=h^0(C,E)=2$ and the inequality $d-2(h^0(C,L)-1)=e-2(h^0(C,E)-1)\le \lfloor \frac{g-1}{2}\rfloor$ that gives the semistability of M_L is in fact an equality. The divisors L and E compute the Clifford index of the curve. This is what restricts the result of (iii) to just one class.

(iv). By [CLV22], the normal bundle of a general canonical curve of genus $g\ge 7$ is semistable, i.e., $M^{(1)}(K_C)$ is semistable. Since $h^1(K_C)=1$, we have $\mu(M^{(1)}(K_C))=-\frac{6g-6}{g-4}$. The assumption $g\ge 7$ also implies that $\mathrm{gon}(C)\ge 5$. It follows that $\omega_C(-2x)$ is globally generated for all $x\in C$. Arguing as in (i) but for $M^{(1)}_{K_C}$ and $p^*\omega_C(-2\Delta)$, we deduce that $p^*K_C+\frac{6g-6}{g-4}f_2-2\delta$ is nef. □

Remark. The Rabindranath–Vojta examples (3.3.1) or the generalized Kouvidakis classes (3.11.1) for $a=2$ give $2f_1+2gf_2-\delta\in \mathrm{Nef}(C\times C)$. Theorem 3.4.(i) for $d=2g$ gives $2gf_1+2f_2-\delta\in \mathrm{Nef}(C\times C)$, which is the same class up to symmetry. For this reason, the divisors in Theorem 3.4.(i) improve the Rabindranath–Vojta examples (3.3.1) only in the range $\lfloor 3g/2\rfloor+1\le d<2g$, which is nonempty when $g \geq 3$. See Figure 3. The Mathematica software suggests that for $g \geq 32$, the nefness of the classes in Theorem 3.4.(i–iii) is a consequence of Theorem 3.4.(iv) and (3.3.1).

We show that Question 1.3 can be restated in terms of the semistability in an asymptotic sense of higher conormal bundles of C. The proof will use Proposition 3.12.(ii).

Theorem 3.6. *Let C be an* arbitrary *smooth projective curve of genus g over $\mathbb{C}$.*

(i) *If $a > 1$ is a rational number, then the class* (3.2.1) $af_1 + \big(1 + \frac{g}{a-1}\big)f_2 - \delta$ *is nef if and only if the sheaves $M^{(n-1)}(nL)$ are* asymptotically semi-stable, *i.e.,*

$$\lim_{n\to\infty} \frac{1}{n}\mu_{\min}\big(M^{(n-1)}(nL)\big) = \lim_{n\to\infty} \frac{1}{n}\mu\big(M^{(n-1)}(nL)\big),$$

where L is an arbitrary $\mathbb{Q}$-divisor on C with $\deg L = a$, *and n is sufficiently divisible.*

(ii) *If $0 \leq a < 1$ is rational, then $-af_1 + \big(-1 + \frac{g}{1-a}\big)f_2 + \delta$ is nef if and only if the sheaves $T^{n-1}(-nL)$ are asymptotically semi-stable.*

Proof. (i). Consider the q-ample class $af_1 - \delta$. Since the class (3.2.1) has self-intersection zero, it is nef if and only if $\sup\big\{t \,\big|\, af_1 - tf_2 - \delta \in \mathrm{Nef}(X)\big\} = -\big(1 + \frac{g}{a-1}\big)$. By Proposition 3.12.(ii), this holds if and only if $\frac{\mu_{\min}(M^{(n-1)}(nL))}{n}$ limits to $-1 - \frac{g}{a-1}$. Recall that L is a $\mathbb{Q}$-divisor on C of degree a. When computing the limit we restrict ourselves to n such that $na \in \mathbb{Z}$. When $a > 1$, for large divisible n (e.g., $n \geq (2g-1)/(a-1)$ by [EN18, Proposition 2.10.(1)]) we have exact sequences

$$0 \longrightarrow M^{(n-1)}(nL) \longrightarrow H^0\big(C, \mathcal{O}(nL)\big) \otimes \mathcal{O}_C \longrightarrow P^{n-1}\mathcal{O}(nL) \longrightarrow 0.$$

Recall that if $\mathcal{L}$ is a line bundle, then $P^{n-1}\mathcal{L}$ denotes the bundle of principal parts $q_*(p^*\mathcal{L} \otimes \mathcal{O}_{n\Delta})$. It is a rank n vector bundle with a natural filtration with quotients $\mathcal{L}$, $\mathcal{L} \otimes \omega_C$, ..., $\mathcal{L} \otimes \omega_C^{\otimes(n-1)}$. From this, one computes

$$\mu\big(M^{(n-1)}(nL)\big) = -n\bigg(1 + \frac{ng}{na + 1 - g - n}\bigg).$$

As n grows, $\frac{1}{n}\mu(M^{(n-1)}(nL))$ approaches $-\big(1+\frac{g}{a-1}\big)$. In particular, the nefness of $af_1+(1+\frac{g}{a-1})f_2-\delta$ is equivalent to the asymptotic semistability of $M^{(n-1)}(nL)$.

(ii). Assume now $0 \leq a < 1$, and consider the q-ample class $-af_1+\delta$. For large divisible n, pushing forward the exact sequence $0 \to p^*\mathcal{O}(-nL) \to p^*\mathcal{O}(-nL) \otimes \mathcal{O}(n\Delta) \to p^*\mathcal{O}(-nL) \otimes \mathcal{O}(n\Delta)|_{n\Delta} \to 0$ by q, we obtain an exact sequence

$$0 \longrightarrow T^{n-1}(-nL) \longrightarrow q_*\big(p^*\mathcal{O}(-nL) \otimes \mathcal{O}(n\Delta)|_{n\Delta}\big) \longrightarrow H^1\big(C, \mathcal{O}(-nL)\big) \otimes \mathcal{O}_C \longrightarrow 0.$$

From Riemann–Roch and by considering the surjections

$$q_*\big(p^*\mathcal{O}(-nL) \otimes \mathcal{O}(n\Delta)|_{(i+1)\Delta}\big) \twoheadrightarrow q_*\big(p^*\mathcal{O}(-nL) \otimes \mathcal{O}(n\Delta)|_{i\Delta}\big)$$

whose kernels are isomorphic to $\mathcal{O}(-nL) \otimes \omega_C^{\otimes(i-n)}$, one computes

$$\frac{1}{n}\mu\big(T^{n-1}(-nL)\big) = \frac{-n^2a - \binom{n+1}{2}(2g-2)}{n^2(1-a) + n(1-g)}.$$

This limits to $-\big(\frac{g}{1-a} - 1\big)$ as n grows. □

Remark. For C an *arbitrary* curve of genus g, [EN18] prove that $M^{(k)}(L)$ is semi-stable if $\deg L$ is exactly equal to $(k^2+2k+2)g + k$. This can be used to reprove the nefness of the divisors (3.3.1) when $b = 1 + \frac{1}{k+1}$ with $k \geq 0$ an integer.

3.4. **Nef classes for very general curves.** Our main result for very general curves constructs one optimal non-symmetric class, answering Question 1.3 in the affirmative for $a = 2$. The proof will use Proposition 3.8 which in turn relies on Lemma 3.9.

Theorem 3.7. *Let C be a very general smooth complex projective curve of genus $g \neq 2$. Then*

$$2f_1 + (1+g)f_2 - \delta \in \mathrm{Nef}(C \times C).$$

When $g = 2$, the class $2f_1 + (1+g)f_2 - \delta$ is not nef. It has negative intersection with the class $2f_1 + 2f_2 - \delta$ of the graph of the hyperelliptic involution.

Proof. If $g = 0$, then $2f_1 + (1+g)f_2 - \delta = f_1$ is nef (and not ample). If $g = 1$, then the class is on the boundary of the nef cone by Remark 3.2. We may assume then $g \geq 3$. The idea is to deform C to a rational curve C_0 with g simple nodes in general position. Since nefness is a very general condition in families, it is enough to prove a nefness statement for C_0. The complication introduced by the nodes is that the positivity problem to be solved is on the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $2g$ points. The construction comes from [Ros07]. We apply it to the non-symmetric situation.

Let C_0 be an irreducible rational curve with g simple nodes in general position. There exists a projective flat family $\mathcal{C} \to T$ over a disc T, relatively smooth with fibers of genus g over the punctured disk, and with central fiber C_0. We may also assume that $\mathcal{C}$ has smooth total space (e.g., by using the Hilbert scheme constructions of [DM69]). For $1 \leq i \leq g$, denote by x_i, y_i the preimages of each node in the normalization $\mathbb{P}^1$ of C_0. Let $L \subset \mathcal{C}$ be a section of $\mathcal{C} \to T$. It avoids the nodes of C_0.

We would like to construct a Cartier divisor on $\mathcal{C} \times_T \mathcal{C}$ that restricts to the general fiber with class $2f_1 + (1+g)f_2 - \delta$. It is clear that for f_1 and f_2 we will use the pullbacks of L by the two projections. However $\mathcal{C} \times_T \mathcal{C}$ is singular at the g^2 pairs of nodes, and the diagonal is not a Cartier divisor at the g diagonal pairs. Instead we blow-up $\mathcal{Y} \to \mathcal{C} \times_T \mathcal{C}$ at the g^2 pairs of nodes. This resolves the singularities, in particular those along the diagonal. Let $\mathcal{D} \subset \mathcal{Y}$ be the strict transform of the diagonal.

For $t \neq 0$ in the disk T, the fiber $\mathcal{Y}_t = C_t \times C_t$ is the self-product of a genus g curve. For $t = 0$, the fiber $\mathcal{Y}_0$ has g^2 exceptional $\mathbb{P}^1 \times \mathbb{P}^1$ components, and a component F, the strict transform of $C_0 \times C_0$. Let $\nu \colon \widetilde{F} \to F$ be the normalization. As a variety, $\widetilde{F}$ is isomorphic to the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the $4g^2$ ordered pairs of points from the list $\{x_1, y_1, \ldots, x_g, y_g\} \subset \mathbb{P}^1$. See [Ros07, Lemma 3.1] for the proofs. Denote the classes of the exceptional $\mathbb{P}^1$'s over the corresponding points by $e_{x_i x_j}, e_{x_i y_j}, e_{y_i x_j}, e_{y_i y_j}$. Let π be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$.

Let E be the sum of the g exceptional $\mathbb{P}^1 \times \mathbb{P}^1$'s sitting over diagonal pairs of nodes (p,p). Over each of the g components of E, the divisor $\mathcal{D}$ restricts with class f_1 in $N^1(\mathbb{P}^1 \times \mathbb{P}^1)$, while E restricts with class $-2f_1 - 2f_2$. By [Ros07, Lemma 3.2], we have $\nu^*(\mathcal{D}|_F) = \pi^*\Delta_{\mathbb{P}^1} - \sum_{i=1}^g (e_{x_i x_i} + e_{y_i y_i})$. Furthermore $\nu^*(E|_F) = \sum_{i=1}^g (e_{x_i x_i} + e_{y_i y_i} + e_{x_i y_i} + e_{y_i x_i})$. With p and q denoting the induced projections from $\mathcal{Y}$ on the factors of $\mathcal{C} \times_T \mathcal{C}$, consider on $\mathcal{Y}$ the Cartier divisor

$$N := p^*2L + q^*(1+g)L - (\mathcal{D} + E).$$

If we prove that its restriction $N|_{\mathcal{Y}_0}$ is nef, then the same holds for the restriction to the very general fiber. Clearly for $t \neq 0$ the fiber restriction has class $2f_1 + (1+g)f_2 - \delta \in N^1(C_t \times C_t)$.

The restriction of N to the exceptional $\mathbb{P}^1 \times \mathbb{P}^1$ components has class $f_1 + 2f_2$, so it is even ample. On the other hand, the class of $\nu^*(N|_F)$ is

$$\begin{aligned} &\pi^*\bigl(2f_1 + (1+g)f_2\bigr) - \left(\pi^*\delta - \sum_{i=1}^g (e_{x_i x_i} + e_{y_i y_i})\right) - \sum_{i=1}^g (e_{x_i x_i} + e_{y_i y_i} + e_{x_i y_i} + e_{y_i x_i}) \\ &= \pi^*(f_1 + g f_2) - \sum_{i=1}^g (e_{x_i y_i} + e_{y_i x_i}) \in N^1(\widetilde{F}). \end{aligned}$$

To settle the nefness of this class, it was enough to blow-up only the $2g$ points (x_i, y_i) and (y_i, x_i) with $1 \leq i \leq g$ on $\mathbb{P}^1 \times \mathbb{P}^1$. The conclusion follows from the result below. □

Proposition 3.8. *Consider general points $z_1, \ldots, z_g \in \mathbb{P}^1 \times \mathbb{P}^1$ with $g \neq 2$. Let $\pi \colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blow-up of the $2g$ points $z_1, \ldots, z_g$ and their reflections $z'_1, \ldots, z'_g$ across the diagonal. Denote by E the exceptional divisor. Then*

$$\pi^*(f_1 + gf_2) - E \in \mathrm{Nef}(X).$$

For all $g \geq 0$, the same nefness result holds if we blow-up $2g$ general points in $\mathbb{P}^1 \times \mathbb{P}^1$.

Proof. `Step 1. The case` $g \in \{0, 1\}$. The case $g = 0$ is trivial. When $g = 1$, then $\pi^*(f_1 + f_2) - E$ is represented by $\overline{F}_1 + \overline{F}_2$, where $\overline{F}_1$ is the strict transform of the fiber of the first projection through z_1, and $\overline{F}_2$ is the strict transform of the fiber of the second projection through z'_1. We have $\overline{F}_1^2 = \overline{F}_2^2 = -1$. We may assume that z_1 is not on the diagonal, hence $\overline{F}_1 \cdot \overline{F}_2 = 1$. In particular $\overline{F}_1 + \overline{F}_2$ has nonnegative (in fact 0) intersection with each of its irreducible components, hence it is nef.

`Step 2. The failure of the case` $g = 2$. $\mathbb{P}^2$ can be identified with the second symmetric power of $\mathbb{P}^1$. The sum map $\sigma \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ given by $\sigma(x, y) = x + y$ is a cover of degree 2, ramified on the diagonal Δ, and $\sigma^*\mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}(1, 1)$. The image $\sigma(\Delta)$ is a smooth conic. From the projection formula we deduce that the vertical and horizontal fibers through $(x, x) \in \mathbb{P}^1 \times \mathbb{P}^1$ are both mapped by σ to the tangent to $\sigma(\Delta)$ at $2x$. In particular if $z = (x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$ is general, then σ maps the horizontal and vertical fibers through z to the two tangents from $\sigma(z) = x + y$ to $\sigma(\Delta)$. The tangents to the conic are the only lines in $\mathbb{P}^2$ that have reducible preimage by σ. If $\ell \subseteq \mathbb{P}^2$ is a general line that meets $\sigma(\Delta)$ at two points, then $\sigma^{-1}\ell$ is a smooth section of $\mathcal{O}(1, 1)$ which intersects Δ at the corresponding points.

In our case, the line through $\sigma(z_1)$ and $\sigma(z_2)$ lifts to a smooth section D of $\mathcal{O}(1, 1)$ through z_1, z'_1, z_2, z'_2. Its strict transform $\overline{D}$ has class $\pi^*(f_1 + f_2) - E$ and has intersection -1 with $\pi^*(f_1 + 2f_2) - E$, so the latter is not nef. The curve $\overline{D}$ is the base locus of the linear system determined by $\pi^*\mathcal{O}(1, 2)(-E)$.

`Step 3. Conclusion of symmetric case.` Assume $g \geq 3$. By Lemma 3.9.(iii), there exists a smooth curve C_g through the g general pairs. In particular it has multiplicity 1 at each point. Its strict transform $\overline{C} \subset X$ is a curve of class $\pi^*(f_1 + gf_2) - E$, which has self intersection zero. Since it is also irreducible, it is nef.

`Step 4. Conclusion of general case.` Assume that the $2g$ points $Z = \{z_1, z_2, \ldots, z_{2g}\}$ are general (including the case $g = 2$). Consider the non-symmetric Cremona transform described in the last paragraph of part 3 in the proof of Lemma 3.9 below. Applying it at the points z_1, z_2, then at the images of z_3, z_4, and so on, reduces $\pi^*(f_1 + gf_2) - E$ to $\pi^* f_1$. By generality, for all $1 \leq i \leq g$, the images of z_{2i-1}, z_{2i} through any composition of the Cremona transforms above are never in the same vertical or horizontal fiber on $\mathbb{P}^1 \times \mathbb{P}^1$. □

Lemma 3.9 (Symmetric interpolation)**.** *Let $Z = \{z_1, z'_1, \ldots, z_m, z'_m\}$ be a set of m general symmetric pairs in $\mathbb{P}^1 \times \mathbb{P}^1$.*

- (i) *If $(n, m) \neq (1, 2)$, then the linear system of sections of $\mathcal{O}(1, n)$ through Z has the expected dimension.*
- (ii) *If $(n, m) \neq (1, 2)$ and r is a nonnegative integer, then the linear system of sections of $\mathcal{O}(1, n)$ through Z and r further general points has the expected dimension.*
- (iii) *If $(n, m) \neq (2, 2)$, and the linear system in* (i) *is nonempty, then the general divisor in this system is irreducible and smooth.*

Proof. Denote by $\mathfrak{b}_m(n)$ the linear system in question. When no confusion is likely, we omit n and denote $\mathfrak{b}_m(n) = \mathfrak{b}_m$.

1. `Continuous variation of` $\mathfrak{b}_m(n)$. For fixed n and m, we show that the linear systems $\mathfrak{b}_m$ vary continuously for general Z. The parameter space $\mathcal{Z}_m$ of ordered m-tuples of ordered symmetric pairs of points in $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to $(\mathbb{P}^1 \times \mathbb{P}^1)^m$. Let $\mathcal{U}_m \subset \mathcal{Z}_m \times (\mathbb{P}^1 \times \mathbb{P}^1)$ be the universal family. The general linear systems $\mathfrak{b}_m$ are captured by the general fibers of the sheaf $pr_{1*}(pr_2^*\mathcal{O}(1,n) \otimes \mathcal{I}_{\mathcal{U}_m})$ (or of its projectivization). Here $\mathcal{I}_{\mathcal{U}_m}$ is the ideal sheaf of $\mathcal{U}_m$. In particular, the dimension of the general $\mathfrak{b}_m$ is constant, depending only on m. By considering the ranks of subsheaves $pr_{1*}(pr_2^*\mathcal{O}(1,a) \otimes \mathcal{I}_{\mathcal{U}_m})$ and $pr_{1*}(pr_2^*\mathcal{O}(0,a) \otimes \mathcal{I}_{\mathcal{U}_m})$ for $a \le n$, one shows that the divisorial components of the base loci of $\mathfrak{b}_m$ also vary continuously for general Z.

2. `The cases` $n = 0$ `and` $n = 1$. If $n = 0$, then the projective dimension of $\mathfrak{b}_m$ is 1 for $m = 0$, and 0 for all $m > 0$. One vertical fiber cannot contain a general symmetric pair. If $n = 1$, then $\mathfrak{b}_0$ consists of plane sections of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. It has projective dimension 3. As in Step 2 of Proposition 3.8, for $Z = \{z_1, z_1'\}$, the system $\mathfrak{b}_1$ is the pullback by σ of the pencil of lines in $\mathbb{P}^2$ through $\sigma(z_1) = \sigma(z_1')$. In particular $\dim \mathfrak{b}_1 = 1$. When $m = 2$, then $\mathfrak{b}_2$ is (unexpectedly) one point, corresponding to the pullback of the line through $\sigma(z_1)$ and $\sigma(z_2)$. For $m > 2$, the system $\mathfrak{b}_m$ is empty as expected.

For general choices of Z, the divisors constructed above are irreducible.

3. `A Cremona transform on` $\mathbb{P}^1 \times \mathbb{P}^1$. Let $z \in \mathbb{P}^1 \times \mathbb{P}^1$ be a point, not on the diagonal, and let $z' \neq z$ be its reflection. Let $\rho\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be the projection from $\sigma(z) \in \mathbb{P}^2$, where $\sigma\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ is the quotient map, identified with the sum map to $\mathrm{Sym}^2\mathbb{P}^1 = \mathbb{P}^2$. Let $pr_2\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the second projection. Consider the rational map

$$Cr\colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$
$$Cr = (\rho \circ \sigma, pr_2)$$

We study some of its properties:

(1) Cr is undefined at z and z'. Indeed ρ is only undefined at $\sigma(z) = \sigma(z')$.
(2) If C is a section of $\mathcal{O}(1,1)$ through z, z', then $\rho \circ \sigma$ is constant on $C \setminus \{z, z'\}$. For this, note that $C = \sigma^{-1}L$, where L is a line through $\sigma(z)$.
(3) In particular Cr contracts the 2 fibers of pr_2 that pass through z and z' respectively. Clearly pr_2 contracts them. Let $F_{2,z}$ be the corresponding fiber of pr_2, and let $F_{1,z'}$ be the fiber of pr_1 through z'. Then $F_{1,z'} + F_{2,z}$ is a section of $\mathcal{O}(1,1)$ through z, z', hence $\rho \circ \sigma$ is constant on it. In particular it is constant on $F_{2,z}$. (Note that Cr does not also contract $F_{1,z'}$ since pr_2 does not contract it.)
(4) If $x \in \mathbb{P}^1 \times \mathbb{P}^1$ is any point different from z and z', then $Cr(x)$ and $Cr(x')$ are in the same fiber of pr_1. This is because z, z', x, x' are contained in some section of $\mathcal{O}(1,1)$.
(5) Cr is birational. For general $x \in \mathbb{P}^1 \times \mathbb{P}^1$, $\rho(\sigma(x))$ determines the section of $\mathcal{O}(1,1)$ that passes through z,z', and through x, while $pr_2(x)$ determines the fiber $F_{2,x}$. Clearly the section and the fiber meet in one point unless the fiber is a component of the section, which is not the general situation.

Finally we resolve Cr. Let $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blow-up of z and z' with exceptional divisors E and E'. Contracting the strict transforms F of $F_{2,z}$ and F' of $F_{2,z'}$, gives a morphism $\gamma\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ which is also the blow-up of two points. We have $Cr = \gamma \circ \pi^{-1}$.

Because of the two blow-up structures of X, the Néron–Severi space $N^1(X)$ has two sets of bases $(\pi^*f_1, \pi^*f_2, E, E')$ and $(\gamma^*f_1, \gamma^*f_2, F, F')$. The following relations are easy consequences of the properties of Cr.

$$\pi^*f_2 = E + F = E' + F'$$
$$\gamma^*f_2 = E + F = E' + F'$$
$$\gamma^*f_1 = \pi^*(f_1 + f_2) - E - E'$$

In particular, the change of coordinates matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}.$$

The matrix is self-inverse, though Cr is not immediately self-inverse because the source $\mathbb{P}^1 \times \mathbb{P}^1$ and the target $\mathbb{P}^1 \times \mathbb{P}^1$ are not canonically identified.

The construction of Cr also works in a less-symmetric situation. If z_1, z_2 are points not on the same horizontal or vertical fiber, then blowing-up the points and contracting the strict transforms of vertical (or of horizontal) fibers through the points gives birational $Cr\colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. In fact one can find an automorphism that fixes z_1 and sends z_2 to z_1'. When the two points are say in the same vertical fiber, and we contract the strict transforms of horizontal fibers, then the target of Cr is naturally the Hirzebruch surface $\mathbb{F}_2 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-2))$, not $\mathbb{P}^1 \times \mathbb{P}^1$.

`4. The cases` $m \in \{0,1,2\}$. Assume $n > 1$. The complete linear system $\mathfrak{b}_0$ has the expected dimension $2n+1$ and irreducible general term. For $m \in \{1,2\}$, we perform a Cremona transform Cr on $\mathbb{P}^1 \times \mathbb{P}^1$ centered at z_1, z_1'. The linear system $\mathfrak{b}_1$ corresponds to sections of $\pi^*\mathcal{O}(1,n)(-E-E') = \gamma^*\mathcal{O}(1, n-1)$ on X, so to sections of $\mathcal{O}(1, n-1)$. This gives the expected dimension of $\mathfrak{b}_1$, and the irreduciblity of a general divisor. The linear system $\mathfrak{b}_2$ similarly corresponds to sections of $\mathcal{O}(1, n-1)$ through $Cr(z_2)$ and $Cr(z_2')$. These two points live on the same vertical fiber. When $n = 2$, the sections of $\mathcal{O}(1, n-1) = \mathcal{O}(1,1)$ through $Cr(z_2), Cr(z_2')$ all contain the vertical fiber through the two points, and an arbitrary horizontal fiber, giving $\dim \mathfrak{b}_2 = 1$ as expected. We also see how irreducibility failed in this case. When $n > 2$, a general section of $\mathcal{O}(1, n-1)$ intersects the vertical fiber containing the two points in $n - 1 \geq 2$ distinct points. Using the PGL(2) action on the second component, we can arrange that one of these points is $Cr(z_2)$, but none of the others is $Cr(z_2')$, and vice versa. Thus $\dim \mathfrak{b}_2 = 2n-3$ as expected. By a similar construction, we can see that there exist irreducible sections of $\mathcal{O}(1, n-1)$ through $Cr(z_2)$ and $Cr(z_2')$, and avoiding the indeterminacy points of Cr^{-1}, hence this is the general situation.

`5. Conclusion of part` (i). Assume $n > 1$ and $m \geq 3$. For fixed $3 \leq m \leq n+1$, assume $\mathfrak{b}_{m-1}$ has the expected dimension $2(n-m+1)+1 \geq 1$, but $\dim \mathfrak{b}_m = 2(n-m+1)$ for every general choice of Z. This is indeed the only choice other than the expected dimension, easily verified by picking z_m outside the base locus of $\mathfrak{b}_{m-1}$. For any such z_m sufficiently general (so that $\dim \mathfrak{b}_m = 2(n-m+1)$ for example), consider $T \in \mathfrak{b}_{m-1}$ passing through z_m. Let C be an irreducible curve in the support of T that passes through z_m.

By our assumption that $\dim \mathfrak{b}_m = 2(n-m+1)$, for all general w in C, by replacing z_m with w, it holds that T also passes through w'. Then T contains C, but also its reflection C'. If C is symmetric, then it has class $cf_1 + cf_2$ for some $c \geq 1$. Since T has class $f_1 + nf_2$, then necessarily $c = 1$. If C is not symmetric, then similarly $C + C'$ has class $f_1 + f_2$, so C is a vertical or horizontal fiber. In both cases, denote $\widetilde{C} = C \cup C'$ (set theoretic union). It is a reduced effective symmetric cycle of class $f_1 + f_2$, a section of $\mathcal{O}(1,1)$ through z_m and z_m'. We can write $T = \widetilde{C} + F_{n-1}$, where F_{n-1} is a sum of $n-1$ horizontal fibers (each of class f_2). This is an equality of cycles, and so $\widetilde{C}$ is uniquely determined by T. Such a decomposition exists for all sufficiently general choices of the m pairs. Permuting the pairs will also produce a general ordered m-tuple of pairs, without changing $\mathfrak{b}_m$. In particular, since $\widetilde{C}$ passes through z_m and z_m', it passes through all the m pairs. This is impossible for $m \geq 3$ general pairs by the case $n = 1$.

Since $\mathfrak{b}_{n+1}$ is empty, so are $\mathfrak{b}_m$ for all $m > n$.

`6. Conclusion of part` (ii). By (i), the system $\mathfrak{b}_m$ has the expected dimension. For any nonempty linear system, passing through one general point (e.g., not in the base locus) is a codimension 1 condition. By iterating, we obtain the claim.

`7. Conclusion of part` (iii). In all cases where irreducibility holds, smoothness is automatic. Indeed any irreducible curve C of class f_1+nf_2 satisfies $C\cdot f_2=1$, hence it is mapped isomorphically by the second projection onto $\mathbb{P}^1$.

We assume $n>1$ and $m\geq 3$. By part (i), $\mathfrak{b}_m$ is nonempty precisely when $m\leq n$. In this case its dimension is $2(n-m)+1$. To prove the irreducibility of the general member of $\mathfrak{b}_m$, it is enough to prove that $\mathfrak{b}_m$ contains one irreducible curve. Since the first $m-1$ pairs in a set of m sufficiently general pairs are also general, we have $\mathfrak{b}_{m-1}\supset\mathfrak{b}_m$. It is then enough to prove that $\mathfrak{b}_n$ has an irreducible curve.

If every $C_n\in\mathfrak{b}_n$ is reducible, it is necessarily of form $C_n=C_{n-1}+F$, where C_{n-1} is a (potentially reducible) section of $\mathcal{O}(1,n-1)$, and F is a fiber of the second projection. By part (ii), the section C_{n-1} contains at most $2n-1$ of the points of Z, while F contains at most one point in Z. Since C_n passes through all of Z, the bounds must be sharp. By part (ii), for every $z\in Z$, there is exactly one section of $\mathcal{O}(1,n-1)$ that contains $Z\setminus\{z\}$. Clearly there exists exactly one F through z. There are then at most $2n$ choices for $C_n\in\mathfrak{b}_n$. This contradicts the equality $\dim\mathfrak{b}_n=1$ from (i). □

Remark (Blow-ups of $\mathbb{P}^2$)**.** Let $g\geq 1$ and let $\pi\colon X\to\mathbb{P}^2$ be the blow-up of $2g$ general points $z_1,\dots,z_{2g}$ with exceptional divisors $E_1,\dots,E_{2g}$ respectively. Let H be the class of a line in $\mathbb{P}^2$. Then

$$\pi^*gH-(g-1)E_1-E_2-\cdots-E_{2g}\in\mathrm{Nef}(X).$$

Indeed this class can be reduced by a sequence of Cremona transforms to π^*H-E_1. This result is equivalent to the general case of Proposition 3.8 via the isomorphism between $\mathbb{P}^2$ blown-up at $2g+1$ points $z_0,\dots,z_{2g}$ and $\mathbb{P}^1\times\mathbb{P}^1$ blown-up at $2g$ points. The exceptional divisor E_0 over z_0 is considered with coefficient 0 in the class above, so it can be blown-down.

See also [CHS08, Proposition 3.4] for an interesting result of similar shape that uses a degeneration construction of [Yan07].

We now construct nef classes on $C\times C$ by degenerating from simple covers of $\mathbb{P}^1$. This follows an idea of [Kou93]. Recall that a finite map $f\colon C\to\mathbb{P}^1$ is called a *simple branched cover* if any fiber of f has at most one ramification point c, and if f is given locally around any such c by the map $x\mapsto x^2$. For example hyperelliptic pencils are simple.

Example 3.10 (Simple branched covers)**.** Let C be a curve of genus $g\geq 1$. Assume that C admits a simple branched cover $f\colon C\to\mathbb{P}^1$ of degree $2\leq d\leq\lfloor\sqrt{g}\rfloor+1$.

$$\text{If } a,b\geq d,\text{ then } af_1+bf_2-\delta \text{ is nef if and only if } a+b\geq\frac{2g}{d-1}+2$$

(Following [Kou93] and [Laz04a, Theorem 1.5.8], consider T the closure of the complement of the diagonal in $C\times_{\mathbb{P}^1}C$. It is irreducible of class $df_1+df_2-\delta$ in $C\times C$. Its self-intersection is $2\cdot((d-1)^2-g)\leq 0$. For example if f is a hyperelliptic pencil, then T is the graph of the induced hyperelliptic involution. For $A,B,C\geq 0$,

$$(A+dC)f_1+(B+dC)f_2-C\delta=Af_1+Bf_2+C[T]$$

is nef if and only if the intersection with T is nonnegative, i.e., $(d-1)(A+dC+B+dC-2C)\geq 2gC$. If $C>0$, after setting $a=\frac{A}{C}+d$ and $b=\frac{B}{C}+d$, we obtain the claim.)

When $d>\lfloor\sqrt{g}\rfloor+1$, and $a,b\geq d$, the class $af_1+bf_2-\delta$ is ample. □

We obtain the following extension of the result of Kouvidakis [Kou93, Theorem 2].

Corollary 3.11. *Let C be a* very general *curve of genus $g\geq 1$. If $2\leq d\leq\lfloor\sqrt{g}\rfloor+1$ is an integer and $a,b\geq d$ satisfy $a+b\geq 2+\frac{2g}{d-1}$, then $af_1+bf_2-\delta$ is nef. In particular*

$$df_1+\left(2+\frac{2g}{d-1}-d\right)f_2-\delta\in\mathrm{Nef}(C\times C)\tag{3.11.1}$$

Proof. By the Riemann existence theorem, for any degree $d \geq 2$ there exists a curve C_d of genus g admitting a simple branched cover $C_d \to \mathbb{P}^1$ of degree d. From the previous example we deduce that $af_1 + bf_2 - \delta$ is nef. Since nefness is a very general condition in families, the result extends to very general curves. For the last statement set $a = d$ and note that $2 + \frac{2g}{d-1} - d \geq d$. □

Remark. When C is *very general*, the nefness of the classes in Theorem 3.4.(i) can be deduced from Corollary 3.11. Some of the classes in Theorem 3.4.(ii) are better, e.g., when $d = 2g-2$. However, for large g (the Mathematica software suggests $g \geq 15$), they are all in the convex span of the classes in (3.3.1) and those in Corollary 3.11.

It is conceivable that for some countable union of families of curves inside $\mathscr{M}_g$ Corollary 3.11 fails, while Theorem 3.4 does not.

3.5. **General technical results used in our proofs.**

Proposition 3.12. *Let $\rho\colon X \to C$ be a flat surjective morphism between projective varieties, where C is a nonsingular projective curve over an algebraically closed field. Let $\mathcal{L}$ be a line bundle on X, and let f be the class of a fiber of ρ.*

(i) *If $\mathcal{L}$ is nef on every fiber of ρ and relatively globally generated on a general fiber of ρ, then*
$$c_1(\mathcal{L}) - \big(\overline{\mu}_{\min}(\rho_*\mathcal{L})\big) \cdot f \ \text{ is nef on } X.$$

(ii) *If $\mathcal{L}$ is ρ-ample, then*

$$\sup\big\{t \bigm| c_1(\mathcal{L}) - tf \ \text{ is nef}\big\} = \lim_{n\to\infty} \frac{\overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes n})}{n}. \tag{3.12.1}$$

Proof. (i). The assumption implies by cohomology and base change that the natural map $\rho^*\rho_*\mathcal{L} \to \mathcal{L}$ is surjective on the general fiber of ρ. We thus have an exact complex
$$\rho^*\rho_*\mathcal{L} \longrightarrow \mathcal{L} \longrightarrow Q \longrightarrow 0,$$
where Q is supported in at most finitely many fibers of ρ. Since $\mathcal{L}$ is nef on the fibers and Q is a direct sum of quotients of $\mathcal{L}$ restricted to fibers, we deduce that Q is a nef coherent sheaf. It is invariant under twisting by classes of form ρ^*D with D an $\mathbb{R}$-divisor on C, since these are trivial on fibers. The twisted sheaf $\rho^*\rho_*\mathcal{L}\langle -\rho^*\overline{\mu}_{\min}(\rho_*\mathcal{L})f\rangle$ is nef by Lemma 2.1. The same is true of its (twisted) image in $\mathcal{L}\langle -\rho^*\overline{\mu}_{\min}(\rho_*\mathcal{L})f\rangle$. We deduce that the latter is an extension of nef twisted coherent sheaves. [FM21, Remark 3.4] and [FM21, Lemma 3.31] prove that such extensions are nef.

One can also argue by blowing-up the ideal sheaf $\mathcal{I}$ on X such that $\mathcal{I} \otimes \mathcal{L}$ is the image of the natural map $\rho^*\rho_*\mathcal{L} \to \mathcal{L}$.

(ii). We first show that the right-hand side of (3.12.1) is indeed a limit. Let n_0 be an integer such that $\rho_*\mathcal{L}^{\otimes n}$ is a vector bundle for every $n \geq n_0$, and such that the natural maps

$$\rho_*\mathcal{L}^{\otimes n} \otimes \rho_*\mathcal{L}^{\otimes m} \longrightarrow \rho_*\mathcal{L}^{\otimes(n+m)} \tag{3.12.2}$$

are surjective for all $n \geq n_0$, $m \geq n_0$. Note that such an n_0 exists by cohomology and base change and by [Laz04a, Example 1.8.4.(ii)], respectively. We then have
$$\overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes n}) + \overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes m}) = \overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes n} \otimes \rho_*\mathcal{L}^{\otimes m}) \leq \overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes(n+m)})$$
for all $n \geq n_0$, $m \geq n_0$. The equality holds by [Miy87, Corollary 3.7 and p. 464]. For the inequality, in characteristic zero use (3.12.2) and (2.0.1). In positive characteristic, the same argument works to show the inequality above after taking a large enough Frobenius pullback of the quotient map by [Lan04, Theorem 2.7]. Finally, the sequence $\{-\overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes n})\}_{n=1}^{\infty}$ is a sequence satisfying the hypothesis of de Bruijn and Erdős's version of Fekete's lemma [dBE52, Theorem 23] for the constant function
$$\varphi(t) = \max\bigg\{0, \max_{1\leq n,m\leq n_0}\Big\{\overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes n}) + \overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes m}) - \overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes(n+m)})\Big\}\bigg\}$$

in their notation, hence the right-hand side of (3.12.1) is indeed a limit.

We now show that

$$\sup\bigl\{t \bigm| c_1(\mathcal{L}) - tf \text{ is nef}\bigr\} \geq \frac{\overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes n})}{n}$$

for all n sufficiently large. By [Laz04a, Theorem 1.7.6.(iii)] and the ρ-ampleness of $\mathcal{L}$, we may assume that n is sufficiently large such that the natural map $\rho^*\rho_*\mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n}$ is surjective. We may also assume that $\rho_*\mathcal{L}^{\otimes n}$ is locally free. Choose a closed point $c \in C$. Letting $a \coloneqq -\overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes n})$, we see that

$$\sup\left\{t \;\middle|\; c_1(\mathcal{L}) + \frac{a}{n}f - tf \text{ is nef}\right\} = \frac{a}{n} + \sup\bigl\{t \bigm| c_1(\mathcal{L}) - tf \text{ is nef}\bigr\}$$

$$\frac{\overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes n}\langle ac\rangle)}{n} = \frac{a}{n} + \frac{\overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes n})}{n} = 0$$

hence it suffices to show that

$$\sup\left\{t \;\middle|\; c_1(\mathcal{L}) + \frac{a}{n}f - tf \text{ is nef}\right\} \geq 0. \tag{3.12.3}$$

Since $\overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes n}\langle ac\rangle) = 0$, we see that $\rho_*\mathcal{L}^{\otimes n}\langle ac\rangle$ is a nef twisted bundle by Lemma 2.1. Using the surjection $\rho^*\rho_*\mathcal{L}^{\otimes n} \twoheadrightarrow \mathcal{L}^{\otimes n}$, we have the commutative diagram

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}\bigl(\rho_*(\mathcal{L}^{\otimes n})\langle ac\rangle\bigr) \\ & \overset{\rho}{\searrow} & \downarrow \pi \\ & & C \end{array}$$

where $\mathcal{O}_{\mathbb{P}(\rho_*(\mathcal{L}^{\otimes n})\langle ac\rangle)}(1)$ is nef, hence so is $\mathcal{O}(1)|_X = \mathcal{L}^{\otimes n}\langle af\rangle$, and (3.12.3) follows.

It remains to show that the inequality $\leq$ holds in (3.12.1). Choose a closed point $c \in C$, and let a be sufficiently large such that $\mathcal{L}\langle af\rangle$ is ample. We then see that replacing $\mathcal{L}$ by $\mathcal{L}\langle ac\rangle$ results in both sides of (3.12.1) increasing by a, and it therefore suffices to consider the case when $\mathcal{L}$ is ample. Now let $t_0 > 0$ be the value of the supremum on the left-hand side of (3.12.1), and fix a real number $\varepsilon > 0$. Choose integers $u, v \geq 1$ such that $u/v + \varepsilon > t_0 > u/v$. We then see that

$$\frac{\overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes vn}(-unc))}{n} = -u + v \cdot \frac{\overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes n})}{n}.$$

Since $\mathcal{L}^{\otimes v}(-uf)$ is ample, Lemma 3.13 below implies

$$\lim_{n\to\infty} \frac{\overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes vn}(-unc))}{n} \geq 0$$

by the fact that $\rho_*\mathcal{L}^{\otimes vn}(-unc)$ is nef for n sufficiently large, and then by Lemma 2.1. We therefore have

$$\lim_{n\to\infty} \frac{\overline{\mu}_{\min}(\rho_*\mathcal{L}^{\otimes n})}{n} \geq \frac{u}{v} > t_0 - \varepsilon.$$

Since ε was arbitrary, the inequality $\leq$ holds in (3.12.1). □

Lemma 3.13. *Let $\rho\colon Y \to X$ be a morphism of projective schemes, and let $\mathcal{L}$ be an ample invertible sheaf on Y. Let $\mathcal{F}$ be a coherent sheaf on X. Then $\mathcal{F} \otimes \rho_*\mathcal{L}^{\otimes n}$ is ample and globally generated for all n sufficiently large.*

Proof. Let A be a very ample divisor on X such that there exists a surjection $\bigoplus \mathcal{O}_X(-A) \twoheadrightarrow \mathcal{F}$. Since ampleness and global generation descend to quotients, it is enough to prove the lemma for $\mathcal{F} = \mathcal{O}_X(-A)$. With the usual arguments of Castelnuovo–Mumford regularity [Laz04a, Theorem 1.8.5], it is enough to prove that if A is a very ample divisor on X, then $\rho_*\mathcal{L}^{\otimes n}$ is -2-regular with respect to A, i.e., $H^i\bigl(X, \rho_*\mathcal{L}^{\otimes n}(-(2+i)A)\bigr) = 0$ for all $i > 0$ for all n sufficiently large. This is because in this case $\rho_*\mathcal{L}^{\otimes n}(-2A)$ is globally generated, hence $\rho_*\mathcal{L}^{\otimes n}(-A)$ is ample and globally generated.

Since $\mathcal{L}$ is ample, it is in particular also ρ-ample. Hence for n large, we have $R^i\rho_*\mathcal{L}^{\otimes n} = 0$ for all $i > 0$. The Leray spectral sequence and the projection formula show that $H^i\big(X, \rho_*\mathcal{L}^{\otimes n}(-(2+i)A)\big) = H^i\big(Y, \mathcal{L}^{\otimes n} \otimes \rho^*(-(2+i)A)\big)$. The ampleness of $\mathcal{L}$ and Serre vanishing show that these cohomology groups are 0. □

Finally, we show that Proposition 3.12.(ii) extends to a more general setting:

Proposition 3.14. *Let $\rho\colon Y \to X$ be a morphism of projective schemes over an algebraically closed field. Let $\mathcal{L}$ be a ρ-ample line bundle on Y. For $\mathcal{F}$ a coherent sheaf on X, and H an ample line bundle on X, denote $\nu_H(\mathcal{F}) \coloneqq \sup\{t \mid \mathcal{F}\langle -tH\rangle \text{ is nef}\}$. Then*

$$\sup\big\{t \,\big|\, c_1(\mathcal{L}) - t\rho^*H \text{ is nef}\big\} = \lim_{n\to\infty} \frac{\nu_H(\rho_*\mathcal{L}^{\otimes n})}{n}.$$

Proof. The sequence $\nu_H(\rho_*\mathcal{L}^{\otimes n}) > -\infty$ is superadditive (in the weaker sense in the proof of Proposition 3.12.(ii)) by ρ-ampleness, hence the limit exists by de Bruijn and Erdős's version of Fekete's lemma [dBE52, Theorem 23]. Since $\mathcal{L}$ is ρ-ample, for sufficiently large n, we have inclusions $Y \hookrightarrow \mathbb{P}(\rho_*\mathcal{L}^{\otimes n})$ such that $\mathcal{O}_{\mathbb{P}_X(\rho_*\mathcal{L}^{\otimes n})}(1)|_Y = \mathcal{L}^{\otimes n}$. It follows that the inequality "$\geq$" holds.

For the reverse inequality, note as in Proposition 3.12.(ii) that both sides translate by t_0 when replacing $\mathcal{L}$ by $\mathcal{L}\langle t_0\rho^*H\rangle$ for $t_0 \in \mathbb{Q}$ (with the understanding that we only consider sufficiently divisible n in the right-hand side). Without loss of generality, we may assume that $\mathcal{L}$ is ample on Y. As in Proposition 3.12, we reduce to proving that $\rho_*\mathcal{L}^{\otimes n}$ is globally generated for large n, which follows from Lemma 3.13. □

4. Products of arbitrarily many factors

In this section we work over $\mathbb{C}$. Let $n \geq 2$ be an integer. It is also interesting to study $\mathrm{Nef}(C^n)$. To our knowledge, no conjecture on the shape of this cone has been made in the literature for $n \geq 3$. In fact it is quite a large cone.

Lemma 4.1. *If $g \geq 1$ and C is very general in moduli, then $N^1(C^n)$ has dimension $\binom{n+1}{2}$. A basis is given by the class f_i of the fibers of the projection from C^n onto the i-th factor for every i, and by the classes δ_{ij} of the big diagonals $\Delta_{ij} = \{(x_1,\dots,x_n) \in C^n \,\big|\, x_i = x_j\}$.*

Pulling back by the projections $pr_{ij}\colon C^n \to C^2$ gives nef classes on C^n.

Example 4.2. With assumptions as in Theorem 3.4.(i), on general curves we have

$$\frac{(n-1)d}{d-g} f_1 + d f_2 + \cdots + d f_n - \delta_{12} - \cdots - \delta_{1n} = \sum_{i=2}^{n} \bigg(\Big(1 + \frac{g}{d-g}\Big) f_1 + d f_i - \delta_{1i} \bigg) \in \mathrm{Nef}(C^n).$$

Indeed, each summand on the right is the pullback via pr_{1i} of a nef class on $C \times C$. On an arbitrary curve, using (3.3.1), we obtain $\sum_{i=2}^n \big(\big(1 + \frac{g}{a-1} + (g-1)(a-1)\big) f_1 + a f_i - \delta_{1i}\big) \in \mathrm{Nef}(C^n)$ for all $a > 1$. □

Our main result of the section is the following

Theorem 4.3. *Let C be a smooth complex projective curve of genus $g > 0$ and let $n \geq 2$. Then*

$$\frac{(n-1)d}{d-g} f_1 + d \cdot \sum_{i=2}^{n} f_i - \sum_{1\leq i<j\leq n} \delta_{ij} \in \mathrm{Nef}(C^n)$$

if one of the following holds:

(i) *$d \geq 2g + n$, or $d \geq \max\{2n + g, 2g\}$, or*
(ii) *$n \geq 2g$ and $d \geq g + n - 1$.*

The proof will make use of the rich geometry of the symmetric products of C. We recall some notation and classical results about these. Denote by

$$C_n := \operatorname{Hilb}^n C = C^n/\mathfrak{S}_n$$

the n-th symmetric product of C with quotient map $\pi\colon C^n \to C_n$. It is also the space of effective divisors D on C with $\deg D = n$. Let $\Delta = \Delta_n$ be the big diagonal on C_n, the image through π of Δ_{ij} for any $i < j$. It is the ramification locus of π, hence there exists a (non-effective) Cartier divisor on C_n denoted $\frac{\Delta_n}{2}$ such that $\pi^*\frac{\Delta_n}{2}$ is the branching divisor $\sum_{i<j}\Delta_{ij}$ and $2\cdot\frac{\Delta_n}{2} = \Delta_n$.

Let x and δ be the classes of $c_0 + C_{n-1}$ and of Δ_n respectively in $N^1(C_n)$. If C is very general, then x and δ are a basis of $N^1(C_n)$.

The cone $\operatorname{Nef}(C_n)$ has been previously studied by [Pac03]. Our methods do not offer improvements here, since we usually exploit the existence of a nonconstant map to a curve. This is in line with our results above in the case $n = 2$. Instead we focus on $\operatorname{Nef}(C\times C_{n-1})$. Any of the nef classes that we construct lift to $\mathfrak{S}_{n-1}$-symmetric (but not necessarily $\mathfrak{S}_n$-symmetric) nef classes on C^n. Consider the diagram

$$\begin{array}{ccccc} Z_{n-1} & \hookrightarrow & C\times C_{n-1} & \xrightarrow{q} & C_{n-1} \\ & & \downarrow{\scriptstyle p} & & \\ & & C & & \end{array}$$

where p and q are the two projections, and $Z = Z_{n-1}$ is the universal family $\{(c,D) \mid c \in \operatorname{Supp} D\}$. Denote by z its class in $N^1(C\times C_{n-1})$.

Lemma 4.4. *If $n \geq 3$, if $g \geq 1$ and C is very general in moduli, then $N^1(C\times C_{n-1})$ is 4-dimensional, generated by the fiber f of the first projection p, by q^*x and $q^*\delta$, and by z.*

Proof. Modulo pullbacks from either factor, a divisor on $C\times C_{n-1}$ can be identified with a morphism $C_{n-1} \to J(C)$. From the universality property of the Albanese variety, this is equivalent to an element of $\operatorname{Hom}(\operatorname{Alb}(C_{n-1}), J(C)) = \operatorname{End}(J(C))$. The latter is $\mathbb{Z}$ for C very general. We used $\operatorname{Alb}(C_{n-1}) = J(C)$. □

For L a divisor on C, consider the tautological divisors (cf. [She21, §3.1]) on C_{n-1}:

- $T_L := \pi(pr_i^*L)$, where $pr_i\colon C^{n-1} \to C$ is any of the projections. We have $\pi^*T_L = L^{\boxtimes(n-1)} := \sum_i pr_i^*L$. If $L = c_0$ is a point, then $T_L = \{c_0 + D \mid D \in C_{n-2}\} = c_0 + C_{n-2} \subset C_{n-1}$.
- $N_L := T_L - \frac{\Delta_{n-1}}{2}$. It is the determinant of the tautological bundle $E_L := q_*\mathcal{O}_{Z_{n-1}}(p^*L)$ (not to be confused with the bundle $E_L = q_*(p^*\mathcal{O}_C(L)\otimes\mathcal{O}_{C\times C}(\Delta))$ from [EL92] appearing in the previous section). For example, $K_{C_n} = N_{K_C}$.

Part (i) in the next result is an important computation for the proof of Theorem 4.3. Part (ii) will not be used, but we find that the diagram used in its proof (taken from [EL15]) is instructive. See Definition 3.5 for the definition of $M^{(m-1)}(L)$.

Lemma 4.5. *Let C be a smooth complex projective curve, and let $n \geq 2$ and $m \geq 1$. If L is a divisor on C, then*

$$pr_{1*}\mathcal{O}\Big(pr_{23\ldots n}^*L^{\boxtimes(n-1)} - m\cdot\sum_{i=2}^{n}\Delta_{1i}\Big) = \bigotimes^{n-1} M^{(m-1)}(L).$$

Consequently,

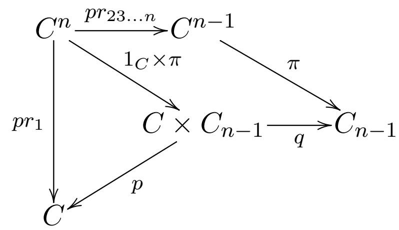
(i) $p_*\mathcal{O}(q^*T_L - mZ) = \operatorname{Sym}^{n-1} M^{(m-1)}(L)$ *and* $p_*\mathcal{O}(q^*N_L - mZ) = \bigwedge^{n-1} M^{(m-1)}(L)$.
(ii) $p_*\mathcal{O}_Z(q^*T_L) = \operatorname{Sym}^{n-2} H^0(C,L)\otimes\mathcal{O}_C(L)$ *and* $p_*\mathcal{O}_Z(q^*N_L) = \bigwedge^{n-2} M_L\otimes\mathcal{O}_C(L)$.

Proof. We can see C^n as

$$\underbrace{(C\times C)\times_C\ldots\times_C(C\times C)}_{n-1\text{ times}},$$

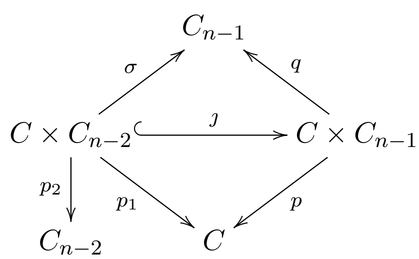
where the fiber product is always over the first projection. Then $pr_{23\ldots n}^*L^{\boxtimes(n-1)}-m\cdot\sum_{i=2}^n\Delta_{1i}$ is the relative box product pullback of $pr_2^*L-m\Delta$ from each $C\times C$ factor. Since $pr_{1*}\mathcal{O}\big(pr_2^*L-m\Delta\big)=M^{(m-1)}(L)$, the claim follows by the projection formula.

For (i) consider the diagram



The permutation group $\mathfrak{S}_{n-1}$ acts naturally on the last $n-1$ components of C^n, and trivially on the first component C and on the quotient C_{n-1}. Then the maps $1_C\times\pi$, p, and pr_1 are $\mathfrak{S}_{n-1}$-equivariant. We have $pr_{23\ldots n}^*L^{\boxtimes(n-1)}-m\cdot\sum_{i=2}^n\Delta_{1i}=(1_C\times\pi)^*(q^*T_L-mZ)$. The associated line bundle can be linearized in two natural ways via the identity and alternating representation. With these linearizations, it descends to $C\times C_{n-1}$ as q^*T_L-mZ and q^*N_L-mZ respectively. Part (i) follows from [She21, Proposition 2.3] since $\operatorname{Sym}^{n-1}M^{(m-1)}(L)$ and $\bigwedge^{n-1}M^{(m-1)}(L)$ are the respective invariants for $pr_{1*}\mathcal{O}\big(pr_{23\ldots n}^*L^{\boxtimes(n-1)}-m\sum_{i=2}^n\Delta_{1i}\big)=\bigotimes^{n-1}M^{(m-1)}(L)$ under these actions.

For part (ii), consider the commutative diagram



where $\sigma(c,D_{n-2})=c+D_{n-2}$, where p_1 and p_2 are the first and second projection, and $\jmath(c,D_{n-2})=(c,c+D_{n-2})$. The map $\jmath$ can be identified with the inclusion of the universal family Z. We compute that $\sigma^*(c_0+C_{n-2})=p_1^*(c_0)+p_2^*(c_0+C_{n-3})$ which extends to $\sigma^*T_L=p_1^*L+p_2^*T_L$, and hence

$$p_*\mathcal{O}_Z(q^*T_L)=p_{1*}\mathcal{O}(\sigma^*T_L)=H^0(C_{n-2},T_L)\otimes\mathcal{O}_C(L)$$

by the projection formula and flat base change for cohomology.

For $n=3$, we observe $\sigma^*(\frac{\Delta}{2})=\Delta_{12}=Z_1$, hence

$$p_*\mathcal{O}_Z(q^*N_L)=p_{1*}\mathcal{O}\big(p_2^*L-\Delta_{12}\big)\otimes\mathcal{O}_C(L)=M_L\otimes\mathcal{O}_C(L).$$

For $n>3$ we compute $\sigma^*(\frac{\Delta}{2})=Z_{n-2}+p_2^*\frac{\Delta_{n-2}}{2}$. Then by part (i),

$$p_*\mathcal{O}_Z(q^*N_L)=p_{1*}\mathcal{O}\left(p_2^*N_L-Z_{n-2}\right)\otimes\mathcal{O}_C(L)=\bigwedge^{n-2}M_L\otimes\mathcal{O}_C(L).\qquad\square$$

Starting with a divisor $L=L_d$ of degree d on C, the classical approach for constructing nef divisors on C_n has been to exploit the properties of tautological sheaves $E_L=q_*\mathcal{O}_Z(p^*L)$ and the divisors T_L and N_L by working with them on C_n directly (see [EL15, She21]). The semistability of E_{L_d} is known for large d (see [Mis19]). What we are missing is an understanding of the positivity of twists of E_L as is necessary for Proposition 3.14. In our approach, where we work with vector bundles on C instead, the positivity of twists is completely determined by semistability.

Proof of Theorem 4.3. In all cases we prove that

$$q^*\Big(dx - \frac{\delta}{2}\Big) - z + \frac{(n-1)d}{d-g} f \in \operatorname{Nef}(C \times C_{n-1}). \tag{4.5.1}$$

Via π^*, it then lifts to $\frac{(n-1)d}{d-g} f_1 + d f_2 + \cdots + d f_{n+1} - \sum_{1 \le i < j \le n} \delta_{ij} \in \operatorname{Nef}(C^n)$.

Recall that a divisor D on a smooth projective curve C is called k-very ample if the evaluation map $H^0(C, D) \to H^0(Y_{k+1}, D|_{Y_{k+1}})$ is surjective for all effective divisors Y_{k+1} of degree $k+1$ on C. Equivalently, the tautological bundle $E_D = q_*\mathcal{O}_{Z_{k+1}}(p^*D)$ on C_{k+1} is globally generated. In particular, if D is k-very ample, then N_D is globally generated on C_{k+1}. [CG90] prove that if D is furthermore $k+1$-very ample, then N_D is in fact very ample on the same C_{k+1}.

It is immediate that if $\deg D \ge 2g + p$, then D is p-very ample.

(i). Let L be a divisor of degree $d \ge 2g + n$. Then $L - c_0$ is $n-1$-very ample for all $c_0 \in C$, hence $N_{L-c_0} = N_L - (c_0 + C_{n-2})$ is very ample on C_{n-1}. By Lemma 4.5,

$$p_*\mathcal{O}(q^*N_L - Z) = \bigwedge^{n-1} M_L.$$

Since M_L is semistable for $d \ge 2g$ (see [EL92, Proposition 3.2]), so are its tensor powers $\bigotimes^{n-1} M_L$ by [Laz04b, Corollary 6.4.14], hence so are any direct summands of these tensor powers, e.g., $\bigwedge^{n-1} M_L$. We conclude that $\bigwedge^{n-1} M_L$ is semistable of slope $(n-1)\cdot\mu(M_L) = -\frac{(n-1)d}{d-g}$. From Proposition 3.12.(i) we find that $q^*N_L - Z + \frac{(n-1)d}{d-g}\cdot f$ is nef. Note that $A = N_L$ has class $dx - \frac{\delta}{2}$, which gives (4.5.1).

Let L be a *general* divisor of degree $d \ge \max\{2n+g, 2g\}$. Then M_L is semistable of slope $-\frac{d}{d-g}$, in particular $\bigwedge^{n-1} M_L$ is semistable of slope $-\frac{(n-1)d}{d-g}$. It remains to prove that $L - c_0$ is $n-1$-very ample for general $c_0 \in C$. This follows from Lemma 4.6 below.

(ii). Let $L = L_d$ be a divisor on C of degree d. Fix $c_0 \in C$. We want to show that $q^*N_L - Z + \frac{(n-1)d}{d-g} p^*c_0$ is nef on $C \times C_{n-1}$. We begin with a reminder on the Abel–Jacobi map:

$$\begin{aligned} u_m \colon C_m &\longrightarrow \operatorname{Pic}^0(C) = J(C) \\ D &\longmapsto \mathcal{O}_C\big(D - mc_0\big) \end{aligned} \tag{4.5.2}$$

It is a projective bundle for $m \ge 2g-1$ [ACGH85, p. 309, Proposition 2.1(i)]. Here, $c_0 \in C$ is our fixed point.

Let θ be the principal polarization on $J(C)$ obtained as the image through the Abel–Jacobi map u_{g-1}. Denoting by τ_α the translation by $\alpha \in J(C)$, it induces an isomorphism

$$\begin{aligned} \kappa \colon J(C) &\longrightarrow \operatorname{Pic}^0(J(C)) \\ \alpha &\longmapsto \tau_\alpha^*\theta - \theta \end{aligned}$$

whose inverse is $u_1^* \colon \operatorname{Pic}^0(J(C)) \to \operatorname{Pic}^0(C) = J(C)$. We make the following claims:

`Main claim.` For some large $N > 0$ and sufficiently general $\alpha_i \in \operatorname{Pic}^0(C)$ with $i \in \{1, 2, \ldots, N\}$, the natural map

$$\bigoplus_{i=1}^{N} p^*p_*\mathcal{O}(q^*N_{L+\alpha_i} - Z) \otimes \mathcal{O}(q^*T_{\alpha_i^\vee}) \longrightarrow \mathcal{O}(q^*N_L - Z) \tag{4.5.3}$$

is surjective along the general fiber of p. The map is obtained by summing the compositions $p^*p_*\mathcal{O}(q^*N_{L+\alpha_i} - Z) \otimes \mathcal{O}(q^*T_{\alpha_i^\vee}) \to \mathcal{O}(q^*N_{L+\alpha_i} - Z) \otimes \mathcal{O}(q^*T_{\alpha_i^\vee}) = \mathcal{O}(q^*N_L - Z)$.

`Claim 2.` If $\alpha \in J(C)$, then T_α is numerically trivial. In fact $T_\alpha = u^*(\kappa(\alpha))$, where $u = u_{n-1}$. (Since u is a projective bundle map, u^* is an isomorphism at the level of numerically trivial divisors. Consider the morphism

$$\begin{aligned} \imath\colon C &\longrightarrow C_{n-1} \\ c &\longmapsto c + (n-2)c_0 \end{aligned}$$

which satisfies $u_{n-1} \circ \imath = u_1$. The pullback $\imath^*\colon \mathrm{Pic}^0(C_{n-1}) \to J(C)$ is an isomorphism with inverse $\alpha \mapsto T_\alpha$. See [She21, §3.2]. The class $u^*\kappa(\alpha)$ is a numerically trivial divisor η on C_{n-1} such that $\imath^*\eta = \alpha$. The claim follows.)

`Claim 3.` If E is a divisor of degree e on C with $e \geq 2g-1$, then $R^i u_* \mathcal{O}_{C_{n-1}}(N_E) = 0$ for $i > 0$. (The pullback $u^*\theta$ has class $(g+n-2)x - \frac{\delta}{2}$, e.g., by [Pac03, Lemma 2.1]. By the projection formula, it is enough to prove that if F is a divisor of degree $f \geq -(n-1-g)$ on C, then $R^i u_* \mathcal{O}_{C_{n-1}}(T_F) = 0$ for $i > 0$. This is true because the fibers of u are projective spaces $\mathbb{P}^{n-1-g}$, and T_F restricts as $\mathcal{O}_{\mathbb{P}^{n-1-g}}(f)$ on them. These line bundles have no higher cohomology in the stated range.)

`Claim 4.` If E is a divisor of degree e on C with $e \geq 2g-1$, then $h^i(C_{n-1}, N_E) = 0$ for $i > 0$.(We have $K_{C_{n-1}} = N_{K_C}$. Then $N_E = N_{K_C} + T_{E-K_C}$. The claim follows by Kodaira vanishing.)

Let us assume for the moment that the main claim is proved. By Lemma 4.5.(i), we have $p_*\mathcal{O}(q^*N_{L+\alpha_i} - Z) = \bigwedge^{n-1} M_{L+\alpha_i}$. This is semistable of slope $-\frac{(n-1)d}{d-g}$ since $d \geq g+n-1 \geq 2g$ by [EL92, Proposition 3.2], hence $p_*\mathcal{O}(q^*N_{L+\alpha_i} - Z)\langle \frac{(n-1)d}{d-g} c_0 \rangle$ is nef by Lemma 2.1. From Claim 2, the line bundles $\mathcal{O}(q^*T_{\alpha_i^\vee})$ are numerically trivial. The twist of the LHS of (4.5.3) by $p^*\frac{(n-1)d}{d-g}c_0$ is then nef. The conclusion follows as in the proof of Proposition 3.12.(i).

`The proof of the main claim.` The surjectivity of (4.5.3) along the general fiber is implied by surjectivity on the fiber over c_0. Note that $Z|_{\{c_0\}\times C_{n-1}} = c_0 + C_{n-2} = T_{c_0}$. By cohomology and base change (using Claim 4), the fiber map is

$$\bigoplus_{i=1}^{N} H^0\big(C_{n-1}, N_{L-c_0+\alpha_i}\big) \otimes T_{\alpha_i^\vee} \longrightarrow \mathcal{O}_{C_{n-1}}(N_{L-c_0}). \tag{4.5.4}$$

The natural relative evaluation map $u^*u_*\mathcal{O}(N_{L-c_0}) \to \mathcal{O}(N_{L-c_0})$ is surjective: arguing as in Claim 3, the divisor $L - c_0$ restricts as $\mathcal{O}(d-1-(g+n-2))$ on the fibers of u, and $d-1-(g+n-2) \geq 0$. Using Claim 2, the surjectivity of (4.5.4) follows after pullback from the surjectivity of

$$\bigoplus_{i=1}^{N} H^0\big(J(C), u_*\mathcal{O}(N_{L-c_0+\alpha_i})\big) \otimes \kappa(\alpha_i^\vee) \longrightarrow u_*\mathcal{O}(N_{L-c_0}).$$

From Claim 2 and from the projection formula, this is equivalent to the surjectivity of the map

$$\bigoplus_{i=1}^{N} H^0\big(J(C), u_*\mathcal{O}(N_{L-c_0}) \otimes \kappa(\alpha_i)\big) \otimes \kappa(\alpha_i^\vee) \longrightarrow u_*\mathcal{O}(N_{L-c_0}).$$

In this form, the result is a direct application of [Par00, Corollary 2.4] if we prove that

$$H^i\big(J(C), u_*\mathcal{O}(N_{L-c_0}) \otimes \kappa(\alpha)\big) = 0$$

for all $i > 0$ and all $\alpha \in J(C)$. This is a consequence of Claims 3 and 4, and the Leray spectral sequence. □

Remark. Using the three parts of Theorem 3.4, the proof of Theorem 4.3.(i) gives slightly better results when C is a general curve.

Lemma 4.6. *Let C be a smooth projective curve of genus g, and let $0 \leq n \leq g$. Let D be a divisor of degree d on C such that $\mathcal{O}_C(D)$ is general in $\mathrm{Pic}^d(C)$. If $d \geq 2n+g+1$, then D is n-very ample.*

We follow an idea of [KM21] where the case $n = 0$ was treated.

Proof. If $d \geq 2g+n$, then any D is n-very ample by Riemann–Roch or Kodaira vanishing. Assume $d \leq 2g+n-1$ and D is not n-very ample. Then there exists Z_{n+1} an effective divisor of degree $n+1$ such that $h^1(C, D-Z_{n+1}) \neq 0$, in particular $h^0(C, K_C - D + Z_{n+1}) \neq 0$. Note that $\deg(K_C - D + Z_{n+1}) = 2g+n-1-d \geq 0$. We have $K_C - D + Z_{n+1} \sim E$, for some effective E of degree $2g+n-1-d$. The pairs (Z_{n+1}, E) move in an $2g+2n-d$-dimensional family, and $2g+2n-d \leq g-1$. For a general choice of $\mathcal{O}_C(D)$ we have $K_C - D \not\sim E - Z_{n+1}$ for any such pair (Z_{n+1}, E), which is a contradiction. □

5. Other considerations

5.1. **Product of very general distinct curves.** If C_1 and C_2 are smooth projective curves, then $\operatorname{Pic}(C_1 \times C_2) \simeq \operatorname{Pic}(C_1) \oplus \operatorname{Pic}(C_2) \oplus \operatorname{Hom}(J(C_1), J(C_2))$. If the curves are sufficiently general, there exists no nontrivial morphism between their Jacobians. In particular, $N^1(C_1 \times C_2)$ is 2-dimensional, generated by the classes f_1 and f_2 of the fibers of each projection. In this case $\operatorname{Nef}(C_1 \times C_2) = \overline{\operatorname{Eff}}(C_1 \times C_2) = \langle f_1, f_2 \rangle$.

5.2. **Restricting from larger ambient spaces.**

Remark 5.1 (Restricting from the Jacobian)**.** Let C be an *arbitrary* curve of genus $g \geq 1$. Let $J(C)$ be the Jacobian of C. The "canonical part" of the nef cone of $J(C) \times J(C)$ is essentially known by [DELV11]. By restricting these classes to $C \times C$, one finds

$$a f_1 + \left(1 + \frac{g^2}{a-1}\right) f_2 - \delta \in \operatorname{Nef}(C \times C) \ (\forall)\ a > 1.$$

See Figure 1 for a comparison with Remark 3.3.

Remark (Restricting from Hilbert schemes of K3 surfaces)**.** For (S, H) a polarized K3 surface of degree $2t$ and Picard number 1, [BM14] compute the Nef cone of $\operatorname{Hilb}^2 S$ in terms of solutions to the Pell equations $x^2 - 4ty^2 = 5$ and $x^2 - ty^2 = 1$. If $C \subset S$ is a smooth curve, one can pullback nef classes from $\operatorname{Hilb}^2 S$ to $C \times C$. The resulting examples are roughly of form $2\sqrt{g}(f_1 + f_2) - \delta$, off by a factor of 2 from Conjecture 1.2.

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