

POSITIVE WEIGHTS AND SELF-MAPS

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ABSTRACT. Spaces with positive weights are those whose rational homotopy type admits a large family of “rescaling” automorphisms. We show that finite complexes with positive weights have many genuine self-maps. We also fix the proofs of some previous related results.

1. MAIN RESULT

Following [4], who attribute the term to Morgan and Sullivan, we say that a simply connected space has *positive weights* if its rational homotopy type has a one-parameter family of “rescaling” automorphisms. A given space will often have many such families. A precise definition is given in §3.

The main result of this paper is that of any such family consisting of a $\mathbb{Q}$ ’s-worth of rational automorphisms, a $\mathbb{Z}$ ’s-worth of them can be realized as self-maps of any finite complex of that homotopy type.

Theorem A. *Let Y be a finite simply connected CW complex with positive weights, as witnessed by a one-parameter family of homomorphisms $\lambda_t : Y_{(0)} \rightarrow Y_{(0)}$. Let $\ell : Y \rightarrow Y_{(0)}$ be the rationalization map. Then there is an integer $t_0 \geq 1$ such that for every $z \in \mathbb{Z}$, there is a genuine map $f_z : Y \rightarrow Y$ whose rationalization is λ_{zt_0} , that is, such that $\ell \circ f_z \simeq \lambda_{zt_0} \circ \ell$.*

The class of spaces with positive weights is large; for example, it includes all formal spaces [20, Thm. 12.7], homogeneous spaces [4, Prop. 3.7], and smooth complex algebraic varieties [18]. Indeed, it is somewhat nontrivial to find a simply connected space which does not have positive weights. The lowest-dimensional nonexample, as far as we know, is a complex given in [17, §4] which is constructed by attaching a 12-cell to $S^3 \vee \mathbb{CP}^2$; other, much higher-dimensional non-examples are given in [2, 7, 6, 1].

We state a corollary for formal spaces which follows immediately by [20, Theorem 12.7], which states that every formal rational homotopy type has a one-parameter family of automorphisms which induces the grading automorphisms on $H^*(Y; \mathbb{Q})$ which send a class $\alpha \in H^n(Y; \mathbb{Q})$ to $t^n \alpha$.

Corollary 1.1. *Let Y be a simply connected formal finite complex. Then there is an integer $t_0 \geq 1$ such that for every $z \in \mathbb{Z}$, there is a map $f_z : Y \rightarrow Y$ which induces multiplication by $(zt_0)^n$ on $H^n(Y; \mathbb{Q})$ for every n .*

While this paper is motivated by an application of this corollary to quantitative homotopy theory, the author hopes that it will be of wider interest.

2. PRIOR WORK

The statement of Theorem A is not quite present in the literature, although a number of prior papers state similar results and give arguments which would imply this theorem. However, the author was unable to fill in the details of these arguments; this is the major motivation for this short paper.

- A slight weakening of Corollary 1.1 was originally stated by Shiga [19]. However, his proof has significant issues. In particular, the argument on the bottom of p. 432 seems to rely on the claim that, e.g., in the equation $\mathbb{Q}^2 = \mathbb{Q}(1, 1) \oplus \mathbb{Q}(-1, 1)$ one can replace $\mathbb{Q}$ with $\mathbb{Z}$, and the author was not able to fix the argument to avoid this.
- A number of similar results are discussed in [4]. The main result of that paper is that for finite complexes, the positive weight condition is equivalent to *p-universality* for any p a prime or zero. A space is *p-universal*, a notion introduced by Mimura and Toda [17], if for every $q \neq p$ it has a self-map that induces isomorphisms on mod- p homology (rational in the case $p = 0$) and the zero map on mod- q homology. In particular, 0-universality is closely related to the conclusion of Theorem A, and Proposition 3.3 and Lemma 3.4 of [4] are similar to Lemmas 5.3 and 5.1 below. However, in this case the author was again unable to complete the argument as written: the map f_q constructed in the inductive step of Lemma 3.4 depends on a choice of homotopy and it's unclear how or whether one can pick a version that would satisfy the claimed conditions. In this paper, we give an alternate proof of Proposition 3.3 of [4].
- Amann [1, Theorem 4.2] asserts a result similar to Shiga's, but for spaces with positive weights in general. However, the proof again contains a mistake: an obstruction lies in the cohomology of the wrong space. Amann has pointed out to the author that this mistake is similar to that in the published proof of [3, Lemma B.1], which has been fixed on the arXiv, and can be fixed in a similar way. This is different from our method, but could be used to give a slightly weaker form of Theorem A.

Our proof method has major similarities to those of [4] (in overall strategy) and [1] (in the use of the Moore–Postnikov tower of the rationalization map $Y \rightarrow Y_{(0)}$).

3. POSITIVE WEIGHTS

We assume knowledge of Sullivan's model of rational homotopy theory; the reader is referred to [9] or [10] for the basics.

Let Y be a simply connected space of finite type, and denote its Sullivan minimal DGA by $\mathcal{M}_Y^*$. Then Y has *positive weights* if there is a set $\{x_i\}$ of indecomposable generators of $\mathcal{M}_Y^*$ and corresponding integers $n_i \geq 1$ such that for each $t \in \mathbb{Q}$, there is a homomorphism $\lambda_t : \mathcal{M}_Y^* \rightarrow \mathcal{M}_Y^*$ such that $\lambda_t(x_i) = t^{n_i}x_i$.

Notice that when $t \neq 0$, this λ_t is an automorphism; the set $\{\lambda_t : t \in \mathbb{Q}^\times\}$ is a subgroup of the automorphism group of $\mathcal{M}_Y^*$ and is called a *one-parameter subgroup* or *family*.

Since there is an equivalence of homotopy categories between rational spaces¹ of finite type and their minimal DGAs, such an automorphism λ_t induces a homotopy automorphism of the rationalization $Y_{(0)}$, which by an abuse of notation we may also call λ_t .

Note that there are often many possible choices of basis and of the n_i . For example, given one such family λ_t any other automorphism φ of $\mathcal{M}_Y^*$, one can get a new family by conjugating λ_t by φ . Concretely, let $Y = S^2 \times S^3$, and choose:

- λ_t to be the product of degree t maps on $S_{(0)}^2$ and $S_{(0)}^3$;
- φ to be the rationalization of the map

$$S^2 \times S^3 \rightarrow S^2 \times S^3$$

which sends S^2 to itself and S^3 to $S^2 \vee S^3$ via Hopf + id_{S^3} .²

¹That is, simply connected CW complexes whose homotopy groups are rational vector spaces.

²Such a map exists because the Whitehead product $[\text{id}_{S^2}, \text{Hopf} + \text{id}_{S^3}]$ is zero in $S^2 \times S^3$.

Then $\varphi^{-1}\lambda_t\varphi$ and λ_t are different families of automorphisms.

It is clear from the definition that λ_t induces diagonalizable automorphisms on $\pi_n(Y) \otimes \mathbb{Q}$. The same is true for homology and cohomology:

Proposition 3.1. *If $\lambda_t : \mathcal{M}_Y^* \rightarrow \mathcal{M}_Y^*$ is a one-parameter family of automorphisms, then there are also bases for $H^*(Y; \mathbb{Q})$ and $H_*(Y; \mathbb{Q})$ consisting of eigenvectors of the maps induced by λ_t .*

Proof. The action of λ_t on $\mathcal{M}_Y^*$ is diagonalizable. Since λ_t sends cocycles in $\mathcal{M}_Y^*$ to cocycles, they form an invariant subspace, which is therefore also diagonalizable. This diagonalization passes to the quotient by coboundaries, giving the result for cohomology. Dualizing gives us the same result for homology. $\square$

In [4, Theorem 2.7], it is shown that the positive weight condition is independent of coefficients: a minimal $\mathbb{Q}$ -DGA has positive weights if and only if its tensor product with $\mathbb{R}$ or another larger field does. Many additional topological and algebraic properties of the positive weight condition are discussed in [8], including closure under operations such as wedge and product. Most interestingly, the condition is its own Eckmann–Hilton dual.

Remark on the definition. In the definition of a space with positive weights, the assignment of “weights”, $x_i \mapsto n_i$, extends uniquely to a second grading on $\mathcal{M}_Y^*$ that respects the multiplication. Then a space with positive weights is one which has such a second grading with respect to which the differential has degree zero. This obviously equivalent definition is the one more often given, e.g. in [4, Definition 2.1].

In [4, Proposition 2.3], this equivalence is shown over other coefficient fields.

4. COROLLARIES AND RELATED RESULTS

A useful result closely related to Theorem A shows that there are many maps between two spaces of the same positive-weight rational homotopy type:

Theorem B. *Let Y and Y' be two rationally equivalent simply connected finite complexes, with rationalizations $\ell : Y \rightarrow Y_{(0)}$ and $\ell' : Y' \rightarrow Y_{(0)}$. Let $\lambda_t : Y_{(0)} \rightarrow Y_{(0)}$ be a one-parameter family of homotopy automorphisms. Then there are maps $f : Y \rightarrow Y'$ and $g : Y' \rightarrow Y$ and $t \in \mathbb{Z}$ such that $\ell g f \simeq \lambda_t \ell$ and $\ell' f g \simeq \lambda_t \ell'$.*

We prove this along with Theorem A in the next section.

A manifold is *flexible* in the sense of Crowley and Löh [7] if it has self-maps of infinitely many degrees (or equivalently, at least one degree other than 0 and ± 1). An immediate corollary of Theorem A and Proposition 3.1 is the following result:

Corollary 4.1. *Manifolds with positive weights are flexible.*

This was previously essentially stated by Amann [1, Theorem 4.2]. Another, quicker proof is implicit in a recent paper of Costoya, Muñoz, and Viruel [5, Theorem 3.2].

Finally, we explore simple quantitative implications of our results. Given finite complexes X and Y with a piecewise Riemannian metric, the *growth function* $g_{[X,Y]}(L)$ of the set $[X, Y]$ of homotopy classes of maps $X \rightarrow Y$ is the number of classes that have representatives of with Lipschitz constant at most L , as a function of L . This notion was first studied by Gromov [11, 12, 13]. While the definition uses the metrics on X and Y , the asymptotics of this function depend only on the homotopy types of the two spaces. Indeed, in [15, §6] it was

shown, based on the results of [4], that if X and Y have positive weights, then the growth function only depends on their rational homotopy type.

In fact, $g_{[X,Y]}(L)$ is always bounded by a polynomial in L when Y is simply connected or, more generally, nilpotent [15, Corollary 4.7]. On the other hand, since $[X, Y]$ is more or less the set of solutions to a system of diophantine equations, general lower bounds are hard to come by. However, for spaces with positive weights, Theorem A provides such a lower bound:

Theorem C. *Suppose that Y is a simply connected finite complex with positive weights. Then the growth function $g_{[X,Y]}(L)$ is bounded below by L^r for some rational r .*

Proof. By Theorem A there is a sequence of maps $f_z : Y \rightarrow Y$ realizing $\lambda_{zt_0} : \mathcal{M}_Y^* \rightarrow \mathcal{M}_Y^*$ for every $z \in \mathbb{Z}$. The latter induce maps on the $\mathbb{R}$ -minimal DGA of Y which we likewise call λ_{zt_0} . Let $m_Y : \mathcal{M}_Y^*(\mathbb{R}) \rightarrow \Omega^*Y$ be a minimal model for the differential forms on Y . By the shadowing principle [14, Theorem 4–1], we can find a map homotopic to f_z with Lipschitz constant controlled by a notion of “size” of the homomorphism $m_Y \lambda_{zt_0} : \mathcal{M}_Y^*(\mathbb{R}) \rightarrow \Omega^*Y$. Specifically, put a norm on the vector space $V_k = \text{Hom}(\pi_k(Y), \mathbb{R})$ of indecomposables in $\mathcal{M}_Y^*(\mathbb{R})$ for each $k \leq \dim Y$, and for every $\varphi : \mathcal{M}_Y^*(\mathbb{R}) \rightarrow \Omega^*Y$ let

$$\text{Dil}(\varphi) = \max_{k \in \{2, \dots, \dim Y\}} \|\varphi|_{V_k}\|_{\text{op}}^{1/k}.$$

This measurement depends on the choices of which elements we consider indecomposable, of norms and of m_Y , but only up to a multiplicative constant. In particular,

$$\text{Dil}(m_Y \lambda_{zt_0}) \leq \max\{C(Y)(zt_0)^{n_i/\dim x_i} \mid \deg(x_i) \leq \dim Y\}.$$

By the shadowing principle, this means that we can choose f_z so that

$$\text{Lip } f_z \leq C'(Y) [\max\{(zt_0)^{n_i/\dim x_i} \mid \deg(x_i) \leq \dim Y\} + 1],$$

and so $g_{[Y,Y]}(L) \geq L^{\min\{\dim x_i/n_i \mid \deg(x_i) \leq \dim Y\}}$. $\square$

5. PROOF OF THEOREMS A AND B

In this section, let Y be a simply connected finite complex equipped with a rationalization map $\ell : Y \rightarrow Y_{(0)}$ and a one-parameter family of automorphisms $\lambda_t : \mathcal{M}_Y^* \rightarrow \mathcal{M}_Y^*$ which induce maps $Y_{(0)} \rightarrow Y_{(0)}$ which we also call λ_t .

We prove Theorems A and B using a series of lemmas.

Lemma 5.1. *For every n , there is a complex K_n and a rational equivalence $q_n : K_n \rightarrow Y_{(0)}$ with the following properties:*

- (i) *For $m \leq n$, $\pi_m(K_n)$ is free abelian.*
- (ii) *For each prime p , there is a map $r_{p,n} : K_n \rightarrow K_n$ such that $q_n \circ r_{p,n} \simeq \lambda_p \circ q_n$. Moreover, the induced map on $\pi_m(K_n)$, $m \leq n$, has a $\mathbb{Z}$ -eigenbasis.*
- (iii) *For $m > n$, $\pi_m(K_n)$ is a $\mathbb{Q}$ -vector space, and therefore $q_{n*} : \pi_m(K_n) \rightarrow \pi_m(Y_{(0)})$ is an isomorphism.*

Proof. We will construct the K_n as successive stages of a Moore–Postnikov tower with base $Y_{(0)}$. That is, we take $K_1 = Y_{(0)}$ (in which case the base case is trivially true) and then construct a tower

$$\begin{array}{ccc} K_{n+1} & & \\ \downarrow f_{n+1} & \searrow q_{n+1} & \\ K_n & \xrightarrow{q_n} & Y_{(0)} \end{array}$$

such that the homotopy fiber of f_{n+1} is a $K(\pi, n)$ (note the nonstandard indexing). In particular, any such construction automatically satisfies (iii).

Now suppose we have constructed K_n and maps $q_n : K_n \rightarrow Y_{(0)}$ and $r_{p,n} : K_n \rightarrow K_n$ satisfying (i)–(iii). We will now construct the next stage of the Moore–Postnikov tower, as well as maps $r_{p,n+1}$ which are lifts of $r_{p,n}$ along f_{n+1} , in the sense that

$$f_{n+1} \circ r_{p,n+1} \simeq r_{p,n} \circ f_n.$$

Since $\pi_n(K_n)$ is free abelian and we would like the same for $\pi_{n+1}(K_{n+1})$, we get

$$\pi_{n+1}(K_n, K_{n+1}) \cong \mathbb{Q}^d / \mathbb{Z}^d,$$

where d is the rank of $\pi_{n+1}(Y)$. Therefore, to fix K_{n+1} , it suffices to specify a k -invariant $\kappa \in H^{n+1}(K_n; (\mathbb{Q}/\mathbb{Z})^d)$ for the pullback diagram

$$\begin{array}{ccc} K_{n+1} & \longrightarrow & \mathcal{P}K((\mathbb{Q}/\mathbb{Z})^d, n+1) \\ f_{n+1} \downarrow & & \downarrow \\ K_n & \xrightarrow{\kappa} & K((\mathbb{Q}/\mathbb{Z})^d, n+1). \end{array}$$

Since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$ -module,

$$H^{n+1}(K_n; \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}(H_{n+1}(K_n), \mathbb{Q}/\mathbb{Z}).$$

Therefore we can think of κ as a homomorphism $H_{n+1}(K_n) \rightarrow (\mathbb{Q}/\mathbb{Z})^d$. Moreover, the composition $\kappa \circ h : \pi_{n+1}(K_n) \rightarrow (\mathbb{Q}/\mathbb{Z})^d$, where h is the Hurewicz homomorphism, fits into the long exact sequence of homotopy groups

$$\cdots \rightarrow \pi_{n+1}(K_n) \xrightarrow{\kappa \circ h} (\mathbb{Q}/\mathbb{Z})^d \rightarrow \pi_n(K_{n+1}) \rightarrow \pi_n(K_n) \rightarrow 0 \rightarrow \cdots.$$

Since we would like $\pi_n(K_{n+1}) \cong \pi_n(K_n)$, $\kappa \circ h$ needs to be surjective.

Denote the n th Postnikov stage of K_n by $(K_n)_n$. To compute $H_{n+1}(K_n)$, we apply the Serre spectral sequence to the map $K_n \rightarrow (K_n)_n$, whose homotopy fiber is an n -connected rational space W with $H_{n+1}(W) \cong \mathbb{Q}^d$. This gives us a short exact sequence

$$0 \rightarrow A \rightarrow H_{n+1}(K_n) \rightarrow H_{n+1}((K_n)_n) \rightarrow 0$$

where $A = \text{coker}(d : H_{n+2}((K_n)_n) \rightarrow H_{n+1}(W))$. Since $(K_n)_n$ is of finite type, the first term is $\mathbb{Q}^d$ modulo a finitely generated subgroup, and the last term is finitely generated. In particular, the first term is an injective $\mathbb{Z}$ -module, so the sequence splits.

Now in order to pick the desired κ , we would like to understand the action of the various $r_{p,n}$ on $H_{n+1}(K_n)$. Specifically, we would like to pick κ so that $\text{im}(\kappa \circ r_{p,n*}) \subseteq \text{im}(\kappa)$. Then since $\kappa \circ f_{n+1*} = 0$ by construction, we also get $\kappa \circ r_{p,n*} \circ f_{n+1*} = 0$, and therefore there is a lift

$$\begin{array}{ccccc} K_{n+1} & \xrightarrow{r_{p,n+1}} & K_{n+1} & \longrightarrow & \mathcal{P}K((\mathbb{Q}/\mathbb{Z})^d, n+1) \\ f_{n+1} \downarrow & & f_{n+1} \downarrow & & \downarrow \\ K_n & \xrightarrow{r_{p,n}} & K_n & \xrightarrow{\kappa} & K((\mathbb{Q}/\mathbb{Z})^d, n+1). \end{array}$$

The automorphism λ_t induces diagonalizable linear transformations with eigenvalues t^α for various integers α on both $\pi_*(K_n) \otimes \mathbb{Q}$ and $H_*(K_n; \mathbb{Q})$. In particular, we can choose a basis of eigenvectors x_i for $\mathbb{Q}^d$, as well as additional eigenvectors $y_j \in H_{n+1}(K_n)$ which, together with those x_i whose images in $A \otimes \mathbb{Q}$ are nonzero, form a basis for $H_{n+1}(K_n; \mathbb{Q})$. The y_j , together with a choice of splitting for the torsion elements, determine a splitting

$s : H_{n+1}((K_n)_n) \rightarrow H_{n+1}(K_n)$. The eigendecomposition determines the action of $r_{p,n}$ on homology except for its action on the torsion subgroup $B \subseteq H_{n+1}((K_n)_n)$. However, the image of B will always be contained in the finite subgroup $B \oplus \{a \in A : |B|a = 0\}$.

Now we fix κ . Write the codomain as $\bigoplus_{i=1}^d (\mathbb{Q}/\mathbb{Z})e_i$. Then we set $\kappa \circ s = 0$ and, for $q \in \mathbb{Q}$, $\kappa(qx_i) = Nqe_i$, where N is large enough that $\{a \in A : |B|a = 0\}$ is sent to zero. As a result,

$$\kappa \circ r_{p,n*}(s(H_{n+1}((K_n)_n))) = 0 \quad \text{and} \quad \kappa \circ r_{p,n*}(x_i) = p^{n_i}x_i,$$

and therefore $\text{im}(\kappa \circ r_{p,n*}) \subseteq \text{im}(\kappa)$. This shows that $r_{p,n} \circ f_{n+1}$ lifts to a map $r_{p,n+1}$. Moreover, the generators x_i/N of

$$\pi_{n+1}(K_{n+1}) = \ker(\kappa \circ h) = \bigoplus_i \langle \frac{1}{N}x_i \rangle$$

form a $\mathbb{Z}$ -eigenbasis for $(r_{p,n+1})_* : \pi_{n+1}(K_{n+1}) \rightarrow \pi_{n+1}(K_{n+1})$. $\square$

Lemma 5.2. *Given K_n and $r_{p,n}$ as in Lemma 5.1, there is a power of $r_{p,n}$ which induces the zero map on $H_*(K_n; \mathbb{Z}/p\mathbb{Z})$ for all $* \leq n$.*

Proof. This is essentially the direction (b') $\Rightarrow$ (b) of [16, Theorem 2.1]. $\square$

Lemma 5.3. *There is a finite complex K and a rational equivalence $q : K \rightarrow Y_{(0)}$ with the following properties:*

- (i) *For each $m \leq \dim Y$, $\pi_m(K)$ is free abelian, and for each $m > \dim Y$, $H_m(K) = 0$.*
- (ii) *For each prime p , there is a map $r_p : K \rightarrow K$ such that $q \circ r_p \simeq \lambda_p \circ q$. Moreover, for every prime $p' \neq p$, r_p is a p' -equivalence, i.e. it induces isomorphisms on $H^*(K; \mathbb{Z}/p'\mathbb{Z})$.*
- (iii) *For each prime p , there is a power s_p of r_p which induces the zero map on $H_*(K; \mathbb{Z}/p\mathbb{Z})$.*

Proof. Let n be the dimension of Y , and let K_n be as in Lemma 5.1. Since $(K_n)_n$ is of finite type, we can build a finite n -complex K' with a map $\iota' : K' \rightarrow K_n$ which induces isomorphisms on H_m for every $m < n$ and a surjection on H_n . In particular, there is no obstruction to homotoping the map $r_{p,n} \circ \iota' : K' \rightarrow K_n$ so that its image lands $\iota'(K')$; we can then lift this to a map $r'_p : K' \rightarrow K'$.

Now, by the Hurewicz theorem, since the homotopy fiber of ι' is $(n-1)$ -connected, the Hurewicz map $\pi_{n+1}(K_n, K') \rightarrow H_{n+1}(K_n, K')$ is surjective. Therefore we can add $(n+1)$ -cells to K' which kill the kernel of $\iota'_* : H_n(K') \rightarrow H_n(K_n)$. Moreover, since K' is n -dimensional, $H_n(K')$ is free abelian, and so is this kernel; thus we can do this without adding any homology in degree $n+1$. We call the resulting $(n+1)$ -complex K ; by construction it satisfies (i).

Note that $H^{n+1}(K; \pi)$ is zero integrally and rationally, but may be nontrivial with torsion coefficients. In particular, ι' extends uniquely over the $(n+1)$ -cells of K , since $\pi_{n+1}(K_n)$ is a $\mathbb{Q}$ -vector space. This gives us a map $\iota : K \rightarrow K_n$, and we then set $q = q_n \circ \iota$. This is a rational equivalence since ι induces an isomorphism on H_m for $m \leq n$ and both K and $Y_{(0)}$ are homologically trivial in degrees $> n$. Finally, r'_p extends to a map $r_p : K \rightarrow K$, and this extension is unique up to torsion; therefore $q \circ r_p \simeq \lambda_p \circ q$.

Since the homotopy groups of K_n are either rational or free abelian, and the maps induced by $r_{p,n}$ on the free abelian groups $\pi_m(K_n)$, $m \leq n$, have an eigenbasis whose vectors are multiplied by powers of p , $r_{p,n}$ induces isomorphisms on $\pi_*(K_n) \otimes \mathbb{Z}/p'\mathbb{Z}$ for every prime $p' \neq p$. By the mod p' Hurewicz theorem, $r_{p,n}$ also induces isomorphisms on $H_*(K_n; \mathbb{Z}/p'\mathbb{Z})$. Now, ι' induces isomorphisms on mod p' homology in degrees $m < n$ and a surjection in degree n , and factors into the inclusion $K \hookrightarrow K_n$ and the homology isomorphism ι . Since r'_p induces isomorphisms in degrees $m < n$ and on the quotient of the surjection in degree n , r_p induces isomorphisms on all mod p' homology and cohomology groups.

Condition (iii) comes directly from Lemma 5.2, since r_p induces the same map on $H_{\leq n}$ as $r_{p,n}$, and $H_{>n}(K) = 0$. $\square$

Lemma 5.4. *There is a map $f : Y \rightarrow K$ which commutes with λ_t after rationalization; more precisely, $q \circ f \simeq \lambda_t \circ \ell$ for some t .*

Proof. Let Z be an infinite telescope of mapping cylinders build using copies of K and maps

$$r_2, r_2, r_3, r_2, r_3, r_5, \dots$$

Now consider the map $\hat{q} : Z \rightarrow Y_{(0)}$ extending q on the first copy of K . Let's call the inclusion map of this copy i_1 , so then $\hat{q} \circ i_1 \simeq q$.

By Lemma 5.3(i) and (ii), $\hat{q}$ induces isomorphisms on π_m for $m \leq n$. Thus, by the Hurewicz theorem, $\hat{q}$ also induces isomorphisms on H_m for $m \leq n$. On the other hand, for $m > n$,

$$H_m(Z) \cong H_m(Y_{(0)}) \cong 0.$$

Therefore $\hat{q}$ is a homotopy equivalence, and so there is a map $\hat{\ell} : Y \rightarrow Z$ such that $\hat{q} \circ \hat{\ell} \simeq \ell$. Since Y is compact, this map lands in a finite set of mapping cylinders, and therefore we can homotope it into a single copy of K . Let's call the inclusion map of this copy i_t , where $t = p_1 \cdots p_N$ such that

$$i_t \simeq R_t \circ i_1 = r_{p_N} \circ \cdots \circ r_{p_1} \circ i_1.$$

The resulting map is f . To see that $q \circ f \simeq \lambda_t \circ \ell$, consider the diagram

$$\begin{array}{ccccc} & & \ell & & \\ & & \curvearrowright & & \\ Y & \xrightarrow{\hat{\ell}} & Z & \xrightarrow{\hat{q}} & Y_{(0)} \\ & \nearrow i_t & \uparrow i_1 & \nearrow q & \downarrow \lambda_t \\ & K & & & \\ f \downarrow & \nwarrow R_t & & & \\ K & \xrightarrow{q} & Y_{(0)} & & \end{array}$$

The triangles all commute up to homotopy by construction, and the bottom right square by Lemma 5.3(ii). Moreover, every map in the diagram is a rational equivalence; in other words, after rationalization, every arrow is reversible. This implies that the outer square commutes rationally. But since the target $Y_{(0)}$ is a rational space, it commutes integrally as well. $\square$

Lemma 5.5. *There is a map $g : K \rightarrow Y$ such that $g \circ f$ realizes λ_{t_0} for some integer t_0 , i.e. $\ell \circ g \circ f \simeq \lambda_{t_0} \circ \ell$.*

Proof. We again use the proof of [16, Theorem 2.1]. That theorem asserts the equivalence of several conditions for a finite simply connected CW complex K , including:

- (b) For any prime p , there is a map $s_p : K \rightarrow K$ which induces the zero map on $H^*(K; \mathbb{Z}/p\mathbb{Z})$.
- (a) Given a rational equivalence $f : Y \rightarrow X$ between two CW complexes and a map $h : K \rightarrow X$, there are maps $g : K \rightarrow Y$ and $k : K \rightarrow K$ completing the diagram

$$\begin{array}{ccc} K & \xrightarrow{k} & K \\ \downarrow g & & \downarrow h \\ Y & \xrightarrow{f} & X. \end{array}$$

We showed in Lemma 5.3 that K satisfies (b) and that we can take s_p to be a power of r_p . The proof that (b) implies (a) goes through several steps, but the resulting map $k : K \rightarrow K$ is always a composition of s_p for various p . Applying (a) with $X = K$, $h = \text{id}$, and f the map from Lemma 5.4, we get a map $g : K \rightarrow Y$ such that $f \circ g : K \rightarrow K$ is a composition of various r_p , whose product is, let's say, t_0 . Then

$$\ell g f \simeq \lambda_{t_0} \ell g f \simeq \lambda_{t_0} \lambda_{t_0} \ell g f \simeq \lambda_{t_0} \lambda_{t_0} \lambda_{t_0} \ell g f \simeq \lambda_{t_0} \ell. \quad \square$$

Proof of Theorem A. For any $z \in \mathbb{Z}$, let $r_z : K \rightarrow K$ be the composition of the r_p 's in its prime decomposition. Then $g \circ r_z \circ f$ is a map realizing the automorphism λ_{zt_0} . $\square$

Proof of Theorem B. Using Lemmas 5.4 and 5.5, we construct maps

$$\begin{array}{ccc} Y & \xrightarrow{f} & K \xrightarrow{g} Y \\ Y' & \xrightarrow{f'} & K \xrightarrow{g'} Y' \end{array}$$

such that $\ell g f \simeq \lambda_t \ell$ and $\ell' g' f' \simeq \lambda_{t'} \ell'$. Then $g' f : Y \rightarrow Y'$ and $f g' : Y' \rightarrow Y$ are the desired maps. $\square$

Finally, the fact that the r_p are p' -equivalences for every $p \neq p'$ completes an alternate proof of Proposition 3.3 in [4]. This can be used to recover the theorem that spaces with positive weights are p -universal for every p .

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