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Linearized boundary control method for an acoustic inverse boundary value problem

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Abstract

We develop a linearized boundary control method for the inverse boundary value problem of determining a potential in the acoustic wave equation from the Neumann-to-Dirichlet map. When the linearization is at the zero potential, we derive a reconstruction formula based on the boundary control method and prove that it is of Lipschitz-type stability. When the linearization is at a nonzero potential, we prove that the problem is of Hölder-type stability in two and higher dimensions. The proposed reconstruction formula is implemented and evaluated using several numerical experiments to validate its feasibility.

Keywords: acoustic wave equation, inverse boundary value problem, Neumann-to-Dirichlet map, boundary control method

(Some figures may appear in colour only in the online journal)

1. Introduction

The paper is concerned with the linearized inverse boundary value problem (IBVP) for the acoustic wave equation with a potential. The goal is to derive stability estimates and reconstruction procedures to numerically compute a small perturbation of a known underlying potential, given the Neumann-to-Dirichlet map on the boundary of the domain.

Formulation. Specifically, let $T > 0$ be a constant and $\Omega \subset \mathbb{R}^n$ be a bounded open subset with smooth boundary $\partial\Omega$. Consider the following boundary value problem for the acoustic

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wave equation with potential:

$$\begin{cases} \square_{c,q}u(t, x) = 0, & \text{in } (0, 2T) \times \Omega \\ \partial_\nu u = f, & \text{on } (0, 2T) \times \partial\Omega \\ u(0, x) = \partial_t u(0, x) = 0 & x \in \Omega. \end{cases} \quad (1)$$

Here $\square_{c,q}$ is a linear wave operator defined as

$$\square_{c,q}u(t, x) := \partial_t^2 u(t, x) - c^2(x)\Delta u(t, x) + q(x)u(t, x);$$

$c(x) \in C^\infty(\overline{\Omega})$ is a smooth wave speed that is strictly positive, $q(x) \in L^\infty(\Omega)$ is a real-valued function. We denote this solution by $u^f(t, x)$ when it is necessary to specify the Neumann data.

Given $f \in C_c^\infty((0, 2T) \times \partial\Omega)$, the well-posedness of this problem is ensured by the standard theory for second order hyperbolic partial differential equations [20]. As a result, the following Neumann-to-Dirichlet (ND) map is well defined:

$$\Lambda_{c,q}f := u^f|_{(0,2T) \times \partial\Omega}. \quad (2)$$

The IBVP concerns recovery of the wave speed $c(x)$ and/or the potential $q(x)$ from knowledge of the ND map $\Lambda_{c,q}$, that is to invert the nonlinear map $(c, q) \mapsto \Lambda_{c,q}$.

Literature. The IBVP has been studied in the mathematical literature for a long time. The first coefficient determination result for a multidimensional wave equation, given the ND map, is by Rakesh and Symes [31]. They proved that a spatially varying potential is uniquely determined by the ND map in the case that $c = 1$. To our knowledge their method has never been implemented computationally.

For variable c and $q = 0$, Belishev [3] proved that the leading order coefficient c is uniquely determined using the boundary control (BC) method combined with Tataru's unique continuation result [37]. The method has since been extended to many wave equations. We mention [7] for a generalization to Riemannian manifolds, and [23] for a result covering all symmetric time-independent lower order perturbations of the wave operator. Non-symmetric, time-dependent and matrix-valued lower order perturbations were recovered in [19, 24] and [25], respectively. For a review of the method, we refer to [5, 21].

The BC method has been implemented numerically to reconstruct the wave speed in [4], and subsequently in [6, 16, 30, 38]. The implementations [4, 6, 16] involve solving unstable control problems, whereas [30, 38] are based on solving stable control problems but with target functions exhibiting exponential growth or decay. The exponential behaviour leads to instability as well. On the other hand, the linearized approach introduced in the present paper is stable. It should be mentioned that the BC method can be implemented in a stable way in the one-dimensional case, see [22]. For an interesting application of a variant of the method in the one-dimensional case, see [11] on detection of blockage in networks.

Under suitable geometric assumptions, it can be proven that the problem to recover the speed of sound is Hölder stable [32, 33], even when the speed is given by an anisotropic Riemannian metric. Moreover, a low-pass version of c can be recovered in a Lipschitz stable manner [27]. The problem to recover q is Hölder stable assuming, again, that the geometry is nice enough [8, 28, 36]. To our knowledge, the method, based on using high frequency solutions to the wave equation and yielding the latter three results, has not been implemented computationally. Stability results applicable to general geometries have been proven using the BC method in [2], with an abstract modulus continuity, and very recently in [12, 13], with a doubly logarithmic modulus of continuity.

Finally, let us mention that recently there has been a lot of activity related to recovery of time-dependent coefficients in wave equations, see for example [1, 34], and the references therein for up-to-date results. The first result in the time-dependent context was obtained in [35].

Linearization. Throughout the paper, we will assume the wave speed $c = c_0(x)$ is known. Here $c_0(x) \in C^\infty(\bar{\Omega})$ is a smooth wave speed that is strictly positive. Then the IBVP only concerns recovery of the potential q .

We will study the linearization of this problem by linearizing the map $q \mapsto \Lambda_{c_0, q}$. Here, we describe a formal way to derive the linearized IBVP. The rigorous proof is included in the appendix, where we show the nonlinear map $q \mapsto \Lambda_q$ is indeed Frechét differentiable and the Frechét derivative agrees with the result of the formal process.

For the formal derivation, we write

$$q(x) = q_0(x) + \epsilon \dot{q}(x), \quad u(t, x) = u_0(t, x) + \epsilon \dot{u}(t, x)$$

where q_0 is a known background potential and u_0 is the background solution. Insert these into (1). Equating the $O(1)$ -terms gives

$$\begin{cases} \square_{c_0, q_0} u_0(t, x) = 0, & \text{in } (0, 2T) \times \Omega \\ \partial_\nu u_0 = f, & \text{on } (0, 2T) \times \partial\Omega \\ u_0(0, x) = \partial_t u_0(0, x) = 0, & x \in \Omega. \end{cases} \quad (3)$$

Equating the $O(\epsilon)$ -terms gives

$$\begin{cases} \square_{c_0, q_0} \dot{u}(t, x) = u_0(t, x) \dot{q}(x), & \text{in } (0, 2T) \times \Omega \\ \partial_\nu \dot{u} = 0, & \text{on } (0, 2T) \times \partial\Omega \\ \dot{u}(0, x) = \partial_t \dot{u}(0, x) = 0 & x \in \Omega. \end{cases} \quad (4)$$

Write the ND map $\Lambda_q = \Lambda_{q_0} + \epsilon \dot{\Lambda}_{\dot{q}}$, where Λ_{q_0} is the ND map for the unperturbed boundary value problem (3), and $\dot{\Lambda}_{\dot{q}}$ is defined as

$$\dot{\Lambda}_{\dot{q}} : f \mapsto \dot{u}|_{(0, 2T) \times \partial\Omega}. \quad (5)$$

Note that the unperturbed problem (3) can be explicitly solved to obtain u_0 and Λ_{q_0} , since c_0 and q_0 are known. As before, we will write $\dot{u} = \dot{u}^f$ if it is necessary to specify the Neumann data f . Then the linearized IBVP concerns recovery of the potential $\dot{q}$ from $\dot{\Lambda}_{\dot{q}}$.

Main results. The main contribution of this paper consists of several results regarding reconstruction of $\dot{q}$ based on the BC method. For constant c_0 and $q_0 = 0$, we derive a reconstruction formula for $\dot{q}$ in theorem 5, and prove that the reconstruction is of Lipschitz-type stability in theorem 7. For constant c_0 and $q_0 \neq 0$, we derive a reconstruction formula in theorem 9, and prove that the reconstruction is of Hölder-type stability in theorem 10. The formula is implemented and validated using numerical experiments, showing reliable reconstructions.

Paper structure. The paper is organized as follows. Section 2 reviews a few fundamental concepts and results in the BC theory. Section 3 consists of an integral identity and a controllability result that are essential to the development of our method. Section 4 establishes stability estimates and reconstruction formulae for the linearized IBVP, which are the central results of the paper. Section 5 is devoted to numerical implementation of the reconstruction formulae as well as numerical experiments to assess performance of the proposed reconstructions.

2. Preliminaries

Introduce some notations: given a function $u(t, x)$, we write $u(t) = u(t, \cdot)$ for the spatial part as a function of x . Introduce the time reversal operator $R : L^2([0, T] \times \partial\Omega) \rightarrow L^2([0, T] \times \partial\Omega)$,

$$Ru(t, \cdot) := u(T - t, \cdot), \quad 0 < t < T; \quad (6)$$

and the low-pass filter $J : L^2([0, 2T] \times \partial\Omega) \rightarrow L^2([0, T] \times \partial\Omega)$

$$Jf(t, \cdot) := \frac{1}{2} \int_t^{2T-t} f(\tau, \cdot) d\tau, \quad 0 < t < T. \quad (7)$$

We write $P_T : L^2((0, 2T) \times \partial\Omega) \rightarrow L^2((0, T) \times \partial\Omega)$ for the orthogonal projection via restriction, that is, for a function $f \in L^2((0, 2T) \times \partial\Omega)$, $P_T f = f|_{(0, T) \times \partial\Omega}$ is the restriction of f onto $(0, T) \times \partial\Omega$. The adjoint operator $P_T^* : L^2((0, T) \times \partial\Omega) \rightarrow L^2((0, 2T) \times \partial\Omega)$ is the extension by zero from $(0, T)$ to $(0, 2T)$. Let $\mathcal{T}_D$ and $\mathcal{T}_N$ be the Dirichlet and Neumann trace operators respectively, that is,

$$\mathcal{T}_D u(t, \cdot) = u(t, \cdot)|_{\partial\Omega}, \quad \mathcal{T}_N u(t, \cdot) = \partial_\nu u(t, \cdot)|_{\partial\Omega}.$$

Introduce the *connecting operator*

$$K := J\Lambda_q P_T^* - R\Lambda_{q,T} RJP_T^* \quad (8)$$

where $\Lambda_{q,T} f := P_T(\Lambda_q f)$. Then the following Blagoveščenskiĭ's identity holds [9, 10, 15, 29]. Here, the weighted space $L^2(\Omega, c_0^{-2} dx)$ is defined as

$$L^2(\Omega, c_0^{-2} dx) := \left\{ u : \int_{\Omega} |u(x)|^2 c_0^{-2}(x) dx < \infty \right\}.$$

Proposition 1. *Let u^f, u^h be the solutions of (1) with Neumann traces $f, h \in L^2((0, T) \times \partial\Omega)$, respectively. Then*

$$(u^f(T), u^h(T))_{L^2(\Omega, c_0^{-2} dx)} = (f, Kh)_{L^2((0, T) \times \partial\Omega)} = (Kf, h)_{L^2((0, T) \times \partial\Omega)}. \quad (9)$$

We remark that the functions $f, h \in L^2((0, T) \times \partial\Omega)$ in this proposition need to be extended as zero from $(0, T)$ to $(0, 2T)$ when solving the boundary value problem (1). Similar convention will be applied henceforth for imposed Neumann data.

Proof. We first prove this for $f, h \in C_c^\infty((0, T) \times \partial\Omega)$. Define

$$I(t, s) := (u^f(t), u^h(s))_{L^2(\Omega, c_0^{-2} dx)}.$$

We compute

$$\begin{aligned} & (\partial_t^2 - \partial_s^2)I(t, s) \\ &= ((\Delta + qc_0^{-2})u^f(t), u^h(s))_{L^2(\Omega)} - (u^f(t), (\Delta + qc_0^{-2})u^h(s))_{L^2(\Omega)} \\ &= (f(t), \Lambda_q P_T^* h(s))_{L^2(\partial\Omega)} - (\Lambda_q P_T^* f(t), h(s))_{L^2(\partial\Omega)}, \end{aligned} \quad (10)$$

where the last equality follows from integration by parts. On the other hand, $I(0, s) = \partial_t I(0, s) = 0$ since $u^f(0, x) = \partial_t u^f(0, x) = 0$. Solve the inhomogeneous 1D wave equation (10)

together with these initial conditions to obtain

$$\begin{aligned} I(T, T) &= \frac{1}{2} \int_0^T \int_t^{2T-t} [(f(t), \Lambda_q P_T^* h(\sigma))_{L^2(\partial\Omega)} - (\Lambda_q P_T^* f(t), h(\sigma))_{L^2(\partial\Omega)}] d\sigma dt \\ &= \int_0^T \left[\left(f(t), \frac{1}{2} \int_t^{2T-t} \Lambda_q P_T^* h(\sigma) d\sigma \right)_{L^2(\partial\Omega)} - \left(\Lambda_q P_T^* f(t), \frac{1}{2} \int_t^{2T-t} h(\sigma) d\sigma \right)_{L^2(\partial\Omega)} \right] dt \\ &= (f, J\Lambda_q P_T^* h)_{L^2((0,T) \times \partial\Omega)} - (P_T(\Lambda_q f), JP_T^* h)_{L^2((0,T) \times \partial\Omega)}. \end{aligned}$$

Using the relations $P_T(\Lambda_q f) = \Lambda_{q,T} f$ and $\Lambda_{q,T}^* = R\Lambda_{q,T}R$ in $L^2((0, T) \times \partial\Omega)$, we have

$$\begin{aligned} I(T, T) &= (f, J\Lambda_q P_T^* h)_{L^2((0,T) \times \partial\Omega)} - (\Lambda_{q,T} f, JP_T^* h)_{L^2((0,T) \times \partial\Omega)} \\ &= (f, J\Lambda_q P_T^* h)_{L^2((0,T) \times \partial\Omega)} - (f, R\Lambda_{q,T}RJP_T^* h)_{L^2((0,T) \times \partial\Omega)} \\ &= (f, Kh)_{L^2((0,T) \times \partial\Omega)} \end{aligned}$$

For general $f, h \in L^2((0, T) \times \partial\Omega)$, simply notice that K is a continuous operator and that compactly supported smooth functions are dense in L^2 . The proof is completed. $\square$

Corollary 2. Suppose $f, h \in C_c^\infty((0, T] \times \partial\Omega)$. Then

$$(c_0^2 \Delta u^f(T) - qu^f(T), u^h(T))_{L^2(\Omega, c_0^{-2} dx)} = (\partial_t^2 f, Kh)_{L^2((0,T) \times \partial\Omega)} = (K\partial_t^2 f, h)_{L^2((0,T) \times \partial\Omega)}. \quad (11)$$

Proof. Replacing f by $\partial_t^2 f$ in (9), we get

$$(u^{\partial_t^2 f}(T), u^h(T))_{L^2(\Omega, c_0^{-2} dx)} = (\partial_t^2 f, Kh)_{L^2((0,T) \times \partial\Omega)} = (K\partial_t^2 f, h)_{L^2((0,T) \times \partial\Omega)}.$$

As both $u^{\partial_t^2 f}$ and $\partial_t^2 u^f$ satisfy (1) with f replaced by $\partial_t^2 f$, they must be equal thanks to the well-posedness of the boundary value problem. We conclude

$$u^{\partial_t^2 f} = \partial_t^2 u^f = c_0^2 \Delta u^f - qu^f.$$

$\square$

Recall that we write $\Lambda_q = \Lambda_{q_0} + \epsilon \dot{\Lambda}_{\dot{q}}$ in the linearization setting. Accordingly, we write $K = K_0 + \epsilon \dot{K}$. Here K_0 is the connecting operator for the background medium:

$$K_0 := J\Lambda_{q_0} P_T^* - R\Lambda_{q_0, T} RJP_T^*. \quad (12)$$

K_0 can be explicitly computed since Λ_{c, q_0} is known. $\dot{K}$ is the resulting perturbation in the connecting operator:

$$\dot{K} := J\dot{\Lambda}_{\dot{q}} P_T^* - R\dot{\Lambda}_{\dot{q}, T} RJP_T^*. \quad (13)$$

$\dot{K}$ can be explicitly computed once $\dot{\Lambda}_{\dot{q}}$ is given.

3. Integral identity and controllability

First, we derive an integral identity that is essential to the development of the reconstruction procedure. We write $\dot{\Lambda}$ for $\dot{\Lambda}_{\dot{q}}$ when there is no risk of confusion.

Proposition 3. Let $\lambda \in \mathbb{R}$ be a real number. If $f, h \in C_c^\infty((0, T] \times \partial\Omega)$ satisfy

$$[\Delta - q_0 c_0^{-2} + \lambda c_0^{-2}]u_0^f(T) = [\Delta - q_0 c_0^{-2} + \lambda c_0^{-2}]u_0^h(T) = 0 \quad \text{in } \Omega, \quad (14)$$

then the following identity holds:

$$-(\dot{q}c_0^{-2}u_0^f(T), u_0^h(T))_{L^2(\Omega)} = (\partial_t^2 f + \lambda f, \dot{K}h)_{L^2((0,T) \times \partial\Omega)} + (\dot{\Delta}f(T), h(T))_{L^2(\partial\Omega)} \quad (15)$$

Proof. For $f, h \in C_c^\infty((0, T) \times \partial\Omega)$, we will make use of (9) and (11) to obtain some identities. First, let $\epsilon = 0$ in (9) to obtain

$$(u_0^f(T), u_0^h(T))_{L^2(\Omega, c_0^{-2} dx)} = (f, K_0 h)_{L^2((0,T) \times \partial\Omega)} = (K_0 f, h)_{L^2((0,T) \times \partial\Omega)}.$$

Next, differentiate (9) in ϵ and let $\epsilon = 0$ to get

$$\begin{aligned} (f, \dot{K}h)_{L^2((0,T) \times \partial\Omega)} &= (\dot{K}f, h)_{L^2((0,T) \times \partial\Omega)} \\ &= (i^f(T), u_0^h(T))_{L^2(\Omega, c_0^{-2} dx)} + (u_0^f(T), i^h(T))_{L^2(\Omega, c_0^{-2} dx)}. \end{aligned} \quad (16)$$

Similarly, let $\epsilon = 0$ in (11) we obtain

$$\begin{aligned} (c_0^2 \Delta u_0^f(T) - q u_0^f(T), u_0^h(T))_{L^2(\Omega, c_0^{-2} dx)} &= (\partial_t^2 f, K_0 h)_{L^2((0,T) \times \partial\Omega)} \\ &= (K_0 \partial_t^2 f, h)_{L^2((0,T) \times \partial\Omega)}. \end{aligned}$$

Next, differentiating (11) in ϵ and let $\epsilon = 0$, we obtain

$$\begin{aligned} &(\partial_t^2 f, \dot{K}h)_{L^2((0,T) \times \partial\Omega)} \\ &= (\dot{K} \partial_t^2 f, h)_{L^2((0,T) \times \partial\Omega)} = (c_0^2 \Delta i^f(T) - \dot{q} u_0^f(T) - q_0 i^f(T), u_0^h(T))_{L^2(\Omega, c_0^{-2} dx)} \\ &\quad + (c_0^2 \Delta u_0^f(T) - q_0 u_0^f(T), i^h(T))_{L^2(\Omega, c_0^{-2} dx)} \\ &= \underbrace{(\Delta i^f(T), u_0^h(T))_{L^2(\Omega)} - (q_0 c_0^{-2} i^f(T), u_0^h(T))_{L^2(\Omega)}}_{:= I_1} - (\dot{q} c_0^{-2} u_0^f(T), u_0^h(T))_{L^2(\Omega)} \\ &\quad + \underbrace{(\Delta u_0^f(T), i^h(T))_{L^2(\Omega)} - (q_0 c_0^{-2} u_0^f(T), i^h(T))_{L^2(\Omega)}}_{:= I_2} \end{aligned} \quad (17)$$

where $L^2(\Omega) = L^2(\Omega, dx)$ is the L^2 -space equipped with the usual Lebesgue measure.

For I_1 , we integrate the first term by parts and use the fact that $\partial_\nu \dot{u} = 0$ and $i^f|_{(0,2T) \times \partial\Omega} = \dot{\Delta}f$ to get

$$\begin{aligned} I_1 &= (\Delta i^f(T), u_0^h(T))_{L^2(\Omega)} - (q_0 c_0^{-2} i^f(T), u_0^h(T))_{L^2(\Omega)} \\ &= (i^f(T), \Delta u_0^h(T))_{L^2(\Omega)} - (\dot{\Delta}f(T), \partial_\nu u_0^h(T))_{L^2(\partial\Omega)} - (q_0 c_0^{-2} i^f(T), u_0^h(T))_{L^2(\Omega)} \\ &= (i^f(T), [\Delta - q_0 c_0^{-2}]u_0^h(T))_{L^2(\Omega)} - (\dot{\Delta}f(T), h(T))_{L^2(\partial\Omega)}. \end{aligned}$$

On the other hand, combining the terms in I_2 we have

$$I_2 = ([\Delta - q_0 c_0^{-2}]u_0^f(T), i^h(T))_{L^2(\Omega)}.$$

Insert these expressions for I_1 and I_2 into (17), then add (16) multiplied by $\lambda \in \mathbb{R}$ to get

$$\begin{aligned}
& (\partial_t^2 f + \lambda f, \dot{K}h)_{L^2((0,T) \times \partial\Omega)} + (\dot{\Delta} f(T), h(T))_{L^2(\partial\Omega)} \\
&= (\dot{u}^f(T), [\Delta - q_0 c_0^{-2} + \lambda c_0^{-2}] u_0^h(T))_{L^2(\Omega)} - (\dot{q} c_0^{-2} u_0^f(T), u_0^h(T))_{L^2(\Omega)} \\
&+ ([\Delta - q_0 c_0^{-2} + \lambda c_0^{-2}] u_0^f(T), \dot{u}^h(T))_{L^2(\Omega)}. \tag{18}
\end{aligned}$$

If $[\Delta - q_0 c_0^{-2} + \lambda c_0^{-2}] u_0^f(T) = [\Delta - q_0 c_0^{-2} + \lambda c_0^{-2}] u_0^h(T) = 0$ in Ω , the first term and last term on the right-hand side vanish, resulting in (15). $\square$

Next, we establish a BC estimate. Given a strictly positive $c_0 \in C^\infty(\overline{\Omega})$, we will write $g := c_0^{-2} dx^2$ for the Riemannian metric associated to c_0 , and denote by $S\overline{\Omega}$ the unit sphere bundle over the closure $\overline{\Omega}$ of Ω .

Proposition 4. *Let $c_0 \in C^\infty(\overline{\Omega})$ be strictly positive and $q_0 \in C^\infty(\overline{\Omega})$. Suppose that all maximal³ geodesics on $(\overline{\Omega}, g)$ have length strictly less than some fixed $T > 0$. Then for any $\phi \in C_c^\infty(\overline{\Omega})$ there is $f \in C_c^\infty((0, T] \times \partial\Omega)$ such that*

$$u_0^f(T) = \phi \quad \text{in } \Omega, \tag{19}$$

where u_0 is the solution of (3). Moreover, there is $C > 0$, independent of ϕ , such that

$$\|f\|_{H^2((0,T) \times \partial\Omega)} \leq C \|\phi\|_{H^4(\Omega)}. \tag{20}$$

Proof. There is small $\delta > 0$ such that the maximal geodesics have length less than $T^* := T - 2\delta$. We extend c_0 and q_0 smoothly to $\mathbb{R}^n$. Then there is a compact domain $\mathcal{K}$ with smooth boundary such that $\overline{\Omega}$ is contained in the interior of $\mathcal{K}$ and that the extended tensor $g = c_0^{-2} dx^2$ gives a Riemannian metric on $\mathcal{K}$. Let $x \in \overline{\Omega}$ and let $\xi \in S_x \overline{\Omega}$, that is, ξ is a unit vector with respect to g at x . Write $\gamma_{x,\xi} : [a, b] \rightarrow \overline{\Omega}$ for the maximal geodesic with the initial data (x, ξ) . Then $b < T^*$. We extend $\gamma_{x,\xi}$ as the maximal geodesic in $\mathcal{K}$ and write $\gamma_{x,\xi} : [\hat{a}, \hat{b}] \rightarrow \mathcal{K}$. Then there are $t_j > b$ such that $t_j \rightarrow b$ and $\gamma_{x,\xi}(t_j) \notin \overline{\Omega}$. If $\hat{b} < T^*$ we extend $\gamma_{x,\xi}$ to $[\hat{a}, T^*]$ by setting $\gamma_{x,\xi}(t) = \gamma_{x,\xi}(\hat{b})$ for $t > \hat{b}$. Then the function

$$\rho(x, \xi) := \max\{d(\gamma_{x,\xi}(t), \overline{\Omega}) : t \in [0, T^*]\}$$

satisfies $\rho(x, \xi) > 0$. Let us show that ρ is continuous. It follows from the smoothness of the geodesic flow and the triangle inequality that the function

$$S\overline{\Omega} \times [0, T^*] \times \overline{\Omega} \ni (x, \xi, t, y) \mapsto d(\gamma_{x,\xi}(t), y)$$

is uniformly continuous. By lemma 12 in the appendix, the function

$$S\overline{\Omega} \times [0, T^*] \ni (x, \xi, t) \mapsto d(\gamma_{x,\xi}(t), \overline{\Omega}),$$

is uniformly continuous, and so is ρ .

We have shown that the continuous function ρ is strictly positive on the compact set $S\overline{\Omega}$. Thus there is an open set ω_0 such that $\overline{\omega_0} \subset \mathcal{K} \setminus \overline{\Omega}$ and that all geodesics $\gamma_{x,\xi}$ with $(x, \xi) \in S\overline{\Omega}$ intersect ω_0 in time T^* .

We choose $\eta \in C_c^\infty(0, T)$ taking values in $[0, 1]$ and satisfying $\eta = 1$ on $[\delta, T - \delta]$, and $\chi \in C_c^\infty(\mathcal{K} \setminus \overline{\Omega})$ satisfying $\chi = 1$ on ω_0 . Finally, we extend ϕ (the extension will be denoted

³ For a maximal geodesic $\gamma : [a, b] \rightarrow \overline{\Omega}$ there may exist $t \in (a, b)$ such that $\gamma(t) \in \partial\Omega$. The geodesics are maximal on the closed set $\overline{\Omega}$.

by ϕ again) so that $\phi \in C_c^\infty(\mathcal{K})$. Now [18, theorem 5.1] implies that there is $Y \in C^\infty((0, T) \times (\mathcal{K} \setminus \bar{\Omega}))$ and $v \in C^\infty((0, T) \times \mathcal{K})$ such that

$$\begin{cases} \square_{c_0, q_0} v = \eta \chi Y & \text{in } (0, T) \times \text{int } \mathcal{K} \\ v|_{x \in \partial \mathcal{K}} = 0, \\ v|_{t=T} = \phi, \quad \partial_t v|_{t=T} = 0, \\ v|_{t=0} = 0, \quad \partial_t v|_{t=0} = 0. \end{cases} \quad (21)$$

We extend v to $(0, 2T) \times \mathcal{K}$ by solving

$$\begin{cases} \square_{c_0, q_0} v = 0 & \text{in } (T, 2T) \times \text{int } \mathcal{K} \\ v|_{x \in \partial \mathcal{K}} = 0, \\ v|_{t=T} = \phi, \quad \partial_t v|_{t=T} = 0, \end{cases}$$

and set $f = \partial_\nu v|_{x \in \partial \Omega}$. As $\eta(t) = 0$ when t is close to zero and as v satisfies vanishing initial conditions at $t = 0$, also $f(t, x) = 0$ for t near zero.

It remains to show estimate (20). The extension of ϕ can be chosen so that

$$\|\phi\|_{H^4(\mathcal{K})} \leq C \|\phi\|_{H^4(\Omega)}.$$

By [18, theorem 5.1] there is a map

$$G: H^4(\mathcal{K}) \rightarrow H^3(\mathcal{K}) \times H^2(\mathcal{K}), \quad G(\phi) = (Y|_{t=0}, \partial_t Y|_{t=0}), \quad (22)$$

and Y satisfies

$$\begin{cases} \square_{c_0, q_0} Y = 0 & \text{in } (0, T) \times \text{int } \mathcal{K} \\ Y|_{x \in \partial \mathcal{K}} = 0. \end{cases}$$

As continuity of (22) is not explicitly stated in [18], we will show the continuity for the convenience of the reader.

Due to the closed graph theorem it is enough to show that if $\phi_j \rightarrow \phi$ in $H^4(\mathcal{K})$ and $G(\phi_j) \rightarrow (Y_0, Y_1)$ in $H^3(\mathcal{K}) \times H^2(\mathcal{K})$ then $G(\phi) = (Y_0, Y_1)$. We define v_j and Y_j by

$$\begin{cases} \square_{c_0, q_0} v_j = \eta \chi Y_j, & \text{in } (0, T) \times \text{int } \mathcal{K} \\ v_j|_{x \in \partial \mathcal{K}} = 0, \\ v_j|_{t=0} = 0, \quad \partial_t v_j|_{t=0} = 0, \end{cases} \quad \begin{cases} \square_{c_0, q_0} Y_j = 0 & \text{in } (0, T) \times \text{int } \mathcal{K} \\ Y_j|_{x \in \partial \mathcal{K}} = 0, \\ (Y_j|_{t=0}, \partial_t Y_j|_{t=0}) = F(\phi_j), \end{cases}$$

Then $v_j \rightarrow v$ and $Y_j \rightarrow Y$ where v and Y solve

$$\begin{cases} \square_{c_0, q_0} v = \eta \chi Y & \text{in } (0, T) \times \text{int } \mathcal{K} \\ v|_{x \in \partial \mathcal{K}} = 0, \\ v|_{t=0} = 0, \quad \partial_t v|_{t=0} = 0, \end{cases} \quad \begin{cases} \square_{c_0, q_0} Y = 0, & \text{in } (0, T) \times \text{int } \mathcal{K} \\ Y|_{x \in \partial \mathcal{K}} = 0, \\ (Y|_{t=0}, \partial_t Y|_{t=0}) = (Y_0, Y_1), \end{cases}$$

and also

$$v_j|_{t=T} = \lim_j v_j|_{t=T} = \lim_j \phi_j = \phi, \quad \partial_t v|_{t=T} = \lim_j \partial_t v_j|_{t=T} = 0.$$

But the control given by [18, theorem 5.1] for ϕ is characterized by the following system

$$\begin{cases} \square_{c_0, q_0} v = \eta \chi Y & \text{in } (0, T) \times \text{int } \mathcal{K} \\ v|_{x \in \partial \mathcal{K}} = 0, \\ v|_{t=0} = \phi, \quad \partial_t u|_{t=0} = 0, \\ v|_{t=T} = 0, \quad \partial_t u|_{t=T} = 0, \end{cases} \quad \begin{cases} \square_{c_0, q_0} Y = 0 & \text{in } (0, T) \times \text{int } \mathcal{K} \\ Y|_{x \in \partial \mathcal{K}} = 0, \end{cases}$$

see [14]. Hence the control Y that we obtained as the limit coincides with that given by [18, theorem 5.1], and $G(\phi) = (Y_0, Y_1)$.

By [18, theorem 5.1] the initial data $G(\phi)$ satisfies suitable compatibility conditions so that $Y \in H^3((0, T) \times \mathcal{K})$. Moreover, solving (21) from the initial data at $t = 0$, while discarding the constant at $t = T$, gives

$$\|v\|_{H^4((0, T) \times \mathcal{K})} \leq C \|Y\|_{H^3((0, T) \times \mathcal{K})} \leq C \|G(\phi)\|_{H^3(\mathcal{K}) \times H^2(\mathcal{K})} \leq C \|\phi\|_{H^4(\mathcal{K})}.$$

□

Remark. Solvability of an equation like (19) is a central question in the BC theory. There are other results which ensure the solvability in other function spaces. For example, if we define

$$W_0 : L^2((0, T) \times \partial \Omega) \rightarrow L^2(\Omega, c_0^{-2} dx), \quad W_0 f = u^f(T).$$

This clearly is a bounded linear operator. Moreover, if $T > 0$ is large enough, the range $\text{Ran}(W_0)$ is dense in $L^2(\Omega)$ by Tataru's unique continuation [37]. If we further assume that the continuous observability condition [27] holds for the background wave solution, then W_0 is surjective, that is, $\text{Ran}(W_0) = L^2(\Omega)$. This ensures the existence of solutions $f, h \in L^2((0, T) \times \partial \Omega)$ for the equation (23). However, such L^2 -regularity is insufficient for our purpose, as we need $\partial_t^2 f$ to exist, see (24).

4. Stability and reconstruction

We will make use of propositions 3 and 4 to derive stability estimate and reconstruction formulae for $\dot{q}$, on the premise that the wave speed is constant. Without loss of generality, we take $c_0 = 1$. The discussion is separate for vanishing and non-vanishing background potentials q_0 .

4.1. Case 1: $c_0 = 1$ and $q_0 = 0$

We take $\lambda \geq 0$ and dimension $n \geq 1$, then the equation (14) for $u_0^f(T)$ and $u_0^h(T)$ becomes the Helmholtz equation

$$(\Delta + \lambda)u_0^f(T) = (\Delta + \lambda)u_0^h(T) = 0 \quad \text{in } \Omega.$$

A class of Helmholtz solutions are the plane waves $e^{i\sqrt{\lambda}\theta \cdot x}$ where $\theta \in \mathbb{S}^{n-1}$ is an arbitrary unit vector. Moreover, proposition 4 ensures the existence of $f, h \in C_c^\infty((0, T] \times \partial \Omega)$ such that

$$u_0^f(T) = u_0^h(T) = e^{i\sqrt{\lambda}\theta \cdot x}. \quad (23)$$

Theorem 5. Suppose $c_0 = 1$ and $q_0 = 0$. Then the Fourier transform $\hat{q}$ of $\dot{q}$ can be reconstructed as follows:

$$\hat{q}(2\sqrt{\lambda}\theta) = -(\partial_t^2 f + \lambda f, \dot{K}h)_{L^2((0, T) \times \partial \Omega)} - (\dot{\Delta} f(T), h(T))_{L^2(\partial \Omega)} \quad (24)$$

where $f, h \in C_c^\infty((0, T] \times \partial\Omega)$ are solutions to (23).

Proof. The formula is obtained by inserting $u_0^f(T) = u_0^h(T) = e^{i\sqrt{\lambda}\theta \cdot x}$ into (15). As $\lambda \geq 0$ and θ are arbitrary, it recovers the Fourier transform of $\dot{q}$ everywhere. $\square$

Remark 6. An explicit procedure to solve for f and h from (23) is explained in section 5.1.

Next, we show that the reconstruction above has Lipschitz-type stability. As the inverse problem is linear, it suffices to bound $\dot{q}$ using continuous functions of $\dot{\Lambda}$.

Theorem 7. Suppose $c_0 = 1$ and $q_0 = 0$. There exists a constant $C > 0$, independent of λ , such that

$$\left| \hat{q}(\sqrt{2\lambda}\theta) \right| \leq C(2 + \lambda)\lambda^4 \left(\|\dot{K}\|_{L^2((0,T) \times \partial\Omega) \rightarrow L^2((0,T) \times \partial\Omega)} + \|\dot{\Lambda}\|_{H^2((0,T) \times \partial\Omega) \rightarrow H^3((0,T) \times \partial\Omega)} \right)$$

Here $\dot{K}$ is viewed as a linear function of $\dot{\Lambda}$ as is defined in (13).

Proof. For a bounded linear operator $T : \mathcal{X} \rightarrow \mathcal{Y}$ between two Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$, we write $\|T\|_{\mathcal{X} \rightarrow \mathcal{Y}}$ for the operator norm of T . Let $f, h \in C_c^\infty((0, T] \times \partial\Omega)$ be solutions of (23) obtained from proposition 4. We employ (24) to estimate:

$$\begin{aligned} \left| \mathcal{F}[\dot{q}](\sqrt{2\lambda}\theta) \right| &\leq \|\partial_t^2 f + \lambda f\|_{L^2((0,T) \times \partial\Omega)} \|h\|_{L^2((0,T) \times \partial\Omega)} \|\dot{K}\|_{L^2((0,T) \times \partial\Omega) \rightarrow L^2((0,T) \times \partial\Omega)} \\ &\quad + \|\dot{\Lambda}f(T)\|_{L^2(\partial\Omega)} \|h(T)\|_{L^2(\partial\Omega)} \\ &\leq (1 + \lambda) \|f\|_{H^2((0,T) \times \partial\Omega)} \|h\|_{L^2((0,T) \times \partial\Omega)} \|\dot{K}\|_{L^2((0,T) \times \partial\Omega) \rightarrow L^2((0,T) \times \partial\Omega)} \\ &\quad + \|\dot{\Lambda}f\|_{H^1((0,T) \times \partial\Omega)} \|h\|_{H^1((0,T) \times \partial\Omega)} \end{aligned}$$

by the continuity of the trace operator.

It remains to estimate $\|\dot{\Lambda}f\|_{H^1((0,T) \times \partial\Omega)}$. For this purpose, we extend $f \in H^2((0, T) \times \partial\Omega)$ to a function $F \in H^{2+\frac{3}{2}}((0, T) \times \Omega)$ so that $\partial_\nu F|_{(0,T) \times \partial\Omega} = f$ and $F(t, x) = 0$ for $x \in \Omega$ and t close to 0 (recall that $f(t, x) = 0$ for t near 0). Such F can be chosen to fulfill

$$\|F\|_{H^{3+\frac{1}{2}}((0,T) \times \Omega)} \leq C \|f\|_{H^2((0,T) \times \partial\Omega)}$$

Set $v := F - u_0$ where u_0 is the solution of (3), then v satisfies

$$\begin{cases} \square_{c_0, q_0} v &= \square_{c_0, q_0} F, & \text{in } (0, 2T) \times \Omega \\ \partial_\nu v &= 0, & \text{on } (0, 2T) \times \partial\Omega \\ v|_{t=0} = \partial_t v|_{t=0} &= 0, & x \in \Omega. \end{cases}$$

The regularity estimate for the wave equation [20] implies

$$\|v\|_{H^{2+\frac{1}{2}}((0,T) \times \Omega)} \leq C \|\square_{c_0, q_0} F\|_{H^{1+\frac{1}{2}}((0,T) \times \Omega)} \leq C \|F\|_{H^{3+\frac{1}{2}}((0,T) \times \Omega)}$$

We conclude $u_0 \in H^{2+\frac{1}{2}}((0, T) \times \Omega)$ and $\dot{q}u_0 \in H^{2+\frac{1}{2}}((0, T) \times \Omega)$. The same regularity estimate for the wave equation applied to (4) implies

$$\|\dot{u}\|_{H^{3+\frac{1}{2}}((0,T)\times\Omega)} \leq C\|\dot{q}u_0\|_{H^{2+\frac{1}{2}}((0,T)\times\Omega)} \leq C\|u_0\|_{H^{2+\frac{1}{2}}((0,T)\times\Omega)}$$

These inequalities together with the trace estimate yield

$$\|\dot{\Delta}f\|_{H^3((0,T)\times\partial\Omega)} \leq C\|\dot{u}\|_{H^{3+\frac{1}{2}}((0,T)\times\Omega)} \leq C\|f\|_{H^2((0,T)\times\partial\Omega)}$$

where the constant $C > 0$ is independent of f . Hence $\dot{\Delta} : H^2((0, T) \times \partial\Omega) \rightarrow H^3((0, T) \times \partial\Omega)$ is a bounded linear operator.

Finally, we can complete the stability estimate:

$$\begin{aligned} & \left| \mathcal{F}[\dot{q}](\sqrt{2\lambda}\theta) \right| \\ & \leq (1 + \lambda)\|f\|_{H^2((0,T)\times\partial\Omega)}\|h\|_{H^1((0,T)\times\partial\Omega)}\|\dot{K}\|_{L^2((0,T)\times\partial\Omega)\rightarrow L^2((0,T)\times\partial\Omega)} \\ & \quad + \|\dot{\Delta}f\|_{H^3((0,T)\times\partial\Omega)}\|h\|_{H^1((0,T)\times\partial\Omega)} \\ & \leq (1 + \lambda)\|f\|_{H^2((0,T)\times\partial\Omega)}\|h\|_{H^1((0,T)\times\partial\Omega)}\|\dot{K}\|_{L^2((0,T)\times\partial\Omega)\rightarrow L^2((0,T)\times\partial\Omega)} \\ & \quad + \|f\|_{H^2((0,T)\times\partial\Omega)}\|\dot{\Delta}\|_{H^2((0,T)\times\partial\Omega)\rightarrow H^3((0,T)\times\partial\Omega)}\|h\|_{H^1((0,T)\times\partial\Omega)} \\ & \leq C_\lambda \left(\|\dot{K}\|_{L^2((0,T)\times\partial\Omega)\rightarrow L^2((0,T)\times\partial\Omega)} + \|\dot{\Delta}\|_{H^2((0,T)\times\partial\Omega)\rightarrow H^3((0,T)\times\partial\Omega)} \right) \end{aligned}$$

where the constant C_λ satisfies (see (20))

$$\begin{aligned} C_\lambda &:= (2 + \lambda)\|f\|_{H^2((0,T)\times\partial\Omega)}\|h\|_{H^1((0,T)\times\partial\Omega)} \\ &\leq C(2 + \lambda)\|e^{i\sqrt{\lambda}\theta\cdot x}\|_{H^4(\Omega)}^2 \leq C(2 + \lambda)\lambda^4. \end{aligned}$$

for some constant $C > 0$ independent of λ . □

4.2. Case 2: $c_0 = 1$ and $q_0 \neq 0$

Let $\lambda \geq 0$ again, then the equations for $u_0^f(T)$ and $u_0^h(T)$ become the perturbed Helmholtz equation

$$[\Delta + \lambda - q_0]u_0^f(T) = [\Delta + \lambda - q_0]u_0^h(T) = 0 \quad \text{in } \Omega.$$

A class of solutions are total waves of the form

$$\phi(x) = e^{i\sqrt{\lambda}\theta\cdot x} + r(x; \lambda) \tag{25}$$

with $\theta \in \mathbb{S}^{n-1}$ and the scattered wave $r(x; \lambda)$ satisfying

$$\|r\|_{H^s(\mathbb{R}^n)} = O\left(\lambda^{\frac{s-1}{2}}\right) \quad \text{as } \lambda \rightarrow \infty \tag{26}$$

for any $s \geq 0$. Indeed, r is the unique outgoing solution to (37), see lemma 13 in the appendix for the construction and property of r .

One dimension. In one dimension (1D), $\theta = \pm 1$. Let us take $\theta = 1$. Insert $u_0^f(T) = u_0^h(T) = \phi$ into (15) to get

$$\begin{aligned} & -\hat{q}(2\sqrt{\lambda}) - 2(\dot{q}e^{i\sqrt{\lambda}\theta\cdot x}, r)_{L^2(\Omega)} - (\dot{q}r, r)_{L^2(\Omega)} \\ & = (\partial_t^2 f + \lambda f, \dot{K}h)_{L^2((0,T)\times\partial\Omega)} + (\dot{\Delta}f(T), h(T))_{L^2(\partial\Omega)} \end{aligned} \tag{27}$$

In the high-frequency regime, one may neglect the small terms to get an approximate reconstruction of $\hat{q}$.

Proposition 8. Suppose $c_0 = 1$ and $q_0 \in C^\infty(\overline{\Omega})$ is not identically zero. For a constant $M > 0$, denote $Q_M := \{\dot{q} \in L^\infty(\Omega) : \|\dot{q}\|_{L^\infty(\Omega)} \leq M\}$. Then the Fourier transform $\hat{q}$ of $\dot{q} \in Q_M$ satisfies

$$\begin{aligned} \hat{q}(2\sqrt{\lambda}\theta) &= -(\partial_t^2 f + \lambda f, \dot{K}h)_{L^2((0,T) \times \partial\Omega)} \\ &\quad - (\dot{\Lambda}f(T), h(T))_{L^2(\partial\Omega)} + O\left(\lambda^{-\frac{1}{2}}\right) \quad \text{as } \lambda \rightarrow \infty \end{aligned} \quad (28)$$

where $f, h \in C_c^\infty((0, T] \times \partial\Omega)$ satisfies $u_0^f(T) = u_0^h(T) = \phi$ (see (26)), and $O(\lambda^{-\frac{1}{2}})$ is a constant that decays on the order of $\lambda^{-\frac{1}{2}}$. Moreover, the following approximate stability estimate holds:

$$\begin{aligned} |\hat{q}(\sqrt{2\lambda}\theta)| &\leq C(2 + \lambda)\lambda^4 \left(\|\dot{K}\|_{L^2((0,T) \times \partial\Omega) \rightarrow L^2((0,T) \times \partial\Omega)} \right. \\ &\quad \left. + \|\dot{\Lambda}\|_{H^2((0,T) \times \partial\Omega) \rightarrow H^3((0,T) \times \partial\Omega)} \right) + O\left(\lambda^{-\frac{1}{2}}\right). \end{aligned}$$

Proof. The standard estimates

$$|\langle \dot{q} e^{i\sqrt{\lambda}\theta \cdot x}, r \rangle_{L^2(\Omega)}| \leq CM \|r\|_{L^2(\Omega)}, \quad |\langle \dot{q} r, r \rangle_{L^2(\Omega)}| \leq CM \|r\|_{L^2(\Omega)}^2.$$

show that the last two terms on the left-hand side of (27) decay like $O(\lambda^{-\frac{1}{2}})$ as $\lambda \rightarrow \infty$ according to (26). The approximate reconstruction follows. The approximate stability estimate can be proved in the same way as in the proof of theorem 7. The only difference is the addition of the error term $O(\lambda^{-\frac{1}{2}})$. $\square$

Two and higher dimensions. In dimension $n \geq 2$, more freedom is available in choosing the wave vectors. Let $\theta, \omega \in \mathbb{R}^n$ be two vectors such that $\theta \perp \omega$. We take the following solutions:

$$\begin{aligned} \phi(x) &:= \phi_0(x) + r_1(x; \lambda), & \phi_0(x) &:= e^{i(k\theta + l\omega) \cdot x} \\ \psi(x) &:= \psi_0(x) + r_2(x; \lambda), & \psi_0(x) &:= e^{i(k\theta - l\omega) \cdot x} \end{aligned}$$

where r_1, r_2 satisfy (26). Choose $k^2 + l^2 = \lambda$ so that $(\Delta + \lambda)\phi_0 = (\Delta + \lambda)\psi_0 = 0$. Proposition 4 asserts that there are $f, h \in C_c^\infty((0, T] \times \partial\Omega)$ such that

$$u_0^f(T) = \phi = \phi_0 + r_1, \quad u_0^h(T) = \psi = \psi_0 + r_2. \quad (29)$$

Theorem 9. Suppose $c_0 = 1$, $q_0 \in C^\infty(\overline{\Omega})$ and q_0 is not identically zero. Then the Fourier transform $\hat{q}$ can be reconstructed as follows:

$$\hat{q}(2k\theta) = -\lim_{l \rightarrow \infty} \left[(\partial_t^2 f + (k^2 + l^2)f, \dot{K}h)_{L^2((0,T) \times \partial\Omega)} + (\dot{\Lambda}f(T), h(T))_{L^2(\partial\Omega)} \right].$$

where $f, h \in C_c^\infty((0, T] \times \partial\Omega)$ are solutions to (29), respectively.

Proof. Inserting $u_0^f(T) = \phi$ and $u_0^h(T) = \psi$ into (15) gives

$$\begin{aligned} &-\hat{q}(2k\theta) - \langle \dot{q} e^{i(k\theta + l\omega) \cdot x}, r_2 \rangle_{L^2(\Omega)} - \langle \dot{q} e^{i(k\theta - l\omega) \cdot x}, r_1 \rangle_{L^2(\Omega)} - \langle \dot{q} r_1, r_2 \rangle_{L^2(\Omega)} \\ &= (\partial_t^2 f + (k^2 + l^2)f, \dot{K}h)_{L^2((0,T) \times \partial\Omega)} + (\dot{\Lambda}f(T), h(T))_{L^2(\partial\Omega)} \end{aligned} \quad (30)$$

If we fix k and let $l \rightarrow \infty$, then $\lambda \rightarrow \infty$ and $\|r_1\|_{L^2(\Omega)}, \|r_2\|_{L^2(\Omega)} \rightarrow 0$ due to its decay property (26). We obtain the reconstruction formula for any $k \geq 0$ and any $\theta \in \mathbb{S}^{n-1}$. Note that f and h depend on l . $\square$

We can also obtain a Hölder-type stability estimate for $\|\dot{q}\|_{H^{-s}(\mathbb{R}^n)}$, where $s > 0$ is an arbitrary real number and $H^{-s}(\mathbb{R}^n)$ is the L^2 -based Sobolev space of order $-s$ over $\mathbb{R}^n$.

Theorem 10. Suppose $c_0 = 1$, $q_0 \in C^\infty(\bar{\Omega})$ and q_0 is not identically zero. For any $s > 0$, there exists a constant $C > 0$ such that

$$\|\dot{q}\|_{H^{-s}(\mathbb{R}^n)} \leq C \left(\|\dot{K}\|_{L^2((0,T) \times \partial\Omega) \rightarrow L^2((0,T) \times \partial\Omega)} + \|\dot{\Lambda}\|_{H^2((0,T) \times \partial\Omega) \rightarrow H^3((0,T) \times \partial\Omega)} \right)^{\frac{2s}{11(n+2s)}}.$$

Proof. Write $\xi := 2k\theta$ and

$$\delta := \left(\|\dot{K}\|_{L^2((0,T) \times \partial\Omega) \rightarrow L^2((0,T) \times \partial\Omega)} + \|\dot{\Lambda}\|_{H^2((0,T) \times \partial\Omega) \rightarrow H^3((0,T) \times \partial\Omega)} \right).$$

Let $\rho > 0$ be a sufficiently large number that is to be determined. We decompose

$$\|\dot{q}\|_{H^{-s}(\mathbb{R}^n)}^2 = \int_{|\xi| \leq \rho} \frac{|\hat{q}(\xi)|^2}{(1 + |\xi|^2)^s} d\xi + \int_{|\xi| > \rho} \frac{|\hat{q}(\xi)|^2}{(1 + |\xi|^2)^s} d\xi.$$

For the integral over high frequencies, we have

$$\int_{|\xi| > \rho} \frac{|\hat{q}(\xi)|^2}{(1 + |\xi|^2)^s} d\xi \leq \frac{1}{(1 + \rho^2)^s} \int_{|\xi| > \rho} |\hat{q}(\xi)|^2 d\xi \leq \frac{\|\dot{q}\|_{L^2(\mathbb{R}^n)}^2}{(1 + \rho^2)^s} \leq C \frac{1}{\rho^{2s}}.$$

For the integral over low frequencies, it is easy to see that:

$$\int_{|\xi| \leq \rho} \frac{|\hat{q}(\xi)|^2}{(1 + |\xi|^2)^s} d\xi \leq \int_{|\xi| \leq \rho} |\hat{q}(\xi)|^2 d\xi \leq C \rho^n \|\dot{q}\|_{L^\infty(B(0,\rho))}^2.$$

The norm $\|\hat{q}\|_{L^\infty(B(0,\rho))}$ can be estimated using (30). Indeed, for $|\xi| \leq \rho$, we have

$$\begin{aligned} |\hat{q}(\xi)| &\leq |(\partial_t^2 f + (k^2 + l^2)f, \dot{K}h)_{L^2((0,T) \times \partial\Omega)} + (\dot{\Lambda}f(T), h(T))_{L^2(\partial\Omega)}| + \frac{C}{\sqrt{\lambda}} \\ &\leq C(2 + \lambda) \|\phi\|_{H^4(\Omega)} \|\psi\|_{H^4(\Omega)} \delta + \frac{C}{\sqrt{\lambda}} \\ &\leq C(2 + \lambda) (\|\phi_0\|_{H^4(\Omega)} + \|r_1\|_{H^4(\Omega)}) (\|\psi_0\|_{H^4(\Omega)} + \|r_2\|_{H^4(\Omega)}) \delta + \frac{C}{\sqrt{\lambda}} \\ &\leq C(2 + \lambda) \left(\lambda^2 + \lambda^{\frac{3}{2}} \right)^2 \delta + \frac{C}{\sqrt{\lambda}} \end{aligned}$$

where the first inequality is a consequence of the L^2 -resolvent estimate for r_1, r_2 , the second inequality follows from the proof of proposition 7, and the last inequality utilizes the resolvent estimate for higher-order derivatives of r_1, r_2 . Utilizing the relation $\lambda = k^2 + l^2$, we conclude

$$\|\hat{q}\|_{L^\infty(B(0,\rho))}^2 \leq C \left[(2+\lambda)^2 \lambda^6 (1+\sqrt{\lambda})^4 \delta^2 + \frac{1}{\lambda} \right] \leq C \left[(\rho^2 + l^2)^{10} \delta + \frac{1}{l^2} \right]$$

provided $\rho > 0$ is sufficiently large. Combining these estimates, we see that

$$\|\dot{q}\|_{H^{-s}(\mathbb{R}^n)}^2 \leq C \left[\rho^n (\rho^2 + l^2)^{10} \delta^2 + \frac{\rho^n}{l^2} + \frac{1}{\rho^{2s}} \right].$$

Choosing $l = \rho^{n+2s}$ and $\rho = \delta^{-\frac{2}{21(n+2s)}}$ yields

$$\|\dot{q}\|_{H^{-s}(\mathbb{R}^n)}^2 \leq C \delta^{\frac{4s}{11(n+2s)}}$$

as long as δ is sufficiently small. $\square$

5. Numerical experiments

This section is devoted to numerical implementation and validation of the reconstruction formula (24) in one dimension (1D) when $c_0 = 1$ and $q_0 = 0$.

5.1. Computing boundary controls with time reversal

A crucial step of the BC method is solving the equations (23) or (29) for the boundary controls f and h . For our purpose, a highly accurate numerical solver is needed due to the appearance of $\partial_t^2 f$ in (24), where the second-order temporal differentiation tends to amplify the numerical error in f . When the spatial dimension n is odd with $c_0 = 1$ and $q_0 = 0$, such scheme can be obtained using time reversal. Indeed, for a prescribed ϕ on Ω , one can construct an extension, named $\tilde{\phi}$, such that $\tilde{\phi} = \phi$ in Ω and $\tilde{\phi}$ is supported in a neighborhood of Ω . Let v be the solution of the following backward initial value problem

$$\begin{cases} \square_{1,0} v(t, x) = 0, & \text{in } (0, T) \times \mathbb{R}^n \\ v(T) = \tilde{\phi}, & \text{in } \mathbb{R}^n \\ \partial_t v(T) = 0, & \text{in } \mathbb{R}^n. \end{cases}$$

If $T > 0$ is sufficiently large and n is odd, we would have $v(0) = \partial_t v(0) = 0$ by the Huygen's principle. This implies

$$u_0^{f-\partial_\nu v}(T) = u_0^f(T) - v(T) = 0 \quad \text{in } \Omega.$$

As a result, we can take $f = \partial_\nu v|_{[0,T] \times \partial\Omega}$. Note that v can be explicitly expressed using the Kirchhoff's formula [20], thus $\partial_t^2 f = \partial_\nu \partial_t^2 v|_{[0,T] \times \partial\Omega}$ can be analytically computed.

We will demonstrate the numerical implementations in dimension $n = 1$ with $\Omega = [a, b]$. In this case, D'Alembert's formula gives

$$v(t, x) = \frac{1}{2} [\tilde{\phi}(x + t - T) + \tilde{\phi}(x + T - t)].$$

Therefore,

$$\partial_t^2 f = \partial_\nu \partial_t^2 v|_{[0,T] \times \partial\Omega} = \pm \frac{1}{2} [\tilde{\phi}'''(x + t - T) + \tilde{\phi}'''(x + T - t)]|_{x=a,b}$$

where we take $+$ when $x = b$ and $-$ when $x = a$. We choose the following extension:

$$\tilde{\phi} := \begin{cases} \phi & x \in [a, b], \\ \phi \cdot \exp\left\{1 - \frac{1}{1 - (x - a)^{2p}}\right\} & x \in (a - 1, a), \\ \phi \cdot \exp\left\{1 - \frac{1}{1 - (x - b)^{2p}}\right\} & x \in (b, b + 1), \\ 0 & x \notin (a - 1, b + 1), \end{cases}$$

where $p \geq 1$ is a positive integer. It is easy to see that $\tilde{\phi}$ is C^{2p-1} at $x = a, b$ and is C^∞ at other points. We take $p \geq 2$ to guarantee the existence of the second derivative of f .

5.2. Numerical experiments

The computational setup is as follows: $\Omega = [-1, 1]$, $c_0 = 1$, $q_0 = 0$, and $T = 5$. The forward problem (4) is solved using the second order central difference scheme on a temporal-spatial grid of size $24\,999 \times 501$ to obtain the linearized ND map (5). Then various $\phi = u_0^f(T)$ are inserted into (24) to recover the Fourier transform of $\dot{q}$ at $2k\theta$, where f and h are computed using the time reversal method as is explained at the beginning of this section. The basis functions for the prescribed Helmholtz solution ϕ in our experiments are

$$1, \sin\left(\frac{\pi}{2}x\right), \cos\left(\frac{\pi}{2}x\right), \dots, \sin\left(\frac{N\pi}{2}x\right), \cos\left(\frac{N\pi}{2}x\right)$$

with $N = 10$. They correspond to Helmholtz solutions with $k = 0, \frac{\pi}{2}, \dots, \frac{N\pi}{2}$.

Experiment 1. The first experiment aims to reconstruct the following smooth $\dot{q}$ using the formula (24):

$$\dot{q} = \sin(\pi x) + 2 \cos(2\pi x) + 4 \sin(4\pi x) - 3.$$

The graph of $\dot{q}$ is shown in figure 1. The measurement $\hat{\Lambda}_{\dot{q}}$ is added with 0%, 1%, and 5% of Gaussian noise, respectively. The reconstructions and the corresponding errors are illustrated in figure 2. Notice that the reconstruction error with 5% noise is relatively larger, as can be expected. When multiple measurements are available, we can repeat the reconstruction several times and then take the average. This strategy effectively reduces the error, since the inverse problem is linear and the Gaussian noise has zero mean, see figure 3.

Experiment 2. The second experiment tests reconstruction of a discontinuous $\dot{q}$. We choose $\dot{q}$ to be the Heaviside function

$$H(x) = \begin{cases} 1 & x \geq 0, \\ 0 & x < 0. \end{cases}$$

The Fourier series of $H(x)$ on $\Omega = [-1, 1]$ is

$$H(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi} \sin((2n-1)\pi x).$$

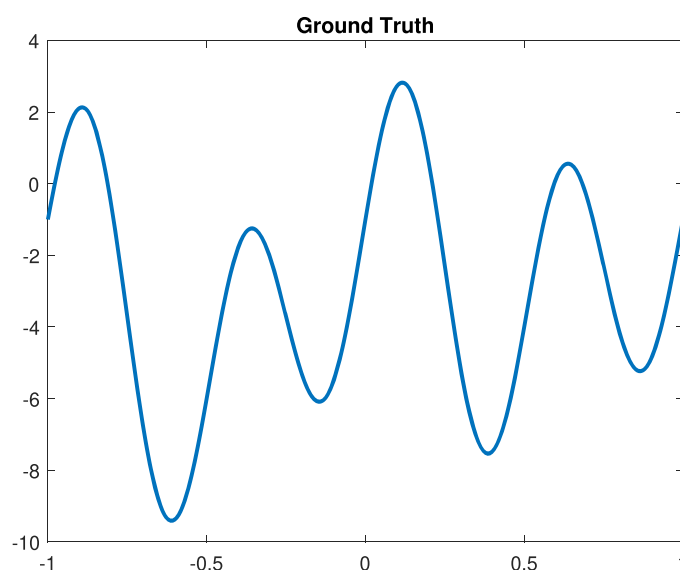


Figure 1. Ground truth $\dot{q} = \sin(\pi x) + 2 \cos(2\pi x) + 4 \sin(4\pi x) - 3$.

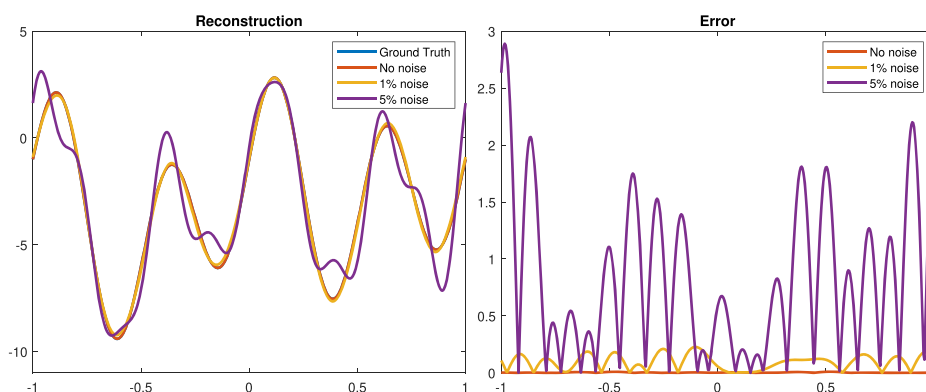


Figure 2. Left: reconstructed $\dot{q}$ with 0%, 1%, 5% Gaussian noise and the ground truth. Right: the corresponding error between the reconstruction and the ground truth. The relative L^2 -errors are 0.1%, 2.5%, and 23.9% respectively.

With the choice of the finite computational basis, we can only expect to reconstruct the following orthogonal projection:

$$H_N(x) := \frac{1}{2} + \sum_{n=1}^{\left\lceil \frac{N}{2} \right\rceil} \frac{2}{(2n-1)\pi} \sin((2n-1)\pi x),$$

see figure 4 for the graph of $H(x)$ and $H_N(x)$. The reconstruction formula (24) is implemented with 0%, 1%, and 5% of Gaussian noise added to $\hat{\Lambda}_{\dot{q}}$, respectively. The reconstructions and

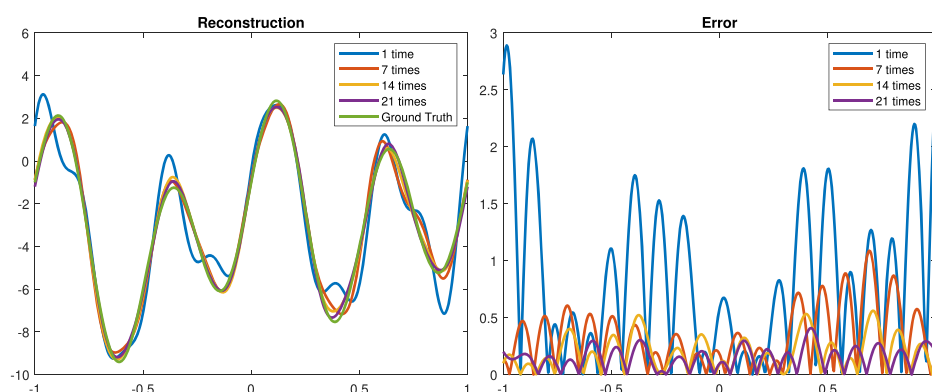


Figure 3. Left: reconstructed $\hat{q}$ under 1, 7, 14, 21 times repetition with 5% Gaussian noise and the ground truth. Right: the corresponding error between the reconstruction and the ground truth. The relative L^2 -errors are 23.9%, 9.4%, 5.5%, and 4.0% respectively.

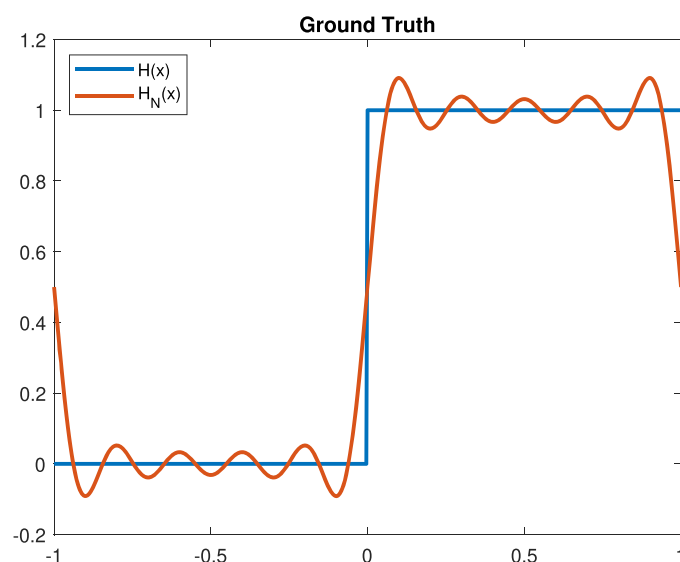


Figure 4. Ground truth $\hat{q} = H(x)$ and its projection $H_N(x)$.

corresponding errors with a single measurement are illustrated in figure 5. The averaged reconstruction with 5% of Gaussian noise and multiple repeated measurements are illustrated in figure 6.

Experiment 3. This experiment aims to test the reconstruction in the case $c_0 = 1$ and a small $q_0 \neq 0$. We choose

$$q_0 = \frac{1}{10} \sin(\pi x),$$

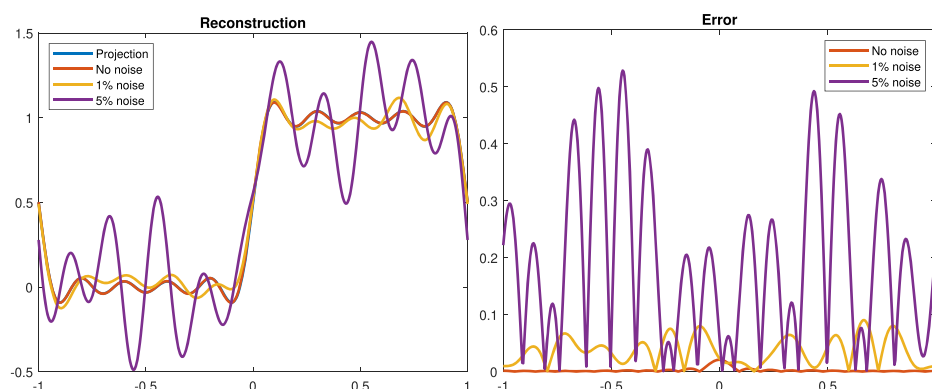


Figure 5. Left: reconstructed $\dot{q}$ with 0%, 1%, 5% Gaussian noise and the projection of the ground truth. Right: the corresponding error between the reconstruction and the projection of the ground truth. The relative L^2 -errors are 0.6%, 6.2%, and 33.8%, respectively.

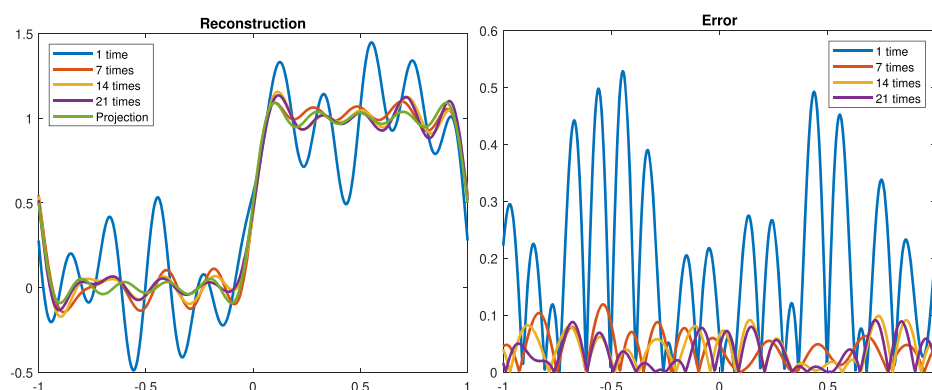


Figure 6. Left: reconstructed $\dot{q}$ under 1, 7, 14, 21 times repetition with 5% Gaussian noise and the projection of the ground truth. Right: the corresponding error between the reconstruction and the projection of the ground truth. The relative L^2 -errors are 33.8%, 7.2%, 6.9%, and 6.1% respectively.

and $\dot{q}$ to be the same Heaviside function as in experiment 2, see figure 7. We attempt to reconstruct an approximate $\dot{q}$ based on (27) by neglecting the terms involving r . A computational challenge is that we cannot find explicit form of $\partial_t^2 f$ when $q_0 \neq 0$. Instead, we make use of the smallness of q_0 to approximately construct $\partial_t^2 f$ as if $q_0 = 0$. In the meanwhile, the operator $\tilde{K}$ and $\tilde{\Lambda}_{\dot{q}}$ are still implemented using the exact q_0 and $\dot{q}$. The reconstructions and corresponding errors with a single measurement under 0%, 1%, and 5% of Gaussian noise are illustrated in figure 8. The averaged reconstruction with 5% of Gaussian noise and multiple repeated measurements are illustrated in figure 9. This experiment confirms that approximate reconstruction using (27) remains possible for $q_0 \neq 0$ as long as it is small.

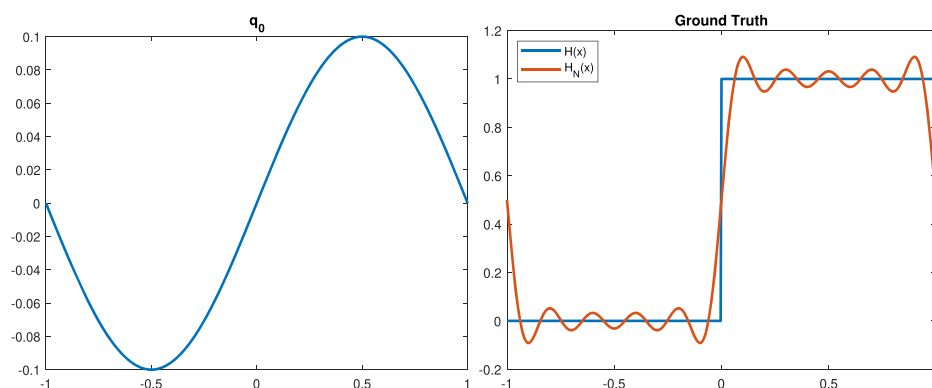


Figure 7. Left: $q_0 = \frac{1}{10} \sin(\pi x)$. Right: ground truth $\dot{q} = H(x)$ and its projection $H_N(x)$.

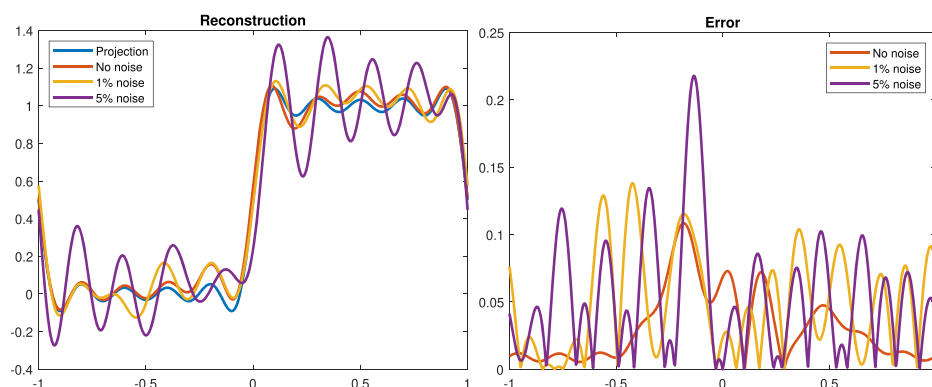


Figure 8. Left: reconstructed $\dot{q}$ with 0%, 1%, 5% Gaussian noise and the projection of the ground truth. Right: the corresponding error between the reconstruction and the projection of the ground truth. The relative L^2 -errors are 5.8%, 8.9%, and 24.2%, respectively.

Experiment 4. In this experiment, we apply the reconstruction formula (24) to measurement from the non-linear IBVP. Specifically, we choose $c_0 = 1$ and the potential

$$q = q_0 + \varepsilon \dot{q} + \varepsilon^2 \ddot{q}$$

where $\varepsilon > 0$ is a small number, the background potential $q_0 = 0$, and the perturbations

$$\dot{q} = \sin(\pi x) + 2 \cos(2\pi x) + 4 \sin(4\pi x) - 3, \quad \ddot{q} = 20 \cos(20\pi x).$$

We apply the reconstruction formula (24), with $\dot{\Lambda}_{\dot{q}} f$ replaced by $\Lambda_q f - \Lambda_{q_0} f$ (where $\Lambda_q f$ and $\Lambda_{q_0} f$ are computed by solving the boundary value problem (1)), and then add q_0 to the reconstruction to obtain an approximation of q . The rationale is that, when ε is small, we have the following approximation

$$\Lambda_q - \Lambda_{q_0} \approx \varepsilon \dot{\Lambda}_{\dot{q}} = \dot{\Lambda}_{\varepsilon \dot{q}}.$$

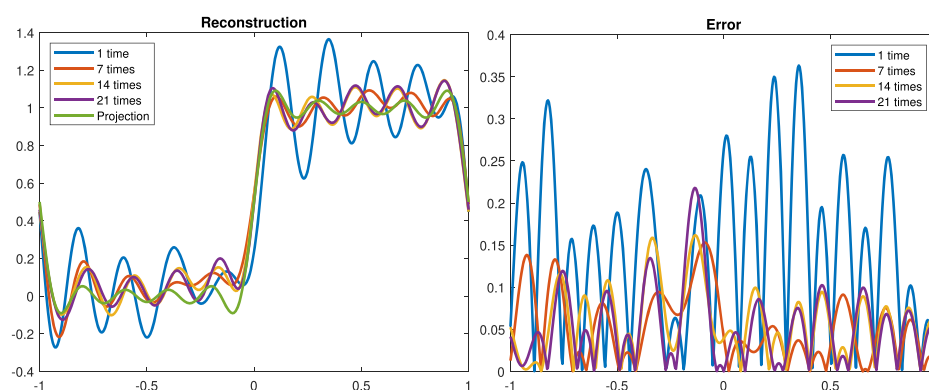


Figure 9. Left: reconstructed $\dot{q}$ under 1, 7, 14, 21 times repetition with 5% Gaussian noise and the projection of the ground truth. Right: the corresponding error between the reconstruction and the projection of the ground truth. The relative L^2 -errors are 24.2%, 9.38%, 9.87% and 9.93%, respectively.

Therefore, the resulting reconstruction from (24) is approximately $\epsilon\dot{q}$, and adding q_0 to it yields a linear approximation of q . We choose $\epsilon = 0.1$. The ground truth is illustrated in figure 10.

Adding noise is a bit delicate in this experiment. The noise to be added is additive and proportional to the magnitude of the signal. We tested two ways of adding noise: (1) adding noise to the difference $(\Lambda_q - \Lambda_{q_0})f$; (2) adding noises to $\Lambda_q f$ and $\Lambda_{q_0} f$ respectively, then subtract to find the difference. The resulting numerical performances are different, and it turns out the former introduces much less error than the latter. This is because the numerical values in the discretization of $(\Lambda_q - \Lambda_{q_0})f$ are much smaller, hence the proportional noise is relatively small. In contrast, the numerical values in the discretization of $\Lambda_q f$ and $\Lambda_{q_0} f$ are larger, hence the proportional noise is relatively large. Comparison of the reconstruction errors are shown in figures 11 and 12. Note that the noise added in figure 12 (0.1% and 0.5%) is only a tenth of that in figure 11 (1% and 5%).

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

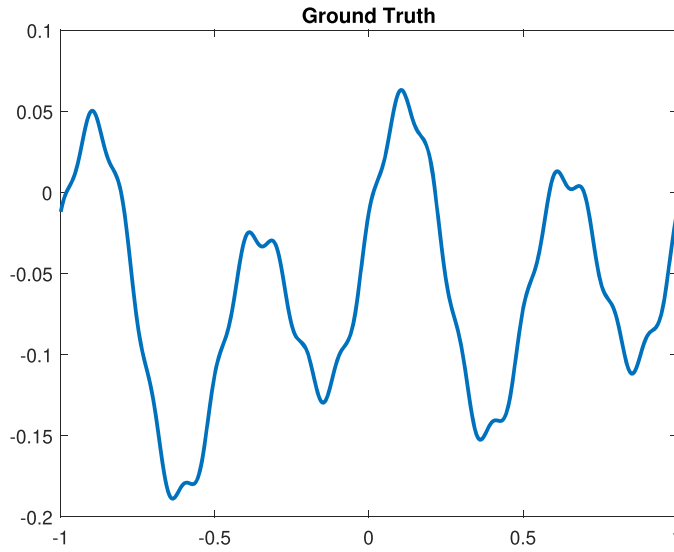


Figure 10. Ground truth $q = q_0 + \varepsilon \dot{q} + \varepsilon^2 \ddot{q}$, where $\dot{q} = \sin(\pi x) + 2 \cos(2\pi x) + 4 \sin(4\pi x) - 3$, $\ddot{q} = 20 \cos(20\pi x)$, $\varepsilon = 0.02$.

Appendix A

In the appendices, we collect a few results that are used in the main text. First, we provide the rigorous justification for the formal linearization process in the introduction to derive (3)–(5). Recall that $c_0 \in C^\infty(\overline{\Omega})$.

For $f \in L^2((0, 2T) \times \partial\Omega)$, the solution $u = u^f$ of the boundary value problem (1) satisfies $u \in C([0, 2T]; H^{5/6-\epsilon}(\Omega))$ for any $\epsilon > 0$ with the norm estimate [26]

$$\|u\|_{C([0, 2T]; H^{5/6-\epsilon}(\Omega))} \leq C \|f\|_{L^2((0, 2T) \times \partial\Omega)} \quad (31)$$

where $\|u^f\|_{C([0, 2T]; H^{5/6-\epsilon}(\Omega))} := \text{ess sup}_{0 \leq t \leq 2T} \|u(t)\|_{H^{5/6-\epsilon}(\Omega)}$. As a result, the ND map $\Lambda_q : L^2((0, 2T) \times \partial\Omega) \rightarrow L^2((0, 2T) \times \partial\Omega)$ is a bounded linear operator.

Denote by $\mathcal{L}(L^2((0, 2T) \times \partial\Omega), L^2((0, 2T) \times \partial\Omega))$ the Banach space of bounded linear operators over $L^2((0, 2T) \times \partial\Omega)$. The IBVP aims to invert the following nonlinear map

$$\mathcal{F} : q \in L^\infty(\Omega) \mapsto \Lambda_q \in \mathcal{L}(L^2((0, 2T) \times \partial\Omega), L^2((0, 2T) \times \partial\Omega))$$

Suppose $q = q_0 + \dot{q}$ with $q_0 \in C^\infty(\overline{\Omega})$. Define a linear operator (which will turn out to be the Frechét differentiation of $\mathcal{F}$):

$$\text{d}\mathcal{F} : \dot{q} \in L^\infty(\Omega) \mapsto \dot{\Lambda}_{\dot{q}} \in \mathcal{L}(L^2((0, 2T) \times \partial\Omega), L^2((0, 2T) \times \partial\Omega)).$$

where $\dot{\Lambda}_{\dot{q}}$ is the operator defined in (5).

Proposition 11. *The nonlinear map $\mathcal{F}$ is Frechét differentiable at a fixed $q_0 \in C^\infty(\overline{\Omega})$, and the Frechét derivative along $\dot{q} \in L^\infty(\Omega)$ is $\dot{\Lambda}_{\dot{q}}$.*

Proof. It suffices to show that as $\|\dot{q}\|_{L^\infty(\Omega)} \rightarrow 0$, we have

$$\|\mathcal{F}(q) - \mathcal{F}(q_0) - \text{d}\mathcal{F}(\dot{q})\|_{\mathcal{L}(L^2((0, 2T) \times \partial\Omega), L^2((0, 2T) \times \partial\Omega))} = O(\|\dot{q}\|_{L^\infty(\Omega)}^2)$$

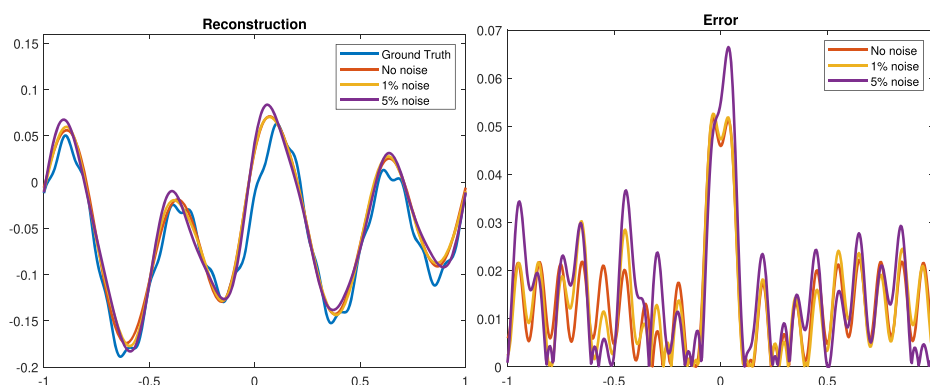


Figure 11. Left: reconstructed q with 0%, 1%, 5% Gaussian noise added to $\Lambda_q - \Lambda_{q_0}$ and the ground truth. Right: the corresponding error between the reconstruction and the ground truth. The relative L^2 -errors are 19.9%, 20.4%, and 23.5%, respectively.

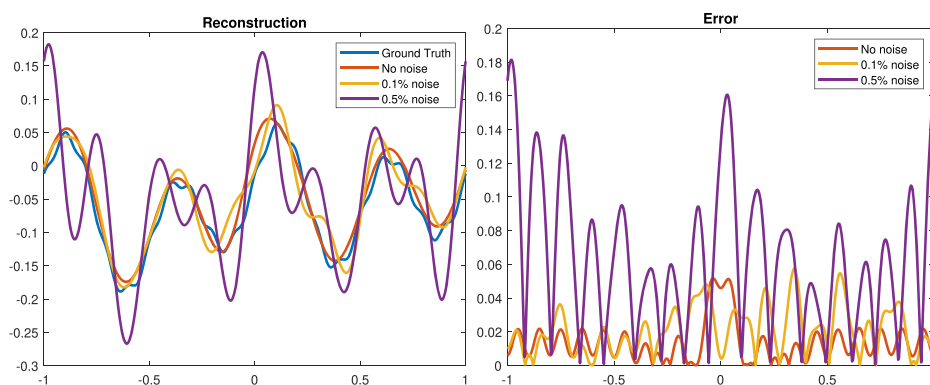


Figure 12. Left: reconstructed q with 0%, 0.1%, 0.5% Gaussian noise added to Λ_q and Λ_{q_0} independently, and the ground truth. Right: the corresponding error between the reconstruction and the ground truth. The relative L^2 -errors are 19.9%, 28.1%, and 86.2%, respectively.

(or equivalently, $\|\Lambda_q - \Lambda_{q_0} - \dot{\Lambda}_{\dot{q}}\|_{\mathcal{L}(L^2((0,2T) \times \partial\Omega), L^2((0,2T) \times \partial\Omega))} = O(\|\dot{q}\|_{L^\infty(\Omega)}^2)$) to justify that $d\mathcal{F}$ is indeed the Frechét differentiation of $\mathcal{F}$. To this end, we will prove for any $f \in L^2((0,2T) \times \partial\Omega)$ that $\square$

$$\|\Lambda_q f - \Lambda_{q_0} f - \dot{\Lambda}_{\dot{q}} f\|_{L^2((0,2T) \times \partial\Omega)} \leq C \|\dot{q}\|_{L^\infty}^2 \|f\|_{L^2((0,2T) \times \partial\Omega)} \quad (32)$$

for some constant $C > 0$ that is independent of f . For ease of notation, we will denote all the constants independent of f by C .

We continue to denote the solutions of (1) and (3) by u and u_0 , respectively. Write $u = u_0 + \delta u$. Then δu satisfies $\delta u|_{[0,2T] \times \partial\Omega} = \Lambda_q f - \Lambda_{q_0} f$ and

$$\begin{cases} \square_{c_0, q_0} \delta u(t, x) &= -u \dot{q}, & \text{in } (0, 2T) \times \Omega \\ \partial_\nu \delta u &= 0, & \text{on } (0, 2T) \times \partial\Omega \\ \delta u(0, x) = \partial_t \delta u(0, x) &= 0 & x \in \Omega. \end{cases} \quad (33)$$

Using the regularity estimate for the wave equation [20] and the trace theorem, we obtain

$$\|\delta u\|_{H^1((0,2T) \times \Omega)} \leq C \|u \dot{q}\|_{L^2((0,2T) \times \Omega)} \leq C \|u\|_{L^2((0,2T) \times \Omega)} \|\dot{q}\|_{L^\infty(\Omega)}. \quad (34)$$

Next, set $w := \delta u - \dot{u}$, then $w|_{[0,2T] \times \partial\Omega} = \Lambda_q f - \Lambda_{q_0} f - \dot{\Lambda}_{\dot{q}} f$, and w satisfies

$$\begin{cases} \square_{c_0, q_0} w(t, x) &= -\dot{q} \delta u, & \text{in } (0, 2T) \times \Omega \\ \partial_\nu w &= 0, & \text{on } (0, 2T) \times \partial\Omega \\ w(0, x) = \partial_t w(0, x) &= 0 & x \in \Omega. \end{cases} \quad (35)$$

Applying the regularity estimate for the wave equation again yields

$$\begin{aligned} \|\Lambda_q f - \Lambda_{q_0} f - \dot{\Lambda}_{\dot{q}} f\|_{L^2((0,2T) \times \partial\Omega)} &\leq \|w\|_{H^1((0,2T) \times \Omega)} \\ &\leq C \|\dot{q} \delta u\|_{L^2((0,2T) \times \Omega)} \\ &\leq \|\dot{q}\|_{L^\infty(\Omega)} \|\delta u\|_{L^2((0,2T) \times \Omega)}. \end{aligned} \quad (36)$$

Combining the estimates (31), (34) and (36) yields the desired estimate (32).

Appendix B

We present a lemma regarding uniform continuity of min/max functions, which is used in the proof of proposition 4.

Lemma 12. *Let X, Y be compact metric spaces and suppose that $f : X \times Y \rightarrow \mathbb{R}$ is uniformly continuous. Then the function $F : X \rightarrow \mathbb{R}$*

$$F(x) := \min\{f(x, y) : y \in Y\}$$

is uniformly continuous. The same is true if min is replaced by max.

Proof. Let $\epsilon > 0$ and let $x, x' \in X$. We may assume without loss of generality that $F(x) \geq F(x')$. Let $y, y' \in Y$ be such that $F(x) = f(x, y)$ and $F(x') = f(x', y')$, then

$$|F(x) - F(x')| = f(x, y) - f(x', y') \leq f(x, y') - f(x', y')$$

Due to uniform continuity there is $\delta > 0$ such that

$$|f(x, y') - f(x', y')| < \epsilon$$

whenever $d(x, x') < \delta$. The proof when ‘min’ is replaced by ‘max’ is similar. $\square$

Appendix C

We construct a Helmholtz solution of the form (25), with r satisfying the asymptotic condition (26).

Lemma 13. *Let $\lambda \geq 0$. The perturbed Helmholtz equation $(\Delta + \lambda - q_0)\phi = 0$ admits solutions of the form*

$$\phi(x) = e^{i\sqrt{\lambda}\theta \cdot x} + r(x; \lambda)$$

for any $\theta \in \mathbb{S}^{n-1}$, with $\|r\|_{H^s(\mathbb{R}^n)} = O(\lambda^{\frac{s-1}{2}})$ as $\lambda \rightarrow \infty$ for any $s \geq 0$.

Proof. If $\phi(x) = e^{i\sqrt{\lambda}\theta \cdot x} + r(x; \lambda)$ solves the perturbed Helmholtz equation and $\theta \in \mathbb{S}^{n-1}$, then r has to satisfy

$$(\Delta + \lambda - q_0)r = q_0 e^{i\sqrt{\lambda}\theta \cdot x} \quad \text{in } \Omega. \quad (37)$$

Such r can be constructed as follows. Let $R(\lambda) := (\Delta + \lambda - q_0)^{-1}$ be the outgoing resolvent of the perturbed Helmholtz operator. For any compactly supported smooth function $\chi \in C_c^\infty(\mathbb{R}^n)$, the following L^2 -resolvent estimate holds [17, theorem 3.1]:

$$\|\chi R(\lambda) \chi\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq \frac{C}{\sqrt{\lambda}} \quad (38)$$

where $C > 0$ is a constant independent of λ . In particular, we choose $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $\chi = 1$ on $\overline{\Omega}$ and set $r := \chi R(\lambda) \chi (q_0 e^{i\sqrt{\lambda}\theta \cdot x})$, then r satisfies (37), r is compactly supported, and

$$\|r\|_{L^2(\mathbb{R}^n)} \leq \frac{C}{\sqrt{\lambda}}. \quad (39)$$

This proves the claim when $s = 0$.

We proceed by induction. Suppose it has been proved that r is compactly supported and satisfies $\|r\|_{H^s(\mathbb{R}^n)} = O(\lambda^{\frac{s-1}{2}})$ as $\lambda \rightarrow \infty$ for some integer $s \geq 0$. Let Ω_1, Ω_2 be two open sets such that $\Omega \subset \text{supp } \chi \subset \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. We apply the interior regularity estimate [20, section 6.3, theorem 2] to the equation $(\Delta - q_0)r = q_0 e^{i\sqrt{\lambda}\theta \cdot x} - \lambda r$ to obtain

$$\begin{aligned} \|r\|_{H^{s+2}(\mathbb{R}^n)} &= \|r\|_{H^{s+2}(\Omega_1)} \leq C \left(\|q_0 e^{i\sqrt{\lambda}\theta \cdot x} - \lambda r\|_{H^s(\Omega_2)} + \|r\|_{H^s(\Omega_2)} \right) \\ &\leq C(1 + \lambda \|r\|_{H^s(\mathbb{R}^n)} + \|r\|_{H^s(\mathbb{R}^n)}) \\ &\leq C \left(1 + \lambda^{\frac{s+1}{2}} \right). \end{aligned}$$

This completes the inductive step, hence the claim holds for all even integers s . The general claim is a consequence of interpolation. $\square$

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