Discrete Applied Mathematics 320 (2022) 150–169

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Truncated metric dimension for finite graphs

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ARTICLE INFO

Article history:
Received 28 July 2021
Received in revised form 26 April 2022
Accepted 28 April 2022
Available online 13 June 2022

Keywords:
k-truncated resolving set
k-truncated metric dimension
Distance-k dimension
Truncated metric dimension
Metric dimension
Adjacency dimension

ABSTRACT

Let G be a graph with vertex set $V(G)$, and let $d(x, y)$ denote the length of a shortest path between nodes x and y in G. For a positive integer k and for distinct $x, y \in V(G)$, let $d_k(x, y) = \min\{d(x, y), k + 1\}$ and $R_k\{x, y\} = \{z \in V(G) : d_k(x, z) \neq d_k(y, z)\}$. A subset $S \subseteq V(G)$ is a *k-truncated resolving set* of G if $|S \cap R_k\{x, y\}| \geq 1$ for any pair of distinct $x, y \in V(G)$. The *k-truncated metric dimension*, $\dim_k(G)$, of G is the minimum cardinality over all k-truncated resolving sets of G, and the usual metric dimension is recovered when $k + 1$ is at least the diameter of G. We obtain some general bounds for k-truncated metric dimension. For all $k \geq 1$, we characterize connected graphs G of order n with $\dim_k(G) = n - 2$ and $\dim_k(G) = n - 1$. For all $j, k \geq 1$, we find the maximum possible order, degree, clique number, and chromatic number of any graph G with $\dim_k(G) = j$. We determine $\dim_k(G)$ when G is a cycle or a path. We also examine the effect of vertex or edge deletion on the truncated metric dimension of graphs, and study various problems related to the truncated metric dimension of trees.

1. Introduction

Let G be a finite, simple, undirected, and connected graph with vertex set $V(G)$ and edge set $E(G)$. The *(geodesic) distance* between two vertices $x, y \in V(G)$, denoted by $d(x, y)$, is the length of a shortest path between x and y in G.

Metric dimension, introduced by Slater [31] and by Harary and Melter [21], is a graph parameter that has been studied extensively (see also, e.g., [3,6,7,15,26,28,38]). For distinct $x, y \in V(G)$, let $R\{x, y\} = \{z \in V(G) : d(x, z) \neq d(y, z)\}$. A subset $S \subseteq V(G)$ is a *resolving set* of G if $|S \cap R\{x, y\}| \geq 1$ for any pair of distinct vertices x and y in G. The *metric dimension* of G, denoted by $\dim(G)$, is the minimum cardinality over all resolving sets of G. It is NP-hard in general to compute $\dim(G)$ [16,26].

Khuller et al. [26] considered robot navigation as one of the applications of metric dimension, where a robot that moves from node to node knows its distances to a set of landmarks, which are placed on the elements of the resolving set. The traditional definition of metric dimension assumes knowledge of all pairwise distances between vertices. This assumption allows any individual vertex v of a resolving set to play a key role in distinguishing any pair of vertices, including pairs very far from v. In practice, however, computing pairwise distances between all pairs of vertices in a

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https://doi.org/10.1016/j.dam.2022.04.021
0166-218X/

large network may be costly, and the quality of pairwise distance measurements may degrade with increasing distance. Indeed, computing the distance matrix of a dense graph has time complexity $O(|V|^3)$ [14], whereas for a sparse graph the complexity is $O(|V||E| + |V|^2 \ln |V|)$ [25]. On the other hand, various models of epidemic spread over networks assume that transmission times across edges are independent random variables [27,34]. As a consequence, transmission times between vertices with many intermediate edges can have high variance. In this setting, resolving sets using expected transmission time between pairs of vertices as the metric may not be effective in identifying the source of the infection (aka ground zero).

The above factors motivate metrics in graphs that only rely on local vertex information in the graph. Assuming that a sensor that can detect long distances to landmarks can be costly, Jannesari and Omoomi [23] consider the situation where a robot can only detect landmarks that are adjacent to it. They define the adjacency dimension, $\mathrm{adim}(G)$, of G to be the minimum number of such landmarks that are needed for the robot to determine its position. From a more technical standpoint, [23] introduced adjacency dimension as a tool to study the metric dimension of lexicographic product graphs.

More generally, if the landmark detection range of a robot is $k > 0$, then the minimum number of such landmarks needed to determine the robot's position on the graph is called the k-truncated metric dimension (see [19] which calls it distance-k dimension).

With truncated metric dimension, elements of a resolving set are only able to distinguish vertices up to a certain distance; in particular, computation of the full distance matrix is no longer necessary. In the context of epidemics, by limiting the number of relevant edges on any shortest path between elements of a resolving set and other vertices in the graph, we have better control of the uncertainty of transmission times. Indeed, since the vertices of a graph can be at most distance $k+1$ from the nearest element of a k-truncated resolving set, these elements tend to be spread out over the space defined by the graph.

For a positive integer k and for $x, y \in V(G)$, let $d_k(x, y) = \min\{d(x, y), k+1\}$. We refer to this as the *distance-k* or *k-truncated distance* on G. For distinct $x, y \in V(G)$, let

$$R_k\{x, y\} = \{z \in V(G) : d_k(x, z) \neq d_k(y, z)\}.$$

A subset $S \subseteq V(G)$ is a *k-truncated resolving set* of G if $|S \cap R_k\{x, y\}| \geq 1$ for any pair of distinct vertices x and y in G, and the *k-truncated metric dimension* of G, denoted by $\dim_k(G)$, is the minimum cardinality over all k-truncated resolving sets of G.

For an ordered set $S = \{u_1, u_2, \ldots, u_\beta\} \subseteq V(G)$ of distinct vertices, the metric code and the k-truncated metric code, respectively, of $v \in V(G)$ with respect to S are the β-vectors

$$d(v|S) = (d(v, u_1), d(v, u_2), \ldots, d(v, u_\beta)),$$
$$d_k(v|S) = (d_k(v, u_1), d_k(v, u_2), \ldots, d_k(v, u_\beta)),$$

where k is any positive integer. Note that a distance-1 resolving set and the distance-1 dimension, respectively, of G corresponds to an *adjacency resolving set* and the *adjacency dimension* of G; in particular, $\dim_1(G) = \mathrm{adim}(G)$.

In this paper, we study the k-truncated metric dimension of graphs. The paper is organized as follows. In Section 2, we obtain some general results on k-truncated metric dimension of graphs that we use in the rest of the paper. In Section 3, we prove characterization results for k-truncated metric dimension. For all positive integers $k \geq 1$, we characterize all connected graphs G of order $n \geq 4$ for which $\dim_k(G)$ equals $n-2$ or $n-1$. In the case that $k = 1$, this solves the problem from [19] of characterizing the graphs G with $\mathrm{adim}(G) = n-2$ when G is connected.

In Section 4, we prove that the maximum possible order of a graph G with $\dim_k(G) = j$ is $(\lfloor \frac{2(k+1)}{3} \rfloor + 1)^j + j \sum_{i=1}^{\lceil \frac{k+1}{3} \rceil} (2i-1)^{j-1}$. For all $j, k \geq 1$, we also determine the maximum possible degree, clique number, and chromatic number of any graph G with $\dim_k(G) = j$.

In Section 5, we examine the relationship between the k-truncated metric dimension and planarity of graphs. In Section 6, we determine $\dim_k(G)$ for some classes of graphs, including paths and cycles.

In Section 7, we examine the effect of vertex or edge deletion on k-truncated metric dimension of graphs. Let v and e, respectively, denote a vertex and an edge of a graph G. For any positive integer $k \geq 1$, we show that $\dim_k(G-v) - \dim_k(G)$ can be arbitrarily large (also see [19] when $k = 1$); for $k \geq 2$, we show that $\dim_k(G) - \dim_k(G-v)$ can be arbitrarily large, whereas it was shown in [19] that $\dim_1(G) - \dim_1(G-v) \leq 1$. It was shown in [19] that $\dim_1(G) - 1 \leq \dim_1(G-e) \leq \dim_1(G) + 1$. We show that $\dim_2(G-e) \leq \dim_2(G) + 1$ and that $\dim_k(G-e) \leq \dim_k(G) + 2$ for $k \geq 3$. Moreover, in contrast to the case of 1-truncated metric dimension, we show that $\dim_k(G) - \dim_k(G-e)$ can be arbitrarily large for $k \geq 2$.

Finally, we focus on trees in Section 8. We define conditions under which finding the exact value of truncated metric dimension is straightforward, present a dynamic program capable of discovering minimum resolving sets on trees when only immediate neighbors are visible, and investigate extreme constructions for graphs in this family.

Related work. The k-truncated metric dimension of graphs was studied in [2], where it was also investigated more generally for metric spaces. The complexity of the problem was studied in [12,13], where it was shown that computing $\dim_k(G)$ is an NP-hard problem for any positive integer k. The graphs G with $\dim_k(G) = 1$ were characterized in [11], which also investigated the problem in a more general setting.

The present paper is a result of merging the preprints [20,37]. The preprint [37] was based on the Ph.D. thesis [35]. A review paper [36] also discussed truncated metric dimension.

Notation. In this paragraph, we introduce some notation that we use in the paper. For $x \in V(G)$ and $S \subseteq V(G)$, let $d(x, S) = \min\{d(x, y) : y \in S\}$. The *diameter*, $\operatorname{diam}(G)$, of G is $\max\{d(x, y) : x, y \in V(G)\}$. The *join* of two graphs H_1 and H_2, denoted by $H_1 + H_2$, is the graph obtained from the disjoint union of two graphs H_1 and H_2 by joining every vertex of H_1 with every vertex of H_2. We denote by P_n, C_n, K_n, and $K_{a,n-a}$ respectively, the path, the cycle, the complete graph, and the complete bipartite graph on n vertices with one part of size a. Suppose $f(x)$ and $g(x)$ are two functions defined for all sufficiently large real numbers x. We write $f(x) = O(g(x))$ if there exist positive constants N and C such that $|f(x)| \le C|g(x)|$ for all $x > N$, $f(x) = \Omega(g(x))$ if $g(x) = O(f(x))$, and $f(x) = \Theta(g(x))$ if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$.

2. General results

In this section, we obtain some general results for k-truncated metric dimension of graphs. In order to state the results in this section, we define some terminology. The *open neighborhood* of a vertex $v \in V(G)$ is $N(v) = \{u \in V(G) : uv \in E(G)\}$. For distinct $u, w \in V(G)$, if $N(u) - \{w\} = N(w) - \{u\}$, then u and w are called *twin vertices* of G. We begin with the following observations from [2,11,22,23] which we use in our proofs.

Observation 2.1. *Let u and w be twin vertices of a graph G, and let k be a positive integer. Then*

(a) ([22]) $S \cap \{u, w\} \neq \emptyset$ for any resolving set S of G;
(b) $S_k \cap \{u, w\} \neq \emptyset$ for any k-truncated resolving set S_k of G.

Observation 2.2 ([2,23]). *Let G be a connected graph of order $n \ge 2$. If $k \ge k'$ are positive integers, then $\dim(G) \le \dim_k(G) \le \dim_{k'}(G)$.*

Observation 2.3 ([11]). *Let G be a connected graph with $\operatorname{diam}(G) = d$, and let k be a positive integer.*

(a) If $d \in \{1, 2\}$, then $\dim_k(G) = \dim(G)$ for any positive integer k.
(b) If $d \ge 2$, then $\dim_k(G) = \dim_{d-1}(G) = \dim(G)$ for any $k \ge d - 1$.

In the next proof, we use a method similar to [7] to obtain a general upper bound on $\dim_k(G)$ in terms of the diameter of G. In Section 3, we use this result to characterize the connected graphs G of order n with $\dim_k(G) = n-2$ for all $k \ge 2$ and $n \ge 4$.

Theorem 2.4. *If G is a connected graph of order $n \ge 2$ and diameter d, then $\dim_k(G) \le n - \min\{d, k + 1\}$ for all $k \ge 1$.*

Proof. Suppose that u and v are vertices in G at distance d, and let $u = v_0, v_1, \ldots, v_d = v$ be a path of order $d + 1$ with endpoints u and v. If $d \le k + 1$, then let $S = V(G) - \{v_1, \ldots, v_d\}$. Note that $d_k(v_0, v_i) = i$ for each $1 \le i \le d$, so S is a k-truncated resolving set for G.

Otherwise $d > k + 1$. In this case, let $S = V(G) - \{v_1, \ldots, v_{k+1}\}$. Note that $d_k(v_0, v_i) = i$ for each $1 \le i \le k + 1$, so S is a k-truncated resolving set for G. □

It was shown in [9] that metric dimension is not a monotone parameter on subgraph inclusion. Moreover, it was shown in [19] that, for two graphs H and G with $H \subset G$, $\frac{\dim(H)}{\dim(G)}$ and $\frac{\dim_1(H)}{\dim_1(G)}$ can be arbitrarily large.

Following [19], for $m \ge 2$, let $H = K_{\frac{m(m+1)}{2}}$; let $V(H)$ be partitioned into $V_1, V_2, \ldots, V_m$ such that $V_i = \{w_{i,1}, w_{i,2}, \ldots, w_{i,i}\}$ with $|V_i| = i$, where $i \in \{1, 2, \ldots, m\}$. Let G be the graph obtained from H and m isolated vertices $u_1, u_2, \ldots, u_m$ such that, for each $i \in \{1, 2, \ldots, m\}$, u_i is joined by an edge to each vertex of $V_i \cup (\cup_{j=i+1}^{m}\{w_{j,i}\})$. Since $\operatorname{diam}(H) = 1$ and $\operatorname{diam}(G) = 2$, by Observation 2.3(a), $\dim(H) = \dim_k(H)$ and $\dim(G) = \dim_k(G)$ for every positive integer k. Note that $H \subset G$, $\dim(H) = \frac{m(m+1)}{2} - 1$ by Theorem 3.1(c), and $\dim(G) \le m$ since $\{u_1, u_2, \ldots, u_m\}$ forms a resolving set of G. So, $\frac{\dim_k(H)}{\dim_k(G)} = \frac{\dim(H)}{\dim(G)} \ge \frac{m^2+m-2}{2m}$ for every positive integer k, which implies the following.

Corollary 2.5. *For all positive integers k and N, there exist connected graphs H and G such that $H \subset G$ and $\frac{\dim_k(H)}{\dim_k(G)} > N$.*

Next, in view of Observation 2.2, we show that $\frac{\dim_k(G)}{\dim(G)}$ and $\frac{\dim_1(G)}{\dim_k(G)}$ can be arbitrarily large with respect to k; thus, $\dim_k(G) - \dim(G)$ and $\dim_1(G) - \dim_k(G)$ can be arbitrarily large with respect to k.

Proposition 2.6 ([6]). *For the grid graph $G = P_m \square P_n$ ($m, n \ge 2$), $\dim(G) = 2$.*

Theorem 2.7 ([19]). *For $m \ge 2$, let $G = P_m \square P_m$. Then $\dim_1(G) = \Theta(m^2)$; thus $\frac{\dim_1(G)}{\dim(G)}$ can be arbitrarily large with respect to m.*

Theorem 2.8. *For any positive integer $k > 1$, let $G = P_{k^2} \square P_{k^2}$. Then $\dim_k(G) = \Theta(k^2)$, and thus $\frac{\dim_k(G)}{\dim(G)}$ and $\frac{\dim_1(G)}{\dim_k(G)}$ can simultaneously be arbitrarily large with respect to k.*

Proof. First, observe that $\dim(G) = 2$ by Proposition 2.6, and $\dim_1(G) = \Theta(k^4)$ by Theorem 2.7. Next, we show that $\dim_k(G) = \Theta(k^2)$. First, note that there are $k^2(k-2)^2$ subgraphs $P_{2k+1}\square P_{2k+1}$ of G, of which at least $\lfloor \frac{k^4}{(2k+1)^2} \rfloor$ are disjoint. Since any k-truncated resolving set of G must contain at least one vertex from every $P_{2k+1}\square P_{2k+1}$ subgraph of G except for at most one such subgraph, $\dim_k(G) \geq \lfloor \frac{k^4}{(2k+1)^2} \rfloor - 1$. On the other hand, if the grid graph $P_{k^2}\square P_{k^2}$ is drawn in the xy-plane with the four corners at $(1, 1)$, $(k^2, 1)$, $(1, k^2)$ and (k^2, k^2) and with horizontal/vertical edges of equal lengths, then $[\cup_{j=0}^{k+1} \cup_{i=0}^{k+1} \{(1 + (k-1)i, 1 + (k-1)j)\}] \cup [\cup_{j=0}^{k} \cup_{i=0}^{k} \{(\lceil \frac{k}{2} \rceil + (k-1)i, \lceil \frac{k}{2} \rceil + (k-1)j)\}]$ forms a k-truncated resolving set of G, and hence $\dim_k(G) \leq (k+2)^2 + (k+1)^2 < 2(k+2)^2$. So, $\dim_k(G) = \Theta(k^2)$. Therefore, $\frac{\dim_k(G)}{\dim(G)}$ and $\frac{\dim_1(G)}{\dim_k(G)}$ can simultaneously be arbitrarily large with respect to k. $\square$

3. Characterizing graphs by their k-truncated metric dimension

It is known that, for any connected graph G of order at least two, $1 \leq \dim(G) \leq |V(G)| - 1$ (see [7]) and $1 \leq \dim_1(G) \leq |V(G)| - 1$ (see [23]). We recall some characterization results on metric dimension and 1-truncated dimension, before proving characterization results about k-truncated metric dimension.

Theorem 3.1 ([7]). *Let G be a connected graph of order $n \geq 2$. Then*

(a) $\dim(G) = 1$ *if and only if* $G = P_n$;
(b) *for* $n \geq 4$, $\dim(G) = n-2$ *if and only if* $G = K_{s,t}$ $(s, t \geq 1)$, $G = K_s + \overline{K_t}$ $(s \geq 1, t \geq 2)$, *or* $G = K_s + (K_1 \cup K_t)$ $(s, t \geq 1)$, *where* $\overline{H}$ *denotes the complement of a graph* H;
(c) $\dim(G) = n-1$ *if and only if* $G = K_n$.

Theorem 3.2 ([23]). *Let G be a connected graph of order $n \geq 2$. Then*

(a) $\dim_1(G) = 1$ *if and only if* $G \in \{P_2, P_3\}$;
(b) $\dim_1(G) = n-1$ *if and only if* $G = K_n$.

More generally, the characterization of graphs G with $\dim_1(G) = \beta$ is provided in [19] (this includes disconnected graphs). Given any graph G_1 on β vertices $v_1, \ldots, v_\beta$ and G_2 on 2^β vertices $\{u_b\}_{b\in\{0,1\}^\beta}$, define the graph $B(G_1, G_2)$ to be obtained by connecting v_i and u_b if and only if the ith digit of b is 1. Moreover, define $\mathcal{B}(G_1, G_2)$ to be the family of induced subgraphs of $B(G_1, G_2)$ that contain every vertex in G_1. Finally, define $\mathcal{H}_0 = \emptyset$ and, for each positive integer β, define $\mathcal{H}_\beta$ to be the family of graphs obtained from taking the union of $\mathcal{B}(G_1, G_2)$ over all graphs G_1 with j vertices $v_1, \ldots, v_j$ and G_2 with 2^j vertices $\{u_b\}_{b\in\{0,1\}^j}$, for each $1 \leq j \leq \beta$.

Theorem 3.3 ([19]). *For each $\beta \geq 1$, the set of graphs G with $\dim_1(G) = \beta$ is $\mathcal{H}_\beta - \mathcal{H}_{\beta-1}$ up to isomorphism.*

By the definition of $\dim_k(G)$, Observation 2.2, and Theorems 3.1 and 3.2, we have the following

Corollary 3.4. *Let G be a connected graph of order $n \geq 2$, and let k be any positive integer. Then $1 \leq \dim_k(G) \leq n-1$, and*

(a) ([11]) $\dim_k(G) = 1$ *if and only if* $G = P_i$ *for some* $i \in \{2, \ldots, k+2\}$,
(b) $\dim_k(G) = n-1$ *if and only if* $G = K_n$.

In the next result, we characterize the connected graphs G of order n with $\dim_k(G) = n-2$ for each $k \geq 2$. Interestingly, these are exactly the same connected graphs G of order n for which $\dim(G) = n-2$.

Theorem 3.5. *Let G be a connected graph of order $n \geq 4$, and let $k \geq 2$. Then $\dim_k(G) = n-2$ if and only if $G = K_{s,t}$ with $s, t \geq 1$, $G = K_s + \overline{K_t}$ with $s \geq 1$ and $t \geq 2$, or $G = K_s + (K_1 \cup K_t)$ with $s, t \geq 1$.*

Proof. First, note that all of the graphs G in the statement of the theorem have $\dim_k(G) = n-2$. This follows immediately from Observation 2.3(a) and Theorem 3.1(b), since all graphs G in the statement of the theorem have diameter 2, and each of these graphs G have $\dim(G) = n-2$. This proves the backward implication of the biconditional.

Now we prove the forward implication. Suppose that G is a connected graph of order $n \geq 4$ with $\dim_k(G) = n-2$. Since $k \geq 2$, the diameter of G must be 2 by Theorem 2.4 and Corollary 3.4(b). Thus $\dim(G) = n-2$ by Observation 2.3(a). Thus the result follows by Theorem 3.1(b). $\square$

It is interesting that the similarity between $\dim(G)$ and $\dim_k(G)$ breaks at $k = 1$. We just showed for $k \geq 2$ that the connected graphs G of order $n \geq 4$ with $\dim(G) = n-2$ are the same as the connected graphs G of order n with $\dim_k(G) = n-2$. However when $k = 1$, observe that $\dim_1(P_4) = 2$ but $\dim(P_4) = 1$, so there exists a connected graph G of order $n = 4$ with $\dim_1(G) = n-2$ and $\dim(G) < n-2$. In the next result, we show that this is the only connected graph G of order n for which $\dim_1(G) = n-2$ and $\dim(G) < n-2$. The next theorem answers an open problem from [19] in the case that G is connected.

Theorem 3.6. *Let G be a connected graph of order $n \geq 4$. Then $\dim_1(G) = n-2$ if and only if $G = K_{s,t}$ with $s, t \geq 1$, $G = K_s + \overline{K_t}$ with $s \geq 1$ and $t \geq 2$, $G = K_s + (K_1 \cup K_t)$ with $s, t \geq 1$, or $G = P_4$.*

Proof. First, note that all of the graphs G in the statement of the theorem have $\dim_1(G) = n-2$. For all of the graphs except for P_4, this follows immediately from Observation 2.3(a) and Theorem 3.1(b), since all graphs G in the statement of the theorem besides P_4 have diameter 2, and each of these graphs G have $\dim(G) = n-2$. In the case of P_4, clearly $\dim_1(P_4) = 2$. This proves the backward implication of the biconditional.

Now we prove the forward implication. Suppose that G is a connected graph of order $n \geq 4$ with $\dim_1(G) = n-2$. Note that G must have diameter at most 3, or else $\dim_1(G) \leq n-3$. To see why this is true, note that if G had two vertices u and v with $d(u, v) = 4$, then there would exist vertices x, y, z in G such that u, x, y, z, v is an induced path of order 5 in G. Then $V(G) - \{u, y, v\}$ would be a 1-truncated resolving set for G, contradicting the fact that $\dim_1(G) = n-2$, so G has diameter at most 3.

For the first case, suppose that G has diameter 3, so there exist vertices u and v in G with $d(u, v) = 3$. Since $d(u, v) = 3$, there must exist vertices $x, y \in V(G)$ such that u, x, y, v form an induced path of order 4 in G.

For contradiction, assume that G has another vertex besides u, v, x, y. Let t be a vertex in G that is not in the copy of P_4 such that t is adjacent to some vertex in the copy of P_4. Note that t must be adjacent to x or y. To see why this is true, note that if t was only adjacent to one of u or v and neither of x nor y, then G would have diameter at least 4. If t was adjacent to both u and v, then $d(u, v) \leq 2$, a contradiction. Thus, t is adjacent to at most one of u or v, and at least one of x or y. Without loss of generality, suppose that t is not adjacent to v.

Since t is adjacent to x or y, and t is not adjacent to v, there are several cases to consider. For the first case, suppose that t is only adjacent to a single vertex among u, v, x, y. This vertex must be x or y. Without loss of generality, let t be adjacent to x. Then $V(G) - \{x, t, v\}$ is a 1-truncated resolving set for G, so $\dim_1(G) \leq n-3$, a contradiction.

Now suppose that t is adjacent to two vertices among u, v, x, y. We know t is not adjacent to v, so either t is adjacent to u and x, t is adjacent to u and y, or t is adjacent to x and y. If t is adjacent to u and x, then $V(G)-\{x, t, v\}$ is a 1-truncated resolving set for G, so $\dim_1(G) \leq n-3$, a contradiction. If t is adjacent to u and y, then $V(G) - \{u, x, y\}$ is a 1-truncated resolving set for G, so $\dim_1(G) \leq n-3$, a contradiction. If t is adjacent to x and y, then $V(G) - \{v, x, y\}$ is a 1-truncated resolving set for G, so $\dim_1(G) \leq n-3$, a contradiction.

Now suppose that t is adjacent to three vertices among u, v, x, y. Since t is not adjacent to v, t must be adjacent to u, x, and y. Then $V(G) - \{v, x, y\}$ is a 1-truncated resolving set for G, so $\dim_1(G) \leq n-3$, a contradiction. This covers all of the possible cases, since t is not adjacent to v. Thus the only vertices in G are u, v, x, y, and these vertices form an induced path, so G is P_4 in the case that G has diameter 3.

Now we can assume that G has diameter 2, since we know that G has diameter at most 3, and we have already considered the case when G has diameter 3. Thus $\dim(G) = n-2$ by Observation 2.3(a) and the result follows by Theorem 3.1(b). □

4. Extremal results for k-truncated metric dimension

In this section, we derive several sharp extremal results about k-truncated metric dimension. First, we recall the following result by Hernando et al.

Theorem 4.1 ([22]). *Let G be a connected graph of order n, diam$(G) = d$, and $\dim(G) = \beta$. Then*

$$n \leq \left(\left\lfloor \frac{2d}{3} \right\rfloor + 1\right)^{\beta} + \beta \sum_{i=1}^{\lceil \frac{d}{3} \rceil} (2i-1)^{\beta-1}.$$

Since $\dim_k(G) = \beta$ implies $\dim(G) \leq \beta$ by Observation 2.2, we have the following.

Corollary 4.2. *For any positive integer k and for any connected graph G with diam$(G) = d$ and $\dim_k(G) = \beta$,*

$$|V(G)| \leq \left(\left\lfloor \frac{2d}{3} \right\rfloor + 1\right)^{\beta} + \beta \sum_{i=1}^{\lceil \frac{d}{3} \rceil} (2i-1)^{\beta-1}.$$

Using a method similar to the one in [22], we find a sharp upper bound on the maximum possible order of a graph G with $\dim_k(G) = j$.

Theorem 4.3. *The maximum possible order of a graph G with $\dim_k(G) = j$ is $(\lfloor \frac{2(k+1)}{3} \rfloor + 1)^j + j \sum_{i=1}^{\lceil \frac{k+1}{3} \rceil} (2i-1)^{j-1}$.*

Proof. First we prove the upper bound. Let G be a graph with $\dim_k(G) = j$. Let S be a k-truncated resolving set for G of size j and let $c \in [0, k]$ be an integer constant that will be chosen at the end. For each $v \in S$ and integer $i \in [0, c]$, define $N_i(v) = \{x \in V(G) : d_k(x, v) = i\}$.

Observe that $|d_k(x,u) - d_k(y,u)| \le 2i$ for any two vertices $x, y \in N_i(v)$ and any vertex $u \in S$, so $d_k(x,v) = i$ and $d_k(x,t)$ has at most $2i+1$ possible values by the triangle inequality for each $t \in S$ such that $t \ne v$. Thus $|N_i(v)| \le (2i+1)^{j-1}$.

Consider $x \in V(G)$ such that $x \notin N_i(v)$ for all $i \in [0,c]$ and $v \in S$, i.e., $c+1 \le d_k(x,v) \le k+1$ for all $v \in S$. Since S is a k-truncated resolving set for G and $|S| = j$, there are at most $(k-c+1)^j$ such vertices. Thus

$$|V(G)| \le (k-c+1)^j + j\sum_{i=0}^{c}(2i+1)^{j-1}.$$

Setting $c = \lceil \frac{k+1}{3} \rceil - 1$ gives the upper bound. To see that the upper bound is sharp, note that the construction in [22] of a graph G of maximum order with diameter $k+1$ and $\dim(G) = j$ must also have $\dim_k(G) = j$ and the same order as the bound we just obtained. $\square$

Remark 4.4. In [19], there is a simple construction of a graph G with $\dim_1(G) = j$ of maximum order $j + 2^j$. For the $k = 2$ case, we also found a simple construction of a graph G with $\dim_2(G) = j$ of maximum order $j + 3^j$, which is similar to a construction in [17]. Start with j copies of K_2, each on vertices a_i and b_i for $i \in \{1,\dots,j\}$. Let c_r for $r \in \{1,\dots,3^j\}$ be labeled with a ternary string of length j. Add an edge from c_r to a_i if the ith digit of c_r is 0. Add an edge from c_r to b_i if the ith digit of c_r is 1. Let $S = \{a_1,\dots,a_j\}$. Remove any c_r with the same 2-truncated vector as b_i with respect to S for each $i \in \{1,\dots,j\}$. The resulting graph G has order $j + 3^j$, and S is a 2-truncated resolving set, so $\dim_2(G) = j$.

For the remaining results in this section, we use some results from [17,18].

Theorem 4.5. *Fix $j \ge 1$. Among all graphs G with $\dim(G) \le j$,*

(a) [17] the maximum possible clique number of G is 2^j;
(b) [18] the maximum possible chromatic number of G is 2^j;
(c) [18] the maximum possible degree of G is $3^j - 1$;
(d) [18] the maximum possible degeneracy of G is $\frac{3^j-1}{2}$;
(e) [18] the maximum possible n for which G contains $K_{n,n}$ as a subgraph is $n = 2^{j-1}$.

Using the last theorem and Observation 2.2, we obtain several sharp extremal results for graphs G with $\dim_k(G) = j$. Our constructions for the remaining results in this section are similar to the paper [18], which defined an infinite family of infinite graphs and used that family to prove extremal results about the standard metric dimension. We define an infinite family of finite graphs $D_{k,j}$ with $k, j \ge 1$ such that $D_{k,j}$ is the graph on the vertex set $\{0, 1, \dots, k+1\}^j$ with edges between points that differ by at most one in each coordinate. Observe that $D_{k,j} \cong \boxtimes_{i=1}^{j} P_{k+2}$ holds. That is, $D_{k,j}$ is a strong product of paths.

Define $C_j(q)$ to be the induced subgraph of $D_{2q-1,j}$ whose vertex set consists of the integer points in the j-dimensional cross polytope centered at $(q,\dots,q)$ having as a face the $(j-1)$-simplex with its corners at the j points with all coordinates equal to q except for one coordinate which is equal to 0. It was proved in [18] that $\dim(C_j(q)) = j$ for all $q, j \ge 1$. Note that $\dim(C_j(q)) = \dim_k(C_j(q))$ for $k \ge 2q-1$ by Observation 2.3(b).

Theorem 4.6. *For all $k, j \ge 1$, the maximum possible clique number of any graph G with $\dim_k(G) = j$ is 2^j.*

Proof. The upper bound follows from Theorem 4.5(a) and Observation 2.2. For the lower bound, consider the graph G_j of order $j + 2^j$ with j vertices $u_1,\dots,u_j$, 2^j vertices v_b with $b \in \{0,1\}^j$, edges between v_b and $v_{b'}$ for all $b, b' \in \{0,1\}^j$ with $b \ne b'$, an edge between u_i and v_b if and only if the ith digit of b is 0, and no edges between vertices u_i and $u_{i'}$ with $i \ne i'$. Then $\{u_1,\dots,u_j\}$ is a k-truncated resolving set for G_j for each $k \ge 1$, so $\dim_k(G_j) \le j$. Moreover G_j contains a clique of size 2^j, so $\dim_k(G_j) \ge j$. Thus $\dim_k(G_j) = j$ and G_j has clique number 2^j. $\square$

We note that the lower bound construction in the last proof was also used for results about metric dimension, edge metric dimension, adjacency dimension, and broadcast dimension in [17,19,40]. We use the same construction to get a sharp result on complete bipartite graphs.

Theorem 4.7. *Fix $j, k \ge 1$. Among all graphs G with $\dim_k(G) \le j$, the maximum possible n for which G contains $K_{n,n}$ as a subgraph is $n = 2^{j-1}$.*

Proof. The upper bound follows from Theorem 4.5(e) and Observation 2.2. For the lower bound, we can take the clique of size 2^j in Theorem 4.6 and assign half of its vertices to the left and the other half to the right to form a $K_{2^{j-1},2^{j-1}}$ subgraph. $\square$

We can also use the result on clique number to obtain a sharp upper bound on the chromatic number.

Theorem 4.8. *For all $k, j \ge 1$, the maximum possible chromatic number of any graph G with $\dim_k(G) = j$ is 2^j.*

Proof. The lower bound follows from Theorem 4.6. The upper bound follows from Theorem 4.5(b) and Observation 2.2. □

Next we determine for all $k \geq 1$ the maximum possible degree of any graph G with $\dim_k(G) = j$. This result was previously known only for $k = 1$ [19].

Theorem 4.9. *Among all graphs G with $\dim_k(G) = j$:*

(a) the maximum possible degree of G is $3^j - 1$ for all $k \geq 2$ and all $j \geq 1$, and
(b) [19] the maximum possible degree of G is $2^j + j - 1$ for $k = 1$ and all $j \geq 1$.

Proof. The upper bound of $3^j - 1$ for (a) follows from Theorem 4.5(c).

For the lower bound of $3^j - 1$ in (a), we split the proof into two parts. For $k \geq 3$, we can take $C_j(2)$ and observe that the center vertex with all coordinates 2 has degree $3^j - 1$. For $k = 2$, we can take the intersection of $C_j(2)$ with $D_{2,j}$. In this graph, the center vertex with all coordinates 2 still has degree $3^j - 1$. □

Our final result in this section is about the degeneracy of a graph G, which is the minimum t such that every subgraph of G has a vertex of degree at most t. Unlike our other extremal results, this result only covers values of k that are sufficiently large with respect to j.

Theorem 4.10. *For all $j \geq 1$, there exists a constant k_j such that for each $k \geq k_j$, the maximum possible degeneracy of any graph G with $\dim_k(G) = j$ is $\frac{3^j-1}{2}$.*

Proof. The upper bound follows from Theorem 4.5(d).

For the lower bound, fix j and let q be sufficiently large so that the ratio of the number of exterior vertices to the total number of vertices in $C_j(q)$ is less than $\frac{2}{3^j-1}$. If we define n to be the number of vertices in $C_j(q)$, then the number of interior vertices in $C_j(q)$ is greater than $(1-\frac{2}{3^j-1})n$. Thus the number of edges in $C_j(q)$ is greater than $\frac{3^j-1}{2}(1-\frac{2}{3^j-1})n = (\frac{3^j-1}{2}-1)n$. Since any graph G with m edges and order n has degeneracy at least $\frac{m}{n}$, $C_j(q)$ must have degeneracy greater than $\frac{3^j-1}{2}-1$. Since degeneracy is an integer, $C_j(q)$ must have degeneracy at least $\frac{3^j-1}{2}$. We can let $k_j = 2q - 1$ since $C_j(q)$ has diameter $2q$. □

5. Planarity and the k-truncated metric dimension

Next, we consider the relation between $\dim_k(G)$ and planarity of G. A graph is *planar* if it can be drawn in a plane without any edge crossing. For two graphs G and H, H is called a *minor* of G if H can be obtained from G by vertex deletion, edge deletion, or edge contraction. We recall some known results on metric dimension and its variations in conjunction with planarity of a graph.

Theorem 5.1 (*[39]*)**.** *A graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G.*

Theorem 5.2 (*[26]*)**.**

(a) A graph G with $\dim(G) = 2$ cannot have K_5 or $K_{3,3}$ as a subgraph.
(b) There exists a non-planar graph G with $\dim(G) = 2$.

Theorem 5.3 (*[19]*)**.**

(a) If $\dim_1(G) = 2$, then G is planar; see Fig. 1(a) for graphs G with $\dim_1(G) = 2$.
(b) For each integer $\beta \geq 3$, there exists a non-planar graph G with $\dim_1(G) = \beta$.

Another variant of metric dimension was introduced by Eroh, Kang, and Yi [10]. A set S of vertices in G is called a *connected resolving set* of G if S resolves G and the subgraph of G induced by S is connected. The *connected metric dimension*, $\mathrm{cdim}(G)$, of G is the minimum cardinality over all connected resolving sets of G. For the characterization of graphs G with $\mathrm{cdim}(G) = 2$, see [10].

Theorem 5.4 (*[10]*)**.**

(a) If $\mathrm{cdim}(G) = 2$, then G is planar. However, there exists a non-planar graph G with $\dim(G) = 2$ and $\mathrm{cdim}(G) > 2$; see Fig. 1(b).
(b) For each integer $\beta \geq 3$, there exists a non-planar graph G with $\mathrm{cdim}(G) = \beta$.

Now, we consider the relation between k-truncated metric dimension and planarity of graphs.

Fig. 1. (a) [19] The graphs G satisfying $\dim_1(G) = 2$, where black vertices must be present, a solid edge must be present whenever the two vertices incident to the solid edge are in the graph, but a dotted edge is not necessarily present; (b) [10] A non-planar graph G with $\dim(G) = 2$ and $\mathrm{cdim}(G) = 3$, where black vertices form a minimum resolving set of G.

Theorem 5.5.

(a) For each $k \geq 2$, there is a non-planar connected graph G with $\dim_k(G) = 2$.
(b) For each $k \geq 1$ and $\beta \geq 3$, there is a non-planar connected graph G with $\dim_k(G) = \beta$.

Proof. For the first part, an example of a non-planar graph G with $\dim_k(G) = 2$ is given in Fig. 1(b), where black vertices form a minimum k-truncated resolving set of G for each $k \geq 2$.

For the second part, let G be a graph obtained from K_{m+2} ($m \geq 3$) by subdividing exactly one edge once; then G is non-planar by Theorem 5.1. It was shown in [10] that $\mathrm{cdim}(G) = \dim(G) = m$. Since $\mathrm{diam}(G) = 2$, $\dim_k(G) = \dim(G) = m$ by Observation 2.3(a). □

6. The k-truncated metric dimension of some classes of graphs

In this section, we determine $\dim_k(G)$ for some classes of graphs. First, we consider graphs G with $\mathrm{diam}(G) \leq 2$. For two graphs H_1 and H_2, $\mathrm{diam}(H_1 + H_2) \leq 2$; thus, by Observation 2.3(a), $\dim(H_1 + H_2) = \dim_k(H_1 + H_2)$ for any positive integer k.

Theorem 6.1 (*[4,30]*)**.** *For $n \geq 3$,*

$$\dim(C_n + K_1) = \begin{cases} 3 & \text{if } n \in \{3, 6\}, \\ \lfloor \frac{2n+2}{5} \rfloor & \text{otherwise.} \end{cases}$$

In [21], Harary and Melter claimed that $\dim(H_1 + H_2) = \dim(H_1) + \dim(H_2)$ for all graphs H_1 and H_2. However, Theorem 6.1 contradicts this claim for $H_1 = C_n$ and $H_2 = K_1$.

Theorem 6.2 (*[5]*)**.** *For $n \geq 1$,*

$$\dim(P_n + K_1) = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n \in \{2, 3\}, \\ 3 & \text{if } n = 6, \\ \lfloor \frac{2n+2}{5} \rfloor & \text{otherwise.} \end{cases}$$

By Observation 2.3(a) and Theorems 6.1 and 6.2, we have the following.

Corollary 6.3. *For any positive integer k and for $n \geq 3$,*

$$\dim_k(C_n + K_1) = \begin{cases} 3 & \text{if } n \in \{3, 6\}, \\ \lfloor \frac{2n+2}{5} \rfloor & \text{otherwise.} \end{cases}$$

Corollary 6.4. *For any positive integer k and for $n \geq 1$,*

$$\dim_k(P_n + K_1) = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n \in \{2, 3\}, \\ 3 & \text{if } n = 6, \\ \lfloor \frac{2n+2}{5} \rfloor & \text{otherwise.} \end{cases}$$

The metric dimension of complete multi-partite graphs was determined in [29].

Theorem 6.5 (*[29]*). *For $m \geq 2$, let $G = K_{a_1,a_2,\ldots,a_m}$ be a complete m-partite graph of order $n = \sum_{i=1}^{m} a_i$. Let s be the number of partite sets of G consisting of exactly one element. Then*

$$\dim(G) = \begin{cases} n - m & \text{if } s = 0, \\ n - m + s - 1 & \text{if } s \neq 0. \end{cases}$$

As an immediate consequence of Observation 2.3(a) and Theorem 6.5, we have the following.

Corollary 6.6. *For $m \geq 2$, let $G = K_{a_1,a_2,\ldots,a_m}$ be a complete m-partite graph of order $n = \sum_{i=1}^{m} a_i$. Let s be the number of partite sets of G consisting of exactly one element. Then, for any positive integer k,*

$$\dim_k(G) = \begin{cases} n - m & \text{if } s = 0, \\ n - m + s - 1 & \text{if } s \neq 0. \end{cases}$$

Note that by Corollary 3.4(b) and Corollary 6.6, the star graph $K_{1,n-1}$ attains the maximum possible value of $\dim_k(T)$ over all trees T of order n. Further, $\dim_k(K_{1,n-1}) = n - 2$ for $n \geq 3$.

Now, we recall the metric dimension of the Petersen graph.

Theorem 6.7 (*[24]*). *For the Petersen graph $\mathcal{P}$, $\dim(\mathcal{P}) = 3$.*

Since $\text{diam}(\mathcal{P}) = 2$, Observation 2.3(a) and Theorem 6.7 imply the following

Corollary 6.8. *For the Petersen graph $\mathcal{P}$ and for any positive integer k, $\dim_k(\mathcal{P}) = 3$.*

Next, we determine the k-truncated metric dimension of cycles. We recall the following results.

Proposition 6.9 (*[28]*). *For $n \geq 3$, $\dim(C_n) = 2$.*

Proposition 6.10 (*[23]*). *For $n \geq 4$, $\dim_1(C_n) = \lfloor \frac{2n+2}{5} \rfloor$.*

Following [4], let M be a set of at least two vertices of C_n, let u_i and u_j be distinct vertices of M, and let P and P' denote the two distinct $u_i - u_j$ paths determined by C_n. If either P or P', say P, contains only two vertices of M (namely, u_i and u_j), then we refer to u_i and u_j as *neighboring vertices* of M and the set of vertices of P that belong to $C_n - \{u_i, u_j\}$ as the *gap* of M (determined by u_i and u_j). The two gaps of M determined by a vertex of M and its two neighboring vertices of M are called *neighboring gaps*. Note that, M has r gaps if $|M| = r$, where some of the gaps may be empty.

Lemma 6.11. *For a positive integer k, let M_k be a minimum k-truncated resolving set of C_n for $n \geq 2k + 3$. Then*

(a) Every gap of M_k contains at most $2k + 1$ vertices. Moreover, at most one gap of M_k contains $2k + 1$ vertices.
(b) If a gap of M_k contains at least $k + 1$ vertices, then any neighboring gaps contain at most k vertices.

Proof. For a positive integer k, let M_k be a minimum k-truncated resolving set of C_n for $n \geq 2k + 3$.

(a) If there is a gap of M_k containing $2k + 2$ consecutive vertices $u_j, u_{j+1}, \ldots, u_{j+2k+1}$ of C_n, where $0 \leq j \leq n - 1$ and the subscript is taken modulo n, then $d_k(u_{j+k}|M_k) = d_k(u_{j+k+1}|M_k)$, a contradiction. If there exist two distinct gaps $\{u_p, u_{p+1}, \ldots, u_{p+2k}\}$ and $\{u_q, u_{q+1}, \ldots, u_{q+2k}\}$ of M_k, then $d_k(u_{p+k}|M_k) = d_k(u_{q+k}|M_k)$, a contradiction.

(b) Suppose a gap of M_k contains at least $k+1$ vertices, and one of its neighboring gaps contains more than k vertices. Then there exist $2k + 3$ consecutive vertices $u_j, u_{j+1}, \ldots, u_{j+2k+2}$ of C_n such that $M_k \cap \{u_j, u_{j+1}, \ldots, u_{j+2k+2}\} = \{u_{j+k+1}\}$ and $d_k(u_{j+k}|M_k) = d_k(u_{j+k+2}|M_k)$, a contradiction. □

Theorem 6.12. *Let $n \geq 3$ and let k be any positive integer.*

(a) If $n \leq 3k + 3$, then $\dim_k(C_n) = 2$.
(b) If $n \geq 3k + 4$, then

$$\dim_k(C_n) = \begin{cases} \lfloor \frac{2n+3k-1}{3k+2} \rfloor & \text{if } n \equiv 0, 1, \ldots, k+2 \pmod{3k+2}, \\ \lfloor \frac{2n+4k-1}{3k+2} \rfloor & \text{if } n \equiv k+3, \ldots, \lceil \frac{3k+5}{2} \rceil - 1 \pmod{3k+2}, \\ \lfloor \frac{2n+3k-1}{3k+2} \rfloor & \text{if } n \equiv \lceil \frac{3k+5}{2} \rceil, \ldots, 3k+1 \pmod{3k+2}. \end{cases}$$

Proof. Let C_n be given by $u_0, u_1, \ldots, u_{n-1}, u_0$ for $n \geq 3$, and let k be a positive integer.

(a) Let $n \leq 3k+3$. Since $\{u_0, u_\alpha\}$, where $\alpha = \min\{2k+2, n-1\}$, forms a k-truncated resolving set of C_n, $\dim_k(C_n) \leq 2$. By Corollary 3.4(a), $\dim_k(C_n) \geq 2$. Thus, $\dim_k(C_n) = 2$ for $n \leq 3k + 3$.

(b) Let $n \geq 3k + 4$; then $\dim_k(C_n) \geq 3$. Since k is a positive integer, we must have $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 2$.

First, we show that $\dim_k(C_n) \geq \lfloor \frac{2n+3k-1}{3k+2} \rfloor$; moreover, we show that $\dim_k(C_n) \geq \lfloor \frac{2n+4k-1}{3k+2} \rfloor$ if $n \equiv k+3, \ldots, \lceil \frac{3k+5}{2} \rceil - 1 \pmod{3k+2}$. Let S_k be a minimum k-truncated resolving set of C_n. If $|S_k| = 2\ell$ for some positive integer ℓ, then at most ℓ gaps contain more than k vertices by Lemma 6.11(b), and those ℓ gaps contain at most $2k$ vertices except possibly one gap containing $2k+1$ vertices by Lemma 6.11(a); thus, the number of vertices belonging to the gaps of S_k is at most $3k\ell+1$, and hence $n-2\ell \le 3k\ell+1$, which implies $|S_k| = 2\ell \ge \lceil \frac{2n-2}{3k+2} \rceil = \lfloor \frac{2n+3k-1}{3k+2} \rfloor$. If $|S_k| = 2\ell+1$ for some positive integer ℓ, then at most ℓ gaps contain more than k vertices by Lemma 6.11(b), and those ℓ gaps contain at most $2k$ vertices except possibly one gap containing $2k+1$ vertices by Lemma 6.11(a); thus, the number of vertices belonging to the gaps of S_k is at most $3k\ell+k+1$, and hence $n-(2\ell+1) \le 3k\ell+k+1$, which implies $|S_k| = 2\ell+1 \ge \lceil \frac{2n+k-2}{3k+2} \rceil = \lfloor \frac{2n+4k-1}{3k+2} \rfloor \ge \lfloor \frac{2n+3k-1}{3k+2} \rfloor$.

Now, suppose $n = (3k+2)x+j$, where $x \ge 1$ and $k+3 \le j \le \lceil \frac{3k+5}{2} \rceil - 1$; notice $k \ge 2$. Then $|S_k| = 2x+2$. To see why $|S_k| \le 2x+2$, note that $R = (\cup_{i=0}^{x-1}\{u_{(3k+2)i}, u_{(3k+2)i+2k+1}\}) \cup \{u_{(3k+2)x}, u_\beta\}$, where $\beta = \min\{(3k+2)x+2k+1, n-1\}$, is a k-truncated resolving set of C_n with $|R| = 2x+2$. To see why $|S_k| \ge 2x+2$, first observe that $|S_k| \ge 2x+1$ follows from the lower bounds that we proved in the last paragraph. However if $|S_k| = 2x+1$, then we proved that $|S_k| \ge \lceil \frac{2n+k-2}{3k+2} \rceil \ge 2x+2$, giving a contradiction. Since $|S_k| = 2x+2$, we have $|S_k| = \lfloor \frac{2n+4k-1}{3k+2} \rfloor$ in this case.

Now we show that $\dim_k(C_n) \le \lfloor \frac{2n+3k-1}{3k+2} \rfloor$ if $n \equiv 0, 1, \dots, k+2 \pmod{3k+2}$ or $n \equiv \lceil \frac{3k+5}{2} \rceil, \dots, 3k+1 \pmod{3k+2}$.

Case 1: $n = (3k+2)x+j$, where $x \ge 1$ and $0 \le j \le 1$. Note that $\lfloor \frac{2n+3k-1}{3k+2} \rfloor = 2x$. Let $S_k = \{u_0, u_{2k+2}\} \cup (\cup_{i=1}^{x-1}\{u_{(3k+2)i+1}, u_{(3k+2)i+2k+2}\})$. Then S_k is a k-truncated resolving set of C_n with $|S_k| = 2x$. So, $\dim_k(C_n) \le |S_k| = 2x = \lfloor \frac{2n+3k-1}{3k+2} \rfloor$.

Case 2: $n = (3k+2)x+j$, where $x \ge 1$ and $2 \le j \le k+2$. Note that $\lfloor \frac{2n+3k-1}{3k+2} \rfloor = 2x+1$. Let $S_k = \{u_0, u_{2k+2}\} \cup (\cup_{i=1}^{x-1}\{u_{(3k+2)i+1}, u_{(3k+2)i+2k+2}\}) \cup \{u_{(3k+2)x+1}\}$. Since S_k is a k-truncated resolving set of C_n with $|S_k| = 2x+1$, $\dim_k(C_n) \le |S_k| = 2x+1 = \lfloor \frac{2n+3k-1}{3k+2} \rfloor$.

Case 3: $n = (3k+2)x+j$, where $x \ge 1$ and $\lceil \frac{3k+5}{2} \rceil \le j \le 3k+1$. Note that $\lfloor \frac{2n+3k-1}{3k+2} \rfloor = 2x+2$. Let $S_k = (\cup_{i=0}^{x-1}\{u_{(3k+2)i}, u_{(3k+2)i+2k+1}\}) \cup \{u_{(3k+2)x}, u_\alpha\}$, where $\alpha = \min\{n-1, (3k+2)x+2k+1\}$. Then S_k is a k-truncated resolving set of C_n with $|S_k| = 2x+2$. So, $\dim_k(C_n) \le |S_k| = 2x+2 = \lfloor \frac{2n+3k-1}{3k+2} \rfloor$. $\square$

Remark 6.13. Note that, for $n \ge 4$, Proposition 6.10 is an immediate corollary of Theorem 6.12 when $k = 1$.

Next, we determine the k-truncated metric dimension of paths. We recall the following result.

Proposition 6.14 ([23]). *For $n \ge 4$, $\dim_1(P_n) = \lfloor \frac{2n+2}{5} \rfloor$.*

Let P_n be an n-path given by $u_0, u_1, \dots, u_{n-1}$, where $n \ge 4$. Similar to the case for C_n, we define gaps and neighboring gaps of a vertex subset M of P_n analogously, where $|M| \ge 2$. If $d(u_0, M) = x$, then the set $\{u_0, u_1, \dots, u_{x-1}\}$ is called the *initial gap* of M; similarly, if $d(u_{n-1}, M) = y$, then the set $\{u_{n-1}, u_{n-2}, \dots, u_{n-y}\}$ is called the *terminal gap* of M. The union of the initial gap and the terminal gap of M is called the *union gap* of M. If $u_0 \in M$ ($u_{n-1} \in M$, respectively), then the initial gap (terminal gap, respectively) is empty. The following lemma is analogous to Lemma 6.11, after adjusting for paths.

Lemma 6.15. *For a positive integer k, let M_k be a minimum k-truncated resolving set of P_n for $n \ge k+3$. Then*

(a) Every gap of M_k contains at most $2k+1$ vertices, the initial gap of M_k contains at most $k+1$ vertices, and the terminal gap of M_k contains at most $k+1$ vertices. Moreover, at most one gap of M_k contains $2k+1$ vertices and the union gap of M_k contains at most $2k+1$ vertices, but not both.

(b) If a gap of M_k contains at least $k+1$ vertices, then any neighboring gaps contain at most k vertices. If the initial gap or the terminal gap of M_k contains at least one vertex, then its neighboring gap contains at most k vertices.

Proof. Let P_n be given by $u_0, u_1, \dots, u_{n-1}$.

(a) If there is a gap of M_k containing $2k+2$ consecutive vertices $u_j, u_{j+1}, \dots, u_{j+2k+1}$ of P_n, where $1 \le j \le n-2k-2$, then $d_k(u_{j+k}|M_k) = d_k(u_{j+k+1}|M_k)$. If the initial gap or the terminal gap of M_k, say the former without loss of generality, contains $k+2$ consecutive vertices $u_0, u_1, \dots, u_{k+1}$, then $d_k(u_0|M_k) = d_k(u_1|M_k)$. If there exist two distinct gaps $u_p, u_{p+1}, \dots, u_{p+2k}$ and $u_q, u_{q+1}, \dots, u_{q+2k}$ of M_k, then $d_k(u_{p+k}|M_k) = d_k(u_{q+k}|M_k)$. If there exists a gap of M_k containing $2k+1$ consecutive vertices, say $u_r, u_{r+1}, \dots, u_{r+2k}$, and the union gap of M_k contains $2k+1$ vertices, say $(\cup_{i=0}^{k}\{u_i\}) \cup (\cup_{j=1}^{k}\{u_{n-j}\})$, then $d_k(u_{r+k}|M_k) = d_k(u_0|M_k)$.

(b) If a gap of M_k contains at least $k+1$ vertices and one of its neighboring gaps contains more than k vertices, then there exist $2k+3$ consecutive vertices $u_j, u_{j+1}, \dots, u_{j+2k+2}$ of P_n such that $M_k \cap \{u_j, u_{j+1}, \dots, u_{j+2k+2}\} = \{u_{j+k+1}\}$ and $d_k(u_{j+k}|M_k) = d_k(u_{j+k+2}|M_k)$. If the initial gap or the terminal gap of M_k, say the former, contains α vertices, where $1 \le \alpha \le k+1$, and its neighboring gap contains more than k vertices, then there exist $k+3$ consecutive vertices $u_{\alpha-1}, u_\alpha, \dots, u_{\alpha+k+1}$ of P_n such that $M_k \cap \{u_{\alpha-1}, u_\alpha, \dots, u_{\alpha+k+1}\} = \{u_\alpha\}$ and $d_k(u_{\alpha-1}|M_k) = d_k(u_{\alpha+1}|M_k)$. $\square$

Theorem 6.16. *Let $n \ge 2$ and let k be any positive integer.*

(a) If $n \le k+2$, then $\dim_k(P_n) = 1$.

(b) If $k+3 \le n \le 3k+3$, then $\dim_k(P_n) = 2$.

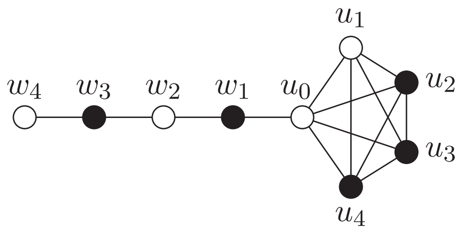

Fig. 2. The graph $U_{5,4}$, where the black vertices form a minimum 1-truncated resolving set.

(c) If $n \ge 3k+4$, then

$$\dim_k(P_n) = \begin{cases} \lfloor \frac{2n+3k-1}{3k+2} \rfloor & \text{if } n \equiv 0, 1, \ldots, k+2 \pmod{3k+2}, \\ \lfloor \frac{2n+4k-1}{3k+2} \rfloor & \text{if } n \equiv k+3, \ldots, \lceil \frac{3k+5}{2} \rceil - 1 \pmod{3k+2}, \\ \lfloor \frac{2n+3k-1}{3k+2} \rfloor & \text{if } n \equiv \lceil \frac{3k+5}{2} \rceil, \ldots, 3k+1 \pmod{3k+2}. \end{cases}$$

Proof. Let $n \ge 2$ and let k be a positive integer.

(a) If $n \le k+2$, then $\dim_k(P_n) = 1$ by Corollary 3.4(a).

(b) Suppose $k+3 \le n \le 3k+3$. If P_n is given by $u_0, u_1, \ldots, u_{n-1}$, then $\{u_k, u_\alpha\}$, where $\alpha = \min\{2k+1, n-1\}$, forms a k-truncated resolving set of P_n; thus $\dim_k(P_n) \le 2$. By Corollary 3.4(a), $\dim_k(P_n) = 2$.

(c) Let $n \ge 3k+4$. First, we show that $\dim_k(P_n) \le \lfloor \frac{2n+3k-1}{3k+2} \rfloor$ if $n \equiv 0, 1, \ldots, k+2 \pmod{3k+2}$ or $n \equiv \lceil \frac{3k+5}{2} \rceil, \ldots, 3k+1 \pmod{3k+2}$, and $\dim_k(P_n) \le \lfloor \frac{2n+4k-1}{3k+2} \rfloor$ if $n \equiv k+3, \ldots, \lceil \frac{3k+5}{2} \rceil - 1 \pmod{3k+2}$. By Lemmas 6.11 and 6.15, for $n \ge 3k+4$ every k-truncated resolving set of P_n and C_n, respectively, has cardinality at least three. Moreover, there exists a minimum k-truncated resolving set S of $C_n = P_n + e$ with a gap containing $2k+1$ vertices $u_j, u_{j+1} \ldots, u_{j+2k}$ in C_n, where $0 \le j \le n-1$ and the subscript is taken modulo n. If $e = u_{j+k}u_{j+k+1}$, then S forms a k-truncated resolving set of P_n; thus $\dim_k(P_n) \le \dim_k(C_n)$ and the desired upper bounds follow from Theorem 6.12.

Second, we show that $\dim_k(P_n) \ge \lfloor \frac{2n+3k-1}{3k+2} \rfloor$; moreover, we show that $\dim_k(P_n) \ge \lfloor \frac{2n+4k-1}{3k+2} \rfloor$ if $n \equiv k+3, \ldots, \lceil \frac{3k+5}{2} \rceil - 1 \pmod{3k+2}$. Let S_k be a minimum k-truncated resolving set of P_n such that the union gap of S_k contains $2k+1$ vertices. If $|S_k| = 2\ell$ for some positive integer ℓ, then at most $\ell - 1$ gaps contain more than k vertices by Lemma 6.15(b) and those $\ell - 1$ gaps contain at most $2k$ vertices by Lemma 6.15(a); thus, the number of vertices belonging to the gaps of S_k or the union gap of S_k is at most $2k(\ell-1) + k\ell + (2k+1) = 3k\ell + 1$, and hence $n - 2\ell \le 3k\ell + 1$, which implies $|S_k| = 2\ell \ge \dim_k(C_n)$. If $|S_k| = 2\ell + 1$ for some positive integer ℓ, then at most $\ell - 1$ gaps contain more than k vertices by Lemma 6.15(b) and those $\ell - 1$ gaps contain at most $2k$ vertices by Lemma 6.15(a); thus, the number of vertices belonging to the gaps of S_k or the union gap of S_k is at most $2k(\ell-1) + k(\ell+1) + (2k+1) = 3k\ell + k + 1$, and hence $n - (2\ell+1) \le 3k\ell + k + 1$, which implies $|S_k| = 2\ell + 1 \ge \dim_k(C_n)$. In each case, $\dim_k(P_n) \ge \dim_k(C_n)$, and thus the desired lower bounds follow from Theorem 6.12. $\square$

Remark 6.17. Note that, for $n \ge 4$, Proposition 6.14 is an immediate corollary of Theorem 6.16 when $k = 1$.

Next, we show a simple upper bound on $\dim_k(G)$ given a fixed diameter.

Lemma 6.18. *For any connected graph G of order n with $diam(G) = \delta$, $\dim_k(G) \le \dim_k(P_{\delta+1}) + (n - (\delta+1))$.*

Proof. Suppose u and v are vertices in G at distance δ, and let $P_{\delta+1}$ be a path of order $\delta+1$ with end points u and v in G. If $R_{\delta+1}$ is a minimum k-truncated resolving set of $P_{\delta+1} \subseteq G$, then $R = (V(G) - V(P_{\delta+1})) \cup R_{\delta+1}$ is a k-truncated resolving set of G. Since $|R| = (n - (\delta+1)) + \dim_k(P_{\delta+1})$, $\dim_k(G) \le \dim_k(P_{\delta+1}) + (n - (\delta+1))$. $\square$

Next, we show the sharpness of the bound in Lemma 6.18. Clearly, the bound in the lemma is achieved for $G \in \{K_n, P_n\}$. For $a \ge 3$ and $b \ge 1$, let $U_{a,b}$ be the graph obtained from the disjoint union of K_a and P_b by joining an edge between a vertex of K_a and a leaf of P_b (see Fig. 2 for $U_{5,4}$). Let $V(K_a) = \{u_0, u_1, u_2, \ldots, u_{a-1}\}$ and let P_b be given by $w_1, w_2, \ldots, w_b$ such that $u_0w_1 \in E(U_{a,b})$; note that $U_{a,b}$ has order $a+b$ and diameter $b+1$.

Lemma 6.19. *For the graph $U_{a,b}$ of order $n = a+b$ and diameter $b+1 \le n-2$, where $a \ge 3$ and $b \ge 1$,*

$$\dim_k(U_{a,b}) = \begin{cases} a-1 & \text{if } b \le 2k+1, \\ a-1+\dim_k(P_{b-2k}) & \text{otherwise.} \end{cases}$$

Proof. Let $U_{a,b}$ be the graph of order $n = a+b$ and diameter $b+1$, where $2 \le b+1 \le n-2$, with the labeling described above (see Fig. 2). Let S be any minimum k-truncating resolving set for $U_{a,b}$. Since any distinct vertices in $\cup_{i=1}^{a-1}\{u_i\}$ are

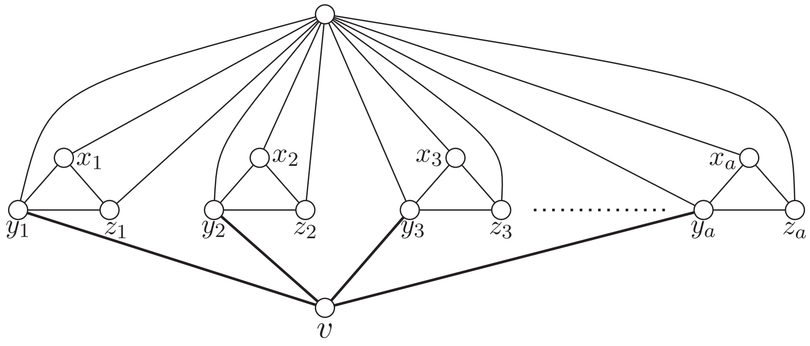

Fig. 3. [19] Graphs G, with $a \geq 2$, such that $\dim(G-v)-\dim(G)=\dim_1(G-v)-\dim_1(G)$ can be arbitrarily large.

twins in $U_{a,b}$, $|S \cap (\cup_{i=1}^{a-1}\{u_i\})| \geq a-2$. Without loss of generality, let $S_0 = \cup_{i=2}^{a-1}\{u_i\} \subseteq S$. Since $d_k(u_0|S_0) = d_k(u_1|S_0)$, at least one vertex lying on the $u_1 - w_b$ path must belong to S. So, $|S| \geq a-1$.

First, suppose $\operatorname{diam}(U_{a,b}) \leq 2(k+1)$; then $b \leq 2k+1$. If $b \geq k$, then $S_0 \cup \{w_k\}$ forms a k-truncated resolving set of $U_{a,b}$. If $1 \leq b \leq k-1$, then $S_0 \cup \{w_b\}$ forms a k-truncated resolving set of $U_{a,b}$. In each case, $|S| \leq |S_0|+1 = a-1$. Since $|S| \geq a-1$, $\dim_k(U_{a,b}) = a-1$.

Second, suppose $\operatorname{diam}(U_{a,b}) > 2(k+1)$; then $b \geq 2k+2$. Let $S_1 = S_0 \cup \{w_k\}$. Then S_1 is a k-truncated resolving set of $U_{a,2k+1} \subseteq U_{a,b}$ and $d_k(w_i|S_1) = (k+1,\ldots,k+1)$ for each $i \in \{2k+1, 2k+2, \ldots, b\}$. If S' is a minimum k-truncated resolving set for the $w_{2k+1}-w_b$ path, then $S_1 \cup S'$ is a k-truncated resolving set of $U_{a,b}$ with $|S_1|+|S'| = a-1+\dim_k(P_{b-2k})$; thus, $\dim_k(U_{a,b}) \leq a-1+\dim_k(P_{b-2k})$. Since any vertex subset $R \subseteq V(U_{a,b})$ with $|R| \leq a-2+\dim_k(P_{b-2k})$ fails to form a k-truncated resolving set of $U_{a,b}$, $|S| \geq a-1+\dim_k(P_{b-2k})$. So, $\dim_k(U_{a,b}) = a-1+\dim_k(P_{b-2k})$. □

It can be shown that $\dim_k(P_{\delta+1}) + (n-(\delta+1)) - \dim_k(U_{n+1-\delta,\delta-1}) \in \{0,1\}$. In particular, this construction achieves the bound in Lemma 6.18 for certain values of n and δ.

Lemma 6.20. *For* $2 \leq \delta \leq n-2$

$$\dim_k(P_{\delta+1}) + (n-(\delta+1)) - \dim_k(U_{n+1-\delta,\delta-1}) \in \{0,1\}.$$

Proof. Suppose that $b = \delta-1 \leq 2k+1$. Then

$$\dim_k(P_{\delta+1}) + (n-(\delta+1)) - \dim_k(U_{n+1-\delta,\delta-1}) = \dim_k(P_{b+2}) - 1 \in \{0,1\}$$

as $\dim_k(P_{b+2})$ is 1 or 2 by Theorem 6.16.

If $b > 2k+1$ instead, we have

$$\dim_k(P_{\delta+1}) + (n-(\delta+1)) - \dim_k(U_{n+1-\delta,\delta-1}) = \dim_k(P_{b+2}) - \dim_k(P_{b-2k}) - 1.$$

In general, $\dim_k(P_n) \leq \dim_k(P_{n-m}) + \dim_k(P_{m+1})$ and $\dim_k(P_n) \geq \dim_k(P_m)$ for $m < n$. Hence, $\dim_k(P_{b+2}) \leq \dim_k(P_{b-2k}) + \dim_k(P_{2k+3})$ and $\dim_k(P_{b+2}) - \dim_k(P_{b-2k}) \leq 2$. Further, $\dim_k(P_{b+2}) > \dim_k(P_{b-2k})$ by Theorem 6.16, and the result follows. □

7. The effect of vertex or edge deletion on the k-truncated metric dimension of graphs

Let v and e, respectively, denote a vertex and an edge of a connected graph G such that both $G-v$ and $G-e$ are connected graphs. First, we consider the effect of vertex deletion on k-truncated metric dimension of graphs. We recall the following results on the effect of vertex deletion on metric dimension and 1-truncated dimension.

Proposition 7.1.

(a) ([4]) $\dim(G) - \dim(G-v)$ *can be arbitrarily large;*
(b) ([8]) $\dim(G-v) - \dim(G)$ *can be arbitrarily large.*

Proposition 7.2 (*[19]*).

(a) For any graph G, $\dim_1(G) \leq \dim_1(G-v)+1$, *where the bound is sharp.*
(b) The value of $\dim_1(G-v) - \dim_1(G)$ *can be arbitrarily large, as* G *varies (see Fig. 3).*

For graphs G in Fig. 3, note that $\operatorname{diam}(G) = \operatorname{diam}(G-v) = 2$, where $a \geq 2$. It was shown in [19] that $\dim(G) = a+1$ and $\dim(G-v) = 2a$. By Observation 2.3(a), for any positive integer k, we have $\dim_k(G) = \dim(G) = a+1$ and $\dim_k(G-v) = \dim(G-v) = 2a$, which implies the following.

Corollary 7.3. *Let k be any positive integer. The value of $\dim_k(G-v)-\dim_k(G)$ can be arbitrarily large, as G varies.*

In contrast to the case for 1-truncated dimension (see Proposition 7.2(a)), we show that $\dim_k(G)-\dim_k(G-v)$ can be arbitrarily large for $k \geq 2$.

Proposition 7.4. *For any positive integer $k \geq 2$, $\dim_k(G)-\dim_k(G-v)$ can be arbitrarily large.*

Proof. Let $k \geq 2$ and $x \geq 1$ be integers. Let $G = C_{5(3k+2)x} + K_1$ with the vertex v in the K_1. Then $\dim_k(G) = \lfloor \frac{10(3k+2)x+2}{5} \rfloor = 2(3k+2)x = 6kx+4x$ by Corollary 6.3, and $\dim_k(G-v) = \lfloor \frac{10(3k+2)x+3k-1}{3k+2} \rfloor = 10x$ by Theorem 6.12(b). So, $\dim_k(G)-\dim_k(G-v) = 6kx+4x-10x = 6(k-1)x \to \infty$ as $x \to \infty$ for $k \geq 2$. □

Next, we consider the effect of edge deletion on k-truncated metric dimension of graphs. Throughout the section, let $d_{H,k}(v_1, v_2)$ denote $d_k(v_1, v_2)$ in a graph H. We recall the following results on the effect of edge deletion on metric dimension and 1-truncated dimension.

Theorem 7.5 ([8]).

(a) For any graph G and any edge $e \in E(G)$, $\dim(G-e) \leq \dim(G)+2$.
(b) The value of $\dim(G)-\dim(G-e)$ can be arbitrarily large.

Theorem 7.6 ([19]). *For any graph G and any edge $e \in E(G)$, $\dim_1(G)-1 \leq \dim_1(G-e) \leq \dim_1(G)+1$.*

The proof for Theorem 7.5(a) in [8], adjusted for the case of k-truncated metric dimension, provides the following result. We include its proof to be self-contained.

Proposition 7.7. *Let $k \geq 3$ be any integer. For any graph G and any edge $e \in E(G)$, $\dim_k(G-e) \leq \dim_k(G)+2$.*

Proof. Let S be a minimum k-truncated resolving set for G, and let $e = uw$. We show that $S \cup \{u, w\}$ is a k-truncated resolving set for $G-e$. Let x and y be distinct vertices in $V(G-e) = V(G)$ such that, for some $z \in S$, $d_{G,k}(x, z) \neq d_{G,k}(y, z)$ and $d_{G-e,k}(x, z) = d_{G-e,k}(y, z)$. We consider two cases.

Case 1: $d_{G,k}(x, z) = d_{G-e,k}(x, z)$ or $d_{G,k}(y, z) = d_{G-e,k}(y, z)$, but not both. Suppose $d_{G,k}(y, z) = d_{G-e,k}(y, z)$. Then $d_{G,k}(y, z) = d_{G-e,k}(y, z) = d_{G-e,k}(x, z) > d_{G,k}(x, z)$, $d_{G,k}(x, z) \leq k$, and the edge e must lie on every $x-z$ geodesic in G. So, up to transposing the labels u and w, we have $d_{G,k}(x, u)+d_{G,k}(u, w)+d_{G,k}(w, z) = d_{G,k}(x, z)$. Notice that $d_{G,k}(x, u) = d_{G-e,k}(x, u)$ since there is an $x-u$ geodesic in G that does not use the edge e. Since $d_{G,k}(x, u) + d_{G,k}(u, z) = d_{G,k}(x, z) < d_{G,k}(y, z) \leq d_{G,k}(y, u) + d_{G,k}(u, z)$, we must have $d_{G,k}(x, u) < d_{G,k}(y, u)$. Then $d_{G-e,k}(x, u) = d_{G,k}(x, u) < d_{G,k}(y, u) \leq d_{G-e,k}(y, u)$ and $d_{G-e,k}(x, u) \leq k-1$.

Case 2: $d_{G,k}(x, z) \neq d_{G-e,k}(x, z)$ and $d_{G,k}(y, z) \neq d_{G-e,k}(y, z)$. In this case, the edge e must lie on every $x-z$ geodesic and on every $y-z$ geodesic in G. Moreover, we must have either $d_{G,k}(x, z) < d_{G,k}(y, z) \leq k$ or $d_{G,k}(y, z) < d_{G,k}(x, z) \leq k$. Notice that if a geodesic from some vertex a to another vertex c traverses the edge e in the order u, w (as apposed to w, u), then a geodesic containing e from any vertex b to c must also traverse e in the order u, w. Suppose that u is traversed before w by an $x-z$ geodesic and a $y-z$ geodesic (directed towards z) in G. Then an $x-u$ geodesic and a $y-u$ geodesic, neither containing the edge e, are obtained by removing a $u-z$ geodesic in G from the $x-z$ geodesic and $y-z$ geodesic respectively. Thus, $d_{G-e,k}(x, u) \neq d_{G-e,k}(y, u)$. □

Remark 7.8. Note that Proposition 7.7 and its proof hold when $k = 1$ or $k = 2$. For $k \in \{1, 2\}$, we obtain the stronger result that $\dim_k(G-e) \leq \dim_k(G)+1$. For $k = 1$ this follows from Theorem 7.6. To see why it is true for $k = 2$, let S be a minimum 2-truncated resolving set of G, let $e = uw$, and let x and y be distinct vertices in $V(G-e) = V(G)$ such that $d_{G,2}(x, z) \neq d_{G,2}(y, z)$ and $d_{G-e,2}(x, z) = d_{G-e,2}(y, z)$ for some $z \in S$; further, suppose that the edge e lies on every $x-z$ geodesic in G and $d_{G,2}(x, u) < d_{G,2}(x, w)$.

First, we consider Case 1. Then $0 < d_{G,2}(x, z) \leq 2$; notice that $x \neq z$ since the edge e lies on every $x-z$ geodesic in G. If $d_{G,2}(x, z) = 1$, then $e = uw = xz$; if $d_{G,2}(x, z) = 2$, then $x = u$ or $xu \in E(G)$. In each case, $S \cup \{u\}$ forms a 2-truncated resolving set for $G-e$. Next, we consider Case 2. Then $d_{G,2}(x, z) < d_{G,2}(y, z) \leq 2$ or $d_{G,2}(y, z) < d_{G,2}(x, z) \leq 2$, say the former; then $e = uw = xz$ and x lies on every $y-z$ geodesic in G. So $S \cup \{u\}$ forms a 2-truncated resolving set for $G-e$. Therefore, $\dim_2(G-e) \leq \dim_2(G)+1$.

For graphs G satisfying $\dim_2(G-e) = \dim_2(G)+1$, see Fig. 4, where $a, b, c \geq 2$; one can easily check that $R = (\cup_{i=1}^{a-1}\{x_i\}) \cup (\cup_{i=1}^{b-1}\{y_i\}) \cup (\cup_{i=1}^{c-1}\{z_i\})$ forms a minimum 2-truncated resolving set of $G-e$ with $|R| = a+b+c-3$ and that $R' = R-\{z_1\}$ forms a minimum 2-truncated resolving set for G with $|R'| = a+b+c-4$.

Remark 7.9. The bound in Proposition 7.7 is sharp. For any integer $k \geq 3$, let G be the graph in Fig. 4 and let $e = x_1z_1$. Let $L_1 = \cup_{i=1}^{a}\{x_i\}$, $L_2 = \cup_{i=1}^{b}\{y_i\}$ and $L_3 = \cup_{i=1}^{c}\{z_i\}$, where $a, c \geq 3$ and $b \geq 2$.

First, we show that $\dim_k(G-e) = a+b+c-3$. Note that any two vertices in L_i are twin vertices in $G-e$, where $i \in \{1, 2, 3\}$. So, for any k-truncated resolving set S of $G-e$, we have $|S \cap L_1| \geq a-1$, $|S \cap L_2| \geq b-1$, and $|S \cap L_3| \geq c-1$

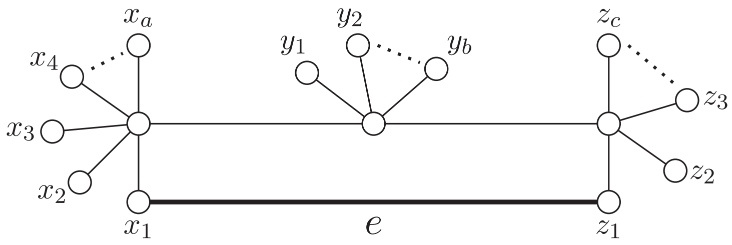

Fig. 4. Graphs G with $\dim_2(G-e) = \dim_2(G)+1$ and $\dim_k(G-e) = \dim_k(G)+2$ for $k \geq 3$.

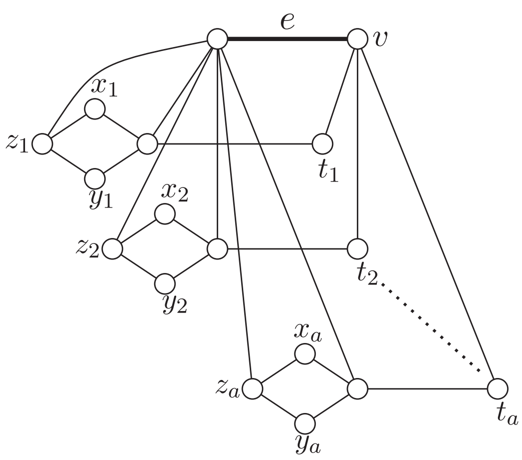

Fig. 5. Graphs G, with $k \geq 2$ and $a \geq 2$, such that $\dim_k(G) - \dim_k(G-e)$ can be arbitrarily large.

by Observation 2.1(b); thus, $\dim_k(G-e) \geq a+b+c-3$. On the other hand, $(L_1 \cup L_2 \cup L_3) - \{x_1, y_1, z_1\}$ forms a k-truncated resolving set of $G-e$, and hence $\dim_k(G-e) \leq a+b+c-3$. Thus, $\dim_k(G-e) = a+b+c-3$.

Second, we show that $\dim_k(G) = a+b+c-5$. For any k-truncated resolving set S' of G, we have $|S' \cap (L_1 - \{x_1\})| \geq a-2$, $|S' \cap L_2| \geq b-1$, and $|S' \cap (L_3 - \{z_1\})| \geq c-2$ by Observation 2.1(b); thus, $\dim_k(G) \geq a+b+c-5$. Since $(L_1 \cup L_2 \cup L_3) - \{x_1, x_2, y_1, z_1, z_2\}$ forms a k-truncated resolving set of G, $\dim_k(G) \leq a+b+c-5$. So, $\dim_k(G) = a+b+c-5$.

Therefore, $\dim_k(G-e) = \dim_k(G)+2$ for $k \geq 3$.

In contrast to Theorem 7.6, we show that $\dim_k(G) - \dim_k(G-e)$ can be arbitrarily large for any integer $k \geq 2$.

Theorem 7.10. *For any integer $k \geq 2$, the value of $\dim_k(G) - \dim_k(G-e)$ can be arbitrarily large.*

Proof. Let G be the graph in Fig. 5. For each $i \in \{1, 2, \ldots, a\}$, $N_G(x_i) = N_G(y_i) = \{z_i, z_i'\} = N_{G-e}(x_i) = N_{G-e}(y_i)$. Let $k \geq 2$ and $a \geq 2$ be any integers. Let S be any minimum k-truncated resolving set for $G-e$, and let S' be any k-truncated resolving set for G. By Observation 2.1(b), $S \cap \{x_i, y_i\} \neq \emptyset$ and $S' \cap \{x_i, y_i\} \neq \emptyset$ for each $i \in \{1, 2, \ldots, a\}$; without loss of generality let $S_0 = \cup_{i=1}^{a}\{x_i\} \subseteq S \cap S'$.

First, we show that $\dim_k(G-e) = a+1$. Since $d_k(z_i|S_0) = d_k(z_i'|S_0)$ for each $i \in \{1, 2, \ldots, a\}$ in $G-e$, $|S| \geq a+1$, and hence $\dim_k(G-e) \geq a+1$. Since $S_0 \cup \{v\}$ forms a k-truncated resolving set of $G-e$, $\dim_k(G-e) \leq a+1$. So, $\dim_k(G-e) = a+1$.

Second, we show that $\dim_k(G) = 2a$. Note that, for each $i \in \{1, 2, \ldots, a\}$, $d_k(z_i|S_0) = d_k(z_i'|S_0)$ in G and $R_k\{z_i, z_i'\} = \{z_i, z_i', t_i\}$; thus, $S' \cap \{z_i, z_i', t_i\} \neq \emptyset$ for each $i \in \{1, 2, \ldots, a\}$. So, $|S'| \geq 2a$, and hence $\dim_k(G) \geq 2a$. Since $S_0 \cup (\cup_{i=1}^{a}\{z_i\})$ forms a k-truncated resolving set of G, $\dim_k(G) \leq 2a$. Thus, $\dim_k(G) = 2a$.

Therefore, $\dim_k(G) - \dim_k(G-e) = 2a - (a+1) = a-1 \to \infty$ as $a \to \infty$. $\square$

8. Trees

Many problems that are NP-complete on arbitrary graphs have efficient solutions when restricted to trees. This phenomenon occurs with traditional metric dimension [21,31]. In this section, we present some preliminary results regarding the behavior of truncated metric dimension on trees. In particular, we define a family of trees for which finding exact truncated metric dimension is straightforward and describe a polynomial time algorithm for determining the 1-truncated metric dimension of trees.

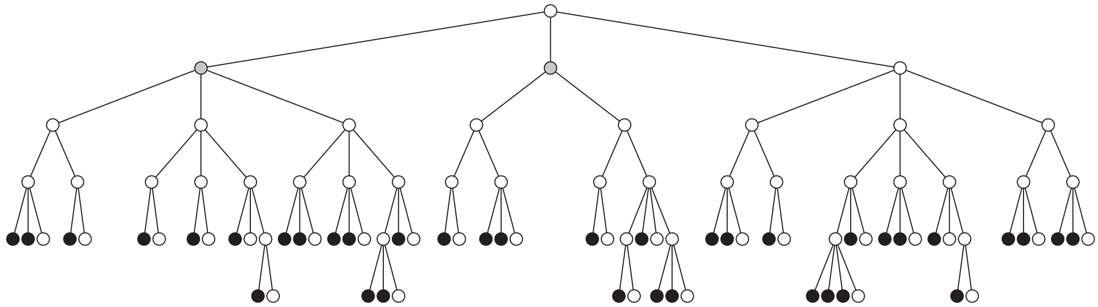

Fig. 6. An element of $\mathbb{T}_2$. The black and gray vertices constitute a minimum 2-truncated resolving set.

Let $T = (V, E)$ be a tree and call $v \in V$ an *exterior major vertex* if it has degree at least three and there is at least one leaf $u \in V$ for which the path $v, \ldots, u$ contains no vertices, except v, with degree greater than two. Let $\ell(T)$ be the set of leaves on T, $\sigma(T)$ be the set of exterior major vertices on T, and $\Delta(v)$ be the set of leaves associated with a vertex $v \in \sigma(T)$. For any tree T that is not a path, $\dim(T) = |\ell(T)| - |\sigma(T)|$ and $R = \bigcup_{v\in\sigma(T)} \Delta(v) \setminus \{x_v\}$ is a minimum resolving set where x_v is any element of $\Delta(v)$ [7]. As the proof that R is minimum relies on having access to full distance information, the precise relationship between this construction and truncated metric dimension on arbitrary trees is unclear. Nevertheless, certain aspects of this proof, along with constraints placed on paths by tree structures, suggest that there may be efficient means of finding minimum k-truncated resolving sets on arbitrary trees.

8.1. The $\mathbb{T}_k$ family of trees

First, we define a class of trees for which we can find minimum k-truncated resolving sets using the construction described above for traditional metric dimension directly.

Let $\mathbb{T}_k$ be a family of trees defined recursively as follows. Let the empty tree, $T = (\{\}, \{\})$, be in $\mathbb{T}_k$. Then $T = (V, E) \in \mathbb{T}_k$ if four conditions hold.

1. T is connected.
2. There are no vertices of degree two in V.
3. For all minimum non-truncated resolving sets R of T and for all $v \in V$, the vector of distances $d_k(v|R)$ is unique or $(k + 1, \ldots, k + 1)$ (the vector of all $(k + 1)$'s).
4. For all minimum non-truncated resolving sets R of T:

$$T' = T \setminus \{v \text{ such that } (\forall u \in V \setminus \{v\}) : d_k(v|R) \neq d_k(u|R)\} \in \mathbb{T}_k.$$

$\mathbb{T}_k$ for $k > 1$ includes, for example, disjoint unions of three or more perfect m-ary trees with $m \geq 2$ of the same height, with an additional vertex acting as a common root.

Condition (3) may seem difficult to verify at first glance; however, condition (2) significantly restricts the set of minimum non-truncated resolving sets—as seen in the next result.

Lemma 8.1 (*[28]*)**.** *If $T \in \mathbb{T}_k$ then every minimum non-truncated resolving set R of T must have the form $\bigcup_{v\in\sigma(T)} \Delta(v) \setminus \{x_v\}$, where x_v is any element of $\Delta(v)$.*

The definition of $\mathbb{T}_k$ suggests an iterative method to find minimum k-truncated resolving sets for trees in this family. Intuitively, we can find minimum k-truncated resolving sets of $T \in \mathbb{T}_k$ by constructing a minimum non-truncated resolving set, removing vertices that this set resolves with k-truncated distances, and repeating until we are left with a tree with at most one vertex. For example, when this approach is applied to the tree in Fig. 6, black vertices are selected for a 2-truncated resolving set on the first iteration and gray vertices are selected on the second iteration. More precisely, we have the following result.

Theorem 8.2. *Let $T_0 \in \mathbb{T}_k$ and R_0 be a minimum non-truncated resolving set on T_0. Further, let the sequence of pairs $(T_1, R_1), \ldots, (T_n, R_n)$ be generated by repeated application of condition (4) in the definition of $\mathbb{T}_k$. In particular, R_j is a minimum non-truncated resolving set of T_j and $T_j = T_{j-1} \setminus \{v \text{ such that } (\forall u \in V \setminus \{v\}) : d_k(v|R_{j-1}) \neq d_k(u|R_{j-1})\}$. If $n > 0$ and $|T_n| \leq 1$, then $\dim_k(T_0) = \sum_{j=0}^{n-1} |R_j|$. Otherwise, $\dim_k(T_0) = \sum_{j=0}^{n} |R_j|$.*

Proof. First, notice that R_j must be of the form $\bigcup_{v\in\sigma(T_j)} \Delta(v) \setminus \{x_v\}$, with $x_v \in \Delta(v)$ for all $0 \leq j \leq n$. Then, based on the definition of T_j, $\bigcup_{j=0}^{n} R_j$ is a k-truncated resolving set of T_0. Further, for any k-truncated resolving set R of T_0 there may be a single vertex $v \in T_0$ such that $d_k(v, r) = k + 1$ for all $r \in R$. In particular, if $n > 0$ and $|T_n| \leq 1$, $d_k(v, r) = k + 1$

where $v \in T_n$ for all $r \in \bigcup_{j=0}^{n-1} R_j$. Since, for all other vertices $u \in T_0 \setminus T_n$, there must be at least one $r \in \bigcup_{j=0}^{n-1} R_j$ such that $d_k(u, r) < k+1$, this is a unique representation and $\bigcup_{j=0}^{n-1} R_j$ is a k-truncated resolving set of T_0. Otherwise, R_n is required to differentiate the vertices of T_n. As a result, $\dim_k(T_0) \leq \sum_{j=0}^{n-1} |R_j|$ if $n > 0$ and $|T_n| \leq 1$ and $\dim_k(T_0) \leq \sum_{j=0}^{n} |R_j|$ otherwise.

To see that these are also lower bounds on $\dim_k(T_0)$, suppose that R' is a k-truncated resolving set of T_0 taking a different form than that described above. In particular, suppose that there is at least one $0 < j \leq n$ such that there is no subset $R'_j \subseteq R'$ of the form $\bigcup_{v\in\sigma(T_j)} \Delta(v) \setminus \{x_v\}$, with $x_v \in \Delta(v)$. Let i be one such value of j.

We note that $R_0 \subseteq R'$ as at least two leaves of T_0 would be indistinguishable otherwise. Similarly, in order to distinguish vertices in T_i, R' must include at least one vertex from all but one of the subtrees rooted at vertices in $\Delta(v)$ where $v \in \sigma(T_i)$. The distance from these vertices to the associated element of $\sigma(T_i)$ cannot exceed $k+1$, otherwise at least two leaves of T_i would be indistinguishable. This means that, if $n > 0$ and $|T_n| \leq 1$, $|R'| \geq \sum_{j=0}^{n-1} |R_j|$ and $\dim_k(T_0) = \sum_{j=0}^{n-1} |R_j|$. Otherwise, $|R'| \geq \sum_{j=0}^{n} |R_j|$ and $\dim_k(T_0) = \sum_{j=0}^{n} |R_j|$. $\square$

8.2. Adjacency dimension on trees

In this section we focus our attention on $k = 1$, and present an algorithm for finding minimum 1-truncated resolving sets on trees in polynomial time.

Let $T = (V, E)$ be a tree with at least two vertices and an arbitrary root. For all $v \in V$, let $C(v)$ be the set of children of v. We call $R \subseteq V$ a *locating dominating set* when R is a 1-truncated resolving set and, for each $v \in V$, there is $r \in R$ such that $d_1(v, r) \leq 1$. Put another way, each $v \in V \setminus R$ must be adjacent to a unique non-empty subset of vertices in R.

There exists an algorithm for finding minimum locating dominating sets on trees [32,33]—though it is not obvious how this approach might be modified to find minimum 1-truncated resolving sets. In this section, we describe a novel dynamic programming based algorithm for computing adjacency dimension exactly on trees.

For all $v \in V$, let T_v be the subtree of T rooted at v. Consider the following definitions:

- $R(v)$ is the size of a minimum locating dominating set R_v of T_v.
- $R'(v)$ is the size of minimum locating dominating sets $R \subseteq V(T_v)$ for $T_v \setminus \{v\}$ such that there is at least one $r \in R$ with $d_1(r, v) \leq 1$. R'_v is one such set.
- $R''(v)$ is the size of minimum locating dominating sets $R \subseteq V(T_v)$ for $T_v \setminus \{v\}$. R''_v is one such set.
- $R'''(v)$ is the size of minimum locating dominating sets $R \subseteq V(T_v) \setminus \{v\}$ for T_v. R'''_v is one such set.

It is easy to see that, for all $v \in \ell(T)$ except possibly the root, $R(v) = 1$, $R'(v) = 1$, and $R''(v) = 0$. Note that $R'''(v)$ is undefined for leaves but, as we will see shortly, it can be defined non-recursively. We describe expressions for each of these quantities before presenting the algorithm itself.

Assume that we have $R(u)$, $R'(u)$, $R''(u)$, and $R'''(u)$ for all $u \in C(v)$ for some $v \in V$. Consider a locating dominating set R_v of T_v. Either $v \in R_v$ or $v \notin R_v$. In the first case, all children of v are adjacent to at least one element of R_v, namely v. To guarantee that each child is adjacent to a different non-empty subset of R_v, there may be a single $u \in C(v)$ adjacent only to $v \in R_v$ while all other $w \in C(v) \setminus \{u\}$ must be adjacent to at least one other vertex of R_v. Consequently, $R_v = \{v\} \cup R''_u \cup (\bigcup_{w\in C(v)\setminus\{u\}} R'_w)$ for some choice of $u \in C(v)$. Then, in this case and taking each $u \in C(v)$ into account, $R(v) = 1 + \min_{u\in C(v)}\{R''(u) + \sum_{w\in C(v)\setminus\{u\}} R'(w)\}$ (Eq. (1)).

If $v \notin R_v$ instead, there are two additional possibilities. Either only one or at least two children of v are included in R_v. Suppose $u \in C(v)$ is the only element of R_v in $C(v)$. Since v is not adjacent to another vertex in R_v, all children of u must be adjacent to at least one other element of R_v. The remaining children of v must be located and dominated without v. Thus, $R_v = \{u\} \cup (\bigcup_{w\in C(u)} R'_w) \cup (\bigcup_{w\in C(v)\setminus\{u\}} R_w)$ for some choice of $u \in C(v)$ and $R(v) = \min_{u\in C(v)}\{1 + \sum_{w\in C(u)} R'(w) + \sum_{w\in C(v)\setminus\{u\}} R(w)\}$ (Eq. (2)).

Next, suppose $u, w \in C(v)$ are both in R_v. Because v is the only vertex that can possibly be adjacent to both u and w, we follow an argument identical to when $v \in R_v$ but focus instead on u and w. In particular, to guarantee that each child of u is adjacent to a different non-empty set of R_v, there may be a single $x \in C(u)$ adjacent only to $u \in R_v$ while all $y \in C(u) \setminus \{x\}$ must be adjacent to at least one other vertex of R_v. A symmetric argument applies to children of w. The remaining children of v must be located and dominated without v. As a result, R_v can be expressed as the union of the three sets

$$R_u = \{u\} \cup R''_{x_u} \cup \left(\bigcup_{y\in C(u)\setminus\{x_u\}} R'_y \right)$$

$$R_w = \{w\} \cup R''_{x_w} \cup \left(\bigcup_{y\in C(w)\setminus\{x_w\}} R'_y \right)$$

$$R_z = \bigcup_{x\in C(v)\setminus\{u,w\}} R_x,$$

for some choice of the pair $u, w \in C(v)$ and for some choice of x_u and x_w. Thus, in this case, $R(v)$ can be described with Eqs. (3)–(5), and Eqs. (1)–(5) fully describe $R(v)$:

$$R(v) = \min \left\{ \quad 1 + \min_{u \in C(v)} \left\{ R''(u) + \sum_{w \in C(v) \setminus \{u\}} R'(w) \right\}, \right. \tag{1}$$

$$\min_{u \in C(v)} \left\{ 1 + \sum_{w \in C(u)} R'(w) + \sum_{w \in C(v) \setminus \{u\}} R(w) \right\}, \tag{2}$$

$$\min_{u,w \in C(v)} \left\{ 2 + \min_{x \in C(u)} \left\{ R''(x) + \sum_{y \in C(u) \setminus \{x\}} R'(y) \right\} \right. \tag{3}$$

$$+ \min_{x \in C(w)} \left\{ R''(x) + \sum_{y \in C(w) \setminus \{x\}} R'(y) \right\} \tag{4}$$

$$\left. \left. + \sum_{x \in C(v) \setminus \{u,w\}} R(x) \right\} \quad \right\}. \tag{5}$$

$R'(v)$ is nearly identical to $R(v)$. The only difference occurs when $v \notin R'_v$. Since we are not concerned with ensuring that v is adjacent to a different non-empty subset of R'_v as compared to all other vertices, we can focus on the case when at least one child of v is in R'_v. Suppose $u \in C(v)$ is in R'_v. Following an argument similar to when $v \in R_v$, to guarantee that each child of u is adjacent to a different non-empty subset of R'_v, there may be a single $w \in C(u)$ adjacent only to u while all other $x \in C(u)$ must be adjacent to at least one other vertex of R'_v. In this case, the remaining children of v must be located and dominated without v. This yields Eqs. (7) and (8) and a full definition of $R'(v)$ below.

$$R'(v) = \min \left\{ \quad 1 + \min_{u \in C(v)} \left\{ R''(u) + \sum_{w \in C(v) \setminus \{u\}} R'(w) \right\}, \right. \tag{6}$$

$$\min_{u \in C(v)} \left\{ 1 + \min_{w \in C(u)} \left\{ R''(w) + \sum_{x \in C(u) \setminus \{w\}} R'(x) \right\} \right. \tag{7}$$

$$\left. \left. + \sum_{w \in C(v) \setminus \{u\}} R(w) \right\} \quad \right\}. \tag{8}$$

For $R''(v)$, we do not require that v be adjacent to any element of R''_v. However, all children of v must be both located and dominated. So, if $v \notin R''_v$, we need sets R_u for each $u \in C(v)$ (Eq. (10)). Again, the case when $v \in R''_v$ is identical to the corresponding cases for $R(v)$ and $R'(v)$ (Eq. (9)):

$$R''(v) = \min \left\{ \quad 1 + \min_{u \in C(v)} \left\{ R''(u) + \sum_{w \in C(v) \setminus \{u\}} R'(w) \right\}, \right. \tag{9}$$

$$\left. \sum_{u \in C(v)} R(u) \quad \right\}. \tag{10}$$

Finally, $R'''(v)$ follows directly from $R(v)$ when $v \notin R_v$ (Eqs. (11)–(14)). We note here that, when $u \in C(v)$, $R'''(u)$ forces $d_1(v, r) = 2$ for all $r \in R'''_u$:

$$R'''(v) = \min \left\{ \quad \min_{u \in C(v)} \left\{ 1 + \sum_{w \in C(u)} R'(w) + \sum_{w \in C(v) \setminus \{u\}} R(w) \right\}, \right. \tag{11}$$

$$\min_{u,w \in C(v)} \left\{ 2 + \min_{x \in C(u)} \left\{ R''(x) + \sum_{y \in C(u) \setminus \{x\}} R'(y) \right\} \right. \tag{12}$$

$$+ \min_{x \in C(w)} \left\{ R''(x) + \sum_{y \in C(w) \setminus \{x\}} R'(y) \right\} \tag{13}$$

$$\left. \left. + \sum_{x \in C(v) \setminus \{u,w\}} R(x) \right\} \quad \right\}. \tag{14}$$

We are now ready to define an algorithm for finding the adjacency dimension of trees. Intuitively, Algorithm 1 determines the size of minimum locating dominating sets on T and then considers the possibility that each $v \in V$ may be the only vertex not adjacent to any element of a 1-truncated resolving set. In particular, suppose $v \in V$ is to be this vertex. Since every $u \in C(v)$ must be located and dominated, but cannot be included in any 1-truncated resolving set, we

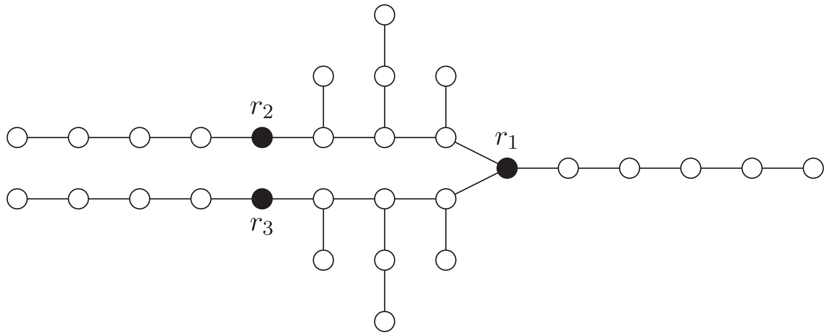

Fig. 7. A visualization of $\widetilde{S}_{3,4}$. Black vertices form a minimum 4-truncated resolving set.

are interested in R_u''' for every u. Then, $R = \bigcup_{u\in C(v)} R_u'''$ locates and dominates every vertex $w \in V \setminus \{v\}$, guaranteeing that $d_1(v, r) = 2$ for all $r \in R$. On the other hand, picking any $v \in V$, R_v is a locating dominating set of minimum size for T. Since a 1-truncated resolving set of minimum size must either leave one vertex at distance 2 from all elements of the set or dominate all vertices, taking the minimum over the sizes of these sets gives us $dim_1(T)$.

We note that by using well established methods for keeping track of vertices solving the minimizations in $R(v)$ and $R'''(v)$, Algorithm 1 can be modified to return a minimum 1-truncated resolving set of T.

Algorithm 1 Minimum 1-Truncated Resolving Sets on Trees

Input: $T = (V, E)$, a tree with $|V| > 2$
Output: Adjacency dimension of T
function $\dim_1(T)$
  $S \leftarrow \{R(v)\}$, with any $v \in V$ as the root
  for all $v \in V$ **do**
    $S \leftarrow S \cup \{\sum_{u\in C(v)} R'''(u)\}$
  return $\min(S)$

8.3. Extreme tree constructions

We end our exploration of truncated metric dimension on trees by examining structures in this family with extreme values of $\dim_k$. We observed earlier that for each $k \geq 1$, the maximum possible value of $\dim_k(T)$ over trees T of order $n \geq 3$ is $n-2$, attained by $K_{1,n-1}$.

Next, we describe a family of trees $\widetilde{S}_{\beta,k}$ such that $\dim_k(\widetilde{S}_{\beta,k}) = \beta$ and $\dim_k(\widetilde{S}_{\beta,k}) \leq \dim_k(T)$ for any tree T with $|\widetilde{S}_{\beta,k}|$ vertices. This construction is from [1] and improves upon a conjecture made in [37].

Let $R = \{r_1, \ldots, r_\beta\}$ and, for each $r_j \in R$, construct a path of length k with r_j as an endpoint. Include a single extra vertex at the end of the path associated with r_1. This vertex will have truncated distance $k+1$ to all elements of R. Now, for each $r_j \in R \setminus \{r_1\}$, add a path to r_1 of length $(2k+\ell)/3$ where $\ell = k \bmod 3$. For each vertex v on the path connecting r_1 and r_j, add a new path of length $k - \max\{d(v, r_1), d(v, r_j)\}$ with v as an endpoint. In particular, the other endpoint of these paths will be at distance k from at least one of r_1 and r_j. The number of vertices on these paths, including those on the path between r_1 and r_j, is $(k^2+k+1)/3$ if $k \bmod 3 = 1$ and $(k^2+k)/3$ otherwise. As a result, $\widetilde{S}_{\beta,k}$ has order

$$|\widetilde{S}_{\beta,k}| = 1 + \beta(k+1) + (\beta-1)\frac{(k^2+k+[\![k \bmod 3 = 1]\!])}{3}.$$

$\widetilde{S}_{3,4}$ is given as an example in Fig. 7.

Observe that R is a k-truncated resolving set of $\widetilde{S}_{\beta,k}$. Indeed, since $d(r_i, r_j) \geq (2k+\ell)/3$ where $\ell = k \bmod 3$ for each distinct pair $r_i, r_j \in R$, each individual element of R resolves its associated path of length k (or $k+1$ for r_1) while $r_j \in R\setminus\{r_1\}$ and r_1 together resolve all vertices v such that $0 < d(v, r_1), d(v, r_j) \leq k$.

To see that $\dim_k(\widetilde{S}_{\beta,k}) = \beta$, note that, for any set of vertices R' in $\widetilde{S}_{\beta,k}$ such that $|R'| < \beta$, there must be at least two vertices u and v with $d_k(u|R') = d_k(v|R') = (k+1, \ldots, k+1)$. Thus, $\dim_k(\widetilde{S}_{\beta,k}) \geq \beta$.

9. Conclusion

Truncated metric dimension restricts the ability of individual vertices to accurately assess distances to far away points in a graph. This variation on the traditional definition somewhat forces resolving sets to take a local perspective, and has the potential to provide more useful distance constrained resolving sets in a number of real world scenarios.

In this work, we explored connections to the traditional definition as well as behavior on paths, cycles, and certain types of trees. Regarding the latter, it remains to answer: *can the k-truncated metric dimension of arbitrary trees be determined efficiently?*

Trees T with $\dim_1(T) = \dim(T)$ were characterized in [19]. For each $k > 1$, it remains to characterize trees T for which $\dim_k(T) = \dim(T)$. *More generally, which connected graphs G satisfy* $\dim_k(G) = \dim(G)$ *for each k?*

We also investigated graph constructions achieving upper and lower bounds in different circumstances. For all $k \geq 1$, we determined the connected graphs G of order n with $\dim_k(G) = n - 1$ and $\dim_k(G) = n - 2$. The graphs G of order n with $\dim_k(G) = 1$ were found in [11]. These results lead to a natural question: *which connected graphs G of order n have* $\dim_k(G) = \beta$, *for each* $k \geq 1$ *and* $\beta \in \{2, 3, \ldots, n - 3\}$*?*

For all $j, k \geq 1$ we determined the maximum possible order, degree, clique number, and chromatic number of any graph G with $\dim_k(G) = j$. There are other natural problems in this direction. For example, *what is the maximum possible degeneracy of any graph G with* $\dim_k(G) = j$*? We determined that the answer is* $\frac{3^j-1}{2}$ *when k is sufficiently large with respect to j, but how large must k be as a function of j for the degeneracy to be* $\frac{3^j-1}{2}$*?*

We determined the maximum n for which there exists a graph G with $\dim_k(G) = j$ which contains the complete bipartite graph $K_{n,n}$ as a subgraph. There are other natural problems in this direction, such as maximizing the size of other complete bipartite subgraphs besides $K_{1,n}$ and $K_{n,n}$. Another problem is to find the maximum n for which there exists a graph G with $\dim_k(G) = j$ which contains the wheel W_n as a subgraph, as well as the maximum n for which there exists a graph G with $\dim_k(G) = j$ which contains the n-cube Q_n as a subgraph. These are analogues of some questions that were investigated for metric dimension and edge metric dimension in [17,18].

A variety of other interesting questions remain open. For instance, *can approximations of k-truncated metric dimension be used to effectively approximate traditional metric dimension? How effective a tool are k-truncated resolving sets for mitigating problems associated with the accumulation of variance in transmission networks in different types of applications?* There are many avenues for future exploration related to these ideas.

Acknowledgments

We thank the referees for their helpful comments. This research was partially funded by NSF IIS, USA grant 1836914.

References

[1] Z. Bartha, J. Komjáthy, J. Raes, Sharp bound on the threshold metric dimension of trees, 2021, https://arxiv.org/abs/2111.08813.
[2] A.F. Beardon, J.A. Rodríguez-Velázquez, On the k-metric dimension of metric spaces, Ars Math. Contemp. 16 (2019) 25–38.
[3] L. Beaudou, P. Dankelmann, F. Foucaud, M.A. Henning, A. Mary, A. Parreau, Bounding the order of a graph using its diameter and metric dimension: A study through tree decompositions and VC dimension, SIAM J. Discrete Math. 32 (2018) 902–918.
[4] P.S. Buczkowski, G. Chartrand, C. Poisson, P. Zhang, On k-dimensional graphs and their bases, Period. Math. Hungar. 46 (2003) 9–15.
[5] J. Cáceres, C. Hernando, M. Mora, I.M. Pelayo, M.L. Puertas, C. Seara, On the metric dimension of some families of graphs, Electron. Notes Discrete Math. 22 (2005) 129–133.
[6] J. Cáceres, C. Hernando, M. Mora, I.M. Pelayo, M.L. Puertas, C. Seara, D.R. Wood, On the metric dimension of cartesian products of graphs, SIAM J. Discrete Math. 21 (2) (2007) 423–441.
[7] G. Chartrand, L. Eroh, M.A. Johnson, O.R. Oellermann, Resolvability in graphs and the metric dimension of a graph, Discrete Appl. Math. 105 (2000) 99–113.
[8] L. Eroh, P. Feit, C.X. Kang, E. Yi, The effect of vertex or edge deletion on the metric dimension of graphs, J. Comb. 6 (4) (2015) 433–444.
[9] L. Eroh, C.X. Kang, E. Yi, Metric dimension and zero forcing number of two families of line graphs, Math. Bohem. 139 (3) (2014) 467–483.
[10] L. Eroh, C.X. Kang, E. Yi, The connected metric dimension at a vertex of a graph, Theoret. Comput. Sci. 806 (2020) 53–69.
[11] A. Estrada-Moreno, On the (K, T)-Metric Dimension of a Graph (Doctoral Thesis), 2016.
[12] A. Estrada-Moreno, I.G. Yero, J.A. Rodríguez-Velázquez, On the (k, t)-metric dimension of graphs, Comput. J. 64 (2021) 707–720.
[13] H. Fernau, J.A. Rodríguez-Velázquez, On the (adjacency) metric dimension of corona and strong product graphs and their local variants: combinatorial and computational results, Discrete Appl. Math. 236 (2018) 183–202.
[14] R.W. Floyd, Algorithm 97: shortest path, Commun. ACM 5 (6) (1962) 345.
[15] F. Foucaud, G.B. Mertzios, R. Naserasr, A. Parreau, P. Valicov, Identification, location-domination and metric dimension on interval and permutation graphs, i. bounds, Theor. Comput. Sci. 668 (2017) 43–58.
[16] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman, New York, 1979.
[17] J. Geneson, Metric dimension and pattern avoidance in graphs, Discrete Appl. Math. 284 (2020) 1–7.
[18] J. Geneson, S. Kaustav, A. Labelle, Extremal results for graphs of bounded metric dimension, Discrete Appl. Math. 309 (2022) 123–129.
[19] J. Geneson, E. Yi, Broadcast dimension of graphs, 2020, https://arxiv.org/abs/2005.07311.
[20] J. Geneson, E. Yi, The Distance-K Dimension of Graphs, 2021, https://arXiv:2106.08303.
[21] F. Harary, R.A. Melter, On the metric dimension of a graph, Ars Combin. 2 (1976) 191–195.
[22] C. Hernando, M. Mora, I.M. Pelayo, C. Seara, D.R. Wood, Extremal graph theory for metric dimension and diameter, Electron. J. Combin. 17 (2010) #R30.
[23] M. Jannesari, B. Omoomi, The metric dimension of the lexicographic product of graphs, Discrete Math. 312 (2012) 3349–3356.
[24] I. Javaid, M.T. Rahim, K. Ali, Families of regular graphs with constant metric dimension, Util. Math. 75 (2008) 21–33.
[25] D.B. Johnson, Efficient algorithms for shortest paths in sparse networks, J. ACM 24 (1) (1977) 1–13.
[26] S. Khuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, Discrete Appl. Math. 70 (1996) 217–229.
[27] P.C. Pinto, P. Thiran, M. Vetterli, Locating the source of diffusion in large-scale networks, Phys. Rev. Lett. 109 (6) (2012) 068702.
[28] C. Poisson, P. Zhang, The metric dimension of unicyclic graphs, J. Combin. Math. Combin. Comput. 40 (2002) 17–32.
[29] S.W. Saputro, E.T. Baskoro, A.N.M. Salman, D. Suprijanto, The metric dimension of a complete n-partite graph and its Cartesian product with a path, J. Combin. Math. Combin. Comput. 71 (2009) 283–293.
[30] B. Shanmukha, B. Sooryanarayana, K.S. Harinath, Metric dimension of wheels, Far East J. Appl. Math. 8 (2002) 217–229.

[31] P.J. Slater, Leaves of trees, Congr. Numer. 14 (1975) 549–559.
[32] P.J. Slater, Domination and location in acyclic graphs, Networks 17 (1987) 55–64.
[33] P.J. Slater, Dominating and reference sets in graphs, J. Math. Phys. Sci. 22 (1988) 445–455.
[34] B. Spinelli, L.E. Celis, P. Thiran, Observer placement for source localization: The effect of budgets and transmission variance, in: 2016 54th Annual Allerton Conference on Communication, Control, and Computing (Allerton), 2016, pp. 743–751.
[35] R.C. Tillquist, Low-Dimensional Embeddings for Symbolic Data Science (Ph.D. thesis), University of Colorado, Boulder, 2020.
[36] R.C. Tillquist, R.M. Frongillo, M.E. Lladser, Getting the lay of the land in discrete space: A survey of metric dimension and its applications, 2021, https://arxiv.org/abs/2104.07201.
[37] R.C. Tillquist, R.M. Frongillo, M.E. Lladser, Truncated metric dimension for finite graphs, 2021, https://arxiv.org/abs/2106.14314.
[38] R.C. Tillquist, M.E. Lladser, Low-dimensional representation of genomic sequences, J. Math. Biol. 79 (2019) 1–29.
[39] K. Wagner, Über eine eigenschaft der ebenen komplexe, Math. Ann. 114 (1937) 570–590.
[40] N. Zubrilina, On the edge dimension of a graph, Discrete Math. 341 (2018) 2083–2088.