COUNTING MINIMAL SURFACES IN NEGATIVELY CURVED 3-MANIFOLDS

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Abstract
We introduce an asymptotic quantity that counts area-minimizing surfaces in negatively curved closed 3-manifolds and show that quantity to only be minimized, among all metrics of sectional curvature ≤ -1, by the hyperbolic metric.

1. Introduction

A classical and beautiful result in geometry says that if (M, h_0) is a closed locally symmetric Riemannian manifold with strictly negative curvature (i.e., quotients of either hyperbolic space, complex hyperbolic space, quaternionic hyperbolic space, or Cayley plane) and h is another negatively curved Riemannian metric on M with the same volume as h_0, then the quantity

$$\delta(h) := \lim_{L\to\infty} \frac{\ln \#\{\text{length}_h(\gamma) \leq L : \gamma \text{ closed geodesic in } (M, h)\}}{L}$$

satisfies $\delta(h) \geq \delta(h_0)$ and equality implies that h is isometric to h_0.

This follows from combining a theorem of Margulis [23] which identified the right-hand side in the inequality above as the topological entropy for negatively curved metrics, a theorem of Manning [22] which says that the volume entropy and topological entropy coincide for negatively curved metrics, and a theorem of Besson, Courtois, and Gallot [5] which says that g_0 minimizes the volume entropy among all metrics with the same volume.

Closed geodesics are a particular case of minimal surfaces, and in recent years great progress has been made regarding the existence of minimal hypersurfaces. For instance, for a closed Riemannian manifold M of dimension between 3 and 7, Irie and the last two authors [14] showed that, for generic metrics, the set of all closed embedded minimal hypersurfaces is dense in M; jointly with Song [25], the last two authors showed that, for generic metrics, there is a sequence of closed embedded

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minimal hypersurfaces that becomes equidistributed; Song [34] showed that for every Riemannian metric on M, there are always infinitely many distinct closed embedded minimal hypersurfaces; and Zhou [37] solved the Multiplicity One Conjecture made by the last two authors, which when combined with [24] implies that, for generic metrics, there is a closed embedded minimal hypersurface of Morse index p for every $p \in \mathbb{N}$.

The purpose of this paper is to study minimal surfaces in a closed orientable 3-manifold in the spirit of the entropy functional mentioned at the beginning of the introduction.

Before we state the main theorem we need to introduce some concepts. Throughout this paper, M will denote a closed orientable 3-manifold that admits a hyperbolic metric. A closed immersed genus g surface $\Sigma \subset M$ is *essential* if the immersion $\iota : \Sigma \to M$ injects $\pi_1(\Sigma)$ into $\pi_1(M)$. In this case, the group $G = \iota_*(\pi_1(\Sigma))$ is called a *surface subgroup of genus* g, and surface subgroups of immersions homotopic to ι are in one-to-one correspondence with conjugates of G by an element of $\pi_1(M)$.

Let $S(M, g)$ denote the set of surface subgroups of genus at most g of $\pi_1(M)$ modulo the equivalence relation of conjugacy. We abuse notation and see an element $\Pi \in S(M, g)$ as being either all subgroups of $\pi_1(M)$ that are conjugate to a fixed surface group of genus at most g or the set of all essential immersions of surfaces $\iota : \Sigma \to M$ for which $\iota_*(\pi_1(\Sigma)) \in \Pi$. Kahn and Markovic [16], [17] showed that surface subgroups exist for all large genera and estimated the cardinality of $S(M, g)$.

Consider a Riemannian metric h on M, and denote the hyperbolic metric by $\bar{h}$. Given $\Pi \in S(M, g)$ we define

$$\operatorname{area}_h(\Pi) = \inf\left\{\operatorname{area}_h(\Sigma) : \Sigma \in \Pi\right\},$$

where $\operatorname{area}_h(\Sigma)$ denotes the area computed with respect to the metric $\iota^* h$.

Given $\varepsilon \geq 0$, we define $S(M, g, \varepsilon)$ to be the conjugacy classes in $S(M, g)$ whose limit set is a $(1+\varepsilon)$-circle (see Definition 2.3) and set

$$S_\varepsilon(M) = \bigcup_{g \in \mathbb{N}} S(M, g, \varepsilon).$$

We are interested in the following geometric quantity:

$$E(h) = \lim_{\varepsilon \to 0} \liminf_{L \to \infty} \frac{\ln \#\{\operatorname{area}_h(\Pi) \leq 4\pi(L-1) : \Pi \in S_\varepsilon(M)\}}{L \ln L}. \tag{1}$$

Note that if $\varepsilon < \varepsilon'$, then $S_\varepsilon(M) \subset S_{\varepsilon'}(M)$, and so the limit in the ε-variable is well defined. In this paper we show the following result.

THEOREM 1.1
Given a Riemannian metric h on M with volume entropy denoted by $E_{\text{vol}}(h)$*, we have* $E(h) \leq 2E_{\text{vol}}(h)^2$.

If the sectional curvature of h is less than or equal to -1*, then*

$$E(h) \geq E(\bar{h}) = 2 \tag{2}$$

with equality if and only if h is the hyperbolic metric.

As far as we know, this is the first result giving asymptotic rigidity for the areas of minimal surfaces—meaning that if there are sufficiently many (in a precise sense to be made in Section 5) minimal surfaces Σ_i with genus g_i so that $\text{area}_h(\Sigma_i)/(4\pi(g_i - 1)) \to 1$, then the metric h is hyperbolic.

One obvious challenge is that the results in [5], [22], and [23] rely on the dynamical properties of the geodesic flow, which have no analogue for minimal surfaces. For this reason we restricted our asymptotic counting invariant to the homotopy classes in $S_\varepsilon(M)$ so that the dynamical properties of the geodesic flow can be of use.

The fact that one can compute $E(\bar{h})$ follows from [16] and from the work of Uhlenbeck in [36]. The inequality (2) in Theorem 1.1 is a consequence of the Gauss–Bonnet theorem. The statement in Theorem 1.1 that only $\bar{h}$ has $E(\bar{h}) = 2$ will follow in two steps. First we combine minimal surface theory with the strong rigidity properties of totally geodesic disks proven independently by Shah [33] and Ratner [30] to find, for every $v \in T_pM$, a totally geodesic hyperbolic disk in (M, g) containing (p, v) in its tangent space. This will occupy most of the proof. Then we use the ergodicity of the frame flow due to Brin and Gromov [9] to show that the sectional curvature of every plane is -1.

We now briefly review some previous results related to our work.

Shah [33] and Ratner [30] showed that a totally geodesic immersion of $\mathbb{H}^2$ in a compact hyperbolic manifold has its image either dense or a closed surface. McMullen, Mohammadi, and Oh [26] recently generalized this result to the noncompact case.

McReynolds and Reid [27] showed that arithmetic hyperbolic 3-manifolds which have the same (nonempty) set of totally geodesic surfaces are commensurable, that is, covered by a common closed 3-manifold. It is not expected that the areas of all totally geodesic surfaces will determine the commensurability class of the arithmetic hyperbolic 3-manifolds (see [19]). Jung [15] studied the asymptotic behavior of the areas of totally geodesic surfaces for some arithmetic hyperbolic 3-manifolds.

Totally geodesic surfaces in hyperbolic manifolds have the attractive feature that they are preserved by the geodesic flow, but their existence is not guaranteed. For instance, there are closed hyperbolic 3-manifolds which admit no totally geodesic immersed closed surface (see [21, Chapter 5.3]) and even finite-volume hyperbolic 3-manifolds which admit no totally geodesic immersed finite-area surfaces either (see [10]). Recently it was shown that a closed hyperbolic 3-manifold having infinitely many totally geodesic surfaces is arithmetic (see [3], [29]).

Finally, it was shown in [28] that the commensurability class of closed hyperbolic 3-manifolds is determined by their surface groups.

2. Notation and preliminaries

We set up the basic notation and then discuss several results, all of which are well known among experts.

There is a discrete subgroup $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}(2,\mathbb{C})$ so that $M = \mathbb{H}^3 \setminus \Gamma$ is a closed orientable 3-manifold and we fix an isomorphism between $\pi_1(M)$ and Γ. A Riemannian metric on M is denoted by h, and the hyperbolic metric is denoted by $\bar{h}$. Geometric quantities with respect to the metric h will usually have the subscript h, while the same quantities will have no subscript if computed with respect to the metric $\bar{h}$. For instance, the distance between two points p, q, the area of an immersed surface $\phi : \Sigma \to M$, or the Hausdorff distance between sets A, B with respect to the metric $\phi^*\bar{h}$ and $\phi^* h$, respectively, are denoted by $d(p,q)$, $d_h(p,q)$, $\mathrm{area}(\Sigma)$, $\mathrm{area}_h(\Sigma)$, or $d_H(A,B)$, $d_{H,h}(A,B)$. Note that if Σ is a k-cover of a surface $\tilde{\Sigma}$, then $\mathrm{area}_h(\Sigma) = k\,\mathrm{area}_h(\tilde{\Sigma})$.

Let (B^3, h) denote the universal cover of (M,h), and let S^2_∞ denote its sphere at infinity, which is defined as the set of all asymptote classes of geodesic rays, where two geodesic rays $\gamma_i : [0,+\infty) \to B^3$, $i = 1,2$, define the same asymptote class, denoted by $\gamma_1(+\infty)$, if $\lim_{t\to\infty} d_h(\gamma_1(t), \gamma_2(t)) < +\infty$. There is a natural topology on $\overline{B}^3 := B^3 \cup S^2_\infty$, the cone topology (see, e.g., [1]), for which $\bar{B}^3$ is homeomorphic to a 3-ball. Given a set $\Omega \subset B^3$ we denote by $\overline{\Omega}$ its closure in $\overline{B}^3$ and $\partial_\infty \Omega$ stands for $\overline{\Omega} \cap S^2_\infty$. We follow convention and denote $(B^3, \bar{h})$ simply by $\mathbb{H}^3$.

An essential immersion $\phi : \Sigma \to M$ must have genus ≥ 2 (by the Preissman theorem), and thus ϕ admits a lift $\bar{\phi} : D \to \mathbb{H}^3$ from a disk D onto $\mathbb{H}^3$. To ease notation, we will often identify the immersions of Σ or D with its images in M or $\mathbb{H}^3$, respectively. This will create an ambiguity when Σ is a k-cover of another surface $\tilde{\Sigma}$, but it will be clear from the context whether we are referring to the immersion (when we compute area, for instance) or to the image set in M (when we compute Hausdorff distances, for instance). Given an essential surface $\Sigma \subset M$ with surface group $G < \Gamma$, there is a lift $D \subset \mathbb{H}^3$ that is invariant under G. Any other disk $D' \subset \mathbb{H}^3$ lifting Σ is invariant under a group $G' < \Gamma$ that is conjugate to G. Necessarily we have (with an obvious abuse of notation) $D \setminus G = D' \setminus G' = \Sigma$.

The Grassmanian bundle of unoriented 2-planes in M or $\mathbb{H}^3$ is denoted by $\mathrm{Gr}_2(M)$ or $\mathrm{Gr}_2(\mathbb{H}^3)$, respectively. An immersed surface Σ in M (or its lift D in $\mathbb{H}^3$) induces a natural immersion into $\mathrm{Gr}_2(M)$ (or $\mathrm{Gr}_2(\mathbb{H}^3)$) via the map $p \mapsto (p, T_p\Sigma)$ (or $p \mapsto (p, T_pD)$).

2.1. *Fundamental domains and Cayley graphs*

Given a subgroup $G < \mathrm{PSL}(2,\mathbb{C})$ acting properly discontinuous on $\mathbb{H}^3$, a *fundamental domain* $\Delta \subset \mathbb{H}^3$ for $\mathbb{H}^3 \setminus G$ is a closed region so that

(i) $\bigcup_{\phi\in G} \phi(\Delta) = \mathbb{H}^3$;

(ii) $\phi \in G$ and $\phi(\Delta) \cap \operatorname{int}\Delta \neq \emptyset \implies \phi = \mathrm{Id}$.

Because the manifold M is compact, we can choose its fundamental domain Δ to be a convex polyhedron with finitely many totally geodesic faces. Such domains are called *Dirichlet fundamental domain*. Each compact set $K \subset \mathbb{H}^3$ intersects only finitely many elements of $\{\phi(\Delta)\}_{\phi\in\Gamma}$.

Given a subgroup $G < \Gamma$, we consider the set $\Gamma \setminus G = \{\phi G : \phi \in \Gamma\}$ and pick a representative $\underline{\phi}$ in each coset ϕG.

LEMMA 2.1

$\Delta_G = \bigcup_{\underline{\phi}\in\Gamma\setminus G} \underline{\phi}^{-1}(\Delta)$ is a fundamental domain for $\mathbb{H}^3 \setminus G$.

Proof

The reader can check that Δ_G is closed and that $\bigcup_{\phi\in G} \phi(\Delta_G) = \mathbb{H}^3$. Suppose there is $\psi \in G$ and $x \in \psi(\Delta_G) \cap \operatorname{int}\Delta_G$. Because $x \in \operatorname{int}\Delta_G$ we can find a finite set $A \subset \Gamma \setminus G$ and an open set U so that $x \in U \subset \bigcup_{\underline{\phi}\in A} \underline{\phi}^{-1}(\Delta)$. Likewise, we have $x \in \psi(\underline{\sigma}^{-1}(\Delta))$ for some $\underline{\sigma} \in \Gamma \setminus G$. We must have

$$\psi\big(\underline{\sigma}^{-1}(\operatorname{int}\Delta)\big) \cap \Big(\bigcup_{\underline{\phi}\in A} \underline{\phi}^{-1}(\Delta)\Big) \neq \emptyset,$$

and thus $(\underline{\sigma}\psi^{-1})^{-1} = \underline{\phi}^{-1}$ for some $\underline{\phi} \in A$. Hence $\underline{\sigma} = \underline{\phi}$ and $\psi = \mathrm{Id}$. □

Fix $p \in \mathbb{H}^3$. Choosing R large enough, the set $A = \{\phi \in \Gamma : d(p, \phi(p)) \leq R\}$ generates Γ. The *Cayley graph* $\mathrm{Gr}(\Gamma, A)$ of Γ generated by A is defined as having vertices $\{\phi(p)\}_{\phi\in\Gamma}$, and two vertices $\psi(p), \phi(p)$ are connected by an edge if $\phi\psi^{-1} \in A$. The graph $\mathrm{Gr}(\Gamma, A)$ admits a distance function d, where $d(\phi, \psi)$ is the word length of $\phi\psi^{-1}$, and the norm of $\phi \in \Gamma$ is given by $|\phi| = d(\phi, \mathrm{Id})$. The Hausdorff distance between two sets A_1, A_2 is denoted by

$$d_H(A_1, A_2) := \max\Big\{\sup_{x\in A_1} \inf_{y\in A_2} d(x,y),\ \sup_{y\in A_2} \inf_{x\in A_1} d(x,y)\Big\}.$$

We will need the following lemma (where $B_r(p)$ denotes the geodesic ball of radius r centered at $p \in \mathbb{H}^3$).

LEMMA 2.2
There is a constant $c > 0$ depending on the Dirichlet domain Δ containing p so that

$$B_{n/c-c}(p) \subset \bigcup_{|\phi|\leq n} \phi(\Delta) \subset B_{nc+c}(p) \quad \text{for all } n \in \mathbb{N}.$$

Proof
The Švarc–Milnor lemma says that the map $\Gamma \to \mathbb{H}^3$, $\phi \mapsto \phi(p)$ is a quasi-isometry, meaning there is a constant K so that
(i) $\mathbb{H}^3 = \bigcup_{\phi\in\Gamma} B_K(\phi(p))$,
(ii) for all $\psi, \phi \in \Gamma$,

$$K^{-1} d\big(\phi(p), \psi(p)\big) - K \leq d(\phi, \psi) \leq K d\big(\phi(p), \psi(p)\big) + K,$$

and there are constants $n_1 \in \mathbb{N}$, $K_1 > 0$ so that $B_{K_1}(p) \subset \bigcup_{|\phi|\leq n_1} \phi(\Delta)$ and $\Delta \subset B_{K_1}(p)$. The constant c can be computed in terms of n_1, K, K_1, and we leave it to the reader. □

The Švarc–Milnor lemma mentioned above also says that choice of a generating set or different basepoints would give another Cayley graph that is quasi-isometric to $\mathrm{Gr}(\Gamma, A)$. We abuse notation and simply denote the Cayley graph by Γ.

2.2. *Morse's lemma*

A curve $\gamma : \mathbb{R} \to \mathbb{H}^3$ is a (K, c) *quasi-geodesic* if

$$K^{-1} d\big(\gamma(t), \gamma(s)\big) - c \leq |t - s| \leq K d\big(\gamma(t), \gamma(s)\big) + c \quad \text{for all } s, t \in \mathbb{R}.$$

A geodesic in B^3 with respect to the metric h is a $(K, 0)$ quasi-geodesic for some $K = K(h)$.

Morse's lemma (see, e.g., [18, Theorem 2.3]) gives the existence of $r_0 = r_0(K, c)$ such that for every (K, c) quasi-geodesic γ in $\mathbb{H}^3$ there is a unique (up to reparameterization) geodesic σ in $\mathbb{H}^3$ so that the Hausdorff distance between $\sigma(\mathbb{R})$ and $\gamma(\mathbb{R})$ is bounded by r_0.

2.3. *Limit sets and quasi-Fuchsian manifolds*

Given a discrete subgroup acting properly discontinuously $G < \mathrm{PSL}(2, \mathbb{C})$, the *limit set* $\Lambda(G) \subset S^2_\infty$ is defined as being the set of accumulation points in S^2_∞ of the orbit Gx, where $x \in \mathbb{H}^3$. It is well known that the definition is independent of the point $x \in \mathbb{H}^3$ chosen and that the limit set is closed. Elements $\phi \in \mathrm{PSL}(2, \mathbb{C})$ induce conformal maps of S^2_∞, and one has that $\Lambda(\phi G \phi^{-1}) = \phi(\Lambda(G))$. From this, one deduces that $\Lambda(G)$ is G-invariant.

Quasiconformal maps $F : S^2_\infty \to S^2_\infty$ with dilation bounded by some $K \in [1, +\infty)$ (see [35, Section 5.9] for a definition) have the property that are Lipschitz, and at points p of differentiability, DF_p sends circles into ellipses whose eccentricity (ratio between major axis and minor axis) is bounded by K. Furthermore, conformal maps are quasi-conformal maps with $K = 1$.

If $\Lambda(G)$ is a geometric circle, then G is called a *Fuchsian* group, and if $\Lambda(G)$ is a Jordan curve, then G is called a *quasi-Fuchsian* group. In this case, it is known (see [35, Proposition 8.7.2]) that $\Lambda(G)$ is a K-quasicircle, meaning there is a quasiconformal map F with dilation bounded by K that maps the equator to $\Lambda(G)$.

Definition 2.3
A discrete subgroup acting properly discontinuously $G < \mathrm{PSL}(2,\mathbb{C})$ is ε-*Fuchsian* if $\Lambda(G)$ is a $(1+\varepsilon)$-quasicircle. This notion is invariant under conjugacy.

The normal bundle of an orientable surface $S \subset M$ is denoted by $T^\perp S \simeq S \times \mathbb{R}$. When the background metric is the hyperbolic metric, Uhlenbeck proved the following result in [36, Theorem 3.3].

THEOREM 2.4
Let $S \subset M$ be an orientable minimal surface with principal curvatures $|\lambda(x)| \le \lambda_0 \le 1$ for all $x \in S$. Then:

(i) *The exponential map $exp : T^\perp S \to M$ is a covering map, and thus $G := \exp_*(\pi_1(S))$ is a surface group.*
(ii) *G is a quasi-Fuchsian group and $N := \mathbb{H}^3 \setminus G \simeq T^\perp S$ is a complete hyperbolic manifold.*
(iii) *S is embedded, area-minimizing, and the only closed minimal surface in N.*
(iv) *For all $t > \tanh^{-1}(\lambda_0)$, the region $S \times [-t,t] \subset N$ is strictly convex and its boundary has principal curvatures bounded from above by*

$$\frac{\sinh t + \cosh t \lambda_0}{\cosh t + \sinh t \lambda_0}.$$

The last property is not explicitly stated in [36, Theorem 3.3], but from its proof one sees that the surface $S \times \{t\} \subset N$ has principal curvatures

$$\lambda^t_\pm(x) = \frac{\sinh t \pm \cosh t \lambda(x)}{\cosh t \pm \sinh t \lambda(x)},$$

which readily implies property (iv).

2.4. Totally geodesic planes

Consider $\mathcal{C}$ to be the set of all geometric circles (of varying radii) in S^2_∞. This set is noncompact and in one-to-one correspondence with the totally geodesic disks in $\mathbb{H}^3$ because given any $\gamma \in \mathcal{C}$ there is exactly one totally geodesic disk $C(\gamma) \subset \mathbb{H}^3$ such that $\partial_\infty C(\gamma) := S^2_\infty \cap \overline{C(\gamma)}$ is identical to γ.

Every $\phi \in \mathrm{PSL}(2,\mathbb{C})$ induces a map from $\mathcal{C}$ to $\mathcal{C}$ (still denoted by ϕ) such that $\phi(C(\gamma)) = C(\phi(\gamma))$. Hence, the group Γ acts naturally on $\mathcal{C}$.

The following result was proven independently by Ratner [30] and Shah [33].

THEOREM 2.5

Given $\gamma \in \mathcal{C}$, either $C(\gamma)$ covers a closed surface in $M = \mathbb{H}^3 \setminus \Gamma$ or its natural immersion into $\mathrm{Gr}_2(\mathbb{H}^3)$ *projects to a dense set in* $\mathrm{Gr}_2(M)$.

Given $\gamma \in \mathcal{C}$, consider the orbit $\Gamma\gamma := \{\phi(\gamma) : \phi \in \Gamma\} \subset \mathcal{C}$.

Using the fact that $\{\gamma_i\}_{i\in\mathbb{N}} \in \mathcal{C}$ converges to $\gamma \in \mathcal{C}$ if and only if $C(\gamma_i)$ converges to $C(\gamma)$ on compact sets of $\mathbb{H}^3$, we leave to the reader to check that $\Gamma\gamma$ is dense in $\mathcal{C}$ if and only if the natural immersion of $C(\gamma)$ into $\mathrm{Gr}_2(\mathbb{H}^3)$ projects to a dense set in $\mathrm{Gr}_2(M)$.

The next theorem was essentially proved in [26, Theorem 11.1]. We provide the modifications that need to be made.

THEOREM 2.6

Consider $\mathcal{L} \subset \mathcal{C}$ a closed set that is Γ-invariant.

Suppose that no element in $\mathcal{L}$ has a dense Γ-orbit in $\mathcal{C}$. Then every $\gamma \in \mathcal{L}$ is isolated and has $C(\gamma)$ projecting to a closed surface in M.

Proof

Every $\gamma \in \mathcal{L}$ must have $C(\gamma)$ projecting to a closed surface in M, because otherwise Theorem 2.5 would say that $\Gamma\gamma$ is dense in $\mathcal{C}$.

We argue by contradiction and suppose there is $\gamma_i \in \mathcal{L}$ converging to γ in $\mathcal{L}$ as $i \to \infty$ with $\gamma_i \neq \gamma$. Set

$$\Gamma^\gamma = \{\phi \in \Gamma : \phi(\gamma) = \gamma\}.$$

The action of Γ^γ preserves $C(\gamma)$, and $C(\gamma) \setminus \Gamma^\gamma$ corresponds to a closed surface because $C(\gamma)$ projects to a closed surface in M.

Choose a disk $\Omega \subset S^2_\infty$ so that $\partial\Omega = \gamma$. Either Γ^γ preserves Ω or it contains a normal subgroup of index 2 that preserves Ω. If the latter occurs, relabel Γ^γ to be that subgroup. By swapping Ω with its complement in S^2_∞ if necessary and after possibly passing to a subsequence of $\{\gamma_i\}_{i\in\mathbb{N}}$, we can assume that $\gamma_i \cap \Omega \neq \emptyset$ for all $i \in \mathbb{N}$.

The disk D carries a natural hyperbolic metric h_Ω conformal to the round metric in S^2_∞, and each map in Γ^γ is an orientation-preserving isometry of Ω with respect to the metric h_Ω. Finally, $\Omega \setminus \Gamma^\gamma$ is isometric to $C(\gamma) \setminus \Gamma^\gamma$ and so the group Γ^γ is a nonelementary, convex, cocompact Fuchsian group as defined in [26, Section 3]. Hence we can apply Corollary 3.2 of [26], which says that if we consider the set $\mathcal{H}(D)$ of all horocycles in (Ω, h_Ω), that is,

$$\mathcal{H}(\Omega) = \{\sigma \in \mathcal{C} : \sigma \subset \overline{\Omega}, \sigma \cap \partial\Omega \neq \emptyset\},$$

then the closure of $\bigcup \Gamma^\gamma \gamma_i$, and hence $\mathcal{L}$, contains $\mathcal{H}(\Omega)$.

From [26, Theorem 4.1] there exists a dense set $\Lambda_0 \subset S^2_\infty$ such that if $\sigma \in \mathcal{C}$ intersects Λ_0, then $\Gamma\sigma$ is dense in $\mathcal{C}$. Necessarily, Λ_0 must intersect some element of $\mathcal{H}(\Omega)$, and so there is $\sigma \in \mathcal{L}$ for which $\Gamma\sigma$ is dense in $\mathcal{C}$. Thus $C(\sigma)$ does not project to a closed surface in M, which is a contradiction. ☐

2.5. Frame flow

We denote the bundle of oriented orthonormal frames of M with respect to $\bar{h}$ or h by $\mathcal{F}(M)$ and $\mathcal{F}(M)(h)$, respectively.

The *frame flow* $F_t : \mathcal{F}(M)(h) \to \mathcal{F}(M)(h)$ is defined in the following way: given an oriented frame (e_1, e_2, e_3) for T_pM,

$$F_t\big(p, (e_1, e_2, e_3)\big) = \big(\gamma(t), \big(\gamma'(t), e_2(t), e_3(t)\big)\big),$$

where $\gamma(t) = \exp_p(te_1)$, and $e_2(t)$, $e_3(t)$ denote the parallel transport of e_2, e_3 along γ. An important result which we will use, due to Brin and Gromov [9], says that when (M, h) is negatively curved, the frame flow is ergodic and, in particular, has a dense orbit in $\mathcal{F}(M)(h)$.

3. Convex hulls

In this section we assume that (M, h) has sectional curvature less than or equal to -1.

Given a closed set $\Lambda \subset S^2_\infty$, its *convex hull* $C_h(\Lambda) \subset \bar{B}^3$ denotes the smallest geodesically closed set of $\bar{B}^3$ (with respect to the metric h) that contains Λ.

The goal of this section is to prove the following result.

THEOREM 3.1

Let $S \subset M$ be a minimal surface (with respect to $\bar{h}$) with principal curvatures $|\lambda(x)| \leq \lambda_0 < 1$ for all $x \in S$, and let $\Sigma \subset M$ be a minimal surface with respect to h in the homotopy class of S. Then, denoting by $D, \Omega \subset \mathbb{H}^3$ the lifts of S and Σ, respectively, that are invariant by the same surface group, we have

$$d_H(D, \Omega) \leq R$$

for some constant $R = R(h, \lambda_0)$.

Bangert and Lang proved similar results to the theorem above (see [4] and references therein) under the conditions that D and Ω are quasi-minimizing. While that will be true for D, it is not necessarily true for Ω, and so the result cannot be straightforwardly applied. It is conceivable that their proof could be extended to our setting, but we choose a different argument.

Given $p \in B^3$, the cone over Λ centered at p with respect to the metric h is given by

$$\mathrm{Co}_p(\Lambda) := \mathrm{clo}\big\{\gamma(t) : \gamma \text{ a geodesic with } \gamma(0) = p, \gamma(\infty) \in \Lambda, 0 \le t < \infty\big\},$$

where the closure is taken with respect to the cone topology. One has $\mathrm{Co}_p(\Lambda) \cap S^2_\infty = \Lambda$.

The space (B^3, h) has sectional curvature less than or equal to -1 and is thus $\bar{\delta}$-hyperbolic for some universal constant $\bar{\delta}$, meaning that a side in any geodesic triangle (with vertices possibly in S^2_∞) is contained in the $\bar{\delta}$-neighborhood of the union of the other two sides. Thus if $p, q \in B^3$, $x \in S^2_\infty$, and γ, σ denote geodesic rays (with respect to h) staring at p, q, respectively, with $\gamma(\infty) = \sigma(\infty) = x$, then, with l denoting the geodesic connecting p to q, we have that γ is contained in the $\bar{\delta}$-neighborhood of the union of σ and l. Therefore

$$d_{H,h}\big(\mathrm{Co}_p(\Lambda), \mathrm{Co}_q(\Lambda)\big) \le \bar{\delta} + d_h(p,q). \tag{3}$$

Likewise, $\mathrm{Co}_p(\Lambda)$ is $\bar{\delta}$-quasiconvex, meaning that given any x, y in $\mathrm{Co}_p(\Lambda)$, the geodesic connecting x to y is contained in a $\bar{\delta}$-neighborhood of $\mathrm{Co}_p(\Lambda)$.

PROPOSITION 3.2
There is $R = R(h)$ so that given a closed set $\Lambda \subset S^2_\infty$, we have $C_h(\Lambda) \cap S^2_\infty = \Lambda$ and

$$d_H\big(C_h(\Lambda), C_{\bar{h}}(\Lambda)\big) \le R.$$

Proof
The key step in the proof is the following claim, which was proved in Proposition 2.5.4 of [7] using the existence of certain convex sets constructed by Anderson in [2].

CLAIM 3.3
There is $R = R(h)$ so that for every $p \in C_h(\Lambda)$,

$$d_{H,h}\big(C_h(\Lambda), \mathrm{Co}_p(\Lambda)\big) \le R.$$

In particular, $C_h(\Lambda) \cap S^2_\infty = \Lambda$.

If $\gamma, \bar{\gamma}$ are two geodesics with respect to h and $\bar{h}$, respectively, that connect $p \in \mathbb{H}^3$ (or $y \in S^2_\infty$) to $x \in S^2_\infty$, then Morse's lemma gives the existence of a constant r_0 depending only on h so that $d_H(\gamma, \bar{\gamma}) \leq r_0$. From this we deduce that

$$\operatorname{dist}\big(C_h(\Lambda), C_{\bar{h}}(\Lambda)\big) \leq r_0 \qquad \text{and} \qquad d_H\big(\mathrm{Co}_p(\Lambda, h), \mathrm{Co}_p(\Lambda, \bar{h})\big) \leq r_0,$$

where $\mathrm{Co}_p(\Lambda, h)$ denotes the cone with respect to h. Combining these inequalities with (3) and Claim 1 we deduce the desired result at once. □

Let G be a quasi-Fuchsian surface group, and set $N := \mathbb{H}^3 \setminus G$. Because $\Lambda(G) \subset S^2_\infty$ is G-invariant, $C_h(\Lambda(G))$ is also G-invariant and $C_h(N) := C_h(\Lambda(G)) \setminus G$ is a compact subset of the N (see [35, Section 8.2]).

PROPOSITION 3.4
Every closed immersed minimal surface in (N, h) is contained in $C_h(N)$.

Proof
Let $\tilde{d} : N \to [0, \infty)$ be the distance function to $C_h(N)$. If π denotes the covering map from (B^3, h) to (N, h), then we have that $\pi^{-1}(C_h(N)) = C_h(\Lambda(G))$ is a geodesically convex set, and so Proposition 4.7 in [6] says that $\tilde{d}$ is a continuous convex function (Theorem 4.7 in [6] is misstated because it requires the subset of N to be geodesically convex instead of requiring the inverse image of the set under the covering map to be geodesically convex).

Given Σ a closed connected minimal immersion, there is $l > 0$ so that $\Sigma \subset \tilde{d}^{-1}[0, l)$, and we set $K = \tilde{d}^{-1}[0, l + 1]$.

The function $\tilde{d}$ does not have to be smooth, but we can apply [11, Theorem 2] to obtain a sequence of smooth functions $\{\phi_i\}_{i \in \mathbb{N}}$ so that ϕ_i tends to $\tilde{d}$ uniformly in K as $i \to \infty$ and, setting

$$\lambda(\phi_i) = \min\big\{D^2\phi_i(v, v) : x \in K, v \in T_x N, |v| = 1\big\},$$

we have $\liminf_{i \to \infty} \lambda(\phi_i) \geq 0$. Hence $\Delta_\Sigma \phi_i \geq \lambda(\phi_i)$ on Σ because Σ is a minimal surface.

Set $\phi_i^+ = \max\{\phi_i, 0\}$. We have

$$\int_{\{x \in \Sigma : \phi_i \geq 0\}} |\nabla \phi_i|^2 \, dA_h = -\int_\Sigma \phi_i^+ \Delta \phi_i \, dA_h \leq -\lambda(\phi_i) \int_\Sigma \phi_i^+ \, dA_h,$$

and so

$$\lim_{i \to \infty} \int_{\{x \in \Sigma : \phi_i \geq 0\}} |\nabla \phi_i|^2 \, dA_h = 0.$$

Suppose that $\Sigma \cap \tilde{d}^{-1}\{\delta\} \neq \emptyset$ for some $l > \delta > 0$. Note that $\tilde{d} \in W^{1,2}(\Sigma)$ and the functions ϕ_i converge weakly to $\tilde{d}$ in $W^{1,2}(\Sigma)$ as $i \to \infty$, and so

$$\int_{\tilde{d}^{-1}[\delta,l]\cap\Sigma} |\nabla d|^2 \, dA_h \leq \liminf_{i\to\infty} \int_{\{x\in\Sigma:\phi_i\geq 0\}} |\nabla \phi_i|^2 \, dA_h = 0.$$

Thus there is some $t \geq \delta$ so that $\Sigma \subset \tilde{d}^{-1}(t)$. An inspection of the proof of Proposition 4.7 of [6] shows that $\{\tilde{d} \leq t\}$ is actually geodesically strictly convex, because the ambient curvature is strictly negative and so it cannot contain the minimal surface Σ in its boundary $\partial\{\tilde{d} \leq t\} = \tilde{d}^{-1}(t)$. ☐

Proof of Theorem 3.1
Without loss of generality we can assume that S is orientable.

Let G be the surface group that preserves both D and Ω so that $S = D \setminus G$ and $\Sigma = \Omega \setminus G$. Set Λ to be the Jordan curve $\Lambda(G)$. From Theorem 2.4 there is $\bar{t} = \bar{t}(\lambda_0)$ so that

$$d_H\big(C_{\bar{h}}(\Lambda), D\big) \leq \bar{t}, \tag{4}$$

and, for all $x \in D$, if γ_x denotes the unit speed hyperbolic geodesic with $\gamma_x(0) = x$ and $\gamma_x'(0)$ orthogonal to $T_x D$, then we have

$$\operatorname{dist}\big(\gamma(t), C_{\bar{h}}(\Lambda)\big) \geq R + 1 \quad \text{for all } |t| \geq \bar{t} + R, \tag{5}$$

where $R = R(h)$ is the constant given by Proposition 3.2.

From Proposition 3.4 we have that $\Omega \subset C_h(\Lambda)$, and thus we obtain from (4) and Corollary 3.2 that Ω is contained in the $(\bar{t} + R)$-neighborhood of D.

To deduce the other inclusion, pick $x \in D$. We have that $\gamma_x(+\infty)$, $\gamma_x(-\infty)$ lie in different connected components of $S^2_\infty \setminus \Lambda$. Because $\overline{\Omega} \subset \overline{B}^3$ is a disk with the same boundary as $\overline{D}$, γ_x must intersect Ω in at least one point $\gamma(t) \in \Omega \cap \gamma$. From (5) and Proposition 3.2 we have that $|t| \leq \bar{t} + R$, and so $d(x, \Omega) \leq \bar{t} + R$. ☐

4. Almost-Fuchsian surface groups

Let $s(M, g, \varepsilon)$ denote the cardinality of $S(M, g, \varepsilon)$, the set of ε-Fuchsian surface subgroups of genus at most g, modulo the equivalence relation of conjugacy. Recall that we defined $S_\varepsilon(M) = \bigcup_{g\in\mathbb{N}} S(M, g, \varepsilon)$ and that A denotes the second fundamental form of a surface of M.

PROPOSITION 4.1
Suppose we have a sequence $\Pi_i \in S_{\delta_i}(M)$, where $\delta_i \to 0$ as $i \to \infty$. For each $i \in \mathbb{N}$, there is an essential minimal surface S_i in the homotopy class Π_i so that $\operatorname{area}(S_i) =$

area(Π_i) *and*

$$\lim_{i\to\infty} \|A\|^2_{L^\infty(S_i)} = 0. \tag{6}$$

Moreover, if D_i is a disk lifting S_i to $\mathbb{H}^3$ that is preserved by the surface group $G_i < \Gamma$ induced by S_i and intersecting a fixed compact set in $\mathbb{H}^3$ for all $i \in \mathbb{N}$, then there is a totally geodesic disk $D \subset \mathbb{H}^3$ such that, after passing to a subsequence, D_i converges smoothly to D on compact sets and $\Lambda(G_i)$ converges in Hausdorff distance to $\partial_\infty D$ in S^2_∞.

Proof

For each $i \in \mathbb{N}$, consider the essential immersion $S_i \subset M$ that minimizes area with respect to the hyperbolic metric in the homotopy class Π_i (using [31], for instance). If D_i is the minimal disk lifting S_i to $\mathbb{H}^3$ that is preserved by the surface group $G_i < \Gamma$ induced by S_i, then we have from Theorem 1 in [32] that $||A||^2_{L^\infty(D_i)}$ tends to zero as $i \to \infty$. Actually, in our setting, we only need to apply [32, Theorem 1] to surfaces which minimize area in their homotopy class, and so the same result could be obtained applying simpler arguments.

Assume that all the disks D_i intersect a compact set. We now argue that, after passing to a subsequence, the disks D_i converge to a totally geodesic disk with multiplicity 1. From Theorem 2.4 we know that, for all i sufficiently large, D_i is embedded and that S_i is the unique closed embedded minimal surface in $M_i = \mathbb{H}^3 \setminus G_i \simeq T^\perp S_i$ and therefore area-minimizing in M_i among all mod 2 cycles representing the same element in $H_2(M_i;\mathbb{Z}_2)$. As a result, D_i is locally area-minimizing among mod 2 cycles as well. Pick $p_i \in D_i$ which converges, after passing to a subsequence, to some $p \in \mathbb{H}^3$. From the fact that for all i sufficiently large, the embedded disks D_i are locally area-minimizing among mod 2 cycles, we obtain from standard compactness theory for minimal surfaces the existence of a totally geodesic disk $D \subset \mathbb{H}^3$ containing p such that, after passing to another subsequence, D_i converges graphically to D on compact sets.

Consider $q_i \in \Lambda(G_i)$, $\sigma_i \subset \mathbb{H}^3$ the geodesic ray with $\sigma_i(0) = p_i$, $\sigma_i(+\infty) = q_i$, and $\gamma_i \subset D_i$ the geodesic ray (for the induced metric on D_i) with $\gamma_i(0) = p_i$, $\gamma_i(+\infty) = q_i$. The geodesic curvature of γ_i in $\mathbb{H}^3$ is a fixed amount below 1 for all i sufficiently large, and so, using tubular neighborhoods of σ_i as barriers, we deduce the existence of $r > 0$ so that γ_i is contained in an r-tubular neighborhood of σ_i for all $i \in \mathbb{N}$. Thus, after passing to a subsequence, both curves converge on compact sets to the same geodesic ray $\sigma \subset D$. Using this fact, the reader can deduce that $\Lambda(G_i)$ converges in Hausdorff distance to $\partial_\infty D$ in S^2_∞. □

Using the above proposition, we now show the following improvement to the main results of [16] and [17].

THEOREM 4.2
There are positive constants $c_1 = c_1(M,\varepsilon)$*,* $c_2 = c_2(M)$*, and* $k = k(M,\varepsilon) \in \mathbb{N}$ *so that for all* $g \geq k$ *we have*

$$(c_1 g)^{2g} \leq s(M,g,\varepsilon) \leq (c_2 g)^{2g}.$$

Moreover, there is a subset $G(M,g,\varepsilon) \subset S(M,g,\varepsilon)$ *with more than* $(c_1 g)^{2g}$ *elements so that any sequence of homotopy classes* $\Pi_i \in G(M,g_i,1/i)$*,* $i \in \mathbb{N}$*, has a representative* $\phi : S_i \to M$ *so that*

(a) S_i *is a minimal immersion with* $\operatorname{area}(S_i) = \operatorname{area}(\Pi_i)$ *and*

$$\lim_{i\to\infty} \sup_{S_i} |A| = 0;$$

(b) *after passing to a subsequence, the Radon measure*

$$f \in C^0(M) \mapsto \mu_i(f) = \frac{1}{\operatorname{area}(S_i)} \int_{S_i} f \circ \phi \, dA$$

converges to a measure ν *which is positive on every open set of* M*.*

Proof
If $s(M,g)$ denotes the cardinality of $S(M,g)$, then, as shown in [16, Theorem 1.1], $c_2 > 0$ exists so that $s(M,g) \leq (c_2 g)^{2g}$ for all g large. Since $s(M,g,\varepsilon) \leq s(M,g)$, the upper bound is verified.

We now verify the lower bound. In [17] the authors show that for all $\varepsilon > 0$ there is a Fuchsian group K (preserving a totally geodesic plane $C(\gamma)$ for some geodesic circle γ) and a $(1+\varepsilon)$-quasiconformal map $\Phi : S^2_\infty \to S^2_\infty$ so that $G = \Phi \circ K \circ \Phi^{-1}$ is a surface subgroup of Γ. The map Φ admits an extension $F : \mathbb{H}^3 \to \mathbb{H}^3$ that is equivariant with respect to K and G and a $(1+o_\varepsilon(1), o_\varepsilon(1))$-quasi-isometry, where $o_\varepsilon(1)$ denotes a quantity depending only on M and ε that tends to zero as $\varepsilon \to 0$. As a result, the essential surface $\Sigma_\varepsilon = F(C(\gamma) \setminus K) \subset M$ induces an element of $S_\varepsilon(M)$. Σ_ε has the property that geodesics with respect to the intrinsic distance are $(1+\varepsilon,\varepsilon)$-quasigeodesics, and we denote such surfaces by $(1+\varepsilon)$*-quasigeodesic surfaces*.

Let g_0 denote the genus of Σ_ε. If Σ_n denotes a degree n cover of Σ_ε, then its genus is $g = n(g_0 - 1) + 1$ and so Σ_n induces an element in $s(M,g,\varepsilon)$. The Müller–Puchta formula says that the number of index n subgroups of a genus g_0 orientable surface grows like $2n(n!)^{2g_0-2}(1+o(1))$, and so (using Stirling's approximation) we get the estimate

$$s(M,g,\varepsilon) \geq (c_1 g)^{2g},$$

where $c_1 > 0$ depends on g_0, which in turn depends only on M and ε.

We set $G(M,g,\varepsilon)$ to be the homotopy classes that come from finite covers of Σ_ε and have genus less than or equal to g.

Description of Σ_ε

We now describe in more detail the properties of Σ_ε. In [13] Hamenstadt extended the results of [17] to some rank 1 locally symmetric spaces, building on the work of Kahn and Markovic. We follow the geometric description and the notation of [13]. The words in italics have precise definitions in [13].

The basic building blocks are called (R,δ)-*geometric skew-pants* P (or simply *geometric skew-pants*), and they are defined in Sections 4 and 6 of [13]. The boundary of P consists of three closed geodesics in M, and P decomposes into five polygon regions with geodesic boundary (two *center triangles* and three *twisted bands* using the notation in [13, Section 6]). Each polygon is a smooth immersion whose principal curvatures depend uniformly on (R,δ) and can be made arbitrarily small by choosing R sufficiently large and δ sufficiently small. Regions that share a common geodesic side have the property that the corresponding conormals make an angle as close to π as desired by choosing R large and δ small. Given any $0 < \eta < 1$ there is $d > 0$ (independent of R and δ) so that the set of points K^P in P that are at an intrinsic distance less than or equal to d from one of the center triangles has

$$(1-\eta)2\pi \leq \operatorname{area}(K^P) \leq \operatorname{area}(P) \leq (1+\eta)2\pi \tag{7}$$

for all R large and δ small. The *seams* of a geometric skew-pants P are three shortest geodesic arcs in M (in the homotopy class defined by P) that connect the three boundary geodesics of P. The endpoints of the seams define two distinguished points on the geodesic geodesic boundary of P and are called the *feet* of P.

For our purposes it will be important as well to control the location of the geometric skew-pants in M. Given a point $x = (p,(e_1,e_2,e_3)) \in \mathcal{F}(M)$ we get a natural orientation in the 2-plane $V = \operatorname{span}\{e_1,e_2\} \subset T_pM$ and an oriented ideal triangle $T \subset V$ whose vertices are the endpoints of the geodesic ray based at p with initial velocity e_1 and its $2\pi/3$ consecutive rotations in V (see [13, Section 4] for definitions: in the codimension 1 setting, framed tripods and frames can be identified). This ideal triangle T contains in its interior an equilateral geodesic triangle T_x (called *center triangle*) whose vertices are the projection of the ideal vertices of T onto its opposite sides (see [13, p. 849]).

In [13, Section 4] it is defined what it means for two frames $x, y \in \mathcal{F}(M)$ to be (R,δ)-*well connected*. When that occurs, in Sections 5 and 6 of [13] a (R,δ)-geometric skew-pants $P(x,y)$ is constructed such that its center triangles can be made uniformly close to the two center triangles T_x, T_y. In particular, for all R sufficiently large and δ sufficiently small, we have that if $x, y, z \in M$ and $r = d(x,z)/2$, then

$$\operatorname{area}\big(P(x,y) \cap B_{2r}(z)\big) \geq \frac{\operatorname{area}(T_x \cap B_{2r}(z))}{2} \geq \omega_0 r^2, \tag{8}$$

where ω_0 is constant depending only on M.

An *oriented* (R,δ)*-skew-pants* is defined as being the homotopy class of some oriented (R,δ)-geometric skew-pants immersion $f : P \to M$, where the homotopies preserve the image and orientation of the boundary geodesics. The space $\mathcal{P}(R,\delta)$ of all such homotopy classes contains only finitely many elements. Given $x, y \in \mathcal{F}(M)$ it is possible that they are (R,δ)-well connected in several different ways which would give rise to geometric skew-pants in different homotopy classes. On the other hand, for all R large enough and δ small, it is shown in Lemma 7.4 of [13] (combined with Lemma 4.3 [13]) that every pair $(x,y) \in \mathcal{F}(M)^2$ is (R,δ)-well connected and that, even if there are several parameters involved in the construction of the correspondent (R,δ)-geometric skew-pants, their homotopy class only depends on (x,y), R, and δ. As a result, we obtain a map

$$\hat{P} : \mathcal{F}(M)^2 \to \mathcal{P}(R,\delta),$$

where $\hat{P}(x,y)$ denotes the homotopy class of any of the (R,δ)-geometric skew-pants $P(x,y)$ given by [13, Lemma 7.4].

Let λ^2 denote the normalized Lebesgue measure in $\mathcal{F}(M)^2$. For each (R,δ) consider the measure μ in $\mathcal{F}(M)^2$ that is obtained by integrating $d\mu$ defined in [13, p. 849] along the fiber $\mathcal{F}(M)^3$. From Lemma 7.4 of [13] we have that μ is absolutely continuous with respect to λ^2, and its Radon–Nikodym derivative has order $1 + O(1/R)$. In particular, for every open set $\Omega \subset \mathcal{F}(M)^2$ and every R large enough we have

$$\mu(\Omega) \geq \frac{\lambda^2(\Omega)}{2}. \tag{9}$$

For each $P \in \mathcal{P}(R,\delta)$, set $h(P) = \mu(\hat{P}^{-1}(P))$.

In Lemma 7.2 and Proposition 7.3 of [13] the quasigeodesic surface Σ_ε is constructed by attaching several elements of $\mathcal{P}(R,\delta)$ along a common boundary geodesic. Moreover, if n_P denotes the number of times that $P \in \mathcal{P}(R,\delta)$ appears in Σ_ε, then

$$n_P \geq \frac{h(P)}{2} \sum_{Q \in P(R,\delta)} n_Q. \tag{10}$$

The attaching of the (R,δ)-geometric skew-pants is made so that if $P, P' \in \mathcal{P}(R,\delta)$ share a common boundary geodesic β (with opposite induced orientations), then the tangent planes of P and P' along β can be made uniformly close to each other as $R \to \infty$ and $\delta \to 0$, and the distance between the feet of P and P' that belong to β is close to 1. This last property is important to ensure that a surface constructed this way will be $(1+\varepsilon)$-quasigeodesic if R and δ are, respectively, very large and very small (see [13, Proposition 6.2]). Note that necessarily $\text{area}(\Sigma_\varepsilon) \simeq 4\pi(g-1)$, where g is the genus of Σ_ε, and that in [13, Lemma 3.1] it is shown that Σ_ε is a locally CAT$(-1/2)$ space for all ε sufficiently small.

The pants decomposition of Σ_ε is *centrally c_0-thick* for some universal constant c_0 (see [13, Section 3]) for definition and proof of [13, Proposition 6.2]), and this implies the existence of $\underline{r} > 0$ (depending only on c_0) so that for all $d > 0$, all R sufficiently large, and all δ sufficiently small (both depending on d), if $x \in \Sigma_\varepsilon$ is at distance d from any of the center triangles coming from the geometric skew-pants, then the intrinsic ball $\hat{B}_{\underline{r}}(x)$ in Σ_ε of radius $\underline{r}$ centered at x intersects at most a finite number of the polygonal regions with geodesic boundary. In particular, by making R large and δ small, we have $\hat{B}_{\underline{r}}(x)$ arbitrarily close to a totally geodesic disk.

With $0 < \eta < 1$ fixed, choose d so that (7) holds, and consider the set of points K_η in Σ_ε that are at an intrinsic distance less than or equal to d from any of the center triangles coming from the geometric skew-pants. We have

$$\text{area}(K_\eta) \geq (1-\eta)(1+\eta)^{-1}\,\text{area}(\Sigma_\varepsilon). \tag{11}$$

Consider the minimal representatives S_i in the homotopy class $\Pi_i \in G(M, g_i, 1/i)$, $i \in \mathbb{N}$, given by Proposition 4.1, from which Theorem 4.2(a) follows immediately.

Proof of Theorem 4.2(b)
Each S_i is homotopic to a $(1+1/i)$-quasigeodesic surface Σ_i, and we choose disks $D_i, \Omega_i \subset \mathbb{H}^3$ that cover S_i and Σ_i, respectively, and such that $\partial_\infty D_i = \partial_\infty \Omega_i$. For all i sufficiently large, Ω_i is a CAT$(-1/2)$ space (see [8, Theorem II.4.1]) for which every geodesic arc can be extended (see [8, Proposition II.5.10]). Combining with the fact that the principal curvatures of S_i tend to zero and that geodesics in Σ_i lift to $(1+o_i(1))$-quasigeodesics in $\mathbb{H}^3$, we obtain

$$d_H(D_i, \Omega_i) \to 0 \quad \text{as } i \to \infty. \tag{12}$$

Let μ_i, ν_i denote the unit Radon measure of M induced by integration over S_i and Σ_i, respectively. After passing to a subsequence we can assume that both measures converge.

LEMMA 4.3
$\lim_{i\to\infty} \mu_i = \lim_{i\to\infty} \nu_i$.

Proof
Fix $0 < \eta < 1$. We saw in Proposition 4.1 that D_i is locally area-minimizing mod 2 in $\mathbb{H}^3$, and so we have from (12) that for all i sufficiently large and every geodesic ball $B \subset \mathbb{H}^3$ of small radius,

$$\operatorname{area}(D_i \cap B) \leq (1+\eta)\operatorname{area}(\Omega_i \cap B),$$

and thus from the fact that $\operatorname{area}(S_i)\operatorname{area}(\Sigma_i)^{-1} \to 1$ as $i \to \infty$ we have

$$\lim_{i\to\infty} \mu_i \leq (1+\eta) \lim_{i\to\infty} \nu_i.$$

Denote the set $K_{\eta,i} \subset \Sigma_i$ simply by K_i, and let $\hat{K}_i \subset \Omega_i$ denote its preimage. We have from (11) that for all i sufficiently large, $\nu_i(M \setminus K_i) \leq 2\eta$. From the definition of $\underline{r}$ and (12) we have that for all i sufficiently large and all $x \in \hat{K}_i$, $B_{\underline{r}}(x) \cap \Omega_i$ is very close to a geodesic disk of radius $\underline{r}$ in D_i. Thus for every geodesic ball $B \subset M$ of radius smaller than $\underline{r}/2$ we obtain

$$\lim_{i\to\infty} \mu_i(B) \geq \lim_{i\to\infty} \mu_i(B \cap K_i) = \lim_{i\to\infty} \nu_i(B \cap K_i) \geq \lim_{i\to\infty} \nu_i(B) - 2\eta.$$

Making $\eta \to 0$, we deduce the result. □

The claim below and Lemma 4.3 prove Theorem 4.2(b).

CLAIM 4.4
For every geodesic ball $B \subset M$, we have $\liminf_{i\to\infty} \nu_i(B) > 0$.

It suffices to consider the case where each Σ_i is one of the surfaces constructed in [13] (the finite covering case follows immediately). If r denotes the radius of B, then choose $\tilde{B} \subset M$ a geodesic ball with the same center as B but radius $r/2$. Consider the open set U of all frames in $\mathcal{F}(M)$ with basepoint in $\tilde{B}$. From (8) we have that for all R large and δ small,

$$\operatorname{area}\big(P(x,y) \cap B\big) \geq \omega_0 r^2/4 \quad \text{for all } x \in U, y \in \mathcal{F}(M). \tag{13}$$

Set

$$\Lambda = \big\{P \in \mathcal{P}(R,\delta) : \hat{P}^{-1}(P) \cap \big(U \times \mathcal{F}(M)\big) \neq \emptyset\big\}.$$

Each time $P \in \Lambda$ choose its geometric representative to be $P(x,y)$, where $x \in U$. Therefore, for all i sufficiently large, we have using (13) that

$$\operatorname{area}(\Sigma_i \cap B) \geq \sum_{P\in\Lambda} n_P \operatorname{area}(P \cap B) \geq \frac{\omega_0 r^2}{4} \sum_{P\in\Lambda} n_P.$$

From (7) we have that for all R sufficiently large and δ sufficiently small, $\operatorname{area}(P) \geq 3\pi$, and so $\sum_{Q\in\mathcal{P}(R,\delta)} n_Q \geq \operatorname{area}(\Sigma_i)/3\pi$, which when combined with (10), the way $h(P)$ was defined, and (9) implies that

$$\begin{aligned}\operatorname{area}(\Sigma_i \cap B) &\geq \frac{\omega_0 r^2}{8} \sum_{P\in\Lambda} h(P) \sum_{Q\in\mathcal{P}(R,\delta)} n_Q \\ &\geq \frac{\omega_0 r^2}{24\pi} \operatorname{area}(\Sigma_i) \sum_{P\in\Lambda} h(P) \\ &\geq \frac{\omega_0 r^2}{24\pi} \operatorname{area}(\Sigma_i)\mu\big(U \times \mathcal{F}(M)\big) \\ &\geq \frac{\omega_0 r^2}{48\pi} \operatorname{area}(\Sigma_i)\lambda^2\big(U \times \mathcal{F}(M)\big).\end{aligned}$$

Thus for all i sufficiently large we have

$$\nu_i(B) \geq \frac{\omega_0 r^2}{48\pi}\lambda^2\big(U \times \mathcal{F}(M)\big) > 0,$$

which proves the claim. □

5. Asymptotic inequality

Consider $\{S_i\}_{i\in\mathbb{N}}$ to be a sequence of minimal essential immersions given by Theorem 4.2, each inducing a surface group $G_i < \Gamma$. For each $i \in \mathbb{N}$ consider as well the minimal essential immersion $\Sigma_i \subset M$ that minimizes area with respect to the metric h in the homotopy class of S_i (using [31], for instance).

The goal of this section (and the next) is to prove the following result.

THEOREM 5.1

Assume the metric h has sectional curvature ≤ -1. Then

$$\limsup_{i\to\infty} \frac{\operatorname{area}_h(\Sigma_i)}{\operatorname{area}(S_i)} \leq 1.$$

If equality holds, then the metric h is hyperbolic.

Proof

Let g_i denote the genus of S_i. From the Gauss equation we have that

$$\operatorname{area}_h(\Sigma_i) = 4\pi(g_i - 1) + \int_{\Sigma_i} (K_{12} + 1)\, dA_h - \frac{1}{2}\int_{\Sigma_i} |A|^2\, dA_h, \tag{14}$$

where $K_{12}(x)$ is the ambient sectional curvature of $T_x\Sigma_i$. Using the fact that $K_{12} \leq -1$ and (6) we have that

$$\limsup_{i\to\infty} \frac{\text{area}_h(\Sigma_i)}{\text{area}(S_i)} \le 1,$$

with equality implying that

$$\lim_{i\to\infty} \frac{1}{\text{area}_h(\Sigma_i)} \int_{\Sigma_i} |A|^2 - (K_{12}+1)\, dA_h = 0.$$

Consider the nonnegative smooth function $f_i = |A|^2 - (K_{12}+1)$ on Σ_i. We then have

$$\lim_{i\to\infty} \frac{1}{\text{area}_h(\Sigma_i)} \int_{\Sigma_i} |f_i|\, dA_h = 0.$$

In Section 6 we show (see Corollary 6.2) the existence of a group $H_i < \Gamma$ conjugate to G_i so that if D_i, Ω_i denote, respectively, the lifts of S_i and Σ_i to $\mathbb{H}^3$ that are preserved by H_i we have, after passing to a subsequence, that

(i) $\Lambda(H_i)$ converges in Hausdorff distance, as $i \to \infty$, to $\gamma \in \mathcal{C}$ with $\Gamma\gamma$ dense in $\mathcal{C}$, and

(ii) for all $R > 0$,

$$\lim_{i\to\infty} \int_{\Omega_i \cap B_R(p)} |f_i|\, dA_h = 0. \tag{15}$$

From (i) we have that all D_i's must intersect a compact set in $\mathbb{H}^3$, and so Proposition 4.1 implies that $\{D_i\}_{i\in\mathbb{N}}$ converges to a totally geodesic disk D for the hyperbolic metric with $\partial_\infty D = \gamma$.

Because the ambient curvature is negative, Σ_i is negatively curved and so, in virtue of being essential, its injectivity radius has a uniform lower bound for all $i \in \mathbb{N}$. Hence, standard stability estimates imply that the second fundamental form of Σ_i is uniformly bounded for all $i \in \mathbb{N}$ along with all its derivatives. As a result, we have from (15) that

$$\lim_{i\to\infty} \sup\big\{|A|(x) + \big|K_{12}(x)+1\big| : x \in \Omega_i \cap B_R(p)\big\} = 0, \quad \text{all } R > 0. \tag{16}$$

We recall for the reader that a smooth surface has vanishing second fundamental form if and only if intrinsic geodesics coincide with extrinsic geodesics (i.e., is totally geodesic).

PROPOSITION 5.2

There is a totally geodesic disk Ω in (B^3, h) with $\partial_\infty \Omega = \gamma$ and such that the sectional curvature of $T_x\Omega$ is -1 for all $x \in \Omega$.

Proof
From Theorem 3.1 we obtain the existence of a compact set K that intersects Ω_i for all $i \in \mathbb{N}$. Choose $x_i \in \Omega_i \cap K$ and denote by $B^i_R(x_i) \subset \Omega_i$ the intrinsic ball of radius R centered at x_i. Note that Ω_i is negatively curved, and thus $B^i_R(x_i)$ is diffeomorphic to a disk for all i sufficiently large. Standard compactness of minimal surfaces with uniform bounds on the second fundamental form gives the existence of a complete minimal surface $\Omega \subset B^3$ so that, after passing to a subsequence, intrinsic disks in Ω_i centered at x_i converge strongly to intrinsic disks in Ω. Furthermore, from (16), we have that $|A| = 0$ on Ω (thus being totally geodesic), and the sectional curvature of $T_x\Omega$ is -1 for all $x \in \Omega$. As a result, Ω is diffeomorphic to a disk. We have from Proposition 3.4 that $\Omega_i \subset C_h(\Lambda(H_i))$ for all i sufficiently large and so $\partial_\infty \Omega \subset \gamma$. On the other hand, $\partial_\infty \Omega$ is homeomorphic to a circle and so it must be equal to γ. □

Consider the following circle bundles:

$$S_1^D := \{(p,v) : p \in D, v \in T_p D, \bar{h}(v,v) = 1\}$$

and

$$S_1^\Omega := \{(p,v) : p \in \Omega, v \in T_p \Omega, h(v,v) = 1\}.$$

Denote by $S_1 M(\bar{h})$ and $S_1 M(h)$ the unit tangent bundle of M with respect to $\bar{h}$ and h, respectively, and let $S_1^D(M) \subset S_1 M(\bar{h})$, $S_1^\Omega(M) \subset S_1 M(h)$ denote, respectively, the projection to $S_1 M(\bar{h})$ and $S_1 M(h)$ of S_1^D and S_1^Ω. From (i) we have that $S_1^D(M)$ is dense in $S_1 M(\bar{h})$.

We now argue that the sectional curvature of every 2-plane in (M,h) is -1.

CLAIM 5.3
For every $(p,v) \in S_1(M)(h)$ there is a totally geodesic hyperbolic disk $\Omega_{(p,v)}$ in (B^3,h) whose projection in M contains the geodesic passing through p with direction v.

From the geodesic rigidity proven in Gromov [12] there is a homeomorphism T from $S_1 M(\bar{h})$ to $S_1 M(h)$ that maps geodesics onto geodesics, meaning that if γ is a geodesic in $(M,\bar{h})$, then there is a geodesic σ in (M,h) so that for all $t \in \mathbb{R}$ there is $s \in \mathbb{R}$ so that $T(\gamma(t),\gamma'(t)) = (\sigma(s),\sigma'(s))$. Moreover, from its proof (see, e.g., [18, Theorem 2.12]), T can be chosen so that if $\gamma(+\infty), \gamma(-\infty) \in S^2_\infty$ are the asymptotes of γ, then σ has the same asymptotes in S^2_∞. Thus, from the fact that $\partial_\infty \Omega = \partial_\infty D$ and that both D and Ω are totally geodesic, we have that T is also a homeomorphism from $S_1^D(M)$ onto $S_1^\Omega(M)$. Therefore, because $S_1^D(M)$ is dense in $S_1 M(\bar{h})$ we obtain that $S_1^\Omega(M)$ is also dense in $S_1 M(h)$. As a result, for every

$(p,v) \in S_1M(h)$ we can find a sequence of points $\{\omega_i\}_{i\in\mathbb{N}}$ in S_1^{Ω} whose projection to $S_1M(h)$ converges to (p,v), and so applying the same reasoning as in Proposition 4.1 to a suitable sequence $\{\phi_i(\Omega)\}_{i\in\mathbb{N}}$, where $\phi_i \in \Gamma$, we obtain a totally geodesic hyperbolic disk $\Omega_{(p,v)} \subset B^3$ whose projection in M contains the geodesic passing through p with direction v.

Recalling the discussion in Section 2.4, choose $(p,(e_1,e_2,e_3)) \in \mathcal{F}(M)(h)$ whose orbit under the frame flow

$$F_t\big((p,(e_1,e_2,e_3))\big) = \big(\gamma(t),\big(\gamma'(t),e_2(t),e_3(t)\big)\big), \quad t \geq 0,$$

is dense in $\mathcal{F}(M)(h)$. We abuse notation and denote the lift of γ to B^3 by γ. By applying a rotation if necessary, we can prescribe the vector e_2 to be any unit vector orthogonal to e_1 that we still obtain a dense orbit in $\mathcal{F}(M)(h)$. Hence we assume that $\{e_1,e_2\}$ span $T_{\gamma(0)}\Omega_{(p,e_1)}$, in which case the fact that $\Omega_{(p,e_1)}$ is totally geodesic implies that $\operatorname{span}\{\gamma'(t),e_2(t)\} = T_{\gamma(t)}\Omega_{(p,e_1)}$ for all $t \geq 0$. Therefore, the set of 2-planes with sectional curvature -1 is dense in $\mathrm{Gr}_2(M)$, and this implies the desired result. $\square$

6. Nearly totally geodesic minimal surfaces

We continue assuming the setup of the last section. Namely, we have a sequence of minimal essential immersions $\{S_i\}_{i\in\mathbb{N}}$ given by Theorem 4.2, each inducing a surface group $G_i < \Gamma$ and lifting to a disk $D_i \subset \mathbb{H}^3$ that is preserved by G_i.

For each $i \in \mathbb{N}$ consider as well the minimal essential immersion $\Sigma_i \subset M$ that minimizes area with respect to the metric h in the homotopy class of S_i with the smooth function f_i defined in the previous section so that

$$\lim_{i\to\infty} \frac{1}{\operatorname{area}_h(\Sigma_i)} \int_{\Sigma_i} |f_i|\,dA_h = 0. \tag{17}$$

Let Ω_i denote the disk lifting Σ_i to B^3 that is preserved by $G_i < \Gamma$, $i \in \mathbb{N}$. To make notation easier, it is understood that the function f_i on $\phi(\Omega_i)$, $\phi \in \Gamma$, means $f_i \circ \pi_{\Omega_i} \circ \phi^{-1}$, where π_{Ω_i} is the projection from Ω_i to Σ_i.

Fix $p \in \mathbb{H}^3$, consider for every $\varepsilon, R > 0$

$$F_i(\varepsilon,R) = \Big\{\phi \in \Gamma : \int_{\phi(\Omega_i)\cap B_R(p)} |f_i|\,dA_h \leq \varepsilon\Big\},$$

and define $\mathcal{L} \subset \mathcal{C}$ as

$$\mathcal{L} = \big\{\gamma \in \mathcal{C} : \exists \phi_i \in F_i(\varepsilon_i,R_i) \text{ with } \varepsilon_i \to 0, R_i \to \infty \text{ so that,}$$
$$\text{after passing to a subsequence, } \Lambda(\phi_i G_i \phi_i^{-1}) \text{ converges to } \gamma\big\}.$$

The goal of this section is to show the following.

THEOREM 6.1
$\mathcal{L} = \mathcal{C}$, and so there is $\gamma \in \mathcal{L}$ so that $\Gamma\gamma$ is dense in $\mathcal{C}$.

This result has the following corollary.

COROLLARY 6.2
There is a conjugate group $H_i = \phi_i G_i \phi_i^{-1}$, $\phi_i \in \Gamma$, so that, after passing to a subsequence,

(a) *$\Lambda(H_i)$ converges in Hausdorff distance, as $i \to \infty$, to $\gamma \in \mathcal{C}$ with $\Gamma\gamma$ dense in $\mathcal{C}$, and*

(b) *for all $R > 0$,*

$$\lim_{i\to\infty} \int_{\phi_i(\Omega_i)\cap B_R(p)} |f_i| dA_h = 0.$$

Proof of Theorem 6.1
We start by showing the following lemma.

LEMMA 6.3
The set $\mathcal{L}$ is closed and Γ-invariant.

Proof
The fact that it is closed follows by extracting a diagonal subsequence.

With $\psi \in \Gamma$, set $\alpha = d(p, \psi(p))$. Using the fact that $\psi^{-1}(B_{R-\alpha}(p)) \subset B_R(p)$ the reader can check that, for all $R > 0$ and all $\varepsilon > 0$,

$$\phi \in F_i(\varepsilon, R) \implies \psi\phi \in F_i(\varepsilon, R-\alpha). \tag{18}$$

Combining this with the fact that $\psi(\Lambda(H)) = \Lambda(\psi H \psi^{-1})$ for every discrete subgroup $H \subset \Gamma$, it follows at once that if $\gamma \in \mathcal{C}$, then $\psi(\gamma) \in \mathcal{C}$. □

Hence, it suffices to find $\gamma \in \mathcal{L}$ so that $\Gamma\gamma$ is dense in $\mathcal{C}$. Before we provide the details we describe first the general idea. The key step is to show that for every compact set $K \subset \mathbb{H}^3$ there is $\gamma \in \mathcal{L}$ so that $C(\gamma)$ intersects K. Indeed, if no dense orbit exists, then every point in $\mathcal{L}$ is isolated (Theorem 2.6), and so we can find a compact set K so that $C(\gamma)$ never intersects K for all $\gamma \in \mathcal{L}$, which is a contradiction.

Consider a Dirichlet fundament domain $p \in \Delta$ for M so that $\partial\Delta$ is transverse to both $\phi(D_i)$ and $\phi(\Omega_i)$ for all $\phi \in \Gamma$. We now consider Γ^{S_i}, $\Gamma^{S_i}(K)$ to be the set of all lifts of S_i that intersect Δ, K, respectively, Γ^{Σ_i} to be the set of all lifts of Σ_i that intersect Δ, and $\Gamma^{\Sigma_i}(\varepsilon, R)$ to be the lifts in Γ^{Σ_i} for which the function $|f_i|$ is small in L^1 on a ball of radius R. More precisely,

$$\Gamma^{S_i} = \{\phi \in \Gamma : \phi(D_i) \cap \Delta \neq \emptyset\},$$
$$\Gamma^{S_i}(K) = \{\phi \in \Gamma : \phi(D_i) \cap K \neq \emptyset\},$$
$$\Gamma^{\Sigma_i} = \{\phi \in \Gamma : \phi(\Omega_i) \cap \Delta \neq \emptyset\},$$
$$\Gamma^{\Sigma_i}(\varepsilon, R) = F_i(\varepsilon, R) \cap \Gamma^{\Sigma_i}.$$

We want to find $\varepsilon_i \to 0$, $R_i \to \infty$, so that $\Gamma^{S_i}(K) \cap F(\varepsilon_i, R_i)$ is always nonempty.

The strategy is the following: The sets described above are all invariant by right multiplication with G_i because G_i preserves both D_i and Ω_i. We denote the projection of these sets in $\Gamma \setminus G_i$ by $\underline{\Gamma}^{S_i}$, $\underline{\Gamma}^{\Sigma_i}$, $\underline{\Gamma}^{S_i}(K)$, and $\underline{\Gamma}^{\Sigma_i}(\varepsilon, R)$. We will see that, for all i very large, $\#\underline{\Gamma}^{S_i}$ is proportional to area(S_i), use the fact that $d_H(\Omega_i, D_i)$ is bounded to conclude that Γ^{S_i} and Γ^{Σ_i} are at a finite Hausdorff distance from each other, deduce from Theorem 4.2(b) that $\frac{\#\underline{\Gamma}^{S_i}(K)}{\#\underline{\Gamma}^{S_i}}$ is bounded below away from zero, and use (17) to deduce that $\frac{\#\underline{\Gamma}^{\Sigma_i}(\varepsilon,R)}{\#\underline{\Gamma}^{\Sigma_i}} \simeq 1$. Putting all these facts together one can then conclude that $\Gamma^{S_i}(K) \cap F(\varepsilon, R) \neq \emptyset$ for all i very large. We now provide the details.

Referring to the notation set in Section 2.1, we fix a representative $\underline{\phi}$ for each coset $\phi G_i \in \Gamma \setminus G_i$. Recall that ν is the measure given by Theorem 4.2(b)

PROPOSITION 6.4
There are constants $n = n(M,h) \in \mathbb{N}$, $\alpha = \alpha(M) > 0$, and $\beta = \beta(\nu, K) > 0$ so that for all i sufficiently large,
(a) $d_H(\Gamma^{S_i}, \Gamma^{\Sigma_i}) \leq n$;
(b) $\alpha^{-1}\mathrm{area}(S_i) \leq \#\underline{\Gamma}^{S_i} \leq \alpha\,\mathrm{area}(S_i)$;
(c) $\liminf_{i\to\infty} \frac{\#\underline{\Gamma}^{S_i}(K)}{\#\underline{\Gamma}^{S_i}} \geq \beta$.

Proof
From Theorem 3.1 we have the existence of $c_1 = c_1(h)$ so that $d_H(\phi(D_i), \phi(\Omega_i)) \leq c_1$ for all $\phi \in \Gamma$ for all i sufficiently large, and from Lemma 2.2 we have the existence of $n = n(M, c_1) \in \mathbb{N}$ so that $B_{c_1}(x) \subset \bigcup_{|\phi|\leq n} \phi(\Delta)$ for all $x \in \Delta$.

Choose $\psi \in \Gamma^{S_i}$, and pick $x \in \psi(D_i) \cap \Delta$. There is $y \in \psi(\Omega_i) \cap B_{c_1}(x)$, and thus some $\phi \in \Gamma$ with $|\phi| \leq n$ for which $\phi^{-1}(\psi(\Omega_i)) \cap \Delta \neq \emptyset$. Hence Γ^{S_i} is in a n-neighborhood of Γ^{Σ_i} (for the distance d), and reversing the roles of Σ_i and S_i proves (a).

Recall from Section 2.1 that for all $\psi \in \Gamma$, $\Delta_i = \bigcup_{\underline{\phi} \in \Gamma\setminus G_i} \underline{\phi}^{-1}(\psi(\Delta))$ is a fundamental domain for $\mathbb{H}^3 \setminus G_i$. Thus,

$$\operatorname{area}(S_i) = \sum_{\underline{\phi}\in\Gamma\backslash G_i} \operatorname{area}\big(\underline{\phi}(D_i)\cap\psi(\Delta)\big). \tag{19}$$

Choose $A \subset \Gamma$ a finite set so that a neighborhood of radius 1 of Δ is contained in the interior of $\bigcup_{\psi\in A}\psi(\Delta)$. If $x \in \underline{\phi}(D_i)\cap\Delta$, then we have from the monotonicity formula that, for some $c_2 = c_2(M)$,

$$c_2 \le \operatorname{area}\big(\underline{\phi}(D_i)\cap B_1(x)\big) \le \sum_{\psi\in A}\operatorname{area}\big(\underline{\phi}(D_i)\cap\psi(\Delta)\big),$$

and so, using (19),

$$\begin{aligned} c_2\#\underline{\Gamma}^{S_i} &\le \sum_{\psi\in A}\sum_{\underline{\phi}\in\underline{\Gamma}^{S_i}}\operatorname{area}\big(\underline{\phi}(D_i)\cap\psi(\Delta)\big)\\ &\le \sum_{\psi\in A}\sum_{\underline{\phi}\in\Gamma\backslash G_i}\operatorname{area}\big(\underline{\phi}(D_i)\cap\psi(\Delta)\big)\\ &= \#A\operatorname{area}(S_i). \end{aligned}$$

Applying Proposition 4.1 to any sequence $\underline{\phi}_i(D_i)$ with $\underline{\phi}_i \in \underline{\Gamma}^{S_i}$, we obtain the existence of a constant $c_3 = c_3(M)$ so that for all i sufficiently large, we have

$$\operatorname{area}\big(\underline{\phi}(D_i)\cap\Delta\big) \le c_3 \quad \text{for all } \underline{\phi}\in\underline{\Gamma}^{S_i}. \tag{20}$$

Thus,

$$\operatorname{area}(S_i) = \sum_{\underline{\phi}\in\Gamma\backslash G_i}\operatorname{area}\big(\underline{\phi}(D_i)\cap\Delta\big) = \sum_{\underline{\phi}\in\underline{\Gamma}^{S_i}}\operatorname{area}\big(\underline{\phi}(D_i)\cap\Delta\big) \le c_3\#\underline{\Gamma}^{S_i},$$

and hence for all i sufficiently large,

$$\frac{1}{c_3}\operatorname{area} S_i \le \#\underline{\Gamma}^{S_i} \le \frac{\#A}{c_2}\operatorname{area} S_i.$$

This proves (b).

Let $f \in C^0(M)$ be a function with $0 \le f \le 1$ and support contained in K. Using (20) we have that for all i sufficiently large,

$$\int_{S_i} f\,dA = \sum_{\underline{\phi}\in\Gamma\backslash G_i}\int_{\underline{\phi}(D_i)\cap\Delta} f\,dA = \sum_{\underline{\phi}\in\underline{\Gamma}^{S_i}(K)}\int_{\underline{\phi}(D_i)\cap\Delta} f\,dA \le c_3\#\underline{\Gamma}^{S_i}(K),$$

which means that

$$c_3\frac{\#\underline{\Gamma}^{S_i}(K)}{\#\underline{\Gamma}^{S_i}} \ge \frac{1}{\operatorname{area}(S_i)}\int_{S_i} f\,dA,$$

and this proves (c). □

In light of Proposition 6.4(a) we can construct, for all i sufficiently large, a map $P_i : \Gamma^{S_i} \to \Gamma^{\Sigma_i}$ so that

(i) $d(P_i(\phi), \phi) \le n$ for all $\phi \in \Gamma^{S_i}$;

(ii) $P_i(\phi g) = P_i(\phi) g$ for all $\phi \in \Gamma^{S_i}$, $g \in G_i$.

Set $\Gamma^{S_i}(\varepsilon, R) = P_i^{-1}(\Gamma^{\Sigma_i}(\varepsilon, R))$. Because the map P_i is G_i-invariant, we have that $\Gamma^{S_i}(\varepsilon, R)$ is also G_i-invariant, and P_i descends to map $\underline{P}_i : \underline{\Gamma}^{S_i} \to \underline{\Gamma}^{\Sigma_i}$.

PROPOSITION 6.5

For all $\varepsilon > 0$, $R > 0$,

$$\liminf_{i \to \infty} \frac{\#\underline{\Gamma}^{S_i}(\varepsilon, R)}{\#\underline{\Gamma}^{S_i}} = 1.$$

Proof

Due to the fact that both S_i and Σ_i minimize area in their homotopy class, there is a constant $c_1 = c_1(h)$ so that

$$c_1^{-1} \operatorname{area}_h(\Sigma_i) \le \operatorname{area}(S_i) \le c_1 \operatorname{area}_h(\Sigma_i)$$

for all $i \in \mathbb{N}$, and so we deduce from Proposition 6.4(b) the existence of $c_2 = c_2(h, M)$ so that, for all i sufficiently large,

$$c_2^{-1} \operatorname{area}_h(\Sigma_i) \le \#\underline{\Gamma}^{S_i} \le c_2 \operatorname{area}_h(\Sigma_i). \tag{21}$$

Set $L^{\Sigma_i}(\varepsilon, R) := \Gamma^{\Sigma_i} - \Gamma^{\Sigma_i}(\varepsilon, R)$, $i \in \mathbb{N}$, and denote its projection to $\Gamma \setminus G_i$ by $\underline{L}^{\Sigma_i}(\varepsilon, R)$. From Lemma 2.2 there is $n_R = n_R(R, M)$ so that

$$B_R(p) \subset \bigcup_{|\psi| \le n_R} \phi(\Delta),$$

and set $c_3 = \#\{\psi \in \Gamma : |\psi| \le n_R\}$. Then, recalling that

$$\Delta_i = \bigcup_{\underline{\phi} \in \Gamma \setminus G_i} \underline{\phi}^{-1}\big(\psi(\Delta)\big)$$

is a fundamental domain for $\mathbb{H}^3 \setminus G_i$ for all $\psi \in \Gamma$, we have

$$\begin{aligned} c_3 \int_{\Sigma_i} |f_i| \, dA_h &= \sum_{|\psi| \le n_R} \sum_{\underline{\phi} \in \Gamma \setminus G_i} \int_{\underline{\phi}(\Omega_i) \cap \psi(\Delta)} |f_i| \, dA_h \\ &= \sum_{\underline{\phi} \in \Gamma \setminus G_i} \sum_{|\psi| \le n_R} \int_{\underline{\phi}(\Omega_i) \cap \psi(\Delta)} |f_i| \, dA_h \end{aligned}$$

$$\geq \sum_{\underline{\phi}\in\Gamma\backslash G_i} \int_{\underline{\phi}(\Omega_i)\cap B_R(p)} |f_i| dA_h$$

$$\geq \sum_{\underline{\phi}\in \underline{L}^{\Sigma_i}(\varepsilon,R)} \int_{\underline{\phi}(\Omega_i)\cap B_R(p)} |f_i| dA_h$$

$$\geq \varepsilon \# \underline{L}^{\Sigma_i}(\varepsilon, R).$$

Hence,

$$\frac{\#\underline{L}^{\Sigma_i}(\varepsilon, R)}{\text{area}_h(\Sigma_i)} \leq \frac{c_3}{\varepsilon\, \text{area}_h(\Sigma_i)} \int_{\Sigma_i} |f_i| dA_h,$$

and we deduce from (17) and (21) that

$$\liminf_{i\to\infty} \frac{\#\underline{L}^{\Sigma_i}(\varepsilon, R)}{\#\underline{\Gamma}^{S_i}} = 0.$$

Set, for all i sufficiently large, $L^{S_i}(\varepsilon, R) = P_i^{-1}(L^{\Sigma_i}(\varepsilon, R))$, which has its projection to $\Gamma \setminus G_i$ satisfying $\underline{L}^{S_i}(\varepsilon, R) = \underline{P}_i^{-1}(\underline{L}^{\Sigma_i}(\varepsilon, R))$.

Define $c_4 = \#\{\phi \in \Gamma : |\phi| \leq n\}$, where n is the constant in Proposition 6.4(a). From property (i) of the map P_i we have that $\#P_i^{-1}(\psi) \leq c_4$ for all $\psi \in \Gamma^{\Sigma_i}$. Hence from property (ii) we deduce that $\#\underline{L}^{S_i}(\varepsilon, R) \leq c_4 \#\underline{L}^{\Sigma_i}(\varepsilon, R)$, and we obtain

$$\liminf_{i\to\infty} \frac{\#\underline{L}^{S_i}(\varepsilon, R)}{\#\underline{\Gamma}^{S_i}} = 0.$$

The desired result follows because the reader can check that $\underline{\Gamma}^{S_i}(\varepsilon, R) = \underline{\Gamma}^{S_i} - \underline{L}^{S_i}(\varepsilon, R)$. □

This proposition allows us to choose $\varepsilon_i \to 0$ and $R_i \to \infty$ as $i \to 0$ so that

$$\liminf_{i\to\infty} \frac{\#\underline{\Gamma}^{S_i}(\varepsilon_i, R_i)}{\#\underline{\Gamma}^{S_i}} = 1. \tag{22}$$

LEMMA 6.6
There is a constant $c = c(M, h)$ so that for every compact set K contained in Δ we can find $\{\phi_i\}_{i\in\mathbb{N}} \subset \Gamma$ so that for all i sufficiently large, $\phi_i \in \Gamma^{S_i}(K) \cap F_i(\varepsilon_i, R_i - c)$.

Proof
From Proposition 6.4(c) and (22) we can choose $\{\phi_i\}_{i\in\mathbb{N}} \subset \Gamma$ so that for all i sufficiently large,

$$\phi_i \in \Gamma^{S_i}(\varepsilon_i, R_i) \cap \Gamma^{S_i}(K).$$

Thus, from the definition of P_i there is $g_i \in \Gamma$ with $|g_i| \le n$ so that $g_i\phi_i \in F_i(\varepsilon_i, R_i)$. Set $c = \max\{d(p,\phi(p)) : |\phi| \le n\}$. Then from (18) we have that $\phi_i \in F_i(\varepsilon_i, R_i - c)$ for all i sufficiently large. □

Suppose that $\mathcal{L}$ has no element with a dense Γ-orbit in $\mathcal{C}$. Then Theorem 2.6 implies that every point in $\mathcal{L}$ is isolated, and so the set

$$\{\gamma \in \mathcal{L} : C(\gamma) \cap \Delta \neq \emptyset\}$$

is finite. Thus, because every $\gamma \in \mathcal{L}$ has $C(\gamma)$ projecting to a closed surface in M, we can choose a compact set $K \subset \Delta$ so that $C(\gamma) \cap K = \emptyset$ for all $\gamma \in \mathcal{L}$. On the other hand, applying Theorem 4.1 to $\phi_i(D_i)$, where the sequence $\{\phi_i\}_{i\in\mathbb{N}} \subset \Gamma$ is the one given by Lemma 6.6, we obtain $\gamma \in \mathcal{L}$ for which $C(\gamma) \cap K \neq \emptyset$, which is a contradiction. □

7. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. Given a closed Riemannian manifold (N, g), denote by $\widehat{B}_R(p)$ and $\hat{d}$, respectively, the geodesic balls and distance function induced by g in the universal cover $\hat{N}$ of N. The following limit exists (as first observed in [22]) and defines the volume entropy of (N, g):

$$E_{\mathrm{vol}}(g) = \lim_{R\to\infty} \frac{\ln \mathrm{vol}(\widehat{B}_R(x))}{R}.$$

Let $\Delta \subset \hat{N}$ be a Dirichlet domain of N containing $x \in \hat{N}$ with diameter d. We have

$$B_{R-d}(x) \subset \bigcup_{\{\gamma\in\pi_1(N):\hat{d}(x,\gamma(x))\le R\}} \gamma(\Delta) \subset B_{R+d}(x),$$

and this implies that

$$E_{\mathrm{vol}}(g) = \lim_{R\to\infty} \frac{\ln \mathrm{vol}(\widehat{B}_R(x))}{R} = \lim_{R\to\infty} \frac{\ln \#\{\gamma \in \pi_1(N) : \hat{d}(x,\gamma(x)) \le R\}}{R}.$$

Proof that $E(h) \le 2E_{\mathrm{vol}}(h)^2$
Suppose we have an essential immersion $\Sigma \subset M$ which lifts to a disk Ω in the universal cover $\hat{M}$ of M. In this case $\pi_1(\Sigma)$ acts naturally by isometries in $\hat{M}$, and if $\hat{d}_\Omega$

denotes the intrinsic distance in Ω, then we have $\hat{d}(x,y) \leq \hat{d}_\Omega(x,y)$ for all $x, y \in \Omega$. Thus,

$$\begin{aligned}\#\{\gamma \in \pi_1(\Sigma) : \hat{d}_\Omega(x,\gamma(x)) \leq R\} &\leq \#\{\gamma \in \pi_1(\Sigma) : \hat{d}(x,\gamma(x)) \leq R\} \\ &\leq \#\{\gamma \in \pi_1(M) : \hat{d}(x,\gamma(x)) \leq R\}.\end{aligned}$$

Hence $E_{\mathrm{vol}}(h_\Sigma) \leq E_{\mathrm{vol}}(h)$. From [5] we have $E_{\mathrm{vol}}(h_\Sigma)^2 \operatorname{area}_h(\Sigma) \geq 4\pi(g-1)$, where g is the genus of Σ, and so by minimizing area in the homotopy class Π of Σ we deduce that

$$\operatorname{area}_h(\Pi) \geq E_{\mathrm{vol}}(h)^{-2} 4\pi(g-1).$$

Thus, denoting by $\lfloor x \rfloor$ the integer part of x,

$$\operatorname{area}_h(\Pi) \leq 4\pi(L-1) \implies \Pi \in S(M, \lfloor E_{\mathrm{vol}}(h)^2 L \rfloor),$$

and so, for all $\varepsilon > 0$ and all L sufficiently large, we have from Theorem 4.2 that

$$\begin{aligned}\ln \#\{\operatorname{area}_h(\Pi) \leq 4\pi(L-1) : \Pi \in S_\varepsilon(M)\} &\leq \ln s(M, \lfloor E_{\mathrm{vol}}(h)^2 L \rfloor, \varepsilon) \\ &\leq 2E_{\mathrm{vol}}(h)^2 L \ln(c_2 E_{\mathrm{vol}}(h)^2 L),\end{aligned}$$

which implies that $E(h) \leq 2E_{\mathrm{vol}}(h)^2$.

Proof that $E(\bar{h}) = 2$

Given $\Pi \in S_\varepsilon(M)$, consider the essential minimal surface $S \in \Pi$ so that $\operatorname{area}(S) = \operatorname{area}(\Pi)$. From Theorem 4.1 we have $|A|^2_{L^\infty(S)} = o_\varepsilon(1)$, meaning that if ε is very small, then the quantity on the left-hand side will also be small. Let g be the genus of S. The integrated form of Gauss's equation (14) gives

$$\operatorname{area}(S) = 4\pi(g-1) + o_\varepsilon(1)\operatorname{area}(S),$$

and so for all ε uniformly small we have

$$\operatorname{area}(S) = 4\pi(g-1)\bigl(1 + o_\varepsilon(1)\bigr). \tag{23}$$

One immediate consequence is that, given $\delta > 0$, for all ε small and all L large (depending on δ but independently of Π), we have both

$$\begin{aligned}\operatorname{area}(\Pi) \leq 4\pi(L-1) \quad \text{and} \quad \Pi \in S_\varepsilon(M) &\implies \Pi \in S(M, \lfloor (1+\delta)L \rfloor, \varepsilon), \\ \Pi \in S(M, \lfloor (1-\delta)L \rfloor, \varepsilon) &\implies \operatorname{area}(\Pi) \leq 4\pi(L-1),\end{aligned}$$

and so, recalling the notation set in Section 4,

$$\begin{aligned}\ln s\big(M, \lfloor (1-\delta)L \rfloor, \varepsilon\big) &\le \ln \#\big\{\text{area}(\Pi) \le 4\pi(L-1) : \Pi \in S_\varepsilon(M)\big\}\\ &\le \ln s\big(M, \lfloor (1+\delta)L \rfloor, \varepsilon\big).\end{aligned}$$

Combining with Theorem 4.2 we deduce that for all ε small,

$$2(1-\delta) \le \liminf_{L\to\infty} \frac{\ln \#\{\text{area}(\Pi) \le 4\pi(L-1) : \Pi \in S_\varepsilon(M)\}}{L \ln L} \le 2(1+\delta).$$

The arbitrariness of δ shows that $E(\bar{h}) = 2$.

Proof that $E(h) \ge E(\bar{h})$

Suppose now that the sectional curvature of h is less than or equal to -1. From the integrated form of Gauss's equation (14), we have that every genus g minimal surface has $\text{area}_h(\Sigma) \le 4\pi(g-1)$. Thus $\Pi \in S(M, \lfloor L \rfloor, \varepsilon)$ implies that $\text{area}_h(\Pi) \le 4\pi(L-1)$. Hence,

$$\#\big\{\text{area}_h(\Pi) \le 4\pi(L-1) : \Pi \in S_\varepsilon(M)\big\} \ge s\big(M, \lfloor L \rfloor, \varepsilon\big),$$

and so Theorem 4.2 implies that $E(h) \ge 2 = E(\bar{h})$.

Proof that $E(h) = E(\bar{h}) \implies h = \bar{h}$

Suppose now that $E(h) = E(\bar{h}) = 2$. Consider the set $G(M, g, \varepsilon) \subset S(M, g, \varepsilon)$ given by Theorem 4.2.

CLAIM 7.1

For all $\delta > 0$*, there is* $j \in \mathbb{N}$ *so that for all* $i \ge j$ *we can find* $g \in \mathbb{N}$ *and* $\Pi \in G(M, g, 1/i)$ *so that*

$$\text{area}_h(\Pi) > 4\pi\big((1+\delta)^{-1} g - 1\big).$$

Suppose not. In that case there is an increasing sequence of integers $\{i_j\}_{j\in\mathbb{N}}$ so that for all $g \in \mathbb{N}$ and $\Pi \in G(M, g, i_j^{-1})$, we have

$$\text{area}_h(\Pi) \le 4\pi\big((1+\delta)^{-1} g - 1\big),$$

and hence, for all $L \ge 0$,

$$\Pi \in G\big(M, \lfloor (1+\delta)L \rfloor, i_j^{-1}\big) \implies \text{area}_h(\Pi) \le 4\pi(L-1).$$

Thus, for all $j \in \mathbb{N}$,

$$\liminf_{L\to\infty} \frac{\ln \#\{\text{area}_h(\Pi) \le 4\pi(L-1) : \Pi \in S_{i_j^{-1}}(M)\}}{L \ln L}$$

$$\geq \liminf_{L\to\infty} \frac{\ln \# G(M, \lfloor (1+\delta)L\rfloor, i_j^{-1})}{L \ln L}$$
$$\geq 2(1+\delta),$$

which contradicts $E(h) = 2$.

Therefore, we can find an increasing sequence of integers $\{j_i\}_{i\in\mathbb{N}}$ and a sequence $\Pi_i \in G(M, g_i, j_i^{-1})$, $i \in \mathbb{N}$, so that

$$\text{area}_h(\Pi_i) \geq 4\pi\big((1 - 1/i)g_i - 1\big) \quad \text{for all } i \in \mathbb{N}. \tag{24}$$

Denote by S_i, Σ_i the minimal surfaces that minimize area in the homotopy class Π_i with respect to h and $\bar{h}$, respectively. We have $\text{area}(S_i) \leq 4\pi(g_i - 1)$, and so we deduce from (24) that

$$\liminf_{i\to\infty} \frac{\text{area}_h(\Sigma_i)}{\text{area}(S_i)} \geq \liminf_{i\to\infty} \frac{4\pi((1-1/i)g_i - 1)}{4\pi(g_i - 1)} = 1. \tag{25}$$

Thus Theorem 5.1 implies that h is hyperbolic, and so $h = \bar{h}$ from the Mostow rigidity theorem.

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