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Research Article

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On the equivalence between weak BMO and the space of derivatives of the Zygmund class

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Abstract: In this paper, we will discuss the space of functions of weak bounded mean oscillation. In particular, we will show that this space is the dual space of the special atom space, whose dual space was already known to be the space of derivative of functions (in the sense of distribution) belonging to the Zygmund class of functions. We show, in particular, that this proves that the Hardy space H^1 strictly contains the special atom space.

Keywords: BMO, derivative, distributions, Zygmund class

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1 Introduction

The space of functions of bounded means oscillation has taken central stage in the mathematical literature after the work of Charles Fefferman [1], where he showed that it is the dual space of the real Hardy space $\mathbb{H}^1$, a long sought-after result. Right after, Ronald Coifman [2] showed this result using a different method. Essentially, he showed that $\mathbb{H}^1$ has an atomic decomposition. De Souza [3] showed there is a subset B^1 of $\mathbb{H}^1$, formed by special atoms that is contained in $\mathbb{H}^1$. This space B^1 has the particularity that it contains some functions whose Fourier series diverge. The question of whether B^1 is equivalent to $\mathbb{H}^1$ was never truly answered explicitly, but it was always suspected that the inclusion $B^1 \subset \mathbb{H}^1$ was strict, that is, there must be at least one function in $\mathbb{H}^1$ that is not in B^1. However, such a function had neither been constructed nor given. Since the dual space $(\mathbb{H}^1)^*$ of $\mathbb{H}^1$ is BMO and $B^1 \subset \mathbb{H}^1$, it follows that the dual space $(B^1)^*$ of B^1 must be a superset of BMO. A natural superset candidate of BMO is therefore the space BMO^w since $\mathrm{BMO} \subset \mathrm{BMO}^w$. So in essence, that BMO^w is the dual of B^1 would also prove that $B^1 \subset \mathbb{H}^1$ with a strict inclusion. Moreover, it was already proved that $(B^1)^* \cong \Lambda_*'$, where Λ_*' is the space of derivative (in the sense of distributions) of functions in the Zygmund class Λ_*, see, for example, [3] and [4]. Therefore, if $(B^1)^* \cong \mathrm{BMO}^w$, then by transition, we would have $\Lambda_*' \cong \mathrm{BMO}^w$.

Henceforth, we will adopt the following notations: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk and let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit sphere. For an integrable function f on a measurable set A, and the Lebesgue measure λ on A, we will write $⨍_A f(\xi)\,d\lambda(\xi) := \frac{1}{\lambda(A)}\int_A f(\xi)\,d\lambda(\xi)$. We will start by recalling the necessary definitions and important results. The interested reader can see, for example, [5] for more information.

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Definition 1. Let $0 < p < \infty$ be a real number. The Hardy Space $\mathbb{H}^p := \mathbb{H}^p(\mathbb{D})$ is the space of holomorphic functions f defined on $\mathbb{D}$ and satisfying

$$\|f\|_{\mathbb{H}^p} := \sup_{0\le r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\xi})|^p\,d\lambda(\xi)\right)^{\frac{1}{p}} < \infty.$$

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, Q be a hypercube in $\mathbb{R}^n$, and λ be the Lebesgue measure on $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

Put

$$f_Q^{\#} = \fint_Q f(\xi)\,d\lambda(\xi), \quad f_Q = \left|\fint_Q f(\xi)\,d\lambda(\xi)\right|.$$

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we define

$$M^{\#}(f)(x) = \sup_{Q\ni x}\fint_Q |f(\xi) - f_Q^{\#}|\,d\lambda(\xi), \tag{1.1}$$

$$M(f)(x) = \sup_{Q\ni x}\left|\fint_Q [f(\xi) - f_Q]\,d\lambda(\xi)\right|, \tag{1.2}$$

$$mf(x) = \sup_{Q\ni x} f_Q, \tag{1.3}$$

where the supremum is taken over all hypercubes Q containing x. Now, we can define the space of functions of bounded mean oscillation and its weak counterpart.

Definition 2. The space of functions of bounded mean oscillation is defined as the space of locally integrable functions f for which the operator $M^{\#}$ is bounded, that is,

$$\mathrm{BMO}(\mathbb{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : M^{\#}(f) \in L^\infty(\mathbb{R}^n)\}.$$

We can endowed $\mathrm{BMO}(\mathbb{R}^n)$ with the norm

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} = \|M^{\#}(f)\|_\infty := \sup_{x\in\mathbb{R}^n} M^{\#}(f)(x).$$

The space of functions of weak bounded mean oscillation is defined as the space of locally integrable functions f for which the operator M is bounded, that is,

$$\mathrm{BMO}^w(\mathbb{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : Mf \in L^\infty(\mathbb{R}^n)\}.$$

Remark 1.1. It follows from the above definitions that $\mathrm{BMO}(\mathbb{R}^n) \subseteq \mathrm{BMO}^w(\mathbb{R}^n)$.

Let us recall the definition of the space of functions of vanishing mean oscillation $\mathrm{VMO}(\mathbb{R}^n)$ and introduce the space of functions of weak vanishing mean oscillations $\mathrm{VMO}^w(\mathbb{R}^n)$.

Definition 3. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

$$f \in \mathrm{VMO}(R^n) \quad \text{if} \quad \lim_{\lambda(Q)\to 0}\fint_Q |f(\xi) - f_Q|\,d\lambda(\xi) \quad = 0.$$

$$f \in \mathrm{VMO}^w(\mathbb{R}^n) \quad \text{if} \quad \lim_{\lambda(Q)\to 0}\left|\fint_Q [f(\xi) - f_Q]\,d\lambda(\xi)\right| = 0.$$

Remark 1.2. It follows from the above definition that $\mathrm{VMO}(\mathbb{R}^n)$ is a subspace of $\mathrm{VMO}^w(\mathbb{R}^n)$ which is itself a subspace of $\mathrm{BMO}^w(\mathbb{R}^n)$. We will show (see Theorem 2.8 below) that VMO^w is in fact closed subspace of BMO^w.

Henceforth, $\mathrm{BMO}(\mathbb{R}^n)$, $\mathrm{BMO}^w(\mathbb{R}^n)$, and $\mathrm{VMO}^w(\mathbb{R}^n)$ will simply be referred to as BMO, BMO^w, and VMO^w.

Now, we consider $A(\mathbb{D})$ as the space of analytic functions defined on the unit disk $\mathbb{D}$. Following the work of Girela in [6] on the space of analytic functions of bounded means oscillations, we introduce their weak counterpart. Before, we recall that the Poisson Kernel is defined as

$$P_r(\theta) = \operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right).$$

Definition 4. The space of analytic functions of bounded mean oscillation is defined as

$$\mathrm{BMOA}(\mathbb{D}) = \left\{F \in A(\mathbb{D});\ \exists f \in \mathrm{BMO}(\mathbb{T}) : F(re^{i\theta}) = \frac{1}{2\pi}\int_{\mathbb{T}} P_r(\theta-\xi)f(e^{i\xi})\,d\lambda(\xi)\right\}.$$

We endow $\mathrm{BMO}(\mathbb{D})$ with the norm

$$\|F\|_{\mathrm{BMO}A(\mathbb{D})} := |F(0)| + \sup_{\substack{0\le r<1\\ \theta\in\mathbb{T}}}\left(\frac{1}{2\pi}\int_{\mathbb{T}} P_r(\theta-\xi)|f(e^{i\xi}) - F(re^{i\theta})|\,d\lambda(\xi)\right) < \infty.$$

In other words, $\mathrm{BMOA}(\mathbb{D})$ is the space of Poisson integrals of functions in $\mathrm{BMO}(\mathbb{T})$.

We can now define the space BMOA^w of analytic function of weak bounded mean oscillation.

Definition 5. An analytic function F on $\mathbb{D}$ is said to be of weak bounded mean oscillation if there exists $f \in \mathrm{BMO}^w(\mathbb{T})$ such that

$$F(re^{i\theta}) = \frac{1}{2\pi}\int_{\mathbb{T}} P_r(\theta-\xi)f(e^{i\xi})\,d\lambda(\xi).$$

We endow $\mathrm{BMOA}^w(\mathbb{D})$ with the norm

$$\|F\|_{\mathrm{BMO}A^w(\mathbb{D})} := |F(0)| + \sup_{\substack{0\le r<1\\ \theta\in\mathbb{T}}}\left(\frac{1}{2\pi}\left|\int_{\mathbb{T}} P_r(\theta-\xi)[f(e^{i\xi}) - F(re^{i\theta})]\,d\lambda(\xi)\right|\right) < \infty.$$

We recall the definition of special atom space B^1, see [7].

Definition 6. For $n \ge 1$, we consider the hypercube of $\mathbb{R}^n$ given as $J = \prod_{j=1}^n [a_j - h_j, a_j + h_j]$ where a_j, h_j are real numbers with $h_j > 0$. Let $\phi \in L^1(J)$ with $\phi(J) = \int_J \phi(\xi)\,d\lambda(\xi)$.

The special atom (of **type 1**) is a function $b : I \subseteq J \to \mathbb{R}$ such that

$$b(\xi) = 1 \text{ on } J\backslash I \text{ or}$$
$$b(\xi) = \frac{1}{\phi(I)}[\chi_R(\xi) - \chi_L(\xi)],$$

where

- $R = \bigcup_{j=1}^{2^{n-1}} I_{i_j}$ for some $i_1, i_2, \dots, i_{2^{n-1}} \in \{1, 2, \dots, 2^n\}$ with $i_1 < i_2 < \cdots < i_{2^{n-1}}$ and $L = I\backslash R$.
- $\{I_1, I_2, \dots, I_{2^n}\}$ is the collection of sub-cubes of I, cut by the hyperplanes $x_1 = a_1, x_2 = a_2, \dots, x_n = a_n$.
- χ_A represents the characteristic function of set A.

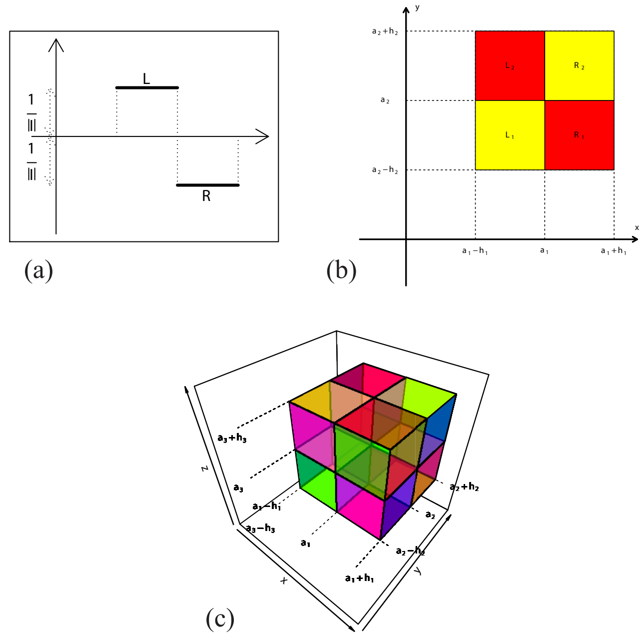

Figure 1: Illustration of the special atom, for $n = 1$ in (a), $n = 2$ in (b), and $n = 3$ in (c).

Definition 7. The special atom space is defined as

$$B^1 = \left\{ f : J \to \mathbb{R};\, f(\xi) = \sum_{n=0}^{\infty} \alpha_n b_n(\xi);\ \sum_{n=0}^{\infty} |\alpha_n| < \infty \right\},$$

where the b_n's are special atoms of **type 1**.

B^1 is endowed with the norm $\|f\|_{B^1} = \inf \sum_{n=0}^{\infty} |\alpha_n|$, where the infimum is taken over all representations of f (Figure 1).

Now, we define the Zygmund class of functions.

Definition 8. Let $k \in \mathbb{N}$. A function f is said to be in the Zygmund class $\Lambda_*^k(\mathbb{R}^n)$ of functions of order k if $f \in C^{k-1}(\mathbb{R}^n)$ and

$$\|f\|_{\Lambda_*^k(\mathbb{R}^n)} = \sum_{|\alpha|=k} \sup_{x,h} \frac{|\partial^\alpha f(x+h) + \partial^\alpha f(x-h) - 2\partial^\alpha f(x)|}{|h|} < \infty.$$

In particular, for $k = 1$, we have $\Lambda_* := \Lambda_*^1(\mathbb{R}^n)$, and hence

$$\Lambda_* = \left\{ f \in C^0(\mathbb{R}^n) : \|f\|_{\Lambda_*} := \sup_{x,h>0} \frac{|f(x+h) + f(x-h) - 2f(x)|}{2h} < \infty \right\}.$$

One important note about the space Λ_* is that it contains the so-called Weierstrass functions that are known to be continuous everywhere but nowhere differentiable. Therefore, the space Λ_*^k is the space of derivatives of functions in Λ_*^{k-1}, where the derivative is taken in the sense of distributions. Another equivalent way to see Λ_*^k is to consider functions of Λ_*^{k-1} that are either differentiable or limits of convolutions with the Poisson kernel, that is, $f(\xi) = \lim_{r\to 1} (f * P_r)(\xi)$ where $P_r(\theta)$ is the Poisson kernel.

2 Main results

Our first result is about the constant f_Q in the definition of BMO^w. If fact, the constant f_Q can be replaced with any non-negative constant. The same can be said about BMO as well.

Proposition 2.1. *Let $f \in L^1_{\mathrm{loc}}$.*

(1) *For any non-negative real number α, we have*

$$\sup_{Q\ni x}\left|\fint_Q [f(\xi)-f_Q]\,d\lambda(\xi)\right| \le 2\sup_{Q\ni x}\left|\fint_Q [f(\xi)-\alpha]\,d\lambda(\xi)\right|. \tag{2.1}$$

(2) *$f \in \mathrm{BMO}^w$ if and only if for any $x \in \mathbb{R}^n$ and any cube $Q \ni x$, there exists a non-negative number $\alpha \in Q$ such that*

$$\sup_{Q\ni x}\left|\fint_Q [f(\xi)-\alpha]\,d\lambda(\xi)\right| < \infty. \tag{2.2}$$

Proof. To prove assertion (1), fix $x \in \mathbb{R}^n$ and a cube $Q \subseteq \mathbb{R}^n$ containing x. Observe that for every non-negative α we have,

$$\left|\fint_Q [f(\xi)-f_Q]\,d\lambda(\xi)\right| \le 2\left|\fint_Q [f(\xi)-\alpha]\,d\lambda(\xi)\right|.$$

Indeed, for any non-negative real number α, we have

$$\begin{aligned}
\left|\fint_Q [f(\xi)-f_Q]\,d\lambda(\xi)\right| &= \left|\fint_Q [f(\xi)-\alpha]\,d\lambda(\xi) + \fint_Q [\alpha-f_Q]\,d\lambda(\xi)\right| \\
&\le \left|\fint_Q [f(\xi)-\alpha]\,d\lambda(\xi)\right| + \left|\fint_Q [\alpha-f_Q]\,d\lambda(\xi)\right| \\
&\le \left|\fint_Q [f(\xi)-\alpha]\,d\lambda(\xi)\right| + \left|\left|\fint_Q \alpha\,d\lambda(\xi)\right| - \left|\fint_Q f(\xi)\,d\lambda(\xi)\right|\right| \\
&\le \left|\fint_Q [f(\xi)-\alpha]\,d\lambda(\xi)\right| + \left|\fint_Q [\alpha-f(\xi)]\,d\lambda(\xi)\right| \le 2\left|\fint_Q [f(\xi)-\alpha]\,d\lambda(\xi)\right|.
\end{aligned}$$

To conclude, we take the supremum over of all cubes $Q \ni x$.

Assertion (2) follows immediately from (1). □

We can define two equivalent norms on BMO^w and prove that endowed with these norms, BMO^w is in fact a Banach space.

Proposition 2.2. *Consider the following: for every $f \in \mathrm{BMO}^w$, we put*

$$\|f\|_{\mathrm{BMO}^w} = \|mf\|_\infty + \|Mf\|_\infty \quad \text{and} \quad \|f\|^*_{\mathrm{BMO}^w} = \|mf\|_\infty + 2\sup_{Q\ni x}\inf_{\alpha>0}\left|\fint_Q [f(\xi)-f_Q]\,d\lambda(\xi)\right|.$$

Then,

$$\|f\|_{\mathrm{BMO}^w} \leq \|f\|^*_{\mathrm{BMO}^w} \leq 2\,\|f\|_{\mathrm{BMO}^w}.$$

Proof. The proof is an immediate consequence of Proposition 2.1. We note that the proof can also be obtained from the closed-graph theorem, but that will require to first prove that endowed with the two norms, BMO^w is a Banach space. □

Theorem 2.3. *The space* $(\mathrm{BMO}^w, \|\cdot\|_{\mathrm{BMO}^w})$ *is complete.*

Proof.

(1) In the proof that $\|\cdot\|_{\mathrm{BMO}^w}$ is a norm, homogeneity and the triangle inequality are easy to prove. As for positivity, we note that $\|f\|_{\mathrm{BMO}^w} = 0 \Leftrightarrow \sup\limits_{Q \ni x} f_Q = 0$ and $f(\xi) = f_Q$ on all cubes $Q \ni x$. It follows immediately that $f = 0$.

(2) Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in BMO^w. Let $\epsilon > 0$ and $N \in \mathbb{N}$ such that $\forall n, m \in N$, we have $\|f_n - f_m\|_{\mathrm{BMO}^w} < \epsilon$. That is,

$$\sup_{Q\ni x}\left[(f_n - f_m)_Q + \left|\fint_Q [(f_n - f_m)(\xi) - (f_n - f_m)_Q]\,d\lambda(\xi)\right|\right] < \epsilon. \tag{2.3}$$

In particular, from (2.3), we have that $\sup\limits_{Q\ni x}(f_n - f_m)_Q < \epsilon$, therefore,

$$|f_{n,Q} - f_{m,Q}| = \left|\left|\fint_Q f_n(\xi)\,d\lambda(\xi)\right| - \left|\fint_Q f_m(\xi)\,d\lambda(\xi)\right|\right| \leq \left|\fint_Q [f_n - f_m](\xi)\,d\lambda(\xi)\right| = (f_n - f_m)_Q \leq \sup_{Q\ni x}(f_n - f_m)_Q < \epsilon.$$

Hence, for fixed Q, $\{f_{n,Q}\}$ is Cauchy sequence in $\mathbb{R}$. Let $f_Q = \lim\limits_{n\to\infty} f_{n,Q}$. We note from the above inequalities that

$$(f_n - f_m)_Q = \left|\fint_Q [f_n - f_m](\xi)\,d\lambda(\xi)\right| \geq |f_{n,Q} - f_{m,Q}| \geq f_{n,Q} - f_{m,Q}.$$

Therefore, given a cube $Q \subset \mathbb{R}^n$ containing x, we have

$$\begin{aligned}
|Mf_n(x) - Mf_m(x)| &\leq \sup_{Q\ni x}\left|\fint_Q [(f_n(\xi) - f_m(\xi)) - (f_{n,Q} - f_{m,Q})]\,d\lambda(\xi)\right| \\
&\leq \sup_{Q\ni x}\left|\fint_Q [(f_n(\xi) - f_m(\xi)) - (f_n - f_m)_Q]\,d\lambda(\xi)\right| + \sup_{Q\ni x}|(f_n - f_m)_Q - (f_{n,Q} - f_{m,Q})| \\
&\leq \|f_n - f_m\|_{\mathrm{BMO}^w} + \sup_{Q\ni x}|(f_{n,Q} - f_{m,Q})| < 2\epsilon.
\end{aligned}$$

It follows that $\{Mf_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in L^∞_{loc}. Let $h = \lim\limits_{n\to\infty} Mf_n$.

Since $L^\infty_{\mathrm{loc}} \subset L^1_{\mathrm{loc}}$, we have $h \in L^1_{\mathrm{loc}}$.

$$h(x) = \lim_{n\to\infty} Mf_n(x) = \lim_{n\to\infty}\sup_{Q\ni x}\left|\fint_Q [f_n(\xi) - f_{n,Q}]\,d\lambda(\xi)\right| = \sup_{Q\ni x}\left|\fint_Q [\lim_{n\to\infty} f_n(\xi) - f_Q]\,d\lambda(\xi)\right|.$$

Since $h(x)$ is finite on any cube $Q \ni x$, that follows that $f(\xi) = \lim\limits_{n\to\infty} f_n(\xi)$ is finite a.e. on Q. Thus, $h = Mf$, for some $f \in \mathrm{BMO}^w$ and $\lim\limits_{n\to\infty}\|f_n - f\|_{\mathrm{BMO}^w} \to 0$. □

Theorem 2.4. (Hölder's type inequality) *Let* $g \in \mathrm{BMO}^w$ *and a hyper-cube* $J \subset \mathbb{R}^n$. *Consider the following operator* $T_g : B^1 \to \mathbb{R}$ *given by* $T_g(f) = \int_J f(\xi)g(\xi)\,d\lambda(\xi)$. *Then,* $T_g \in (B^1)^*$ *with* $\|T_g\|_{(B^1)^*} \le \|g\|_{\mathrm{BMO}^w}$.

Moreover, the operator $H : \mathrm{BMO}^w \to (B^1)^*$ *defined as* $H(g) = T_g$ *is onto.*

Proof. By linearity of the integral, T_g is a linear. To start, we study the action of this operator on special atoms. Indeed, let $\xi \in I \subseteq J$ and suppose $f(\xi) = b(\xi) = \frac{1}{\phi(I)}[\chi_R(\xi) - \chi_L(\xi)]$, where R, L are sub-cubes of I such that $I = R \cup L$ and $R \cap L = \varnothing$. Therefore, we have:

$$T_g(b) = \int_J b(\xi)g(\xi)\,d\lambda(\xi) = \int_I b(\xi)g(\xi)\,d\lambda(\xi) = \int_I b(\xi)\,[g(\xi) - g_I]\,d\lambda(\xi), \quad \text{since} \quad \int_I b(\xi)g_I\,d\lambda(\xi) = 0.$$

Taking the supremum over all $I \ni x$ and using the fact that $b(\xi) \le \frac{1}{\phi(I)}$, we have

$$|T_g(b)| \le \sup_{\substack{I \ni x \\ I \subseteq J}} \inf_{\alpha > 0} \left| \fint_I [g(\xi) - \alpha]\,d\lambda(\xi) \right| \le \|g\|_{\mathrm{BMO}^w}. \tag{2.4}$$

Now suppose $f(\xi) = \sum_{n=1}^{\infty} c_n b_n(\xi)$ with $\sum_{n=1}^{\infty}|c_n| < \infty$ is an element of B^1, where the b_n's are special atoms defined on sub-cubes I_n of J with $I_n = R_n \cup L_n$ and $R_n \cap L_n = \varnothing$. Let $I = \bigcup_{n=1}^{\infty} I_n$. We have for $\alpha > 0$

$$\begin{aligned} T_g(f) &= \int_J \left(\sum_{n=1}^{\infty} c_n b_n(\xi) \right) g(\xi)\,d\lambda(\xi) = \int_I \left(\sum_{n=1}^{\infty} c_n b_n(\xi) \right) [g(\xi) - \alpha]\,d\lambda(\xi) \\ &= \sum_{n=1}^{\infty} c_n \int_I b_n(\xi)\,[g(\xi) - \alpha]\,d\lambda(\xi) = \sum_{n=1}^{\infty} c_n \int_{I_n} b_n(\xi)\,[g(\xi) - \alpha]\,d\lambda(\xi). \end{aligned}$$

It follows from (2.4) that

$$|T_g(f)| \le \left(\sum_{n=1}^{\infty} |c_n| \right) \cdot \|g\|_{\mathrm{BMO}^w}.$$

Talking the infimum over all representations of f yields as follows:

$$|T_g(f)| \le \|f\|_{B^1} \cdot \|g\|_{\mathrm{BMO}^w}.$$

Therefore, T_g is a bounded linear operator on B^1 with

$$\|T_g\|_{(B^1)^*} = \sup_{\|f\|_{B^1}=1} |T_g(f)| \le \|g\|_{\mathrm{BMO}^w}. \tag{2.5}$$

Now suppose that T is a bounded linear functional on B^1, that is, $T \in (B^1)^*$. We want to show that there exists a function $g \in \mathrm{BMO}^w$ such that $T(f) = T_g(f) = \int_J f(\xi)g(\xi)\,d\lambda(\xi)$. That $T \in (B^1)^*$ implies the existence of an absolute constant C such that

$$|T(f)| \le C\|f\|_{B^1}, \quad \forall f \in B^1. \tag{2.6}$$

Recall that a function $G : J \to \mathbb{R}$ is said to be absolutely continuous on J if for every positive number ϵ, there exists a positive number δ, such that for a finite sequence of pairwise disjoint sub-cubes $I_n = (x_n, y_n)$ of J,

$$\sum_{n=1}^{\infty} (y_n - x_n) < \delta \;\Rightarrow\; \sum_{n=1}^{\infty} |G(y_n) - G(x_n)| < \epsilon. \tag{2.7}$$

Suppose such a sequence of sub-cubes exists. Now, consider $G(x) = T(\chi_{[x-h, x+h)})$ for some real number $h > 0$. Then, given an cube I_n and a real number h_n, we have by linearity of T that

$$G(y_n) - G(x_n) = T\left(\left[\chi_{[y_n - h_n, y_n + h_n)} - \chi_{[x_n - h_n, x_n + h_n)} \right] \right).$$

Now, if we define $L_n = [x_n - h_n, x_n + h_n)$ and $R_n = [y_n - h_n, y_n + h_n)$, we see that $R_n \cap L_n = \varnothing$ and their union forms a single cube J_n if we choose $h_n = (y_n - x_n)/2$. Moreover, in that case, the length of the cube J_n is $y_n + h_n - x_n + h_n = 2(y_n - x_n)$. Therefore, by linearity

$$G(y_n) - G(x_n) = 2(y_n - x_n)T\left(\frac{1}{2(y_n - x_n)}\left[\chi_{[y_n-h_n,y_n+h_n)} - \chi_{[x_n-h_n,x_n+h_n)}\right]\right).$$

We observe that $b_n(\xi) = \frac{1}{2(y_n - x_n)}\left[\chi_{[y_n-h_n,y_n+h_n)}(\xi) - \chi_{[x_n-h_n,x_n+h_n)}(\xi)\right]$ is a special atom with norm in B^1 equal to 1. It follows that using (2.6)

$$|G(y_n) - G(x_n)| \leq 2(y_n - x_n)|T(b_n)| \leq 2C(y_n - x_n)\|b_n\|_{B^1} = 2C(y_n - x_n).$$

Hence,

$$\sum_{n=1}^{\infty}|G(y_n) - G(x_n)| \leq 2C\sum_{n=1}^{\infty}(y_n - x_n).$$

We conclude by noting that given $\epsilon > 0$, (2.7) is satisfied if we choose $\delta = \frac{\epsilon}{2C}$. We conclude that the function G is absolutely continuous on J. Therefore, G is differentiable almost everywhere, that is, there exists $g \in L^1$ such that $G(x) = \int_a^x g(\xi)d\lambda(\xi)$ for all cubes $I = [a, b] \subseteq J$. Let $I \ni x$ be a sub-cube of J. Therefore, $\sup_{I\ni x}\frac{1}{\lambda(I)}\left|\int_I g(\xi)d\lambda(\xi)\right| < \infty$ because an absolutely continuous function is a function with bounded variation. Since $g \in L^1$, we have that $g_I = \frac{1}{\lambda(I)}\left|\int_I g(\xi)d\lambda(\xi)\right| < \infty$. It follows that

$$Mg(x) = \sup_{I\ni x}\frac{1}{\lambda(I)}\left|\int_I (g(\xi) - g_I)d\lambda(\xi)\right| \leq \sup_{I\ni x}\frac{1}{\lambda(I)}\left|\int_I g(\xi)d\lambda(\xi)\right| + \sup_{I\ni x} g_I < \infty.$$

This proves that $g \in \mathrm{BMO}^w$. That is, $H : \mathrm{BMO}^w \to (B^1)^*$ is onto with $H(g) = T = T_g$. Identifying T with the g, it follows from (2.5) that

$$\|g\|_{(B^1)^*} = \|T\|_{(B^1)^*} = \|T_g\|_{(B^1)^*} \leq \|g\|_{\mathrm{BMO}^w}. \tag{2.8}$$

□

Remark 2.5. We observe that the above result can be obtained differently. In fact, we recall that it was proved in [3] that the dual space of B^1 is equivalent to Λ_*^2. Let $x \in J$, $h > 0$, $I = [x - h, x + h] \subseteq J$, $L = [x - h, x)$, and $R = [x, x + h]$. Let $b(\xi) = \frac{1}{2h}[\chi_R(\xi) - \chi_L(\xi)]$. We have

$$\|f\|_{(B^1)^*} \cong \|f\|_{\Lambda_*^2} = \sup_{\substack{I\ni x\\ h>0}}\left|\int_I b(\xi)\partial f(\xi)d\lambda(\xi)\right| = \sup_{\substack{I\ni x\\ h>0}}\left|\int_I b(\xi)(\partial f(\xi) - \partial f_I)d\lambda(\xi)\right| \leq \|f\|_{\mathrm{BMO}^w}.$$

This shows that $\mathrm{BMO}^w \subseteq \Lambda_*^2 \cong (B^1)^*$.

Theorem 2.6. *The dual space $(B^1)^*$ of B^1 is BMO^w with $\|g\|_{\mathrm{BMO}^w} \cong \|g\|_{(B^1)^*}$.*

To prove Theorem 2.6, we recall that $B^1(\mathbb{T})$ has an analytic extension $B_A^1(\mathbb{D})$ to the unit disk. In fact, in [8], it was shown that $B_A^1(\mathbb{D})$ consists of functions F that are Poisson integrals of functions in $B^1(\mathbb{T})$, that is, $F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1 + e^{-i\xi}z}{1 - e^{-i\xi}z}f(e^{i\xi})d\lambda(\xi)$ where $f \in B^1(\mathbb{T})$. Moreover, the norm $\|F\|_{B_A^1(\mathbb{D})} = \int_0^1\int_{-\pi}^{\pi}|F'(z)|dz$ is equivalent to the norm $\|f\|_{B^1(\mathbb{T})}$. This allows to identify $B_A^1(\mathbb{D})$ with $B^1(\mathbb{T})$ and thus the following:

Proposition 2.7. *$B_A^1(\mathbb{D})$ can be identified with a closed subspace of $L^1(\mathbb{T})$.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $B^1(\mathbb{T})$ that converges to f in $L^1(\mathbb{T})$. We need to show that the Poisson integral of f is in $B^1_A(\mathbb{D})$. Since $f_n \in B^1(\mathbb{T})$, $\forall n \in \mathbb{N}$, then $F_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1+e^{-i\xi}z}{1-e^{-i\xi}z}f_n(e^{i\xi})d\lambda(\xi)$ belongs to $B^1_A(\mathbb{D})$. Let $F(z)$ be the Poisson integral of f. We note that if $z = e^{i\theta}$, then $(\xi-\theta)^2 - \frac{(\xi-\theta)^4}{2} \le |1-e^{-i\xi}z|^2 \le (\xi-\theta)^2$. Therefore, we have that $F'(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2e^{-i\xi}}{(1-e^{-i\xi}z)^2}f(e^{i\xi})d\lambda(\xi)$ and $|F'(z)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2}{|1-e^{-i\xi}z|^2}|f(e^{i\xi})|d\lambda(\xi) \le C\,\|f\|_{L^1(\mathbb{T})}$. It follows that,

$$\|F_n - F\|_{B^1_A} = \int_0^1\int_{-\pi}^{\pi}|F_n'(z) - F'(z)|dz \le C\|f_n - f\|_{L^1(\mathbb{T})}.$$

Since $f_n \to f$ in $L^1(\mathbb{T})$, it follows that $F \in B^1_A$ and $F_n \to F$ in $B^1_A(\mathbb{D})$. The result follows by identifying $B^1_A(\mathbb{D})$ and $B^1(\mathbb{T})$. □

We note that there is an extension of this result to the polydisk $\mathbb{D}^n$ and polysphere $\mathbb{T}^n$, see [9].

Proof of Theorem 2.6. It is sufficient to show that there exists a constant $M > 0$ such that $\|T\|_{(B^1(\mathbb{T}))^*} \ge M\,\|g\|_{\mathrm{BMO}^w(\mathbb{T})}$. The extension to $\mathbb{R}^n$ is natural, using the results in [7,9]. With Proposition 2.7, the proof follows along the lines of the proof of Proposition 7.3 in [6].

Let $T \in (B^1(\mathbb{T}))^*$. Since $B^1(\mathbb{T})$ is a closed subspace of $L^1(\mathbb{T})$ by Proposition 2.7, then by Hahn-Banach Theorem, T can be extended to a bounded linear operator $T' \in (L^1(\mathbb{T}))^*$ with $\|T\|_{(B^1(\mathbb{T}))^*} = \|T'\|_{(L^1(\mathbb{T}))^*}$. Since $(L^1(\mathbb{T}))^* = L^\infty(\mathbb{T})$, then there exists $g_0 \in L^\infty(\mathbb{T})$ such that $\|g\|_{L^\infty(\mathbb{T})} = \|T'\|_{(L^1)^*} = \|T\|_{(B^1)^*}$ and $T(f) = \frac{1}{2\pi}\int_{\mathbb{T}}\overline{g_0(e^{i\xi})}f(e^{i\xi})d\lambda(\xi)$, for all $f \in B^1_A(\mathbb{D})$. Note that here, we identify f with its correspondent in $B^1_A(\mathbb{D})$. Now, let $\sum_{n\in\mathbb{N}}A_ne^{in\xi}$ be the Fourier series of g_0. Since $g_0 \in L^\infty(\mathbb{T}) \subset L^2(\mathbb{T})$, we have that $\sum_{n\in\mathbb{Z}}|A_n|^2 < \infty$. This means that g_0 is holomorphic. Let

$$g(z) = \frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{g_0(e^{i\xi})}{e^{i\xi}-z}d(e^{i\xi}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{g_0(e^{i\xi})}{1-e^{-i\xi}z}d\xi.$$

Since $\frac{1}{1-e^{i\xi}z} = \sum_{n\in\mathbb{N}}e^{-in\xi}z^n$ and $A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-in\xi}g_0(e^{i\xi})d\lambda(\xi)$, we have

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{g_0(e^{i\xi})}{1-e^{-i\xi}z}d\xi = \sum_{n\in\mathbb{N}}A_nz^n.$$

This implies that $g \in H^2(\mathbb{D})$. For $\theta \in \mathbb{R}$, $\overline{g(e^{i\theta})} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\overline{g_0(e^{i\xi})}}{1-e^{-i\xi}e^{i\theta}}d\lambda(\xi)$. Moreover, given $f \in B^1_A(\mathbb{D}) \subseteq H^1(\mathbb{D})$, and using the Cauchy integral formula, we have:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{g(e^{i\theta})}f(e^{i\theta})d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{g_0(e^{i\theta})}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(e^{i\xi})}{1-e^{-i\xi}e^{i\theta}}d\lambda(\xi)\right)d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{g_0(e^{i\theta})}f(e^{i\theta})d\theta = T(f).$$

We also observe that

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left[\frac{1+e^{-i\xi}z}{1-e^{-i\xi}z}+1\right]g_0(e^{i\xi})d\xi = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1+e^{-i\xi}z}{1-e^{-i\xi}z}G_0(e^{i\xi})d\lambda(\xi), \tag{2.9}$$

where $G_0(e^{i\xi}) = \frac{1}{2}\left(g_0(e^{i\xi}) + \frac{1}{2\pi}\int_{-\pi}^{\pi}g_0(e^{i\xi})d\lambda(\xi)\right) = \frac{1}{2}(g_0(e^{i\xi}) + A_0)$.

Put $G_1 = \mathrm{Re}(G_0)$, $G_2 = \mathrm{Im}(G_0)$, and

$$U(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1+e^{-i\xi}z}{1-e^{-i\xi}z}G_1(e^{i\xi})d\lambda(\xi); \quad V(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1+e^{-i\xi}z}{1-e^{-i\xi}z}G_2(e^{i\xi})d\lambda(\xi).$$

Then $g = U + iV$. Moreover, U and V are analytic in $\mathbb{D}$ since they represent the Poisson integral of $G_1, G_2 \in L^\infty(\mathbb{T}) \subseteq B^1(\mathbb{T})$. Observe that BMO $\subseteq$ BMO^w. Moreover, Theorem 3.2 in [6] shows that $\|g\|_{\mathrm{BMO}} \cong \|g\|_{\mathrm{BMO}A}$. It therefore follows that there exists a constant $C > 0$ such that

$$C\|g\|_{\mathrm{BMO}^w} \le C\|g\|_{\mathrm{BMO}} \le \|g\|_{\mathrm{BMO}A} \le \|T'\|_{(L^1)^*} = \|T\|_{(B^1)^*}. \qquad \square$$

2.1 Discussion

We note that BMO$^w \subseteq \Lambda'_*$ with $\|f\|_{\Lambda'_*} \le C\|f\|_{\mathrm{BMO}^w}$, where $\|g\|_{\Lambda'_*} = \|f\|_{\Lambda_*}$ with $g' = f$ in the sense of distributions. Since $(B^1)^* \cong \Lambda'_*$, and from Theorem 2.6 above $(B^1)^* \cong$ BMO^w, it follows that BMO$^w \cong \Lambda'_*$. The consequence is that there exists $c > 0$ such that $c\|f\|_{\mathrm{BMO}^w} \le \|f\|_{\Lambda'_*}$, that is, those two norms are equivalent. We finish by noting Λ'_* has an analytic characterization, so we would expect the analytic characterization of BMO^w to also be equivalent to that of Λ'_*.

Another takeaway from Theorem 2.6 is that it provides another proof that BMO is strictly contained in BMO^w otherwise we would have had $B^1 \cong \mathbb{H}^1$, which is not true. In other words, there exists $f \in \mathbb{H}^1$ such that $f \notin B^1$.

Our next result is about the closeness of VMO^w in BMO^w.

Theorem 2.8. VMO^w *is a closed subspace of* BMO^w.

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in VMO^w that converges to $f \in$ BMO^w. Let us prove that $f \in$ VMO^w. That $f_n \to f$ as $n \to \infty$ in BMO^w is equivalent to $\lim_{n\to\infty} \|f_n - f\|_{\mathrm{BMO}^w} = 0$. The latter is also equivalent, by definition of the norm in BMO^w to

$$\lim_{n\to\infty} \sup_{Q\ni x} (f_n - f)_Q = 0 \quad \text{and} \quad \lim_{n\to\infty} M(f_n - f)(x) = 0.$$

Since $f \in$ BMO^w, then $\sup_{Q\ni x} \left| \fint_Q [f(\xi) - f_Q]\, d\lambda(\xi) \right| < \infty$. Therefore,

$$\begin{aligned}
\left| \fint_Q [f(\xi) - f_Q]\, d\lambda(\xi) \right| &= \left| \fint_Q [f(\xi) - f_n(\xi) + f_n(\xi) - f_{n,Q} + f_{n,Q} - f_Q]\, d\lambda(\xi) \right| \\
&\le \left| \fint_Q [f(\xi) - f_n(\xi)]\, d\lambda(\xi) \right| + \left| \fint_Q [f_n(\xi) - f_{n,Q}]\, d\lambda(\xi) \right| + \left| \fint_Q [f_{n,Q} - f_Q]\, d\lambda(\xi) \right| \\
&\le \left| \fint_Q [f(\xi) - f_n(\xi)]\, d\lambda(\xi) \right| + \left| \fint_Q [f_n(\xi) - f_{n,Q}]\, d\lambda(\xi) \right| + |f_{n,Q} - f_Q| \\
&\le \left| \fint_Q [f(\xi) - f_n(\xi)]\, d\lambda(\xi) \right| + \|f_n - f\|_{\mathrm{BMO}^w} \le \|f_n - f\|_{L^1_{\mathrm{loc}}} + \|f_n - f\|_{\mathrm{BMO}^w}.
\end{aligned}$$

Since $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of function is BMO^w, we have that $\lim_{n\to\infty} \|f_n - f\|_{\mathrm{BMO}^w} = 0$. Since BMO$^w \subset L^1_{\mathrm{loc}}$, we have that $\lim_{n\to\infty} \|f_n - f\|_{L^1_{\mathrm{loc}}} = 0$. Hence, VMO^w is a closed subspace of BMO^w. $\square$

Remark 2.9. We note that it was proved in [6] that VMO is a closed subspace of BMO. A stronger result even showed that if we restrict ourselves to $\mathbb{T}$, then the space of complex-valued and continuous functions $C(\mathbb{T}) \subseteq$ VMO$(\mathbb{T})$ and the BMO$(\mathbb{T})$-closure of $C(\mathbb{T})$ is precisely VMO$(\mathbb{T})$. It turns out this result is also true for VMO^w.

Theorem 2.10. *The* BMO^w*-closure of* $C(\mathbb{T})$ *is precisely* $\mathrm{VMO}^w(\mathbb{T})$, *that is*

$$\overline{C(\mathbb{T})}^{\mathrm{BMO}^w} = \mathrm{VMO}^w(\mathbb{T}).$$

Proof. We observe that $C(\mathbb{T}) \subseteq \mathrm{VMO}(\mathbb{T}) \subseteq \mathrm{VMO}^w(\mathbb{T}) \subseteq \mathrm{BMO}^w(\mathbb{T})$. Therefore, since $\mathrm{VMO}^w(\mathbb{T})$ is closed in $\mathrm{BMO}^w(\mathbb{T})$, we have that

$$\overline{C(\mathbb{T})}^{\mathrm{BMO}^w(\mathbb{T})} \subseteq \overline{\mathrm{VMO}^w(\mathbb{T})}^{\mathrm{BMO}^w(\mathbb{T})} = \mathrm{VMO}^w(\mathbb{T}).$$

For the other direction, let $f \in \mathrm{VMO}^w(\mathbb{T})$, and consider the sequence $\{R_\epsilon(f)\}_{\epsilon>0}$ of rotations of f by angle ϵ, defined on $\mathbb{T}$ as $R_\epsilon(f)(e^{i\theta}) = f(e^{i(\theta-\epsilon)})$; $\theta \in \mathbb{R}$. Then, from Theorem 2.1 in [6], we have that for all $\epsilon > 0$, $R_\epsilon(f) \in C(\mathbb{T})$ and $\lim\limits_{\epsilon\to 0} \|R_\epsilon(f) - f\|_{\mathrm{BMO}(\mathbb{T})} = 0$. Since $\mathrm{BMO}(\mathbb{T}) \subseteq \mathrm{BMO}^w(\mathbb{T})$, we also have that $\lim\limits_{\epsilon\to 0} \|R_\epsilon(f) - f\|_{\mathrm{BMO}^w(\mathbb{T})} = 0$. That is, $f \in \overline{C(\mathbb{T})}^{\mathrm{BMO}^w(\mathbb{T})}$. □

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