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Regularized Trace for Operators on a Separable Banach Space

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Abstract. In this paper we consider a Sturm–Liouville type differential operator with unbounded operator coefficients given on a finite interval, with values in a separable Banach space $\mathcal{B}$. In the past, problems of this type have been mainly studied on Hilbert space. Kuelbs (J Funct Anal 5:354–367, 1970) has shown that every separable Banach space $\mathcal{B}$ can be continuously embedded in a separable Hillbert space $\mathcal{H}$. Given this result, we first prove that there always exists a separable Banach space $\mathcal{B}_z^* \subset \mathcal{H}^*$ as a continuous embedding, which is a (conjugate) isometric isomorphic copy of $\mathcal{B}$. This space generates a semi-inner product structure for $\mathcal{B}$ and is the tool we use to develop our theory. We are able to obtain a regularized trace formula for the above differential operator when the problem is posed on $\mathcal{B}$. We also provide a few examples illustrating the scope and implications of our approach.

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Introduction

The construction of a regularized trace formula for the Sturm–Liouville operator

$$L := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x) \qquad y'(0) = y'(\pi) = 0$$

with $q(x) \in C^1[0,1]$ was first studied by Gelfand and Levitan [2]. Later, Dikii [3] obtained trace formulas using a different method. In their studies, Buslaev and Faddev [4] obtained a number of important expressions for the trace of the negative part of integer and half-integer powers of a singular differential operator. These investigations have branched in different directions leading to many related results. For example, Pushnitski and Sorrell [5] have found trace formulas for the perturbed harmonic oscillator, while Makin [6] has found a regularized formula for the Sturm–Liouville operator with irregular boundary conditions. Recently, new trace formulas for the Lasso-Graph have been found by Guan and Yang [7] and some regularized trace formulas have been found using the essentially isospectral transformation by Guliyev [8]. Additional

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studies related to trace formulas for the scalar or differentiable operators (with bounded or unbounded coefficients) can be found in the following: [9–17] and references therein.

In a series of interesting recent papers Hu, Bondarenko and Yang consider the following Sturm–Liouville-type problem:

$$L := -\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} + q(x)y(a) = \lambda y\left(x\right), \;\; x \in \left(0, \pi\right), \text{ with } \; y^{(a)}(0) = y^{(b)}(\pi) = 0,$$

where $\alpha, \beta \in \{0, 1\}$, λ is the spectral parameter, $a \in (0, \pi)$, and $q(x)$ is a real-valued function in $W_2^1(0, \pi)$. They call these Sturm–Liouville-type problems with a frozen argument. It turns out that these problems are nonlocal and belong to a special class of functional differential equations (see [18] and references therein).

In another direction, Hu, Bondarenko, Shieh and Yang obtain regularized trace formulas for a class of Dirac-type integro-differential operators on graphs. Operators of this type have become important because they are used to describe the motion of quantum particles confined to a class of thin atomic structures (see [19] and references therein).

1. Background

Let $\mathcal{H}$ be a separable Hilbert space and let operators A and Q satisfy the following conditions:

1. $A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is a self adjoint, with $A \geq I$ and $A^{-1} \in \mathbb{S}_\infty(\mathcal{H})$, the compact operators on $\mathcal{H}$.
2. For every $x \in [0, \pi]$, $Q(x) : \mathcal{H} \longrightarrow \mathcal{H}$ is a self-adjoint trace class operator (i.e., $Q(x) \in \mathbb{S}_1[\mathcal{H}]$).
3. The functions $\|Q^{(i)}(x)\|_{\mathbb{S}_1[\mathcal{H}]}$ $(i = 0, 1, 2)$ are bounded and measurable in the interval $[0, \pi]$, where $Q^{(i)}(x)$ is the ith derivative of $Q(x)$.
4. For every $f \in \mathcal{H}$, $\int_0^\pi (Q(x)f, f)_\mathcal{H} \mathrm{d}x = 0$.

Let

$$L = L_0 + Q\,, \qquad L_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + A \quad \text{with } \; y(0) = y'(\pi) = 0.$$

A regularized trace for L was obtained on the space $\mathcal{H}_1 = L^2(\mathcal{H}; [0, \pi])$ by Gül [20]. The purpose of this paper is to use some recent results by Gill and Gül [21] to consider the same problem when $\mathcal{H}$ is replaced by an arbitrary separable Banach space $\mathcal{B}$. More precisely, let $\mathcal{B}$ be a separable Banach space and let $\mathcal{B}_1 = S^2(\mathcal{B}; [0, \pi])$ denote the set of all measurable functions f with values in $\mathcal{B}$ such that:

$$\int_0^\pi \|f(x)\|_\mathcal{B}^2 \mathrm{d}x < \infty.$$

We consider the operators L_0 and L in $\mathcal{B}_1$ with the identical boundary conditions as follows:

$$L = L_0 + Q\,, \qquad L_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + A \quad \text{with } y(0) = y'(\pi) = 0.$$

Here A and $Q(x)$ satisfy the following conditions:

1. $A : D(A) \longrightarrow \mathcal{B}$ is a self adjoint operator with:
   (a) $[Au, u]_z \geq [u, u]_z$,
   (b) $A^{-1} \in \mathbb{S}_\infty[\mathcal{B}]$,
   where self adjoint for operators on $\mathcal{B}$ and the bracket $[\,\cdot\,,\cdot\,]_z$ is defined in the next section, while $\mathbb{S}_\infty[\mathcal{B}]$ is the set of compact operators on $\mathcal{B}$.
2. For every $x \in [0, \pi]$, $Q(x) : \mathcal{B} \longrightarrow \mathcal{B}$ is a self-adjoint compact operator. It is also a trace class operator ($Q(x) \in \mathbb{S}_1[\mathcal{B}]$).
3. The functions $\|Q^{(i)}(x)\|_{\mathbb{S}_1[\mathcal{B}]}$ $(i = 0, 1, 2)$ are bounded and measurable in the interval $[0, \pi]$, where $Q^{(i)}(x)$ is the ith derivative with respect to x.
4. For every $f \in \mathcal{B}$, $\int_0^\pi [Q(x)f, f]_z \mathrm{d}x = 0$.

1.1. Summary

The following section is devoted to some new background material on Banach space operator theory. In particular, we define notions of self-adjointness and trace class for operators on a separable Banach space. In the third section, we obtain a few relations for the resolvent using the self-adjoint property of L. In section four, we derive our regularized trace formula for L and in section five, we give a few examples.

2. Adjoints and Trace Class Operators for a Banach Space

Let $\mathcal{B}$ be a separable Banach space with dual space $\mathcal{B}^*$, let $\mathcal{C}[\mathcal{B}]$ be the closed densely defined linear operators and $\mathcal{L}[\mathcal{B}]$ be the bounded linear operators on B. The following lemma is the important part of a theorem due to Kuelbs [1].

Lemma 2.1. (Kuelbs Lemma) *Let $\mathcal{B}$ be a separable Banach space. Then, there exist a separable Hilbert space $\mathcal{H}$ such that $\mathcal{B} \subset \mathcal{H}$ as continuous dense embedding.*

If T is an operator, we let $\sigma(T)$ denote the spectrum of T and $\sigma_p(T) \subset \sigma(T)$ denote the point spectrum of T. The following theorem is due to Lax [22].

Theorem 2.2. (Lax's Theorem) *Let $\mathcal{B}$ be a separable Banach space that is continuously and densely embedded in a Hilbert space $\mathcal{H}$, and let T be a bounded linear operator on B that is symmetric with respect to the inner product of $\mathcal{H}$* (i.e., $(Tu, v)_{\mathcal{H}} = (u, Tv)_{\mathcal{H}}$ for all $u, v \in \mathcal{B}$). *Then,*

1. *T is bounded with respect to the $\mathcal{H}$ norm, and*

$$\|T^*T\|_{\mathcal{H}} = \|T\|_{\mathcal{H}}^2 \leq k\,\|T\|_{\mathcal{B}}^2\,,$$

   where k is a positive constant.
2. *$\sigma(T)$ relative to $\mathcal{H}$ is a subset of $\sigma(T)$ relative to $\mathcal{B}$.*
1. *$\sigma_p(T)$ relative to $\mathcal{H}$ is equal to $\sigma_p(T)$ relative to $\mathcal{B}$.*

Let $\mathbf{J}$ be the (conjugate) isometric isomorphism of $\mathcal{H} \to \mathcal{H}^*$, and let $\mathbf{J}_\mathcal{B} = \mathbf{J}|_\mathcal{B}$ (restriction). Since $\mathcal{B}$ is densely and continuously embedded in $\mathcal{H}$, $\mathbf{J}_\mathcal{B}$ is a (conjugate) bijective mapping of $\mathcal{B}$ onto $\mathbf{J}_\mathcal{B}(\mathcal{B}) \subset \mathcal{H}^*$ as a continuous dense embedding. We want to define a norm on $\mathbf{J}_\mathcal{B}(\mathcal{B})$, so that it becomes a continuous dense embedding with the same relation to $\mathcal{H}^*$ as $\mathcal{B}$ has to $\mathcal{H}$.

Definition 2.3. For $u \in \mathcal{B}$ let $u_h^* = \mathbf{J}_\mathcal{B}(u)$ and let $u_z^* = \frac{\|u\|_\mathcal{B}^2}{\|u\|_\mathcal{H}^2} u_h^*$ and define $\mathcal{B}_z^* = \{u_z^* : u \in \mathcal{B}\}$, with norm $\|u_z^*\|_{\mathcal{B}_z^*} = \|u\|_\mathcal{B}$. We call $\mathcal{B}_z^*$ the Zachary representation for $\mathcal{B}$ in $\mathcal{H}^*$.

Remark 2.4. Recall that a semi-inner product on a separable Banach space $\mathcal{B}$ is a mapping $[\cdot,\cdot]$ on $\mathcal{B} \times \mathcal{B}$ such that:

1. $[au + v, w] = a\,[u, w] + [v, w]$,
2. $[u, u] = \|u\|_\mathcal{B}^2$ and
3. $|\,[u, v]\,|^2 \le [u, u]\,[v, v]$.

Theorem 2.5. *Define the functional on* $[\,\cdot,\cdot\,]$ *on* $\mathcal{B}\times\mathcal{B}$*, by* $[v, u] = u_z^*\,(v)$*, then this defines a semi-inner product structure on* $\mathcal{B}$*.*

Proof. The proof of (1) and (2) are clear, so we only need to prove (3). To prove (3),

$$\begin{aligned}
|[v,u]|^2 &= \left[\frac{\|u\|_\mathcal{B}^2}{\|u\|_\mathcal{H}^2}\right]^2 |(v,u)_\mathcal{H}|^2 \le \left[\frac{\|u\|_\mathcal{B}^2}{\|u\|_\mathcal{H}^2}\right]^2 \|v\|_\mathcal{H}^2\,\|u\|_\mathcal{H}^2 \\
&= \left\{\frac{\|u\|_\mathcal{B}^2}{\|u\|_\mathcal{H}^2}\frac{\|v\|_\mathcal{H}^2}{\|v\|_\mathcal{B}^2}\right\}\left\{\tfrac{\|v\|_\mathcal{B}^2}{\|v\|_\mathcal{H}^2}(v,v)_\mathcal{H}\right\}\left\{\tfrac{\|u\|_\mathcal{B}^2}{\|u\|_\mathcal{H}^2}(u,u)_\mathcal{H}\right\} \\
&= \left\{\frac{\|u\|_\mathcal{B}^2}{\|u\|_\mathcal{H}^2}\frac{\|v\|_\mathcal{H}^2}{\|v\|_\mathcal{B}^2}\right\}[v,v]\,[u,u]\,.
\end{aligned}$$

For the terms in braces we have $\frac{\|u\|_\mathcal{B}^2}{\|u\|_\mathcal{H}^2} \ge 1$ and $\frac{\|v\|_\mathcal{H}^2}{\|v\|_\mathcal{B}^2} \le 1$, for all $u, v \in \mathcal{B}$, including $u = v$. It follows that the inequality holds when this term is replaced by 1. □

Theorem 2.6. *The separable Banach space* $\mathcal{B}_z^* \subset \mathcal{H}^*$ *is a continuous dense embedding and a* (conjugate) *isometric isomorphic copy of* $\mathcal{B}$*.*

Proofs of the following can be found in Gill [23].

Theorem 2.7. *If* $A \in \mathcal{C}(\mathcal{B})$ *and* A' *its dual mapping on* $\mathcal{B}^*$*, then there is a unique operator* $A^* = \mathbf{J}_\mathcal{B}^{-1} A' \mathbf{J}_\mathcal{B} \in \mathcal{C}[\mathcal{B}]$ *that satisfies the following:*

1. $(aA)^* = \bar{a}A^*$*;*
2. $A^{**} = A$*;*
3. $(A + B)^* = A^* + B^*$*;*
4. $(AB)^* = B^*A^*$ *on* $D(A^*) \bigcap D(B^*)$*;*
5. *if* $A \in \mathcal{L}[\mathcal{B}]$*, then* $\|A^*A\|_\mathcal{B} \le M\|A\|_\mathcal{B}^2$*, for some constant* M *and it has a bounded extension to* $\mathcal{L}[\mathcal{H}]$*.*

Definition 2.8. Let U be bounded and $A \in \mathcal{C}[\mathcal{B}]$. Then:

1. U is unitary if $UU^* = U^*U = I$.
2. A is said to be self-adjoint if $D(A) = D(A^*)$ and $A = A^*$.
3. A is said to be normal if $D(A) = D(A^*)$ and $AA^* = A^*A$.
4. $\mathcal{U}$ is $\perp$ to $\mathcal{V}$ if and only, for each $\forall v \in \mathcal{V}$ and $\forall u \in \mathcal{U}$, $u_z^*(v) = v_z^*(u) = 0$.

Theorem 2.9. (Polar Representation) *Let $\mathcal{B}$ be a separable Banach space. If $A \in \mathcal{C}[\mathcal{B}]$, then there exists a partial isometry U and a self-adjoint operator T, with $D(T) = D(A)$ and $A = UT$. Furthermore, $T = [A^*A]^{1/2}$ in a well-defined sense.*

Theorem 2.10. *For every $\phi \in \mathcal{B}$, there exists a $\varphi_\phi^* \in \mathcal{B}^*$ and a constant $c_\phi > 0$ depending on ϕ such that $(f, \phi)_{\mathcal{H}} = c_\phi^{-1}\left\langle f, \varphi_\phi^*\right\rangle_{\mathcal{B}^*}$ for all $f \in \mathcal{B}$.*

Let $\mathbb{S}_\infty[\mathcal{B}]$ be the set of compact operators on $\mathcal{B}$, let $A = U[A^*A]^{1/2} \in \mathbb{S}_\infty[\mathcal{B}]$ and let $\bar{A} = \bar{U}[\bar{A}^*\bar{A}]^{1/2}$ be it's extension to $\mathcal{H}$. By Lax's theorem, the point spectrum of $\bar{A}$ is unchanged by the extension, so that $\bar{A}$ is also compact. Thus, without loss of generality there exists a orthonormal family $\{\phi_n \mid n \in \mathbb{N}\} \subset \mathcal{B}$ such that

$$\bar{A} = \sum_{n=1}^{\infty} s_n(\bar{A})\,(\cdot\,, \phi_n)_{\mathcal{H}}\,\bar{U}\phi_n.$$

From Theorem 2.10 and the fact that $s_n(\bar{A}) = s_n(A)$ by Lax's theorem, we can write A as follows:

$$A = \sum_{n=1}^{\infty} s_n(A) c_n^{-1}\langle \cdot, \varphi_{\phi_n}^* \rangle_{\mathcal{B}^*} U\phi_n.$$

If $\bar{A} \in \mathbb{S}_p[\mathcal{H}]$ (the Schatten class of order p in $\mathcal{L}[\mathcal{H}]$), its norm can be represented as follows:

$$\begin{aligned}\|\bar{A}\|_p^{\mathcal{H}} = \left\{\mathrm{tr}\left[\bar{\mathrm{A}}^*\bar{\mathrm{A}}\right]^{\mathrm{p}/2}\right\}^{1/p} &= \left\{\sum_{n=1}^{\infty}\left(\bar{A}^*\bar{A}\phi_n, \phi_n\right)_{\mathcal{H}}^{p/2}\right\}^{1/p} \\ &= \left\{\sum_{n=1}^{\infty}\left|s_n\left(\bar{A}\right)\right|^p\right\}^{1/p}.\end{aligned}$$

Definition 2.11. We define $\mathbb{S}_p[\mathcal{B}]$, the Schatten class of order p in $\mathcal{L}[\mathcal{B}]$, as follows:

$$\mathbb{S}_p[\mathcal{B}] = \left\{A \in \mathbb{S}_\infty[\mathcal{B}] : \|A\|_p^{\mathcal{B}} = \left\{\sum_{n=1}^{\infty}\left|s_n\left(A\right)\right|^p\right\}^{1/p} < \infty\right\}.$$

Since $s_n(A) = s_n(\bar{A})$, we have the following:

Corollary 2.12. *If $A \in \mathbb{S}_p[\mathcal{B}]$, then $\bar{A} \in \mathbb{S}_p[\mathcal{H}]$ and $\|A\|_p^{\mathcal{B}} = \|\bar{A}\|_p^{\mathcal{H}}$.*

If $A \in \mathbb{S}_\infty[\mathcal{B}]$ then, by the polar representation theorem, A^*A is a non-negative self-adjoint operator and $|A| = [A^*A]^{1/2} \in \mathbb{S}_\infty[\mathcal{B}]$ where A^* is the adjoint of A. Let $s_1(A) \geq s_2(A) \geq \cdots \geq s_k(A)$ $(1 \leq k \leq \infty)$ be the non-zero eigenvalues of $|A|$ with each eigenvalue is repeated as many times as its multiplicity (s-numbers). When $k < \infty$, we assume that $s_j(A) = 0$ for $j = k+1, k+2, \ldots$.

If $A \in \mathbb{S}_p[\mathcal{B}]$, $1 \leq p < \infty$, then by Corollary 2.12, A extends to $\bar{A} \in \mathbb{S}_p[\mathcal{H}]$ with $\|A\|_p^{\mathcal{B}} = \|\bar{A}\|_p^{\mathcal{H}}$. If $A \in \mathbb{S}_1[\mathcal{B}]$, we called it a trace class (or nuclear) operator on $\mathcal{B}$.

3. Resolvents and Regularized Trace

In this section, we assume that $\mathcal{B}$ is fixed separable Banach space and $\mathcal{H} = \mathcal{H}_\mathcal{B}$ is our fixed Hilbert space constructed via Kuelbs lemma and $\mathcal{H}_1 = L^2[\mathcal{H}; [0,\pi]]$. Since $\mathcal{B}$ is a continuous dense embedding in $\mathcal{H}$, $\mathcal{B}_1 = S^2(\mathcal{B};\ [0,\pi])$ is a continuous dense embedding in $\mathcal{H}_1$. With this setup, our approach is to first pose our problem on $\mathcal{B}_1$, then use the known properties of $\mathcal{H}_1$ to solve it and restrict back to $\mathcal{B}_1$ to obtain the same result.

By assumption, A is an unbounded self adjoint operator with A^{-1} is compact. Thus A^{-1} and its extension $\bar{A}^{-1}$ have the same eigenvalues $\gamma_1^{-1} \geq \gamma_2^{-1} \geq \cdots$. Let $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n \leq \cdots$ be the eigenvalues of A and its extension $\bar{A}$ to $\mathcal{H}$. Let $\{\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots\} \subset \mathcal{B}$ be the corresponding orthonormal eigenvectors.

Let D_0 denote the set of the functions $y(x)$ in $\mathcal{B}_1$ satisfying the conditions:

(1^o) $y(x)$ has second order continuous derivatives with respect to the $\mathcal{B}$ norm on $[0,\pi]$.

(2^o) $Ay(x)$ is continuous with respect to the $\mathcal{B}$ norm.

(3^o) $y(0) = y'(\pi) = 0$.

The set D_0 is dense in $\mathcal{B}_1$ and the operator $L_0' : D_0 \to \mathcal{H}_1$ defined by $L_0' = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + A$ has a closed extension to $\mathcal{H}_1$. Since A is self adjoint, L_0' is symmetric, as

$$(L_0'y, z)_{\mathcal{H}_1} = \int_0^\pi (L_0'y, z)_\mathcal{H}\, \mathrm{d}x = \int_0^\pi (y, L_0'z)_\mathcal{H}\, \mathrm{d}x = (y, L_0'z)_{\mathcal{H}_1}$$

for each $y, z \in D_0$. The eigenvalues of L_0' and its extension are of the form $(\frac{1}{2}+k)^2 + \gamma_j \quad (k = 0,1,2,\ldots; j = 1,2,\ldots)$. The corresponding orthonormal eigenvectors have the form $M_k \sin(k+\frac{1}{2})x \cdot \varphi_j \quad (k = 0,1,2,\ldots; j = 1,2,\ldots)$, where $M_k = \sqrt{\frac{2}{\pi}}$ for $k = 0,1,2,\ldots$.

We assume that the family of eigenvectors for the extension of L_0' to $\mathcal{H}_1$ is a complete basis for $\mathcal{H}_1$. Since, the closed extension of $L_0 = \overline{L_0'}$ to $\mathcal{H}_1$ is self-adjoint, it follows that its restriction to $\mathcal{B}_1$ is also self-adjoint.

By the condition (3) on $Q(x)$, for every $x \in [0,\pi]$, there is a $c > 0$ such that

$$\|Q(x)\|_\mathcal{B} \leq c.$$

In this case, for every $y = y(x) \in \mathcal{B}_1$ we see that

$$\|Qy\|_1^2 = \int_0^\pi \|Q(x)y(x)\|_{\mathcal{B}}^2 \, \mathrm{d}x \le \int_0^\pi \|Q(x)\|_{\mathcal{B}}^2 \|y(x)\|_{\mathcal{B}}^2 \, \mathrm{d}x$$
$$\le c^2 \int_0^\pi \|y(x)\|_{\mathcal{B}}^2 \, \mathrm{d}x = c^2 \|y(x)\|_1^2$$

or

$$\|Qy\|_1 \le c \|y\|_1 .$$

This means that Q is a bounded operator from $\mathcal{B}_1$ to $\mathcal{B}_1$. Its self-adjointness on $\mathcal{B}_1$ follows from the same argument used to establish the self-adjointness of L_0'. Hence, we conclude that the operator $L = L_0 + Q$ is a self-adjoint operator from $D(L) = D(L_0)$ to $\mathcal{B}_1$.

Let R_λ^0 and R_λ be resolvents of the operators L_0 and L respectively and let $\mu_1 \le \mu_2 \le \cdots \le \mu_n \le \cdots$ be the eigenvalues of operator L_0. Here each eigenvalue is accounted as many times as its multiplicity number. Since the eigenvalues of operator L_0 are $(\frac{1}{2}+k)^2+\gamma_j \quad (k = 0, 1, 2, \ldots; j = 1, 2, \ldots)$ and $\lim_{j\to\infty} \gamma_j = \infty$, we have $\lim_{n\to\infty} \mu_n = \infty$.

This means that

$$\lim_{n\to\infty} \frac{1}{\mu_n - \mu} = 0 \qquad (\mu \ne \mu_n; \quad n = 1, 2, \ldots).$$

On the other hand, for every real μ which is not an eigenvalue of L_0 , the operator R_μ^0 is self-adjoint and the system of orthonormal eigenfunctions, $M_k \sin(k + \frac{1}{2})x \cdot \varphi_j \quad (k = 0, 1, 2, \ldots; j = 1, 2, \ldots)$ is complete. In this case, it is known that R_μ^0 is a compact operator. By the Hilbert identity

$$R_\lambda^0 - R_\mu^0 = (\lambda - \mu) R_\lambda^0 R_\mu^0$$

it follows that the operator R_λ^0 is also compact for every real number $\lambda \ne \mu_n \quad (n = 1, 2, \ldots)$. Therefore, the operator L_0 has pure discrete spectrum. Again, since the operator Q is a bounded self-adjoint operator, the spectrum of operator $L = L_0 + Q$ is also pure discrete, (see [24]).

Let $\lambda_1 \le \lambda_1 \le \cdots \le \lambda_1 \le \cdots$ be the eigenvalues of operator L. In a similar way above we find that the self-adjoint operator R_λ is a compact operator for every $\lambda \ne \lambda_n \ (n = 1, 2, \ldots)$. Now, if $\gamma_j \sim a j^\alpha \ (a, \alpha > 0)$ then it is not difficult to see that as $n \to \infty$

$$\mu_n, \lambda_n \sim d_0 n^{\frac{2\alpha}{2+\alpha}}, \tag{3.1}$$

where d_0 is a constant. Using these asymptotically approaches (3.1), it follows that the sequence $\{\mu_n\}_{n=1}^\infty$ has a subsequence $\{\mu_{n_m}\}_{m=1}^\infty$ such that

$$\mu_k - \mu_{n_m} \ge d_1 \left(k^{\frac{2\alpha}{2+\alpha}} - n_m^{\frac{2\alpha}{2+\alpha}} \right), \quad (k = n_m, n_m + 1, n_m + 2, \ldots)$$

where d_1 is a positive constant.

With this property, the limit

$$\lim_{m\to\infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k)$$

is called the regularized trace of operator L. Note that the operator functions $(QR_\lambda^0)^j$ for $j = 1, 2, \ldots$, are analytic with respect to the norm of $\mathbb{S}_1[\mathcal{H}_1]$ in the resolvent region $\rho(L_0)$ of the operator L_0 (see [20]). From here, by the relation

$$\mathrm{tr}(R_\lambda - R_\lambda^0) = \mathrm{tr}R_\lambda - \mathrm{tr}R_\lambda^0 = \sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k - \lambda} - \frac{1}{\mu_k - \lambda}\right)$$

of the trace class operators R_λ^0 and R_λ we conclude that

$$\sum_{k=1}^{n_m}(\lambda_k - \mu_k) = \sum_{j=1}^{p} D_{mj} + D_m^{(p)}, \tag{3.2}$$

where

$$D_{mj} = \frac{(-1)^j}{2\pi i j}\int\limits_{|\lambda|=b_m} \mathrm{tr}[(QR_\lambda^0)^j]\,\mathrm{d}\lambda \tag{3.3}$$

and

$$D_m^{(p)} = \frac{(-1)^p}{2\pi i}\int\limits_{|\lambda|=b_m} \lambda\,\mathrm{tr}[R_\lambda(QR_\lambda^0)^{p+1}]\,\mathrm{d}\lambda. \tag{3.4}$$

Let $\{\psi_q(x)\}_{q=1}^{\infty}$ be the system of orthonormal eigenvectors corresponding to the eigenvalues $\{\mu_q(x)\}_{q=1}^{\infty}$ of operator L_0 respectively. Since for $k = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots$

$$M_k \sin\left(\frac{1}{2} + k\right)x \cdot \varphi_j$$

is the system of orthonormal eigenvectors corresponding to the eigenvalues $\left(\frac{1}{2} + k\right)^2 + \gamma_j$ of operator L_0 respectively, we have

$$\psi_q(x) = M_{k_q} \sin\left(\frac{1}{2} + k_q\right)x \cdot \varphi_{j_q} \ \ (q = 1, 2, \ldots). \tag{3.5}$$

Theorem 3.1. *If the operator function $Q(x)$satisfies the conditions (2–4) and if as $j \to \infty$ $\gamma_j \sim a_j^\alpha$ $(0 < a < \infty,\ 2 < \alpha < \infty)$ then*

$$\lim_{m\to\infty} D_{m1} = -\frac{1}{4}[trQ(0) - trQ(\pi)]. \tag{3.6}$$

Proof. According to the relation (3.3) we just get:

$$D_{m1} = \frac{-1}{2\pi i}\int\limits_{|\lambda|=b_m} \mathrm{tr}(QR_\lambda^0)\,\mathrm{d}\lambda.$$

Since the functions $\{\psi_q(x)\}_{q=1}^{\infty}$ given by (3.5) is an orthonormal basis of the space H_1 then, for every $\lambda \in \rho(L_0)$, the equality

$$\mathrm{tr}(QR_\lambda^0) = \sum_{q=1}^{\infty}(QR_\lambda^0\psi_q, \psi_q)_1$$

holds. Considering this equality, a straight-forward calculation gives

$$\begin{aligned} D_{m1} &= \sum_{q=1}^{\infty}(Q\psi_q,\psi_q)_1 \frac{1}{2\pi i}\int\limits_{|\lambda|=b_m}\frac{\mathrm{d}\lambda}{\lambda-\mu_q} = \sum_{q=1}^{n_m}(Q\psi_q,\psi_q)_1 \\ &= \sum_{q=1}^{n_m}\int\limits_0^{\pi}(Q(x)\psi_q(x),\psi_q(x))_z\,\mathrm{d}x \\ &= \frac{1}{2}\sum_{q=1}^{n_m}M_{k_q}^2\int\limits_0^{\pi}(1-\cos(2k_q+1)x)(Q(x)\varphi_{j_q},\varphi_{j_q})_z\,\mathrm{d}x. \end{aligned}$$

From here, it follows that

$$\begin{aligned} \lim_{m\to\infty} D_{m1} =& \frac{-1}{\pi}\sum_{r=1}^{\infty}\sum_{j=1}^{\infty}\int\limits_0^{\pi}(Q(x)\varphi_j,\varphi_j)_z\cos rx\,\mathrm{d}x \\ &+\frac{1}{2\pi}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left[\int\limits_0^{\pi}(Q(x)\varphi_j,\varphi_j)_z\cos kx\,\mathrm{d}x\right. \\ &\left.+(-1)^k\int\limits_0^{\pi}(Q(x)\varphi_j,\varphi_j)_z\cos kx\,\mathrm{d}x\right] \end{aligned}$$

or

$$\begin{aligned} \lim_{m\to\infty} D_{m1} =& -\frac{1}{2}\sum_{j=1}^{\infty}\left\{\sum_{r=1}^{\infty}\left[\frac{2}{\pi}\int\limits_0^{\pi}(Q(x)\varphi_j,\varphi_j)_z\cos rx\,\mathrm{d}x\right]\cos k0\right\} \\ &+\frac{1}{4}\sum_{j=1}^{\infty}\left\{\sum_{k=1}^{\infty}\left[\frac{2}{\pi}\int\limits_0^{\pi}(Q(x)\varphi_j,\varphi_j)_z\cos kx\,\mathrm{d}x\right]\cos k0\right. \\ &\left.+\sum_{k=1}^{\infty}\left[\frac{2}{\pi}\int\limits_0^{\pi}(Q(x)\varphi_j,\varphi_j)_z\cos kx\,\mathrm{d}x\right]\cos k\pi\right\}. \end{aligned}$$

The terms in second $\{\cdot\}$ above are the values of the Fourier series of the function $(Q(x)\varphi_j,\varphi_j)_z$ having the derivative of second order continuous derivatives at the points 0 and π respectively, with respect to the functions $\{\cos kx\}_{k=0}^{\infty}$ in $[0,\pi]$. The term in first $\{\cdot\}$ is the value at 0 in the same sense. Therefore, we have:

$$\lim_{m\to\infty} D_{m1} = -\frac{1}{2}\sum_{j=1}^{\infty}[(Q(0)\varphi_j,\varphi_j)_z] + \frac{1}{4}\sum_{j=1}^{\infty}[(Q(0)\varphi_j,\varphi_j)_z + (Q(\pi)\varphi_j,\varphi_j)_z].$$

This gives the limit Eq. (3.6). □

4. Regularized Trace Formula of the Operator L

In this section, we obtain a formula for the limit $\{\lim_{m\to\infty}\sum_{k=1}^{n_m}(\lambda_k-\mu_k)\}$ that we called the regularized trace of the operator L in the previous section. Following Gül [20], it is not difficult to show that

$$D_{mj}=\frac{(-1)^j}{2\pi ij}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_j=1}^{\infty}*\Bigg[\Bigg(\int\limits_{|\lambda|=b_m}\prod_{q=1}^{j}(\mu_{k_q}-\lambda)^{-1}\mathrm{d}\lambda\Bigg)\cdot\prod_{q=1}^{j}(Q\psi_{k_q},\psi_{k_{\rho(q)+1}})_1\Bigg] \tag{4.1}$$

where the symbol "$*$" denotes that there are numbers among $\mu_{k_1},\mu_{k_2},\ldots,\mu_{k_j}$ less than or greater than b_m.

Observe that the Eq. (4.1) gives the estimations for D_{m2} and D_{m3}, respectively

$$|D_{m2}|\le\|Q\|_1^2\Omega_m, \tag{4.2}$$

$$|D_{m3}|\le\|Q\|_1^3\Omega_m(\Omega_m+4d_1^{-1}n_m^{1-\delta}), \tag{4.3}$$

where

$$\Omega_m\sum_{j=n_m+1}^{\infty}(\mu_j-\mu_{n_m})^{-1},\ d_1=\frac{d_0}{4}\text{ and }\ \delta=\frac{\alpha-2}{\alpha+2}.$$

Moreover, if $\gamma_j\sim aj^{\alpha}$ as $j\to\infty\ (0<a<\infty,\ 2<\alpha<\infty)$ then the inequality

$$\|R_\lambda^0\|_{\mathbb{S}_1(H_1)}<\text{const.}n_m^{1-\delta}\qquad(\delta=\frac{\alpha-2}{\alpha+2}) \tag{4.4}$$

holds on the circle $|\lambda|=b_m$.

Theorem 4.1. *Suppose that* $\gamma_j\sim aj^{\alpha}$ *as* $j\to\infty\ (0<a<\infty,\ 2<\alpha<\infty)$. *Then, with the conditions* (3.2) *and* (3.3) *on the operator function* $Q(x)$, *for* $j\ge 2$ *we have* $\lim_{m\to\infty}D_{mj}=0$.

Proof. From Eq. (3.3) we find:

$$|D_{mj}|\le\frac{1}{2\pi j}\int\limits_{|\lambda|=b_m}\|Q\|_1^j\|R_\lambda^0\|_{\mathbb{S}_1(H_1)}\|R_\lambda^0\|_{\mathbb{S}_1(H_1)}^{j-1}|\mathrm{d}\lambda|. \tag{4.5}$$

Since, in the case $Q(x)\equiv 0$, $R_\lambda=R_\lambda^0$ then we have

$$\|R_\lambda\|_1<\frac{d_1}{4}n_m^{-\delta},\qquad\left(\delta=\frac{\alpha-2}{\alpha+2}\right). \tag{4.6}$$

This last inequality (4.6) together with the inequalities (4.4) and (4.5) implies that

$$|D_{mj}|\le\text{const.}\int\limits_{|\lambda|=b_m}n_m^{1-\delta}n_m^{-\delta(j-1)}|\mathrm{d}\lambda|\le\text{const. }\mu_{n_m}n_m^{1-\delta j}.$$

Since $\mu_{n_m} \leq \text{const.}\, n_m^{1+\delta}$ we have

$$|D_{mj}| \leq \text{const.}\, n_m^{2-\delta(j-1)}.$$

Clearly, if $j > 1 + 2\delta^{-1}$ then

$$\lim_{m\to\infty} D_{mj} = 0.$$

For $j = 2$ since $\lim_{m\to\infty} \Omega_m = 0$, from (4.2) we obtain that

$$\lim_{m\to\infty} D_{m2} = 0.$$

Similarly, from (4.3) we see that

$$\lim_{m\to\infty} D_{m3} = 0.$$

It follows that for $j = 2, 3, \ldots, |2\delta^{-1}| + 1$

$$\lim_{m\to\infty} D_{mj} = 0.$$

Now, we state the regularized trace formula of operator L on Banach space $\mathcal{B}$ with the next theorem. □

Theorem 4.2. *Suppose that* $\gamma_j \sim aj^{\alpha}$ *as* $j \to \infty$ $(0 < a < \infty,\ 2 < \alpha < \infty)$. *Then, with the conditions* (3.2)–(3.4) *on* $Q(x)$, *we have the formula*

$$\lim_{m\to\infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k) = -\frac{1}{4}[tr\, Q(0) - tr\, Q(\pi)]. \tag{4.7}$$

Proof. By Theorems 3.1 and 4.1, we write

$$\lim_{m\to\infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k) = -\frac{1}{4}[\text{tr}\, Q(0) - \text{tr}\, Q(\pi)] + \lim_{m\to\infty} D_m^{(p)}. \tag{4.8}$$

Since

$$D_m^{(p)} = \frac{(-1)^p}{2\pi i} \int\limits_{|\lambda|=b_m} \lambda \text{tr}\, [R_\lambda (Q R_\lambda^0)^{p+1}]\, \mathrm{d}\lambda$$

we can have shortly

$$|D_m^{(p)}| \leq b_m \int\limits_{|\lambda|=b_m} \|R_\lambda\|_1 \|Q\|_1^p \|R_\lambda^0\|_1^p \|Q\|_1 \|R_\lambda^0\|_{\mathbb{S}_1(H_1)}\, |\mathrm{d}\lambda|.$$

Thus, taking into account inequalities (4.4) and (4.6) we obtain

$$|D_m^{(p)}| \leq \text{const.}\, b_m^2 n_m^{-(p+1)\delta} n_m^{1-\delta}.$$

Since $b_m \leq \text{const.}\, n_m^{1+\delta}$ then we have

$$|D_m^{(p)}| \leq \text{const.}\, n_m^{-(p+2)\delta+1} n_m^{2(1+\delta)} = \text{const.}\, n_m^{3-p\delta}.$$

It follows that for $p > 3\delta^{-1}$

$$\lim_{m\to\infty} D_m^{(p)} = 0.$$

Substituting this result into Eq. (4.8) one obtains the required formula (4.7). □

5. Examples

In this section, We will consider a few examples related to our theory.

Example 1. Let $\mathcal{B} = C[0,\pi]$ where $C[0,\pi]$ is the set of continuous functions on $[0,\pi]$ and let $D(A)$ denote the set of all functions $u = u(t)$ satisfying the following conditions:

- $u(t) \in C^2[0,\pi]$,
- $u(0) = u(\pi) = 0$,
- $u''(t) \in \mathcal{B}$.

Define the operator $A : D(A) \to \mathcal{B}$ by $Au(t) = u''(t)$. It is easy to check that A is self-adjoint, with eigenvalues $\gamma_n = -n^2$ $(n = 1, 2, \dots)$ and corresponding orthonormal eigenvectors $\psi_n(t) = \sqrt{\frac{2}{\pi}} \sin nt$, $t \in [0,\pi]$. In this case, for every $y = y(x) = y(x,t) \in D_0$, the operator L_0' can be defined by:

$$L_0'(y) = -\frac{\partial^2 y(x,t)}{\partial x^2} + \frac{\partial^2 y(x,t)}{\partial t^2}.$$

Example 2. Take $\mathcal{B}$ as a continuous dense embedding in $l_2 = \{\{f_n\} : \sum_{n=1}^{\infty} |f_n|^2 < \infty\}$ and let $D(A)$ denote the set of vectors $f = \{f_n\} \in l_2$ with the condition $\sum_{n=1}^{\infty} n^2 |f_n|^2 < \infty$. Define the operator $A : D(A) \to \mathcal{B}$ by

$$Af = \{nf_n\} = \{f_1, 2f_2, 3f_3, \dots\}.$$

We can check that A is self-adjoint, with eigenvalues $\gamma_n = n$ $(n = 1, 2, \dots)$ and corresponding orthonormal eigenvectors

$$\psi_n = \{0, 0, \dots\ 0, \overbrace{\ 1\ }^{n^{\text{th}} slot}\ 0, \cdots\}, \; n \in \mathbb{N}.$$

In this case, for every $\{y_n(x)\} \in D_0$, the operator L_0' can be defined by:

$$L_0'(y) = -y''(x) + Ay(x) = -\{y_n''(x)\} + \{ny_n(x)\}.$$

Example 3. Take $\mathcal{B}_1 = S^2(\mathcal{B}; [0,\pi])$ where $\mathcal{B}$ is separable Banach space which is continuous dense embedding in a separable Hilbert space $\mathcal{H}$. Consider the operator function $Q(t) = \pi^{-1} tT$, $t \in [0,\pi]$, where for every $x \in \mathcal{B}$, $T : \mathcal{B} \longrightarrow \mathcal{B}$ is given by

$$Tx = \sum_{i=1}^{\infty} i^{-2} (x, \phi_i)_z \phi_i$$

with the o.n.b. $\{\phi_i\}_{i \geq 1}$ in $\mathcal{H}$. Here $(\cdot,\cdot)_z$ is the Zachary functional on $\mathcal{B}$. We first want to show that for every $t \in [0,\pi]$ the operator function $Q(t)$ is a trace class operator on $\mathcal{B}$. To see this, it is enough to show that T is a trace class operator. For every $x, y \in \mathcal{B}$ we have

$$\begin{aligned}(Tx, y)_z &= \left(\sum_{i=1}^{\infty} i^{-2} (x, \phi_i)_z \phi_i, y \right)_z = \sum_{i=1}^{\infty} i^{-2} (x, \phi_i)_z (\phi_i, y)_z \\ &= \sum_{i=1}^{\infty} \big(x, i^{-2}, (y, \phi_i)_z \phi_i\big)_z = \left(x, \sum_{i=1}^{\infty} i^{-2} (y, \phi_i)_z \phi_i \right)_z \\ &= (x, Ty)_z.\end{aligned}$$

Since T has eigenvalues $\lambda_i = s_i = i^{-2}$ $(i = 1, 2, 3, \ldots)$, T is a trace class operator. Moreover, $Q'(t) = \pi^{-1}T$ and $Q^{(i)}(t) = 0$ for $i \geq 2$ with respect to the norm of $\mathbb{S}_1[\mathcal{B}]$. This implies the self-adjointness of $Q^{(i)}(t)$ for $(i = 0, 1, 2, \ldots)$, that is,

$$[Q^{(i)}(t)]^* = Q^{(i)}(t) \ ; \quad (i = 0, 1, 2, \ldots).$$

Here, we can also see that Q is a self adjoint, trace class operator from $\mathcal{B}_1$ to $\mathcal{B}_1$.

Example 4. Let $\mathcal{B} = C[0, 1]$ and $\mathcal{H} = L^2[0, 1]$ and take $D(A)$ as a set of functions $\phi(t)$ satisfying the following conditions:

(i) $\phi'''(t)$ is absolutely continuous on $[0, 1]$ and $\phi^{(iv)}(t) \in C[0, 1]$,

(ii) $\phi(0) = \phi''(0) = \phi(3.1) = \phi''(3.1) = 0$.

Define the operator $A : D(A) \longrightarrow \mathcal{B}$ by $A\phi(t) = d^4\phi(t)/\mathrm{d}t^4$. Then A is an operator such that $A = A^* \geq I$ and $A^{-1} \in \mathbb{S}_\infty[\mathcal{B}]$. Its eigenvalues are of the form $\gamma_j = (j\pi)^4$ $(j = 1, 2, 3, \ldots)$,with the corresponding orthonormal eigenfunctions $\phi_j(t) = \sqrt{2} \sin j\pi t$. Consider the operator function $Q(x)$, for every $x \in [0, \pi]$, from $\mathcal{B}$ to $\mathcal{B}$ defined by

$$Q(x)\phi(t) = \cos x \int_0^1 (t + s)^2 \phi(s) \mathrm{d}s.$$

One can check that this operator satisfies all conditions in our main problem. Moreover, we notice that $\mathcal{B}_1 = S^2(\mathcal{B}; [0, \pi]) = C([0, \pi] \times [0, 1])$ as continuous dense embedding in $\mathcal{H}_1 = L^2(L^2[0, 1]; [0, \pi]) = L^2([0, \pi] \times [0, 1])$. Now, by letting $D(L_0) = D(L) = \mathcal{B}_1$ we define the self-adjoint linear operators L_0 and L, respectively, as

$$L_0 = -\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial t^4},$$
$$L = L_0 + \cos x \int_0^1 (t + s)^2 \phi(s) \mathrm{d}s$$

with the identity boundary conditions

$$u_x(0, t) = u'_x(0, t) = 0,$$
$$u(x, 0) = u''_{tt}(x, 0) = u(x, 1) = u''_{tt}(x, 1) = 0.$$

By applying our current theory here, we find the right side of the regularized trace formula to be zero.

6. Conclusions

In this paper, we have introduced a new tool for the study of differential equations on Banach spaces. In particular, we have extended the study and computation of regularized trace formulas to Banach spaces. We have also provided a few interesting examples to show our approach works in the new setting.

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