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MANUSCRIPT

Simultaneous Development of Shocks and Cusps for 2D Euler with Azimuthal Symmetry from Smooth Data

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Abstract

A fundamental question in fluid dynamics concerns the formation of discontinuous shock waves from smooth initial data. We prove that from smooth initial data, smooth solutions to the 2d Euler equations in azimuthal symmetry form a first singularity, the so-called $C^{\frac{1}{3}}$ *pre-shock*. The solution in the vicinity of this pre-shock is shown to have a fractional series expansion with coefficients computed from the data. Using this precise description of the pre-shock, we prove that a *discontinuous shock* instantaneously develops after the pre-shock. This *regular shock solution* is shown to be unique in a class of entropy solutions with azimuthal symmetry and regularity determined by the pre-shock expansion. Simultaneous to the development of the shock front, two other characteristic surfaces of cusp-type singularities emerge from the pre-shock. These surfaces have been termed *weak discontinuities* by Landau & Lifschitz [12, Chapter IX, §96], who conjectured some type of singular behavior of derivatives along such surfaces. We prove that along the slowest surface, all fluid variables except the entropy have $C^{1,\frac{1}{2}}$ one-sided cusps from the shock side, and that the normal velocity is decreasing in the direction of its motion; we thus term this surface a *weak rarefaction wave*. Along the surface moving with the fluid velocity, density and entropy form $C^{1,\frac{1}{2}}$

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one-sided cusps while the pressure and normal velocity remain C^2; as such, we term this surface a *weak contact discontinuity*.

Keywords Shock formation · Shock development · Weak contact · Weak rarefaction · Pre-shock · Compressible euler

Contents

1 Introduction

We consider the simultaneous development of *shock waves* and *weak singularities* (contact and rarefaction cusps) from smooth initial data, for the two-dimensional compressible Euler equations in azimuthal symmetry. This problem consists of:

- the *shock formation* process, in which we start from smooth initial data and construct the first singularity, the so-called *pre-shock*;
- the *shock development* process, in which the pre-shock instantaneously evolves into a discontinuous entropy producing shock wave, and two other families of weak characteristic singularities (cusps).

1.1 The Compressible Euler Equations

For shock development, it is essential to write the Euler equations in conservation form, so as to ensure the physical jump conditions (conserving total mass, momentum and energy) are satisfied. The system reads

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + pI) = 0\,, \tag{1.1a}$$

$$\partial_t \rho + \operatorname{div}(\rho u) = 0\,, \tag{1.1b}$$

$$\partial_t E + \operatorname{div}((p+E)u) = 0\,, \tag{1.1c}$$

where $u : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ denotes the velocity vector field, $\rho : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}_+$ denotes the strictly positive density, $E : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ denotes the total energy, and $p : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ denotes the pressure function which is related to (u, ρ, E) by the identity

$$p = (\gamma - 1)\left(E - \tfrac{1}{2}\rho\, |u|^2\right)\,,$$

where $\gamma > 1$ denotes the adiabatic exponent. For smooth solutions, the conservation of energy equation (1.1c) can be replaced by the transport of (specific) entropy $\partial_t S + u \cdot \nabla S = 0$, where $S : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ denotes the entropy function, and the pressure has the equivalent form

$$p(\rho, S) = \tfrac{1}{\gamma}\rho^\gamma e^S\,. \tag{1.2}$$

We consider solutions to the Euler equations (1.1) which start from smooth *non-degenerate* initial data at time T_0, form a first singularity or *pre-shock* at time T_1, and simultaneously develop a discontinuous shock wave and surfaces of weak characteristic discontinuities on the time interval $(T_1, T_2]$. Solutions on the time interval $[T_0, T_1)$ are *classical solutions* to (1.1), and only the continuation of these solutions past T_1 requires the introduction of the Rankine-Hugoniot jump conditions.

Suppose that for $t \in (T_1, T_2]$, the shock front $\mathcal{S} \subset \mathbb{R}^d \times (T_1, T_2]$ is an orientable space-time hypersurface across which the velocity $u^\pm$, density $\rho^\pm$, and energy $E^\pm$ jump. We consider the case where this surface is given by $\mathcal{S} := \{\mathfrak{s}(t, x_1, x_2, \ldots x_d) =$

$0\}$ with spacetime normal $-(\dot{\mathfrak{s}}, \nabla_x \mathfrak{s})|_{\mathcal{S}} := (-\dot{\mathfrak{s}}, n)$. We assume that $(u^\pm, \rho^\pm, E^\pm)$ are defined in the sets $\Omega^\pm(t) \subset \mathbb{R}^2$ separated by the shock front at time t. Let $n(\cdot, t)$ point from $\Omega^-(t)$ to $\Omega^+(t)$, which is in the direction of propagation of the shock front. In two-dimensions, we let $\tau(\cdot, t) = n(\cdot, t)^\perp$ denote the tangent vector. We denote $[\![f]\!] = f^- - f^+$ where $f^\pm$ (sometimes denoted $f_\pm$) are the traces of f along $\mathcal{S}$ in the regions $\Omega^\pm$ respectively, and $u_n = u \cdot n|n|^{-1}$, $u_\tau = u \cdot \tau|\tau|^{-1}$. The shock speed is denoted by $\dot{\mathfrak{s}}$. The Rankine-Hugoniot jump conditions state that the shock speed $\dot{\mathfrak{s}}$ along with the jumps of the fields across $\mathcal{S}$ must simultaneously satisfy

$$\dot{\mathfrak{s}}|n|^{-1}[\![\rho u_n]\!] = [\![\rho u_n^2 + pI]\!], \tag{1.3a}$$

$$\dot{\mathfrak{s}}|n|^{-1}[\![\rho]\!] = [\![\rho u_n]\!], \tag{1.3b}$$

$$\dot{\mathfrak{s}}|n|^{-1}[\![E]\!] = [\![(p + E)u_n]\!], \tag{1.3c}$$

where we have used $[\![u_\tau]\!] = 0$ for a shock discontinuity.

Definition 1.1 (*Regular shock solution*) We say that $(u, \rho, E, \mathfrak{s})$ is a *regular shock solution* on $\mathbb{R}^d \times [T_1, T_2]$ if the following conditions hold:

(i) (u, ρ, E) is a weak solution of (1.1) and $\rho \geqslant \rho_{\min} > 0$;
(ii) the shock front $\mathcal{S} \subset \mathbb{R}^d \times \mathbb{R}_+$ is an orientable hypersurface;
(iii) (u, ρ, E) are Lipschitz continuous in space and time on the complement of the shock surface $(\mathbb{R}^d \times [T_1, T_2]) \setminus \mathcal{S}$;
(iv) (u, ρ, E) have discontinuities across the shock which satisfy the Rankine-Hugoniot conditions (1.3).

Furthermore, the solution has a *weak shock* if

$$\sup_{t\in[T_1,T_2]} \left(|[\![u(t)]\!]| + |[\![\rho(t)]\!]| + |[\![E(t)]\!]|\right) \ll 1\,.$$

1.2 Prior Results in Shock Development Problem for Euler

For hyperbolic systems in one space dimension, existence (and in some cases uniqueness) of global weak solutions is well understood using either the Glimm scheme or compensated compactness techniques (see e.g. [8]). Unfortunately, these methods cannot provide a description of the surfaces across which weak and strong singularities propagate. In multiple space dimensions, Majda [14, 15] establishes the short-time evolution (and stability) of a shock front. This is a free-boundary problem in which the parameterized shock surface moves with the shock speed given by the Rankine-Hugoniot conditions. In this problem, the initial data consists of a shock surface and discontinuous (u, ρ, E) which are smooth on either side of the shock. As such, this framework does not include the shock development problem, in which the surface of discontinuity must evolve from a Hölder pre-shock.

There are very few results on the formation and development of shocks. For the one-dimensional p-system (which models 1d isentropic Euler), Lebaud [13] was the first to prove shock formation and development. Following [13], Chen & Dong [4] and

Kong [11] also proved formation and development of shocks for the 1d p-system with slightly more general initial data. However, because entropy is created at the shock, the use of the isentropic 2×2 p-system cannot produce weak solutions to the 1d Euler equations.[1] Yin [17] was the first to consider the formation and development problem for the non-isentropic 3×3 Euler equations in spherical symmetry. Independently, shock development for the barotropic Euler equations under spherical symmetry was established by Christodoulou & Lisbach [7]. The use of the isentropic model or the assumption of an irrotational flow in higher dimensions cannot produce weak solutions to the Euler equations, and as such has been termed the *restricted shock development*. Christodoulou [6] has established restricted shock development for the irrotational and isentropic Euler equations in three spatial dimensions and completely outside of symmetry. Yin & Zhu [18] have recently established shock development in two dimensions for a scalar conservation law.

As previously noted by Landau & Lifschitz in [12, Chapter IX, §96], at the same time that the discontinuous shock wave develops, other surfaces of singularities are expected to simultaneously form. Landau & Lifschitz termed these surfaces *weak discontinuities*. In the restricted shock development problem, Christodoulou [6, Page 3] constructs $C^{1,\frac{1}{2}}$ cusp singularities along the characteristic of the fluid velocity minus the sound speed, emanating from the first singularity (akin to the $\mathfrak{s}_1$ curve in Theorem 3.2). For the full Euler system (with or without symmetry, even in one dimension) the analysis of these surfaces of weak discontinuity has been heretofore nonexistent. In this paper we prove that two such surfaces of weak singularities emerge from the pre-shock and move with the slower sound-speed characteristic and the fluid velocity respectively. We shall refer to these two surfaces as a *weak contact* ($\mathfrak{s}_2$), respectively a *weak rarefaction* ($\mathfrak{s}_1$). We call the curve $\mathfrak{s}_2$ a weak contact because it moves with the fluid velocity, and both the normal velocity and the pressure are one degree smoother than the density and entropy. The curve $\mathfrak{s}_1$ is called a weak rarefaction because the normal velocity to this curve is decreasing in the direction of its motion – see Section 7.

1.3 Statement of the Main Results

The goal of this paper is to prove the following (we refer to Theorems 7.1 and 7.2 for a precise statement):

Theorem 1.2 (Main result for 2D Euler – abbreviated version) *From smooth isentropic initial data with azimuthal symmetry, at time T_0, there exist smooth solutions to the 2d Euler equations (1.1) that form a pre-shock singularity at a time $T_1 > T_0$. The first singularity occurs along a half-infinite ray and the blowup is asymptotically self-similar, exhibiting a $C^{\frac{1}{3}}$ cusp in the angular velocity and mass density, and a $C^{1,\frac{1}{3}}$ cusp in the radial velocity. Moreover, the blowup is given by a series expansion whose coefficients are computed as a function of the initial data.*

[1] We emphasize that the Rankine-Hugoniot jump conditions are not satisfied under the isentropic assumption, see Lemma 2.1.

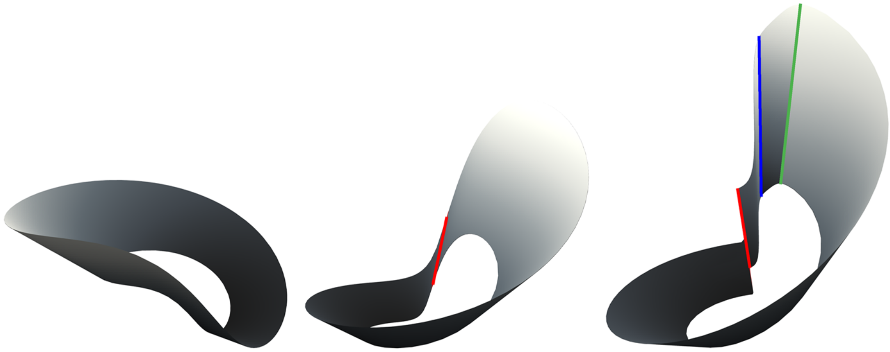

Fig. 1 The images represent values of the density written in polar coordinates $\rho(r, \theta, t)$, and plotted for $r \in [1, 2]$. The image on the left represents the smooth data at time T_0. The center image shows the pre-shock formed at time T_1, at one specific value of the angular coordinate; we marked the corresponding line in red. The image on the right represents the density at time T_2, where we have represented in red the line along which the shock discontinuity occurs, in blue the line containing the weak contact, and in green the line corresponding to the weak rarefaction

Past the pre-shock, the solution is continued on $(T_1, T_2]$, *as an entropy–producing regular shock solution of the full 2d non-isentropic Euler equations (1.1). The solution is unique in the class of entropy producing weak solutions with azimuthal symmetry, with a certain weak shock structure and suitable regularity off the shock (see Definition 5.3 below). The following properties are established:*

- *Across the shock curve, all the state variables jump:*

$$[\![u_\theta]\!] \sim (t - T_1)^{\frac{1}{2}}, \qquad [\![\rho]\!] \sim (t - T_1)^{\frac{1}{2}},$$
$$[\![\partial_\theta u_r]\!] \sim (t - T_1)^{\frac{1}{2}}, \qquad [\![S]\!] \sim (t - T_1)^{\frac{3}{2}}.$$

- *Across the characteristic emanating from the pre-shock and moving with the fluid velocity, the entropy, density and radial velocity all have a* $C^{1,\frac{1}{2}}$ *one-sided cusp from the right, while from the left, they are all* C^2 *smooth. The second derivative of the angular velocity and of the pressure is bounded across this curve for* $t \in (T_1, T_2]$.
- *Across the characteristic emanating from the pre-shock and moving with sound speed minus the fluid velocity, the entropy is zero while the angular velocity and density have* $C^{1,\frac{1}{2}}$ *one-sided cusps from the right, while from the left, they are all* C^2 *smooth. The second derivative of the radial velocity is bounded across this curve for* $t \in (T_1, T_2]$.

We thereby obtain a full propagation of singularities result for regular shock solutions, capturing both the jump discontinuity and the weak singularities emanating from the initial cusp in the pre-shock (Fig. 1).

Remark 1.3 (*Anomalous entropy production*) In analogy with Onsager's conjecture on anomalous dissipation of kinetic energy by weak solutions of incompressible

Euler, entropy can be anomalously produced by singular inviscid solutions of the compressible Euler equations. Theorem 3 of [9] establishes the following L^3-based Onsager-criterion: if $u, \rho, E \in L^\infty(0, T; (B^{1/3+}_{3,\infty} \cap L^\infty)_{\mathrm{loc}}(\mathbb{R}^d))$ then there is no entropy production. Our Theorem 1.2 provides an example of an entropy producing weak solution resulting from continuing past a finite time singularity. In fact, the solution we construct lies in $u, \rho, E \in (BV \cap L^\infty)_{\mathrm{loc}} \subset (B^{1/p}_{p,\infty})_{\mathrm{loc}}$, for every $p \geqslant 1$, illustrating the sharpness of the Onsager criterion in this context.

Remark 1.4 (*Uniqueness and entropy*) With regards to the question of uniqueness, the recent work [10] established that infinitely many entropy-producing weak solutions emanating from 1d Riemann data exist (see also the references therein for the rich history of such convex-integration constructions going back to [5]). The solutions in [10] break the 1d symmetry and are in general just bounded, and show that the usual entropy condition cannot ensure uniqueness in the class of bounded weak Euler solutions. By contrast, we establish uniqueness in a class of weak solutions with azimuthal symmetry, exhibiting *weak shock structure*, and which have regularity consistent with the fact that they emanate from a $C^{\frac{1}{3}}$ pre-shock (see Definition 5.3).

2 Jump Conditions and Entropy Conditions

2.1 The Rankine-Hugoniot Jump Conditions for the Euler Equations

We now return to the Rankine-Hugoniot conditions (1.3). The weak shock regime is relevant to the development of a discontinuous shock wave from Hölder continuous data (the pre-shock). A key feature of a regular shock solution to the Euler equations is the production of entropy along the shock surface.

In order to best exemplify this entropy production, we shall set

$$S_+ = 0\,. \tag{2.1}$$

We then define

$$v = u_n - \dot{\mathfrak{s}}\,. \tag{2.2}$$

Then noting that $u_n^2 = (v + \dot{\mathfrak{s}})^2 = v^2 + 2\dot{\mathfrak{s}}v + \dot{\mathfrak{s}}^2$ and $\dot{\mathfrak{s}}u_n = \dot{\mathfrak{s}}v + \dot{\mathfrak{s}}^2$, the jump conditions (1.3) become

$$0 = [\![\rho v^2 + p]\!]\,, \tag{2.3a}$$

$$0 = [\![\rho v]\!]\,, \tag{2.3b}$$

$$0 = [\![v^2 + \tfrac{2\gamma}{\gamma-1}\tfrac{p}{\rho}]\!]\,, \tag{2.3c}$$

From (2.3b), we know that the mass flux is continuous $\rho_- v_- = \rho_+ v_+ =: j$. For a shock discontinuity $j \neq 0$ implying the tangential velocity is continuous across the shock $[\![u_\tau]\!] = 0$. In our setup, mass is crossing the shock from the '+' phase to the

'−' phase, so the shock is traveling from '−' to '+'. With our choice of orientation for the normal, this fixes $j < 0$, which implies that

$$u^- \cdot n < \dot{\mathfrak{s}}, \qquad u^+ \cdot n < \dot{\mathfrak{s}}. \tag{2.4}$$

Thus, the shock speed is greater than the normal velocity of the fluid on both sides of the shock, consistent with that mass flux being negative $j < 0$. We will refer to '−' state as behind the shock and the '+' state as the front.

2.2 Second Law of Thermodynamics and the Physical Entropy Condition

We now explain the meaning and consequences of the physical entropy condition. The motion of a viscous compressible fluid in d-spatial dimensions is, to good approximation, governed by the Navier-Stokes system. In that system, any non-trivial state has the property that net entropy is increasing

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho S \,\mathrm{d}x > 0, \tag{2.5}$$

provided u is tangent to Ω (and the boundaries $\partial\Omega$ are insulating if the thermal diffusivity is non-vanishing). Namely, the second law of thermodynamics holds. For the Euler equations, the entropy satisfies

$$\partial_t(\rho S) + \nabla\cdot(\rho u S) = 0 \tag{2.6}$$

and is thus has conserved average for smooth solutions. We recall here the following classical result

Lemma 2.1 *Let (u, ρ, E) be a weak shock solution. Then, entropy is produced (2.5) if and only if $[\![S]\!] > 0$. Moreover, provided that the specific volume $V := 1/\rho$ and enthalpy $h = \frac{p}{\rho} + e$ when viewed as a functions of pressure and entropy are C^4, then the following leading order description of the entropy jump holds*

$$[\![S]\!] = \frac{1}{12}\frac{1}{T_+}\left(\frac{\partial^2 V}{\partial p_+^2}\right)\bigg|_S [\![p]\!]^3 + \mathcal{O}([\![p]\!]^4). \tag{2.7}$$

The notation "$f(x) = \mathcal{O}(x)$" means, as usual, $|f(x)| \leqslant (\text{const.})|x|$ for all sufficiently small x. An immediate implication of equation (2.7) is that entropy variation is produced once a shock is formed, even if the flow was initially isentropic.

Remark 2.2 (*Equations of State*) Although we only require finite regularity in Lemma 2.1, away from phase transitions, all thermodynamic functions are smooth in their arguments. Thus, the specific volume $V := V(\rho, S)$ and the enthalpy $h := h(\rho, S)$ which are used in the subsequent proof are smooth functions of p and S. As such, our assumption physically is that our medium is far from criticality. Moreover, strict convexity $\partial^2 V/\partial p^2 > 0$ is a material property. For example, for a ideal gas (the family we consider) we have explicitly

$$\left(\frac{\partial^2 V}{\partial p^2}\right)\bigg|_S = (1+\gamma^{-1})\frac{V^2}{p} > 0 \tag{2.8}$$

which can be obtained by differentiating the relationship $pV^\gamma = (\text{const.})$ (equation (1.2)). The thermodynamic temperature appearing in (2.7) can also be explicitly related to ρ and p in this setting. Specifically, for the ideal gas law $T = e/c_v$ where the internal energy $e = \frac{p}{(\gamma-1)\rho}$, we have the following explicit formula $\frac{1}{T} = \frac{c_v(\gamma-1)\rho}{p}$.

Remark 2.3 (*Correlations in jumps*) One consequence of Lemma 2.1 is that, if V is a strictly convex function of the pressure (as it is for the ideal gas), then positive entropy production implies positivity of the jumps $[\![p]\!] > 0$, $[\![\rho]\!] > 0$ and $[\![u_n]\!] > 0$. This conclusion is simply the well known fact that pressure and mass density trailing the shock exceed their values at the front, due to compression. See Landau and Lifshitz [12], Chapter IX for an extended discussion.

Proof of Lemma 2.1 Integrating the entropy balance (2.6) over the domain one finds

$$\dot{\mathfrak{s}}[\![\rho S]\!] - [\![\rho u S]\!] = -j[\![S]\!]\,. \tag{2.9}$$

Thus we must have $-j[\![S]\!] > 0$, to be consistent with the second law of thermodynamics (2.5) imposed, for example, by the effects of infinitesimal viscosity. Recalling that, with our conventions the mass flux $j = \rho v$ is negative (mass is passing the shock from $+$ to $-$), we see that the *physical entropy condition* (2.9) is equivalent to the condition

$$[\![S]\!] > 0\,. \tag{2.10}$$

We now derive the consequences of (2.10) for *weak shocks*. In what follows, we will show that $[\![S]\!] = \mathcal{O}([\![p]\!]^3)$. In the calculations below, we anticipate this result in our expansions. It is convenient to work with the enthalpy $h = \frac{p}{\rho} + e$. We regard $h = h(p, S)$ and Taylor expand to obtain

$$\begin{aligned}[\![h]\!] &= \left(\frac{\partial h}{\partial S_+}\right)\bigg|_p [\![S]\!] + \left(\frac{\partial h}{\partial p_+}\right)\bigg|_S [\![p]\!] \\ &\quad + \frac{1}{2}\left(\frac{\partial^2 h}{\partial p_+^2}\right)\bigg|_S [\![p]\!]^2 + \frac{1}{6}\left(\frac{\partial^3 h}{\partial p_+^3}\right)\bigg|_S [\![p]\!]^3 + \mathcal{O}([\![S]\!][\![p]\!], [\![p]\!]^4, [\![S]\!]^2).\end{aligned}$$

Recalling the first law of thermodynamics in the form

$$\mathrm{d}h = T\mathrm{d}S + V\mathrm{d}p, \tag{2.11}$$

where $V := 1/\rho$ is the specific volume, we find that

$$\left(\frac{\partial h}{\partial S}\right)\bigg|_p = T, \qquad \left(\frac{\partial h}{\partial p}\right)\bigg|_S = V. \tag{2.12}$$

Thus, the Taylor expansion of the enthalpy becomes

$$\begin{aligned}[\![h]\!] = T_+[\![S]\!] + V_-[\![p]\!] + \frac{1}{2}\left(\frac{\partial V}{\partial p_+}\right)\bigg|_S [\![p]\!]^2 + \frac{1}{6}\left(\frac{\partial^2 V}{\partial p_+^2}\right)\bigg|_S [\![p]\!]^3 \\ + \mathcal{O}([\![S]\!][\![p]\!], [\![p]\!]^4, [\![S]\!]^2).\end{aligned} \tag{2.13}$$

Recalling that the mass flux j is continuous across the shock, we note that by (2.3a) that

$$[\![p]\!] = -[\![\rho v^2]\!] = -j[\![v]\!] = -j^2[\![V]\!], \tag{2.14}$$

which implies $j^2 = -[\![p]\!]/[\![V]\!]$. Moreover, from (2.3c), we have

$$[\![h]\!] = -[\![\tfrac{1}{2}v^2]\!] = -\frac{j^2}{2}[\![V^2]\!] = \frac{1}{2}[\![p]\!]\frac{[\![V^2]\!]}{[\![V]\!]} = [\![p]\!]V_{\text{ave}}, \tag{2.15}$$

where $V_{\text{ave}} = \frac{1}{2}(V_- + V_+)$. Combining with (2.13), after some manipulation we find

$$\begin{aligned}T_+[\![S]\!] = \tfrac{1}{2}[\![V]\!][\![p]\!] - \tfrac{1}{2}\left(\tfrac{\partial V}{\partial p_+}\right)\bigg|_S [\![p]\!]^2 - \tfrac{1}{6}\left(\tfrac{\partial^2 V}{\partial p_+^2}\right)\bigg|_S [\![p]\!]^3 \\ + \mathcal{O}([\![S]\!][\![p]\!], [\![p]\!]^4, [\![S]\!]^2).\end{aligned} \tag{2.16}$$

Finally, Taylor expanding the specific volume yields

$$[\![V]\!] = \left(\frac{\partial V}{\partial p_+}\right)\bigg|_S [\![p]\!] + \frac{1}{2}\left(\frac{\partial^2 V}{\partial p_+^2}\right)\bigg|_S [\![p]\!]^2 + \mathcal{O}([\![p]\!]^3, [\![S]\!]). \tag{2.17}$$

Upon substitution into (2.16), we obtain the relation (2.7). Note that provided $\partial^2 V/\partial p^2 > 0$, for weak shocks $[\![p]\!] \ll 1$, equation (2.7) shows that $[\![p]\!] > 0$. Hence, by (2.14), we have $[\![\rho]\!] > 0$ and $[\![u_n]\!] > 0$. □

2.3 Lax Geometric Entropy Conditions and Determinism of Shock Development

In this section, we show that the entropy condition implies that the shock discontinuity is supersonic relative to the state ahead ('+' phase) and subsonic relative to the state behind ('−' phase)

$$u^+ \cdot n + c^+ < \dot{\mathfrak{s}} < u^- \cdot n + c^-, \tag{2.18}$$

where c^- and c^+ are the sound speeds behind and at the front of the shock. In this way, the $\{t = 0\}$ hypersurface is the Cauchy surface for the state ahead (+) whereas $\{t = 0\}$ together with the shock front serve as the Cauchy surface for the state behind (−). The region behind the shock is thus determined by the initial conditions together with data along the shock front which are determined by enforcing Rankine-Hugoniot conditions.

Equations (2.18) (together with (2.4)) are called *Lax's geometric entropy conditions*. We now show that the Lax geometric entropy conditions are *equivalent* to the physical entropy condition (2.10), at least for weak shocks.

Lemma 2.4 *In the setting of Lemma 2.1, the physical entropy condition (2.5) holds if and only if the geometric Lax entropy conditions (2.18) and (2.4) hold.*

Proof of Lemma 2.4 Conditions (2.4) hold since the mass flux $j < 0$. Using $u_\pm \cdot n - \dot{\mathfrak{s}} = jV^\pm$, (2.18) becomes

$$\tfrac{c^+}{V^+} < -j < \tfrac{c^-}{V^-}. \tag{2.19}$$

Thus, when the jump conditions, the Lax geometric conditions hold provided

$$[\![c/V]\!] > 0. \tag{2.20}$$

We now show how this is implied by $[\![S]\!] > 0$ in the weak shock regime. Letting $w := V^2/c^2$, we have

$$[\![w]\!] = \left(\tfrac{1}{(c/V)^-} + \tfrac{1}{(c/V)^+}\right)[\![V/c]\!] = -\left(\tfrac{1}{(c/V)^-} + \tfrac{1}{(c/V)^+}\right)\tfrac{[\![c/V]\!]}{(c/V)^-(c/V)^+}. \tag{2.21}$$

Thus, verifying condition (2.20) and thus (2.19) is equivalent to showing $[\![w]\!] < 0$. To verify this note first, that viewing $\rho := \rho(p, S)$, as an application of the chain rule we have

$$\frac{1}{c^2} = \left(\frac{\partial\rho}{\partial p}\right)\bigg|_S = -\frac{1}{V^2}\left(\frac{\partial V}{\partial p}\right)\bigg|_S, \tag{2.22}$$

which yields $w = -\left(\frac{\partial V}{\partial p}\right)\Big|_S$. Appealing to the leading order entropy jump (2.7) of Lemma 2.1, we obtain

$$[\![w]\!] = -\left(\frac{\partial^2 V}{\partial p_+^2}\right)\bigg|_S [\![p]\!] + \mathcal{O}([\![p]\!]^2) = -\frac{12T_+}{[\![p]\!]^2}[\![S]\!] + \mathcal{O}([\![p]\!]^2). \tag{2.23}$$

Thus, we see that $[\![S]\!] > 0$ if and only if $[\![w]\!] < 0$ which in turn implies the Lax conditions (2.18), (2.4). □

Remark 2.5 (*Determinism of shock development and entropy conditions*) We now discuss an interpretation of the Lax geometric inequalities as they pertain to the issue of determinism of the shock development problem. To simplify ideas, we specialize to 1D setting in which the spacetime shock curve is given by $\{x = \mathfrak{s}(t)\}$. The spacetime normal to the shock curve is $\mathfrak{n} = (-\dot{\mathfrak{s}}, 1)$. With the notation $\nabla_{t,x} = (\partial_t, \partial_x)$, the transport operators for the Riemann invariants are

$$(1, u - c) \cdot \nabla_{t,x}, \qquad (1, u) \cdot \nabla_{t,x}, \qquad (1, u + c) \cdot \nabla_{t,x}. \tag{2.24}$$

See equations (3.5b), (3.5c) and (3.5a) respectively. In the front of the shock (+ phase), the Lax inequalities (2.18) read

$$u^+ - c^+ < \dot{\mathfrak{s}}, \qquad u^+ < \dot{\mathfrak{s}}, \qquad u^+ + c^+ < \dot{\mathfrak{s}} \tag{2.25}$$

all of which follow directly using the fact that the sounds speed is positive. Geometrically, these translate to

$$\mathfrak{n} \cdot (1, u^+ - c^+) < 0, \qquad \mathfrak{n} \cdot (1, u^+) < 0, \qquad \mathfrak{n} \cdot (1, u^+ + c^+) < 0, \tag{2.26}$$

showing that all the associated characteristics in front of the shock (+ phase) impinge on the shock front, carrying with they Cauchy data from the $\{t = 0\}$ hypersurface. This ensures that the front of the shock is causally isolated from shock and determined solely from initial conditions. On the other hand, behind the shock (− phase) we have from (2.18) and (2.4) that

$$u^- - c^- < \dot{\mathfrak{s}}, \qquad u^- < \dot{\mathfrak{s}}, \qquad u^- + c^- > \dot{\mathfrak{s}} \tag{2.27}$$

which has the geometric meaning of

$$\mathfrak{n} \cdot (1, u^- - c^-) < 0, \qquad \mathfrak{n} \cdot (1, u^-) < 0, \qquad \mathfrak{n} \cdot (1, u^- + c^-) > 0. \tag{2.28}$$

Unlike the situation in the + phase, we see that two of the characteristics corresponding to wave speeds $u^- - c^-$ and u^- are “exiting the shock", carrying with them data from along shock hypersurface. Only one of the characteristics corresponding to $u^- + c^-$ is impinging on the surface, carrying Cauchy data from $\{t = 0\}$. The significance of this is the following: the data *along the shock front* for the Riemann invariants carried by characteristics leaving the shock are free and will be chosen to enforce two out of the three jump conditions for mass, momentum and energy. The third invariant whose characteristics impinge on the shock enjoys no such freedom – rather the speed of the shock will be designed to arrange for the last jump condition to be satisfied. Simultaneously ensuring these constraints hold define a free boundary problem for the shock development. If additional characteristics were to lack this freedom, the problem would become overdetermined and no solution could be found in general. As such, the entropy condition is precisely what is required for the shock development problem to be “deterministic”.

Remark 2.6 (*Shock speed near formation*) From the Rankine-Hugoniot conditions, it follows that the rate of propagation of weak shock waves (relative to the fluid) is the sound speed, $\dot{\mathfrak{s}} \approx u_n + c$. This follows from the fact that, at the pre-shock, $v_- = v_+$ so

$$v_- = v_+ = v = jV = -\sqrt{-V^2(\partial p/\partial V)|_S} = -\sqrt{(\partial p/\partial \rho)|_S} = -c, \quad (2.29)$$

which follows from the identity $j^2 = -[\![p]\!]/[\![V]\!]$. Since $\dot{\mathfrak{s}} = u_n - v$, the claim follows.

2.4 The Euler System in Terms of Entropy, Velocity, and Sound Speed

In preparation for reducing the equations to a symmetry class and deriving equations of motion for the Riemann variables, we reformulate the two-dimensional non-isentropic compressible Euler equations. First, for classical solutions the energy equation can be replaced by the transport of entropy

$$\partial_t(\rho u) + \operatorname{div}(\rho\, u \otimes u) + \nabla p(\rho, S) = 0\,, \quad (2.30a)$$
$$\partial_t \rho + \operatorname{div}(\rho u) = 0\,, \quad (2.30b)$$
$$\partial_t S + u \cdot \nabla S = 0\,, \quad (2.30c)$$

where $S : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is the (specific) entropy. If the initial entropy is chosen to be a constant $S_0 \in \mathbb{R}$, then the entropy function satisfies $S(\cdot, t) = S_0$ as long as the solution remains smooth. The formulation of Euler given in (2.30) is equivalent to the usual conservation law form (see (1.1)) up to the pre-shock, and will be used for the shock formation process.

We introduce the adiabatic exponent

$$\alpha = \tfrac{\gamma-1}{2}$$

so that the (rescaled) sound speed reads

$$\sigma = \tfrac{1}{\alpha}\sqrt{\partial p/\partial \rho} = \tfrac{1}{\alpha} e^{\frac{S}{2}} \rho^\alpha\,. \quad (2.31)$$

With this notation, the ideal gas equation of state (1.2) becomes

$$p = \tfrac{\alpha^2}{\gamma}\rho\sigma^2\,. \quad (2.32)$$

The Euler equations (2.30) as a system for (u, σ, S) are then given by

$$\partial_t u + (u \cdot \nabla)u + \alpha\sigma\nabla\sigma = \tfrac{\alpha}{2\gamma}\sigma^2\nabla S\,, \quad (2.33a)$$
$$\partial_t \sigma + (u \cdot \nabla)\sigma + \alpha\sigma \operatorname{div} u = 0\,, \quad (2.33b)$$
$$\partial_t S + (u \cdot \nabla)S = 0\,. \quad (2.33c)$$

We let $\omega = \nabla^\perp \cdot u$ denote the scalar vorticity, and define the *specific vorticity* by $\zeta = \frac{\omega}{\rho}$. A straightforward computation shows that ζ is a solution to

$$\partial_t \zeta + (u \cdot \nabla)\zeta = \tfrac{\alpha}{\gamma}\tfrac{\sigma}{\rho}\nabla^\perp \sigma \cdot \nabla S \,. \tag{2.34}$$

The term term $\frac{\alpha}{\gamma}\frac{\sigma}{\rho}\nabla^\perp\sigma \cdot \nabla S$ on the right side of (2.34) can also be written as $\rho^{-3}\nabla^\perp\rho \cdot \nabla p$ and is referred to as *baroclinic torque*.

2.5 Jump Formulas for Ideal Gas Equation of State

In this section, we perform some manipulations of the Rankine-Hugoniot conditions (1.3a)–(1.3c) which will be used later in the paper. Combining (2.3) together with (2.1), we find that

$$[\![p]\!] = -\tfrac{2\rho_+^\gamma}{(\gamma-1)\rho_- - (\gamma+1)\rho_+}[\![\rho]\!] \,. \tag{2.35}$$

We can also compute the jump in pressure as

$$[\![p]\!] = \tfrac{1}{\gamma}(e^{S_-} - 1)\rho_-^\gamma + \tfrac{1}{\gamma}[\![\rho^\gamma]\!] \,. \tag{2.36}$$

Equating (2.35) and (2.36), we see that

$$\rho_-^\gamma(e^{S_-} - 1) = -\tfrac{2\gamma\rho_+^\gamma}{(\gamma-1)\rho_- - (\gamma+1)\rho_+}[\![\rho]\!] - [\![\rho^\gamma]\!] \,, \tag{2.37}$$

where we recall that $S_- = [\![S]\!]$. In order to simplify (2.37), we introduce

$$Q = \frac{\rho_+}{\rho_-} \tag{2.38}$$

which we expect to be close to 1 on the shock curve, for a short time after the pre-shock. Then, (2.37) reads

$$e^{S_-} - 1 = \tfrac{(Q-1)^3}{(\gamma-1)-(\gamma+1)Q}\left(\tfrac{\gamma(\gamma-1)(1+\gamma)}{6} - (Q-1)B_\gamma(Q)\right) \,, \tag{2.39}$$

where $B_\gamma(Q)$ is a smooth function in the neighborhood of $Q = 1$, with $B_\gamma(1) = \frac{1}{12}(\gamma-2)(\gamma-1)\gamma(\gamma+1)$ and $B_\gamma'(1) = \frac{-1}{40}(\gamma-3)(\gamma-2)(\gamma-1)\gamma(\gamma+1)$.

When $\gamma = 2$ and $\alpha = \frac{1}{2}$, the above formulae simplify. First we note that $B_2(Q) = 0$ for all Q, and in that case, (2.39) becomes

$$e^{S_-} - 1 = \frac{(Q-1)^3}{1-3Q} = \frac{[\![\rho]\!]^3}{\rho_-^2(3\rho_+ - \rho_-)} \,. \tag{2.40}$$

From (2.31) and the fact that $S_+ = 0$, we have that

$$\rho_- = \tfrac{1}{4}\sigma_-^2 e^{-S_-} \,, \qquad \rho_+ = \tfrac{1}{4}\sigma_+^2 \,,$$

from which it follows that

$$[\![\rho]\!] = \tfrac{1}{4}e^{-S_-}\left(\sigma_-^2 - e^{S_-}\sigma_+^2\right).$$

This allows (2.40) to be rewritten as

$$(e^{S_-} - 1)\sigma_-^4(3\sigma_+^2 e^{S_-} - \sigma_-^2) = \left(\sigma_-^2 - e^{S_-}\sigma_+^2\right)^3. \tag{2.41}$$

3 Azimuthal Symmetry

3.1 The Euler Equations in Polar Coordinates and Azimuthal Symmetry

The 2D Euler equations (2.33) take the following form in polar coordinates for the variables (u_θ, u_r, ρ, S):

$$\left(\partial_t + u_r\partial_r + \tfrac{1}{r}u_\theta\partial_\theta\right)u_r - \tfrac{1}{r}u_\theta^2 + \alpha\sigma\partial_r\sigma = \tfrac{\alpha}{2\gamma}\sigma^2\partial_r S\,, \tag{3.1a}$$

$$\left(\partial_t + u_r\partial_r + \tfrac{1}{r}u_\theta\partial_\theta\right)u_\theta + \tfrac{1}{r}u_r u_\theta + \alpha\tfrac{\sigma}{r}\partial_\theta\sigma = \tfrac{\alpha}{2\gamma}\tfrac{\sigma^2}{r}\partial_\theta S\,, \tag{3.1b}$$

$$\left(\partial_t + u_r\partial_r + \tfrac{1}{r}u_\theta\partial_\theta\right)\sigma + \alpha\sigma\left(\tfrac{1}{r}u_r + \partial_r u_r + \tfrac{1}{r}\partial_\theta u_\theta\right) = 0\,, \tag{3.1c}$$

$$\left(\partial_t + u_r\partial_r + \tfrac{1}{r}u_\theta\partial_\theta\right)S = 0. \tag{3.1d}$$

We introduce the new variables

$$\begin{aligned} u_\theta(r,\theta,t) &= rb(\theta,t)\,, \quad u_r(r,\theta,t) = ra(\theta,t)\,, \\ \sigma(r,\theta,t) &= rc(\theta,t), \quad S(r,\theta,t) = k(\theta,t)\,. \end{aligned} \tag{3.2}$$

The system (3.1) then takes the form

$$(\partial_t + b\partial_\theta)\,a + a^2 - b^2 + \alpha c^2 = 0 \tag{3.3a}$$

$$(\partial_t + b\partial_\theta)\,b + \alpha c\partial_\theta c + 2ab = \tfrac{\alpha}{2\gamma}c^2\partial_\theta k \tag{3.3b}$$

$$(\partial_t + b\partial_\theta)\,c + \alpha c\partial_\theta b + \gamma ac = 0 \tag{3.3c}$$

$$(\partial_t + b\partial_\theta)\,k = 0\,. \tag{3.3d}$$

For simplicity of presentation we shall henceforth focus on the case

$$\gamma = 2 \quad \text{and} \quad \alpha = \tfrac{1}{2}\,.$$

Note, however, that all the results in this work generalize to the case when $\gamma > 1$.[2] The Riemann functions w and z are defined by

$$w = b + c\,, \qquad z = b - c\,, \tag{3.4a}$$
$$b = \tfrac{1}{2}(w + z)\,, \qquad c = \tfrac{1}{2}(w - z)\,. \tag{3.4b}$$

It is convenient to rescale time, letting $\partial_t \mapsto \frac{3}{4}\partial_{\widetilde{t}}$, and for notational simplicity, we continue to write t for $\widetilde{t}$. With this temporal rescaling employed, the system (3.3c) can be equivalently written as

$$\partial_t w + \lambda_3 \partial_\theta w = -\tfrac{8}{3} a w + \tfrac{1}{24}(w - z)^2 \partial_\theta k\,, \tag{3.5a}$$
$$\partial_t z + \lambda_1 \partial_\theta z = -\tfrac{8}{3} a z + \tfrac{1}{24}(w - z)^2 \partial_\theta k\,, \tag{3.5b}$$
$$\partial_t k + \lambda_2 \partial_\theta k = 0\,, \tag{3.5c}$$
$$\partial_t a + \lambda_2 \partial_\theta a = -\tfrac{4}{3} a^2 + \tfrac{1}{3}(w + z)^2 - \tfrac{1}{6}(w - z)^2\,. \tag{3.5d}$$

where the three wave speeds are given by

$$\lambda_1 = \tfrac{1}{3} w + z\,, \qquad \lambda_2 = \tfrac{2}{3} w + \tfrac{2}{3} z\,, \qquad \lambda_3 = w + \tfrac{1}{3} z\,. \tag{3.6}$$

We note that (3.3c) takes the form

$$\partial_t c + \lambda_2 \partial_\theta c + \tfrac{1}{2} c \partial_\theta \lambda_2 = -\tfrac{8}{3} a c\,. \tag{3.7}$$

Finally, we denote the specific vorticity in azimuthal symmetry by

$$\varpi = 4(w + z - \partial_\theta a) c^{-2} e^k\,, \tag{3.8}$$

which satisfies the evolution equation

$$\partial_t \varpi + \lambda_2 \partial_\theta \varpi = \tfrac{8}{3} a \varpi + \tfrac{4}{3} e^k \partial_\theta k\,. \tag{3.9}$$

We supplement (3.5) with initial conditions

$$w_0(\theta) = w(\theta, T_0)\,, \quad z_0(\theta) = z(\theta, T_0)\,, \quad a_0(\theta) = a(\theta, T_0)\,,$$
$$k_0(\theta) = k(\theta, T_0)\,, \quad \varpi_0(\theta) = \varpi(\theta, T_0)\,.$$

[2] The pre-shock formation for general $\gamma > 1$ in (3.3) was already done in [1] for an open set of smooth isentropic initial data. Using the arguments in [3], the same result may be obtained also for the non-isentropic problem. The more detailed information required for shock-development can be obtained in analogy with the analysis in Section 4. The shock development problem for general $\gamma > 1$ is conceptually the same; see the outline of the proof in Section 3.4. One of the main differences is that the slightly more complicated Rankine-Hugoniot condition (2.39) must be used in place of (2.40). Another difference is that for general $\gamma > 1$, in the formation part the subdominant Riemann variable is not transported and thus cannot be taken to equal a constant up to the pre-shock; this issue was already addressed in [1–3], see also [16].

We shall study the shock formation process for solutions to (3.5) on the time interval $T_0 < t \leqslant T_1$, where T_1 denotes the time of the first singularity, also known as the *pre-shock*. One of our main objectives is to provide a detailed description of the pre-shock $w(\cdot, T_1)$. We shall provide the fractional series expansion of $w(\theta, T_1)$ for θ in a neighborhood of the blowup location θ_*.

For the shock formation process, we choose initial data[3]

$$k_0(\theta) = 0\,, \quad z_0(\theta) = 0\,,$$

which is preserved by the dynamics so that $k(\theta, t) = 0$ and $z(\theta, t) = 0$ for all time t up to the time time of the pre-shock. Thus (3.5) is reduced to a coupled system of equations for a and w, satisfying

$$\partial_t w + w\partial_\theta w = -\tfrac{8}{3} a w\,, \tag{3.10a}$$

$$\partial_t a + \tfrac{2}{3} w\partial_\theta a = -\tfrac{4}{3} a^2 + \tfrac{1}{6} w^2\,. \tag{3.10b}$$

We emphasize however that the theorems in this paper generalize to the case that k_0 and z_0 do not vanish identically, and instead we just assume that z_0 and k_0 are small in the C^5 topology, and for all $\gamma > 1$. We refer to the paper [16] for the detailed analysis of the full system (3.5) during the shock formation process.

3.2 The Rankine-Hugoniot Jump Conditions Under Azimuthal Symmetry

Under the azimuthal symmetry assumptions and using our temporal rescaling $\widetilde{t} \mapsto \frac{3}{4} t$, from (3.2) (fixing $\gamma = 2$), we have that the shock hypersurface is given as the graph $\{(r, \theta, t) \;:\; \theta = \mathfrak{s}(t)\}$. The spacetime normal to this curve is $\mathfrak{n} = (-\dot{\mathfrak{s}}, \frac{1}{r})$. Thus, $\mathfrak{s}$ satisfies the Rankine-Hugoniot conditions (1.3a) and (1.3b)

$$\dot{\mathfrak{s}} = \frac{4}{3} \frac{[\![e^{-k} c^2 b^2 + \frac{1}{8} e^{-k} c^4]\!]}{[\![e^{-k} c^2 b]\!]}\,, \tag{3.11a}$$

$$\dot{\mathfrak{s}} = \frac{4}{3} \frac{[\![e^{-k} c^2 b]\!]}{[\![e^{-k} c^2]\!]}\,. \tag{3.11b}$$

We note that the third Rankine-Hugoniot condition (1.3c) has already been employed to deduce the relation (2.41).

Let us now convert (3.11) and (2.41) into our azimuthal variables as follows. We denote by $w_\pm(\cdot, t)$, $z_\pm(\cdot, t)$, $k_\pm(\cdot, t)$ the limiting values, from the left $(-)$ and right

[3] This choice is made for the following reason: irregardless of the choice of initial entropy function k_0, the Rankine-Hugoniot conditions guarantee that a jump in entropy *must occur* at the shock. As such the choice of $k_0 = 0$ emphasizes the production of entropy in the clearest possible terms. Similarly, the choice of $\gamma = 2$ and that $k_0 = 0$ allows the equation (3.5b) to reduce to a transport-type equation. Just as we did for entropy, we can (in this case) choose $z_0 = 0$ and up to the pre-shock, the sub-dominant Riemann variable z will remain zero. Once again the Rankine-Hugoniot conditions ensure that z must experience a jump discontinuity along the shock, and thus the choice of $z_0 = 0$ allows us to most easily demonstrate this fact.

(+), of the shock curve $\mathfrak{s}(t)$. We also note the fact that $k_+ = 0$ and $z_+ = 0$. Now, from (3.4), the system (3.11) becomes

$$\dot{\mathfrak{s}}(t) = \frac{2}{3}\frac{e^{-k_-}(w_- - z_-)^2(w_- + z_-)^2 + \frac{1}{8}e^{-k_-}(w_- - z_-)^4 - \frac{9}{8}w_+^4}{e^{-k_-}(w_- - z_-)^2(w_- + z_-) - w_+^3}, \tag{3.12a}$$

$$\dot{\mathfrak{s}}(t) = \frac{2}{3}\frac{e^{-k_-}(w_- - z_-)^2(w_- + z_-) - w_+^3}{e^{-k_-}(w_- - z_-)^2 - w_+^2}. \tag{3.12b}$$

We note that the jump conditions (3.12a) and (3.12b) for the mass and the momentum equations are a priori two different equations for the shock speed. To remedy this, we set the right sides of these equations equal to each other, and instead work with one evolution equation for $\dot{\mathfrak{s}}$, namely (3.12b), and one constraint

$$\begin{aligned}&\left((w_- - z_-)^2(w_- + z_-)^2 + \tfrac{1}{8}(w_- - z_-)^4 - \tfrac{9}{8}e^{k_-}w_+^4\right)\left((w_- - z_-)^2 - e^{k_-}w_+^2\right)\\ &\quad = \left((w_- - z_-)^2(w_- + z_-) - e^{k_-}w_+^3\right)^2\end{aligned} \tag{3.13a}$$

Also, we have that (2.41) takes the form

$$(e^{k_-} - 1)(w_- - z_-)^4\left(3w_+^2e^{k_-} - (w_- - z_-)^2\right) = \left((w_- - z_-)^2 - e^{k_-}w_+^2\right)^3. \tag{3.13b}$$

To summarize, we shall first use the system formed by the Eqs. (3.13a) and (3.13b) in order to solve for z_- and k_- in terms of w_- and w_+, and then insert these solutions into (3.12b) and determine an evolution equation for $\mathfrak{s}$, solely in terms of w_- and w_+. This is discussed in Sect. 5.6.

3.3 Main Result in Azimuthal Symmetry

As mentioned in Theorem 1.2, in the *formation part* of our result, i.e. for $t \in [T_0, T_1)$, we have that the solution (w, z, k, a) of the Euler equations in azimuthal symmetry is smooth, so that the notion of solution is the classical one: the system (3.5) is satisfied in the sense of C^1 functions of space and time. On the time interval $[T_1, T_2]$, which covers the *development part* of our result, the notion of *regular shock solution* is used, as defined by Definition 1.1 above. In azimuthal symmetry, this definition becomes:

Definition 3.1 (*Regular azimuthal shock solution*) We say that $(w, z, k, a, \mathfrak{s})$ is a *regular azimuthal shock solution* on $\mathbb{T} \times [T_1, T_2]$ if

(i) (w, z, k, a) are $C^1_{\theta,t}$ smooth, and ϖ is $C^0_{\theta,t}$ smooth, on the complement of the shock curve $\{\theta = \mathfrak{s}(t)\}$;
(ii) on the complement of the shock curve (w, z, k, a) solve the equations (3.5) pointwise, and ϖ solves the integrated form of (3.9);
(iii) (w, z, k) have jump discontinuities across the shock curve which satisfy the algebraic Eqs. (3.13a), (3.13b) arising from the Rankine-Hugoniot conditions;

(iv) the shock location $\mathfrak{s} : [T_1, T_2] \to \mathbb{T}$ is C^1_t smooth and solves (3.12b).

Our main result for the azimuthal 2D Euler equations (3.5) is stated in detail in Theorems 5.5 and 6; here we only give a condensed statement:

Theorem 3.2 (Main result in azimuthal symmetry – abbreviated version) *From smooth isentropic initial data with vanishing subdominant Riemann variable at time T_0, there exist smooth solutions to the azimuthal Euler system (3.5) that form a pre-shock singularity, at a time $T_1 > T_0$. The first singularity occurs at a single point in space, θ_*, and this first singularity is shown to have an asymptotically self-similar shock profile exhibiting a $C^{\frac{1}{3}}$ cusp in the dominant Riemann variable velocity and a $C^{1,\frac{1}{3}}$ in the radial velocity. After the pre-shock, the solution to (3.5) is continued for a short time $(T_1, T_2]$ as a regular azimuthal shock solution (cf. Definition 3.1) with the following properties:*

- *Across the shock curve $\mathfrak{s}$, all the state variables jump*

$$[\![w]\!] \sim (t - T_1)^{\frac{1}{2}}, \qquad [\![\partial_\theta a]\!] \sim (t - T_1)^{\frac{1}{2}}, \qquad [\![z]\!] \sim (t - T_1)^{\frac{3}{2}}, \qquad [\![k]\!] \sim (t - T_1)^{\frac{3}{2}}$$

  for $t \in (T_1, T_2]$.
- *Across the characteristic $\mathfrak{s}_2$ emanating from the pre-shock and moving with the fluid velocity, the Riemann variables and the entropy make $C^{1,\frac{1}{2}}$ cusps approaching from the right side. Approaching from the left side, are these variables are C^2 smooth.*
- *Across the characteristic $\mathfrak{s}_1$ emanating from the pre-shock and moving with the fluid velocity minus the sound speed, the entropy is zero while the subdominant Riemann variable z makes a $C^{1,\frac{1}{2}}$ cusp approaching from the right. Approaching from left, they all variables are C^2 smooth on $(T_1, T_2]$.*

3.4 Outline of the Proof

The proof of Theorem 3.2 consists of five main steps, which we outline next. For simplicity, in this outline we focus only on the intuition behind the result, and skip over the technical difficulties which emerge when we turn this intuition into a complete proof.

Step 1: detailed formation of first singularity, the pre-schock. The formation of the first gradient singularity for the Euler equations, from an open set of smooth initial datum, was previously established in [1–3]. In azimuthal symmetry, [1] shows that that the first singularity is characterized as an asymptotically self-similar $C^{1/3}_\theta$ cusp for the dominant Riemann variable w defined in (3.4); this is the so-called *pre-shock*.

In order to best illustrate a symmetry breaking phenomenon which occurs after the formation of the pre-shock, in this paper we consider smooth initial conditions for (3.5) which are both isentropic ($k|_{t=T_0} \equiv 0$) and have vanishing subdominant Riemann variable ($z|_{t=T_0} \equiv 0$). Both of these conditions are propagated for smooth solutions (the interval $[T_0, T_1]$ in Figure 2), but we shall prove that this symmetry is broken as

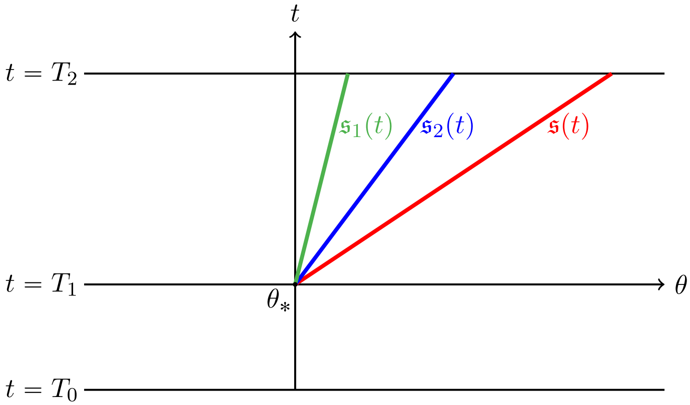

Fig. 2 At time T_0 a smooth datum is given, which forms a first singularity at time T_1, at a single angle θ_*; this is the pre-shock. For $t \in (T_1, T_2]$, we have three curves of singularities emerging from the point (θ_*, T_1): $\mathfrak{s}$ is a classical shock curve across which $(w, z, k, \partial_\theta a)$ jump, and the Rankine-Hugoniot conditions are satisfied; along the characteristic curve $\mathfrak{s}_2$ the quantities (w, z, k) have regularity $C^{1,1/2}$ and no better, while along the characteristic curve $\mathfrak{s}_1$, the function z has regularity $C^{1,1/2}$ and no better

soon as the shock forms (the interval $(T_1, T_2]$ in Figure 2). From such smooth initial data, satisfying in addition a genericity condition on the initial gradient of the dominant Riemann variable, we construct a first singularity occurring at a point (θ_*, T_1). For simplicity of notation, this space-time location of the pre-shock is relabelled as (0, 0), and the solution $(w, z, k, a)|_{t=T_1}$ is denoted as (w_0, z_0, k_0, a_0). From [1] we have that at the pre-shock, the solution takes the form

$$w_0(\theta) = \kappa - \mathsf{b}\theta^{\frac{1}{3}} + \dots, \tag{3.14a}$$

$$a_0(\theta) = \mathsf{a}_0 + \mathsf{a}_1\theta + \mathsf{a}_2\theta^{\frac{4}{3}} + \dots, \tag{3.14b}$$

$$z_0(\theta) = 0, \tag{3.14c}$$

$$k_0(\theta) = 0\,, \tag{3.14d}$$

asymptotically for $|\theta| \ll 1$. We note also that specific vorticity ϖ (see (3.8)) at the pre-shock is Lipschitz continuous; we denote it as ϖ_0.

While for the schematic understanding of shock development the asymptotic expansions in (3.14) are sufficient, in order to rigorously capture the formation of higher order characteristic singularities emerging along the curves $\mathfrak{s}_1$ and $\mathfrak{s}_2$ in Figure 2, a much finer understanding of the pre-shock is required. In particular, we need to show that the equality (3.14a) holds in a C^3 sense; by this we mean that $w_0'(\theta) = -\frac{1}{3}\mathsf{b}\theta^{-\frac{2}{3}} + \dots$, that $w_0''(\theta) = \frac{2}{9}\mathsf{b}\theta^{-\frac{5}{3}} + \dots$, and that $w_0'''(\theta) = -\frac{10}{27}\mathsf{b}\theta^{-\frac{8}{3}} + \dots$, for $|\theta| \ll 1$. This information is not provided by our previous work [1] and is established in Section 4 of this paper; here we combine the information provided by the self-similar analysis in [1] with a Lagrangian perspective in unscaled variables for (3.10), and the characterization of the pre-shock as the point in space time where the characteristic associated with

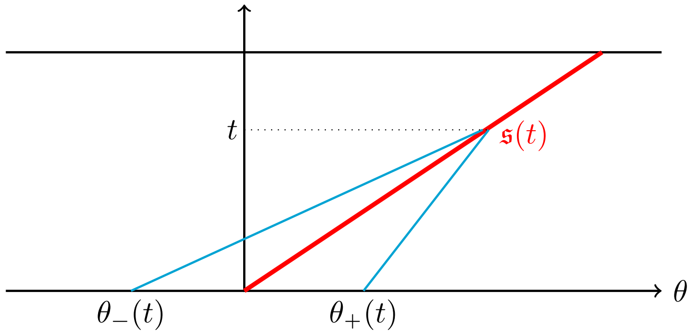

Fig. 3 The shock curve is represented in bold red, while the paths $\{\eta_{\mathsf{B}}(\theta_\pm(t), s)\}_{s\in[0,t]}$ are the cyan paths

the speed λ_1 has a vanishing first and second gradient (with respect to the Lagrangian label).

Step 2: emergence of shock front. By Remark 2.6, for short time $\dot{\mathfrak{s}} \approx u_n + c$. Accounting for the temporal rescaling done in Section 3.1 (see paragraph above (3.5)), this says $\dot{\mathfrak{s}} \approx b + c = w$ close to the pre-shock, so that from (3.14a) we have

$$\mathfrak{s}(t) \approx \kappa t.$$

Entropy is produced as soon as the shock has developed, cf. Lemma 2.1. However, this contribution is small at small times, and thus the dynamics of w (cf. (3.5a)) near the pre-shock can be roughly thought of as

$$\partial_t w + w\partial_\theta w = (\text{small amplitude error involving entropy gradients}), \qquad (3.15a)$$

$$w|_{t=0} = \kappa - \mathsf{b}\theta^{\frac{1}{3}} + (\text{small error near pre-shock}). \qquad (3.15b)$$

Note that the characteristics of this equation, the flow of $\partial_t + w\partial_\theta$, are to leading order in time tangent to the shock, if initiated at the pre-shock location. Otherwise, these characteristics impinge upon the shock from either the left or right sides, since the pre-shock data ensures that the Lax entropy conditions (2.18) are satisfied. As such, we can view the dominant Riemann variable w as being a perturbation of an inviscid Burgers solution:

$$w_{\mathsf{B}}(\eta_{\mathsf{B}}(\theta, t), t) = w_0(\theta), \qquad \eta_{\mathsf{B}}(\theta, t) = \theta + t w_0(\theta). \qquad (3.16)$$

A large part of the proof of Theorem 5.5 is indeed dedicated to proving that the errors made in approximating equation (3.15a) with the Burgers equation can indeed be controlled, in a C^1 topology of a suitable space-time. This part of the analysis uses in a crucial way the specific transport structure of the entropy gradient present on the right side of (3.15a) or (3.5a), and the evolution equations for the good unknowns q^w and q^z defined in (3.29) below, which relate the gradients of entropy to those of the Riemann variables and the sound speed (Fig. 3).

The outcome of this analysis is that indeed we may approximate $[\![w]\!] \approx [\![w_{\mathsf{B}}]\!]$ where

$$[\![w_{\mathsf{B}}]\!](t) = w_0(\theta_-(t)) - w_0(\theta_+(t)), \tag{3.17}$$

where $\theta_\pm(t) = {\eta_{\mathsf{B}}}^{-1}(\mathfrak{s}(t)^\pm, t)$ are the locations of the labels of the particles which fell into the shock at time t. To find how these labels depend on the elapsed time, we use the expression for the Burgers flowmap (3.16) near the pre-shock

$$\eta_{\mathsf{B}}(\theta, t) - \kappa t \approx \theta - \mathsf{b}t\theta^{\frac{1}{3}} \tag{3.18}$$

when $\eta_{\mathsf{B}}(\theta, t) = \mathfrak{s}(t)$. This yields $\theta_\pm(t) \approx \pm (\mathsf{b}t)^{\frac{3}{2}}$ and returning to (3.17) we find

$$[\![w]\!](t) \sim t^{\frac{1}{2}}. \tag{3.19}$$

Step 3: jumps of entropy and the subdominant Riemann variable on the shock front. In analogy to Lemma 2.1, by choosing the smallest root of the system (3.13a)–(3.13b) it can be shown that in the weak shock regime $|[\![w]\!]| \ll 1$ which corresponds to short times after the pre-shock, the Rankine-Hugoniot conditions imply

$$-[\![z]\!](t) \sim [\![w]\!]^3(t) \sim t^{\frac{3}{2}}, \tag{3.20}$$

for the subdominant Riemann variable, and similarly

$$[\![k]\!](t) \sim [\![w]\!]^3(t) \sim t^{\frac{3}{2}}, \tag{3.21}$$

for the jump in entropy along the shock front. As such, entropy and the subdominant Riemann variable are *produced instantaneously* along the shock in order to enforce that mass, momentum and total energy are not lost. This is a manifestation of *symmetry breaking* associated to physical shocks, and emphasizing this point is the reason for the choice (3.14c)–(3.14d).

At this point we note that since a is being forced in (3.5d) by both z and w, which themselves jump across $\mathfrak{s}(t)$, the function a too exhibits a singularity on $\mathfrak{s}(t)$. Ordinarily, this singularity might be expected to appear in a itself, but since the characteristics of a are transversal to the shock, together with the special structure of the specific vorticity evolution (3.9), we prove that a is continuous across the shock, and that its derivative exhibits a jump discontinuity:

$$[\![\partial_\theta a]\!](t) \sim [\![w]\!](t) \sim t^{\frac{1}{2}}. \tag{3.22}$$

An extended discussion of this point will appear in the next step.

Step 4: development of weak singularities. We use equations (3.5) to determine the solution away from the shock curve. In front of the shock (to the right in our case), the solution is determined by its initial data on the Cauchy surface $\{t = 0\}$. This is because

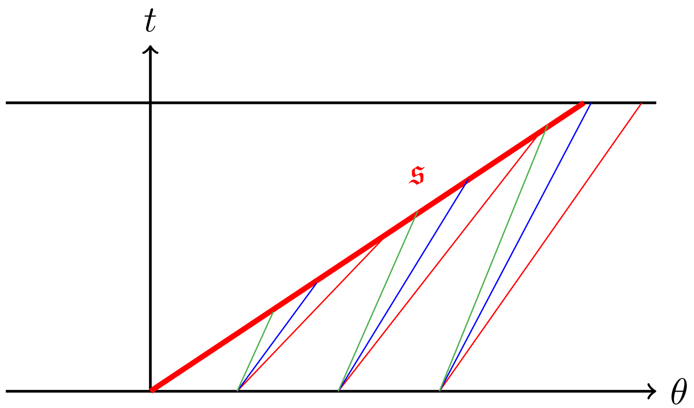

Fig. 4 The characteristic curves of $\lambda_3 = w$ in front of the shock curve are represented in red, those of $\lambda_2 = \frac{2}{3}w$ are in blue, and those of $\lambda_1 = \frac{1}{3}w$ are plotted in green

all of the characteristic curves moving with velocities λ_i, $i = 1, 2, 3$, as defined in (3.6), impinge upon the shock front in that region, since the shock is supersonic there. As such, in that region z and k are identically zero since they are zero initially and (3.5b)–(3.5c) have no forcing when $z = k = 0$ (Fig. 4).

On the other hand, behind the shock (to the left side in our case), this is not the case. As discussed in **Step 3**, along the shock front z and k *must* be produced in order to enforce the three Rankine-Hugoniot jump conditions. These values z_- and k_- are propagated off the shock along their characteristics with speeds λ_1 and λ_2 which are both *slower* than the speed of the shock $\dot{\mathfrak{s}}(t)$. As such, the surface $\{\theta < 0, t = 0\} \cup \{\theta = \mathfrak{s}(t), t > 0\}$ serves as a new Cauchy surface for the z, k, a equations (3.5) once the shock has formed. Schematically, the initial data on this new Cauchy surface is

$$\widetilde{z}_0(\theta) \approx \begin{cases} 0 & \text{on } \{\theta < 0, t = 0\} \\ \widetilde{\mathrm{z}}_0 \theta^{\frac{3}{2}} + \dots & \text{on } \{\theta = \kappa t,\ t \geqslant 0\} \end{cases}, \tag{3.23a}$$

$$\widetilde{k}_0(\theta) \approx \begin{cases} 0 & \text{on } \{\theta < 0, t = 0\} \\ \widetilde{\mathrm{k}}_0 \theta^{\frac{3}{2}} + \dots & \text{on } \{\theta = \kappa t,\ t \geqslant 0\} \end{cases}, \tag{3.23b}$$

$$\widetilde{a}_0(\theta) \approx \begin{cases} \widetilde{\mathrm{a}}_0 \theta + \widetilde{\mathrm{a}}_1 \theta^{\frac{4}{3}} + \dots & \text{on } \{\theta < 0, t = 0\} \\ \text{smooth} & \text{on } \{\theta = \kappa t,\ t \geqslant 0\} \end{cases}, \tag{3.23c}$$

for some constants $\widetilde{\mathrm{z}}_0$, $\widetilde{\mathrm{a}}_0$, $\widetilde{\mathrm{k}}_0$, and for $|\theta|, t \ll 1$. As discussed above, this data is carried away from the shock surface along characteristics which are slower than the shock. The entropy is simply transported cf. (3.5c), whereas the subdominant Riemann variable is transported, self-amplified and forced by the entropy cf. (3.5b), and the radial velocity is forced by a, w, and z cf. (3.5d).

We begin by discussing what happens to the entropy. Since its data (3.23b) is smooth away from the point $\theta = 0$, the solution in the domain of influence of this region is likewise smooth. Only across one single curve can the entropy be non-smooth: the λ_2-

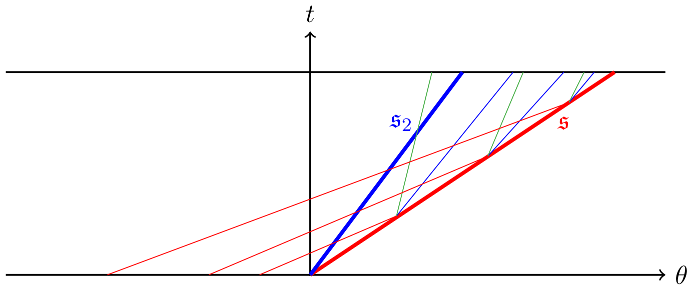

Fig. 5 The entropy k is propagated off the shock curve along the λ_2 characteristics represented by blue curves. The subdominant Riemann variable z is also propagated off the shock curve $\mathfrak{s}$, but along the λ_1 characteristics represented in green. The λ_3 characteristics initiated at $\{t = 0\}$, represented in red, impinge on the shock curve from the left side, determining w in terms of w_0

characteristic curve $\mathfrak{s}_2(t)$ emanating from the pre-shock location $(0, 0)$; see Figure 5. Along this curve, one may expect that the $\frac{3}{2}$–Hölder regularity of the Cauchy data $\widetilde{k}_0$ is transported. Since at the initial time we have $\lambda_2(0) \approx \frac{2}{3} w_0$, due to (3.14a) at short times we expect

$$\mathfrak{s}_2(t) \approx \tfrac{2}{3}\kappa t\,.$$

The entropy exhibits a $C^{1,\frac{1}{2}}$ cusp singularity across $\{\theta = \mathfrak{s}_2(t)\}$, taking the approximate form

$$k(\theta, t) \approx \begin{cases} 0, & \theta < \mathfrak{s}_2(t) \\ 3^{\frac{3}{2}} \widetilde{\mathrm{k}}_0 \left(\theta - \mathfrak{s}_2(t)\right)^{\frac{3}{2}}, & \mathfrak{s}_2(t) < \theta < \mathfrak{s}(t) \\ 0, & \theta > \mathfrak{s}(t) \end{cases}. \tag{3.24}$$

Note that along the shock curve $\mathfrak{s}(t)$ (for $t > 0$) the entropy k smoothly matches its generated values along shock given by (3.23b); this is because $\mathfrak{s}(t) - \mathfrak{s}_2(t) \approx \frac{1}{3}\kappa t$. We emphasize that equation (3.24) gives quite an accurate picture of the entropy for short times, even in the fully nonlinear problem; this fact is established in Sections 5 and 6, and the proof uses a precise understanding of the second derivative of the λ_2 wavespeed in the region between $\mathfrak{s}_2$ and $\mathfrak{s}$.

With the structure of the entropy understood, we can study the behavior of w, z and a which evolve according to (3.5). First note that, since the shock is subsonic relative to the state behind it, the λ_3 characteristics impinge upon the shock front, and therefore the initial data for w is determined entirely by the values on the surface $\{t = 0\}$, i.e. by w_0 as given in (3.14a) (see Figure 5). As such, w is smooth away from the pre-shock and we are able to precisely quantify how the the bounds degenerate as $(\theta, t) \to (0, 0)$. On the other hand, the characteristics of the subdominant Riemann variable and radial velocity are slower than the shock and thus the solutions in the region $\mathfrak{s}_2(t) \leqslant \theta < \mathfrak{s}(t)$ are determined entirely by their data along the shock curve. Near the shock curve $\mathfrak{s}(t)$, approaching from the left, the solution fields z and a smoothly match their values along the shock (see Figure 5).

Since away from $\mathfrak{s}_2(t)$ the entropy given by (3.24) is smooth, in spite of both w and z being forced by an entropy gradient, it can be shown that w and z are smooth away from $\mathfrak{s}_2(t)$; this also uses the fact that both w_0 (see (3.14a)) and $\widetilde{z}_0$ (see (3.23a)) are smooth away from $(0, 0)$.

The most interesting behavior happens along $\mathfrak{s}_2(t)$, from the right side. Here, we have determined that the entropy exhibits a cusp-type Hölder singularity in its derivative; by (3.24) we have that $\partial_\theta k \sim (\theta - \mathfrak{s}_2(t))^{\frac{1}{2}}$. This singularity is seen by the Riemann variables w and z and radial velocity a through their forcing terms $(w-z)^2\partial_\theta k$ which, naively, are just $C^{\frac{1}{2}}$ across $\mathfrak{s}_2$. However, the fact that the entropy has a specific cusp structure (3.24) near the curve $\mathfrak{s}_2$, together with the fact that the wavespeeds of w and z are strictly different from the wavespeed of k, actually provides a *regularization effect* for w and z. The situation with the radial velocity a is more challenging because it shares the same wavespeed as the entropy; here, the evolution for the specific vorticity is used crucially in our analysis.

In order to explain this regularization effect in greater detail, let us denote the λ_i characteristics by

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\eta_i(\theta, t) = \lambda_i(\eta_i(\theta, t), t), \qquad \eta_i(\theta, 0) = \theta\,,$$

for every $i \in \{1, 2, 3\}$ (in the proof, we in fact denote by η the characterstic of λ_3, but for the λ_1 and λ_2 we need to use backwards in time flows, denoted by ψ_t and ϕ_t, see Section 5.7 for details). Since for $|\theta| \ll 1$ the wave speeds at the pre-shock are given by $\lambda_1 \approx \frac{1}{3}\kappa$, $\lambda_2 \approx \frac{2}{3}\kappa$ and $\lambda_3 \approx \kappa$, to leading order in time and for small values of $|\theta|$, we have that

$$\begin{aligned}
&\eta_1(\theta, t) \approx \theta + \lambda_1 t \approx \theta + \tfrac{1}{3}\kappa t\,, \qquad \eta_2(\theta, t) \approx \theta + \lambda_2 t \approx \theta + \tfrac{2}{3}\kappa t\,,\\
&\eta_3(\theta, t) \approx \theta + \lambda_3 t \approx \theta + \kappa t\,.
\end{aligned}$$

We are interested in the behavior near the curve $\mathfrak{s}_2$. Thus, we seek labels $\theta_i(t)$ such that $\eta_i(\theta_i(t), t) = \mathfrak{s}_2(t) + y$, where $0 < y \ll 1$. Since $\mathfrak{s}_2(t) \approx \lambda_2 t$, we have $\theta_i(t) \approx y + (\lambda_2 - \lambda_i)t$. The flowmaps are

$$\eta_i(\theta_i(t), s) \approx y + (\lambda_2 - \lambda_i)t + \lambda_i s, \qquad s \in [0, t], \qquad i = 1, 3. \tag{3.25}$$

Ignoring the integrating factors $e^{\frac{8}{3}\int_0^t a(\eta_3(\theta,\tau),\tau)\mathrm{d}\tau} \approx 1$ at short times, the solutions of (3.5a) and (3.5b) take the form

$$\begin{aligned}
w(\mathfrak{s}_2(t) + y, t) &\approx w_0(y + (\lambda_2 - \lambda_3)t)\\
&\qquad + \tfrac{1}{24}\int_0^t ((w-z)^2\partial_\theta k)(y + (\lambda_2 - \lambda_3)t + \lambda_3 s, s)\mathrm{d}s, \qquad (3.26\text{a})\\
z(\mathfrak{s}_2(t) + y, t) &\approx z_0(y + (\lambda_2 - \lambda_1)t)\\
&\qquad + \tfrac{1}{24}\int_0^t ((w-z)^2\partial_\theta k)(y + (\lambda_2 - \lambda_1)t + \lambda_1 s, s)\mathrm{d}s. \qquad (3.26\text{b})
\end{aligned}$$

As discussed above, since $\lambda_3 \approx \lambda_2 + \frac{1}{3}\kappa > \lambda_2$, the characteristic curves of w impinge on $\mathfrak{s}_2(t)$ from the left, carrying up initial data w_0 from $\{t = 0\}$. On the other hand, the characteristics of z impinge from the right of $\mathfrak{s}_2$ since $\lambda_1 \approx \lambda_2 - \frac{1}{3}\kappa < \lambda_2$. Therefore, the data for z is carried from the shock surface $\{\mathfrak{s}(t) = \theta\}$. Although this data is singular at $(0, 0)$, this point is not sampled by the characteristics above since $t > 0$ is fixed, and thus $(\lambda_2 - \lambda_1)t > 0$. Regarding the forcing terms appearing on the right sides of (3.26), from the asymptotic description of k in (3.24), the approximation $\mathfrak{s}_2(t) \approx \frac{2}{3}\kappa t \approx \lambda_2 t$, and the fact that by (3.4) $w - z$ equals twice the azimuthal sound speed c, which we expect to remain bounded from above and below in terms of κ, we obtain that

$$\int_0^t ((w-z)^2\partial_\theta k)(y + (\lambda_2 - \lambda_i)t + \lambda_i s, s)\mathrm{d}s \sim \int_{t+\frac{y}{\lambda_2-\lambda_i}}^t (y - \lambda_i(t-s))^{\frac{1}{2}}\mathrm{d}s \sim y^{\frac{3}{2}}, \tag{3.27}$$

for $0 < y, t \ll 1$. Thus, the forcing *gains one derivative*, expressed above by an extra power of y, due to the fact that it is integrated along curves which are transversal (since $\lambda_i \neq \lambda_2$) to the characteristics of the entropy (namely, the flow of $\partial_t + \lambda_2\partial_\theta$). Thus, from (3.26) and (3.27), we expect that w and z are both $C^{1,\frac{1}{2}}$ across the curve $\mathfrak{s}_2(t)$, rather than just $C^{\frac{1}{2}}$ which is the naive expectation.

Turning this intuition into a proof requires a C^2-type analysis of the characteristics of $\{\lambda_i\}_{i=1}^3$, including an understanding of the times at which the λ_1 and λ_2 characteristics intersect the shock curve $\mathfrak{s}$; see for instance Lemmas 5.24, 6.7, and 6.9. Additionally, in this stage of the proof we need to analyze the time integrals of $\partial_y w$ and $\partial_{yy} w$ (objects which do blow up rather severely as one approaches the pre-shock) when composed with the flows of λ_1 and λ_2; here the transversality of these flows with respect to $\mathfrak{s}$ plays a crucial role, along with a precise understanding of the function w_{B} in the vicinity of the pre-shock; see Lemmas 5.11, 5.23, and 6.12. This is one of the principal reasons why the pre-shock obtained in **Step 1** needs to be analyzed in a C^3 sense.

The intuition behind the gain of regularity for the radial velocity a is less direct. The data for a along the new Cauchy surface (including the shock curve) is $C^{1,\frac{1}{3}}$ due to the formula (3.23c). Thus, such a singularity would be expected to propagate along its characteristic emanating from the pre-shock location. To see this, we recall that the specific vorticity at the pre-shock is Lipschitz. Since by (3.9) it is transported by the velocity λ_2, it is forced by $\partial_\theta k$, and because the wavespeed for ϖ is the same as that of k, we conclude from (3.9) only that ϖ is $C^{\frac{1}{2}}$ across the curve $\mathfrak{s}_2$. Since k, z and w are all $C^{1,\frac{1}{2}}$ across this curve, by (3.8) we deduce that $\partial_\theta a \in C^{\frac{1}{2}}$, and consequently that $a \in C^{1,\frac{1}{2}}$ across $\mathfrak{s}_2$. Thus, for positive times $t > 0$, the radial velocity becomes smoother than its initial condition ($C^{1,\frac{1}{2}}$ vs $C^{1,\frac{1}{3}}$). This regularization effect is in essence a consequence of Lemmas 6.5 and 6.6.

Finally, we discuss the region to the left of $\mathfrak{s}_2(t)$. In this region the entropy is trivial ($k \equiv 0$) since it is determined solely by its data on the surface $\{\theta < 0,\ t = 0\}$,

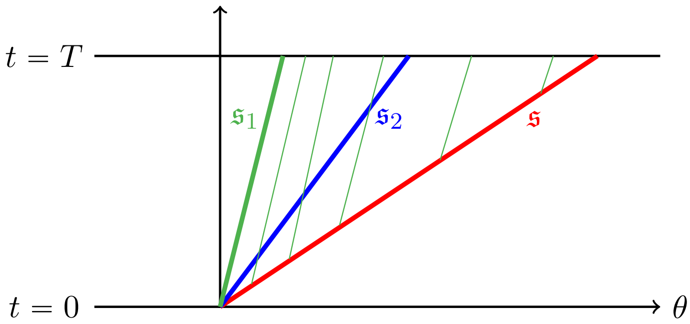

Fig. 6 The λ_1 characteristics, represented here by the green curves, propagate information about z from the shock curve $\mathfrak{s}$ into the region between $\mathfrak{s}_1$ and $\mathfrak{s}$

see (3.24). The equations reduce to

$$
\begin{aligned}
\partial_t w + \lambda_3 \partial_\theta w &= -\tfrac{8}{3} a w\,,\\
\partial_t z + \lambda_1 \partial_\theta z &= -\tfrac{8}{3} a z\,,\\
\partial_t a + \lambda_2 \partial_\theta a &= -\tfrac{4}{3} a^2 + \tfrac{1}{3}(w+z)^2 - \tfrac{1}{6}(w-z)^2\,.
\end{aligned}
$$

The object z has singular data as in (3.23a), which will be propagated along the λ_1-characteristic curve (Fig. 6). Specifically, we have that at the pre-shock $\lambda_1 \approx \frac{1}{3} w_0 \approx \frac{1}{3}\kappa$, so that the curve $\mathfrak{s}_1$ along which z is transported from the pre-shock is given by

$$
\mathfrak{s}_1(t) \approx \tfrac{1}{3}\kappa t.
$$

The $\frac{3}{2}$-Hölder singularity in the Cauchy data for z (3.23b) is morally speaking transported along these λ_1-characteristics for short times $t \ll 1$, resulting in

$$
z(\theta, t) \approx \begin{cases} 0, & \theta < \mathfrak{s}_1(t) \\ \mathrm{z}_0\left(\theta - \mathfrak{s}_1(t)\right)^{\frac{3}{2}}, & \mathfrak{s}_1(t) < \theta \ll \mathfrak{s}_2(t) \end{cases}. \tag{3.28}
$$

The difficulty in showing that the intuitive behavior (3.28) is indeed true lies in the fact that the λ_1-characteristics emanating from the shock curve do spend some time in the region between $\mathfrak{s}_2$ and $\mathfrak{s}$, and in this region the entropy gradient present in (3.5b) causes the first and second derivatives of z to behave badly. By using the transversality of the λ_1 and λ_2 characteristics, we are nonetheless able to show in Section 6.6 that (3.28) is morally correct.

Note that in this region, the relevant initial data for w and a is far away from the pre-shock, and so the fields w and a are as regular as their forcing for short times. This forcing involves the field z, which makes a $C^{1,\frac{1}{2}}$ cusp along $\mathfrak{s}_1(t)$. However, again the wave speeds for w and a are different than that of z, and as such their characteristics are transversal to $\mathfrak{s}_1(t)$. This means that the solution fields gain a derivative relative to the forcing, similar to (3.27). It thus seems reasonable to conjecture that $w, a \in C^{2,\frac{1}{2}}$

on the right side of $\mathfrak{s}_1$. Establishing this fact would in turn require us to show that (3.14a) holds in a $C^{3,\frac{1}{2}}$ sense, a regularity level which we did not pursue in **Step 1**. As such, in this paper we only prove that $w, a \in C^2$ on $\mathfrak{s}_1$, which is nonetheless a better regularity exponent that the naively expected $C^{1,\frac{1}{2}}$.

Step 5: returning to basic fluid variables. There is a certain regularization effect along the curve $\mathfrak{s}_2$, when returning to the original fluid variables, as we now explain. A straightforward calculation shows that the good unknowns

$$q^w := \partial_\theta w - \tfrac{1}{4} c \partial_\theta k, \qquad q^z := \partial_\theta z + \tfrac{1}{4} c \partial_\theta k, \tag{3.29}$$

satisfy the evolution equations

$$(\partial_t + \lambda_3 \partial_\theta) q^w + (\partial_\theta \lambda_3 + \tfrac{8}{3} a) q^w = -\tfrac{8}{3} \partial_\theta a w + \left(\tfrac{4}{3} a c + \tfrac{1}{6} c \partial_\theta \lambda_2\right) \partial_\theta k, \tag{3.30a}$$

$$(\partial_t + \lambda_1 \partial_\theta) q^z + (\partial_\theta \lambda_1 + \tfrac{8}{3} a) q^z = -\tfrac{8}{3} \partial_\theta a z - \left(\tfrac{4}{3} a c + \tfrac{1}{6} c \partial_\theta \lambda_2\right) \partial_\theta k. \tag{3.30b}$$

The remarkable feature of the system (3.30) is that the second derivatives of k do not appear in the equations; indeed, if one naively considers the evolution equation for $\partial_\theta w$ or $\partial_\theta z$ alone, then from (3.5a) and respectively (3.5b) we note the emergence of the forcing term $\frac{1}{24}(w - z)^2 \partial_{\theta\theta} k$. The unknowns q^w and q^z, and the system (3.30), is useful because it involves only $\partial_\theta k$, and this forcing makes a $C^{\frac{1}{2}}$ cusp along the curve $\mathfrak{s}_2$. However, since the characteristic speed of k is λ_2, and the characteristics of q^w and q^z are λ_3 and respectively λ_1, and are thus transversal, we again have a regularization effect akin to (3.27), and we find that the (Lagrangian) force is actually $C^{1,\frac{1}{2}}$ across $\mathfrak{s}_2$. Now, the initial data relevant to the behavior of q^w and q^z comes from different places. For q^w, it originates along the $\{t = 0\}$ surface and so it is easy to see that it is smooth for positive time (away from the pre-shock). On the other hand, the data for q^z originates on the shock curve itself and once again, away from the pre-shock it is smooth. It follows that, for $t > 0$ the regularity is set by the forcing, resulting in bounds consistent with $q^w, q^z \in C^{1,\frac{1}{2}}$. Again, in the proof we only establish the C^1 regularity of q^w and q^z, due to the C^3 expansion of the pre-shock; this argument is made rigorous in Sections 6.5 and 6.6. The outcome is that $q^w + q^z = \partial_\theta z + \partial_\theta w = \frac{2}{3} \partial_\theta u_\theta$ is smoother than the naive expectation $C^{\frac{1}{2}}$: we prove that it lies in C^1 across $\mathfrak{s}_2$ (which translates into C^2 regularity for the angular velocity u_θ), and conjecture that the sharp regularity is $C^{1,\frac{1}{2}}$. Similarly, the improved regularity for q^w and q^z shows that the second derivative of the pressure is bounded on $\mathfrak{s}_2$, see (7.1).

Summary. In terms of the Riemann variables in azimuthal symmetry, we find

- Across the shock curve $\mathfrak{s}(t)$, we have

$$[\![w]\!] \sim t^{\frac{1}{2}}, \qquad [\![\partial_\theta a]\!] \sim t^{\frac{1}{2}}, \qquad [\![z]\!] \sim t^{\frac{3}{2}}, \qquad [\![k]\!] \sim t^{\frac{3}{2}}.$$

- Across the curve $\mathfrak{s}_2(t)$, the functions $\partial_\theta w, \partial_\theta a, \partial_\theta k, \partial_\theta z$ all behave as $C^{\frac{1}{2}}$ cusps approaching $\mathfrak{s}_2$ from the right. Approaching from the left, they are all smooth, in positive time.

- Across the curve $\mathfrak{s}_1(t)$, the entropy is zero, $\partial_\theta w$ and $\partial_\theta a$ are C^1 (expected to be $C^{1,\frac{1}{2}}$) and $\partial_\theta z$ behaves as a $C^{\frac{1}{2}}$ cusp approaching $\mathfrak{s}_1$ from the right. Approaching from the left, $\partial_\theta z$ is C^1 in positive time.

In terms of the physical variables, we find

- Across the shock curve $\mathfrak{s}(t)$, all state variables jump

$$[\![u_\theta]\!] \sim rt^{\frac{1}{2}}, \qquad [\![\rho]\!] \sim r^2 t^{\frac{1}{2}}, \qquad [\![\partial_\theta u_r]\!] \sim rt^{\frac{1}{2}}, \qquad [\![S]\!] \sim t^{\frac{3}{2}}. \tag{3.31}$$

- Across the curve $\mathfrak{s}_2(t)$, the entropy, density and radial velocity derivatives all make $C^{\frac{1}{2}}$ cusps approaching $\mathfrak{s}_2$ from the right. Approaching from the left, they are all smooth. The second derivative of the angular velocity and the pressure are *bounded* for $t > 0$, and are expected to be $C^{\frac{1}{2}}$ smooth.
- Across the curve $\mathfrak{s}_1(t)$, the entropy is zero while the angular velocity and density derivatives make $C^{\frac{1}{2}}$ cusps approaching $\mathfrak{s}_1$ from the right. Approaching from the left, they are all smooth for $t > 0$. The second derivative of the radial velocity is bounded and is expected to make a $C^{\frac{1}{2}}$ cusp.

4 Detailed Shock Formation

In [1], it was established that for an open set of C^4 initial data, solutions to (3.5) form a generic, stable, asymptotically self-similar pre-shock at time $t = T_*$ and that the dominant Riemann variable $w(\cdot, T_*) \in C^{\frac{1}{3}}$. The primary objective of this section is to provide a precise description of $w(\cdot, T_*)$ in the vicinity of the pre-shock. We shall prove the following

Theorem 4.1 (Detailed shock formation) *For $\kappa_0 > 1$ taken sufficiently large and $\varepsilon > 0$ sufficiently small, and for initial data $(w, z, k, a)|_{t=-\varepsilon} = (w_0, 0, 0, a_0) \in C^5(\mathbb{T})$ satisfying (4.17)–(4.26) below, there exists a blowup time $t = T_*$, where $T_* = \mathcal{O}(\varepsilon^{\frac{5}{4}})$, a unique blowup location ξ_*, and unique solutions (w, a) to (3.5) in $C^0([-\varepsilon, T_*), C^4(\mathbb{T})) \cap C^4([-\varepsilon, T_*), C^0(\mathbb{T}))$ such that*

$$w(\cdot, T_*) \in C^{\frac{1}{3}}(\mathbb{T}), \qquad a(\cdot, T_*) \in C^{1,\frac{1}{3}}(\mathbb{T}), \qquad \varpi(\cdot, T_*) \in C^{0,1}(\mathbb{T}). \tag{4.1}$$

Furthermore, there exists a unique blowup label x_* satisfying

$$|x_*| \leqslant 20\kappa_0\varepsilon^4 \quad \text{such that} \quad \lim_{t\to T_*} \eta(x_*, t) = \xi_*,$$

where η is the 3-characteristic defined by (4.40). The pre-shock $w(\cdot, T_*)$ has the fractional series expansion

$$\left|w(\theta, T_*) - \kappa_* - \mathsf{a}_1(\theta - \xi_*)^{\frac{1}{3}} - \mathsf{a}_2(\theta - \xi_*)^{\frac{2}{3}} - \mathsf{a}_3(\theta - \xi_*)\right| \lesssim \left|\theta - \xi_*\right|^{\frac{4}{3}} \tag{4.2}$$

for all $\theta \in \eta(B_{x_*}(\varepsilon^3))$, *where*

$$\kappa_* = e^{-\frac{8}{3}\int_{-\varepsilon}^{T_*} \partial_x(a(\eta(x_*,r),r))dr} w_0(x_*) ,$$

and

$$|\kappa_* - \kappa_0| \leqslant 2\varepsilon\kappa_0 , \quad -\tfrac{6}{5} \leqslant \mathsf{a}_1 \leqslant -\tfrac{4}{5} , \quad \left|\mathsf{a}_2\right| \leqslant \varepsilon^{\frac{1}{10}} , \quad \left|\mathsf{a}_3\right| \leqslant \tfrac{7}{6\varepsilon} . \tag{4.3}$$

In fact, the expansion (4.2) is valid in a C^3*-sense, by which we mean that the bounds*

$$\left|\partial_\theta w(\theta, T_*) - \tfrac{1}{3}\mathsf{a}_1(\theta - \xi_*)^{-\frac{2}{3}} - \tfrac{2}{3}\mathsf{a}_2(\theta - \xi_*)^{-\frac{1}{3}}\right| \lesssim \tfrac{1}{\varepsilon} , \tag{4.4a}$$

$$\left|\partial_\theta^2 w(\theta, T_*) - \tfrac{2}{9}\mathsf{a}_1(\theta - \xi_*)^{-\frac{5}{3}}\right| \lesssim \varepsilon^{-\frac{63}{8}} \left|\theta - \xi_*\right|^{-\frac{4}{3}} , \tag{4.4b}$$

$$\left|\partial_\theta^3 w(\theta, T_*)\right| \lesssim \varepsilon^{-\frac{151}{8}} \left|\theta - \xi_*\right|^{-\frac{8}{3}} , \tag{4.4c}$$

hold for all $\theta \in \eta(B_{x_*}(\varepsilon^3))$. *Moreover, the* C^4 *regularity away from the pre-shock is characterized by*

$$\sup_{t\in[-\varepsilon,T_*)} \max_{\gamma\leqslant 4} \left(\left|\partial_\theta^\gamma a(\eta(x,t),t)\right| + \left|\partial_\theta^\gamma w(\eta(x,t),t)\right|\right)$$
$$\leqslant \begin{cases} C_\varepsilon \left((T_* - t) + 3\varepsilon^{-3}(\varepsilon + t)(x - x_*)^2\right)^{-4} & |x - x_*| \leqslant \varepsilon^2 \\ C_\varepsilon & |x - x_*| \geqslant \varepsilon^2 \end{cases} , \tag{4.5}$$

where $C_\varepsilon > 0$ *is a sufficiently large constant depending on inverse powers of* ε. *Lastly, the specific vorticity satisfies the bounds*

$$\tfrac{10}{\kappa_0} \leqslant \varpi(x,t) \leqslant \tfrac{28}{\kappa_0} , \qquad \left|\partial_x \varpi(x,t)\right| \leqslant \tfrac{70}{\kappa_0^2\varepsilon} , \tag{4.6}$$

for all $x \in \mathbb{T}$ *and* $t \in [-\varepsilon, T_*)$.

The proof of this theorem makes use of detailed estimates for the characteristic families and their derivatives. As we will detail below, we let $\eta(x,t)$ denote the flow w. Here x denotes a particle label, and $\eta(x,t)$ provides the location of x at time t; specifically we have the formula $\eta(x,t) = x + \int_{-\varepsilon}^t w(\eta(x,s),s)ds$. Moreover, we see that $w(\eta(x,t),t) = e^{-\frac{8}{3}\int_{-\varepsilon}^t a(\eta(x,s),s)ds} w_0(x)$ and hence that

$$w(\theta,t) = e^{-\frac{8}{3}\int_{-\varepsilon}^t a(\eta(\eta^{-1}(\theta,t),s),s)ds} w_0(\eta^{-1}(\theta,t)) .$$

It follows that a power series expansion of $w(\theta, T_*)$ about the blowup location $\theta = \xi_*$ requires a series expansion for the inverse flow map $\eta^{-1}(\theta,t)$ about $\theta = \xi_*$. The formula for $\eta^{-1}(\theta, T_*)$ requires us to first compute $\eta(x, T*)$, and then invert the polynomial equation $\eta(x, T_*) = \theta$ for θ in a neighborhood of ξ_*.

We shall write $\eta(x, T_*)$ as a Taylor series about the blowup label x_*. To do so, we prove the existence of a unique blowup trajectory $\eta(x_*, t)$ which converges to ξ_*, and study the behavior of $\partial_x^\gamma \eta(x, t)$, $\gamma \leqslant 4$. Our analysis makes use of self-similar coordinates only for the purpose of isolating the unique blowup trajectory $\eta(x_*, t)$, whereas all of our estimates for $\partial_x^\gamma \eta(x, t)$, $\partial_\theta^\gamma w(\eta(x, t), t)$, and $\partial_\theta^\gamma a(\eta(x, t), t)$ are obtained in physical coordinates. With these bounds in hand, we establish the Taylor expansion for $\eta(x, t)$ about the blowup label x_*, proceed to invert this relation, and then obtain a detailed description of the pre-shock.

4.1 Changing Variables to Modulated Self-similar Variables

We shall make use of self-similar coordinates (y, s) that rely upon time dependent modulation functions $\kappa(t)$, $\xi(t)$ and $\tau(t)$, which are introduced to enforce three pointwise constraints. Specifically, we map the physical coordinates (θ, t) to self-similar coordinates (y, s) by the following transformations:

$$s(t) := -\log(\tau(t) - t)\,, \qquad y(\theta, t) := \tfrac{\theta - \xi(t)}{(\tau(t)-t)^{\frac{3}{2}}} = e^{\frac{3}{2}s}(\theta - \xi(t))\,.$$

It follows that

$$\tau - t = e^{-s}\,,\ \tfrac{ds}{dt} = (1 - \dot\tau)e^s\,, \tag{4.7}$$

and thus

$$\partial_\theta y = e^{\frac{3}{2}s}\,,\ \partial_t y = \tfrac{-\dot\xi}{(\tau-t)^{\frac{3}{2}}} - \tfrac{3(\dot\tau-1)(\theta-\xi)}{2(\tau-t)^{\frac{5}{2}}} = -e^{\frac{3}{2}s}\dot\xi + \tfrac{3}{2}(1 - \dot\tau)ye^s\,. \tag{4.8}$$

We then transform the physical variables (a, w) to self-similar variables (A, W) by

$$w(\theta, t) = e^{-\frac{s}{2}}W(y, s) + \kappa(t)\,, \quad a(\theta, t) = A(y, s)\,. \tag{4.9}$$

Introducing the parameter

$$\beta_\tau = \beta_\tau(t) = \tfrac{1}{1-\dot\tau(t)}\,, \tag{4.10}$$

a simple computation shows that (W, A) solve

$$\partial_s W - \tfrac{1}{2}W + \left(\tfrac{3}{2}y + \beta_\tau W + e^{\frac{s}{2}}\beta_\tau(\kappa - \dot\xi)\right)\partial_y W$$
$$= -e^{-\frac{s}{2}}\beta_\tau\dot\kappa - \tfrac{8}{3}e^{-\frac{s}{2}}\beta_\tau A(e^{-\frac{s}{2}}W + \kappa)\,, \tag{4.11a}$$
$$\partial_s A + \left(\tfrac{3}{2}y + \tfrac{2}{3}\beta_\tau W + e^{\frac{s}{2}}\beta_\tau(\tfrac{2}{3}\kappa - \dot\xi)\right)\partial_y A$$
$$= -\tfrac{4}{3}\beta_\tau e^{-s}A^2 + \tfrac{1}{6}\beta_\tau e^{-s}(e^{-\frac{s}{2}}W + \kappa)^2\,, \tag{4.11b}$$

with initial conditions given at self-similar time $s = -\log\varepsilon$ by

$$W(y, -\log\varepsilon) = \varepsilon^{-\frac{1}{2}}(w_0(\theta) - \kappa_0)\,, \qquad A(y, -\log\varepsilon) = a_0(\theta)\,, \tag{4.12}$$

and

$$\kappa(-\varepsilon) = \kappa_0\,, \qquad \tau(-\varepsilon) = 0\,, \qquad \xi(-\varepsilon) = 0\,. \tag{4.13}$$

For notational brevity, we introduce the transport velocities and forcing functions

$$\begin{aligned}
\mathcal{V}_W &:= \tfrac{3}{2}y + \beta_\tau W + e^{\frac{s}{2}}\beta_\tau(\kappa - \dot{\xi})\,,\\
F_W &:= -\tfrac{8}{3}e^{-\frac{s}{2}}\beta_\tau A(e^{-\frac{s}{2}}W + \kappa) &&\text{(4.14a)}\\
\mathcal{V}_A &:= \tfrac{3}{2}y + \tfrac{2}{3}\beta_\tau W + e^{\frac{s}{2}}\beta_\tau(\tfrac{2}{3}\kappa - \dot{\xi})\,,\\
F_A &:= -\tfrac{4}{3}\beta_\tau e^{-s}A^2 + \tfrac{1}{6}\beta_\tau e^{-s}(e^{-\frac{s}{2}}W + \kappa)^2\,, &&\text{(4.14b)}
\end{aligned}$$

so that (4.11) takes the form

$$\begin{aligned}
\partial_s W - \tfrac{1}{2}W + \mathcal{V}_W\partial_y W &= -\beta_\tau e^{-\frac{s}{2}}\dot{\kappa} + F_W\,, &&\text{(4.15a)}\\
\partial_s A + \mathcal{V}_A\partial_y A &= F_A\,. &&\text{(4.15b)}
\end{aligned}$$

We shall also consider the perturbation of the stable self-similar stationary solution $\overline{W}(y)$ of the Burgers equation[4]; the function $\widetilde{W} = W - \overline{W}$ solves

$$\begin{aligned}
&\partial_s\widetilde{W} + \left(-\tfrac{1}{2} + \beta_\tau\partial_y\overline{W} + \tfrac{8}{3}e^{-s}\beta_\tau A\right)\widetilde{W} + \mathcal{V}_W\partial_y\widetilde{W}\\
&\quad = (1-\beta_\tau)\overline{W}\partial_y\overline{W} - \tfrac{8}{3}e^{-s}\beta_\tau A\overline{W} - e^{-\frac{s}{2}}\beta_\tau\dot{\kappa}\,. &&\text{(4.16)}
\end{aligned}$$

4.2 Bounds on the Solution

In order to obtain the necessary quantitative bounds on characteristics and their derivatives, we shall make use of the bounds on W provided by Theorem 4.4 of [1] for the shock formation process. As such, we give a precise description of the initial data used for the asymptotically self-similar shock formation.

4.2.1 Initial Data in Self-similar Variables

It is convenient to describe the initial data in terms of the self-similar variables $(W(\cdot, -\log\varepsilon), A(\cdot, -\log\varepsilon))$ defined in (4.12), which may be equivalently written as

$$w_0(\theta) = \varepsilon^{\frac{1}{2}}W(y, -\log\varepsilon) + \kappa_0\,, \qquad a_0(\theta) = A(y, -\log\varepsilon)\,. \tag{4.17}$$

4 Recall that $\overline{W}(y)$ is the solution of $-\frac{1}{2}\overline{W} + \left(\frac{3y}{2} + \overline{W}\right)\partial_y\overline{W} = 0$ and has an explicit formula which is obtained by inverting the cubic polynomial $\overline{W}^3 + \overline{W} = -y$.

We choose $w_0 \in C^4(\mathbb{T})$ so that for all $\theta \in \mathbb{T}$:

$$\tfrac{7}{8}\kappa_0 \leqslant w_0(\theta) \leqslant \tfrac{9}{8}\kappa_0 \,, \qquad \text{where} \qquad \kappa_0 \geqslant 3 \,. \tag{4.18}$$

We assume that the initial data $(W(\cdot, -\log\varepsilon), A(\cdot, -\log\varepsilon))$ has compact support in the set

$$\mathcal{X}_0 := \left\{ |y| \leqslant 2\varepsilon^{-1} \right\} .$$

In order to obtain stable shock formation, we require that[5]

$$W(0, -\log\varepsilon) = 0 \,, \quad \partial_y W(0, -\log\varepsilon) = -1 \,, \quad \partial_y^2 W(0, -\log\varepsilon) = 0 \,. \tag{4.19}$$

As in [2], there exists a sufficiently large parameter $M = M(\kappa_0) \geqslant 1$ (which is in particular independent of ε), a small length scale ℓ, and a large length scale $\mathcal{L}$ by

$$\ell = (\log M)^{-5} \,, \qquad \mathcal{L} = \varepsilon^{-\frac{1}{10}} \,. \tag{4.20}$$

The initial datum of $\widetilde{W} = W - \overline{W}$ is given by

$$\widetilde{W}(y, -\log\varepsilon) = W(y, -\log\varepsilon) - \overline{W}(y) = \varepsilon^{-\frac{1}{2}} (w_0(\theta) - \overline{w}_\varepsilon(\theta)) =: \varepsilon^{-\frac{1}{2}} \widetilde{w}_0(\theta) \,,$$

where we have defined $\overline{w}_\varepsilon(\theta) = \varepsilon^{\frac{1}{2}} \overline{W}(\varepsilon^{-\frac{3}{2}}\theta) + \kappa_0$. We consider data such that for $|y| \leqslant \mathcal{L}$,

$$(1 + y^2)^{-\frac{1}{6}} \left| \widetilde{W}(y, -\log\varepsilon) \right| \leqslant \varepsilon^{\frac{1}{10}} \,, \tag{4.21a}$$

$$(1 + y^2)^{\frac{1}{3}} \left| \partial_y \widetilde{W}(y, -\log\varepsilon) \right| \leqslant \varepsilon^{\frac{1}{11}} \,, \tag{4.21b}$$

for $|y| \leqslant \ell$ (equivalently $|\theta| \leqslant \varepsilon^{\frac{3}{2}}\ell$), we assume that

$$\left| \partial_y^4 \widetilde{W}(y, -\log\varepsilon) \right| \leqslant \varepsilon^{\frac{5}{8}} \qquad \Leftrightarrow \qquad \left| \partial_\theta^4 \widetilde{w}_0(\theta) \right| \leqslant \varepsilon^{-\frac{39}{8}} \,, \tag{4.22}$$

and at $y = 0$, we have that

$$\left| \partial_y^3 \widetilde{W}(0, -\log\varepsilon) \right| \leqslant \varepsilon^{\frac{3}{8}} \qquad \Leftrightarrow \qquad \left| \partial_\theta^3 \widetilde{w}_0(0) \right| \leqslant \varepsilon^{-\frac{29}{8}} \,. \tag{4.23}$$

For y in the region $\{|y| \geqslant \mathcal{L}\} \cap \mathcal{X}_0$, we suppose that

$$(1 + y^2)^{-\frac{1}{6}} \left| W(y, -\log\varepsilon) \right| \leqslant 1 + \varepsilon^{\frac{1}{11}} \,, \tag{4.24a}$$

[5] As shown in Corollary 4.7 in [1], the conditions (4.19) on the initial data are satisfied by any data in an open set (within azimuthal symmetry) in the C^4 topology, as long as a global non-degenerate minimal slope is attained at a point.

$$(1+y^2)^{\frac{1}{3}} \left|\partial_y W(y, -\log\varepsilon)\right| \leqslant 1+\varepsilon^{\frac{1}{12}}\,, \tag{4.24b}$$

while for W_y, globally for all $y \in \mathcal{X}(-\log\varepsilon)$ we shall assume that

$$\left|\partial_y W(y, -\log\varepsilon)\right| \leqslant (1+y^2)^{-\frac{1}{3}}\,, \tag{4.25a}$$

$$\left|\partial_y^2 W(y, -\log\varepsilon)\right| \leqslant 7(1+y^2)^{-\frac{1}{3}}\,, \tag{4.25b}$$

$$\left|\partial_y^\gamma W(y, -\log\varepsilon)\right| \lesssim (1+y^2)^{-\frac{1}{3}} \quad \text{for} \quad \gamma = 3, 4, 5\,. \tag{4.25c}$$

For the initial conditions of $A(y, -\log\varepsilon) = a_0(\theta)$, we require that $a_0 \in C^4(\mathbb{T})$, and that

$$\|a_0\|_{C^0} \leqslant \varepsilon\,, \qquad \|\partial_x a_0\|_{C^0} \leqslant \tfrac{\kappa_0}{14}\,, \qquad \|a_0\|_{C^5} \lesssim 1\,. \tag{4.26}$$

4.2.2 Bounds on *W* and *A*

The following facts are established in [2]. The spatial support of (W, A) is the s-dependent ball

$$\mathcal{X}(s) := \left\{|y| \leqslant 2\varepsilon^{\frac{1}{2}} e^{\frac{3}{2}s}\right\} \text{ for all } s \geqslant -\log\varepsilon\,. \tag{4.27}$$

It follows that

$$1+y^2 \leqslant 40\varepsilon e^{3s} \qquad \Rightarrow \qquad (1+y^2)^{\frac{1}{3}} \leqslant 4\varepsilon^{\frac{1}{3}} e^s\,. \tag{4.28}$$

We have the following bounds for $W(y, s)$ for all $y \in \mathbb{R}$ and $s \geqslant -\log\varepsilon$:

$$\left|\partial^\gamma W(y, s)\right| \leqslant \begin{cases} (1+2\varepsilon^{\frac{1}{20}})(1+y^2)^{\frac{1}{6}}, & \text{if } \gamma = 0\,, \\ 2(1+y^2)^{-\frac{1}{3}}, & \text{if } \gamma = 1\,, \\ M^{\frac{1}{3}}(1+y^2)^{-\frac{1}{3}}, & \text{if } \gamma = 2\,. \end{cases} \tag{4.29}$$

For the perturbation function $\widetilde{W}(y, s) = W(y, s) - \overline{W}(y)$ and for $|y| \leqslant \mathcal{L} = \varepsilon^{-\frac{1}{10}}$,

$$\left|\widetilde{W}(y, s)\right| \leqslant 2\varepsilon^{\frac{1}{11}}(1+y^2)^{\frac{1}{6}}\,, \tag{4.30a}$$

$$\left|\partial_y \widetilde{W}(y, s)\right| \leqslant 2\varepsilon^{\frac{1}{12}}(1+y^2)^{-\frac{1}{3}}\,, \tag{4.30b}$$

while for $|y| \leqslant \ell = (\log M)^{-5}$,

$$\left|\partial^\gamma \widetilde{W}(y, s)\right| \leqslant (\log M)^4 \varepsilon^{\frac{1}{10}} |y|^{4-\gamma} + M\varepsilon^{\frac{1}{4}} |y|^{3-\gamma}\,, \qquad \gamma \leqslant 3\,, \tag{4.31a}$$

$$\left|\partial^4 \widetilde{W}(y, s)\right| \leqslant \varepsilon^{\frac{1}{10}}\,, \tag{4.31b}$$

and at $y = 0$,

$$\left|\partial^3 \widetilde{W}(0,s)\right| \leqslant \varepsilon^{\frac{1}{4}}, \tag{4.32}$$

for all $s \geqslant -\log\varepsilon$. With w_0 satisfying (4.18), as shown in [1] via the maximum principle, we have that

$$\tfrac{4\kappa_0}{5} \leqslant w(\theta,t) \leqslant \tfrac{5\kappa_0}{4}, \quad t \in [-\varepsilon, T_*). \tag{4.33}$$

4.2.3 Bootstrap Assumptions on $\partial_\theta^\gamma a$, $\gamma \leqslant 2$

Bounds for a and $\partial_\theta a$ were previously established in [2]. In this paper, we revisit these estimates and establish the following sharp bootstrap bounds

$$|a(\theta,t)| \leqslant 2\kappa_0^2\varepsilon, \tag{4.34a}$$
$$|\partial_\theta a(\theta,t)| \leqslant 2\kappa_0, \tag{4.34b}$$
$$\left|\partial_\theta^2 a(\theta,t)\right| \leqslant 12e^s, \tag{4.34c}$$

for all $\theta \in \mathbb{T}$ and $t \in [-\varepsilon, T_*)$. The bootstrap bounds (4.34) are closed in Section 4.6 below.

4.3 Evolution Equations and Bounds for the Modulation Variables

The modulation variables $\tau(t)$, $\xi(t)$, and $\kappa(t)$ are used to impose the following constraints at $y = 0$

$$W(0,s) = 0, \qquad \partial_y W(0,s) = -1, \qquad \partial_y^2 W(0,s) = 0. \tag{4.35}$$

Imposing $\partial_y W(0,s) = -1$ in the first derivative of (4.11a) at $y = 0$, and using (4.35), shows that

$$\dot{\tau}(t) = e^{-\frac{s}{2}}\tfrac{8}{3}\left(\kappa(t)A_y(0,s) - e^{-\frac{s}{2}}A(0,s)\right). \tag{4.36a}$$

Next, requiring that $\partial_y^2 W(0,s) = 0$ holds, by taking the second derivative of (4.11a), evaluating the resulting equation at $y = 0$ and using (4.35), we obtain

$$\dot{\xi}(t) - \kappa(t) = -\tfrac{8}{3}\tfrac{e^{-s}}{W_{yyy}(0,s)}\left(2e^{-\frac{s}{2}}A_y(0,s) - \kappa A_{yy}(0,s)\right), \tag{4.36b}$$

and finally with $W(0,s) = 0$ used in (4.11a), we find that

$$\begin{aligned}\dot{\kappa}(t) &= -\tfrac{8}{3}\left(\kappa(t)A(0,s) + \kappa\tfrac{A_{yy}(0,s)}{W_{yyy}(0,s)} - 2e^{-\frac{s}{2}}\tfrac{A_y(0,s)}{W_{yyy}(0,s)}\right)\\ &= -\tfrac{8}{3}\kappa(t)A(0,s) - e^s(\dot{\xi} - \kappa)(t).\end{aligned} \tag{4.36c}$$

The equations (4.36) are ODEs for the modulation functions. From (4.34), it follows that for ε taken sufficiently small, for all $t \in [-\varepsilon, T_*)$ we have

$$\left|\dot\tau(t)\right| \leqslant 9\kappa_0^2 \varepsilon e^{-s}\,, \qquad \left|\dot\kappa(t)\right| \leqslant 6\kappa_0^3 \varepsilon\,, \qquad \left|\dot\xi(t)\right| \leqslant \kappa_0 + 8\kappa_0^2\varepsilon^2\,. \tag{4.37}$$

For the last bound, we have used that since $\int_{-\varepsilon}^{T_*}(1-\dot\tau(t'))dt' = \varepsilon$, then

$$|T_*| \leqslant 7\kappa_0^2\varepsilon^3\,. \tag{4.38}$$

It follows that

$$|\kappa - \kappa_0| \leqslant 7\kappa_0^3\varepsilon^2\,, \qquad |\tau| \leqslant 7\kappa_0^2\varepsilon^3\,, \qquad |\xi| \leqslant 2\varepsilon\kappa_0\,, \qquad |1-\beta_\tau| \leqslant 7\kappa_0^2\varepsilon e^{-s}\,. \tag{4.39}$$

4.4 Characteristics in Physical Variables (x, t)

4.4.1 3-Characteristics η Associated to λ_3

We let $\eta(x, t)$ denote the characteristics of $\lambda_3 = w$ so that

$$\partial_t\eta(x,t) = w(\eta(x,t),t) \qquad \text{for } -\varepsilon < t < T_*\,, \tag{4.40a}$$
$$\eta(x,-\varepsilon) = x\,, \tag{4.40b}$$

for all labels x.

4.4.2 2-Characteristics ϕ Associated to λ_2

We let $\phi(x, t)$ denote the characteristics of $\lambda_2 = \frac{2}{3}w$ so that (Fig. 7)

$$\partial_t\phi(x,t) = \tfrac{2}{3}w(\phi(x,t),t) \qquad \text{for } -\varepsilon < t < T_*\,, \tag{4.41a}$$
$$\phi(x,-\varepsilon) = x\,, \tag{4.41b}$$

for all labels x.

4.4.3 Identities Involving the 3-Characteristics η

From (4.40) it follows that

$$\eta(x,t) = x + \int_{-\varepsilon}^{t} w(\eta(x,t'),t')dt' \tag{4.42}$$

and from (3.5a) that

$$\partial_t w(\eta(x,t),t) = -\tfrac{8}{3}a(\eta(x,t),t)w(\eta(x,t),t)\,.$$

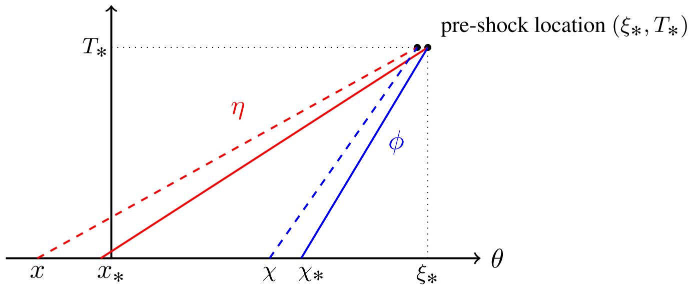

Fig. 7 Characteristic evolution during the pre-shock formation. The blowup point is ξ_*, the blowup time is T_*, and the blowup label x_* satisfies $\eta(x_*, T_*) = \xi_*$. In red, we display the 3-characteristics $\eta(\cdot, t)$ originating from the blowup label x_* and a nearby label x, while in blue we display the 2-characteristics $\phi(\cdot, t)$ originating from the label $\chi_* = \phi^{-1}(\xi_*, T_*)$ and a nearby label χ

We define the integrating factor

$$I_t(x) = e^{-\frac{8}{3}\int_{-\varepsilon}^{t} a(\eta(x,r),r)dr}\,, \tag{4.43}$$

Integration yields

$$w(\eta(x,t),t) = I_t(x)w_0(x)\,. \tag{4.44}$$

We make use of the following identities:

$$I'_\tau I_\tau^{-1} = -\tfrac{8}{3}\int_{-\varepsilon}^{\tau} a' \circ \eta\ \eta_x dr\,, \tag{4.45a}$$

$$I''_\tau I_\tau^{-1} = \Big(\tfrac{8}{3}\int_{-\varepsilon}^{\tau} a' \circ \eta\ \eta_x dr\Big)^2 - \tfrac{8}{3}\int_{-\varepsilon}^{\tau} \big(a'' \circ \eta\ \eta_x^2 + a' \circ \eta\ \eta_{xx}\big)dr \tag{4.45b}$$

$$\begin{aligned} I'''_\tau I_\tau^{-1} &= -\tfrac{512}{27}\Big(\int_{-\varepsilon}^{\tau} a' \circ \eta\ \eta_x\, dr\Big)^3 \\ &\quad + \tfrac{64}{3}\Big(\int_{-\varepsilon}^{\tau} a' \circ \eta\ \eta_x dr\Big)\int_{-\varepsilon}^{\tau}\Big(a'' \circ \eta\ \eta_x^2 + a' \circ \eta\ \eta_{xx}\Big)\, dr \\ &\quad - \tfrac{8}{3}\int_{-\varepsilon}^{\tau}\big(a''' \circ \eta\ \eta_x^3 + 3a'' \circ \eta\ \eta_x\eta_{xx} + a' \circ \eta\ \eta_{xxx}\big)dr\,, \end{aligned} \tag{4.45c}$$

$$\begin{aligned} I''''_\tau I_\tau^{-1} &= \tfrac{4096}{81}\Big(\int_{-\varepsilon}^{\tau} a' \circ \eta\ \eta_x\, dr\Big)^4 \\ &\quad - \tfrac{1024}{9}\Big(\int_{-\varepsilon}^{\tau} a' \circ \eta\ \eta_x dr\Big)^2 \int_{-\varepsilon}^{\tau}\Big(a'' \circ \eta\ \eta_x^2 + a' \circ \eta\ \eta_{xx}\Big)\, dr \\ &\quad + \tfrac{64}{3}\Big(\int_{-\varepsilon}^{\tau}(a'' \circ \eta\eta_x^2 + a' \circ \eta\ \eta_{xx})dr\Big)^2 \\ &\quad + \tfrac{256}{9}\Big(\int_{-\varepsilon}^{\tau} a' \circ \eta\ \eta_x dr\Big)\int_{-\varepsilon}^{\tau}\Big(a''' \circ \eta\ \eta_x^3 + 3a'' \circ \eta\ \eta_x\eta_{xx} + a' \circ \eta\ \eta_{xxx}\Big)\, dr \end{aligned}$$

$$
\begin{aligned}
&- \tfrac{8}{3} \int_{-\varepsilon}^{\tau} \big(a'''' \circ \eta \, \eta_x^4 + 6a''' \circ \eta \, \eta_x^2 \eta_{xx} + 3a'' \circ \eta \, \eta_{xx}^2 \\
&+ 4a'' \circ \eta \, \eta_x \eta_{xxx} + a' \circ \eta \, \eta_{xxxx} \big) dr
\end{aligned} \tag{4.45d}
$$

and from (4.42),

$$
\eta = x + w_0 \int_{-\varepsilon}^{t} I_\tau d\tau \,, \tag{4.46a}
$$

$$
\partial_x \eta = 1 + w_0' \int_{-\varepsilon}^{t} I_\tau d\tau + w_0 \int_{-\varepsilon}^{t} I_\tau' d\tau \,, \tag{4.46b}
$$

$$
\partial_x^2 \eta = w_0'' \int_{-\varepsilon}^{t} I_\tau d\tau + 2w_0' \int_{-\varepsilon}^{t} I_\tau' d\tau + w_0 \int_{-\varepsilon}^{t} I_\tau'' d\tau \,, \tag{4.46c}
$$

$$
\partial_x^3 \eta = w_0''' \int_{-\varepsilon}^{t} I_\tau d\tau + 3w_0'' \int_{-\varepsilon}^{t} I_\tau' d\tau + 3w_0' \int_{-\varepsilon}^{t} I_\tau'' d\tau + w_0 \int_{-\varepsilon}^{t} I_\tau''' d\tau \,, \tag{4.46d}
$$

$$
\begin{aligned}
\partial_x^4 \eta = {}& w_0'''' \int_{-\varepsilon}^{t} I_\tau d\tau + 4w_0''' \int_{-\varepsilon}^{t} I_\tau' d\tau + 6w_0'' \int_{-\varepsilon}^{t} I_\tau'' d\tau \\
&+ 4w_0' \int_{-\varepsilon}^{t} I_\tau''' d\tau + w_0 \int_{-\varepsilon}^{t} I_s'''' ds \,.
\end{aligned} \tag{4.46e}
$$

4.4.4 Identities Involving the 2-Characteristics ϕ

We write (3.5a) as

$$
\partial_t w + \tfrac{2}{3} w \partial_x w + \tfrac{1}{3} w \partial_x w = -\tfrac{8}{3} a w \,, \tag{4.47}
$$

and define the Lagrangian variables

$$
\mathcal{W} = w \circ \phi \,, \quad \mathcal{V} = \tfrac{2}{3} w \circ \phi = \partial_t \phi \,.
$$

Then it follows from the chain-rule that (4.47) can be written as

$$
\partial_t \mathcal{W} + \tfrac{1}{2} (\partial_x \phi)^{-1} \partial_x \mathcal{V} \mathcal{W} = -\tfrac{8}{3} \mathcal{W} a \circ \phi \,. \tag{4.48}
$$

We multiply (4.48) by $(\partial_x \phi)^{\frac{1}{2}}$ to find that

$$
\partial_t \big((\partial_x \phi)^{\frac{1}{2}} \mathcal{W} \big) = -\tfrac{8}{3} \big((\partial_x \phi)^{\frac{1}{2}} \mathcal{W} \big) \, a \circ \phi \,,
$$

and hence that

$$
\partial_x \phi(x,t) = \frac{w_0^2(x)}{w^2(\phi(x,t),t)} e^{-\frac{16}{3} \int_{-\varepsilon}^{t} a(\phi(x,s),s) ds} \,. \tag{4.49}
$$

It follows from (4.18), (4.33), (4.34), and since is ε small enough, that

$$
\tfrac{12}{25} \leqslant \partial_x \phi(x,t) \leqslant 2 \,, \quad \tfrac{1}{2} \leqslant \partial_x \phi^{-1}(x,t) \leqslant \tfrac{25}{12} \,, \qquad t \in [-\varepsilon, T_*) \,. \tag{4.50}
$$

Differentiating (4.41a), we see that $\partial_t \partial_x \phi = \frac{2}{3}\partial_x w \circ \phi \, \partial_x \phi$ and that $\partial_x \phi(x, -\varepsilon) = 1$. Hence we have that

$$\tfrac{12}{25} \leqslant e^{\frac{2}{3}\int_{-\varepsilon}^{t} \partial_\theta w(\phi(x,s),s)ds} \leqslant 2\,, \quad \tfrac{1}{2} \leqslant e^{-\frac{2}{3}\int_{-\varepsilon}^{t} \partial_\theta w(\phi(x,s),s)ds} \leqslant \tfrac{25}{12}\,, \qquad t \in [-\varepsilon, T_*)\,. \tag{4.51}$$

Differentiating (4.49), we have that

$$\begin{aligned} \partial_x^2 \phi(x,t) &= e^{-\frac{16}{3}\int_{-\varepsilon}^{t} a(\phi(x,s),s)ds} \frac{w_0^2(x)}{w^2(\phi(x,t),t)} \\ &\quad \Big(-\tfrac{16}{3}\int_{-\varepsilon}^{t} \partial_\theta a \circ \phi \, \phi_x ds + 2\tfrac{w_0'}{w_0} - 2\tfrac{\partial_\theta w \circ\phi \; \phi_x}{w\circ\phi}\Big) \\ &= \partial_x \phi(x,t)\Big(-\tfrac{16}{3}\int_{-\varepsilon}^{t} \partial_\theta a \circ \phi \, \phi_x ds + 2\tfrac{w_0'}{w_0} - 2\tfrac{\partial_\theta w \circ\phi \; \phi_x}{w\circ\phi}\Big)\,. \end{aligned} \tag{4.52}$$

Using that $|w_0'(x)| \leqslant \varepsilon^{-1}$, and the bounds (4.18), (4.34), and (4.50), we see that

$$\left|\partial_x^2 \phi(x,t)\right| \lesssim \tfrac{1}{\varepsilon} + \left|\partial_\theta w(\phi(x,t),t)\right|. \tag{4.53}$$

Finally, differentiating (4.52),

$$\begin{aligned} \partial_x^3 \phi(x,t) &= \phi_{xx}\Big(-\tfrac{16}{3}\int_{-\varepsilon}^{t} \partial_\theta a \circ \phi \, \phi_x ds + 2\tfrac{w_0'}{w_0} - 2\tfrac{w_\theta \circ\phi \; \phi_x}{w\circ\phi}\Big) \\ &\quad + \phi_x\Big(-\tfrac{16}{3}\int_{-\varepsilon}^{t} \big(\partial_\theta^2 a \circ \phi \, \phi_x^2 + \partial_\theta a \circ \phi \, \phi_{xx}\big) ds \\ &\quad + 2\tfrac{w_0 w_0'' - (w_0')^2}{w_0^2} + 2\left(\tfrac{\partial_\theta w\circ\phi \; \phi_x}{w\circ\phi}\right)^2\Big) \\ &\quad - 2\phi_x \tfrac{\partial_\theta^2 w\circ\phi \; \phi_x^2 + \partial_\theta w\circ\phi \; \phi_{xx}}{w\circ\phi}\,. \end{aligned} \tag{4.54}$$

We will make use of the fact that by (4.7), the change of variables formula, and (4.61),

$$\int_{-\varepsilon}^{t} \left|\partial_\theta w(\phi(x,t,t')\right| dt' = \int_{-\log\varepsilon}^{s} \left|\partial_y W(\Phi_A(y,s'),s')\right| \beta_\tau ds'\,.$$

As we will show in (4.62), $\int_{-\log\varepsilon}^{s} \left|\partial_y W(\Phi_A(y,s'),s')\right| \beta_\tau ds' \lesssim 1$. Together with (4.19), (4.23), and (4.34), we see that

$$\begin{aligned} \left|\partial_x^3 \phi(x,t)\right| &\lesssim \tfrac{1}{\varepsilon^2} + \left|w_0''(x)\right| \\ &\quad + \tfrac{1}{\varepsilon}\left|\partial_\theta w(\phi(x,t),t)\right| + \left|\partial_\theta w(\phi(x,t),t)\right|^2 + \left|\partial_\theta^2 w(\phi(x,t),t)\right| \\ &\quad + \int_{-\varepsilon}^{t} \left|\partial_\theta^2 a(\phi(x,t'),t')\right| dt'\,. \end{aligned} \tag{4.55}$$

4.5 Characteristics in Self-similar Coordinates

4.5.1 3-Characteristics in Self-similar Coordinates

Having defined the 3-characteristics $\eta(x,t)$ in (4.40), we now let $\Phi_W(y,s)$ denote the 3-characteristic of the transport velocity for $\mathcal{V}_W$ which emanates from the label y so that

$$\partial_s \Phi_W(y,s) = \mathcal{V}_W(\Phi_W(y,s),s) \qquad \text{for} \quad -\log\varepsilon < s < \infty\,, \tag{4.56a}$$

$$\Phi_W(y,-\log\varepsilon) = y\,, \tag{4.56b}$$

where the velocity $\mathcal{V}_W$ is defined in (4.14a). Before stating the next lemma, we recall from (4.13) that $\xi(-\varepsilon) = 0$ and that particle labels are assigned at $t = -\varepsilon \Leftrightarrow s = -\log\varepsilon$.

Lemma 4.2 (3-characteristics in physical and self-similar coordinates) *With particle labels related by*

$$x = \varepsilon^{\frac{3}{2}} y\,, \tag{4.57}$$

we have that

$$\eta(x,t) = e^{-\frac{3}{2}s}\Phi_W(y,s) + \xi(t)\,, \tag{4.58}$$

or equivalently

$$\Phi_W(y,s) = e^{\frac{3}{2}s}\left(\eta(x,t) - \xi(t)\right)\,. \tag{4.59}$$

Proof of Lemma 4.2 From (4.56a), we have that

$$\partial_s\left(e^{-\frac{3}{2}s}\Phi_W(y,s)\right) = \left(e^{-\frac{s}{2}}W(\Phi_W(y,s),s) + \kappa - \dot{\xi}\right)\beta_\tau e^{-s}\,.$$

Using (4.7) and (4.10), we see that

$$\partial_t\left(e^{-\frac{3}{2}s}\Phi_W(y,s) + \xi\right) = e^{-\frac{s}{2}}W(\Phi_W(y,s),s) + \kappa\,.$$

Then, from (4.9), we have that $e^{-\frac{s}{2}}W(y,s) + \kappa(t) = w(e^{-\frac{3}{2}s}y + \xi(t),t)$, and hence

$$\partial_t\left(e^{-\frac{3}{2}s}\Phi_W(y,s) + \xi\right) = w\left(e^{-\frac{3}{2}s}\Phi_W(y,s) + \xi(t),t\right)\,.$$

On the other hand, from (4.40a) we have $\partial_t\eta(x,t) = w(\eta(x,t),t)$, which then proves the identity (4.58). □

4.5.2 2-Characteristics Φ_A in Self-similar Coordinates

Having defined the 2-characteristics ϕ in (x, t) coordinates, we now define their self-similar counterparts in (y, s) coordinates. We define the 2-characteristics Φ_A by

$$\partial_s \Phi_A(y, s) = \mathcal{V}_A(\Phi_A(y, s), s) \qquad \text{for } -\log\varepsilon < s < \infty\,, \tag{4.60a}$$

$$\Phi_A(y, -\log\varepsilon) = y\,. \tag{4.60b}$$

where the transport velocity $\mathcal{V}_A$ is given in (4.14b). In the same way that we established (4.59), we have that

$$\Phi_A(y, s) = e^{\frac{3}{2}s}\left(\phi(x, t) - \xi(t)\right)\,, \tag{4.61}$$

where $x = \varepsilon^{\frac{3}{2}} y$. The following integral bound was proven in Corollary 8.4 in [2]:

$$\sup_{y\in\mathcal{X}(-\log\varepsilon)} \int_{-\log\varepsilon}^{s} \left|W_y(\Phi_A(y, s'), s')\right| ds' \lesssim 1\,. \tag{4.62}$$

4.5.3 The Unique Blowup Trajectory Associated to 3-Characteristics

A basic advantage of the use of self-similar coordinates is that the blowup trajectory can be isolated. In particular, all but one of the trajectories $\Phi_W(y, s)$ "eventually escape" exponentially fast towards infinity.

Lemma 4.3 (The unique blowup trajectory) *There exists a unique blowup label y_* such that*

$$\Phi_W(y_*, s) = e^{\frac{3}{2}s}\left(\eta(x_*, t) - \xi(t)\right)$$

is the unique trajectory which converges to $y = 0$ as $s \to \infty$. Moreover,

$$|\Phi_W(y_*, s)| \leqslant 20\kappa_0 e^{-\frac{5}{2}s} \qquad \text{for all } s \geqslant -\log\varepsilon\,, \tag{4.63}$$

and

$$|y_*| \leqslant 20\kappa_0\varepsilon^{\frac{5}{2}} \qquad \Leftrightarrow \qquad |x_*| \leqslant 20\kappa_0\varepsilon^4\,. \tag{4.64}$$

Proof of Lemma 4.3 Using (4.56a), we can write the evolution equation for Φ_W as

$$\partial_s \Phi_W(y, s) = \mathcal{V}_W \circ \Phi_W = \tfrac{1}{2}\Phi_W(y, s) + G_\Phi(y, s) + h(s)\,, \tag{4.65}$$

where

$$G_\Phi = G \circ \Phi_W\,, \tag{4.66a}$$

$$G = \left(\overline{W} + y\right) + (1 - \beta_\tau)\overline{W} + \beta_\tau \widetilde{W}\,, \tag{4.66b}$$

$$h = e^{\frac{s}{2}} \beta_\tau (\kappa - \dot{\xi})\,. \tag{4.66c}$$

The particular form of G_Φ in (4.66b) is chosen to make use of the fact that for all y,

$$\left| y + \overline{W}(y) \right| \leqslant |y|^3\,, \tag{4.67}$$

which follows from the identity $|y + \overline{W}(y)| = |\overline{W}(y)|^3$ and the bound $|\overline{W}(y)| \leqslant |y|$.

Hence, we integrate (4.65) to obtain

$$\Phi_W(y_*, s) = e^{\frac{s}{2}} \varepsilon^{\frac{1}{2}} y_* + e^{\frac{s}{2}} \int_{-\log\varepsilon}^{s} e^{-\frac{s'}{2}} \left(G_\Phi(y_*, s') + h(s') \right) ds'\,. \tag{4.68}$$

If $e^{-\frac{s'}{2}} \left(G_\Phi(y_*, s') + h(s') \right)$ is integrable on $[-\log\varepsilon, \infty)$ then, we can rewrite (4.68) as

$$\begin{aligned} \Phi_W(y_*, s) = e^{\frac{s}{2}} &\left(\varepsilon^{\frac{1}{2}} y_* + \int_{-\log\varepsilon}^{\infty} e^{-\frac{s'}{2}} \left(G_\Phi(y_*, s') + h(s') \right) ds' \right) \\ &- e^{\frac{s}{2}} \int_s^\infty e^{-\frac{s'}{2}} \left(G_\Phi(y_*, s') + h(s') \right) ds'\,. \end{aligned} \tag{4.69}$$

Together with (4.32), (4.34), and (4.90), the identity (4.36b) shows that

$$\left| \dot{\xi} - \kappa \right| \leqslant 38\kappa_0 e^{-3s}\,, \tag{4.70}$$

so that using (4.66c) and (4.70), we have the bound

$$|h(s)| \leqslant 39\kappa_0 e^{-\frac{5}{2}s}\,, \tag{4.71}$$

so the integrability of $e^{-\frac{s'}{2}} G_\Phi(y_*, s')$ will be of paramount importance.

We additionally note that since the first term on the right side of (4.69) is a constant multiplying $e^{\frac{s}{2}}$, in order for $\Phi_W(y_*, s) \to 0$ as $s \to \infty$, this constant must vanish, and thus, we must insist that

$$y_* = -\varepsilon^{-\frac{1}{2}} \int_{-\log\varepsilon}^{\infty} e^{-\frac{s'}{2}} \left(G_\Phi(y_*, s') + h(s') \right) ds'\,, \tag{4.72a}$$

which then implies

$$\Phi_W(y_*, s) = -e^{\frac{s}{2}} \int_s^\infty e^{-\frac{s'}{2}} \left(G_\Phi(y_*, s') + h(s') \right) ds'\,. \tag{4.72b}$$

Notice that (4.72) implies that as long as $e^{-\frac{s'}{2}} G_\Phi(y_*, s')$ is integrable,

$$\Phi_W(y_*, -\log\varepsilon) = y_*\,, \quad \text{and} \quad \lim_{s\to\infty} \Phi_W(y_*, s) = 0\,.$$

We shall now establish the existence of a unique trajectory $\Phi_W(y_*, s)$ solving (4.72b). We define the set

$$\mathcal{T} = \{\varphi \in C^0([-\log\varepsilon, \infty)) \ : \ |\varphi(s)| \leqslant 20\kappa_0 e^{-\frac{5}{2}s}\},$$

with norm given by $\|\varphi\|_{\mathcal{T}} := \sup_{s\in[-\log\varepsilon,\infty)} e^{\frac{5}{2}s}\,|\varphi(s)|$, and consider the map Ψ, which maps $\overline{\varphi} \in \mathcal{T}$ to φ, given by

$$\varphi(s) = \Psi(\overline{\varphi}(s)) := -e^{\frac{s}{2}} \int_s^\infty e^{-\frac{s'}{2}} \left(G_{\overline{\varphi}}(s') + h(s')\right) ds' .$$

We note that for $\overline{\varphi} \in \mathcal{T}$, $|\overline{\varphi}| \leqslant \alpha\kappa_0\varepsilon^{\frac{5}{2}} \leqslant \ell$ for ε small enough, so that we may apply the bounds (4.31a) to the function $G_{\overline{\varphi}}(s')$. Doing so, we see that the bounds (4.67), (4.31a) and (4.39) show that for ε taken small enough,

$$\begin{aligned}\left|G_{\overline{\varphi}}(s)\right| &\leqslant |\overline{\varphi}(s)|^3 + (1+6\varepsilon^2)\left((\log M)^4 \varepsilon^{\frac{1}{10}}\,|\overline{\varphi}(s)|^4 + M\varepsilon^{\frac{1}{3}}\,|\overline{\varphi}(s)|^3\right) + 6\varepsilon e^{-s}\overline{\varphi}(s)\\ &\leqslant (20\kappa_0)^3 e^{-\frac{15}{2}s} + \varepsilon^{\frac{1}{12}}\varepsilon^{3\alpha}(20\kappa_0)^4 e^{-10s} + 120\kappa_0\varepsilon e^{-\frac{7}{2}s} \leqslant 122\kappa_0\varepsilon e^{-\frac{7}{2}s} .\end{aligned}$$

Together with (4.71), we have that

$$e^{-\frac{s'}{2}}\left(\left|G_{\overline{\varphi}}(s')\right| + \left|h(s')\right|\right) \leqslant 40\kappa_0 e^{-3s'} .$$

By the fundamental theorem of calculus, $s \mapsto \varphi(s)$ is continuous, and satisfies the bound

$$|\varphi(s)| \leqslant 18\kappa_0 e^{-\frac{5}{2}s} \quad \text{for all } s \geqslant -\log\varepsilon .$$

Therefore, $\Psi : \mathcal{T} \to \mathcal{T}$.

Let us now prove that Ψ is a contraction. Suppose that $\varphi_1 = \Psi(\overline{\varphi}_1)$ and $\varphi_2 = \Psi(\overline{\varphi}_2)$. We then have

$$|\varphi_1(s) - \varphi_2(s)| \leqslant e^{\frac{s}{2}} \int_s^\infty e^{-\frac{s'}{2}} \left|G_{\overline{\varphi}_1}(s') - G_{\overline{\varphi}_2}(s')\right| ds' . \tag{4.73}$$

From the identity in footnote 4 (in a similar fashion to (4.67)), we have that

$$\left|\overline{W}(y_1) + y_1 - \overline{W}(y_2) - y_2\right| \leqslant \left|y_1^3 - y_2^3\right| ,$$

so that

$$\begin{aligned}&\left|\left(\overline{W}(\overline{\varphi}_1(s)) + \overline{\varphi}_1(s)\right) - \left(\overline{W}(\overline{\varphi}_2(s)) + \varphi_2(s)\right)\right|\\ &\quad \leqslant \left|\overline{\varphi}_1^3(s) - \overline{\varphi}_2^3(s)\right|\end{aligned}$$

$$\begin{aligned}
&\leqslant \left|\overline{\varphi}_1(s)^2 + \overline{\varphi}_1(s)\overline{\varphi}_2(s) + \overline{\varphi}_2(s)^2\right| \left|\overline{\varphi}_1(s) - \overline{\varphi}_2(s)\right| \\
&\leqslant \varepsilon^2 e^{-s} \left|\overline{\varphi}_1(s) - \overline{\varphi}_2(s)\right| , \qquad (4.74)
\end{aligned}$$

where we have used that both $\overline{\varphi}_1$ and $\overline{\varphi}_2$ are in $\mathcal{T}$. Next, since $\left|\overline{W}(y_1) - \overline{W}(y_2)\right| \leqslant |y_1 - y_2|$, by (4.39),

$$|1 - \beta_\tau| \left|\overline{W}(\overline{\varphi}_1(s)) - \overline{W}(\overline{\varphi}_2(s))\right| \leqslant 6\varepsilon e^{-s} \left|\overline{\varphi}_1(s) - \overline{\varphi}_2(s)\right| , \qquad (4.75)$$

and finally, employing the mean value theorem together with the bound (4.31a), for some a function $s \mapsto \alpha(s) \in (0, 1)$ and

$$\begin{aligned}
&|\beta_\tau| \left|\widetilde{W}(\overline{\varphi}_1(s), s) - \widetilde{W}(\overline{\varphi}_2(s), s)\right| \\
&\quad \leqslant 2 \left|\partial_y \widetilde{W}((1 - \alpha(s))\overline{\varphi}_1(s) + \alpha(s)\overline{\varphi}_2(s), s)\right| \left|\overline{\varphi}_1(s) - \overline{\varphi}_2(s)\right| \\
&\quad \leqslant 2 \left((\log M)^4 \varepsilon^{\frac{1}{10}} (20\kappa_0)^3 e^{-\frac{15}{2}s} + M\varepsilon^{\frac{1}{3}} (20\kappa_0)^2 e^{-5s}\right) \left|\overline{\varphi}_1(s) - \overline{\varphi}_2(s)\right| \\
&\quad \leqslant \varepsilon^4 e^{-s} \left|\overline{\varphi}_1(s) - \overline{\varphi}_2(s)\right| . \qquad (4.76)
\end{aligned}$$

Combining the bounds (4.74), (4.75), and (4.76), and taking ε sufficiently small, we have that

$$\left|G_{\overline{\varphi}_1}(s') - G_{\overline{\varphi}_2}(s')\right| \leqslant 7\varepsilon e^{-s} \left|\overline{\varphi}_1(s) - \overline{\varphi}_2(s)\right| ,$$

and thus from (4.73), we see that

$$\begin{aligned}
e^{\frac{5}{2}s} |\varphi_1(s) - \varphi_2(s)| &\leqslant e^{3s} \int_s^\infty e^{-\frac{1}{2}s'} \left|G_{\overline{\varphi}_1}(s') - G_{\overline{\varphi}_2}(s')\right| ds' \\
&\leqslant 7\varepsilon e^{3s} \int_s^\infty e^{-\frac{1}{2}s'} \left|\overline{\varphi}_1(s') - \overline{\varphi}_2(s')\right| ds' \\
&\leqslant 14\varepsilon \sup_{s\in[-\log\varepsilon,\infty)} e^{\frac{5}{2}s} \left|\overline{\varphi}_1(s) - \overline{\varphi}_2(s)\right| ,
\end{aligned}$$

so that

$$\|\varphi_1 - \varphi_2\|_{\mathcal{T}} \leqslant 14\varepsilon \left\|\overline{\varphi}_1 - \overline{\varphi}_2\right\|_{\mathcal{T}} ,$$

which shows that Ψ is a contraction. By the contraction mapping theorem, there exists a unique trajectory $\varphi \in \mathcal{T}$ such that for all $s \geqslant -\log\varepsilon$,

$$\begin{aligned}
\varphi(s) = -e^{\frac{s}{2}} \int_s^\infty e^{-\frac{s'}{2}} &\left(\left(\overline{W}(\varphi(s)) + \varphi(s)\right) + (1 - \beta_\tau)\overline{W}(\varphi(s))\right. \\
&\left. + \beta_\tau \widetilde{W}(\varphi(s), s) + h(s')\right) ds' ,
\end{aligned}$$

or equivalently

$$e^{-\frac{s}{2}}\varphi(s) = -\int_s^\infty e^{-\frac{s'}{2}}\left(\varphi(s) + \beta_\tau W(\varphi(s), s) + h(s')\right) ds' .$$

Differentiating this identity in self-similar time shows that

$$\partial_s \varphi = \mathcal{V}_W \circ \varphi .$$

Setting

$$y_* = -\varepsilon^{-\frac{1}{2}} \int_{-\log\varepsilon}^\infty e^{-\frac{s'}{2}} \Big(\big(\overline{W}(\varphi(s)) + \varphi(s)\big) \\ +(1-\beta_\tau)\overline{W}(\varphi(s)) + \beta_\tau \widetilde{W}(\varphi(s), s) + h(s')\Big) ds' ,$$

we see that $\varphi(-\log\varepsilon) = y_*$ from which it follows that

$$\Phi_W(y_*, s) = \varphi(s) \quad \text{for all } s \geqslant -\log\varepsilon ,$$

and $\Phi_W(y_*, s)$ is a solution to (4.72). Clearly $\left|y_*\right| \leqslant 20\kappa_0\varepsilon^{\frac{5}{2}}$ and by (4.57), it follows that $\left|x_*\right| \leqslant 20\kappa_0\varepsilon^4$.

We next show that y_* is the only blowup label. From (4.14b) and (4.56), we have that

$$\partial_s(\Phi_W(y_*, s) - \Phi_W(y, s)) = \tfrac{3}{2}(\Phi_W(y_*, s) - \Phi_W(y, s)) \\ + \beta_\tau W(\Phi_W(y_*, s), s) - \beta_\tau W(\Phi_W(y, s), s) .$$

Suppose that $y_* \geqslant y$. By the mean value theorem and the bound (4.39), we have that

$$|\beta_\tau W(\Phi_W(y_*, s), s) - \beta_\tau W(\Phi_W(y, s), s)| \leqslant (1+6\varepsilon)(\Phi_W(y_*, s) - \Phi_W(y, s)) .$$

Here we have used the global bound $\left|\partial_y W(y, s)\right| \leqslant 1$ and the fact that characteristics cannot cross so that $\Phi_W(y_*, s) - \Phi_W(y, s) \geqslant 0$. Therefore,

$$\partial_s(\Phi_W(y_*, s) - \Phi_W(y, s)) \geqslant (\tfrac{1}{2} - \varepsilon^{\frac{3}{4}})(\Phi_W(y_*, s) - \Phi_W(y, s)) ,$$

and then

$$\Phi_W(y_*, s) - \Phi_W(y, s) \geqslant \varepsilon^{\frac{1}{2}} e^{(\frac{1}{2}-\varepsilon^{\frac{3}{4}})s}(y_* - y) .$$

If $y \geqslant y_*$, in the same way we, we obtain $\Phi_W(y, s) - \Phi_W(y_*, s) \geqslant \varepsilon^{\frac{1}{2}} e^{(\frac{1}{2}-\varepsilon^{\frac{3}{4}})s}(y - y_*)$. □

4.6 Bounds for $\partial_x^\gamma a, \gamma \leqslant 4$

4.6.1 Improving the Bootstrap Bound for a

We note here that from (3.8) and (3.9), the specific vorticity $\varpi = \frac{16}{w^2}(w - \partial_\theta a)$ solves

$$\partial_t \varpi + \tfrac{2}{3} w \partial_\theta \varpi = \tfrac{8}{3} a \varpi \,, \quad \varpi(x, -\varepsilon) = \varpi_0(x) \,,$$

and hence

$$\varpi(\phi(x,t),t) = e^{\frac{8}{3}\int_{-\varepsilon}^{t} a(\phi(x,t'),t')dt'} \varpi_0(x) \,. \tag{4.77}$$

We also have from (3.10b), that

$$a(\phi(x,t),t) = a_0(x) + \int_{-\varepsilon}^{t} (-\tfrac{4}{3}a^2 + \tfrac{1}{6}w^2) \circ \phi ds \tag{4.78}$$

so that assuming the bootstrap bound $|a(\theta,t)| \leqslant 2\kappa_0^2\varepsilon$ and using (4.26) and (4.33), we find that for ε taken sufficiently small,

$$\|a(\cdot,t)\|_{L^\infty} \leqslant \tfrac{3}{2}\kappa_0^2\varepsilon \,, \quad t \in [-\varepsilon, T_*) \,, \tag{4.79}$$

which improves the bootstrap bound (4.34a).

4.6.2 Improving the Bootstrap Bound for $\partial_\theta a$

From (4.50), we see that $\phi(\cdot,t)$ is a diffeomorphism with a well-defined inverse map, so that for each $t \in [-\varepsilon, T_*)$ and for ε small enough, the identity (4.77) and the bound (4.79) show that

$$(1-\varepsilon)\varpi_0(\theta) \leqslant \varpi(\phi(\theta,t),t) \leqslant (1+\varepsilon)\varpi_0(\theta) \,, \quad t \in [-\varepsilon, T_*) \,, \tag{4.80}$$

From (4.18), $\frac{7}{8}\kappa_0 \leqslant w_0(\theta) \leqslant \frac{9}{8}\kappa_0$. Since $\varpi_0 = \frac{16}{w_0^2}(w_0 - \partial_\theta a_0)$, by (4.26),

$$\frac{101}{10\kappa_0} \leqslant \varpi_0(\theta) \leqslant \frac{27}{\kappa_0} \,,$$

and by (4.80), for ε small enough,

$$\frac{10}{\kappa_0} \leqslant \varpi(\theta,t) \leqslant \frac{28}{\kappa_0} \,, \quad \theta \in \mathbb{T}, t \in [-\varepsilon, T_*) \,. \tag{4.81}$$

Again using that

$$\partial_\theta a = w - \frac{w^2}{16}\varpi \,, \tag{4.82}$$

we then have that

$$\left|\partial_\theta a(\theta,t)\right| = \left|w - \tfrac{w^2}{16}\varpi\right| \leqslant \tfrac{3}{2}\kappa_0\,, \quad \theta\in\mathbb{T}, t\in[-\varepsilon,T_*)\,, \tag{4.83}$$

which improves the bootstrap bound (4.34b).

4.6.3 Improving the Bootstrap Bound for $\partial_\theta^2 a$

Differentiating (4.77), we have that

$$\begin{aligned}\partial_\theta\varpi(\phi(x,t),t) &= (\partial_x\phi(x,t))^{-1}e^{\frac{8}{3}\int_0^t a(\phi(x,t'),t')dt'}\\ &\quad\left(\partial_\theta\varpi_0(x) + \tfrac{8}{3}\varpi_0\int_{-\varepsilon}^t \partial_\theta a(\phi(x,t'),t')\partial_x\phi(x,t')dt'\right)\\ &= (\partial_x\phi(x,t))^{-1}\varpi(\phi(x,t),t)\\ &\quad\left(\tfrac{\partial_\theta\varpi_0(x)}{\varpi_0(x)} + \tfrac{8}{3}\int_{-\varepsilon}^t \partial_\theta a(\phi(x,t'),t')\partial_x\phi(x,t')dt'\right)\,. \end{aligned}\tag{4.84}$$

It follows from (4.50), (4.80), (4.83), and (4.84) that for ε small enough,

$$\left|\partial_\theta\varpi(\phi(x,t),t)\right| \leqslant \tfrac{51}{24}\left|\partial_\theta\varpi_0(x)\right| + 500\varepsilon\,. \tag{4.85}$$

Using the formula

$$\partial_\theta\varpi_0 = \tfrac{16}{w_0^2}(\partial_\theta w_0 - \partial_x^2 a_0) - \tfrac{32}{w_0^3}(w_0 - \partial_\theta a_0)\partial_\theta w_0$$

and the bounds (4.18), $-\frac{1}{\varepsilon} \leqslant \partial_\theta w_0(x)$, and (4.26), we estimate that

$$\left|\partial_\theta\varpi_0(x)\right| \leqslant \tfrac{34}{\kappa_0^2\varepsilon}\,, \tag{4.86}$$

and hence from (4.85),

$$\left|\partial_\theta\varpi(x,t)\right| \leqslant \tfrac{70}{\kappa_0^2\varepsilon}\,, \quad x\in\mathbb{T}, t\in[-\varepsilon,T_*)\,. \tag{4.87}$$

We shall use the fact that

$$\partial_\theta^2 a = \partial_\theta w(1 - \tfrac{1}{8}w\varpi) - \tfrac{w^2}{16}\partial_\theta\varpi\,, \tag{4.88}$$

so that combined with the above estimates,

$$\left|\partial_\theta^2 a(x,t)\right| \leqslant \tfrac{7}{2}\left|\partial_\theta w(x,t)\right| + \tfrac{7}{\varepsilon}\,, \tag{4.89}$$

and hence by (4.9), we have that

$$\left|\partial_y^2 A(y,s)\right| \leqslant \tfrac{7}{2}e^{-2s}\left|\partial_y W(y,s)\right| + e^{-3s}\tfrac{7}{\varepsilon} \leqslant \tfrac{23}{2}e^{-2s}\,, \tag{4.90}$$

where we have used that $\left|\partial_y W(y,s)\right| \leqslant 1$ as proven in [2]. This then implies that

$$\left|\partial_\theta^2 a(x,t)\right| = e^{3s}\left|\partial_y^2 A(y,s)\right| \leqslant \tfrac{23}{2} e^s\,, \quad x \in \mathbb{T}, t \in [-\varepsilon, T_*) \tag{4.91}$$

which improves the bootstrap bound (4.34c).

4.6.4 A Bound for $\partial_\theta^3 a$

We next differentiate (4.84) to obtain

$$\begin{aligned}\partial_\theta^2 \varpi(\phi(x,t),t) &= \left(\phi_x^{-1}\partial_\theta \varpi \circ \phi - \phi_x^{-3}\phi_{xx}\varpi \circ \phi\right)\left(\tfrac{\partial_\theta \varpi_0}{\varpi_0} + \tfrac{8}{3}\int_{-\varepsilon}^t \partial_\theta a \circ \phi\, \phi_x dt'\right)\\ &\quad + \phi_x^{-2}\varpi \circ \phi\Big(\tfrac{\varpi_0 \partial_\theta^2 \varpi_0 - (\partial_\theta \varpi_0)^2}{\varpi_0^2}\\ &\quad + \tfrac{8}{3}\underbrace{\int_{-\varepsilon}^t \left(\partial_\theta^2 a \circ \phi\, \phi_x^2 + \partial_\theta a \circ \phi\, \phi_{xx}\right)dt'}_{\mathcal{R}(x,t)}\Big)\,. \end{aligned}\tag{4.92}$$

We first bound the integral $\mathcal{R}$. By (4.50), (4.53), and (4.89), we have that

$$\left|\mathcal{R}(x,t)\right| \lesssim \int_{-\varepsilon}^t (1 + \left|\partial_\theta a \circ \phi\right|)(\tfrac{1}{\varepsilon} + \left|\partial_\theta w \circ \phi\right|)dt'\,. \tag{4.93}$$

We note that by (4.61),

$$\partial_\theta w(\phi(x,t),t) = e^s W_y(\Phi_A(y,s),s)\,.$$

The identity (4.7) then shows that $dt = \beta_\tau e^{-s} ds$ so that by the change of variables formula, we have that

$$\int_{-\varepsilon}^t \left|\partial_\theta w(\phi(x,t'),t')\right| dt' = \int_{-\log\varepsilon}^s \left|W_y(\Phi_A(y,s'),s')\right|\beta_\tau ds' \lesssim 1\,, \tag{4.94}$$

where we have used (4.62) for the last inequality. Hence, with (4.34) and (4.93), we have that

$$\left|\mathcal{R}(x,t)\right| \lesssim 1\,. \tag{4.95}$$

With (4.95), the formula (4.92) and the bounds (4.34) and (4.86) allow us to estimate $\partial_\theta^2 \varpi \circ \phi$ in the following way:

$$\begin{aligned}\left|\partial_\theta^2 \varpi(\phi(x,t),t)\right| &\lesssim 1 + \tfrac{1}{\varepsilon}\left|\phi_{xx}(x,t)\right| + \left|\partial_\theta^2 \varpi_0(x)\right|\\ &\lesssim \tfrac{1}{\varepsilon^2} + \tfrac{1}{\varepsilon}\left|\partial_\theta w(\phi(x,t),t)\right| + \left|\partial_\theta^2 \varpi_0(x)\right| \end{aligned}\tag{4.96}$$

where we have used (4.53) for the last inequality.

Differentiating (4.88) yields the identity

$$\partial_\theta^3 a = \partial_\theta^2 w(1 - \tfrac{1}{8} w\varpi) - \tfrac{w^2}{16}\partial_\theta^2 \varpi - \tfrac{1}{4} w \partial_\theta w \partial_\theta \varpi - \tfrac{1}{8}\varpi (\partial_\theta w)^2 \tag{4.97}$$

so that

$$\begin{aligned} \left|\partial_\theta^3 a(x,t)\right| &\lesssim \left|\partial_\theta^2 w(x,t)\right| + \left|\partial_\theta^2 \varpi(x,t)\right| + \left|\partial_\theta w(x,t)\right|^2 + \tfrac{1}{\varepsilon}\left|\partial_\theta w(x,t)\right| \\ &\lesssim \tfrac{1}{\varepsilon^2} + \left|\partial_\theta^2 w(x,t)\right| + \left|\partial_\theta^2 \varpi_0(\phi^{-1}(x,t),t)\right| \\ &\quad + \left|\partial_\theta w(x,t)\right|^2 + \tfrac{1}{\varepsilon}\left|\partial_\theta w(x,t)\right| \end{aligned} \tag{4.98}$$

where we have used (4.96) for the last inequality.

Restricting the identity (4.97) to $t = -\varepsilon$, we see that

$$\tfrac{w_0^2}{16}\partial_\theta^2 \varpi_0 = -\partial_\theta^3 a_0 + \partial_\theta^2 w_0(\tfrac{1}{8} w_0 \varpi_0 - 1) - \tfrac{1}{4} w_0 \partial_\theta w_0 \partial_\theta \varpi_0 - \tfrac{1}{8}(\partial_\theta w_0)^2 \varpi_0\,, \tag{4.99}$$

and so

$$\left|\partial_\theta^2 \varpi_0\right| \lesssim \tfrac{1}{\varepsilon^2} + \left|\partial_\theta^3 a_0\right| + \left|\partial_\theta^2 w_0\right| \lesssim \tfrac{1}{\varepsilon^2} + \left|\partial_\theta^2 w_0\right| \tag{4.100}$$

since we assumed that $\left|\partial_\theta^3 a_0(x)\right| \lesssim 1$ in (4.26). Using the bound (4.100) in (4.98) shows that

$$\begin{aligned} &\left|\partial_\theta^3 a(\eta(x,t),t)\right| \\ &\quad\lesssim \tfrac{1}{\varepsilon^2} + \left|\partial_\theta^2 w(\eta(x,t),t)\right| + \left|\partial_\theta^2 w_0(\phi^{-1}(\eta(x,t),t),t)\right| \\ &\quad\quad + \left|\partial_\theta w(\eta(x,t),t)\right|^2 + \tfrac{1}{\varepsilon}\left|\partial_\theta w(\eta(x,t),t)\right|. \end{aligned} \tag{4.101}$$

By (4.25b), we have that for $x \in \mathbb{T}$, $\left|\partial_\theta^2 w_0(x)\right| \lesssim \varepsilon^{-\frac{5}{2}}$ and therefore

$$\left|\partial_\theta^2 w_0(\phi^{-1}(\eta(x,t),t),t)\right| \lesssim \varepsilon^{-\frac{5}{2}}\,. \tag{4.102}$$

Using this bound in (4.101), for all $t \in [-\varepsilon, T_*)$,

$$\left|\partial_\theta^3 a(\eta(x,t),t)\right| \lesssim \varepsilon^{-\frac{5}{2}} + \left|\partial_\theta^2 w(\eta(x,t),t)\right| + \left|\partial_\theta w(\eta(x,t),t)\right|^2 + \tfrac{1}{\varepsilon}\left|\partial_\theta w(\eta(x,t),t)\right|. \tag{4.103}$$

4.6.5 A Bound for $\partial_\theta^4 a$

As we will now explain, the bound for $\partial_\theta^4 a(x,t)$ does not depend on $\partial_x^4 \eta$, $\partial_x^4 \phi$, or $\partial_\theta^4 w$, and as such is merely a consequence of the bounds that have already been established.

To obtain this bound, we make one final differentiation of (4.92) and obtain that

$$
\begin{aligned}
&\partial_\theta^3 \varpi(\phi(x,t),t) \\
&\quad = \Big(\phi_x^{-1}\partial_\theta^2\varpi\circ\phi + 3\phi_x^{-5}\phi_{xx}^2\varpi\circ\phi - \phi_x^{-4}\phi_{xxx}\varpi\circ\phi - 2\phi_x^{-3}\phi_{xx}\partial_\theta\varpi\circ\phi\Big) \\
&\qquad \Big(\tfrac{\partial_\theta\varpi_0}{\varpi_0} + \tfrac{8}{3}\int_{-\varepsilon}^{t}\partial_\theta a\circ\phi\,\phi_x dt'\Big) \\
&\qquad + \Big(2\phi_x^{-2}\partial_\theta\varpi\circ\phi - 3\phi_x^{-4}\phi_{xx}\varpi\circ\phi\Big)\Big(\tfrac{\varpi_0\partial_\theta^2\varpi_0 - (\partial_\theta\varpi_0)^2}{\varpi_0^2} \\
&\qquad + \tfrac{8}{3}\int_{-\varepsilon}^{t}\big(\partial_\theta^2 a\circ\phi\,\phi_x^2 + \partial_\theta a\circ\phi\,\phi_{xx}\big)dt'\Big) \\
&\qquad + \phi_x^{-3}\varpi\circ\phi\Big(\tfrac{\varpi_0^2\partial_\theta^3\varpi_0 - 3\varpi_0\partial_\theta\varpi_0\partial_\theta^2\varpi_0 + 2(\partial_\theta\varpi_0)^3}{\varpi_0^3} \\
&\quad + \underbrace{\tfrac{8}{3}\int_{-\varepsilon}^{t}\big(\partial_\theta^3 a\circ\phi\,\phi_x^3 + 3\partial_\theta^2 a\circ\phi\,\phi_x\phi_{xx} + \partial_\theta a\circ\phi\,\phi_{xxx}\big)dt'\Big)}_{\mathcal{S}(x,t)}.
\end{aligned}
\tag{4.104}
$$

Our goal is to bound $\left|\partial_\theta^3\varpi(\phi(x,t),t)\right|$ using the identity (4.104). The time integral in the first line is $\mathcal{O}(\varepsilon)$ due to (4.34) and (4.50). The time integral in the second line is the term $\mathcal{R}(x,t)$ in (4.92), which was estimated in (4.95). It thus remains to establish the bound for the integral term $\mathcal{S}(x,t)$ on the third line. We write $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3$, where

$$\mathcal{S}_1(x,t) = \int_{-\varepsilon}^{t} \partial_\theta^3 a\circ\phi\,\phi_x^3 dt'\,, \tag{4.105a}$$

$$\mathcal{S}_2(x,t) = \int_{-\varepsilon}^{t} 3\partial_\theta^2 a\circ\phi\,\phi_x\phi_{xx} dt'\,, \tag{4.105b}$$

$$\mathcal{S}_3(x,t) = \int_{-\varepsilon}^{t} \partial_\theta a\circ\phi\,\phi_{xxx} dt'\,, \tag{4.105c}$$

and we shall first estimate the integral $\mathcal{S}_3$. The key idea in estimating $\mathcal{S}_3$ is to use the identity (4.54) for ϕ_{xxx} and isolate the term

$$\partial_\theta^2 w\circ\phi\,\phi_x^2 + \partial_\theta w\circ\phi\,\phi_{xx} =: \partial_x(\partial_\theta w\circ\phi\,\phi_x)\,,$$

and estimate its integral in a very careful manner.

The identity for ϕ_{xxx} in (4.54) and the bound (4.55), together with the estimates (4.89) and (4.94), and the integral bound (4.62), we conclude that

$$\left|\mathcal{S}_3(x,t)\right| \lesssim \tfrac{1}{\varepsilon} + \varepsilon\left|w_0''(x)\right| + \int_{-\varepsilon}^{t}\left|\partial_\theta w(\phi(x,t'),t')\right|^2 dt' + \left|\mathcal{S}_4(x,t)\right|, \tag{4.106}$$

where the term $\mathcal{S}_4$ contains the important term on the last line of (4.54), and is given by

$$\mathcal{S}_4(x,t) = \int_{-\varepsilon}^{t} \partial_x(a\circ\phi)(w\circ\phi)^{-1}\partial_x(\partial_\theta w\circ\phi\,\phi_x)dt' \,. \tag{4.107}$$

We now rewrite the evolution equation (4.47) as $\partial_t(w\circ\phi)+\frac{8}{3}(aw)\circ\phi = -\frac{1}{3}(w\partial_\theta w)\circ\phi$ which yields

$$\partial_\theta w\circ\phi\,\phi_x = -3\phi_x\,(w\circ\phi)^{-1}\partial_t(w\circ\phi) - 8a\circ\phi\,\phi_x \,.$$

Differentiating this equation, we have that

$$\begin{aligned}\partial_x\big(\partial_\theta w\circ\phi\,\phi_x\big) &= -3\phi_x\,(w\circ\phi)^{-1}\partial_t(\partial_\theta w\circ\phi\,\phi_x)\\ &\quad - 3\phi_x^2\tfrac{\partial_\theta w\circ\phi}{w\circ\phi}\left(\tfrac{8}{3}a\circ\phi+\tfrac{1}{3}\partial_\theta w\circ\phi\right)+\phi_{xx}\partial_\theta w\circ\phi - 8\partial_\theta a\circ\phi\,\phi_x^2 \,.\end{aligned}$$

We can then write the term $\mathcal{S}_4$ in (4.107) as $\mathcal{S}_4=\mathcal{S}_{4a}+\mathcal{S}_{4b}$, where

$$\begin{aligned}\mathcal{S}_{4a}(x,t) &= -3\int_{-\varepsilon}^{t}\partial_x(a\circ\phi)\phi_x\,(w\circ\phi)^{-2}\partial_t(\partial_\theta w\circ\phi\,\phi_x)dt' \,,\\ \mathcal{S}_{4b}(x,t) &= -\int_{-\varepsilon}^{t}\partial_x(a\circ\phi)\Big(3\phi_x^2\tfrac{\partial_\theta w\circ\phi}{(w\circ\phi)^2}\left(\tfrac{8}{3}a\circ\phi-\tfrac{1}{3}\partial_\theta w\circ\phi\right)\\ &\qquad -\phi_{xx}\tfrac{\partial_\theta w\circ\phi}{w\circ\phi}+8\tfrac{\partial_\theta a\circ\phi}{w\circ\phi}\phi_x^2\Big)\,dt' \,.\end{aligned}$$

The term $\mathcal{S}_{4a}(x,t)$ requires a careful analysis; meanwhile, the bounds (4.18), (4.33), (4.34), (4.38), (4.50), (4.94) together with (4.53) show that

$$\big|\mathcal{S}_{4b}(x,t)\big| \lesssim \tfrac{1}{\varepsilon}+\int_{-\varepsilon}^{t}\big|\partial_\theta w\circ\phi\big|^2 dt' \,.$$

To estimate $\mathcal{S}_{4a}(x,t)$ we integrate by parts, appeal to the identities (3.10b), (4.41a), and (4.47), to obtain that

$$\begin{aligned}\mathcal{S}_{4a}(x,t) &= 3a_0'w_0^{-2}w_0' - 3\partial_x(a\circ\phi)\phi_x^2\,(w\circ\phi)^{-2}\partial_\theta w\circ\phi\\ &\quad + 4\int_{-\varepsilon}^{t}\partial_x(a\circ\phi)(w\circ\phi)^{-2}(\partial_\theta w\circ\phi)^2\,\phi_x^2 dt'\\ &\quad + 3\int_{-\varepsilon}^{t}\partial_x(-\tfrac{4}{3}a^2\circ\phi+\tfrac{1}{6}w^2\circ\phi)\phi_x^2\,(w\circ\phi)^{-2}\partial_\theta w\circ\phi dt'\\ &\quad + 16\int_{-\varepsilon}^{t}\partial_x(a\circ\phi)\phi_x^2\,(w\circ\phi)^{-2}a\circ\phi\partial_\theta w\circ\phi dt' \,.\end{aligned}$$

From the above identity and the bounds (4.18), (4.33), (4.34), (4.50), (4.94), we obtain that

$$\left|\mathcal{S}_{4a}(x,t)\right| \lesssim \tfrac{1}{\varepsilon} + \left|\partial_\theta w(\phi(x,t),t)\right| + \int_{-\varepsilon}^{t} \left|\partial_\theta w(\phi(x,t'),t')\right|^2 dt' \,.$$

Using the above bound in (4.106) shows that

$$\left|\mathcal{S}_3(x,t)\right| \lesssim \tfrac{1}{\varepsilon} + \varepsilon\left|w_0''(x)\right| + \left|\partial_\theta w(\phi(x,t),t)\right| + \int_{-\varepsilon}^{t} \left|\partial_\theta w(\phi(x,t'),t')\right|^2 dt' \,. \tag{4.108}$$

Having estimated $\mathcal{S}_3$ in (4.105), it remains to bound $\mathcal{S}_1$ and $\mathcal{S}_2$.

For $\mathcal{S}_1$, we return to the identity (4.88) and write

$$\partial_\theta^2 a \circ \phi = \left(\partial_\theta w \circ \phi \phi_x\right) \phi_x^{-1}(1 - \tfrac{1}{8} w\varpi) \circ \phi - \tfrac{w^2\circ\phi}{16} \partial_\theta \varpi \circ \phi \,,$$

so that after differentiation in x

$$\begin{aligned} \partial_\theta^3 a \circ \phi\, \phi_x &= \partial_x\left(\partial_\theta w \circ \phi \phi_x\right) \phi_x^{-1}(1 - \tfrac{1}{8} w\varpi) \circ \phi \\ &\quad - \left(\partial_\theta w \circ \phi \phi_x\right) \phi_x^{-2} \phi_{xx}(1 - \tfrac{1}{8} w\varpi) \circ \phi \\ &\quad - \tfrac{1}{8}\left(\partial_\theta w \circ \phi \phi_x\right) \partial_\theta(w\varpi) \circ \phi - \tfrac{w^2\circ\phi}{16} \partial_\theta^2 \varpi \circ \phi\, \phi_x \\ &\quad - \tfrac{1}{8} w \circ \phi\, \partial_\theta w \circ \phi\, \phi_x \partial_\theta \varpi \circ \phi \,. \end{aligned} \tag{4.109}$$

Due to (4.109), the integrand $\partial_\theta^3 a \circ \phi\, \phi_x^3$ in $\mathcal{S}_1$ has the same structure to the integrand in $\mathcal{S}_3$, with one additional type of term in the form of $-\frac{w^2\circ\phi}{16} \partial_\theta^2 \varpi \circ \phi\, \phi_x$, which requires us to use the already established bounds (4.96) and (4.100). We therefore can show that $\mathcal{S}_1$ is bounded as

$$\left|\mathcal{S}_1(x,t)\right| \lesssim \tfrac{1}{\varepsilon} + \varepsilon\left|w_0''(x)\right| + \left|\partial_\theta w(\phi(x,t),t)\right| + \int_{-\varepsilon}^{t} \left|\partial_\theta w(\phi(x,t'),t')\right|^2 dt' \,. \tag{4.110}$$

The integral $\mathcal{S}_2$ in (4.105) is relatively straightforward to bound. We use the inequalities (4.53) and (4.89) together with (4.62), and find that

$$\left|\mathcal{S}_2(x,t)\right| \lesssim \tfrac{1}{\varepsilon} + \int_{-\varepsilon}^{t} \left|\partial_\theta w(\phi(x,t'),t')\right|^2 dt' \,. \tag{4.111}$$

Combining the bounds (4.108), (4.110), and (4.111), we have shown that the $\mathcal{S}(x,t)$ integral in (4.104) satisfies

$$\left|\mathcal{S}(x,t)\right| \lesssim \tfrac{1}{\varepsilon} + \varepsilon\left|\partial_\theta^2 w_0(x)\right| + \left|\partial_\theta w(\phi(x,t),t)\right| + \int_{-\varepsilon}^{t} \left|\partial_\theta w(\phi(x,t'),t')\right|^2 dt' \,.$$

It thus follows from (4.53), (4.55), (4.89), and (4.104) that

$$\begin{aligned}\left|\partial_\theta^3 \varpi(\phi(x,t),t)\right| \lesssim \tfrac{1}{\varepsilon^3} + \left|\partial_\theta^3 \varpi_0(x)\right| + \tfrac{1}{\varepsilon}\left|\partial_\theta^2 w_0(x)\right| \\ + \tfrac{1}{\varepsilon^2}\left|\partial_\theta w(\phi(x,t),t)\right| + \tfrac{1}{\varepsilon}\left|\partial_\theta w(\phi(x,t),t)\right|^2 \\ + \tfrac{1}{\varepsilon}\left|\partial_\theta^2 w(\phi(x,t),t)\right| + \int_{-\varepsilon}^t \left|\partial_\theta w(\phi(x,t'),t')\right|^2 dt' .\end{aligned}$$

Therefore, we have that

$$\begin{aligned}\left|\partial_\theta^3 \varpi(\eta(x,t),t)\right| \lesssim \tfrac{1}{\varepsilon^3} + \left|\partial_\theta^3 \varpi_0(\phi^{-1}(\eta(x,t),t))\right| + \tfrac{1}{\varepsilon}\left|\partial_\theta^2 w_0(\phi^{-1}(\eta(x,t),t))\right| \\ + \tfrac{1}{\varepsilon^2}\left|\partial_\theta w(\eta(x,t),t)\right| \\ + \tfrac{1}{\varepsilon}\left|\partial_\theta w(\eta(x,t),t)\right|^2 + \tfrac{1}{\varepsilon}\left|\partial_\theta^2 w(\eta(x,t),t)\right| \\ + \int_{-\varepsilon}^t \left|\partial_\theta w(\phi(\phi^{-1}(\eta(x,t),t),t'),t')\right|^2 dt' .\end{aligned} \tag{4.112}$$

In order to bound the first term in the above inequality, we differentiate (4.99) to obtain

$$\begin{aligned}\tfrac{w_0^2}{16}\partial_\theta^3 \varpi_0 = -\tfrac{1}{8} w_0 \partial_\theta w_0 \partial_\theta^2 \varpi_0 - \partial_\theta^4 a_0 + \partial_\theta^3 w_0 (\tfrac{1}{8} w_0 \varpi_0 - 1) + \tfrac{1}{8}\partial_\theta^2 w_0 \partial_\theta (w_0 \varpi_0) \\ - \partial_\theta (\tfrac{1}{4} w_0 \partial_\theta w_0 \partial_\theta \varpi_0 + \tfrac{1}{8}(\partial_\theta w_0)^2 \varpi_0) .\end{aligned}$$

With (4.100), we see that

$$\begin{aligned}\left|\partial_\theta^3 \varpi_0(\theta)\right| &\lesssim \tfrac{1}{\varepsilon^3} + \tfrac{1}{\varepsilon}\left|\partial_\theta^2 w_0(\theta)\right| + \left|\partial_\theta^3 w_0(\theta)\right| + \left|\partial_\theta^4 a_0(\theta)\right| \\ &\lesssim \tfrac{1}{\varepsilon^3} + \tfrac{1}{\varepsilon}\left|\partial_\theta^2 w_0(\theta)\right| + \left|\partial_\theta^3 w_0(\theta)\right| ,\end{aligned} \tag{4.113}$$

where we have used that $\left|\partial_\theta^4 a_0(x)\right| \lesssim 1$ by (4.26). From (4.25c), for all $x \in \mathbb{T}$, $\left|\partial_\theta^3 w_0(x)\right| \lesssim \varepsilon^{-4}$, so that

$$\left|\partial_\theta^3 w_0(\phi^{-1}(\eta(x,t),t),t)\right| \lesssim \varepsilon^{-4} ,$$

and hence by (4.113),

$$\left|\partial_\theta^3 \varpi_0(\phi^{-1}(\eta(x,t),t),t)\right| \lesssim \varepsilon^{-4} .$$

With this bound and using (4.102), estimate (4.112) becomes

$$\begin{aligned}\left|\partial_\theta^3 \varpi(\eta(x,t),t)\right| \lesssim \varepsilon^{-4} + \tfrac{1}{\varepsilon^2}\left|\partial_\theta w(\eta(x,t),t)\right| + \tfrac{1}{\varepsilon}\left|\partial_\theta w(\eta(x,t),t)\right|^2 \\ + \tfrac{1}{\varepsilon}\left|\partial_\theta^2 w(\eta(x,t),t)\right| + \int_{-\varepsilon}^t \left|\partial_\theta w(\phi(\phi^{-1}(\eta(x,t),t),t'),t')\right|^2 dt' .\end{aligned} \tag{4.114}$$

Having established a bound for the third derivative of ϖ, we are now ready to estimate the fourth derivative of a. We differentiate the identity (4.97) and obtain

$$\begin{aligned}\partial_\theta^4 a &= \partial_\theta^3 w(\tfrac{1}{8}w\varpi - 1) + \tfrac{1}{8}\partial_\theta^2 w\partial_\theta(w^2\varpi) - \tfrac{w^2}{16}\partial_\theta^3\varpi \\ &\quad - \tfrac{1}{8}w\partial_\theta w\partial_\theta^2\varpi - \tfrac{1}{8}\partial_\theta\big(2w\partial_\theta w\partial_\theta\varpi + (\partial_\theta w)^2\varpi\big)\,, \end{aligned} \tag{4.115}$$

so that

$$\begin{aligned}\left|\partial_\theta^4 a(\theta,t)\right| &\lesssim \left|\partial_\theta^3\varpi(\theta,t)\right| + \left|\partial_\theta^3 w(\theta,t)\right| + \big(\tfrac{1}{\varepsilon} + \left|\partial_\theta w(\theta,t)\right|\big) \\ &\quad \big(\tfrac{1}{\varepsilon}\left|\partial_\theta w(\theta,t)\right| + \left|\partial_\theta^2 w(\theta,t)\right| + \left|\partial_\theta^2\varpi(\theta,t)\right|\big)\,,\end{aligned}$$

and with (4.114), we have that

$$\begin{aligned}\left|\partial_\theta^4 a(\eta(x,t),t)\right| &\lesssim \varepsilon^{-4} + \tfrac{1}{\varepsilon^2}\left|\partial_\theta w(\eta(x,t),t)\right| + \tfrac{1}{\varepsilon}\left|\partial_\theta w(\eta(x,t),t)\right|^2 + \left|\partial_\theta^3 w(\eta(x,t),t)\right| \\ &\quad + \big(\tfrac{1}{\varepsilon} + \left|\partial_\theta w(\eta(x,t),t)\right|\big)\big(\left|\partial_\theta^2 w(\eta(x,t),t)\right| + \left|\partial_\theta^2\varpi(\eta(x,t),t)\right|\big) \\ &\quad + \int_{-\varepsilon}^t \left|\partial_\theta w(\phi(\phi^{-1}(\eta(x,t),t),t'),t')\right|^2 dt'\,.\end{aligned}$$

We observe that by (4.96), (4.100), and (4.102),

$$\left|\partial_\theta^2\varpi(\eta(x,t),t)\right| \lesssim \varepsilon^{-\frac{5}{2}} + \varepsilon^{-1}\left|\partial_\theta w(\eta(x,t),t)\right|\,,$$

and thus

$$\begin{aligned}\left|\partial_\theta^4 a(\eta(x,t),t)\right| &\lesssim \varepsilon^{-4} + \varepsilon^{-\frac{5}{2}}\left|\partial_\theta w(\eta(x,t),t)\right| + \varepsilon^{-1}\left|\partial_\theta w(\eta(x,t),t)\right|^2 \\ &\quad + \varepsilon^{-1}\left|\partial_\theta^2 w(\eta(x,t),t)\right| \\ &\quad + \left|\partial_\theta w(\eta(x,t),t)\right|\left|\partial_\theta^2 w(\eta(x,t),t)\right| + \left|\partial_\theta^3 w(\eta(x,t),t)\right| \\ &\quad + \int_{-\varepsilon}^t \left|\partial_\theta w(\phi(\phi^{-1}(\eta(x,t),t),t'),t')\right|^2 dt'\,.\end{aligned} \tag{4.116}$$

4.7 Bounds on Derivatives of 3-Characteristics

4.7.1 Identities for $\partial_\theta^\gamma w \circ \eta$

With the integrating factor $I_t(x)$ defined in (4.43), the equation (4.44) is written as $w \circ \eta = I_t w_0$, and differentiation yields

$$\partial_\theta w \circ \eta\ \eta_x = I_t w_0' + I_t' w_0\,, \tag{4.117a}$$

$$\partial_\theta^2 w \circ \eta\ \eta_x^2 = I_t w_0'' + 2I_t' w_0' + I_t'' w_0 - \partial_\theta w \circ \eta \eta_{xx}\,, \tag{4.117b}$$

$$\begin{aligned}\partial_\theta^3 w \circ \eta\ \eta_x^3 &= I_t w_0''' + 3I_t' w_0'' + 3I_t'' w_0' + I_t''' w_0 \\ &\quad - 3\partial_\theta^2 w \circ \eta \eta_x \eta_{xx} - \partial_\theta w \circ \eta \eta_{xxx}\,,\end{aligned} \tag{4.117c}$$

$$\begin{aligned}\partial_\theta^4 w\circ\eta\,\eta_x^4 &= I_t w_0'''' + 4I_t' w_0''' + 6I_t'' w_0'' + 4I_t''' w_0' + I_t'''' w_0\\ &\quad - 6\partial_\theta^3 w\circ\eta\eta_x^2\eta_{xx} - 4\partial_\theta^2 w\circ\eta\eta_x\eta_{xxx}\\ &\quad - 3\partial_\theta^2 w\circ\eta\eta_{xx}^2 - \partial_\theta w\circ\eta\eta_{xxxx}\,. \end{aligned} \tag{4.117d}$$

4.7.2 Bounds for $\partial_x\eta$

We shall now obtain the precise rate at which $\partial_x\eta(x_*,t)\to 0$ as $t\to T_*$, as well as a global bound for $\partial_x\eta(x,t)$.

Lemma 4.4 *For $-\varepsilon\leqslant t\leqslant T_*$, at the blowup label $x_*=\varepsilon^{\frac32}y_*$,*

$$\tfrac{1-\varepsilon}{\varepsilon}e^{-s}\leqslant\partial_x\eta(x_*,t)\leqslant\tfrac{1+\varepsilon}{\varepsilon}e^{-s}\,, \tag{4.118}$$

and for all labels x, we have that

$$\sup_{t'\in[t,T_*)}\partial_x\eta(x,t')\leqslant t(6-\tfrac{1}{\varepsilon})+6\varepsilon\quad\text{for}\quad|x-x_*|\leqslant\varepsilon^2\,, \tag{4.119}$$

and

$$\tfrac14\varepsilon\leqslant\partial_x\eta(x,t)\leqslant 3\quad\text{for}\quad|x-x_*|\geqslant\varepsilon^2\,. \tag{4.120}$$

Proof of Lemma 4.4 *Step 1. Bounds at the blowup label y_*.* From (4.57) and (4.58), we have that

$$\partial_x\eta(x,t)=\varepsilon^{-\frac32}e^{-\frac32 s}\partial_y\Phi_W(y,s)\,,\qquad y=\varepsilon^{-\frac32}x\,. \tag{4.121}$$

We will use the following identity, which may be derived from (4.58), the x-differentiated version of (4.40a), and the y-differentiated version of (4.9):

$$\partial_y\Phi_W(y,s)=e^{\frac32 s}\varepsilon^{\frac32}e^{\int_{-\log\varepsilon}^{s}\beta_\tau\partial_y W(\Phi_W(y,r),r)dr}\,. \tag{4.122}$$

We consider the blowup trajectory $\Phi_W(y_*,s)$. For this, we decompose $\beta_\tau\partial_y W$ as

$$\beta_\tau\partial_y W=\partial_y\overline{W}-(1-\beta_\tau)\partial_y\overline{W}+\beta_\tau\partial_y\widetilde{W}\,. \tag{4.123}$$

By (4.64), $|y_*|\leqslant 20\kappa_0\varepsilon^{\frac52}$ and by (4.63), $|\Phi_W(y_*,s)|\leqslant 20\kappa_0 e^{-\frac52 s}$ and as such, this unique trajectory stays in the Taylor region $|y|\leqslant\ell$ for ε sufficiently small. Using the Taylor remainder theorem, we have that $\partial_y\overline{W}(y)=-1+b_2y^2$, where $b_2=\frac12\partial_y^3\overline{W}(\overline{y})$ for some $\overline{y}$ between 0 and y, so that $\left|b_2-3\right|\leqslant\varepsilon^2$. Substitution of this expansion into (4.123) gives

$$\beta_\tau\partial_y W=-1+b_2y^2-(1-\beta_\tau)\partial_y\overline{W}+\beta_\tau\partial_y\widetilde{W}\,. \tag{4.124}$$

Hence,

$$e^{\int_{-\log\varepsilon}^{s}\beta_\tau\partial_y W(\Phi_W(y_*,r),r)dr}$$
$$=\tfrac{1}{\varepsilon}e^{-s}e^{b_2\int_{-\log\varepsilon}^{s}\Phi_W(y_*,r)^2dr}e^{\int_{-\log\varepsilon}^{s}(\beta_\tau-1)\partial_y\overline{W}(\Phi_W(y_*,r),r)dr}e^{\int_{-\log\varepsilon}^{s}\beta_\tau\partial_y\widetilde{W}(\Phi_W(y_*,r),r)dr}. \tag{4.125}$$

From (4.31a), (4.39), the fact that $\left|\partial_y\overline{W}\right|\leqslant 1$, and (4.63) we have that for ε small enough,

$$1-\varepsilon\leqslant e^{b_2\int_{-\log\varepsilon}^{s}\Phi_W(y_*,r)^2dr}e^{\int_{-\log\varepsilon}^{s}(\beta_\tau-1)\partial_y\overline{W}(\Phi_W(y_*,r),r)dr}e^{\int_{-\log\varepsilon}^{s}\beta_\tau\partial_y\widetilde{W}(\Phi_W(y_*,r),r)dr}$$
$$\leqslant 1+\varepsilon\,,$$

and therefore

$$\tfrac{1-\varepsilon}{\varepsilon}e^{-s}\leqslant e^{\int_{-\log\varepsilon}^{s}\beta_\tau\partial_y W(\Phi_W(y_*,r),r)dr}\leqslant\tfrac{1+\varepsilon}{\varepsilon}e^{-s}\,. \tag{4.126}$$

The bound (4.126) and the identity (4.122) then shows that for ε sufficiently small,

$$(1-\varepsilon)\varepsilon^{\frac12}e^{\frac{s}{2}}\leqslant\partial_y\Phi_W(y_*,s)\leqslant(1+\varepsilon)\varepsilon^{\frac12}e^{\frac{s}{2}}\,. \tag{4.127}$$

It follows from (4.121) that (4.118) holds.
Step 2. A bound for $\partial_x\eta$. The identity (4.46b) together with (4.45a) show that

$$\eta_x=1+\int_{-\varepsilon}^{t}I_\tau d\tau w_0'-\tfrac83 w_0\int_{-\varepsilon}^{t}I_\tau\int_{-\varepsilon}^{\tau}a'\circ\eta\;\eta_x drd\tau\,. \tag{4.128}$$

From (4.34),

$$|a(\theta,t)|\leqslant 2\kappa_0^2\varepsilon\quad\text{and}\quad|\partial_\theta a(\theta,t)|\leqslant 2\kappa_0\,. \tag{4.129}$$

Therefore, for ε taken sufficiently small, we have that

$$1-\varepsilon\leqslant I_\tau(x)\leqslant 1+\varepsilon\,. \tag{4.130}$$

By (4.31a), for ε taken sufficiently small,

$$-\tfrac{1}{\varepsilon}\leqslant w_0'(x)\leqslant-\tfrac{1-4\varepsilon}{\varepsilon}\qquad\text{for}\quad|x-x_*|\leqslant\varepsilon^2\,, \tag{4.131a}$$
$$\left|w_0'(x)\right|\leqslant\tfrac{1}{\varepsilon}\qquad\text{for}\quad|x-x_*|\geqslant\varepsilon^2\,, \tag{4.131b}$$

From (4.38), (4.128)–(4.130), we have that for ε taken sufficiently small,

$$\sup_{t\in[-\varepsilon,T_*)}\eta_x(x,t)\leqslant\tfrac52+7\varepsilon^2\kappa_0^2\sup_{t\in[-\varepsilon,T_*)}\eta_x(x,t)$$

and hence

$$\sup_{t\in[-\varepsilon,T_*)} \eta_x(x,t) \leqslant 3 \tag{4.132}$$

which is the upper bound in (4.120) when $|x-x_*| \geqslant \varepsilon^2$. We also have that from (4.38), (4.128)–(4.130), and (4.132) that for ε taken sufficiently small,

$$\sup_{t'\in[t,T_*)} \eta_x(x,t') \leqslant 1 - \tfrac{1-4\varepsilon}{\varepsilon}(1-\varepsilon)(\varepsilon+t) + 21\varepsilon^2\kappa_0^2\,, \qquad |x-x_*| \leqslant \varepsilon^2\,,$$

and hence

$$\sup_{t'\in[t,T_*)} \eta_x(x,t') \leqslant \begin{cases} t(6-\frac{1}{\varepsilon}) + 6\varepsilon & |x-x_*| \leqslant \varepsilon^2 \\ 3 & |x-x_*| \geqslant \varepsilon^2 \end{cases}, \tag{4.133}$$

which establishes (4.119).

Notice also from (4.128) that with the bound (4.25a), for all $|x-x_*| \geqslant \varepsilon^2$ and for ε taken small enough, $\left|w_0'(x)\right| \leqslant (1-\frac{7\varepsilon}{24})\varepsilon^{-1}$, and hence for all $t \in [-\log\varepsilon, T_*)$, we have the lower bound

$$\partial_x\eta(x,t) \geqslant \tfrac{\varepsilon}{4}\,,$$

which gives the lower bound in (4.120). Note that here we have used that $|I_\tau(x)-1| \lesssim \varepsilon^2$, which follows from (4.43) and (4.34a). □

4.7.3 Bounds for $\partial_x^2\eta$

We establish the rate at which $\partial_x^2\eta(x_*,t) \to 0$ as $t \to T_*$, and obtain bounds for $\partial_x^2\eta(x,t)$ for all labels x.

Lemma 4.5 *For all $-\varepsilon \leqslant t \leqslant T_*$, we have the decay estimate*

$$\left|\partial_x^2\eta(x_*,t)\right| \leqslant 62\kappa_0 e^{-s} \tag{4.134}$$

and for any label x, we have the bound

$$\left|\partial_x^2\eta(x,t)\right| \leqslant \begin{cases} 8\varepsilon^{-1} & |x-x_*| \leqslant \varepsilon^2 \\ 8\varepsilon^{-\frac{3}{2}} & |x-x_*| \geqslant \varepsilon^2 \end{cases}. \tag{4.135}$$

Proof of Lemma 4.5 *Step 1. A bound for $\partial_x^2\eta$ along the blowup label x_*.* Since $\eta_x = e^{\int_{-\varepsilon}^t \partial_\theta w\circ\eta dr}$, we have that

$$\eta_{xx}(x,t) = \eta_x(x,t)\int_{-\varepsilon}^t \partial_\theta^2 w(\eta(x,t'),t')\eta_x(x,t')dt'$$

$$= \eta_x(x,t) \int_{-\log \varepsilon}^{s} e^{\frac{3}{2}s'} \beta_\tau W_{yy}(\Phi_W(y,s'),s') \eta_x(x,t') ds' , \tag{4.136}$$

where we have used the change of variables formula together with the identity (4.7) which shows that $dt' = \beta_\tau e^{-s'} ds'$. By Lemma 4.3, $\left|\Phi_W(y_*,s)\right| \leqslant 20\kappa_0 e^{-\frac{5}{2}s}$ and $|y_*| \leqslant 20\kappa_0 \varepsilon^{\frac{5}{2}}$, so that together with (4.31a), we have that for ε taken small enough and for all $-\log \varepsilon \leqslant s' \leqslant s$,

$$\left|\beta_\tau W_{yy}(\Phi_W(y_*,s'),s')\right| \leqslant 122\kappa_0 e^{-\frac{5s}{2}} . \tag{4.137}$$

Hence, with (4.118) and the identity evaluated at the label x_*, we have that

$$|\eta_{xx}(x_*,t)| \leqslant 62\kappa_0 e^{-s} ,$$

which proves (4.134).
Step 2. A bound for $\partial_x^2\eta$ for all labels x. Using the identity in (4.46c) and (4.45b), we have that

$$\begin{aligned} \partial_x^2\eta &= \int_{-\varepsilon}^{t} I_\tau d\tau\, w_0'' - \tfrac{16}{3} w_0' \int_{-\varepsilon}^{t} I_\tau \int_{-\varepsilon}^{\tau} a' \circ \eta\; \eta_x dr d\tau \\ &\quad + w_0 \int_{-\varepsilon}^{t} I_\tau \left(\left(\tfrac{8}{3} \int_{-\varepsilon}^{\tau} a' \circ \eta\; \eta_x dr \right)^2 - \tfrac{8}{3} \int_{-\varepsilon}^{\tau} \left(a'' \circ \eta\; \eta_x^2 + a' \circ \eta\; \eta_{xx} \right) dr \right) d\tau . \end{aligned} \tag{4.138}$$

From (4.89) and (4.117),

$$\left|a''(\eta(x,t),t)\right| \leqslant \tfrac{7}{2}\left|\partial_\theta w(\eta(x,t),t)\right| + \tfrac{7}{\varepsilon} \leqslant \tfrac{7}{2}(I_t w_0' + I_t' w_0)\eta_x^{-1} + \tfrac{7}{\varepsilon} . \tag{4.139}$$

It follows from (4.133) that

$$\left|a''(\eta(x,t),t)\eta_x^2\right| \leqslant \tfrac{7}{2}\left|I_t w_0' + I_t' w_0\right|\eta_x + \tfrac{7}{\varepsilon}\eta_x^2 \leqslant \begin{cases} \frac{11}{\varepsilon} & |x-x_*| \leqslant \varepsilon^2 \\ \frac{74}{\varepsilon} & |x-x_*| \geqslant \varepsilon^2 \end{cases} . \tag{4.140}$$

By (4.25b) and (4.31a), for ε small enough,

$$\left|w_0''(x)\right| \leqslant \begin{cases} 7\varepsilon^{-2} & |x-x_*| \leqslant \varepsilon^2 \\ 7\varepsilon^{-\frac{5}{2}} & |x-x_*| \geqslant \varepsilon^2 \end{cases} . \tag{4.141}$$

It follows from (4.18), (4.129)–(4.133), (4.138)–(4.141) that

$$(1 - 7\kappa_0^2\varepsilon^2) \sup_{t \in [-\varepsilon, T_*)} \left|\partial_x^2\eta(x,t)\right| \leqslant \begin{cases} \frac{15}{2}\varepsilon^{-1} + \mathcal{O}(\varepsilon) & |x-x_*| \leqslant \varepsilon^2 \\ \frac{15}{2}\varepsilon^{-\frac{3}{2}} + \mathcal{O}(\varepsilon) & |x-x_*| \geqslant \varepsilon^2 \end{cases} ,$$

and thus taking ε sufficiently small,

$$\sup_{t\in[-\varepsilon,T_*)}\left|\partial_x^2\eta(x,t)\right| \leqslant \begin{cases} 8\varepsilon^{-1} & |x-x_*|\leqslant\varepsilon^2 \\ 8\varepsilon^{-\frac{3}{2}} & |x-x_*|\geqslant\varepsilon^2 \end{cases},$$

which proves (4.135). □

Remark 4.6 We have shown in the proof of Lemmas 4.4 and 4.5 that for ε taken sufficiently small,

$$\left|I_t(x)\right| \leqslant 1+\varepsilon\,, \tag{4.142a}$$

$$\left|I_t'\right| \leqslant \begin{cases} 3\kappa_0(\varepsilon+t) & |x-x_*|\leqslant\varepsilon^2 \\ 14\kappa_0\varepsilon & |x-x_*|\geqslant\varepsilon^2 \end{cases}, \tag{4.142b}$$

$$\left|I_t''\right| \leqslant \begin{cases} 64\kappa_0\frac{\varepsilon+t}{\varepsilon} & |x-x_*|\leqslant\varepsilon^2 \\ 40\kappa_0\varepsilon^{-\frac{1}{2}} & |x-x_*|\geqslant\varepsilon^2 \end{cases}. \tag{4.142c}$$

4.7.4 Bounds for $\partial_x^3\eta$

Lemma 4.7 *For all $-\varepsilon\leqslant t\leqslant T_*$, we have that*

$$\sup_{t\in[-\varepsilon,T_*)}\left|\partial_x^3\eta(x,t)\right| \leqslant \begin{cases} \frac{(6+\varepsilon^{\frac{1}{6}})}{\varepsilon^3} & |x-x_*|\leqslant\varepsilon^2 \\ \frac{C}{\varepsilon^4} & |x-x_*|\geqslant\varepsilon^2 \end{cases}, \tag{4.143}$$

and for $|x-x_*|\leqslant\varepsilon^2$,

$$\frac{(\varepsilon+t)(6-\varepsilon^{\frac{1}{6}})}{\varepsilon^4} \leqslant \partial_x^3\eta(x,t) \leqslant \frac{6+\varepsilon^{\frac{1}{6}}}{\varepsilon^3}\,. \tag{4.144}$$

Proof of Lemma 4.7 We first note that the bounds (4.119) and (4.135) show that

$$|\eta_x(x,t)| \leqslant \begin{cases} t(6-\frac{1}{\varepsilon})+6\varepsilon & |x-x_*|\leqslant\varepsilon^2 \\ 3 & |x-x_*|\geqslant\varepsilon^2 \end{cases} \quad \text{and}$$

$$|\eta_{xx}(x,t)| \leqslant \begin{cases} 8\varepsilon^{-1} & |x-x_*|\leqslant\varepsilon^2 \\ 8\varepsilon^{-\frac{3}{2}} & |x-x_*|\geqslant\varepsilon^2 \end{cases}. \tag{4.145}$$

The identities (4.45c) and (4.46d) give

$$\partial_x^3\eta = w_0'''\int_{-\varepsilon}^t I_\tau d\tau + 3w_0''\int_{-\varepsilon}^t I_\tau' d\tau + 3w_0'\int_{-\varepsilon}^t I_\tau'' d\tau$$
$$+ w_0\int_{-\varepsilon}^t I_\tau\Big(-\tfrac{512}{27}\Big(\int_{-\varepsilon}^\tau a'\circ\eta\,\eta_x\,dr\Big)^3$$

$$+\tfrac{64}{3}\left(\int_{-\varepsilon}^{\tau} a'\circ\eta\,\eta_x dr\right)\int_{-\varepsilon}^{\tau}\big(a''\circ\eta\,\eta_x^2+a'\circ\eta\,\eta_{xx}\big)dr$$
$$-\tfrac{8}{3}\int_{-\varepsilon}^{\tau}\big(a'''\circ\eta\,\eta_x^3+3a''\circ\eta\,\eta_x\eta_{xx}+a'\circ\eta\,\eta_{xxx}\big)dr\Big)d\tau\,. \tag{4.146}$$

From (4.117), we have that

$$\partial_\theta^2 w(\eta(x,t),t)=\eta_x^{-2}(I_t w_0''+2I_t'w_0'+I_t''w_0-\eta_x^{-1}(w_0'I_t+I_t'w_0)\eta_{xx})\,. \tag{4.147}$$

From (4.103), (4.117), and (4.147),

$$\begin{aligned}\left|\partial_\theta^3 a(\eta(x,t),t)\right| &\lesssim \varepsilon^{-\frac{5}{2}}+\left|\partial_\theta^2 w(\eta(x,t),t)\right|+\left|\partial_\theta w(\eta(x,t),t)\right|^2+\tfrac{1}{\varepsilon}\left|\partial_\theta w(\eta(x,t),t)\right|\\ &\lesssim \varepsilon^{-\frac{5}{2}}+\eta_x^{-2}(I_t w_0''+2I_t'w_0'+I_t''w_0-\eta_x^{-1}(w_0'I_t+I_t'w_0)\eta_{xx})\\ &\quad+\left|\eta_x^{-1}(I_t w_0'+I_t'w_0)\right|^2+\tfrac{1}{\varepsilon}\left|\eta_x^{-1}(I_t w_0'+I_t'w_0)\right|\,.\end{aligned} \tag{4.148}$$

We will use (4.131), (4.141), and the fact that by (4.23) and (4.25c),

$$\tfrac{6-\varepsilon^{\frac{1}{4}}}{\varepsilon^4}\leqslant w_0'''(\theta)\leqslant\tfrac{6+\varepsilon^{\frac{1}{4}}}{\varepsilon^4}\qquad\text{for}\quad |x-x_*|\leqslant\varepsilon^2\,, \tag{4.149a}$$
$$\left|w_0'''(x)\right|\lesssim\varepsilon^{-4}\qquad\text{for}\quad |x-x_*|\geqslant\varepsilon^2\,. \tag{4.149b}$$

Then, with (4.141), (4.142) and (4.148), we have that

$$\left|a'''(\eta(x,t),t)\eta_x^3\right|\lesssim\begin{cases}\varepsilon^{-2} & |x-x_*|\leqslant\varepsilon^2\\ \varepsilon^{-\frac{5}{2}} & |x-x_*|\geqslant\varepsilon^2\end{cases}. \tag{4.150}$$

With these bounds, and with (4.142), (4.145)–(4.148) applied to (4.146), we have that

$$\sup_{t\in[-\varepsilon,T_*)}\left|\partial_x^3\eta(x,t)\right|$$
$$\leqslant\begin{cases}(\varepsilon+\varepsilon^2)\frac{(6+\varepsilon^{\frac{1}{4}})}{\varepsilon^4}+\frac{C}{\varepsilon}+7\varepsilon^2\kappa_0^2\sup_{t\in[-\varepsilon,T_*)}\left|\partial_x^3\eta(x,t)\right| & |x-x_*|\leqslant\varepsilon^2\\ C\varepsilon^{-4}+7\varepsilon^2\kappa_0^2\sup_{t\in[-\varepsilon,T_*)}\left|\partial_x^3\eta(x,t)\right| & |x-x_*|\geqslant\varepsilon^2\end{cases}. \tag{4.151}$$

It immediately follows that for ε small enough,

$$\sup_{t\in[-\varepsilon,T_*)}|\eta_{xxx}(x,t)|\leqslant\begin{cases}\frac{(6+\varepsilon^{\frac{1}{6}})}{\varepsilon^3} & |x-x_*|\leqslant\varepsilon^2\\ \frac{C}{\varepsilon^4} & |x-x_*|\geqslant\varepsilon^2\end{cases}, \tag{4.152}$$

which establishes (4.143).

For labels $\left|x - x_*\right| \leqslant \varepsilon^2$, we can easily see that $\partial_x^3 \eta(x,t)$ is positive. With (4.149), we have that the first term on the right side of (4.146) has the lower bound

$$\tfrac{(\varepsilon+t)(6-\varepsilon^{\frac{1}{5}})}{\varepsilon^4} \leqslant \int_{-\varepsilon}^{t} I_\tau d\tau w_0''' .$$

Thus, with (4.131), (4.141), (4.142), (4.145)–(4.148), in the same way that we obtained (4.151), we find that

$$\tfrac{(\varepsilon+t)(6-\varepsilon^{\frac{1}{6}})}{\varepsilon^4} \leqslant \partial_x^3 \eta(x,t) \leqslant \tfrac{(6+\varepsilon^{\frac{1}{6}})}{\varepsilon^3} ,$$

which establishes (4.144). □

4.7.5 A Sharp Bound for $\partial_x \eta$ and $\partial_x^2 \eta$

Proposition 4.8 *For $|x - x_*| \leqslant \varepsilon^2$, we have that*

$$\begin{aligned} \tfrac{1-\varepsilon^{\frac{1}{2}}}{\varepsilon}(T_* - t) + \tfrac{(\varepsilon+t)(3-\varepsilon^{\frac{1}{8}})}{\varepsilon^4}(x - x_*)^2 &\leqslant \partial_x \eta(x,t) \\ &\leqslant \tfrac{1+\varepsilon^{\frac{1}{2}}}{\varepsilon}(T_* - t) + \tfrac{(3+\varepsilon^{\frac{1}{8}})}{\varepsilon^3}(x - x_*)^2 , \end{aligned} \tag{4.153}$$

and

$$\begin{aligned} -7\varepsilon^{-2}(T_* - t) + \tfrac{(\varepsilon+t)(6-2\varepsilon^{\frac{1}{8}})}{\varepsilon^4}(x - x_*) &\leqslant \partial_x^2 \eta(x,t) \\ &\leqslant 7\varepsilon^{-2}(T_* - t) + \tfrac{6+2\varepsilon^{\frac{1}{8}}}{\varepsilon^3}(x - x_*) \quad \textit{for} \quad x \geqslant x_* , \end{aligned} \tag{4.154a}$$

$$\begin{aligned} -7\varepsilon^{-2}(T_* - t) + \tfrac{6+2\varepsilon^{\frac{1}{8}}}{\varepsilon^3}(x - x_*) &\leqslant \partial_x^2 \eta(x,t) \\ &\leqslant 7\varepsilon^{-2}(T_* - t) \\ &\quad + \tfrac{(\varepsilon+t)(6-2\varepsilon^{\frac{1}{8}})}{\varepsilon^4}(x - x_*) \quad \textit{for} \quad x \leqslant x_* . \end{aligned} \tag{4.154b}$$

Proof of Proposition 4.8 By Lemma 2.1 in [1], there exists a short time $\overline{T} \geqslant -\varepsilon$, such that (w, a) is a unique solution to (3.10) with initial data (w_0, a_0) and

$$(a, w) \in C^0([-\varepsilon, \overline{T}]; C^4(\mathbb{T})) \cap C^1([-\varepsilon, \overline{T}]; C^3(\mathbb{T})) . \tag{4.155}$$

By the local existence and uniqueness theorem for ODE, $\eta \in C^1([-\varepsilon, \overline{T}]; C^3(\mathbb{T})) \cap C^2([-\varepsilon, \overline{T}]; C^2(\mathbb{T}))$. Given the uniform bounds (4.143) and (4.145), the standard continuation argument shows that

$$\eta \in C^1([-\varepsilon, T_*], C^3(\mathbb{T})) \cap C^2([-\varepsilon, T_*], C^2(\mathbb{T})) .$$

By the Taylor remainder theorem, there exist a point x_1 between x and x_* and a point t_1 between t and T_* such that

$$\begin{aligned}\partial_x \eta(x,t) &= \partial_t \partial_x \eta(x_*, T_*)(t - T_*) + \tfrac{1}{2}\partial_x^3 \eta(x_1, t_1)(x - x_*)^2 \\ &\quad + \tfrac{1}{2}\partial_t^2 \partial_x \eta(x_1, t_1)(t - T_*)^2 \\ &\quad + \partial_t \partial_x^2 \eta(x_1, t_1)(t - T_*)(x - x_*)\,. \end{aligned} \tag{4.156}$$

Note that we have used (4.118) and (4.134) which give

$$\partial_x \eta(x_*, T_*) = 0\,, \quad \partial_x^2 \eta(x_*, T_*) = 0\,. \tag{4.157}$$

From (4.46b), we have that

$$\partial_t \partial_x \eta(x,t) = I_t(x) w_0'(x) + I_t'(x) w_0(x)\,. \tag{4.158}$$

We use the bounds (4.38), (4.129)–(4.133) to find that for ε small enough,

$$-\tfrac{1+\varepsilon^{\frac{3}{4}}}{\varepsilon} \leqslant \partial_t \partial_x \eta(x_*, T_*) \leqslant -\tfrac{1-\varepsilon^{\frac{3}{4}}}{\varepsilon}\,. \tag{4.159}$$

Differentiation of (4.158) with respect to ∂_x yields

$$\partial_t \partial_x^2 \eta = I_t w_0'' + 2 I_t' w_0' + I_t'' w_0\,,$$

while differentiation of (4.158) with respect to ∂_t gives

$$\partial_t^2 \partial_x \eta = I_t' w_0' + I_t'' w_0\,.$$

We again use the bounds (4.38), (4.129)–(4.131), (4.141), and (4.142) to obtain that

$$\left|\partial_t^2 \partial_x \eta(x_1, t_1)\right| \leqslant 50\kappa_0^2\,, \tag{4.160}$$

$$\left|\partial_t \partial_x^2 \eta(x_1, t_1)\right| \leqslant 8\varepsilon^{-2}\,. \tag{4.161}$$

From (4.144), we have that

$$\tfrac{(\varepsilon+t)(3-\varepsilon^{\frac{1}{7}})}{\varepsilon^4} \leqslant \tfrac{1}{2}\partial_x^3 \eta(x_1, t_1) \leqslant \tfrac{(3+\varepsilon^{\frac{1}{7}})}{\varepsilon^3}\,. \tag{4.162}$$

Since $t \leqslant T_*$, $\left|x - x_*\right| \leqslant \varepsilon^2$, and $(T_* - t)^2 \leqslant 2\varepsilon^2$, the bounds (4.159)–(4.162) used in the identity (4.156) show that for ε taken sufficiently small,

$$\begin{aligned}&\tfrac{1-\varepsilon^{\frac{1}{2}}}{\varepsilon}(T_* - t) + \tfrac{(\varepsilon+t)(3-\varepsilon^{\frac{1}{8}})}{\varepsilon^4}(x - x_*)^2 \\ &\quad \leqslant \partial_x \eta(x,t) \leqslant \tfrac{1+\varepsilon^{\frac{1}{2}}}{\varepsilon}(T_* - t) + \tfrac{3+\varepsilon^{\frac{1}{8}}}{\varepsilon^3}(x - x_*)^2\,,\end{aligned}$$

which establishes (4.153).

We can again apply the Taylor remainder theorem to find that for a point $\mathring{x}_1$ between x and x_* and a point $\mathring{t}_1$ between t and T_*,

$$\partial_x^2 \eta(x,t) = \partial_x^3 \eta(\mathring{x}_1, \mathring{t}_1)(x - x_*) + \partial_t \partial_x^2 \eta(\mathring{x}_1, \mathring{t}_1)(t - T_*) \, .$$

It then follows from (4.161) and (4.162) that (4.154) holds. □

4.7.6 Bounds for $\partial_\theta w$

Lemma 4.9 (Bound for $\partial_\theta w$) *For $t \in [-\log \varepsilon, T_*)$,*

$$|\partial_\theta w(\eta(x,t),t)| \leqslant \begin{cases} \frac{2}{(T_*-t)+3\varepsilon^{-3}(\varepsilon+t)(x-x_*)^2} & |x - x_*| \leqslant \varepsilon^2 \\ 5\varepsilon^{-2} & |x - x_*| \geqslant \varepsilon^2 \end{cases} . \tag{4.163}$$

Proof of Lemma 4.9 From (4.117), we have that

$$\partial_\theta w(\eta(x,t),t) = (I_t(x) w_0'(x) + I_t'(x) w_0(x)) \eta_x^{-1}(x,t) \, . \tag{4.164}$$

Using the bounds (4.18), (4.120), (4.131), (4.142), and (4.153), obtain the bound (4.163). □

4.7.7 Bounds for $\partial_x^4 \eta$

In order to obtain a bound for the fourth derivative of η, we shall appeal to the identity (4.46e). Before estimating the terms on the right side of (4.46e), we first record a useful estimate:

Lemma 4.10 *For $|x - x_*| \leqslant \varepsilon^2$ it holds that*

$$\eta_x^4(x,t) \int_{-\varepsilon}^{t} \left| \partial_\theta w(\phi(\phi^{-1}(\eta(x,t),t),t'),t') \right|^2 dt' \lesssim \varepsilon^{-1} \eta_x^2(x,t) \, . \tag{4.165}$$

Proof of Lemma 4.10 Fix a label x which is within ε^2 of x_*, and a time $t \in [-\varepsilon, T_*)$, throughout the proof. In order to estimate the integral in (4.165) we use the bound on $\partial_\theta w$ obtained in (4.163). Note however that this estimate is obtained when we compose with the flow η; as such we first define the label (Fig. 8)

$$\chi(x,t) = \phi^{-1}(\eta(x,t),t) \, , \tag{4.166}$$

and then for each $t' \in [-\varepsilon, t]$, we also define the label

$$q(x,t') = \eta^{-1}(\phi(\chi(x,t),t'),t') \, . \tag{4.167}$$

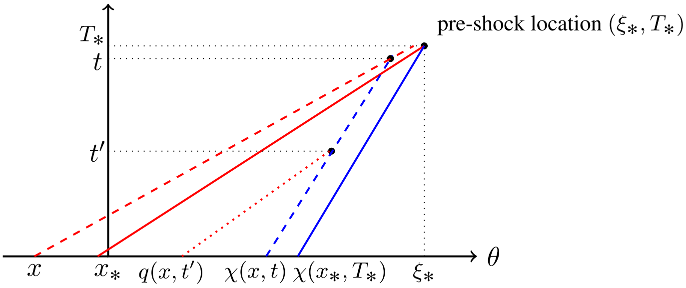

Fig. 8 The identity (4.168) is explained. The 3-characteristics η are shown in red and 2-characteristics ϕ are shown in blue. The worst case scenario is depicted: the label x is to the left of the blowup label x_*. For each such label x and each $t \in [-\varepsilon, T_*)$, $\chi(x,t)$ denotes the label which satisfies $\phi(\chi(x,t),t) = \eta(x,t)$. For each $t' \in [-\varepsilon, t]$, we define the label $q(x,t')$ such that $\eta(q(x,t'),t') = \phi(\chi(x,t),t')$. As $t' \to t$, $q(x,t') \to x$. A particle moving up the dashed blue curve is equivalent to that particle moving by the 3-characteristic but emanating from the moving label $q(x,t')$

The definitions (4.166) and (4.167) show that

$$\partial_\theta w(\phi(\phi^{-1}(\eta(x,t),t),t'),t') = \partial_\theta w(\phi(\chi(x,t),t'),t') = \partial_\theta w(\eta(q(x,t'),t'),t')\,. \tag{4.168}$$

Therefore, $q(x,-\varepsilon) = \chi(x,t)$, $q(x,t') \to x$ from the right as $t' \to t$, while from (4.163) we have that

$$\left|\partial_\theta w(\eta(q(x,t'),t'),t')\right| \leqslant \begin{cases} \frac{2}{(T_*-t')+3\varepsilon^{-3}(\varepsilon+t')(q(x,t')-x_*)^2} & \left|q(x,t')-x_*\right| \leqslant \varepsilon^2 \\ 5\varepsilon^{-2} & \left|q(x,t')-x_*\right| \geqslant \varepsilon^2 \end{cases}. \tag{4.169}$$

We will assume first that $x_* \in [x, \chi(x_*,t)]$. The proof is based on decomposing the interval $[-\varepsilon, t)$ into three different sets

$$I_{\text{start}} = \{t' \in [-\varepsilon,t) \colon |q(x,t')-x_*| \geqslant \varepsilon^2 \text{ or } t' \leqslant -\tfrac{1}{2}\varepsilon\} \tag{4.170a}$$

$$I_{\text{middle}} = \{t' \in [-\tfrac{\varepsilon}{2},t) \colon |q(x,t')-x_*| < \varepsilon^2 \text{ and } x_* - \tfrac{1}{2}(x_*-x) < q(x,t') < x_* + \varepsilon^2\} \tag{4.170b}$$

$$I_{\text{end}} = \{t' \in [-\tfrac{\varepsilon}{2},t) \colon |q(x,t')-x_*| < \varepsilon^2 \text{ and } x \leqslant q(x,t') \leqslant x_* - \tfrac{1}{2}(x_*-x)\}\,. \tag{4.170c}$$

From (4.169) we immediately have that

$$\begin{aligned}\int_{I_{\text{start}}} \left|\partial_\theta w(\eta(q(x,t'),t'),t')\right|^2 dt' &\leqslant \int_{-\varepsilon}^{-\frac{1}{2}\varepsilon} \frac{4}{(T_*-t')^2}dt' + \int_{-\varepsilon}^{T_*} \frac{25}{\varepsilon^4}dt' \\ &\leqslant \frac{4}{T_*+\frac{\varepsilon}{2}} + \frac{25(T_*+\varepsilon)}{\varepsilon^4} \leqslant 50\varepsilon^{-3}\end{aligned} \tag{4.171}$$

since $T_* = \mathcal{O}(\varepsilon^3)$ and ε is sufficiently small.

For the remaining two time intervals, since $\frac{1}{2}\varepsilon \leqslant t' + \varepsilon \leqslant 2\varepsilon$, and $|q(x, t') - x_*| < \varepsilon^2$, we will use that

$$\left|\partial_\theta w(\eta(q(x,t'),t'),t')\right| \leqslant \frac{1}{G(x,t')}, \quad \text{where}$$
$$G(x,t') := (T_* - t') + \tfrac{3}{2}\varepsilon^{-2}(q(x,t') - x_*)^2 . \tag{4.172}$$

The second important fact that we will use frequently is that (4.153) implies

$$\tfrac{1}{3\varepsilon} G(x,t') \leqslant (\partial_x \eta)(q(x,t'),t') \leqslant \tfrac{3}{\varepsilon} G(x,t') . \tag{4.173}$$

The third important ingredient is an estimate for the time derivative of the label $q(x, t')$. Using that η^{-1} solves the transport equation $(\partial_{t'} + w\partial_\theta)\eta^{-1} = 0$, upon differentiating (4.167) with respect to t' we obtain

$$\begin{aligned}
\partial_{t'} q(x,t') &= \partial_{t'}\eta^{-1}(\phi(\chi(x,t),t'),t') + \partial_\theta \eta^{-1}(\phi(\chi(x,t),t'),t')\partial_{t'}\phi(\chi(x,t),t') \\
&= \partial_{t'}\eta^{-1}(\eta(q(x,t'),t'),t') + \tfrac{2}{3}\partial_\theta\eta^{-1}(\eta(q(x,t'),t'),t') w(\eta(q(x,t'),t')) \\
&= -\tfrac{1}{3}\partial_\theta \eta^{-1}(\eta(q(x,t'),t')) w(\eta(q(x,t'),t'),t') \\
&= -\frac{w(\eta(q(x,t'),t'),t')}{3\eta_x(q(x,t'),t')} .
\end{aligned} \tag{4.174}$$

From the above identity, using the bounds (4.33) and (4.173) we conclude that

$$\frac{\varepsilon\kappa_0}{20G(x,t')} \leqslant -\partial_{t'} q(x,t') \leqslant \frac{2\varepsilon\kappa_0}{G(x,t')} . \tag{4.175}$$

With (4.172), (4.173), and (4.175) in hand, we return to the two remaining cases described in (4.170). First, we note that (4.175) shows that the function $q(x, t')$ is strictly decreasing, as a function of t', and thus when ε is sufficiently small there exists a unique time $t^\sharp \in [-\frac{1}{2}\varepsilon, t)$ such that

$$q(x, t^\sharp) = x_* - \tfrac{1}{2}(x_* - x) .$$

As such, $I_{\text{end}} = [t^\sharp, t]$, and $I_{\text{middle}} \subset [-\frac{1}{2}\varepsilon, t^\sharp]$. Since $q(x, t) = x$, the fundamental theorem of calculus, (4.175), and the definition of I_{end} show that

$$\begin{aligned}
\tfrac{1}{2}(x_* - x) = q(x,t^\sharp) - q(x,t) &= \int_{t^\sharp}^{t} (-\partial_{t'} q(x,t'))dt' \\
&\leqslant \int_{t^\sharp}^{t} \frac{2\varepsilon\kappa_0}{G(x,t')} dt' \\
&\leqslant \frac{2\varepsilon\kappa_0 (t - t^\sharp)}{(T_* - t) + \frac{3}{8}\varepsilon^{-2}(x_* - x)^2}
\end{aligned}$$

$$\leqslant \frac{8\varepsilon\kappa_0(t-t^\sharp)}{G(x,t)}\,.$$

The purpose of the above estimate is to provide the lower bound

$$t-t^\sharp \geqslant \frac{1}{16\varepsilon\kappa_0}(x_*-x)G(x,t)\,. \tag{4.176}$$

With (4.172), (4.175) and (4.176), since $I_{\text{middle}} \subset [-\frac{1}{2}\varepsilon, t^\sharp]$ we may then estimate

$$\begin{aligned}
\int_{I_{\text{middle}}} \left|\partial_\theta w(\eta(q(x,t'),t'),t')\right|^2 dt' &\leqslant \int_{I_{\text{middle}}} \frac{1}{G(x,t')^2}dt' \\
&\leqslant \frac{1}{T_*-t^\sharp}\int_{I_{\text{middle}}} \frac{-20\partial_{t'}q(x,t')}{\varepsilon\kappa_0}dt' \\
&\leqslant \frac{20}{\varepsilon\kappa_0}\cdot\frac{(x_*+\varepsilon^2)-(x_*-\frac{1}{2}(x_*-x))}{(T_*-t)+\frac{1}{16\varepsilon\kappa_0}(x_*-x)G(x,t)} \\
&\leqslant \frac{30\varepsilon}{\kappa_0}\cdot\frac{1}{(T_*-t)+\frac{1}{16\varepsilon\kappa_0}(x_*-x)G(x,t)} \\
&\leqslant \frac{30\varepsilon}{\kappa_0}\begin{cases}\frac{1}{G(x,t)}\,, & \text{if } G(x,t)\geqslant 24\kappa_0\varepsilon^{-1}(x_*-x) \\ \frac{400\kappa_0^2}{G(x,t)^2}\,, & \text{if } G(x,t)< 24\kappa_0\varepsilon^{-1}(x_*-x)\end{cases} \\
&\leqslant \frac{12000\varepsilon\kappa_0}{G(x,t)^2} \leqslant \frac{108000\kappa_0}{\varepsilon\eta_x(x,t)^2}\,,
\end{aligned} \tag{4.177}$$

where in the second-to-last inequality we have used that $0 < G(x,t) \leqslant \varepsilon$, and in the last inequality we have appealed to (4.173). Lastly, since $I_{\text{end}} = [t^\sharp, t]$, a similar argument and the bound (4.173) shows that

$$\begin{aligned}
\int_{I_{\text{end}}} \left|\partial_\theta w(\eta(q(x,t'),t'),t')\right|^2 dt' &\leqslant \int_{I_{\text{end}}} \frac{1}{G(x,t')^2}dt' \\
&\leqslant \frac{1}{(T_*-t)+\frac{3}{8}\varepsilon^{-2}(x_*-x)^2}\int_{t^\sharp}^{t} \frac{-20\partial_{t'}q(x,t')}{\varepsilon\kappa_0}dt' \\
&\leqslant \frac{40}{\varepsilon\kappa_0 G(x,t)}\left((x_*-\frac{1}{2}(x_*-x))-x\right) \\
&\leqslant \frac{20(x_*-x)}{\varepsilon\kappa_0 G(x,t)} \leqslant \frac{60(x_*-x)}{\varepsilon^2\kappa_0\eta_x(x,t)} \leqslant \frac{60}{\kappa_0\eta_x(x,t)}\,.
\end{aligned} \tag{4.178}$$

Combining (4.171), (4.177), and (4.178), we arrive at

$$\eta_x^4(x,t)\int_{-\varepsilon}^{t} \left|\partial_\theta w(\phi(\phi^{-1}(\eta(x,t),t),t'),t')\right|^2 dt'$$

$$\lesssim \varepsilon^{-3}\eta_x^4(x,t)+\varepsilon^{-1}\eta_x^2(x,t)+\eta_x^3(x,t)\,,$$

and then by appealing to the first case in (4.119), concludes the proof of the lemma in the case that $x_* > x$.

For the other case, $x_* < x$, we have that $(q(x,t')-x_*)^2 \geqslant (x-x_*)^2$, and then we simply have

$$\begin{aligned}\int_{[-\varepsilon,t)\setminus I_{\text{start}}}\left|\partial_\theta w(\eta(q(x,t'),t'),t')\right|^2 dt' &\leqslant \int_{-\varepsilon}^{t}\frac{1}{((T_*-t')+\frac{3}{2}\varepsilon^{-2}(x_*-x)^2)^2}dt'\\ &\leqslant \frac{1}{(T_*-t)+\frac{3}{2}\varepsilon^{-2}(x_*-x)^2}\\ &\leqslant \frac{3}{\varepsilon\partial_x\eta(x,t)}\end{aligned} \tag{4.179}$$

in light of the definition of G and of (4.173). The estimate (4.165) follows as before (it is in fact better in this case). □

Lemma 4.11 *For labels x, we have that*

$$\sup_{t\in[-\varepsilon,T_*)}\left|\partial_x^4\eta(x,t)\right| \leqslant \begin{cases}3\varepsilon^{-\frac{31}{8}} & |x-x_*|\leqslant\varepsilon^3\\ 363\varepsilon^{-4} & |x-x_*|\leqslant\varepsilon^2\\ C\varepsilon^{-\frac{9}{2}} & |x-x_*|\geqslant\varepsilon^2\end{cases}\,, \tag{4.180}$$

where C_ε denotes a positive constant that depends on inverse powers of ε.

Proof of Lemma 4.11 We shall first consider the case that the label x satisfies $|x-x_*|\leqslant\varepsilon^2$. The identity (4.117) shows that

$$\begin{aligned}\partial_\theta^3 w(\eta(x,t),t) = \eta_x^{-3}&(I_t w_0''' + 3I_s' w_0'' + 3I_t'' w_0' + I_t''' w_0)\\ &- 3\eta_x^{-4}\eta_{xx}(I_t w_0'' + 2I_t' w_0' + I_t'' w_0 - \eta_x^{-1}(w_0' I_t + I_t' w_0)\eta_{xx})\\ &- \eta_x^{-4}\eta_{xxx}(I_t w_0' + I_t' w_0)\,.\end{aligned} \tag{4.181}$$

We next use the inequality (4.116) together with the identities (4.164), (4.181), and (4.147),

$$\begin{aligned}\left|\partial_\theta^4 a(\eta(x,t),t)\right| \lesssim \varepsilon^{-4} &+ \varepsilon^{-2}\left|\eta_x^{-1}(I_t w_0' + I_t' w_0)\right| + \tfrac{1}{\varepsilon}\left|\eta_x^{-1}(I_t w_0' + I_t' w_0)\right|^2\\ &+ \tfrac{1}{\varepsilon}\left|\eta_x^{-2}(I_t w_0'' + 2I_t' w_0' + I_t'' w_0 - \eta_x^{-1}(w_0' I_t + I_t' w_0)\eta_{xx})\right|\\ &+ \left|\eta_x^{-1}(I_t w_0' + I_t' w_0)\right|\left|\eta_x^{-2}(I_t w_0'' + 2I_t' w_0'\right.\\ &\left.+ I_t'' w_0 - \eta_x^{-1}(w_0' I_t + I_t' w_0)\eta_{xx})\right|\\ &+ \left|\ \eta_x^{-3}(I_t w_0''' + 3I_t' w_0'' + 3I_t'' w_0' + I_t''' w_0)\right|\\ &+ 3\left|\eta_x^{-4}\eta_{xx}(I_t w_0'' + 2I_t' w_0' + I_t'' w_0 - \eta_x^{-1}(w_0' I_t + I_t' w_0)\eta_{xx})\right|\\ &+ \left|\eta_x^{-4}\eta_{xxx}(I_t w_0' + I_t' w_0)\right|\end{aligned}$$

$$+ \int_{-\varepsilon}^{t} \left|\partial_\theta w(\phi(\phi^{-1}(\eta(x,t),t),t'),t')\right|^2 dt' \,. \tag{4.182}$$

By (4.22) and (4.25c), for ε sufficiently small, we have that

$$\left|w_0''''(\theta)\right| \leqslant 2\varepsilon^{-\frac{39}{8}} \qquad \text{for} \quad |x-x_*| \leqslant \varepsilon^3 \,, \tag{4.183a}$$

$$\left|w_0''''(\theta)\right| \leqslant 361\varepsilon^{-5} \qquad \text{for} \quad |x-x_*| \leqslant \varepsilon^2 \,, \tag{4.183b}$$

$$\left|w_0''''(x)\right| \lesssim \varepsilon^{-\frac{11}{2}} \qquad \text{for} \quad |x-x_*| \geqslant \varepsilon^2 \,. \tag{4.183c}$$

Using the identity (4.45c) together with (4.140), (4.143), (4.145), (4.150)

$$\left|I_t'''\right| \lesssim \begin{cases} \varepsilon^{-1} & |x-x_*| \leqslant \varepsilon^2 \\ \varepsilon^{-3} & |x-x_*| \geqslant \varepsilon^2 \end{cases} . \tag{4.184}$$

Then, with (4.142) and (4.131), (4.141), (4.149), (4.165), and (4.182), we have that for ε taken sufficiently small,

$$\left|\partial_\theta^4 a(\eta(x,t),t)\eta_x^4\right| \lesssim \begin{cases} \varepsilon^{-4} & |x-x_*| \leqslant \varepsilon^2 \\ \varepsilon^{-7} & |x-x_*| \geqslant \varepsilon^2 \end{cases} . \tag{4.185}$$

Using the identities (4.45d) and (4.46e), we have that

$$\begin{aligned}
\partial_x^4 \eta = {} & w_0'''' \int_{-\varepsilon}^{t} I_\tau d\tau + 4w_0''' \int_{-\varepsilon}^{t} I_\tau' d\tau + 6w_0'' \int_{-\varepsilon}^{t} I_\tau'' d\tau + 4w_0' \int_{-\varepsilon}^{t} I_\tau''' d\tau \\
& + w_0 \int_{-\varepsilon}^{t} I_\tau \Bigg(\tfrac{4096}{81} \Big(\int_{-\varepsilon}^{\tau} a' \circ \eta \; \eta_x \, dr \Big)^4 \\
& - \tfrac{1024}{9} \left(\int_{-\varepsilon}^{\tau} a' \circ \eta \; \eta_x dr \right)^2 \int_{-\varepsilon}^{\tau} \big(a'' \circ \eta \; \eta_x^2 + a' \circ \eta \; \eta_{xx}\big) dr \\
& + \tfrac{64}{3} \left(\int_{-\varepsilon}^{\tau} (a'' \circ \eta \; \eta_x^2 + a' \circ \eta \; \eta_{xx}) dr \right)^2 \\
& + \tfrac{256}{9} \left(\int_{-\varepsilon}^{\tau} a' \circ \eta \; \eta_x dr \right) \left(\int_{-\varepsilon}^{\tau} (a''' \circ \eta \; \eta_x^3 + 3a'' \circ \eta \; \eta_x \eta_{xx} + a' \circ \eta \; \eta_{xxx}) dr \right) \\
& - \tfrac{8}{3} \int_{-\varepsilon}^{\tau} \big(a'''' \circ \eta \; \eta_x^4 + 6a''' \circ \eta \; \eta_x^2 \eta_{xx} + 3a'' \circ \eta \; \eta_{xx}^2 \\
& + 4a'' \circ \eta \; \eta_x \eta_{xxx} + a' \circ \eta \; \eta_{xxxx}\big) dr \Bigg) d\tau \,.
\end{aligned}$$

Notice that from (4.153) and (4.154), for $|x-x_*| \leqslant \varepsilon^2$, we have that

$$\eta_x^{-1} \eta_{xx}^2 \leqslant 100\varepsilon^{-3} \,.$$

Then, together with the bounds (4.139), (4.140), (4.142), (4.145)–(4.150), (4.153), (4.154), (4.183)–(4.185), and with (4.142), (4.145)–(4.148), we find that for ε sufficiently small,

$$\sup_{t\in[-\varepsilon,T_*)}\left|\partial_x^4\eta(x,t)\right| \leqslant \begin{cases} \frac{5}{2}\varepsilon^{-\frac{31}{8}} + 7\varepsilon^2\kappa_0^2 \sup_{t\in[-\varepsilon,T_*)}\left|\partial_x^4\eta(x,t)\right| & |x-x_*|\leqslant\varepsilon^3 \\ 362\varepsilon^{-4} + 7\varepsilon^2\kappa_0^2 \sup_{t\in[-\varepsilon,T_*)}\left|\partial_x^4\eta(x,t)\right| & |x-x_*|\leqslant\varepsilon^2 \\ C\varepsilon^{-\frac{9}{2}} + 7\varepsilon^2\kappa_0^2 \sup_{t\in[-\varepsilon,T_*)}\left|\partial_x^4\eta(x,t)\right| & |x-x_*|\geqslant\varepsilon^2 \end{cases}, \tag{4.186}$$

and hence

$$\sup_{t\in[-\varepsilon,T_*)}\left|\partial_x^4\eta(x,t)\right| \leqslant \begin{cases} 3\varepsilon^{-\frac{31}{8}} & |x-x_*|\leqslant\varepsilon^3 \\ 363\varepsilon^{-4} & |x-x_*|\leqslant\varepsilon^2 \\ C\varepsilon^{-\frac{9}{2}} & |x-x_*|\geqslant\varepsilon^2 \end{cases}, \tag{4.187}$$

which proves (4.180). □

4.8 C^4 Regularity Away from the Blowup

Lemma 4.12 *For labels x, we have that*

$$\begin{aligned} &\sup_{t\in[-\varepsilon,T_*)} \max_{\gamma\leqslant 4}\left(\left|\partial_\theta^\gamma a(\eta(x,t),t)\right| + \left|\partial_\theta^\gamma w(\eta(x,t),t)\right|\right) \\ &\leqslant \begin{cases} C_\varepsilon\left((T_*-t)+3\varepsilon^{-3}(\varepsilon+t)(x-x_*)^2\right)^{-4} & |x-x_*|\leqslant\varepsilon^2 \\ C_\varepsilon & |x-x_*|\geqslant\varepsilon^2 \end{cases}, \end{aligned} \tag{4.188}$$

where C_ε denotes a generic positive constant depending on inverse powers of ε.

Proof of Lemma 4.12 We use the identities (4.117) for $\partial_\theta^\gamma w\circ\eta$. The bounds on the initial data (4.131), (4.141), (4.149), (4.183), the bounds on derivatives of η given in (4.119), (4.135), (4.143), (4.153), (4.154), and (4.180), the bounds on I_t and its derivatives given in (4.142) and (4.184) prove the stated bound for $\partial_\theta^\gamma w\circ\eta$ in (4.188).

The additional inequalities (4.34), (4.139), (4.148), and (4.182) then proved the stated bound for $\partial_\theta^\gamma a\circ\eta$ in (4.188). □

Proposition 4.13 (Taylor expansion for $\eta(x,t)$) *The 3-characteristics η satisfy*

$$\eta\in C^1([-\varepsilon,T_*],C^4(\mathbb{T}))\,,$$

and at the blowup time, $\eta(x,T_*)$ has the Taylor expansion about x_* given by

$$\eta(x,T_*) = \eta(x_*,T_*) + \tfrac{1}{6}\partial_x^3\eta(x_*,T_*)(x-x_*)^3 + \tfrac{1}{6}\partial_x^4\eta(\overline{x},T_*)(x-x_*)^4\,, \tag{4.189}$$

for some $\overline{x}$ between x_* and x.

Proof of Proposition 4.13 By Lemma 2.1 in [1], there exists a short time $\overline{T} \geqslant -\varepsilon$, such that (w, a) is a unique solution to (3.10) with initial data (w_0, a_0) and

$$(a, w) \in C^0([-\varepsilon, \overline{T}]; C^4(\mathbb{T})) . \tag{4.190}$$

for any open set U which does not intersect ξ_*. By the local existence and uniqueness theorem for ODE, $\eta \in C^1([-\varepsilon, \overline{T}]; C^4(\mathbb{T}))$. Given the uniform bounds (4.119), (4.135), (4.143), and (4.180), the standard continuation argument shows that

$$\eta \in C^1([-\varepsilon, T_*], C^4(\mathbb{T})) .$$

The Taylor remainder theorem provides the expansion (4.189). □

4.9 Newton Iteration to Solve Quartic Equations in a Fractional Series

We wish to invert the polynomial equation $\eta(x, T_*) = z$. As given by (4.156), this requires inversion of a quartic polynomial. We shall derive the root that yields a Hölder-$\frac{1}{3}$ solution for $\eta^{-1}(\cdot, T_*)$ and satisfies $\eta^{-1}(\xi_*, T_*) = x_*$.

Lemma 4.14 (Quartic inversion) *If*

$$f(x, y) = -x + a_3 y^3 + a_4 y^4 ,$$

and $a_3 > 0$*, then the solution* $y(x)$ *to* $f(x, y) = 0$ *such that* $y(0) = 0$ *is given by the fractional power-series*

$$y(x) = a_3^{-\frac{1}{3}} x^{\frac{1}{3}} - \tfrac{1}{3} a_4 a_3^{-\frac{5}{3}} x^{\frac{2}{3}} + \tfrac{1}{3} a_3^{-3} a_4^2 x + \mathcal{O}(|x|^{\frac{4}{3}}) . \tag{4.191}$$

Proof of Lemma 4.14 We will first obtain an approximate solution using the Newton polygon method. Each term of the polynomial $f(x, y)$ is written as $cx^a y^b$, and the Newton polygon for $f(x, y)$ is constructed as the smallest convex polygonal set that contains the points $be_1 + ae_2$. This polygon consists of a finite set of segments, and we consider the segment Γ_1, such that each of the points $(b, a) = be_1 + ae_2$ is either above or to the right of this segment.

We will construct a fractional-series solution to $f(x, y) = 0$ as

$$y(x) = c_1 x^{\gamma_1} + c_2 x^{\gamma_1+\gamma_2} + c_3 x^{\gamma_1+\gamma_2+\gamma_3} + \cdots . \tag{4.192}$$

The first fractional power γ_1 is chosen as minus the slope of Γ_1. For $-x+a_3y^3+a_4y^4 = 0$, the points (b, a) are given by (0, 1), (3, 0), and (4, 0), and thus it is easy to see that the two lower segments of the Newton polygon have slopes $-\frac{1}{3}$ and 0, but that the segment with slope 0 exists only if $a_4 \neq 0$. We first consider the segment Γ_1 with slope $-\frac{1}{3}$, in which case $\gamma_1 = \frac{1}{3}$. We thus factor $x^{\frac{1}{3}}$ from (4.192), and write

$$y(x) = x^{\frac{1}{3}}(c_1 + y_1(x)) , \quad y_1(x) = c_2 x^{\gamma_2} + c_3 x^{\gamma_2+\gamma_3} + \cdots .$$

We compute

$$f(x, x^{\frac{1}{3}}(c_1 + y_1)) = -x + a_3 x(c_1 + y_1)^3 + a_4 x^{\frac{4}{3}}(c_1 + y_1)^4 .$$

The coefficient of the monomial x must equal to zero, so we can determine c_1:

$$x(-1 + a_3 c_1^3) = 0 \implies c_1 = a_3^{-1/3} .$$

We next define $f_1(x, y_1) = x^{-\alpha_1} f(x, x^{\frac{1}{3}}(c_1 + y_1))$ where α_1 is the intersection of the segment Γ_1 and the vertical a-axis, so that $\alpha_1 = 1$. We have that

$$\begin{aligned} f_1(x, y_1) &= x^{-1} f(x, x^{\frac{1}{3}}(a_3^{-1/3} + y_1)) \\ &= a_4 a_3^{-4/3} x^{\frac{1}{3}} + 3a_3^{1/3} y_1 + 4a_4 a_3^{-1} x^{\frac{1}{3}} y_1 \\ &\quad + 3a_3^{2/3} y_1^2 + 6a_4 a_3^{-2/3} x^{\frac{1}{3}} y_1^2 + a_3 y_1^3 + 4a_4 a_3^{-\frac{1}{3}} x^{\frac{1}{3}} y_1^3 + a_4 x^{\frac{1}{3}} y_1^4 . \end{aligned}$$

The Newton polygon for $f_1(x, y_1) = 0$ shows that the segment Γ_2, whose slope is equal to minus the exponent γ_2, connects the points $(0, \frac{1}{3})$ and $(1, 0)$, so that $\gamma_2 = \frac{1}{3}$. We next write

$$y_1(x) = x^{\frac{1}{3}}(c_2 + y_2(x)) , \quad y_2(x) = c_3 x^{\gamma_3} + c_4 x^{\gamma_3+\gamma_4} + \cdots .$$

We compute $f_1(x, x^{\frac{1}{3}}(c_2 + y_2))$ and cancel the coefficients in the lowest-order term to find that $c_2 = -\frac{1}{3} a_4 a_3^{-\frac{5}{3}}$. We then define

$$f_2(x, y_2) = x^{-\alpha_2} f_1(x, x^{\frac{1}{3}}(c_2 + y_2)) = x^{-\frac{1}{3}} f_1(x, x^{\frac{1}{3}}(-\tfrac{1}{3} a_4 a_3^{-\frac{5}{3}} + y_2)) ,$$

where $\alpha_2 = \frac{1}{3}$ is the a-intercept for the segment Γ_2. A computation reveals that

$$f_2(x, y_2) = -a_4^2 a_3^{-\frac{8}{3}} x^{\frac{1}{3}} + 3a_3^{\frac{1}{3}} y_2 + o(|x|^{\frac{1}{3}}) ,$$

and the Newton polygon for $f_2(x, y)$ shows that the exponent $\gamma_3 = \frac{1}{3}$, which in turn shows that $y_2(x) = Cx^{\frac{1}{3}} + \cdots$. Continuing one more step in the iteration to $f_3(x, y_3)$ (whose details we omit), we find that $C = \frac{1}{3} a_3^{-3} a_4^2$. We thus determined the first two non-trivial terms of this fractional series expansion (4.191). The result follows by an application of the implicit function theorem to the approximate solution that we have just determined.

We now return to the case in which the first fractional power uses the segment of the Newton polygon with slope 0. In this case, we begin the iteration with $\gamma_1 = 0$, we find that $y(x) = -\frac{a_3}{a_4} - \frac{a_4^2}{a_3^3} x + \mathcal{O}(x^2)$. Note however that $y(0) \neq 0$ in this case. □

4.10 Proof of Theorem 4.1

Having established the expansion for $\eta(x, T_*)$ we can now prove the main result of this section.

Proof of Theorem 4.1 We consider labels x satisfying $|x - x_*| \leqslant \varepsilon^3$. By Proposition 4.13, we have that $\eta(x, T_*)$ has the Taylor series expansion (4.156), which we write again as

$$\eta(x, T_*) = \xi_* + \tfrac{1}{6}\partial_x^3 \eta(x_*, T_*)(x - x_*)^3 + \tfrac{1}{24}\partial_x^4 \eta(\overline{x}, T_*)(x - x_*)^4 , \quad (4.193)$$

where $\xi_* = \eta(x_*, T_*)$, and $\overline{x}$ is a point between x_* and x. By (4.144), the coefficient for the cubic monomial cannot vanish:

$$\tfrac{1}{6}\partial_x^3 \eta(x_*, T_*) \geqslant \tfrac{6 - \varepsilon^{\frac{1}{6}}}{\varepsilon^3} > 0 . \quad (4.194)$$

Setting $\eta(x, T_*) = \theta$, we find that

$$\tfrac{1}{6}(x - x_*)^3 \partial_x^3 \eta(x_*, T_*) + \tfrac{1}{24}(x - x_*)^4 \partial_x^4 \eta(\overline{x}, T_*) = \theta - \xi_* .$$

We define the constants[6]

$$\alpha_1 = \Big(\tfrac{6}{\partial_x^3 \eta(x_*, T_*)}\Big)^{\frac{1}{3}} > 0 , \quad (4.195a)$$

$$\alpha_2 = -\tfrac{1}{3}\tfrac{\partial_x^4 \eta(\overline{x}, T_*)}{24}\Big(\tfrac{6}{\partial_x^3 \eta(x_*, T_*)}\Big)^{\frac{5}{3}} , \quad (4.195b)$$

$$\alpha_3 = \tfrac{1}{3}\Big(\tfrac{6}{\partial_x^3 \eta(x_*, T_*)}\Big)^3 \Big(\tfrac{\partial_x^4 \eta(\overline{x}, T_*)}{24}\Big)^2 , \quad (4.195c)$$

where clearly the positivity condition (4.195a) is merely a restatement of (4.194). Using Lemma 4.14, we have that

$$x - x_* = \alpha_1(\theta - \xi_*)^{\frac{1}{3}} + \alpha_2(\theta - \xi_*)^{\frac{2}{3}} + \alpha_3(\theta - \xi_*) + \mathcal{O}(|\theta - \xi_*|^{\frac{4}{3}}) \quad (4.196)$$

We define the function

$$\mathcal{I}(x) = -\tfrac{8}{3}\int_{-\varepsilon}^{T_*} a(\eta(x, r), r) dr .$$

[6] Note that, as defined by (4.195), α_2 and α_3 actually depend on x through the intermediate point $\overline{x}$, and thus are not truly “constants”. Nevertheless, in our proof we need only upper and lower bounds on α_2 and α_3 which are independent of x, bounds which are indeed available here; no information on the regularity of these functions with respect to x is needed. The same comment applies to b_3 defined in (4.199). It is however crucial that α_1, b_1 and b_2 are independent of x, which holds true. We emphasize that since the initial data (w_0, a_0) is taken to be C^5 smooth instead of just C^4, we may use arguments similar to those in Lemma 4.11 and Lemma 4.12 to show that η, $w \circ \eta$, and $a \circ \eta$ are in fact bounded uniformly in time with values in $W^{5,\infty}$; as such the expansion (4.193) can be developed to *fifth order*, and this does make α_1, α_2, b_3 constant in x. We omit these computations which do not require new ideas but are quite involved, and instead refer to the paper [16] for these details (the paper [16] includes these details even when z and k do not vanish identically).

Taylor expanding $w_0(x)$ about x_* in the identity (4.44), we have that

$$\begin{aligned} w(\eta(x,t),T_*) &= e^{\mathcal{I}(x)} w_0(x) \\ &= e^{\mathcal{I}(x)}\Big(w_0(x_*) + \partial_x w_0(x_*)(x-x_*) + \tfrac12 \partial_x^2 w_0(x_*)(x-x_*)^2 \\ &\quad + \tfrac16 \partial_x^3 w_0(x_*)(x-x_*)^3 + \tfrac{1}{24}\partial_x^4 w_0(\overline{x})(x-x_*)^4\Big)\,, \end{aligned} \tag{4.197}$$

for some $\overline{x}$ between x_* and x.

By Proposition 4.13, $a \circ \eta \in C^4$, so we can apply the Taylor remainder theorem to the function $e^{\mathcal{I}(x)}$, expanding about about x_*, and obtain

$$\begin{aligned} e^{\mathcal{I}(x)} &= e^{\mathcal{I}(x_*)}\Big(1 + \mathcal{I}'(x_*)(x-x_*) + \tfrac12(\mathcal{I}'(x_*)^2 + \mathcal{I}''(x_*))(x-x_*)^2 \\ &\quad + \tfrac16(\mathcal{I}'(\widehat{x})^3 + 3\mathcal{I}'(\widehat{x})\mathcal{I}''(\widehat{x}) + \mathcal{I}'''(\widehat{x}))(x-x_*)^3\Big)\,, \end{aligned} \tag{4.198}$$

where $\widehat{x}$ is a point between x_* and x. To simplify notation, we define the constants

$$b_1 = \mathcal{I}'(x_*)\,, \quad b_2 = \tfrac12(\mathcal{I}'(x_*)^2 + \mathcal{I}''(x_*))\,, \quad b_3 = \tfrac16(\mathcal{I}'(\widehat{x})^3 + 3\mathcal{I}'(\widehat{x})\mathcal{I}''(\widehat{x}) + \mathcal{I}'''(\widehat{x}))\,, \tag{4.199}$$

and write (4.198) as

$$e^{\mathcal{I}(x)} = e^{\mathcal{I}(x_*)}\Big(1 + b_1(x-x_*) + b_2(x-x_*)^2 + b_3(x-x_*)^3\Big)\,. \tag{4.200}$$

From (4.197) and (4.200), we have that

$$\begin{aligned} w(\eta(x,t),T_*) =& e^{\mathcal{I}(x_*)}\Big(1 + b_1(x-x_*) + b_2(x-x_*)^2 + b_3(x-x_*)^3\Big) \\ & \Big(w_0(x_*) + \partial_x w_0(x_*)(x-x_*) \\ & + \tfrac12\partial_x^2 w_0(x_*)(x-x_*)^2 + \tfrac16\partial_x^3 w_0(x_*)(x-x_*)^3 + \tfrac{1}{24}\partial_x^4 w_0(\overline{x})(x-x_*)^4\Big) \\ =& e^{\mathcal{I}(x_*)}\Big(w_0(x_*) + \big(b_1 w_0(x_*) + \partial_x w_0(x_*)\big)(x-x_*) \\ & + \big(b_2 w_0(x_*) + b_1\partial_x w_0(x_*) + \tfrac12\partial_x^2 w_0(x_*)\big)(x-x_*)^2 \\ & + \big(b_3 w_0(x_*) + b_2\partial_x w_0(x_*) + \tfrac12 b_1\partial_x^2 w_0(x_*) \\ & + \tfrac16\partial_x^3 w_0(x_*)\big)(x-x_*)^3\Big) + \mathcal{O}(|x-x_*|^4)\,. \end{aligned} \tag{4.201}$$

We define the constants

$$B_1 = b_1 w_0(x_*) + \partial_x w_0(x_*)\,, \tag{4.202a}$$

$$B_2 = b_2 w_0(x_*) + b_1\partial_x w_0(x_*) + \tfrac12\partial_x^2 w_0(x_*)\,, \tag{4.202b}$$

$$B_3 = b_3 w_0(x_*) + b_2\partial_x w_0(x_*) + \tfrac12 b_1\partial_x^2 w_0(x_*) + \tfrac16\partial_x^3 w_0(x_*)\,, \tag{4.202c}$$

and

$$\kappa_* = e^{\mathcal{I}(x_*)} w_0(x_*) \,,$$

and thus

$$w(\eta(x,t), T_*) = \kappa_* + e^{\mathcal{I}(x_*)}\Big(B_1(x-x_*) + B_2(x-x_*)^2 + B_3(x-x_*)^3\Big) + \mathcal{O}(|x-x_*|^4) \,. \tag{4.203}$$

With $\theta = \eta(x, T_*)$ as before, it follows from (4.196) that

$$w(\theta, T_*) = \kappa_* + e^{\mathcal{I}(x_*)}\Big(\alpha_1 B_1(\theta-\xi_*)^{\frac{1}{3}} + \big(\alpha_2 B_1 + \alpha_1^2 B_2\big)(\theta-\xi_*)^{\frac{2}{3}} + \big(\alpha_3 B_1 + 2\alpha_1\alpha_2 B_2 + \alpha_1^3 B_3\big)(\theta-\xi_*)\Big) + \mathcal{O}(|\theta-\xi_*|^{\frac{4}{3}}) \,, \tag{4.204}$$

We can now define the constants a_1, a_2, and a_3 in (4.2) as follows:

$$\mathsf{a}_1 = e^{\mathcal{I}(x_*)}\alpha_1 B_1 \,, \tag{4.205a}$$

$$\mathsf{a}_2 = e^{\mathcal{I}(x_*)}(\alpha_2 B_1 + \alpha_1^2 B_2) \,, \tag{4.205b}$$

$$\mathsf{a}_3 = e^{\mathcal{I}(x_*)}(\alpha_3 B_1 + 2\alpha_1\alpha_2 B_2 + \alpha_1^3 B_3) \,. \tag{4.205c}$$

We note that by Lemma 4.7,

$$\tfrac{9}{10}\varepsilon \leqslant \alpha_1 \leqslant \tfrac{11}{10}\varepsilon \,, \quad |\alpha_2| \lesssim \varepsilon^{\frac{9}{8}} \,, \quad |\alpha_3| \leqslant \varepsilon^{\frac{5}{4}} \,. \tag{4.206}$$

Furthermore, since by (4.19), $w_0(0) - \kappa_0 = 0$, and we assume the inequality (4.23), we see that since $\left|x_*\right| \leqslant 2\kappa_0\varepsilon^4$, we have that

$$\kappa_0 - 2\varepsilon^{\frac{5}{2}} \leqslant w_0(x_*) \leqslant \kappa_0 + 2\varepsilon^{\frac{5}{2}} \,, \tag{4.207}$$

and from (4.137)

$$-\tfrac{1+\varepsilon}{\varepsilon} \leqslant \partial_x w_0(x_*) \leqslant -\tfrac{1-\varepsilon}{\varepsilon} \,, \quad |\partial_x^2 w_0(x_*)| \leqslant 7\varepsilon^{\frac{1}{2}} \,. \tag{4.208}$$

From (4.199) and (4.26), we see that b_1, b_2, and b_3 are $\mathcal{O}(\varepsilon)$. Using (4.202) together with (4.207) and (4.208), we find that

$$-\tfrac{1+\varepsilon}{\varepsilon} - \varepsilon^{\frac{9}{10}} \leqslant B_1 \leqslant -\tfrac{1-\varepsilon}{\varepsilon} + \varepsilon^{\frac{9}{10}} \,, \quad |B_2| \leqslant 4\varepsilon^{\frac{1}{2}} \,.$$

Together with (4.205) and (4.206), we have that for ε taken small enough,

$$-\tfrac{6}{5} \leqslant \mathsf{a}_1 \leqslant -\tfrac{4}{5} \,, \quad \left|\mathsf{a}_2\right| \lesssim \varepsilon^{\frac{1}{8}} \leqslant \varepsilon^{\frac{1}{10}} \,, \quad \left|\mathsf{a}_3\right| \leqslant \tfrac{7}{6\varepsilon} \,.$$

Let us now follow the same argument that we used above to produce an expansion for $w_x(\eta(x, T_*), T_*)$. We see that

$$\begin{aligned}\partial_\theta w(\eta(x,t), T_*)\eta_x(x, T_*) &= e^{\mathcal{I}(x)}\big(w_0'(x) + \mathcal{I}'(x)w_0(x)\big)\\ &= e^{\mathcal{I}(x)}\Big(\partial_x w_0(x_*)(x-x_*) + \partial_x^2 w_0(x_*)(x-x_*)\\ &\quad + \tfrac{1}{2}\partial_x^3 w_0(x_*)(x-x_*)^2\\ &\quad + \tfrac{1}{6}\partial_x^4 w_0(\overline{x})(x-x_*)^3\Big)\\ &\quad + e^{\mathcal{I}(x)}\mathcal{I}'(x)\Big(w_0(x_*) + \partial_x w_0(x_*)(x-x_*)\\ &\quad + \tfrac{1}{2}\partial_x^2 w_0(x_*)(x-x_*)^2 + \tfrac{1}{6}\partial_x^3 w_0(x_*)(x-x_*)^3\\ &\quad + \tfrac{1}{24}\partial_x^4 w_0(\overline{x})(x-x_*)^4\Big),\end{aligned} \tag{4.209}$$

where $\overline{x}$ lies between x and x_*. In addition to (4.200), we shall need the expansion of $e^{\mathcal{I}(x)}\mathcal{I}'(x)$ and we continue to use b_1, b_2, b_3 defined in (4.199) and write

$$e^{\mathcal{I}(x)}\mathcal{I}'(x) = e^{\mathcal{I}(x_*)}\Big(b_1 + 2b_2(x-x_*) + 3b_3(x-x_*)^2\Big), \tag{4.210}$$

We can then write

$$\begin{aligned}\partial_\theta w(\eta(x,t), T_*)\eta_x(x, T_*) &= e^{\mathcal{I}(x_*)}\Big(b_1 w_0(x_*) + \partial_x w_0(x_*)\\ &\quad + \big(2b_2 w_0(x_*) + 2b_1\partial_x w_0(x_*) + \partial_x^2 w_0(x_*)\big)(x-x_*)\\ &\quad + \tfrac{1}{2}\big(6b_3 w_0(x_*) + 6b_2\partial_x w_0(x_*) + 3b_1\partial_x^2 w_0(x_*) + \partial_x^3 w_0(x_*)\big)\\ &\quad (x-x_*)^2 + \mathcal{O}(|x-x_*|^3)\Big).\end{aligned} \tag{4.211}$$

With the expansion $\eta_x(x, T_*)$ is written as

$$\eta_x(x, T_*) = \tfrac{1}{2}\partial_x^3\eta(x_*, T_*)(x-x_*)^2 + \tfrac{1}{6}\partial_x^4\eta(\mathring{x}, T_*)(x-x_*)^3 \tag{4.212}$$

for some $\mathring{x} \in (x, x_*)$. Therefore, with (4.211), we have that

$$\begin{aligned}\partial_\theta w(\eta(x,t), T_*) &= e^{\mathcal{I}(x_*)}\Big(b_1 w_0(x_*) + \partial_x w_0(x_*)\\ &\quad + \big(2b_2 w_0(x_*) + 2b_1\partial_x w_0(x_*) + \partial_x^2 w_0(x_*)\big)(x-x_*)\\ &\quad + \tfrac{1}{2}\big(6b_3 w_0(x_*) + 6b_2\partial_x w_0(x_*)\\ &\quad + 3b_1\partial_x^2(x_*) + \partial_x^3 w_0(x_*)\big)(x-x_*)^2 + \mathcal{O}(|x-x_*|^3)\Big)\\ &\quad \times\Big(\tfrac{1}{2}\partial_x^3\eta(x_*, T_*)(x-x_*)^2 + \tfrac{1}{6}\partial_x^4\eta(\mathring{x}, T_*)(x-x_*)^2\Big)^{-1}.\end{aligned} \tag{4.213}$$

Another expansion of the right side of (4.213) gives

$$\partial_\theta w(\eta(x,t),T_*) = e^{\mathcal{I}(x_*)}\Big(d_{-2}(x-x_*)^{-2} + d_{-1}(x-x_*)^{-1} + d_0\Big) + \mathcal{O}(|x-x_*|)\,, \tag{4.214}$$

where

$$\begin{aligned}
d_{-2} &= \tfrac{2(b_1 w_0(x_*)+\partial_x w_0(x_*))}{\partial_x^3\eta(x_*,T_*)}\,,\\
d_{-1} &= \tfrac{2\big(2b_2 w_0(x_*)+2b_1\partial_x w_0(x_*)+\partial_x^2 w_0(x_*)\big)}{\partial_x^3\eta(x_*,T_*)}\\
&\quad -\tfrac{2\big(b_1 w_0(x_*)+\partial_x w_0(x_*)\big)\partial_x^4\eta(\mathring{x},T_*)}{\partial_x^3\eta(x_*,T_*)^2}\,,\\
d_0 &= \tfrac{6b_3 w_0(x_*)+6b_2\partial_x w_0(x_*)+3b_1\partial_x^2 w_0(x_*)+\partial_x^3 w_0(x_*)}{\partial_x^3\eta(x_*,T_*)}\\
&\quad -\tfrac{2\big(2b_2 w_0(x_*)+2b_1\partial_x w_0(x_*)+\partial_x^2 w_0(x_*)\big)\partial_x^4\eta(\mathring{x},T_*)}{3\partial_x^3\eta(x_*,T_*)^2}\\
&\quad +\tfrac{2\big(b_1 w_0(x_*)+\partial_x w_0(x_*)\big)\partial_x^4\eta(\mathring{x},T_*)}{9\partial_x^3\eta(x_*,T_*)^3}\,.
\end{aligned}$$

By substituting (4.196) into (4.214), we obtain that

$$\begin{aligned}
&\Big|\partial_\theta w(\theta,T_*) - e^{\mathcal{I}(x_*)}\alpha_1^{-2}d_{-2}z^{-\frac{2}{3}} - e^{\mathcal{I}(x_*)}\big(\alpha_1^{-1}d_{-1} - 2\alpha_1^{-3}\alpha_2 d_{-2}\big)z^{-\frac{1}{3}}\Big|\\
&\quad \leqslant 2e^{\mathcal{I}(x_*)}\Big(d_0 - \alpha_1^{-2}\alpha_2 d_{-1} + (3\alpha_2^2 - 2\alpha_1\alpha_3)\alpha_1^{-4}d_{-2}\Big)\,.
\end{aligned} \tag{4.215}$$

Notice from (4.195a), (4.202a), (4.205a) that since

$$\mathsf{a}_1 = e^{\mathcal{I}(x_*)}\big(\tfrac{6}{\partial_x^3\eta(x_*,T_*)}\big)^{\frac{1}{3}}\big(b_1 w_0(x_*) + \partial_x w_0(x_*)\big)\,,$$

and since

$$\begin{aligned}
e^{\mathcal{I}(x_*)}\alpha_1^{-2}d_{-2} &= 2e^{\mathcal{I}(x_*)}\big(\tfrac{6}{\partial_x^3\eta(x_*,T_*)}\big)^{-\frac{2}{3}}\Big(\tfrac{b_1 w_0(x_*)+\partial_x w_0(x_*)}{\partial_x^3\eta(x_*,T_*)}\Big)\\
&= \tfrac{1}{3}e^{\mathcal{I}(x_*)}\big(\tfrac{6}{\partial_x^3\eta(x_*,T_*)}\big)^{\frac{1}{3}}\big(b_1 w_0(x_*) + \partial_x w_0(x_*)\big) = \tfrac{1}{3}\mathsf{a}_1\,,
\end{aligned}$$

A similar computation shows that

$$e^{\mathcal{I}(x_*)}\big(\alpha_1^{-1}d_{-1} - 2\alpha_1^{-3}\alpha_2 d_{-2}\big) = \tfrac{2}{3}\mathsf{a}_2\,.$$

As such, we have established the inequality

$$\Big|\partial_\theta w(\theta,T_*) - \tfrac{1}{3}\mathsf{a}_1(\theta-\xi_*)^{-\frac{2}{3}} - \tfrac{2}{3}\mathsf{a}_2(\theta-\xi_*)^{-\frac{1}{3}}\Big| \leqslant C_{\mathsf{m}}\,, \tag{4.216}$$

where

$$C_{\mathsf{m}} = 2e^{\mathcal{I}(x_*)}\Big(d_0 - \alpha_1^{-2}\alpha_2 d_{-1} + (3\alpha_2^2 - 2\alpha_1\alpha_3)\alpha_1^{-4} d_{-2}\Big)\,,$$

satisfies $\left|C_{\mathsf{m}}\right| \lesssim \frac{1}{\varepsilon}$. The inequality (4.216) and the bound for C_{m} establishes (4.4a).

From (4.209), we see that

$$\partial_\theta^2 w(\eta(x,T_*),T_*)\eta_x^2(x,T_*) = -\partial_\theta(\eta(x,T_*),T_*)\eta_{xx}(x,T_*) + e^{\mathcal{I}(x)}\big(w_0''(x) + 2I'(x)w'(x) + \mathcal{I}''(x)w_0(x)\big)\,. \tag{4.217}$$

In addition to the expansion (4.212), we shall also need the fact that

$$\eta_{xx}(x,T_*) = \partial_x^3\eta(x_*,T_*)(x-x_*) + \tfrac{1}{2}\partial_x^4\eta(\ddot{\mathring{x}},T_*)(x-x_*)^2$$

for some $\ddot{\mathring{x}} \in (x, x_*)$. After a lengthy computation, we find that

$$\left|\partial_\theta^2 w(\theta,T_*) - \tfrac{2}{9}\mathsf{a}_1(\theta-\xi_*)^{-\frac{5}{3}}\right| \leqslant \overline{C}_{\mathsf{m}}(\theta-\xi_*)^{-\frac{4}{3}}\,, \tag{4.218}$$

where

$$\left|\overline{C}_{\mathsf{m}}\right| \lesssim \varepsilon^{-\frac{63}{8}}\,,$$

which establishes (4.4b).

Finally, from (4.217), we see that

$$\begin{aligned}\partial_\theta^3 w(\eta(x,T_*),T_*)\eta_x^3(x,T_*) = &-3\partial_\theta^2 w(\eta(x,T_*),T_*)\eta_x(x,T_*)\eta_{xx}(x,T_*)\\ &-\partial_\theta w(\eta(x,T_*),T_*)\eta_{xxx}(x,T_*)\\ &+e^{\mathcal{I}(x)}\big(w_0'''(x)\\ &+3I'(x)w''(x) + 3I''(x)w'(x) + \mathcal{I}'''(x)w_0(x)\big)\,.\end{aligned} \tag{4.219}$$

We make use of one further expansion given by

$$\partial_x^3\eta(x,T_*) = \partial_x^3\eta(x_*,T_*) + \partial_x^4\eta(\dddot{\mathring{x}},T_*)(x-x_*)$$

for some $\dddot{\mathring{x}} \in (x, x_*)$. A final lengthy computation shows that

$$\left|\partial_\theta^3 w(\theta,T_*)\right| \lesssim \varepsilon^{-\frac{151}{8}}\left|\theta-\xi_*\right|^{-\frac{8}{3}}\,, \tag{4.220}$$

which establishes (4.4c).

The estimates (4.5) are established by (4.188). The bounds (4.6) for the specific vorticity are established in (4.81) and (4.87). From (4.79) we have that

$\sup_{[0,T_*)} \|a(\cdot,t)\|_{L^\infty} \leqslant \frac{3}{2}\varepsilon$. From (4.2), we have that $w(\cdot, T_*) \in C^{\frac{1}{3}}(\mathbb{T})$; therefore, since $\partial_x a = \frac{w^2}{16}\varpi - w$, by (4.33) and (4.81), we have that $a(\cdot, T_*) \in C^{1,\frac{1}{3}}(\mathbb{T})$ which gives the regularity statement in (4.1). The bounds for ϖ are given in (4.81), and for $\partial_x \varpi$ in (4.87). □

5 Shock Development

In this section we consider the system (3.5)–(3.6), with pre-shock initial datum as obtained in Section 4, and consider the associated *development problem*. The main result is Theorem 5.5 below.

5.1 Initial Data for Shock Development Comes from the *Pre-shock*

Theorem 4.1 guarantees the finite time formation of a first singularity for the (w, z, a, k) system (3.5) at $(\theta, t) = (\xi_*, T_*)$; more precisely, the first Riemann variable w forms a $C^{\frac{1}{3}}$ *pre-shock* as described in (4.2), z and k remain equal to 0 (their initial datum), while the function a retains $C^{1,\frac{1}{3}}$ regularity at the time that the pre-shock forms.

The initial data for the development problem is provided by Theorem 4.1. For the remainder of paper, it is convenient to change coordinates so that the pre-shock occurs at $\theta = 0$ (instead of ξ_*), at time $t = 0$ (instead of T_*). The initial condition for the first Riemann variable thus is $w_0(\theta) = w(\theta - \xi_*, T_*)$, with the latter function being given by (4.2). In particular, we have that w_0 satisfies the quantitative estimates

$$w_0(\theta) \leqslant \mathsf{m} \tag{5.1a}$$

$$w_0(\theta) \geqslant \tfrac{1}{2}\kappa \tag{5.1b}$$

$$\left|w_0(\theta) - \kappa + \mathsf{b}\theta^{\frac{1}{3}} - \mathsf{c}\theta^{\frac{2}{3}}\right| \leqslant \mathsf{m}\,|\theta|\ , \tag{5.1c}$$

$$\left|w_0'(\theta) + \tfrac{1}{3}\mathsf{b}\theta^{-\frac{2}{3}} - \tfrac{2}{3}\mathsf{c}\theta^{-\frac{1}{3}}\right| \leqslant \mathsf{m}\,, \tag{5.1d}$$

$$\left|w_0''(\theta) - \tfrac{2}{9}\mathsf{b}\theta^{-\frac{5}{3}}\right| \leqslant \mathsf{m}|\theta|^{-\frac{4}{3}}\,, \tag{5.1e}$$

$$\left|w_0'''(\theta)\right| \leqslant \mathsf{m}\,|\theta|^{-\frac{8}{3}}\,, \tag{5.1f}$$

for all $\theta \in \mathbb{T}$, where $\kappa, \mathsf{m} \geqslant 1$, $\mathsf{b} > 0$, and $\mathsf{c} \in \mathbb{R}$ are suitable constants given as follows. In light of (4.2) and (4.4), we identify $\kappa = \kappa_*$, $\mathsf{b} = -\mathsf{a}_1$, $\mathsf{c} = \mathsf{a}_2$, while the constant m is taken to be sufficiently large, in terms of the large parameters κ_0 and ε^{-1} from Theorem 4.1. Note however that (4.2) and (4.4) only give the bounds (5.1c)–(5.1f) for θ in a ε-dependent ball around 0 (of radius ε^4, recall that we have mapped $\xi_* \mapsto 0$), whereas in (5.1) we require that these bounds hold for all $\theta \in \mathbb{T}$. We note however that for $|\theta|$ which is at a fixed positive distance away from 0, the bounds (5.1c)–(5.1f) follow once m is chosen to be sufficiently large with respect to κ_0 and ε^{-1}; this is because the bounds (4.5) imply uniform C^4 regularity once a fixed distance from the pre-shock is chosen. Indeed, (4.5), (4.119), (4.120), and (4.153) show that for $|\theta| \geqslant \varepsilon^4$, there exists a constant $C_\varepsilon > 0$ such that $|\partial_\theta^\gamma w_0(\theta)| \leqslant C_\varepsilon$ for $0 \leqslant \gamma \leqslant 4$.

We also note that by (4.37) and (4.3) the coefficients in (5.1) satisfy the conditions

$$|\kappa - \kappa_0| \leqslant \varepsilon^3 , \qquad \tfrac{1}{2} \leqslant \mathsf{b} \leqslant 2 , \qquad |\mathsf{c}| \leqslant \varepsilon^{\frac{1}{2}} ,$$

where we recall that $\kappa_0 > 1$ was chosen sufficiently large. In order to simplify our argument we shall frequently use the relations

$$|\mathsf{c}| \ll \mathsf{b} \leqslant 2 \qquad \text{and} \qquad 4 \leqslant \kappa \ll \mathsf{m} . \tag{5.2}$$

In particular, we shall use that m sufficiently large with respect to κ: if $\mathsf{C} > 0$ is a universal constant (independent of $\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}$), then $\kappa \mathsf{C} \leqslant \mathsf{m}^{\frac{1}{10}}$. Similarly, we shall use that $|\mathsf{c}|$ is sufficiently small with respect to b, so that $\mathsf{C}\mathsf{b}|\mathsf{c}| \leqslant 1$.

The initial conditions for the second Riemann variable and the entropy function are given by

$$z_0(\theta) \equiv 0 , \qquad \text{and} \qquad k_0(\theta) \equiv 0 . \tag{5.3}$$

Lastly, in view of Theorem 4.1 we identify $a_0(\theta) = a(\theta - \xi_*, T_*) \in C^{1,\frac{1}{3}}$ and $\varpi_0(\theta) = \varpi(\theta - \xi_*, T_*) \in C^1$. In particular, due to (4.79) and (4.83),

$$\|a_0\|_{W^{1,\infty}(\mathbb{T})} \leqslant \tfrac{3}{2}\kappa , \tag{5.4}$$

and due to (4.6), we have that

$$\tfrac{10}{\kappa} \leqslant \varpi_0(\theta) \leqslant \tfrac{28}{\kappa} \qquad \text{and} \qquad \left|\varpi_0'(x)\right| \leqslant \mathsf{m} , \tag{5.5}$$

for all $\theta \in \mathbb{T}$.

Remark 5.1 (*The small parameter* $\overline{\varepsilon}$ *and the large constant* C) Throughout Sections 5 and 6, we shall denote by $C = C(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) \geqslant 1$ a generic constant, which only depends on the parameters $\kappa, \mathsf{b}, \mathsf{c}$, and m, which appear in (5.1), and which may increase from line to line. We shall also denote by $\overline{\varepsilon} = \overline{\varepsilon}(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) \in (0, 1]$ a sufficiently small constant, which only depends on the parameters $\kappa, \mathsf{b}, \mathsf{c}$, and m. Note that the parameter $\overline{\varepsilon}$ is not the same as the parameter ε in Section 4.

5.2 Definitions

Definition 5.2 (*Jump, mean, left value, right value, domain*) Given a smooth curve $\mathfrak{s}\colon [0, T] \to \mathbb{T}$, we shall denote

$$\mathcal{D}_T = (\mathbb{T} \times [0, T]) \setminus (\mathfrak{s}(t), t)_{t\in[0,T]} \tag{5.6}$$

the space-time domain which excludes a shock curve. Given any function $f : \mathcal{D}_T \to \mathbb{R}$ we denote the left and right values of f at $\mathfrak{s}$ as

$$f_-(t) = \lim_{\theta \to \mathfrak{s}(t)^-} f(\theta, t) \qquad \text{and} \qquad f_+(t) = \lim_{\theta \to \mathfrak{s}(t)^+} f(\theta, t) \,. \tag{5.7}$$

We denote the *jump of f across $\mathfrak{s}$ by*

$$[\![f]\!] = [\![f(t)]\!] = f_-(t) - f_+(t) \,, \tag{5.8}$$

and the *mean of f at $\mathfrak{s}$ by*

$$\langle\!\langle f \rangle\!\rangle = \langle\!\langle f(t) \rangle\!\rangle = \tfrac{1}{2} \left(f_-(t) + f_+(t) \right) \,, \tag{5.9}$$

for all $t \in [0, T]$. The dependence of f_-, f_+, $[\![f]\!]$, and $\mathcal{D}_T$ on the curve $\mathfrak{s}$ is not displayed.

Next, we define a space $\mathcal{X}_T$ which will be used for the construction of unique solutions.

Definition 5.3 (*Functional space for shock emanating from $C^{1/3}$ pre-shock*) Let $\mathsf{m} > 1$ be as in (5.2). Given $T > 0$ and a curve $\mathfrak{s} \colon [0, T] \to \mathbb{T}$, define the norm

$$\begin{aligned} |\!|\!| (v, z, k, a) |\!|\!|_T = \sup_{(\theta,t) \in \mathcal{D}_T} \max \Big\{ & t^{-1} (50\mathsf{m}^2)^{-1} \left| v(\theta, t) \right| \,, \\ & \mathsf{m}^{-3} \left(\mathsf{b}^3 t^3 + (\theta - \mathfrak{s}(t))^2 \right)^{\frac{1}{6}} \left| \partial_\theta v(\theta, t) \right| \,, \\ & \mathsf{m}^{-1} t^{-\frac{3}{2}} \left| z(\theta, t) \right| \,, \mathsf{m}^{-1} t^{-\frac{1}{2}} \left| \partial_\theta z(\theta, t) \right| \,, \mathsf{m}^{-\frac{1}{2}} t^{-\frac{3}{2}} \left| k(\theta, t) \right| \,, \\ & \mathsf{m}^{-\frac{1}{2}} t^{-\frac{1}{2}} \left| \partial_\theta k(\theta, t) \right| \,, (4\mathsf{m})^{-1} \left| a(\theta, t) \right| \,, (4\mathsf{m})^{-1} \left| \partial_\theta a(\theta, t) \right| \Big\} \end{aligned} \tag{5.10}$$

where $\mathcal{D}_T$ is as defined in (5.6). For $T > 0$ we also define

$$\begin{aligned} \mathcal{X}_T = \Big\{ & (w, z, k, a) \in C^1_{\theta,t}(\mathcal{D}_T) \colon (w, z, k, a)|_{t=0} = (w_0, 0, 0, a_0) \,, \\ & |\!|\!| (w - w_{\mathsf{B}}, z, k, a) |\!|\!|_T \leqslant 1 \Big\} \,, \end{aligned} \tag{5.11}$$

where w_{B} is the solution of the 1D Burgers equation in $\mathcal{D}_T$ with datum w_0, which jumps across the shock curve $\mathfrak{s}$ (see Proposition 5.7 for its precise definition). That is, the role of the dummy variable v in (5.10) is played by $w - w_{\mathsf{B}}$.

In order to state the desired properties for $\mathfrak{s}$, in terms of the parameters κ and b appearing in (5.1c), we define two time-dependent subsets of $\mathbb{T}$. The first set, Σ, will be shown to contain the location of the shock front for w at time t, while the second set, Ω, contains the labels of the two particle trajectories associated with the flow of w, which fall into the shock at time t.

Definition 5.4 (*Regular shock curve*) For every $t \in [0, \kappa \mathsf{m}^{-4}]$, we define

$$\Sigma(t) = \left[\kappa t - \tfrac{1}{2}\mathsf{m}^4 t^2, \kappa t + \tfrac{1}{2}\mathsf{m}^4 t^2\right] \tag{5.12a}$$

$$\Omega(t) = \left[-\tfrac{5}{4}(\mathsf{b}t)^{\frac{3}{2}}, -\tfrac{3}{4}(\mathsf{b}t)^{\frac{3}{2}}\right] \cup \left[\tfrac{3}{4}(\mathsf{b}t)^{\frac{3}{2}}, \tfrac{5}{4}(\mathsf{b}t)^{\frac{3}{2}}\right] \tag{5.12b}$$

extended periodically on the circle $\mathbb{T}$. For a given $T \in (0, \kappa \mathsf{m}^{-4})$, we say that $t \mapsto \mathfrak{s}(t) \colon [0, T] \to \mathbb{T}$ is a *regular shock curve* if it $\mathfrak{s}$ satisfies

$$\mathfrak{s}(t) \in \Sigma(t), \qquad |\dot{\mathfrak{s}}(t) - \kappa| \leqslant \mathsf{m}^4 t, \qquad |\ddot{\mathfrak{s}}(t)| \leqslant 6\mathsf{m}^4, \tag{5.13}$$

for all $t \in (0, T]$.

5.3 The Shock Development Problem in Azimuthal Symmetry

We defined a solution to the development problem in Definition 3.1. The main result of this section is to establish the existence and the uniqueness of such solutions.

Theorem 5.5 (Azimuthal shock development) *Given pre-shock initial data* (w_0, z_0, k_0, a_0) *and* ϖ_0 *satisfying conditions (5.1)–(5.5), there exist:*

(i) $\overline{\varepsilon} = \overline{\varepsilon}(\mathsf{b}, \mathsf{m}, \mathsf{c}, \kappa) > 0$ *sufficiently small;*
(ii) *a* C^2 *regular shock curve* $\mathfrak{s} \colon [0, \overline{\varepsilon}] \to \mathbb{T}$*, in the sense of Defintion 5.4; in particular,* $\mathfrak{s}$ *solves the ordinary differential equation (3.12b), corresponding to Rankine-Hugoniot jump condition;*
(iii) *a* unique *solution* $(w, z, k, a) \in \mathcal{X}_{\overline{\varepsilon}}$ *to the system (3.5), in the sense of Definitions 3.1 and 5.3;*
(iv) *two* C^1 *smooth curves* $\mathfrak{s}_1, \mathfrak{s}_2 \colon [0, \overline{\varepsilon}] \to \mathbb{T}$*, with* $\mathfrak{s}_1(0) = \mathfrak{s}_2(0) = 0$ *and* $\mathfrak{s}_1(t) < \mathfrak{s}_2(t) < \mathfrak{s}(t)$ *for* $t \in (0, \overline{\varepsilon}]$*, such that* $\mathfrak{s}_i$ *is a characteristic curve for the* λ_i *wave-speed,* $i \in \{1, 2\}$*;*

such that the following hold:

(v) *letting* $\mathcal{D}^k_{\overline{\varepsilon}} = \{(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}} \colon \mathfrak{s}_2(t) < \theta < \mathfrak{s}(t)\}$ *we have that* $k \equiv 0$ *on* $(\mathcal{D}^k_{\overline{\varepsilon}})^{\complement}$ *with* $k(\theta, t) = \mathcal{O}((\theta - \mathfrak{s}_2(t))^{\frac{3}{2}})$ *in* $\mathcal{D}^k_{\overline{\varepsilon}}$*, cf. (5.215), and* $\partial_\theta k(\mathfrak{s}_2(t), t) = 0$*;*
(vi) *letting* $\mathcal{D}^z_{\overline{\varepsilon}} = \{(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}} \colon \mathfrak{s}_1(t) < \theta < \mathfrak{s}(t)\}$*, we have that* $z \equiv 0$ *on* $(\mathcal{D}^z_{\overline{\varepsilon}})^{\complement}$ *with* $z(\theta, t) = \mathcal{O}((\theta - \mathfrak{s}_1(t))^{\frac{3}{2}})$ *in* $\mathcal{D}^z_{\overline{\varepsilon}}$*, cf. (5.218), and* $\partial_\theta z(\mathfrak{s}_1(t), t) = 0$*;*
(vii) *on* $\mathfrak{s}(t)$*, the function* $w(\cdot, t)$ *exhibits an* $\mathcal{O}(t^{\frac{1}{2}})$ *jump, cf. (5.63), while the functions* $z(\cdot, t)$ *and* $k(\cdot, t)$ *exhibit* $\mathcal{O}(t^{\frac{3}{2}})$ *jumps, cf. (5.69), and solve the system of algebraic equations (3.13a)-(3.13b);*
(viii) *the specific vorticity* ϖ *(see its definition in (3.8)) solves (3.9) in* $\mathcal{D}_{\overline{\varepsilon}}$*, is uniformly bounded with* $\mathcal{O}(\kappa^{-1})$ *upper and lower (see (5.223)), and is continuous across the shock curve* $\mathfrak{s}(t)$*;*
(ix) *the function* $a(\cdot, t)$ *is continuous across* $\mathfrak{s}(t)$*, while* $\partial_\theta a(\cdot, t)$ *exhibits an* $\mathcal{O}(t^{\frac{1}{2}})$ *jump.*

5.4 A Given Shock Curve Determines *w*, *z*, *k*, and *a*

The goal of this subsection is to show that given a regular shock curve $\{\mathfrak{s}(t)\}_{t\in[0,\overline{\varepsilon}]}$, as in Definition 5.4, we may compute a solution (w, z, k, a) of the system (3.5)–(3.6) with initial datum as described in Section 5.1, and which exhibits *a jump discontinuity across the curve* $\mathfrak{s}(t)$. This statement is summarized in Proposition 5.6 below. Note that at this stage we do not assume that $\mathfrak{s}$ satisfies the ODE which corresponds to the jump conditions in Section 2.1; this will be discussed in Section 5.10.

With the above notation, the main result of this section is:

Proposition 5.6 (Computing w, z, k, and a, in terms of $\mathfrak{s}$) *Consider initial datum* (w_0, z_0, k_0, a_0) *which satisfy conditions (5.1), (5.3), and (5.4). Let* $T_0 > 0$ *be given, and assume that* $\mathfrak{s}\colon [0, T_0] \to \mathbb{T}$ *is a given regular shock curve, as in (5.13). Then, there exists* $\overline{\varepsilon} \in (0, T_0]$*, which is sufficiently small with respect the parameters* κ, b, c, m*, such that the following hold on* $[0, \overline{\varepsilon}]$*:*

- (i) *There exist functions* (w, z, k, a) *which belong to the space* $\mathcal{X}_{\overline{\varepsilon}}$ *defined in (5.11).*
- (ii) *On the spacetime region* $\mathcal{D}_{\overline{\varepsilon}}$*, defined in terms of* $\mathfrak{s}$ *in (5.6), the functions* (w, z, k, a) *solve the azimuthal Euler equations (3.5)–(3.6).*
- (iii) *The function* w *has a jump discontinuity on* $(\mathfrak{s}(t), t)_{t\in(0,\overline{\varepsilon}]}$ *which satisfies (5.63).*
- (iv) *There exist* C^1 *smooth curves* $\mathfrak{s}_1, \mathfrak{s}_2\colon [0, \overline{\varepsilon}] \to \mathbb{T}$ *which are the* λ_1 *and* λ_2 *characteristics through the point shock. They satisfy* $\mathfrak{s}_1(0) = \mathfrak{s}_2(0) = 0$*,* $\mathfrak{s}_1(t) < \mathfrak{s}_2(t) < \mathfrak{s}(t)$ *for all* $t \in (0, \overline{\varepsilon}]$*, and we have the bounds* $|\dot{\mathfrak{s}}_1(t) - \frac{1}{3}\kappa| = \mathcal{O}(t^{\frac{1}{3}})$*, and* $|\dot{\mathfrak{s}}_2(t) - \frac{2}{3}\kappa| = \mathcal{O}(t^{\frac{1}{3}})$.
- (v) *The function* z *has a jump discontinuity on* $(\mathfrak{s}(t), t)_{t\in(0,\overline{\varepsilon}]}$ *which satisfies (5.69a). Moreover, for every* $t \in [0, \overline{\varepsilon}]$ *we have that* $z(\theta, t) = 0$ *for* $\theta \in \mathbb{T}\setminus[\mathfrak{s}_1(t), \mathfrak{s}(t)]$.
- (vi) *The function* k *has a jump discontinuity on* $(\mathfrak{s}(t), t)_{t\in(0,\overline{\varepsilon}]}$ *which satisfies (5.69b). Moreover, for every* $t \in [0, \overline{\varepsilon}]$ *we have that* $k(\theta, t) = 0$ *for* $\theta \in \mathbb{T}\setminus[\mathfrak{s}_2(t), \mathfrak{s}(t)]$.
- (vii) *We have that* (w_-, w_+, z_-, k_-) *satisfy the system of algebraic equations (3.13a)-(3.13b), arising from the Rankine–Hugoniot conditions.*

The proof of Proposition 5.6 is the content of Sections 5.5–5.8, and is summarized in Section 5.9 (Fig. 9).

5.5 Computing *w* When $a = z = k = 0$

In light of (5.10) and (5.11), it is natural to treat z and k as a perturbation of 0. As such, it convenient to first look at the evolution (3.5a) for w, in the case that $a = k = z = 0$. In this case (3.5a) and the definition of λ_3 in (3.6) show that w solves the 1d Burgers equation; to distinguish this solution from the true w, we denote it as w_{B}.

Proposition 5.7 (Burgers solution with a prescribed shock location) *Let* w_0 *be as described in (5.1), and assume that* $\mathfrak{s}\colon [0, T_0] \to \mathbb{T}$ *satisfies (5.13). There exists* $\overline{\varepsilon} \in (0, T_0]$ *and a function* $w_{\mathsf{B}}\colon \mathcal{D}_{\overline{\varepsilon}} \to \mathbb{R}$ *which solves*

$$\partial_t w_{\mathsf{B}} + w_{\mathsf{B}} \partial_\theta w_{\mathsf{B}} = 0\,, \qquad \text{in} \quad \mathcal{D}_{\overline{\varepsilon}}\,, \tag{5.14a}$$

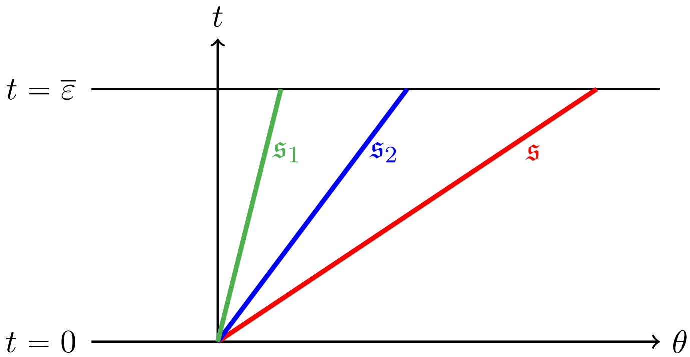

Fig. 9 The curves $\mathfrak{s}_1$, $\mathfrak{s}_2$, and $\mathfrak{s}$ discussed in Proposition 5.6 all originate from the pre-shock

$$w_{\mathsf{B}} = w_0\,, \qquad\qquad on \quad \mathbb{T} \times \{0\}\,, \tag{5.14b}$$

which is C^2 smooth in $\mathcal{D}_{\overline{\varepsilon}}$, and has a jump discontinuity across the curve $(\mathfrak{s}(t), t)_{t\in(0,\overline{\varepsilon}]}$, with jump across $\mathfrak{s}$ and mean at $\mathfrak{s}$ bounded as

$$\left|[\![w_{\mathsf{B}}(t)]\!] - 2\mathsf{b}^{\frac{3}{2}} t^{\frac{1}{2}}\right| \leqslant t\,, \qquad |\langle\!\langle w_{\mathsf{B}}(t)\rangle\!\rangle - \kappa| \leqslant \tfrac{1}{3}\mathsf{m}^4 t\,, \tag{5.15a}$$

$$\left|\tfrac{d}{dt}[\![w_{\mathsf{B}}(t)]\!] - \mathsf{b}^{\frac{3}{2}} t^{-\frac{1}{2}}\right| \leqslant 2\mathsf{m}^4\,, \qquad \left|\tfrac{d}{dt}\langle\!\langle w_{\mathsf{B}}(t)\rangle\!\rangle\right| \leqslant \mathsf{m}^4\,, \tag{5.15b}$$

$$\left|\tfrac{d^2}{dt^2}[\![w_{\mathsf{B}}(t)]\!] + \tfrac{1}{2}\mathsf{b}^{\frac{3}{2}} t^{-\frac{3}{2}}\right| \leqslant 2\mathsf{m}^4 t^{-1}\,, \qquad \left|\tfrac{d^2}{dt^2}\langle\!\langle w_{\mathsf{B}}(t)\rangle\!\rangle\right| \leqslant \mathsf{m}^4 t^{-1}\,. \tag{5.15c}$$

It is important to emphasize that the function w_{B} given by Proposition 5.7 is a solution of the Burgers equation in the region where it is C^2 smooth, i.e., it is not an entropy-producing weak solution of the Burgers equation which contains the shock. Instead, w_{B} takes a given curve $\mathfrak{s}$ as given, and constructs a "good" solution of the Burgers equation to the left and to the right of this curve $\mathfrak{s}$.

In Proposition 5.7 we use the notation from Remark 5.2 and Definition 5.2. Prior to the proof of Proposition 5.7, it is convenient to establish an auxiliary result for the derivatives of w_0 (cf. Lemma 5.8), and a result (cf. Lemma 5.9) which concerns the invertibility of the usual flow map for the Burgers equation:

$$\eta_{\mathsf{B}}(x, t) = x + t w_0(x)\,, \tag{5.16}$$

which is well-defined for every $x \in \mathbb{T}$.[7] We first record a few estimates for w_0, which follow from (5.1):

Lemma 5.8 *There exists $\overline{\varepsilon} \in (0, 1]$ such that for every $t \in (0, \overline{\varepsilon}]$ we have*

$$\left|w_0(x)\right| \leqslant \mathsf{m}\,, \qquad\qquad x \in \mathbb{T}\,, \tag{5.17a}$$

[7] Here and throughout the remainder of the paper we shall denote the Eulerian variable by θ, while for the corresponding Lagrangian label we use x.

$$\left|w_0'(x)\right| \leqslant \tfrac{2}{5}t^{-1}\,, \qquad \tfrac{4}{5}(\mathsf{b}t)^{\frac{3}{2}} \leqslant |x| \leqslant \pi\,, \tag{5.17b}$$

$$\left|w_0''(x)\right| \leqslant \tfrac{1}{3}\mathsf{b}^{-\frac{3}{2}}t^{-\frac{5}{2}}\,, \qquad \tfrac{4}{5}(\mathsf{b}t)^{\frac{3}{2}} \leqslant |x| \leqslant \pi \tag{5.17c}$$

$$\left|w_0'''(x)\right| \leqslant 2\mathsf{m}(\mathsf{b}t)^{-4}\,, \qquad \tfrac{4}{5}(\mathsf{b}t)^{\frac{3}{2}} \leqslant |x| \leqslant \pi \tag{5.17d}$$

$$\left|\frac{w_0'(x)}{1+tw_0'(x)}\right| \leqslant \tfrac{2}{3}t^{-1}\,, \qquad \tfrac{4}{5}(\mathsf{b}t)^{\frac{3}{2}} \leqslant |x| \leqslant \pi\,. \tag{5.17e}$$

Proof of Lemma 5.8 For simplicity, we only give the proof for $x > 0$. The bound (5.17a) follows directly from (5.1a) since (5.1b) implies that w_0 is nonnegative. In order to prove (5.17b) we use assumption (5.1d), which gives

$$\left|w_0'(x)\right|t \leqslant \tfrac{1}{3}\mathsf{b}\left(\tfrac{4}{5}\right)^{-\frac{2}{3}}(\mathsf{b}t)^{-1}t + \tfrac{2}{3}|\mathsf{c}|\left(\tfrac{4}{5}\right)^{-\frac{1}{3}}(\mathsf{b}t)^{-\frac{1}{2}}t + \mathsf{m}t \leqslant \tfrac{1}{3}\left(\tfrac{4}{5}\right)^{-\frac{2}{3}} + Ct^{\frac{1}{2}} \leqslant \tfrac{2}{5}$$

upon choosing $\overline{\varepsilon}$ (and hence t) to be sufficiently small, in terms of κ, b, c, and m. The proof of (5.17c) is similar to the one of (5.17b), except that we appeal to assumption (5.1e) and derive

$$\left|w_0''(x)\right|\mathsf{b}^{\frac{3}{2}}t^{\frac{5}{2}} \leqslant \tfrac{2}{9}\left(\tfrac{4}{5}\right)^{-\frac{5}{3}} + Ct^{\frac{1}{2}} \leqslant \tfrac{1}{3} \tag{5.18}$$

once $\overline{\varepsilon}$ (and hence t) is small enough. The bound (5.17d) immediately follows from (5.1f) and (5.2). Lastly, the estimate (5.17e) is a direct consequence of (5.17b). □

Second, we discuss the invertibility of η_{B}:

Lemma 5.9 (Local inversion of the Burgers flow map) *Let w_0 be as described in (5.1), assume that $\mathfrak{s}$ satisfies (5.13) on $[0, T_0]$, and let η_{B} be defined as in (5.16). Then, there exists a sufficiently small $\overline{\varepsilon} \in (0, T_0]$, which only depends on κ, b, c, m, such that for $t \in (0, \overline{\varepsilon}]$ the following holds. There exists a* largest $x_{\mathsf{B},+} = x_{\mathsf{B},+}(t) > 0$ *and a* smallest $x_{\mathsf{B},-} = x_{\mathsf{B},-}(t) < 0$ *such that*

$$\mathfrak{s}(t) = \eta_{\mathsf{B}}(x_{\mathsf{B},\pm}(t), t) \tag{5.19}$$

and moreover we have

$$\left|x_{\mathsf{B},\pm}(t) \mp (\mathsf{b}t)^{\frac{3}{2}}\right| \leqslant \mathsf{m}^4 t^2 \quad \Rightarrow \quad \tfrac{4}{5}(\mathsf{b}t)^{\frac{3}{2}} < \left|x_{\mathsf{B},\pm}(t)\right| < \tfrac{6}{5}(\mathsf{b}t)^{\frac{3}{2}}\,. \tag{5.20}$$

We also define $x_{\mathsf{B},\pm}(0) = 0$. Note that $x_{\mathsf{B},\pm}(t) \in \Omega(t)$ for all $t \in [0, \overline{\varepsilon}]$. Moreover, defining the set of labels

$$\Upsilon_{\mathsf{B}}(t) = \mathbb{T} \setminus [x_{\mathsf{B},-}(t), x_{\mathsf{B},+}(t)]$$

we have that the map $\eta_{\mathsf{B}}(\cdot, t)\colon \Upsilon_{\mathsf{B}}(t) \to \mathbb{T} \setminus \{\mathfrak{s}(t)\}$ is a bijection satisfying the bounds

$$|\partial_x \eta_{\mathsf{B}}(x, s) - 1| \leqslant \tfrac{2}{5} \tag{5.21a}$$

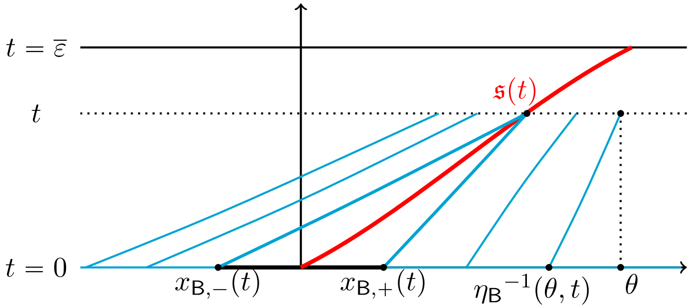

Fig. 10 Several Lagrangian paths $\{\eta_B(x,s)\}_{s\in[0,t]}$ are represented by cyan paths. The extremal points $x_{B,\pm}(t)$ are the two labels which are colliding into the shock curve precisely at time t. All the labels in between them have collided with the shock curve at some time $s \in [0,t)$

$$|\eta_B(x,s) - \mathfrak{s}(s)| \geqslant \tfrac{4}{5} b^{\frac{3}{2}} t^{\frac{1}{2}} (t-s)\,. \tag{5.21b}$$

for all $s \in [0,t)$ and $x \in \Upsilon_B(t)$. The above estimate implies that the trajectory $\{\eta(x,s)\}_{s\in[0,t]}$ can not intersect the shock curve prior to time $s = t$, for every $x \in \Upsilon_B(t)$. Lastly, the inverse map $\eta_B{}^{-1}(\cdot,t)\colon \mathbb{T}\setminus\{\mathfrak{s}(t)\} \to \Upsilon_B(t)$ satisfies the estimates (Fig. 10)

$$\tfrac{5}{7} \leqslant \partial_\theta \eta_B{}^{-1}(\theta,t) \leqslant \tfrac{5}{3} \tag{5.22a}$$

$$\tfrac{4}{5}(bt)^{\frac{3}{2}} + \tfrac{1}{2}|\theta - \mathfrak{s}(t)| \leqslant \left|\eta_B{}^{-1}(\theta,t)\right| \leqslant \tfrac{6}{5}(bt)^{\frac{3}{2}} + 2|\theta - \mathfrak{s}(t)| \tag{5.22b}$$

for all $(\theta,t) \in \mathcal{D}_{\overline{\varepsilon}}$.

Proof of Lemma 5.9 It is convenient to denote

$$g_0(x) = w_0(x) - \kappa + bx^{\frac{1}{3}} - cx^{\frac{2}{3}} \tag{5.23}$$

so that in view of (5.1) we have that $|g_0(x)| \leqslant m|x|$ and $|g_0'(x)| \leqslant m$. For $t > 0$ we let

$$\tau = (bt)^{\frac{1}{2}}, \qquad y = x^{\frac{1}{3}}\tau^{-1}, \qquad \zeta = (\mathfrak{s}(t) - \kappa t)\tau^{-3}\,. \tag{5.24}$$

Note that the condition $\mathfrak{s}(t) \in \Sigma(t)$ in (5.13) together with (5.2) imply that $|\zeta| \leqslant b^{-2}m^4\tau \ll 1$, an in particular $|\zeta| \leqslant \frac{1}{10}$. With this notation, for any $t > 0$ the equation (5.19) is equivalent to

$$\tau^3\zeta + \kappa t = \tau^3 y^3 + t\left(\kappa - b\tau y + c\tau^2 y^2 + g_0(\tau^3 y^3)\right)\,.$$

After collecting terms, and dividing by τ^3, we obtain that the above equality is equivalent to

$$0 = -\zeta + y^3 - y + \underbrace{\mathsf{cb}^{-1}\tau y^2 + \tau^{-1}\mathsf{b}^{-1}g_0(\tau^3 y^3)}_{=:G(y,\tau)} . \tag{5.25}$$

In view of the aforementioned properties of g_0, we have that for all $|y| \leqslant 10$ and all $0 < \tau \leqslant \overline{\varepsilon}$, with $\overline{\varepsilon}$ sufficiently small in terms of κ, b, c, m, we have that

$$\left|G(y,\tau) - \mathsf{cb}^{-1}\tau y^2\right| \leqslant C\tau^2 \tag{5.26a}$$

$$\left|\partial_\tau G(y,\tau) - \mathsf{cb}^{-1} y^2\right| \leqslant C\tau \tag{5.26b}$$

$$\left|\partial_y G(y,\tau) - 2\mathsf{cb}^{-1}\tau y\right| \leqslant C\tau^2 \tag{5.26c}$$

where $C > 0$ only depends on κ, b, c, and m.

Returning to (5.25), we next claim that for every fixed $\zeta \in [-\frac{1}{10}, \frac{1}{10}]$ and any τ sufficiently small, there exists a unique *most negative* root $y_- = y_-(\zeta,\tau)$ and a unique *most positive* root $y_+ = y_+(\zeta,\tau)$ of the implicit equation

$$y^3 - y + G(y,\tau) = \zeta . \tag{5.27}$$

The key observation is that in view of (5.26a), when when $\tau = 0$, the equation in the above display becomes $\zeta = y^3 - y$. For every $\zeta \in (-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}) \supset [-\frac{1}{10}, \frac{1}{10}]$ we introduce two functions $\mathtt{Z}_+(\zeta)$ and $\mathtt{Z}_-(\zeta)$ which are the largest (positive) root and respectively the smallest (negative) root of the equation

$$\zeta = \mathtt{Z}^3 - \mathtt{Z} . \tag{5.28}$$

The power series of these functions is given by

$$\mathtt{Z}_\pm(\zeta) = \pm 1 + \tfrac{1}{2}\zeta \mp \tfrac{3}{8}\zeta^2 + \tfrac{1}{2}\zeta^3 \mp \tfrac{105}{128}\zeta^4 + \tfrac{3}{2}\zeta^5 + \mathcal{O}(|\zeta|^6) \tag{5.29}$$

and is valid for $|\zeta| \ll 1$. In particular, we have

$$\mathtt{Z}_+(\zeta) + \mathtt{Z}_-(\zeta) = \zeta + \zeta^3 + 3\zeta^5 + \mathcal{O}(|\zeta|^7) . \tag{5.30}$$

For later purposes, it is also convenient to note here that

$$|\mathtt{Z}_+(\zeta) + \mathtt{Z}_-(\zeta) - \zeta| \leqslant \tfrac{6}{5}\zeta^3 \quad \text{and} \quad |\mathtt{Z}_+(\zeta) - \mathtt{Z}_-(\zeta) - 2| \leqslant \zeta^2 \tag{5.31}$$

for all $|\zeta| \leqslant \frac{1}{5}$. With this notation, we have thus obtained the desired roots of (5.27) when $\tau = 0$, namely

$$\mathtt{Z}_\pm^3(\zeta) - \mathtt{Z}_\pm(\zeta) + G(\mathtt{Z}_\pm(\zeta), 0) = \zeta .$$

The proof is then completed by an application of the implicit function theorem. This is possible since

$$\partial_y \left(y^3 - y + G(y,\tau) \right) |_{(y,\tau)=(\mathtt{Z}_\pm,0)} = 3\mathtt{Z}_\pm^2 - 1 \neq 0\,.$$

In fact, for every $|\zeta| \leqslant \frac{1}{10}$, one may verify that $\frac{3}{2} \leqslant 3\mathtt{Z}_\pm(\zeta)^2 - 1 \leqslant \frac{5}{2}$, since $\mathtt{Z}_\pm$ are explicit functions. The implicit function theorem guarantees the existence of an $\overline{\varepsilon} > 0$, such that if $\tau \in (0, (\mathsf{b}\overline{\varepsilon})^{\frac{1}{2}}]$ and $|\zeta| \leqslant \frac{1}{10}$, the equation (5.27) has a most negative root $y_-(\zeta,\tau)$ which is $\mathcal{O}(\tau)$-close to $\mathtt{Z}_-(\zeta)$, and a most positive root $y_+(\zeta,\tau)$, which is $\mathcal{O}(\tau)$-close to $\mathtt{Z}_+(\zeta)$. Upon unpacking the definitions in (5.24), we have thus identified

$$x_{\mathsf{B},\pm}^{\frac{1}{3}}(t) = (\mathsf{b}t)^{\frac{1}{2}} y_\pm \left(\frac{\mathfrak{s}(t) - \kappa t}{(\mathsf{b}t)^{\frac{3}{2}}}, (\mathsf{b}t)^{\frac{1}{2}} \right), \tag{5.32}$$

for all $t \in (0, \overline{\varepsilon}]$, which solves (5.19).

Note however that $|\zeta| \leqslant \mathsf{b}^{-2}\mathsf{m}^4\tau$, and that $\tau \leqslant (\mathsf{b}\overline{\varepsilon})^{\frac{1}{2}}$ is taken to be small. In this τ-dependent range for ζ we may obtain a sharper estimate than the $|y_\pm(\zeta,\tau) - \mathtt{Z}_\pm(\zeta)| \leqslant C\tau$ claimed above. Indeed, since the bounds (5.26b)–(5.26c) are available, from the Taylor theorem with remainder applied to (5.27), we may deduce that

$$\left| y_\pm(\zeta,\tau) - \mathtt{Z}_\pm(\zeta) + \tau \mathsf{c}\mathsf{b}^{-1} \frac{\mathtt{Z}_\pm(\zeta)}{3\mathtt{Z}_\pm^2(\zeta) - 1} \right| \leqslant C\tau^2$$

if $\overline{\varepsilon}$ is sufficiently small, for a constant $C = C(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$. Taking into account the power series expansion of $\mathtt{Z}_\pm$ in (5.29), and $\overline{\varepsilon}$ to be sufficiently small (hence τ sufficiently small), we deduce that

$$\left| y_\pm(\zeta,\tau) \mp 1 - \tfrac{1}{2}\zeta \pm \tfrac{\mathsf{c}}{2\mathsf{b}}\tau \right| \leqslant C\tau^2, \qquad \text{for all} \qquad |\zeta| \leqslant \mathsf{b}^{-2}\mathsf{m}^4 \quad \text{and} \quad \tau \leqslant (\mathsf{b}\overline{\varepsilon})^{\frac{1}{2}}\,. \tag{5.33}$$

In particular, keeping in mind (5.24) and (5.32), we deduce from (5.33) the estimate

$$\left| x_{\mathsf{B},\pm}(t)^{\frac{1}{3}} \mp (\mathsf{b}t)^{\frac{1}{2}} - \frac{\mathfrak{s}(t) - \kappa t}{2\mathsf{b}t} \pm \frac{\mathsf{c}t}{2} \right| \leqslant Ct^{\frac{3}{2}}\,, \tag{5.34}$$

for all $t \in (0, \overline{\varepsilon}]$, where $C = C(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$ is a computable constant. The bound (5.20) is an immediate consequence of (5.34), the working assumptions (5.2) and (5.13), upon taking $\overline{\varepsilon}$ to be sufficiently small.

The bound (5.21a) is a direct consequence of (5.17b), (5.20), and the fact that by (5.16) we have $\partial_x \eta_{\mathsf{B}}(x,s) - 1 = s w_0'(x)$. Therefore, the map $\eta_{\mathsf{B}}(\cdot,t)$ is a strictly increasing function on the label $x \in \mathbb{T}$, thus being injective from $\Upsilon_{\mathsf{B}}(t) \mapsto \mathbb{T} \setminus \{\mathfrak{s}(t)\}$. Surjectivity follows from the intermediate value theorem, and fact that by (5.19) we have $\lim_{x \to x_{\mathsf{B},-}(t)^-} \eta_{\mathsf{B}}(x,t) = \mathfrak{s}(t) = \lim_{x \to x_{\mathsf{B},+}(t)^+} \eta_{\mathsf{B}}(x,t)$. In order to show that for every $x \in \Upsilon_{\mathsf{B}}(t)$ the trajectory $\eta_{\mathsf{B}}(x,\cdot)$ does not meet the shock curve prior to time t, by the monotonicity property of η_{B} in the x variable, we only need to show that $\eta_{\mathsf{B}}(x_{\mathsf{B},-}(t), s) < \mathfrak{s}(s)$ and that $\eta_{\mathsf{B}}(x_{\mathsf{B},+}(t), s) > \mathfrak{s}(s)$. These two statements are established in the same way, so we only give the proof for the label $x_{\mathsf{B},-}(t)$. By appealing to (5.19), the $\dot{\mathfrak{s}}$ assumption in (5.13), the w_0 assumption in (5.1), and the previously established estimate (5.34), we have that

$$
\begin{aligned}
\mathfrak{s}(s) - \eta_{\mathsf{B}}(x_{\mathsf{B},-}(t), s) &= -\int_s^t \big(\dot{\mathfrak{s}}(\tau) - (\partial_t \eta_{\mathsf{B}})(x_{\mathsf{B},-}(t), \tau)\big)\, d\tau \\
&= \int_s^t \big(w_0(x_{\mathsf{B},-}(t)) - \kappa\big)\, d\tau - \int_s^t (\dot{\mathfrak{s}}(\tau) - \kappa)\, d\tau \\
&\geqslant \left(-\mathsf{b} x_{\mathsf{B},-}^{\frac{1}{3}}(t) - 2|\mathsf{c}|\mathsf{b}t\right)(t-s) - \tfrac{1}{2}\mathsf{m}^4 (t^2 - s^2) \\
&\geqslant \tfrac{4}{5}\mathsf{b}^{\frac{3}{2}} t^{\frac{1}{2}} (t-s)
\end{aligned}
$$

for any $s \in [0, t)$, with $t \leqslant \bar{\varepsilon}$ which is sufficiently small.

The proof is concluded once we establish (5.22). The bound (5.22a) is an immediate consequence of (5.21a) and the inverse function theorem. For the proof of (5.22b), let us first consider a point θ which is to the left of $\mathfrak{s}(t)$. Then, by the mean value theorem and (5.19), we have that

$$
\eta_{\mathsf{B}}{}^{-1}(\theta, t) - x_{\mathsf{B},-}(t) = \eta_{\mathsf{B}}{}^{-1}(\theta, t) - \eta_{\mathsf{B}}{}^{-1}(\mathfrak{s}(t), t)) = (\theta - \mathfrak{s}(t))(\partial_\theta \eta_{\mathsf{B}}{}^{-1})(\overline{\theta}, t)
$$

for some $\overline{\theta} \in (y, \mathfrak{s}(t))$. The above identity, combined with (5.22a) and the first inequality in (5.20) implies (5.22b), upon taking $\bar{\varepsilon}$ sufficiently small. The proof in the case that y is to the right of $\mathfrak{s}(t)$ is identical. □

Next, we discuss the solution w_{B} to (5.14) and its properties.

Proof of Proposition 5.7 By Lemma 5.9, for all $(\theta, t) \in \mathcal{D}_{\bar{\varepsilon}}$ we may define

$$
w_{\mathsf{B}}(\theta, t) = w_0(\eta_{\mathsf{B}}{}^{-1}(\theta, t)) \,. \tag{5.35}
$$

By the of construction η_{B} and the properties of w_0, the above defined w_{B} is C^2 smooth in $\mathcal{D}_{\bar{\varepsilon}}$ and solves (5.14) in this region. Indeed, differentiating the relation $w_{\mathsf{B}}(\eta_{\mathsf{B}}(x, t), t) = w_0(x)$ and using the definition of η_{B} we have the identities

$$
\partial_\theta w_{\mathsf{B}}(\theta, t) = \frac{w_0'(\eta_{\mathsf{B}}{}^{-1}(\theta, t))}{1 + t w_0'(\eta_{\mathsf{B}}{}^{-1}(\theta, t))} \tag{5.36a}
$$

$$
\partial_\theta^2 w_{\mathsf{B}}(\theta, t) = \frac{w_0''(\eta_{\mathsf{B}}{}^{-1}(\theta, t))}{(1 + t w_0'(\eta_{\mathsf{B}}{}^{-1}(\theta, t)))^3} \tag{5.36b}
$$

for all $\theta \in \mathbb{T} \setminus \{\mathfrak{s}(t)\}$. In particular, combining (5.36a) with (5.22b) and (5.1), gives that

$$
|\partial_\theta w_{\mathsf{B}}(\theta, t)| \leqslant \tfrac{4}{5}\mathsf{b}((\mathsf{b}t)^3 + |\theta - \mathfrak{s}(t)|^2)^{-\frac{1}{3}} \tag{5.37a}
$$

$$\left|\partial_\theta^2 w_{\mathsf{B}}(y,t)\right| \leqslant 2\mathsf{b}((\mathsf{b}t)^3 + |\theta - \mathfrak{s}(t)|^2)^{-\frac{5}{6}} \tag{5.37b}$$

for all $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}$ such that $|\theta - \mathfrak{s}(t)| \leqslant \overline{\varepsilon}^{\frac{1}{2}}$, as soon as $\overline{\varepsilon}$ is sufficiently small.

Next, we we discuss the mean and the jump of w_{B} at the shock curve. We have that

$$\begin{aligned}
[\![w_{\mathsf{B}}(t)]\!] &= w_0(x_{\mathsf{B},-}(t)) - w_0(x_{\mathsf{B},+}(t)) \\
&= \left(x_{\mathsf{B},+}^{\frac{1}{3}}(t) - x_{\mathsf{B},-}^{\frac{1}{3}}(t)\right)\left(\mathsf{b} - \mathsf{c}x_{\mathsf{B},+}^{\frac{1}{3}}(t) - \mathsf{c}x_{\mathsf{B},-}^{\frac{1}{3}}(t)\right) \\
&\quad + g_0(x_{\mathsf{B},-}(t)) - g_0(x_{\mathsf{B},+}(t))
\end{aligned}$$

where we recall the notation from (5.23). Using (5.34), (5.1c), and (5.2), we deduce that

$$\left|[\![w_{\mathsf{B}}(t)]\!] - 2\mathsf{b}^{\frac{3}{2}}t^{\frac{1}{2}}\right| \leqslant 8\mathsf{b}|\mathsf{c}|t \leqslant t$$

upon choosing $\overline{\varepsilon}$ to be sufficiently small with respect to κ, b, c, and m. This proves the first bound in (5.15a). Similarly,

$$\begin{aligned}
\langle\!\langle w_{\mathsf{B}}(t)\rangle\!\rangle &= \tfrac{1}{2}\left(w_0(x_{\mathsf{B},-}(t)) + w_0(x_{\mathsf{B},+}(t))\right) \\
&= \kappa - \tfrac{1}{2}\mathsf{b}\left(x_{\mathsf{B},-}^{\frac{1}{3}}(t) + x_{\mathsf{B},+}^{\frac{1}{3}}(t)\right) + \tfrac{1}{2}\mathsf{c}\left(x_{\mathsf{B},-}^{\frac{2}{3}}(t) + x_{\mathsf{B},+}^{\frac{2}{3}}(t)\right) \\
&\quad + \tfrac{1}{2}\left(g_0(x_{\mathsf{B},-}(t)) - g_0(x_{\mathsf{B},+}(t))\right) .
\end{aligned}$$

From (5.34), (5.20), and (5.1c) we deduce that

$$\left|\langle\!\langle w_{\mathsf{B}}(t)\rangle\!\rangle - \kappa + \frac{\mathfrak{s}(t) - \kappa t}{2t}\right| \leqslant Ct^{\frac{3}{2}} .$$

The second inequality in (5.15a) now follows from (5.13).

Appealing to the definitions (5.19), (5.16), and (5.35), we arrive at

$$\begin{aligned}
\tfrac{d}{dt}\left(w_{\mathsf{B}}(\mathfrak{s}(t)^{\pm}, t)\right) &= \tfrac{d}{dt}\left(w_0(x_{\mathsf{B},\pm}(t))\right) = w_0'(x_{\mathsf{B},\pm}(t))\tfrac{d}{dt}x_{\mathsf{B},\pm}(t) \\
&= w_0'(x_{\mathsf{B},\pm}(t))\frac{\dot{\mathfrak{s}}(t) - w_0(x_{\mathsf{B},\pm}(t))}{1 + tw_0'(x_{\mathsf{B},\pm}(t))} .
\end{aligned}$$

Therefore, using (5.1c), (5.17e), the asymptotic description (5.34) for $x_{\mathsf{B},\pm}(t)$, and the assumption on $\dot{\mathfrak{s}}$ from (5.13), after a tedious computation we obtain

$$\begin{aligned}
\left|\tfrac{d}{dt}\left(w_{\mathsf{B}}(\mathfrak{s}(t)^{\pm}, t)\right) \pm \tfrac{1}{2}\mathsf{b}^{\frac{3}{2}}t^{-\frac{1}{2}}\right| &\leqslant \left|\mathsf{b}x_{\mathsf{B},\pm}(t)^{\frac{1}{3}}\frac{w_0'(x_{\mathsf{B},\pm}(t))}{1 + tw_0'(x_{\mathsf{B},\pm}(t))} \mp \tfrac{1}{2}\mathsf{b}^{\frac{3}{2}}t^{-\frac{1}{2}}\right| \\
&\quad + \frac{t|w_0'(x_{\mathsf{B},\pm}(t))|}{1 + tw_0'(x_{\mathsf{B},\pm}(t))}
\end{aligned}$$

$$\left(\mathsf{m}^4 + \frac{\kappa - w_0(x_{\mathsf{B},\pm}(t)) + \mathsf{b}x_{\mathsf{B},\pm}(t)^{\frac{1}{3}}}{t}\right)$$
$$\leqslant \tfrac{5}{6}\mathsf{m}^4 + \mathsf{b}|\mathsf{c}| + Ct^{\frac{1}{2}} \leqslant \mathsf{m}^4\,.$$

From the above estimate, it is clear that (5.15b) follows. Differentiating once more, we obtain

$$\frac{d^2}{dt^2}\left(w_{\mathsf{B}}(\mathfrak{s}(t)^{\pm}, t)\right) = \ddot{\mathfrak{s}}(t)\frac{w_0'(x_{\mathsf{B},\pm}(t))}{1 + tw_0'(x_{\mathsf{B},\pm}(t))} + w_0''(x_{\mathsf{B},\pm}(t))\frac{(\dot{\mathfrak{s}}(t) - w_0(x_{\mathsf{B},\pm}(t)))^2}{(1 + tw_0'(x_{\mathsf{B},\pm}(t)))^3}$$
$$- \frac{2(w_0'(x_{\mathsf{B},\pm}(t)))^2(\dot{\mathfrak{s}}(t) - w_0(x_{\mathsf{B},\pm}(t)))}{(1 + tw_0'(x_{\mathsf{B},\pm}(t)))^2}$$

and therefore, after an even more tedious computation, we arrive at

$$\left|\frac{d^2}{dt^2}\left(w_{\mathsf{B}}(\mathfrak{s}(t)^{\pm}, t)\right) \mp \tfrac{1}{4}\mathsf{b}^{\frac{3}{2}}t^{-\frac{3}{2}}\right| \leqslant \left(\tfrac{1}{3}\mathsf{m}^4 + 3\mathsf{b}|\mathsf{c}|\right)t^{-1} + Ct^{-\frac{1}{2}} \leqslant \mathsf{m}^4 t^{-1}\,.$$

The claim (5.15c) now follows, thereby completing the proof of the proposition. □

5.5.1 Lagrangian Trajectories for Velocity Fields that are Close to w_{B}

For future purposes, see Section 5.7, at this stage it is convenient to consider velocities $\lambda_3 \colon \mathcal{D}_{\overline{\varepsilon}} \to \mathbb{R}$ which are close to the w_{B} we have constructed in Proposition 5.7, in the sense that $\lambda_3 \in C^1_{\theta,t}(\mathcal{D}_{\overline{\varepsilon}})$, and we have the pointwise bounds

$$|\lambda_3(\theta, t) - w_{\mathsf{B}}(\theta, t)| \leqslant \mathsf{R}_1 t + Ct^{\frac{3}{2}} \tag{5.38a}$$
$$|\partial_\theta\lambda_3(\theta, t) - \partial_\theta w_{\mathsf{B}}(\theta, t)| \leqslant \mathsf{R}_2((\mathsf{b}t)^3 + (\theta - \mathfrak{s}(t))^2)^{-\frac{1}{6}} + Ct^{\frac{1}{2}} \tag{5.38b}$$

for all $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}$, for positive constants $\mathsf{R}_1, \mathsf{R}_2, C$ which only depend on $\kappa, \mathsf{b}, \mathsf{c}$, and m; see (5.142) for the values of $\mathsf{R}_1, \mathsf{R}_2$ which are used in the proof, namely $\mathsf{R}_1 = \mathsf{R}_2 = \mathsf{m}^3$.

Note that in view of (5.35) and (5.37a), assumptions (5.38) imply that λ_3 is C^1 smooth on the complement of the shock curve. In particular, this means that for every label $x \in \mathbb{T} \setminus \{0\}$, we are guaranteed the short time (x-dependent time) unique solvability of the ODE

$$\partial_t\eta(x, t) = \lambda_3(\eta(x, t), t)\,, \qquad \eta(x, 0) = x\,. \tag{5.39}$$

In view of the assumed regularity of λ_3, for a given label x the path $\eta(x, t)$ can be continued on a maximal time interval $[0, T_x)$, where the stopping time T_x is defined as

$$T_x := \min\left\{\overline{\varepsilon}, \sup\{t \in [0, \overline{\varepsilon}] \colon |\eta(x, t) - \mathfrak{s}(t)| > 0\}\right\}. \tag{5.40}$$

That is, if the trajectory $\eta(x,\cdot)$ intersects the shock curve prior to time $\overline{\varepsilon}$, then we record this stopping time in T_x, and in this case we have $\eta(x, T_x) = \mathfrak{s}(T_x)$. Note that since $\mathfrak{s} \in C^1$, and since λ_3 is C^1 smooth on the complement of the shock curve, the stopping time T_x is continuous in x.

Next, for every $t \in (0, \overline{\varepsilon}]$, in analogy to (5.19), we wish to define in a unique way two *extremal* labels $x_\pm(t)$ with the property that

$$\mathfrak{s}(t) = \eta(x_\pm(t), t)\,. \tag{5.41}$$

By (5.40) we have that the above definition is equivalent to $T_{x_\pm(t)} = t$, which then motivates

$$\begin{aligned} &x_-(t) = \inf\{x \in [-\pi, 0) \colon T_x \leqslant t\}\,, \quad x_+(t) = \sup\{x \in (0, \pi] \colon T_x \leqslant t\}\,, \\ &\Upsilon(t) = \mathbb{T} \setminus [x_-(t), x_+(t)]\,. \end{aligned} \tag{5.42}$$

By the continuity of T_x in x, the above inf / sup are in fact min / max. Moreover, for every $x \in \Upsilon(t)$, we know that $T_x \geqslant t$. One of our goals will be to show that $\eta(\cdot, t) \colon \Upsilon(t) \to \mathbb{T} \setminus \{\mathfrak{s}(t)\}$ is a bijection, for every $t \in (0, \overline{\varepsilon}]$.

As mentioned above, $T_x \in (0, \overline{\varepsilon}]$ if $x \neq 0$. Now for fixed x and $t \in [0, T_x)$, by Lemma 5.9 we may define

$$q(t) = \eta_{\mathsf{B}}{}^{-1}(\eta(x,t), t)\,, \tag{5.43}$$

and note that $q(t) \in \Upsilon_{\mathsf{b}}(t)$ and that $q(0) = x$. Since $\eta_{\mathsf{B}}{}^{-1}$ solves the transport equation with speed w_{B}, and η solves (5.39), we have that

$$\tfrac{d}{dt} q = (\partial_t \eta_{\mathsf{B}}{}^{-1}) \circ \eta + (\partial_\theta \eta_{\mathsf{B}}{}^{-1}) \circ \eta \partial_t \eta = (\lambda_3 - w_{\mathsf{B}}) \circ \eta (\partial_\theta \eta_{\mathsf{B}}{}^{-1}) \circ \eta\,.$$

Thus, by also appealing to (5.38a) and (5.22a), we have that

$$\left| \eta_{\mathsf{B}}{}^{-1}(\eta(x,t), t) - x \right| = |q(t) - q(0)| \leqslant \mathsf{R}_1 t^2 \tag{5.44}$$

whenever $t < T_x$, upon taking $\overline{\varepsilon}$ to be sufficiently small. By (5.42), we note that (5.44) in particular holds for all $t \in (0, \overline{\varepsilon}]$, and all $x \in \Upsilon(t)$. Note that from (5.19), (5.41), (5.44), and continuity, we have that

$$\left| x_\pm(t) - x_{\mathsf{B},\pm}(t) \right| = \left| x_\pm(t) - \eta_{\mathsf{B}}{}^{-1}(\eta(x_\pm(t), t)) \right| \leqslant \mathsf{R}_1 t^2$$

for all $t \in (0, \overline{\varepsilon}]$, and thus similarly to (5.20) we have that

$$\left| x_\pm(t) \mp (\mathsf{b}t)^{\frac{3}{2}} \right| \leqslant t^2(\mathsf{m}^4 + \mathsf{R}_1) \qquad \Rightarrow \qquad \tfrac{4}{5}(\mathsf{b}t)^{\frac{3}{2}} < |x_\pm(t)| < \tfrac{6}{5}(\mathsf{b}t)^{\frac{3}{2}}\,. \tag{5.45}$$

upon taking $\overline{\varepsilon}$ to be sufficiently small.

If $T_x < \overline{\varepsilon}$, and $t \in [0, T_x)$, the bound (5.44) and the identity (5.36a) allow us to estimate

$$\begin{aligned}\int_0^t |\partial_\theta w_{\mathsf{B}}(\eta(x,s),s)|\,ds &= \int_0^t \frac{\left|w_0'(q(s))\right|}{1+sw_0'(q(s))}ds \\ &\leqslant \int_0^t \frac{\left|w_0'(x)\right|}{1+sw_0'(x)}ds + \mathsf{R}_1\int_0^t s^2 \sup_{|\overline{x}-x|\leqslant \mathsf{R}_1 s^2} \frac{\left|w_0''(\overline{x})\right|}{(1+sw_0'(\overline{x}))^2}ds \\ &\leqslant \frac{|w_0'(x)|}{w_0'(x)}\log\left(1+tw_0'(x)\right) + \tfrac{8}{5}\mathsf{b}^{-\frac{3}{2}}\mathsf{R}_1\int_0^t s^{-\frac{1}{2}}ds\,.\end{aligned}$$

At this stage we recall that the values of x that we are interested in satisfy $|x| \geqslant (\mathsf{b}t)^{\frac{3}{2}} - t^2(\mathsf{m}^4+\mathsf{R}_1) \geqslant \frac{9}{10}(\mathsf{b}t)^{\frac{3}{2}}$. We distinguish two cases: $\frac{9}{10}(\mathsf{b}t)^{\frac{3}{2}} \leqslant |x| \leqslant \mathsf{b}^{\frac{3}{2}}t$, and $\mathsf{b}^{\frac{3}{2}}t \leqslant |x| \leqslant \pi$. Using assumption (5.1d), in the first case we deduce that

$$0 > tw_0'(x) > -\tfrac{1}{3}\mathsf{b}tx^{-\frac{2}{3}}(1+3|\mathsf{c}|t^{\frac{1}{3}}) > -\tfrac{1}{3}(\tfrac{9}{10})^{-\frac{2}{3}}(1+3|\mathsf{c}|t^{\frac{1}{3}}) > -\tfrac{7}{19}\,.$$

In the other case, we we use that $t \leqslant \overline{\varepsilon} \ll 1$, and thus

$$t|w_0'(x)| \leqslant \tfrac{1}{3}\mathsf{b}t^{\frac{1}{3}} + \tfrac{2}{3}|\mathsf{c}|\mathsf{b}^{-\frac{1}{2}}t^{\frac{2}{3}} + \mathsf{m}t \leqslant \overline{\varepsilon}^{\frac{1}{3}}\,.$$

From the above three inequalities, and the fact that $\operatorname{sgn}(r)\log(1+r) \leqslant \log(\frac{19}{12})$ for all $r \in (-\frac{7}{19}, \overline{\varepsilon}^{\frac{1}{3}})$, we deduce that

$$\int_0^t |\partial_\theta w_{\mathsf{B}}(\eta(x,s),s)|\,ds \leqslant \log(\tfrac{19}{12}) + \tfrac{16}{5}\mathsf{b}^{-\frac{3}{2}}\mathsf{R}_1 t^{\frac{1}{2}} \leqslant \tfrac{19}{40}\,, \tag{5.46}$$

since $t \leqslant \overline{\varepsilon} \ll 1$. As before, we note in particular that (5.46) holds for all $t \in (0, \overline{\varepsilon}]$, and all $x \in \Upsilon(t)$. We note that using (5.36b), (5.44), and (5.17c), in addition to (5.46) we have

$$\begin{aligned}\int_0^t \left|\partial_\theta^2 w_{\mathsf{B}}(\eta(x,s),s)\right| ds &\leqslant \int_0^t \frac{|w_0''(\eta_{\mathsf{B}}^{-1}(\eta(x,s),s))|}{(1+tw_0'(\eta_{\mathsf{B}}^{-1}(\eta(x,s),s)))^3}ds \\ &\leqslant 3(\mathsf{b}t)^{-\frac{3}{2}}\end{aligned} \tag{5.47}$$

whenever $x \in \Upsilon(t)$. Here we have used that $|\eta_{\mathsf{B}}{}^{-1}(\eta(x,s),s)| \geqslant |x| - \mathsf{R}_1 s^2 \geqslant \frac{4}{5}(\mathsf{b}t)^{\frac{3}{2}}$ for $s \leqslant t \leqslant \overline{\varepsilon}$.

With (5.46) in hand, and appealing also to (5.38b), for every $x \in \Upsilon(t)$ we may now have

$$\partial_x\eta(x,t) = \exp\left(\int_0^t (\partial_\theta w_{\mathsf{B}})(\eta(x,s),s)ds\right)\exp\left(\int_0^t (\partial_\theta\lambda_3 - \partial_\theta w_{\mathsf{B}})(\eta(x,s),s)ds\right), \tag{5.48}$$

and thus

$$\tfrac{1}{2} \leqslant \exp(-\tfrac{1}{2} - 4\mathsf{R}_2\mathsf{b}^{-\frac{1}{2}}t^{\frac{1}{2}}) \leqslant \partial_x\eta(x,t) \leqslant \exp(\tfrac{1}{2} + 4\mathsf{R}_2\mathsf{b}^{-\frac{1}{2}}t^{\frac{1}{2}}) \leqslant \tfrac{7}{4} \tag{5.49}$$

since $\overline{\varepsilon}$ is sufficiently small with respect to κ, b, c, and m. This shows that the map $\eta(\cdot,t)$ is strictly monotone (thus injective) on either side of the shock curve; combined with (5.41) and the intermediate function theorem (ensuring surjectivity), we obtain that $\eta(\cdot,t)\colon \Upsilon(t) \to \mathbb{T}\setminus\{\mathfrak{s}(t)\}$ is a bijection, as claimed earlier. Moreover, (5.49) shows that for every $x \in \Upsilon(t)$, the curve $\eta(x,s)$ does not intersect the shock curve prior to time t; in fact, by the monotonicity of η we have that $|\mathfrak{s}(s) - \eta(x,s)| \geqslant |\mathfrak{s}(s) - \eta(x_\pm(t),s)|$, and analogously to (5.21b), using (5.45) we have that

$$\begin{aligned}
\mathfrak{s}(s) - \eta(x_-(t),s) &= -\int_s^t \dot{\mathfrak{s}}(\tau) - \lambda_3(\eta(x_-(t),\tau),\tau)d\tau \\
&= \int_s^t \Big(w_0({\eta_{\mathsf{B}}}^{-1}(\eta(x_-(t),\tau),\tau)) - \kappa\Big)d\tau \\
&\quad + \int_s^t (\kappa - \dot{\mathfrak{s}}(\tau)) + (\lambda_3 - w_{\mathsf{B}})(\eta(x_-(t),\tau),\tau)d\tau \\
&\geqslant (w_0(x_-(t)) - \kappa)(t-s) - \tfrac{1}{2}(\mathsf{m}^3 + 4\mathsf{R}_1)(t^2 - s^2) \\
&\geqslant \tfrac{4}{5}\mathsf{b}^{\frac{3}{2}}t^{\frac{1}{2}}(t-s)
\end{aligned} \tag{5.50}$$

for all $s \in [0,t)$, and all $x \in \Upsilon(t)$. This bound shows that $\Upsilon(s) \supset \Upsilon(t)$ for $s < t$.

Recalling the $\eta_{\mathsf{B}}(x,t)$ is defined by (5.16) for all $x \in \mathbb{T}$, and in particular for $x \in \Upsilon(t)$, from (5.45) and (5.49) we immediately deduce that

$$\big|\eta(x,t) - \eta_{\mathsf{B}}(x,t)\big| \leqslant \tfrac{3}{2}\mathsf{R}_1 t^2, \qquad \text{for all} \quad x \in \Upsilon(t)\,, \tag{5.51a}$$

$$\big|\partial_x\eta(x,t) - \partial_x\eta_{\mathsf{B}}(x,t)\big| \leqslant \Big(16\mathsf{R}_1\mathsf{b}^{-\frac{3}{2}} + 8\mathsf{R}_2\mathsf{b}^{-\frac{1}{2}}\Big)t^{\frac{1}{2}}, \qquad \text{for all} \quad x \in \Upsilon(t)\,, \tag{5.51b}$$

for all $t \in (0,\overline{\varepsilon}]$. The bound (5.51a) follows from (5.44), the mean value theorem, and the fact that by (5.17b) we have that $|\partial_x\eta_{\mathsf{B}}(x,t) - 1| \leqslant \frac{9}{20}$ for all $x \in \Upsilon(t)$ (in analogy to (5.21a)). In order to prove the bound (5.51b), we use

$$\begin{aligned}
\partial_t(\partial_x\eta - \partial_x\eta_{\mathsf{B}}) &= (\partial_\theta w_{\mathsf{B}})\circ\eta\;(\partial_x\eta - \partial_x\eta_{\mathsf{B}}) + (\partial_\theta\lambda_3 - \partial_\theta w_{\mathsf{B}})\circ\eta\;\partial_x\eta \\
&\quad + ((\partial_\theta w_{\mathsf{B}})\circ\eta - (\partial_\theta w_{\mathsf{B}})\circ\eta_{\mathsf{B}})\,\partial_x\eta_{\mathsf{B}}\,,
\end{aligned}$$

and the fact that $\partial_x\eta(x,0) - \partial_x\eta_{\mathsf{B}}(x,0) = 0$. First, we note that due to (5.17c), (5.21a), (5.36a), (5.44), and the mean value theorem, we have that

$$\big|((\partial_\theta w_{\mathsf{B}})\circ\eta - (\partial_\theta w_{\mathsf{B}})\circ\eta_{\mathsf{B}})\,\partial_x\eta_{\mathsf{B}}\big| \leqslant 2\left|\frac{w_0'({\eta_{\mathsf{B}}}^{-1}(\eta(x,t),t))}{1 + tw_0'({\eta_{\mathsf{B}}}^{-1}(\eta(x,t),t))} - \frac{w_0'(x)}{1 + tw_0'(x)}\right|$$

$$\leqslant 4\mathsf{R}_1\mathsf{b}^{-\frac{3}{2}}t^{-\frac{1}{2}}\,. \tag{5.52}$$

Second, by the assumption (5.38b) and the bound (5.49) we know that

$$\left|(\partial_\theta\lambda_3 - \partial_\theta w_{\mathsf{B}})\circ\eta\;\partial_x\eta\right| \leqslant 2\mathsf{R}_2(\mathsf{b}t)^{-\frac{1}{2}}\,. \tag{5.53}$$

Combining the above two estimates with the evolution equation for $\partial_x\eta - \partial_x\eta_{\mathsf{B}}$ and (5.46), we obtain (5.51b).

The results in this section may be summarized as follows:

Lemma 5.10 *Let η be defined by (5.39), with λ_3 satisfying (5.38). Then, by possibly further reducing the value of $\overline{\varepsilon}$, solely in terms of κ, b, c, m, the following hold. With the definition of $\Upsilon(t)$ in (5.42), we have that $\eta(\cdot,t)\colon\Upsilon(t)\to\mathbb{T}\setminus\{\mathfrak{s}(t)\}$ is a bijection. For $x\in\Upsilon(t)$, the curve $\{\eta(x,s)\}_{s\in[0,t]}$ does not intersect the shock curve, and by (5.49), (5.51a), (5.51b), we have the estimates*

$$\tfrac{1}{2} \leqslant \partial_x\eta(x,t) \leqslant \tfrac{7}{4} \tag{5.54a}$$

$$\tfrac{1}{3}\kappa \leqslant \partial_t\eta(x,t) \leqslant \tfrac{3}{2}\mathsf{m} \tag{5.54b}$$

$$\left|\eta(x,t)-\eta_{\mathsf{B}}(x,t)\right| \leqslant \tfrac{3}{2}\mathsf{R}_1 t^2 \tag{5.54c}$$

$$\left|\partial_x\eta(x,t)-\partial_x\eta_{\mathsf{B}}(x,t)\right| \leqslant (16\mathsf{R}_1\mathsf{b}^{-\frac{3}{2}} + 8\mathsf{R}_2\mathsf{b}^{-\frac{1}{2}})t^{\frac{1}{2}} \tag{5.54d}$$

The inverse map $\eta^{-1}\colon\mathcal{D}_{\overline{\varepsilon}}\to\mathbb{T}\setminus\{0\}$ is continuous in space-and-time, with bounds

$$\tfrac{4}{7} \leqslant \partial_\theta\eta^{-1}(\theta,t) \leqslant 2 \tag{5.55a}$$

$$-3\mathsf{m} \leqslant \partial_t\eta^{-1}(\theta,t) \leqslant -\tfrac{1}{4}\kappa \tag{5.55b}$$

for all $(\theta,t)\in\mathcal{D}_{\overline{\varepsilon}}$. Lastly, from (5.46) and (5.47) we have that

$$\int_0^t |\partial_\theta w_{\mathsf{B}}(\eta(x,s),s)|\,ds \leqslant \tfrac{19}{40} \tag{5.56a}$$

$$\int_0^t \left|\partial_\theta^2 w_{\mathsf{B}}(\eta(x,s),s)\right| ds \leqslant 3(\mathsf{b}t)^{-\frac{3}{2}} \tag{5.56b}$$

for all $x\in\Upsilon(t)$, and all $t\in[0,\overline{\varepsilon}]$.

Proof of Lemma 5.10 The only estimates which were not established in the discussion above the lemma are (5.54b) and (5.55). In order to prove (5.54b), we appeal to (5.39), (5.38a), (5.35), (5.1a), (5.1a), and take $\overline{\varepsilon}$ to be sufficiently small:

$$\begin{aligned}\partial_t\eta(x,t) &= \lambda_3(\eta(x,t)) = w_{\mathsf{B}}(\eta(x,t)) + \mathcal{O}(t)\\ &= \underbrace{w_0({\eta_{\mathsf{B}}}^{-1}(\eta(x,t),t))}_{\in[\frac{\kappa}{2},\mathsf{m}]} + \mathcal{O}(t) \in \left[\tfrac{\kappa}{3},\tfrac{3\mathsf{m}}{2}\right].\end{aligned}$$

The bound (5.55a) follows from (5.54a) and the inverse function theorem. Lastly, in order to prove (5.55b), we use that η^{-1} solves the transport equation dual to the ODE (5.39), namely $\partial_t \eta^{-1} + \lambda_3 \partial_y \eta^{-1} = 0$. As such, from (5.55a), (5.38a), (5.35), (5.1a), (5.1a), we obtain that

$$\begin{aligned}\partial_t \eta(\theta, t) &= -w_{\mathsf{B}}(\theta, t)\partial_\theta \eta^{-1} + \mathcal{O}(t)\\ &= -\underbrace{w_0({\eta_{\mathsf{B}}}^{-1}(\theta, t))}_{\in[\frac{\kappa}{2},\mathsf{m}]}\underbrace{\partial_\theta \eta^{-1}}_{\in[\frac{4}{7},2]} + \mathcal{O}(t) \in \left[-3\mathsf{m}, -\tfrac{\kappa}{4}\right]\end{aligned}$$

upon taking $\overline{\varepsilon}$ to be sufficiently small. □

5.5.2 Estimates for Derivatives of w_{B} Along Flows Transversal to the Shock

In analogy to Lemma 5.10, we also have an estimate for the time integral of $\partial_\theta w_{\mathsf{B}}$ along any flow which is transversal to $\mathfrak{s}$. More precisely, we have:

Lemma 5.11 *Fix $t \in (0, \overline{\varepsilon}]$ and $0 \leqslant \theta < \mathfrak{s}(t)$. For some $t' \in [0, t)$, assume that we are given a differentiable curve $\gamma : [t', t] \to \mathcal{D}_{\overline{\varepsilon}}$ which does not intersect the shock curve $\mathfrak{s}$, such that $\gamma(t) = \theta$, and such that $\dot{\gamma}(s) \leqslant \mu\kappa$ for all $s \in [t', t]$, for some $\mu \in [0, 1)$. Then, we have that*

$$\int_{t'}^{t} |\partial_\theta w_{\mathsf{B}}(\gamma(s), s)|\, ds \leqslant \frac{13\mathsf{b}}{(1-\mu)\kappa^{\frac{2}{3}}} t^{\frac{1}{3}}, \tag{5.57a}$$

$$\int_{t'}^{t} \left|\partial_\theta^2 w_{\mathsf{B}}(\gamma(s), s)\right| ds \leqslant \frac{9\mathsf{b}}{(1-\mu)\kappa} \left(\tfrac{1}{2}|\gamma(t') - \mathfrak{s}(t')| + \tfrac{4}{5}(\mathsf{b}t')^{\frac{3}{2}}\right)^{-\frac{2}{3}} \tag{5.57b}$$

$$\leqslant \frac{11}{(1-\mu)\kappa} t'^{-1}. \tag{5.57c}$$

Proof of Lemma 5.11 As in the proof of Lemma 5.10, the goal is to understand the evolution of $x(s) := {\eta_{\mathsf{B}}}^{-1}(\gamma(s), s)$. First, we note that since γ lies on the left side of $\mathfrak{s}$, the point $x(s)$ is well-defined, and satisfies $x(s) \leqslant -\frac{4}{5}(\mathsf{b}s)^{\frac{3}{2}}$. Next, from the definition of η_{B} and its inverse, we have that

$$\begin{aligned}\dot{x}(s) &= (\partial_t {\eta_{\mathsf{B}}}^{-1})(\gamma(s), s) + \dot{\gamma}(s)(\partial_\theta {\eta_{\mathsf{B}}}^{-1})(\gamma(s), s)\\ &= \frac{\dot{\gamma}(s) - (\partial_t \eta_{\mathsf{B}})({\eta_{\mathsf{B}}}^{-1}(\gamma(s), s), s)}{(\partial_x \eta_{\mathsf{B}})({\eta_{\mathsf{B}}}^{-1}(\gamma(s), s), s)}\\ &= \frac{\dot{\gamma}(s) - w_0(x(s))}{1 + s w_0'(x(s))}.\end{aligned} \tag{5.58}$$

Due to the aforementioned lower bound on $|x(s)|$ and the estimate (5.17b), the denominator of the fraction on the right side of (5.58) lies in the interval $[\frac{1}{2}, \frac{3}{2}]$. Furthermore, since $t \leqslant \overline{\varepsilon}$ and $\overline{\varepsilon}$ is sufficiently small, we have that $|x(t)| = |{\eta_{\mathsf{B}}}^{-1}(y, t)|$ is sufficiently small to ensure via (5.1c) that $|w_0(x(t)) - \kappa| \leqslant 2\mathsf{b}|x(t)|^{\frac{1}{3}} \leqslant \frac{1-\mu}{4}\kappa$. Also, from (5.58) we may deduce that $|\dot{x}(s)| \leqslant 4\mathsf{m}$ which implies $|x(s)| \leqslant |x(t)| + 4\mathsf{m}t$; therefore, since $t \leqslant \bar{\varepsilon}$ is sufficiently small, we may show that $|w_0(x(s)) - \kappa| \leqslant \frac{1-\mu}{2}\kappa$ for all $s \in [t', t]$. We then immediately obtain from (5.58) that

$$-(3-\mu)\kappa \leqslant \frac{-w_0(x(s))}{1+sw_0'(x(s))} \leqslant \dot{x}(s) \leqslant \frac{\mu\kappa - w_0(x(s))}{1+sw_0'(x(s))} \leqslant -\frac{(1-\mu)\kappa}{3}. \tag{5.59}$$

Then, using (5.36a) and the fact that $x(s)$ is strictly negative, we obtain that

$$\begin{aligned}\int_{t'}^{t} |\partial_\theta w_{\mathsf{B}}(\gamma(s), s)|\, ds &= \int_{t'}^{t} \frac{\left|w_0'(x(s))\right|}{1+sw_0'(x(s))} ds \leqslant \mathsf{b} \int_{t'}^{t} (x(s))^{-\frac{2}{3}} ds \\ &\leqslant -\tfrac{3\mathsf{b}}{(1-\mu)\kappa} \int_{t'}^{t} \dot{x}(s)(x(s))^{-\frac{2}{3}} ds \\ &= \tfrac{9\mathsf{b}}{(1-\mu)\kappa} \left(x(t')^{\frac{1}{3}} - x(t)^{\frac{1}{3}} \right) \\ &\leqslant \tfrac{9\mathsf{b}}{(1-\mu)\kappa} |x(t)|^{\frac{1}{3}} = \tfrac{9\mathsf{b}}{(1-\theta)\kappa} |\eta_{\mathsf{B}}{}^{-1}(\theta, t)|^{\frac{1}{3}} \\ &\leqslant \tfrac{9\mathsf{b}}{(1-\mu)\kappa} (3\kappa t)^{\frac{1}{3}} \end{aligned} \tag{5.60}$$

In the last inequality we have used that since $0 \leqslant \theta < \mathfrak{s}(t)$ we have that $|\eta_{\mathsf{B}}{}^{-1}(\theta, t)| \leqslant |\eta_{\mathsf{B}}{}^{-1}(0, t)| \leqslant 3\kappa t$ for all $t \leqslant \bar{\varepsilon}$, which is sufficiently small.

The proof of (5.57c) is nearly identical, but instead of (5.36a) we appeal to (5.36b), arriving at

$$\int_{t'}^{t} \left|\partial_\theta^2 w_{\mathsf{B}}(\gamma(s), s)\right| ds \leqslant \tfrac{9\mathsf{b}}{(1-\mu)\kappa} \left(x(t')^{-\frac{2}{3}} - x(t)^{-\frac{2}{3}} \right) \leqslant \tfrac{9\mathsf{b}}{(1-\mu)\kappa} \left(x(t') \right)^{-\frac{2}{3}} \tag{5.61}$$

In order to obtain (5.57b)–(5.57c), we use the above bound and (5.22b), which implies that $|x(t')| = |\eta_{\mathsf{B}}{}^{-1}(\gamma(t'), t')| \geqslant \frac{1}{2}|\gamma(t') - \mathfrak{s}(t')| + \frac{4}{5}(\mathsf{b}t')^{\frac{3}{2}} \geqslant \frac{4}{5}(\mathsf{b}t')^{\frac{3}{2}}$. □

5.6 *z* and *k* on the Shock Curve

For every $t \in (0, \bar{\varepsilon}]$, let us *assume* that we are given a left speed $w_- = w_-(t) = w(\mathfrak{s}(t)^-, t)$ and a right speed $w_+ = w_+(t) = w(\mathfrak{s}(t)^+, t)$ at the point $(\mathfrak{s}(t), t)$. Furthermore, let us assume that w_- and w_+ behave similarly to the solution of the Burgers equation computed in Proposition 5.7; by this we mean that the jump and the mean at $(\mathfrak{s}(t), t)$, defined by

$$[\![w]\!] = [\![w]\!](t) = w_-(t) - w_+(t)\,, \qquad \langle\!\langle w \rangle\!\rangle = \langle\!\langle w \rangle\!\rangle(t) = \tfrac{1}{2}\left(w_-(t) + w_+(t)\right)\,, \tag{5.62}$$

satisfy the bounds

$$\left|[\![w]\!](t) - 2\mathsf{b}^{\frac{3}{2}} t^{\frac{1}{2}}\right| \leqslant \mathsf{R}_j t \qquad \text{and} \qquad |\langle\!\langle w \rangle\!\rangle(t) - \kappa| \leqslant \mathsf{R}_m t\,, \tag{5.63}$$

for all $t \in (0, \overline{\varepsilon}]$, for two constants $\mathsf{R}_j, \mathsf{R}_m > 0$ which only depend on κ, b, c, and m. These bounds are consistent with (5.15a) and (5.145a) (to be established below).

The variables w_{-} and w_+ are the same as those in equations (3.13a)–(3.13b). Our goal in this subsection is to solve the coupled system of equations (3.13a)–(3.13b), for the jumps of z and k at the fixed point $(\mathfrak{s}(t), t)$, as a function the left speed w_{-} and right speed w_+, at this point. Since z and k are equal to 0 on the right side of the shock curve, we note that the jumps of z and k are equal to their values on the *left* of $(\mathfrak{s}(t), t)$; as such, we work with the unknowns

$$z_- = z_-(t) = [\![z]\!](t)\,, \qquad k_- = k_-(t) = [\![k]\!](t)\,. \tag{5.64}$$

In fact, because we expect k_- to be close to 0 (see (2.7)), and since (3.13a)–(3.13b) contain the variables e^{-k_-} and e^{k_-}, which are thus close to 1, it is more convenient to replace k_- with the unknown

$$\mathfrak{e}_- = \mathfrak{e}_-(t) = e^{k_-(t)} - 1\,. \tag{5.65}$$

Then, with this notation the equations (3.13a)–(3.13b) may be rewritten as the system

$$\mathcal{E}_1(w_{-}, w_+, z_-, \mathfrak{e}_-) = 0 \tag{5.66a}$$
$$\mathcal{E}_2(w_{-}, w_+, z_-, \mathfrak{e}_-) = 0 \tag{5.66b}$$

where

$$\begin{aligned}
&\mathcal{E}_1(w_{-}, w_+, z_-, \mathfrak{e}_-)\\
&\quad = \Big((w_{-} - z_-)^2(w_{-} + z_-)^2 + \tfrac{1}{8}(w_{-} - z_-)^4 - \tfrac{9}{8}(1+\mathfrak{e}_-)w_+^4\Big)\\
&\quad\Big((w_{-} - z_-)^2 - (1+\mathfrak{e}_-)w_+^2\Big)\\
&\quad - \Big((w_{-} - z_-)^2(w_{-} + z_-) - (1+\mathfrak{e}_-)w_+^3\Big)^2
\end{aligned} \tag{5.67a}$$

$$\begin{aligned}
&\mathcal{E}_2(w_{-}, w_+, z_-, \mathfrak{e}_-)\\
&\quad = \mathfrak{e}_-(w_{-} - z_-)^4(3w_+^2(1+\mathfrak{e}_-) - (w_{-} - z_-)^2) - \Big((w_{-} - z_-)^2 - (1+\mathfrak{e}_-)w_+^2\Big)^3\,.
\end{aligned} \tag{5.67b}$$

We view (5.66) as a coupled system of equations for the unknowns z_- and $\mathfrak{e}_-$ (or alternatively, k_-), with w_{-} and w_+ given. The correct root of (5.66) is given by:

Lemma 5.12 (Existence and asymptotic formula for z_- and k_-) *Assume that w_{-} and w_+ are such that their jump and mean at $(\mathfrak{s}(t), t)$ satisfy (5.63). Then, the system of equations (5.66) has a smallest (in absolute value) root $(z_-, \mathfrak{e}_-)$, such that z_- and $k_- = \log(\mathfrak{e}_- + 1)$ satisfy the bounds*

$$\left| z_-(t) + \tfrac{9[\![w]\!](t)^3}{16\langle w\rangle(t)^2}\right| \leqslant C_0 t^{\frac{5}{2}}\,. \tag{5.68a}$$

$$\left|k_-(t) - \tfrac{4[\![w]\!](t)^3}{\langle w\rangle(t)^3}\right| \leqslant C_0 t^{\frac{5}{2}}\,, \tag{5.68b}$$

where $C_0 = C_0(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$ *is an explicitly computable constant. In particular, in view of (5.63) we have the estimates*

$$\left|z_-(t) + \tfrac{9\mathsf{b}^{\frac{9}{2}}}{2\kappa^2} t^{\frac{3}{2}}\right| \leqslant Ct^2 \qquad \Rightarrow \qquad |z_-(t)| \leqslant \tfrac{5\mathsf{b}^{\frac{9}{2}}}{\kappa^2} t^{\frac{3}{2}} \tag{5.69a}$$

$$\left|k_-(t) - \tfrac{32\mathsf{b}^{\frac{9}{2}}}{\kappa^3} t^{\frac{3}{2}}\right| \leqslant Ct^2 \qquad \Rightarrow \qquad |k_-(t)| \leqslant \tfrac{40\mathsf{b}^{\frac{9}{2}}}{\kappa^3} t^{\frac{3}{2}} \tag{5.69b}$$

for all $t \in (0, \overline{\varepsilon}]$*, assuming that* $\overline{\varepsilon}$ *is sufficiently small.*

Proof of Lemma 5.12 Throughout the proof, we fix $t \in (0, \overline{\varepsilon}]$, and omit the t dependence of the unknowns. In view of (5.63), we view $[\![w]\!]$ as a small parameter, thus suitable for asymptotic expansions, and $\langle\!\langle w\rangle\!\rangle$ as an $\mathcal{O}(1)$ parameter. As such, in (5.67) we replace

$$w_- = \langle\!\langle w\rangle\!\rangle + \tfrac{1}{2}[\![w]\!]\,, \qquad w_+ = \langle\!\langle w\rangle\!\rangle - \tfrac{1}{2}[\![w]\!]\,.$$

Because we expect $|z_-|, |\mathfrak{e}_-| \ll 1$, we first perform a Taylor series expansion of (5.66), and identify only the linear terms with respect to z_- and $\mathfrak{e}_-$. This becomes

$$\begin{aligned}
&\tfrac{1}{16}[\![w]\!]^4(12\langle\!\langle w\rangle\!\rangle^2 - [\![w]\!]^2) - \tfrac{1}{8}[\![w]\!](32\langle\!\langle w\rangle\!\rangle^4 + 8\langle\!\langle w\rangle\!\rangle^2[\![w]\!]^2 + 6\langle\!\langle w\rangle\!\rangle[\![w]\!]^3 - [\![w]\!]^4)z_- \\
&\quad - \tfrac{1}{64}[\![w]\!](48\langle\!\langle w\rangle\!\rangle^5 + 40\langle\!\langle w\rangle\!\rangle^3[\![w]\!]^2 - 48\langle\!\langle w\rangle\!\rangle^2[\![w]\!]^3 + 3\langle\!\langle w\rangle\!\rangle[\![w]\!]^4 + 4[\![w]\!]^5)\mathfrak{e}_- \\
&\quad = \mathcal{O}(|z_-|^2 + |\mathfrak{e}_-|^2) \\
&\quad - 8\langle\!\langle w\rangle\!\rangle^3[\![w]\!]^3 + 12[\![w]\!]^2(2\langle\!\langle w\rangle\!\rangle^3 + \langle\!\langle w\rangle\!\rangle^2[\![w]\!])z_- \\
&\quad + \tfrac{1}{32}(64\langle\!\langle w\rangle\!\rangle^6 + 240\langle\!\langle w\rangle\!\rangle^4[\![w]\!]^2 - 512\langle\!\langle w\rangle\!\rangle^3[\![w]\!]^3 + 60\langle\!\langle w\rangle\!\rangle^2[\![w]\!]^4 + [\![w]\!]^6)\mathfrak{e}_- \\
&\quad = \mathcal{O}(|z_-|^2 + |\mathfrak{e}_-|^2)\,.
\end{aligned}$$

By dropping the higher order terms in $|[\![w]\!]| \ll 1$, this motivates our definition of the approximate solutions z_-^{app} and $\mathfrak{e}_-^{\text{app}}$ as the solutions of the linear system

$$\begin{pmatrix} 4[\![w]\!]\langle\!\langle w\rangle\!\rangle^4 & \tfrac{3}{4}[\![w]\!]\langle\!\langle w\rangle\!\rangle^5 \\ 24[\![w]\!]^2\langle\!\langle w\rangle\!\rangle^3 & 2\langle\!\langle w\rangle\!\rangle^6 \end{pmatrix} \begin{pmatrix} z_-^{\text{app}} \\ \mathfrak{e}_-^{\text{app}} \end{pmatrix} = \begin{pmatrix} \tfrac{3}{4}[\![w]\!]^4\langle\!\langle w\rangle\!\rangle^2 \\ 8[\![w]\!]^3\langle\!\langle w\rangle\!\rangle^3 \end{pmatrix}. \tag{5.70}$$

This system is uniquely solvable, and yields

$$z_-^{\text{app}} = -\frac{9[\![w]\!]^3}{16\langle\!\langle w\rangle\!\rangle^2} Q_1\left(\frac{[\![w]\!]}{\langle\!\langle w\rangle\!\rangle}\right), \qquad Q_1(x) = \frac{1}{1 - \frac{9}{4}x^2}\,, \tag{5.71a}$$

$$\mathfrak{e}_-^{\text{app}} = \frac{4[\![w]\!]^3}{\langle\!\langle w\rangle\!\rangle^3} Q_2\left(\frac{[\![w]\!]}{\langle\!\langle w\rangle\!\rangle}\right), \qquad Q_2(x) = \frac{1 - \frac{9}{16}x^2}{1 - \frac{9}{4}x^2}\,. \tag{5.71b}$$

In order to apply the implicit function theorem, we at last introduce the variables

$$Z = \frac{z_- - z_-^{\mathrm{app}}}{[\![w]\!]^5} \qquad \text{and} \qquad E = \frac{\mathfrak{e}_- - \mathfrak{e}_-^{\mathrm{app}}}{[\![w]\!]^5} \tag{5.72}$$

and substitute in the system (5.67) the ansatz $z_- = z_-^{\mathrm{app}} + Z[\![w]\!]^5$ and $\mathfrak{e}_- = \mathfrak{e}_-^{\mathrm{app}} + E[\![w]\!]^5$. After some algebraic manipulations, the system of equations (5.67) is rewritten as system

$$\begin{aligned} 0 &= \mathcal{F}_1([\![w]\!], \langle\!\langle w\rangle\!\rangle, Z, E) \\ &= [\![w]\!]^{-6}\mathcal{E}_1(\langle\!\langle w\rangle\!\rangle + \tfrac{1}{2}[\![w]\!], \langle\!\langle w\rangle\!\rangle - \tfrac{1}{2}[\![w]\!], z_-^{\mathrm{app}} + Z[\![w]\!]^5, \mathfrak{e}_-^{\mathrm{app}} + E[\![w]\!]^5) \end{aligned} \tag{5.73a}$$

$$\begin{aligned} 0 &= \mathcal{F}_2([\![w]\!], \langle\!\langle w\rangle\!\rangle, Z, E) \\ &= [\![w]\!]^{-5}\mathcal{E}_2(\langle\!\langle w\rangle\!\rangle + \tfrac{1}{2}[\![w]\!], \langle\!\langle w\rangle\!\rangle - \tfrac{1}{2}[\![w]\!], z_-^{\mathrm{app}} + Z[\![w]\!]^5, \mathfrak{e}_-^{\mathrm{app}} + E[\![w]\!]^5) \end{aligned} \tag{5.73b}$$

for the unknowns Z and E. Defining

$$P_w = (0, \langle\!\langle w\rangle\!\rangle, -\tfrac{27}{64}\langle\!\langle w\rangle\!\rangle^{-4}, -15\langle\!\langle w\rangle\!\rangle^{-5}),$$

we observe that

$$\begin{aligned} &\mathcal{F}_1(P_w) = 0, \qquad \partial_Z\mathcal{F}_1(P_w) = -4\langle\!\langle w\rangle\!\rangle^4, \qquad \partial_Z\mathcal{F}_2(P_w) = 0, \\ &\mathcal{F}_2(P_w) = 0, \qquad \partial_E\mathcal{F}_1(P_w) = 0, \qquad \partial_E\mathcal{F}_2(P_w) = 2\langle\!\langle w\rangle\!\rangle^6. \end{aligned}$$

Thus, the Jacobian determinant associated to $(\mathcal{F}_1, \mathcal{F}_2)(\cdot, \cdot, Z, E)$ evaluated at P_w equals to $-8\langle\!\langle w\rangle\!\rangle^{10} \neq 0$. Here we are using that by (5.63) we have that $|\langle\!\langle w\rangle\!\rangle - \kappa| \leqslant \frac{\kappa}{2}$, and thus $\langle\!\langle w\rangle\!\rangle \neq 0$. Thus, by the implicit function theorem, there exists a $J_0 = J_0(\langle\!\langle w\rangle\!\rangle) > 0$, such that for all $|[\![w]\!]| \leqslant J_0$, we have a unique solution $Z = Z([\![w]\!], \langle\!\langle w\rangle\!\rangle)$ and $E = E([\![w]\!], \langle\!\langle w\rangle\!\rangle)$ of (5.73), with $Z(0, \langle\!\langle w\rangle\!\rangle) = -\frac{27}{64}\langle\!\langle w\rangle\!\rangle^{-4}$ and $E(0, \langle\!\langle w\rangle\!\rangle) = -15\langle\!\langle w\rangle\!\rangle^{-6}$. To conclude, we note that since J_0 depends only on $\langle\!\langle w\rangle\!\rangle$, it may be estimated solely in terms of κ; and since by (5.63) we have that $|[\![w]\!]| \leqslant 3\mathsf{b}^{\frac{3}{2}}\bar{\varepsilon}^{\frac{1}{2}}$ with $\bar{\varepsilon}$ which is sufficiently small in terms of κ and b, we deduce that the condition $|[\![w]\!]| \leqslant J_0$ is automatically guaranteed.

As a consequence, from the above discussion we deduce that for all $t \leqslant \bar{\varepsilon}$, we have

$$\left|z_- - z_-^{\mathrm{app}}\right| \leqslant C_0[\![w]\!]^5, \qquad \text{and} \qquad \left|\mathfrak{e}_- - \mathfrak{e}_-^{\mathrm{app}}\right| \leqslant C_0[\![w]\!]^5, \tag{5.74}$$

where $C_0 > 0$ is a constant which only depends on κ.

The proof of the bounds (5.68a)–(5.68b) are now essentially completed, upon combining (5.63), (5.71), and (5.74). To see this, note that the rational function Q_1 appearing in the definition (5.71a) satisfies $|Q_1(x) - 1| \leqslant 3x^2$ for all $x \leqslant \frac{1}{10}$. Thus, we obtain that

$$\left| z_-^{\rm app} + \frac{9[\![w]\!]^3}{16\langle\!\langle w\rangle\!\rangle^2}\right| \leqslant C_0[\![w]\!]^5$$

since $\langle\!\langle w\rangle\!\rangle \geqslant \frac{\kappa}{2}$ when $\overline{\varepsilon}$ is sufficiently small. The bound (5.68a) follows from the above estimate, (5.63), and (5.74). Similarly, by using that the rational function Q_2 appearing in the definition (5.71a) satisfies $|Q_2(x) - 1| \leqslant 2x^2$ for all $x \leqslant \frac{1}{10}$, we obtain the bound

$$\left| \mathfrak{e}_-^{\rm app} - \frac{4[\![w]\!]^3}{\langle\!\langle w\rangle\!\rangle^3}\right| \leqslant C_0[\![w]\!]^5 ,$$

which may be combined with (5.63) and (5.74), to establish

$$\left| \mathfrak{e}_- - \frac{4[\![w]\!]^3}{\langle\!\langle w\rangle\!\rangle^3}\right| \leqslant C_0[\![w]\!]^5 \tag{5.75}$$

with $C_0 > 0$ a constant which depends only on κ and b. The bound (5.68b) now follows because $k_- = \log(1 + \mathfrak{e}_-)$, and $|\log(1 + \mathfrak{e}_-) - \mathfrak{e}_-| \leqslant 2\mathfrak{e}_-^2$ for $|\mathfrak{e}_-| \leqslant \frac{1}{2}$; clearly, $|\mathfrak{e}_-| = \mathcal{O}(t^{\frac{3}{2}}) \leqslant \frac{1}{2}$ in view of (5.75).

The bounds (5.69a)–(5.69b) follow from (5.68a)–(5.68b), (5.63), and the fact that $t \leqslant \overline{\varepsilon}$, which in turn may be made arbitrarily small with respect to κ and b. □

Let us further assume that w_+ and w_- are differentiable with respect to ξ and t for all $(\xi, t) \in \Omega_{\overline{\varepsilon}}$. By implicitly differentiating (5.66a)–(5.66b), we may then deduce:

Lemma 5.13 (Lipschitz bounds for z_- and k_-) *For $t \in (0, \overline{\varepsilon}]$, assume that w_- and w_+ are such that their jump and mean at $(\mathfrak{s}(t), t)$ satisfy (5.63), and further assume that $\langle\!\langle w\rangle\!\rangle$ and $[\![w]\!]$ are differentiable with respect to t. Then, the smallest roots of the system of equations (5.66) are such that z_- and $k_- = \log(\mathfrak{e}_- + 1)$ satisfy the pointwise estimates*

$$\left| \tfrac{d}{dt} z_-(t) + \tfrac{d}{dt}\left(\tfrac{9[\![w]\!](t)^3}{16\langle w\rangle(t)^2}\right)\right| \leqslant C_0 t^2\Big(|\tfrac{d}{dt}[\![w]\!](t)| + |\tfrac{d}{dt}\langle\!\langle w\rangle\!\rangle(t)|\Big) \tag{5.76a}$$

$$\left| e^{k_-(t)}\tfrac{d}{dt} k_-(t) - \tfrac{d}{dt}\left(\tfrac{4[\![w]\!](t)^3}{\langle w\rangle(t)^3}\right)\right| \leqslant C_0 t^2\Big(|\tfrac{d}{dt}[\![w]\!](t)| + |\tfrac{d}{dt}\langle\!\langle w\rangle\!\rangle(t)|\Big) \tag{5.76b}$$

where the constant $C_0 > 0$ only depends on κ, b, and m.

Proof of Lemma 5.13 From the definition $k_- = \log(1 + \mathfrak{e}_-)$ we obtain that $\frac{d}{dt}k_- = e^{-k_-}\frac{d}{dt}\mathfrak{e}_-$, and thus, in order to prove the lemma it is sufficient to obtain derivative bounds for z_- and $\mathfrak{e}_-$.

Implicitly differentiating (5.66) we arrive at

$$\frac{d}{dt}\begin{pmatrix} z_- \\ \mathfrak{e}_- \end{pmatrix} = -\begin{pmatrix} \partial_{z_-}\mathcal{E}_1 & \partial_{\mathfrak{e}_-}\mathcal{E}_1 \\ \partial_{z_-}\mathcal{E}_2 & \partial_{\mathfrak{e}_-}\mathcal{E}_2 \end{pmatrix}^{-1} \begin{pmatrix} \partial_{w_-}\mathcal{E}_1 & \partial_{w_+}\mathcal{E}_1 \\ \partial_{w_-}\mathcal{E}_2 & \partial_{w_+}\mathcal{E}_2 \end{pmatrix} \frac{d}{dt}\begin{pmatrix} w_- \\ w_+ \end{pmatrix}, \tag{5.77}$$

pointwise for $t \in (0, \bar{\varepsilon}]$, where we recall that the functions $\mathcal{E}_1$ and $\mathcal{E}_2$ are defined in (5.67). In order to evaluate these Jacobi matrices, we resort to the notation in (5.62) and rewrite $w_- = \langle\!\langle w \rangle\!\rangle + \frac{1}{2}[\![w]\!]$ and $w_+ = \langle\!\langle w \rangle\!\rangle - \frac{1}{2}[\![w]\!]$; furthermore, we write $z_- = z_-^{\rm app} + \mathcal{O}([\![w]\!]^5)$ and $\mathfrak{e}_- = \mathfrak{e}_-^{\rm app} + \mathcal{O}([\![w]\!]^6)$ as justified by (5.72), with $z_-^{\rm app}$ defined by (5.71a), and $\mathfrak{e}_-^{\rm app}$ given by (5.71b). We emphasize that the implicit constants in the $\mathcal{O}([\![w]\!]^5)$ and $\mathcal{O}([\![w]\!]^6)$ symbols only depend on κ and b, since the bounds on the solutions Z and E of (5.73) only depend on κ and b. After some tedious computations, we arrive at

$$-\begin{pmatrix} \partial_{z_-}\mathcal{E}_1 & \partial_{\mathfrak{e}_-}\mathcal{E}_1 \\ \partial_{z_-}\mathcal{E}_2 & \partial_{\mathfrak{e}_-}\mathcal{E}_2 \end{pmatrix}^{-1} \begin{pmatrix} \partial_{w_-}\mathcal{E}_1 & \partial_{w_+}\mathcal{E}_1 \\ \partial_{w_-}\mathcal{E}_2 & \partial_{w_+}\mathcal{E}_2 \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{27[\![w]\!]^2}{16\langle w\rangle^2} + \frac{9[\![w]\!]^3}{16\langle w\rangle^3} & \frac{27[\![w]\!]^2}{16\langle w\rangle^2} + \frac{9[\![w]\!]^3}{16\langle w\rangle^3} \\ \frac{12[\![w]\!]^2}{\langle w\rangle^3} - \frac{6[\![w]\!]^3}{\langle w\rangle^4} & -\frac{12[\![w]\!]^2}{\langle w\rangle^3} - \frac{6[\![w]\!]^3}{\langle w\rangle^4} \end{pmatrix} + \mathcal{O}([\![w]\!]^4)\,, \tag{5.78}$$

where the implicit constant only depends on κ and b. From (5.63), (5.77), (5.78), and recalling that $\frac{d}{dt}w_- = \frac{d}{dt}\langle\!\langle w \rangle\!\rangle + \frac{1}{2}\frac{d}{dt}[\![w]\!]$ and $\frac{d}{dt}w_+ = \frac{d}{dt}\langle\!\langle w \rangle\!\rangle - \frac{1}{2}\frac{d}{dt}[\![w]\!]$, we deduce that there exists a constant $C_0 > 0$, which only depends on κ and b, such that

$$\left|\tfrac{d}{dt}z_- + \tfrac{d}{dt}\left(\tfrac{9[\![w]\!]^3}{16\langle w\rangle^2}\right)\right| + \left|\tfrac{d}{dt}\mathfrak{e}_- - \tfrac{d}{dt}\left(\tfrac{4[\![w]\!]^3}{\langle w\rangle^3}\right)\right| \leqslant C_0[\![w]\!]^4\left(\left|\tfrac{d}{dt}[\![w]\!]\right| + \left|\tfrac{d}{dt}\langle\!\langle w \rangle\!\rangle\right|\right) \tag{5.79}$$

The bounds (5.76) follow from (5.79), upon recalling that $[\![w]\!] = \mathcal{O}(t^{\frac{1}{2}})$. □

A direct consequence of Lemmas 5.7, 5.12, and 5.13 is the following statement, which will be useful in the proof of Proposition 5.6.

Corollary 5.14 *In addition to the assumption of Lemmas 5.7, assume that $[\![w]\!]$ and $\langle\!\langle w \rangle\!\rangle$ satisfy the bounds (5.63). Let $z_-(t)$ and $k_-(t)$ be as defined in Lemma 5.12. In addition, suppose that there exists $\mathsf{R} = \mathsf{R}(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$ such that for all $t \in (0, \bar{\varepsilon}]$ we have*

$$\left|\tfrac{d}{dt}[\![w]\!](t) - \tfrac{d}{dt}[\![w_{\mathsf{B}}]\!](t)\right| \leqslant 2\mathsf{R}, \qquad \left|\tfrac{d}{dt}\langle\!\langle w \rangle\!\rangle(t) - \tfrac{d}{dt}\langle\!\langle w_{\mathsf{B}} \rangle\!\rangle(t)\right| \leqslant \mathsf{R}\,. \tag{5.80}$$

Then, assuming that $\bar{\varepsilon}$ is sufficiently small with respect to κ, b, c and m, we have that

$$\left|\tfrac{d}{dt}z_-(t) + \tfrac{27\mathsf{b}^{\frac{9}{2}}}{4\kappa^2}t^{\frac{1}{2}}\right| \leqslant Ct\,, \qquad \left|\tfrac{d}{dt}k_-(t) - \tfrac{48\mathsf{b}^{\frac{9}{2}}}{\kappa^3}t^{\frac{1}{2}}\right| \leqslant Ct\,, \tag{5.81}$$

for all $t \in (0, \bar{\varepsilon}]$, where $C = C(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$ is a constant.

In addition to (5.80), if we are also given that

$$\left|\tfrac{d^2}{dt^2}[\![w]\!](t) - \tfrac{d^2}{dt^2}[\![w_{\mathsf{B}}]\!](t)\right| \leqslant 2\mathsf{R}^* t^{-1}, \qquad \left|\tfrac{d^2}{dt^2}\langle\!\langle w \rangle\!\rangle(t) - \tfrac{d^2}{dt^2}\langle\!\langle w_{\mathsf{B}} \rangle\!\rangle(t)\right| \leqslant \mathsf{R}^* t^{-1}\,, \tag{5.82}$$

for a constant $\mathsf{R}^* = \mathsf{R}^*(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$. *Then, by possibly further reducing the value of* $\overline{\varepsilon}$ *we also have the estimates*

$$\left|\tfrac{d^2}{dt^2} z_-(t) + \tfrac{27\mathsf{b}^{\frac{9}{2}}}{8\kappa^2} t^{-\frac{1}{2}}\right| \leqslant C\,, \qquad \left|\tfrac{d^2}{dt^2} k_-(t) - \tfrac{24\mathsf{b}^{\frac{9}{2}}}{\kappa^3} t^{-\frac{1}{2}}\right| \leqslant C\,, \tag{5.83}$$

where $C = C(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$ *is a constant.*

Proof of Corollary 5.14 Recall that by assumption the bound (5.63) holds, and thus by Lemma 5.12 we have the estimate (5.69). The assumption (5.80) and the bound (5.15b) imply that

$$\left|\tfrac{d}{dt}[\![w]\!] - \mathsf{b}^{\frac{3}{2}} t^{-\frac{1}{2}}\right| + \left|\tfrac{d}{dt}\langle\!\langle w\rangle\!\rangle\right| \leqslant 3\mathsf{m}^4 + 3\mathsf{R}\,, \tag{5.84}$$

and thus the right sides of (5.76a) and (5.76b) are $\mathcal{O}(t^{\frac{3}{2}})$. For the bound on the time derivative of z_-, we appeal to (5.76a), which gives

$$\left|\tfrac{d}{dt} z_- + \tfrac{27[\![w]\!]^2}{16\langle w\rangle^2}\tfrac{d}{dt}[\![w]\!] - \tfrac{9[\![w]\!]^3}{8\langle w\rangle^3}\tfrac{d}{dt}\langle\!\langle w\rangle\!\rangle\right| \leqslant C t^{\frac{3}{2}}\,.$$

Incorporating into the above estimate the bounds (5.84) and (5.63), we arrive at the z_- bound in (5.81). The time derivative of k_- is bounded by appealing to (5.76b), which yields

$$\left|e^{k_-}\tfrac{d}{dt} k_- - \tfrac{12[\![w]\!]^2}{\langle w\rangle^3}\tfrac{d}{dt}[\![w]\!] + \tfrac{12[\![w]\!]^3}{\langle w\rangle^4}\tfrac{d}{dt}\langle\!\langle w\rangle\!\rangle\right| \leqslant C t^{\frac{3}{2}}\,.$$

Using (5.84), (5.69b), and (5.63), the k_- bound in (5.81) now follows.

In order to prove (5.83), we first note that assumption (5.82) and the bound (5.15c) imply that

$$\left|\tfrac{d^2}{dt^2}[\![w]\!] + \tfrac{1}{2}\mathsf{b}^{\frac{3}{2}} t^{-\frac{3}{2}}\right| + \left|\tfrac{d^2}{dt^2}\langle\!\langle w\rangle\!\rangle\right| \leqslant 3\left(5\mathsf{m}^4 + \mathsf{R}^*\right) t^{-1}\,. \tag{5.85}$$

Next, we implicitly differentiate (5.66) a second time, to obtain

$$\begin{aligned}
&\frac{d^2}{dt^2}\begin{pmatrix} z_- \\ \mathfrak{e}_- \end{pmatrix} + \begin{pmatrix} \partial_{z_-}\mathcal{E}_1 & \partial_{\mathfrak{e}_-}\mathcal{E}_1 \\ \partial_{z_-}\mathcal{E}_2 & \partial_{\mathfrak{e}_-}\mathcal{E}_2 \end{pmatrix}^{-1} \begin{pmatrix} \partial_{w_-}\mathcal{E}_1 & \partial_{w_+}\mathcal{E}_1 \\ \partial_{w_-}\mathcal{E}_2 & \partial_{w_+}\mathcal{E}_2 \end{pmatrix} \frac{d^2}{dt^2}\begin{pmatrix} w_- \\ w_+ \end{pmatrix} \\
&\quad + \begin{pmatrix} \partial_{z_-}\mathcal{E}_1 & \partial_{\mathfrak{e}_-}\mathcal{E}_1 \\ \partial_{z_-}\mathcal{E}_2 & \partial_{\mathfrak{e}_-}\mathcal{E}_2 \end{pmatrix}^{-1} \begin{pmatrix} \partial_{w_-w_-}\mathcal{E}_1 & \partial_{w_-w_+}\mathcal{E}_1 & \partial_{w_+w_+}\mathcal{E}_1 \\ \partial_{w_-w_-}\mathcal{E}_2 & \partial_{w_-w_+}\mathcal{E}_2 & \partial_{w_+w_+}\mathcal{E}_2 \end{pmatrix} \begin{pmatrix} (\frac{d}{dt} w_-)^2 \\ 2\frac{d}{dt} w_- \frac{d}{dt} w_+ \\ (\frac{d}{dt} w_+)^2 \end{pmatrix} \\
&= -\begin{pmatrix} \partial_{z_-}\mathcal{E}_1 & \partial_{\mathfrak{e}_-}\mathcal{E}_1 \\ \partial_{z_-}\mathcal{E}_2 & \partial_{\mathfrak{e}_-}\mathcal{E}_2 \end{pmatrix}^{-1} \begin{pmatrix} \partial_{z_-z_-}\mathcal{E}_1 & \partial_{z_-k_-}\mathcal{E}_1 & \partial_{k_-k_-}\mathcal{E}_1 \\ \partial_{z_-z_-}\mathcal{E}_2 & \partial_{z_-k_-}\mathcal{E}_2 & \partial_{k_-k_-}\mathcal{E}_2 \end{pmatrix} \begin{pmatrix} (\frac{d}{dt} z_-)^2 \\ 2\frac{d}{dt} z_- \frac{d}{dt} k_- \\ (\frac{d}{dt} k_-)^2 \end{pmatrix}
\end{aligned}$$

$$
-2\begin{pmatrix} \partial_{z_-}\mathcal{E}_1 & \partial_{\mathfrak{e}_-}\mathcal{E}_1 \\ \partial_{z_-}\mathcal{E}_2 & \partial_{\mathfrak{e}_-}\mathcal{E}_2 \end{pmatrix}^{-1} \begin{pmatrix} \partial_{z_-w_-}\mathcal{E}_1 & \partial_{z_-w_+}\mathcal{E}_1 & \partial_{k_-w_-}\mathcal{E}_1 & \partial_{k_-w_+}\mathcal{E}_1 \\ \partial_{z_-w_-}\mathcal{E}_2 & \partial_{z_-w_+}\mathcal{E}_2 & \partial_{k_-w_-}\mathcal{E}_2 & \partial_{k_-w_+}\mathcal{E}_2 \end{pmatrix} \begin{pmatrix} \frac{d}{dt}z_- \frac{d}{dt}w_- \\ \frac{d}{dt}z_- \frac{d}{dt}w_+ \\ \frac{d}{dt}k_- \frac{d}{dt}w_- \\ \frac{d}{dt}k_- \frac{d}{dt}w_+ \end{pmatrix} . \tag{5.86}
$$

By appealing to (5.78), (5.76), and (5.63), similarly to (5.79) we deduce that the right side of (5.86) equals

$$
\begin{aligned}
&\begin{pmatrix} \frac{3\langle w\rangle}{16[\![w]\!]} \frac{d}{dt}z_- \frac{d}{dt}k_- + \frac{\langle w\rangle^2}{16[\![w]\!]}(\frac{d}{dt}k_-)^2 \\ 0 \end{pmatrix} + \mathcal{O}\left(\left(|\tfrac{d}{dt}z_-| + |\tfrac{d}{dt}k_-|\right)^2\right) \\
&+ \begin{pmatrix} -\frac{8}{\langle w\rangle}\frac{d}{dt}\langle\!\langle w\rangle\!\rangle + \frac{2[\![w]\!]^2-2\langle w\rangle^2}{\langle w\rangle^2[\![w]\!]}\frac{d}{dt}[\![w]\!] & \frac{27[\![w]\!]^2-12\langle w\rangle^2}{32[\![w]\!]\langle w\rangle}\frac{d}{dt}[\![w]\!] + \frac{3}{8}\frac{d}{dt}\langle\!\langle w\rangle\!\rangle \\ -\frac{24[\![w]\!]}{\langle w\rangle^3}\frac{d}{dt}[\![w]\!] & -\frac{21[\![w]\!]}{2\langle w\rangle^2}\frac{d}{dt}[\![w]\!] - \frac{12}{\langle w\rangle}\frac{d}{dt}\langle\!\langle w\rangle\!\rangle \end{pmatrix} \begin{pmatrix} \frac{d}{dt}z_- \\ \frac{d}{dt}k_- \end{pmatrix} \\
&+ \mathcal{O}\left([\![w]\!]^2\left(\left|\tfrac{d}{dt}[\![w]\!]\right| + \left|\tfrac{d}{dt}\langle\!\langle w\rangle\!\rangle\right|\right)\left(|\tfrac{d}{dt}z_-| + |\tfrac{d}{dt}k_-|\right)\right) .
\end{aligned} \tag{5.87}
$$

Similarly, one may verify that the sum of the last two terms on the left side of (5.86) is given by

$$
\begin{aligned}
&\begin{pmatrix} \frac{27[\![w]\!]^2}{16\langle w\rangle^2}\frac{d^2}{dt^2}[\![w]\!] - \frac{9[\![w]\!]^3}{8\langle w\rangle^3}\frac{d^2}{dt^2}\langle\!\langle w\rangle\!\rangle \\ -\frac{12[\![w]\!]^2}{\langle w\rangle^3}\frac{d^2}{dt^2}[\![w]\!] + \frac{12[\![w]\!]^3}{\langle w\rangle^4}\frac{d^2}{dt^2}\langle\!\langle w\rangle\!\rangle \end{pmatrix} + \begin{pmatrix} \frac{9[\![w]\!]}{4\langle w\rangle^2}(\frac{d}{dt}[\![w]\!])^2 + \frac{27[\![w]\!]^2}{2\langle w\rangle^3}\frac{d}{dt}[\![w]\!]\frac{d}{dt}\langle\!\langle w\rangle\!\rangle \\ -\frac{24[\![w]\!]}{\langle w\rangle^3}(\frac{d}{dt}[\![w]\!])^2 + \frac{72[\![w]\!]^2}{\langle w\rangle^4}\frac{d}{dt}[\![w]\!]\frac{d}{dt}\langle\!\langle w\rangle\!\rangle \end{pmatrix} \\
&+ \mathcal{O}\left([\![w]\!]^3\left(\left|\tfrac{d}{dt}[\![w]\!]\right| + \left|\tfrac{d}{dt}\langle\!\langle w\rangle\!\rangle\right|\right)\right)
\end{aligned} \tag{5.88}
$$

where the implicit constants only depend on κ, b, c, and m.

To conclude we use the bounds (5.63), (5.84), (5.85), (5.69), and (5.81) in the equality given by (5.86), (5.87), and (5.88), to arrive at

$$
\left|\frac{d^2}{dt^2}z_- + \frac{27[\![w]\!]}{16\langle w\rangle^2}\left(2\left(\tfrac{d}{dt}[\![w]\!]\right)^2 + [\![w]\!]\tfrac{d^2}{dt^2}[\![w]\!]\right)\right| \leqslant C \tag{5.89}
$$

and by also appealing to $\frac{d^2}{dt^2}k_- = e^{-k_-}\frac{d^2}{dt^2}\mathfrak{e}_- - (\frac{d}{dt}k_-)^2$ we obtain

$$
\left|e^{k_-}\frac{d^2}{dt^2}k_- - \frac{12[\![w]\!]}{\langle w\rangle^3}\left(2\left(\tfrac{d}{dt}[\![w]\!]\right)^2 + [\![w]\!]\tfrac{d^2}{dt^2}[\![w]\!]\right)\right| \leqslant Ct^{\frac{1}{2}} , \tag{5.90}
$$

where $C = C(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$. To conclude, we combine (5.89)–(5.90) with the precise estimates for $[\![w]\!]$ and its first two time derivatives, cf. (5.63), (5.84), and (5.85) and arrive at (5.83). □

5.7 Transport Structure, Spacetime Regions, and Characteristic Families

5.7.1 A New Form of the *w* and *z* Equations

We first observe that using (3.5c) and recalling that $c = \frac{1}{2}(w - z)$, we can write the system (3.5) as

$$\partial_t w + \lambda_3 \partial_\theta w = -\tfrac{8}{3} a w + \tfrac{1}{4} c (\partial_t k + \lambda_3 \partial_\theta k) , \tag{5.91a}$$

$$\partial_t z + \lambda_1 \partial_\theta z = -\tfrac{8}{3} a z - \tfrac{1}{4} c (\partial_t k + \lambda_1 \partial_\theta k) , \tag{5.91b}$$

$$\partial_t k + \lambda_2 \partial_\theta k = 0 , \tag{5.91c}$$

$$\partial_t a + \lambda_2 \partial_\theta a = -\tfrac{4}{3} a^2 + \tfrac{1}{3} (w + z)^2 - \tfrac{1}{6} (w - z)^2 , \tag{5.91d}$$

Our iteration scheme will be based on (5.91), and in particular on the estimates for $\partial_\theta w$ that the specific form of the equations (5.91a) and (5.91b) provide. It will be convenient to introduce the vector of unknowns

$$U = (w, z, k, a) . \tag{5.92}$$

5.7.2 Characteristic Families, Shock-Intersection Times, Spacetime Regions

Recalling the definition of the wave speeds (3.6), we let η denote the 3-characteristic which satisfies

$$\partial_t \eta(x, t) = \lambda_3(\eta(x, t), t) , \qquad \eta(x, 0) = x , \tag{5.93a}$$

for $t \in (0, \overline{\varepsilon})$. We also define the 1- and 2-characteristics as

$$\partial_s \psi_t(\theta, s) = \lambda_1(\psi_t(\theta, s), s) , \qquad \psi_t(\theta, t) = \theta , \tag{5.93b}$$

$$\partial_s \phi_t(\theta, s) = \lambda_2(\phi_t(\theta, s), s) , \qquad \phi_t(\theta, t) = \theta , \tag{5.93c}$$

for $s \in (0, t)$. We note that η has a prescribed initial datum at time 0, while ϕ_t and ψ_t have a prescribed terminal datum, at time t. Moreover, note that as opposed to η, the characteristics ϕ_t and ψ_t may cross the shock curve $(\mathfrak{s}(t), t)_{t \in [0, \overline{\varepsilon}]}$ in a continuous fashion; this will be shown to be possible because λ_1 and λ_2 have bounded *one-sided* derivatives on the shock.

Definition 5.15 For $(\theta, t) \in \mathbb{T} \times [0, \overline{\varepsilon}]$ consider the integral curves $\psi_t(\theta, s)$ and $\phi_t(\theta, s)$ defined by the ODEs (5.93b)–(5.93c). If the curves $(\psi_t(\theta, s), s)_{s \in [0,t]}$ and $(\mathfrak{s}(s), s)_{s \in [0,t]}$, respectively $(\phi_t(\theta, s), s)_{s \in [0,t]}$ and $(\mathfrak{s}(s), s)_{s \in [0,t]}$, intersect then we define the *shock-intersection times* $\daleth(\theta, t)$ and $\beth(\theta, t)$ as the (largest) time at which

$$\psi_t(\theta, \beth(\theta, t)) = \mathfrak{s}(\beth(\theta, t)) , \qquad \text{and}$$
$$\phi_t(\theta, \daleth(\theta, t)) = \mathfrak{s}(\daleth(\theta, t)) . \tag{5.94}$$

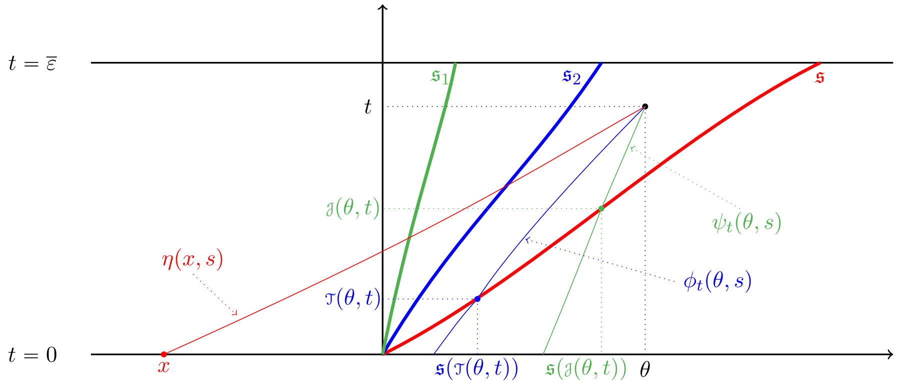

Fig. 11 Fix a spatial location (θ, t), just to the left of the given shock curve $\mathfrak{s}$, which is represented in red. The flow $\eta(x, s)$ defined in (5.93a), and the label x such that $\eta(x, t) = \theta$, are also represented in red. The flow $\phi_t(\theta, s)$ defined in (5.93c), its associated shock-intersection time $\mathcal{T}(\theta, t)$ from (5.94), and the curve $\mathfrak{s}_2$ from (5.95), are represented in blue. The flow $\psi_t(\theta, s)$ defined in (5.93b), its associated shock-intersection time $\jmath(\theta, t)$ from (5.94), and the curve $\mathfrak{s}_2$ from (5.95), are represented in green

If the curves $(\psi_t(\theta, s), s)_{s\in[0,t]}$ and $(\mathfrak{s}(s), s)_{s\in[0,t]}$, respectively $(\phi_t(\theta, s), s)_{s\in[0,t]}$ and $(\mathfrak{s}(s), s)_{s\in[0,t]}$, do not intersect, then we overload notation and define $\jmath(\theta, t) = \overline{\varepsilon}$, respectively $\mathcal{T}(\theta, t) = \overline{\varepsilon}$.

Implicit in the above definition is the assumption that if the characteristics $\psi_t(\theta, \cdot)$ or $\phi_t(\theta, \cdot)$ intersect the shock curve, then they do so only once; we will indeed prove this holds, due to the *transversality* of these characteristics (Fig. 11).

Definition 5.16 Define $\mathring{\theta}_1, \mathring{\theta}_2 \in \mathbb{T}$ implicitly by the equations $\jmath(\mathring{\theta}_1, \overline{\varepsilon}) = 0$ and $\mathcal{T}(\mathring{\theta}_2, \overline{\varepsilon}) = 0$. For all $t \in [0, \overline{\varepsilon}]$ we define

$$\mathfrak{s}_1(t) = \psi_{\overline{\varepsilon}}(\mathring{\theta}_1, t), \qquad \text{and} \qquad \mathfrak{s}_2(t) = \phi_{\overline{\varepsilon}}(\mathring{\theta}_2, t). \tag{5.95}$$

In particular, $\mathfrak{s}_1(0) = \mathfrak{s}_2(0) = 0$, and $\jmath(\mathfrak{s}_1(t), t) = \mathcal{T}(\mathfrak{s}_2(t), t) = 0$. The spacetime curves $\mathfrak{s}_1(t)$, $\mathfrak{s}_2(t)$, and $\mathfrak{s}(t)$, divide the spacetime region $\mathcal{D}_{\overline{\varepsilon}}$ into four regions with distinct behavior. We also define the sets

$$\mathcal{D}^z_{\overline{\varepsilon}} = \{(\theta, s) \in \mathcal{D}_{\overline{\varepsilon}} \colon \mathfrak{s}_1(s) < \theta < \mathfrak{s}(s)\,, s \in (0, \overline{\varepsilon}]\}\,,$$
$$\mathcal{D}^k_{\overline{\varepsilon}} = \{(\theta, s) \in \mathcal{D}_{\overline{\varepsilon}} \colon \mathfrak{s}_2(s) < \theta < \mathfrak{s}(s)\,, s \in (0, \overline{\varepsilon}]\}\,.$$

Implicit in the above definition is the assumption that the points $\mathring{\theta}_1$ and $\mathring{\theta}_2$ exist, and are uniquely defined; we will indeed prove that this holds, due to the *monotonicity* of $\psi_t(\theta, s)$ and $\phi_t(\theta, s)$ with respect θ, and the the regularity of these curves with respect to y and s.

Definition 5.17 It is convenient to define the vectors

$$\mathcal{U} = (w, z, k, c, a) \qquad \text{and}$$

$$\mathcal{U}_{\mathsf{L}}(t) = (w, z, k, c, a)(\mathfrak{s}(t)^-, t) = (w_-, z_-, k_-, c_-, a_-)(t)\,. \tag{5.96}$$

Remark 5.18 (*Notation for derivatives*) Throughout the remainder of manuscript we shall interchangeably use the following notations for the derivatives of various functions f with respect to the Lagrangian label x or the Eulerian variable θ: $\partial_x f \leftrightarrow f_x, \partial_x^2 f \leftrightarrow f_{xx}, \partial_\theta f \leftrightarrow f_\theta, \partial_\theta^2 f \leftrightarrow f_{\theta\theta}$. Similarly, we shall sometimes denote time derivatives as $\partial_t f \leftrightarrow f_t$. Derivatives for function restricted to the shock curve, shall be denoted as $\frac{d}{dt}(f(\mathfrak{s}(t), t)) = \dot{f}|_{(\mathfrak{s}(t),t)}$; this notation for instance shall be used for the function $\mathcal{U}_{\mathsf{L}}$ defined in (5.96).

5.7.3 Identitities Up to the First Derivative for *w*, *z*, *k*, and *a*

There are particularly useful forms of the equations for w, z, k, and a and their first derivatives. These identities will be used both for designing a simple iteration scheme for the construction of unique solutions, and also for second derivative estimates in Section 6.

Identies for w. Equation (5.91a) can then be written as

$$\tfrac{d}{dt}(w\circ\eta) = \tfrac{1}{4}c\circ\eta\tfrac{d}{dt}(k\circ\eta) - \tfrac{8}{3}(aw)\circ\eta\,. \tag{5.97}$$

Differentiating this equation, we find that

$$\begin{aligned}
\tfrac{d}{dt}(w_\theta\circ\eta\,\eta_x) &= \tfrac{1}{4}c\circ\eta\tfrac{d}{dt}(k_\theta\circ\eta\,\eta_x) + \tfrac{1}{4}c_\theta\circ\eta\,\eta_x(k_t+\lambda_3 k_\theta)\circ\eta - \tfrac{8}{3}\partial_\theta(aw)\circ\eta\,\eta_x\\
&= \tfrac{1}{4}\tfrac{d}{dt}(c\circ\eta\,k_\theta\circ\eta\,\eta_x) - \tfrac{1}{4}(c_t+\lambda_3 c_\theta)\circ\eta\,k_\theta\circ\eta\,\eta_x\\
&\quad + \tfrac{1}{4}c_\theta\circ\eta\,\eta_x(\partial_t k+\lambda_3\partial_\theta k)\circ\eta - \tfrac{8}{3}\partial_\theta(aw)\circ\eta\,\eta_x\\
&= \tfrac{1}{4}\tfrac{d}{dt}\big((ck_\theta)\circ\eta\,\eta_x\big) + \tfrac{1}{6}\big(ck_\theta(z_\theta+c_\theta+4a)\big)\circ\eta\,\eta_x - \tfrac{8}{3}(aw)_\theta\circ\eta\,\eta_x\,.
\end{aligned} \tag{5.98}$$

To obtain the last equality, we have used that (3.7) can be written as

$$\partial_t c + \lambda_3\partial_\theta c = -\tfrac{2}{3}c\partial_\theta z - \tfrac{8}{3}ca\,,$$

and that $\partial_t k = -\lambda_2\partial_\theta k$ with the fact that $\lambda_3 - \lambda_2 = \frac{2}{3}c$. Integrating (5.98) in time, we obtain that

$$w_\theta\circ\eta = \tfrac{w_0'}{\eta_x} + \tfrac{1}{4}(ck_y)\circ\eta + \tfrac{1}{\eta_x}\int_0^t\Big(\tfrac{1}{6}ck_\theta(z_\theta+c_\theta+4a) - \tfrac{8}{3}\partial_\theta(aw)\Big)\circ\eta\,\eta_x dt'\,. \tag{5.99}$$

We wish to emphasize that although (3.5a) appears to have derivative loss on the right side, the structure of (5.91a) leads to the identity (5.99) which shows that there is, in fact, no such loss incurred.

Notice that by expanding the time derivative in (5.97) and using (5.91c), we find that

$$\partial_t w \circ \eta = -w_\theta \circ \eta\, \lambda_3 \circ \eta + \tfrac{1}{6} c^2 k_\theta \circ \eta - \tfrac{8}{3} a w \circ \eta$$

It follows that

$$\begin{aligned}(\partial_t w + \dot{\mathfrak{s}} \partial_\theta w) \circ \eta &= (\dot{\mathfrak{s}} - \lambda_3 \circ \eta) w_\theta \circ \eta + \tfrac{1}{6} c^2 k_\theta \circ \eta - \tfrac{8}{3} a w \circ \eta \\ &= \tfrac{w_0'}{\eta_x} (\dot{\mathfrak{s}} - \lambda_3 \circ \eta) + \tfrac{1}{4} (c k_\theta) \circ \eta (\dot{\mathfrak{s}} - \lambda_3 \circ \eta) \\ &\quad + \tfrac{1}{6} c^2 k_\theta \circ \eta - \tfrac{8}{3} a w \circ \eta \\ &\quad + \tfrac{(\dot{\mathfrak{s}} - \lambda_3 \circ \eta)}{\eta_x} \int_0^t \Big(\tfrac{1}{6} c k_\theta (z_\theta + c_\theta + 4a) - \tfrac{8}{3} \partial_\theta (a w) \Big) \circ \eta\, \eta_x dt' \,. \end{aligned} \tag{5.100}$$

Identies for z and k. Equation (5.91b) can then be written as

$$\tfrac{d}{ds} (z \circ \psi_t) = -\tfrac{1}{4} c \circ \psi_t \tfrac{d}{ds} (k \circ \psi_t) - \tfrac{8}{3} (a z) \circ \psi_t \,. \tag{5.101}$$

Differentiating (5.101), a similar identity to (5.99) holds for $\partial_\theta z$. The analogous computation to (5.98) shows that

$$\begin{aligned}\tfrac{d}{ds} (z_\theta \circ \psi_t \partial_\theta \psi_t) &= -\tfrac{1}{4} \tfrac{d}{ds} \big((c\, k_\theta) \circ \psi_t \partial_\theta \psi_t \big) \\ &\quad - \Big(\tfrac{1}{12} c k_\theta (w_\theta + z_\theta + 8a) + \tfrac{8}{3} \partial_\theta (a z) \Big) \circ \psi_t\ \partial_\theta \psi_t \,, \end{aligned} \tag{5.102}$$

and thus, upon integration in time from $\mathfrak{z}(\theta, t)$ to t, we find that

$$\begin{aligned} z_\theta (y, t) &= \Big(\big(z_\theta (\mathfrak{s}(\mathfrak{z}), \mathfrak{z}) + \tfrac{1}{4} (c k_\theta)(\mathfrak{s}(\mathfrak{z}), \mathfrak{z}) \big) \partial_\theta \psi_t (\mathfrak{s}(\mathfrak{z}), \mathfrak{z}) \\ &\quad + \mathcal{F}_{z_\theta} (U, \psi_t, \mathfrak{z}) \Big) (y, t) \,, \end{aligned} \tag{5.103a}$$

$$\begin{aligned} \mathcal{F}_{z_\theta} (U, \psi_t, \mathfrak{z}) &= -\tfrac{1}{4} (c k_\theta)(\theta, t) \\ &\quad - \int_{\mathfrak{z}(\theta,t)}^t \Big(\tfrac{1}{12} c k_\theta (w_\theta + z_\theta + 8a) + \tfrac{8}{3} \partial_\theta (a z) \Big) \circ \psi_t\ \partial_\theta \psi_t dt' \,. \end{aligned} \tag{5.103b}$$

Again, the identity (5.103) shows that no derivative loss occurs for $\partial_\theta z$ as well. This formula is not yet in its final form. We shall view the given shock curve $(\mathfrak{s}(t), t)$ as a Cauchy surface for both z and k. As such, we shall write the first term on the right in (5.103) in terms of the differentiated *data on the shock curve*, which we now make precise.

The transport equation (5.91c) allows us to write $\frac{d}{ds}(k \circ \phi_t) = 0$, so that integration from $\mathfrak{T}(\theta, t)$ to t shows that for all $(\theta, t) \in \mathcal{D}^k_{\bar{\varepsilon}}$,

$$k(\theta, t) = k(\mathfrak{s}(\mathfrak{T}(\theta, t)), \mathfrak{T}(\theta, t)) \,. \tag{5.104}$$

Differentiation then gives

$$\tfrac{d}{ds}(\partial_\theta k\circ\phi_t\ \partial_\theta\phi_t)=0\,, \tag{5.105}$$

and integration using (5.93c) and (5.94) shows that

$$\partial_\theta k(\theta,t)=\partial_\theta k(\mathfrak{s}(\mathcal{T}(\theta,t)),\mathcal{T}(\theta,t))\ \partial_\theta\phi_t(\mathfrak{s}(\mathcal{T}(\theta,t)),\mathcal{T}(\theta,t))\,. \tag{5.106}$$

Letting $\dot{k_-}(t):=\frac{d}{dt}k_-(t)$ denote differentiation along the shock curve, from (5.91c) we have the coupled system

$$\dot{k_-}(t)=\partial_t k(\mathfrak{s}(t),t)+\dot{\mathfrak{s}}(t)\partial_\theta k(\mathfrak{s}(t),t)\,, \tag{5.107a}$$

$$0=\partial_t k(\mathfrak{s}(t),t)+\lambda_2(\mathfrak{s}(t),t)\partial_\theta k(\mathfrak{s}(t),t)\,. \tag{5.107b}$$

We see that

$$\partial_\theta k(\mathfrak{s}(t),t)=\frac{\dot{k_-}(t)}{\dot{\mathfrak{s}}(t)-\lambda_2(\mathfrak{s}(t),t)}\,, \tag{5.108}$$

and thus with (5.94),

$$\partial_\theta k(\mathfrak{s}(\mathcal{T}(\theta,t)),\mathcal{T}(\theta,t))=\frac{\dot{k_-}(\mathcal{T}(\theta,t)))}{\dot{\mathfrak{s}}(\mathcal{T}(\theta,t)))-\partial_s\phi_t(\theta,\mathcal{T}(\theta,t))}\,. \tag{5.109}$$

Substitution of (5.109) into (5.106) shows that for all $(\theta,t)\in\mathcal{D}^k_{\bar\varepsilon}$,

$$\partial_\theta k(\theta,t)=\frac{\dot{k_-}(\mathcal{T}(\theta,t)))}{\dot{\mathfrak{s}}(\mathcal{T}(\theta,t)))-\partial_s\phi_t(\theta,\mathcal{T}(\theta,t))}\ \partial_\theta\phi_t(\mathfrak{s}(\mathcal{T}(\theta,t)),\mathcal{T}(\theta,t))\,. \tag{5.110}$$

Once again, we let $\dot{z_-}(t)$ denote differentiation along the shock curve so that using (3.5b), we obtain the coupled system

$$\dot{z_-}(t)=\partial_t z(\mathfrak{s}(t),t)+\dot{\mathfrak{s}}(t)\partial_\theta z(\mathfrak{s}(t),t)\,, \tag{5.111a}$$

$$(\tfrac16 c^2\partial_\theta k-\tfrac83 az)(\mathfrak{s}(t),t)=\partial_t z(\mathfrak{s}(t),t)+\lambda_1(\mathfrak{s}(t),t)\partial_\theta z(\mathfrak{s}(t),t)\,. \tag{5.111b}$$

Thus,

$$\partial_\theta z(\mathfrak{s}(t),t)=\frac{\dot{z_-}(t)-\frac16(c^2\partial_\theta k)(\mathfrak{s}(t),t)+\frac83(az)(\mathfrak{s}(t),t)}{\dot{\mathfrak{s}}(t)-\lambda_1(\mathfrak{s}(t),t)}\,, \tag{5.112}$$

and hence with (5.108),

$$\partial_\theta z(\mathfrak{s}(\mathfrak{J}),\mathfrak{J})=\frac{\dot{z_-}(\mathfrak{J})-\frac16\frac{c_-^2(\mathfrak{J})\dot{k_-}(\mathfrak{J})}{\dot{\mathfrak{s}}(\mathfrak{J})-\lambda_2(\mathfrak{s}(\mathfrak{J}),\mathfrak{J})}+\frac83 a_-(\mathfrak{J})z_-(\mathfrak{J})}{\dot{\mathfrak{s}}(\mathfrak{J})-\partial_s\psi_t(\theta,\mathfrak{J})}\,, \tag{5.113}$$

where $\jmath = \jmath(\theta, t)$. We can now substitute (5.109) and (5.113) into (5.103) to conclude that

$$\partial_\theta z = \left(\frac{\dot{z_-}(\jmath) - \frac{1}{6}\frac{c_-^2(\jmath)\dot{k_-}(\jmath)}{\dot{\mathfrak{s}}(\jmath) - \lambda_2(\mathfrak{s}(\jmath),\jmath)} + \frac{8}{3}a_-(\jmath)z_-(\jmath)}{\dot{\mathfrak{s}}(\jmath) - \partial_s\psi_t(\mathfrak{s}(\jmath),\jmath)} + \frac{1}{4}\frac{c_-(\jmath)\dot{k_-}(\jmath)}{\dot{\mathfrak{s}}(\jmath) - \lambda_2(\mathfrak{s}(\jmath),\jmath)} \right) \partial_\theta\psi_t(\mathfrak{s}(\jmath),\jmath) + \mathcal{F}_{z_\theta}\,, \tag{5.114}$$

for any $(\theta, t) \in \mathcal{D}^z_{\overline{\varepsilon}}$. We define

$$\mathcal{H}_{z_\theta}(\mathcal{U}_\mathsf{L}, \dot{\mathcal{U}}_\mathsf{L}, \psi_t, \jmath) := \left(\frac{\dot{z_-}(\jmath) - \frac{1}{6}\frac{c_-^2(\jmath)\dot{k_-}(\jmath)}{\dot{\mathfrak{s}}(\jmath) - \lambda_2(\mathfrak{s}(\jmath),\jmath)} + \frac{8}{3}a_-(\jmath)z_-(\jmath)}{\dot{\mathfrak{s}}(\jmath) - \partial_s\psi_t(\mathfrak{s}(\jmath),\jmath)} + \frac{1}{4}\frac{c_-(\jmath)\dot{k_-}(\jmath)}{\dot{\mathfrak{s}}(\jmath) - \lambda_2(\mathfrak{s}(\jmath),\jmath)} \right) \partial_\theta\psi_t(\mathfrak{s}(\jmath),\jmath)\,, \tag{5.115}$$

so that (5.114) is concisely written as

$$\partial_\theta z = \mathcal{H}_{z_\theta}(\mathcal{U}_\mathsf{L}, \dot{\mathcal{U}}_\mathsf{L}, \psi_t, \jmath) + \mathcal{F}_{z_\theta}(U, \psi_t, \jmath)\,, \tag{5.116}$$

with $\mathcal{F}_{z_\theta}$ and $\mathcal{H}_{z_\theta}$ given by (5.103b) and (5.115), respectively.

Identies for a. We next obtain identities for $\partial_\theta a$, first in $\mathcal{D}_{\overline{\varepsilon}}$. We write (5.91d) as $\partial_t a + \lambda_2\partial_\theta a = -\frac{4}{3}a^2 + \frac{1}{6}(w^2 + z^2) + wz$. We consider this equation along the characteristics ϕ_t and integrate from time $s \in [0, t]$ to t to find that

$$a(\theta, t) = a(\phi_t(\theta, s), s) + \int_s^t \left(-\tfrac{4}{3}a^2 + \tfrac{1}{6}w^2 + \tfrac{1}{6}z^2 + wz\right) \circ \phi_t dr\,. \tag{5.117}$$

Differentiation shows that

$$\partial_\theta a(\theta, t) = \partial_\theta a(\phi_t(\theta, s), s)\partial_\theta\phi_t(\theta, s) + \int_s^t \partial_\theta\left(-\tfrac{4}{3}a^2 + \tfrac{1}{6}w^2 + \tfrac{1}{6}z^2 + wz\right) \circ \phi_t\ \partial_\theta\phi_t dr\,. \tag{5.118}$$

5.8 Construction of Solutions by an Iteration Scheme

5.8.1 Wave Speeds, Characteristics, and Stopping Times

For each $n \geqslant 1$, the three wave speeds are given by

$$\lambda_1^{(n)} = \tfrac{1}{3}w^{(n)} + z^{(n)}\,, \qquad \lambda_2^{(n)} = \tfrac{2}{3}w^{(n)} + \tfrac{2}{3}z^{(n)}\,, \qquad \lambda_3^{(n)} = w^{(n)} + \tfrac{1}{3}z^{(n)}\,. \tag{5.119}$$

For $n \geqslant 1$, we define $\psi_t^{(n)}$ and $\phi_t^{(n)}$ as flows solving

$$\partial_s \psi_t^{(n)}(\theta, s) = \lambda_1^{(n)}(\psi_t^{(n)}(\theta, s), s)\,, \qquad \psi_t^{(n)}(\theta, t) = \theta\,, \tag{5.120a}$$

$$\partial_s \phi_t^{(n)}(\theta, s) = \lambda_2^{(n)}(\phi_t^{(n)}(\theta, s), s)\,, \qquad \phi_t^{(n)}(\theta, t) = \theta\,. \tag{5.120b}$$

We next define $\eta^{(n)}$ to be the solution of

$$\partial_s \eta^{(n)}(x, s) = \lambda_3^{(n)}(\eta^{(n)}(x, s), s)\,, \qquad \eta^{(n)}(x, 0) = x\,. \tag{5.121}$$

Using the characteristics $\phi_t^{(n)}$ and $\psi_t^{(n)}$, we define the shock-intersection times $\mathcal{T}^{(n)}(\theta, t)$ and $\mathcal{J}^{(n)}(\theta, t)$ as in Definition 5.15. Similarly, the curves $\mathfrak{s}_1^{(n)}(t)$ and $\mathfrak{s}_2^{(n)}(t)$ and the spacetime regions $\mathcal{D}_{\overline{\varepsilon}}^{z,(n)}$ and $\mathcal{D}_{\overline{\varepsilon}}^{k,(n)}$ are defined just as in Definition 5.16. The rigorous justification of these definitions is provided in Lemma 5.24.

5.8.2 Specification of the First Iterates

We begin by defining the first iterate $\eta^{(1)}$ associated to the 3-characteristic and $w^{(1)}$ as follows. First, we set

$$\eta^{(1)}(x, s) = \eta_{\mathsf{B}}(x, s) = x + s w_0(x)\,, \tag{5.122}$$

and then define

$$w^{(1)}(\theta, t) = w_{\mathsf{B}}(\theta, t) = w_0(\eta_{\mathrm{inv}}^{(1)}(\theta, t))\,, \qquad z^{(1)} = 0\,, \qquad k^{(1)} = 0\,, \qquad a^{(1)} = a_0\,, \tag{5.123}$$

where $\eta_{\mathrm{inv}}^{(1)} := (\eta^{(1)})^{-1} = \eta_{\mathsf{B}}{}^{-1}$. We also define $\psi_t^{(1)}$ and $\phi_t^{(1)}$ via (5.120a)–(5.120b) as the characteristic flows of the velocity fields $\frac{1}{3} w_{\mathsf{B}}$ and respectively $\frac{2}{3} w_{\mathsf{B}}$.

5.8.3 The Iteration Scheme for $w^{(n+1)}$

We can now state the iteration scheme for all $n \geqslant 1$. We set

$$c^{(n)} = \tfrac{1}{2}(w^{(n)} + z^{(n)})\,,$$

and define $w^{(n+1)}$ as the solution to

$$\tfrac{d}{dt}(w^{(n+1)} \circ \eta^{(n)}) = -\tfrac{8}{3}(a^{(n)} w^{(n)}) \circ \eta^{(n)} + \tfrac{1}{4} c^{(n)} \circ \eta^{(n)} \tfrac{d}{dt}(k^{(n)} \circ \eta^{(n)})\,, \tag{5.124}$$

with initial condition $w^{(n+1)} \circ \eta^{(n)}(x, 0) = w_0(x)$. Integrating in time shows that

$$w^{(n+1)}(\eta^{(n)}(x, t), t)$$

$$= w_0(x) - \tfrac{8}{3}\int_0^t (a^{(n)}w^{(n)})(\eta^{(n)}(x,t'),t')dt' \\ + \tfrac{1}{4}\int_0^t c^{(n)}(\eta^{(n)}(x,t'),t')\tfrac{d}{dt'}\Big(k^{(n)}(\eta^{(n)}(x,t'),t')\Big)\,dt'\,. \tag{5.125}$$

It follows that for all $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}$, $w^{(n+1)}$ is the solution to

$$\partial_s w^{(n+1)} + \lambda_3^{(n)}\partial_\theta w^{(n+1)} = -\tfrac{8}{3}a^{(n)}w^{(n)} + \tfrac{1}{4}c^{(n)}\big(\partial_t k^{(n)} + \lambda_3^{(n)}\partial_\theta k^{(n)}\big)\,, \tag{5.126a}$$

$$w^{(n+1)}(x,0) = w_0(x)\,. \tag{5.126b}$$

In terms of the restrictions of $w^{(n+1)}$ on the left and right sides of shock curve, i.e. $w_-^{(n+1)}(t) = \lim_{\theta\to\mathfrak{s}(t)^-} w^{(n+1)}(\theta,t)$ and respectively $w_+^{(n+1)}(t) = \lim_{\theta\to\mathfrak{s}(t)^+} w^{(n+1)}(\theta,t)$, via Lemma 5.12 we define the functions $z_-^{(n+1)}(t)$ and $k_-^{(n+1)}(t)$ as the solutions of the system of equations (5.67)

$$\mathcal{E}_1(w_-^{(n+1)}, w_+^{(n+1)}, z_-^{(n+1)}, \mathfrak{e}_-^{(n+1)}) = \mathcal{E}_2(w_-^{(n+1)}, w_+^{(n+1)}, z_-^{(n+1)}, \mathfrak{e}_-^{(n+1)}) = 0 \tag{5.127}$$

and $k_-^{(n+1)} = \log(1 + \mathfrak{e}_-^{(n+1)})$.

5.8.4 The Iteration Scheme for $a^{(n+1)}$

For all $n \geqslant 1$ and $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}$, we define $a^{(n+1)}$ to be the solution of the Cauchy problem

$$\partial_t a^{(n+1)} + \lambda_2^{(n)}\partial_\theta a^{(n+1)} = -\tfrac{4}{3}(a^{(n)})^2 + \tfrac{1}{3}(w^{(n)})^2 \\ + \tfrac{1}{3}(z^{(n)})^2 + w^{(n)}z^{(n)}\,, \tag{5.128a}$$

$$a^{(n+1)}(x,0) = a_0(x)\,. \tag{5.128b}$$

In view of (5.117), this function is explicitly given by

$$a^{(n+1)}(\theta,t) = a_0(\phi_t^{(n)}(\theta,0)) \\ + \int_0^t \Big(-\tfrac{4}{3}(a^{(n)})^2 + \tfrac{1}{6}(w^{(n)})^2 + \tfrac{1}{6}(z^{(n)})^2 + w^{(n)}z^{(n)}\Big)(\phi_t^{(n)}(\theta,s),s)ds\,. \tag{5.129}$$

5.8.5 The Iteration Scheme for $z^{(n+1)}$

For all $n \geqslant 1$, and for all $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}^{z,(n)}$ we define $z^{(n+1)}$ to be the solution of the ODE

$$\tfrac{d}{ds}\big(z^{(n+1)} \circ \psi_t^{(n)}\big) = -\tfrac{8}{3}\big(a^{(n)}z^{(n)}\big) \circ \psi_t^{(n)} - \tfrac{1}{4}c^{(n)} \circ \psi_t^{(n)}\tfrac{d}{ds}\big(k^{(n)} \circ \psi_t^{(n)}\big)\,, \tag{5.130a}$$

for all $s \in (\jmath^{(n)}(\theta, t), t]$, with Cauchy data defined on the shock curve by

$$z^{(n+1)}(\psi_t^{(n)}(\theta, \jmath^{(n)}(\theta, t)), \jmath^{(n)}(\theta, t)) = z^{(n+1)}(\mathfrak{s}(\jmath^{(n)}(\theta, t))^-, \jmath^{(n)}(\theta, t)) \\ = z_-^{(n+1)}(\jmath^{(n)}(\theta, t)) \tag{5.130b}$$

where the function z_-^{n+1} is defined on the shock curve $(\mathfrak{s}(t), t)_{t\in[0,\bar{\varepsilon}]}$ as the correct root of (5.127) given by Lemma 5.12. In Eulerian variables, we note that the equation (5.130a) is merely

$$\partial_t z^{(n+1)} + \lambda_1^{(n)} \partial_\theta z^{(n+1)} = -\tfrac{8}{3} a^{(n)} z^{(n)} - \tfrac{1}{4} c^{(n)} (\partial_t k^{(n)} + \lambda_1^{(n)} \partial_\theta k^{(n)}) \tag{5.131}$$

for $(\theta, t) \in \mathcal{D}_{\bar{\varepsilon}}^{z,(n)}$. On the other hand, for $(\theta, t) \in (\mathcal{D}_{\bar{\varepsilon}}^{z,(n)})^{\complement}$, we simply define

$$z^{(n+1)}(\theta, t) = 0 \tag{5.132}$$

which corresponds to the solution of (5.130a) with $k^{(n)} \equiv 0$, and Cauchy data at $t = 0$ given by $z_0 \equiv 0$.

5.8.6 The Iteration Scheme for $k^{(n)}$

Having defined $w^{(n+1)}$ and $z^{(n+1)}$, we solve for $\phi_t^{(n+1)}$ using (5.120b). In turn, this defines the curve $\mathfrak{s}_2^{(n+1)}$, the shock intersection times $\mathcal{T}^{(n+1)}(\theta, t)$, and the region $\mathcal{D}_{\bar{\varepsilon}}^{k,(n+1)}$.

For $n \geqslant 1$ and $(\theta, t) \in \mathcal{D}_{\bar{\varepsilon}}^{k,(n+1)}$, we define $k^{(n+1)}$ to be the solution of

$$\tfrac{d}{ds}\left(k^{(n+1)} \circ \phi_t^{(n+1)}\right) = 0\,, \tag{5.133a}$$

for all $s \in (\mathcal{T}^{(n+1)}(\theta, t), t]$, with Cauchy data defined on the shock curve by

$$k^{(n+1)}(\phi_t^{(n+1)}(\theta, \mathcal{T}^{(n+1)}(\theta, t)), \mathcal{T}^{(n+1)}(\theta, t)) \\ = k^{(n+1)}(\mathfrak{s}(\mathcal{T}^{(n+1)}(\theta, t))^-, \mathcal{T}^{(n+1)}(\theta, t)) = k_-^{(n+1)}(\mathcal{T}^{(n+1)}(\theta, t)) \tag{5.133b}$$

where the function $k_-^{n+1} = \log(1+\mathfrak{e}_-^{(n+1)})$ is defined on the shock curve $(\mathfrak{s}(t), t)_{t\in[0,\bar{\varepsilon}]}$ as the correct root of (5.127) given by Lemma 5.12. In Eulerian variables, we note that the equation (5.133a) is the same as

$$\partial_t k^{(n+1)} + \lambda_2^{(n+1)} \partial_\theta k^{(n+1)} = 0 \tag{5.134}$$

for all $(\theta, t) \in \mathcal{D}_{\bar{\varepsilon}}^{k,(n+1)}$. On the other hand, for $(\theta, t) \in (\mathcal{D}_{\bar{\varepsilon}}^{k,(n+1)})^{\complement}$, we define

$$k^{(n+1)}(\theta, t) = 0\,, \tag{5.135}$$

which is the solution of (5.133a) with Cauchy data at time $t = 0$ given by $k_0 \equiv 0$.

5.8.7 Alternative Forms of the Iteration for $w^{(n+1)}$, $z^{(n+1)}$, and $c^{(n+1)}$

Using that $\partial_t k^{(n)} = -\lambda_2^{(n)} \partial_\theta k^{(n)}$, we can also write (5.126a) and (5.131) as

$$\partial_t w^{(n+1)} + \lambda_3^{(n)} \partial_\theta w^{(n+1)} = -\tfrac{8}{3} a^{(n)} w^{(n)} + \tfrac{1}{6} (c^{(n)})^2 \partial_\theta k^{(n)} , \tag{5.136a}$$

$$\partial_t z^{(n+1)} + \lambda_1^{(n)} \partial_\theta z^{(n+1)} = -\tfrac{8}{3} a^{(n)} z^{(n)} + \tfrac{1}{6} (c^{(n)})^2 \partial_\theta k^{(n)} , \tag{5.136b}$$

and therefore

$$\partial_t c^{(n+1)} = -\tfrac{1}{2} \lambda_3^{(n)} \partial_\theta w^{(n+1)} - \tfrac{1}{2} \lambda_1^{(n)} \partial_\theta z^{(n+1)} - \tfrac{8}{3} a^{(n)} c^{(n)} , \tag{5.137}$$

which has the equivalent forms

$$\partial_t c^{(n+1)} + \lambda_2^{(n)} \partial_\theta c^{(n+1)} + \tfrac{1}{2} c^{(n)} \partial_\theta \lambda_2^{(n+1)} = -\tfrac{8}{3} a^{(n)} c^{(n)} , \tag{5.138a}$$

$$\partial_t c^{(n+1)} + \lambda_3^{(n)} \partial_\theta c^{(n+1)} + \tfrac{2}{3} c^{(n)} \partial_\theta z^{(n+1)} = -\tfrac{8}{3} a^{(n)} c^{(n)} , \tag{5.138b}$$

$$\partial_t c^{(n+1)} + \lambda_1^{(n)} \partial_\theta c^{(n+1)} + \tfrac{2}{3} c^{(n)} \partial_\theta w^{(n+1)} = -\tfrac{8}{3} a^{(n)} c^{(n)} . \tag{5.138c}$$

Although it is not necessary to obtain any estimates, we record at this stage the evolution equation for the specific vorticity given according to (3.8) by $\varpi^{(n)} = 4(w^{(n)} + z^{(n)} - \partial_\theta a^{(n)})(c^{(n)})^{-2} e^{k^{(n)}}$. By combining (5.128a), (5.134), (5.136a), (5.136b), and (5.138a), we obtain

$$\begin{aligned}
&\partial_t \varpi^{(n+1)} + \lambda_2^{(n)} \partial_\theta \varpi^{(n+1)} - \tfrac{8}{3} \tfrac{c^{(n)}}{c^{(n+1)}} a^{(n)} \varpi^{(n+1)} - \tfrac{4}{3} \left(\tfrac{c^{(n)}}{c^{(n+1)}}) \right)^2 \partial_\theta k^{(n)} e^{k^{(n+1)}} \\
&\quad = \left(\tfrac{8}{3} a^{(n)} + \partial_\theta \lambda_2^{(n)} \right) \tfrac{c^{(n)}}{c^{(n+1)}} \left(\varpi^{(n+1)} - \varpi^{(n)} \right) \\
&\qquad + \left(\tfrac{8}{3} a^{(n)} + \partial_\theta \lambda_2^{(n)} \right) \tfrac{c^{(n)}}{(c^{(n+1)})^2} \varpi^{(n)} e^{k^{(n+1)}} \left(c^{(n+1)} e^{-k^{(n+1)}} - c^{(n)} e^{-k^{(n)}} \right) \\
&\qquad + \left(\tfrac{c^{(n)}}{c^{(n+1)}} - \partial_\theta k^{(n+1)} \right) \varpi^{(n+1)} \partial_\theta \left(\lambda_2^{(n+1)} - \lambda_2^{(n)} \right) \\
&\qquad - \tfrac{16}{3} \tfrac{c^{(n)}}{(c^{(n+1)})^2} e^{k^{(n+1)}} \partial_\theta \left(c^{(n+1)} - c^{(n)} \right) + 4 \tfrac{1}{(c^{(n+1)})^2} e^{k^{(n+1)}} \partial_\theta \lambda_2^{(n)} \left(a^{(n+1)} - a^{(n)} \right) .
\end{aligned} \tag{5.139}$$

At this stage we only remark that if $(w, z, k, a, \varpi)^{(n)}$ were to equal $(w, z, k, a, \varpi)^{(n+1)}$, then the right side of (5.139) vanishes, as is natural.

5.8.8 The Iteration Space

We will prove stability under iteration $n \mapsto n + 1$ of the following bound

$$\|\!|(w^{(n)} - w^{(1)}, z^{(n)}, k^{(n)}, a^{(n)})\|\!|_{\overline{\varepsilon}} \leqslant 1 \tag{5.140}$$

where the norm $\|\!|\cdot|\!\|_{\overline{\varepsilon}}$ is as defined in (5.10). For convenience of the reader, we recall that (5.140) means

$$\left|w^{(n)}(\theta,t)-w^{(1)}(\theta,t)\right|\leqslant R_1 t \tag{5.141a}$$

$$\left|\partial_\theta w^{(n)}(\theta,t)-\partial_\theta w^{(1)}(\theta,t)\right|\leqslant R_2\left(\mathsf{b}^3t^3+(\theta-\mathfrak{s}(t))^2\right)^{-\frac16} \tag{5.141b}$$

$$\left|z^{(n)}(\theta,t)\right|\leqslant R_3 t^{\frac32} \tag{5.141c}$$

$$\left|\partial_\theta z^{(n)}(\theta,t)\right|\leqslant R_4 t^{\frac12} \tag{5.141d}$$

$$\left|k^{(n)}(\theta,t)\right|\leqslant R_5 t^{\frac32} \tag{5.141e}$$

$$\left|\partial_\theta k^{(n)}(\theta,t)\right|\leqslant R_6 t^{\frac12} \tag{5.141f}$$

$$\left|a^{(n)}(\theta,t)\right|+\left|\partial_\theta a^{(n)}(\theta,t)\right|\leqslant R_7\,, \tag{5.141g}$$

for all $(\theta,t)\in\mathcal{D}_{\overline{\varepsilon}}$, where

$$R_1=50\mathsf{m}^2\,,\quad R_2=\mathsf{m}^3\,,\quad R_3=R_4=\mathsf{m}\,,\quad R_5=R_6=\mathsf{m}^{\frac12}\,,\quad R_7=4\mathsf{m}\,. \tag{5.142}$$

Lemma 5.19 *Assume that* $(w^{(n)},z^{(n)},k^{(n)},a^{(n)})\in\mathcal{X}_{\overline{\varepsilon}}$. *Then for all* $(\theta,t)\in\mathcal{D}_{\overline{\varepsilon}}$,

$$\left|\partial_t w^{(n)}(\theta,t)-\partial_t w^{(1)}(\theta,t)\right|\leqslant 3\mathsf{m}^4\left((\theta-\mathfrak{s}(t))^2+t^3\right)^{-\frac16}\,. \tag{5.143}$$

Proof of Lemma 5.19 Using the identity (5.136a) and the fact that $\partial_t w^{(1)}+w^{(1)}\partial_\theta w^{(1)}=0$, we have that

$$\begin{aligned}\partial_t w^{(n)}-\partial_t w^{(1)}&=-\lambda_3^{(n-1)}(\partial_\theta w^{(n)}-\partial_\theta w^{(1)})-(\lambda_3^{(n-1)}-w^{(1)})\partial_\theta w^{(1)}\\&\quad-\tfrac83 a^{(n-1)}w^{(n-1)}+\tfrac16(c^{(n-1)})^2\partial_\theta k^{(n-1)}\,,\end{aligned} \tag{5.144}$$

Now from (5.37a) and (5.141), we have that for $\overline{\varepsilon}$ taken sufficiently small,

$$\begin{aligned}&\left|(\lambda_3^{(n-1)}-w^{(1)})\partial_\theta w^{(1)}\right|\leqslant R_1\,,\quad \tfrac83\left|a^{(n-1)}w^{(n-1)}\right|\leqslant 3\mathsf{m}R_7\,,\quad\text{and}\\&\tfrac16\left|(c^{(n-1)})^2\partial_\theta k^{(n-1)}\right|\lesssim t^{\frac12}\,.\end{aligned}$$

Then from (5.141b) and with $\overline{\varepsilon}$ taken even small, we have that

$$\begin{aligned}\left|\partial_t w^{(n)}-\partial_t w^{(1)}\right|&\leqslant 2\mathsf{m}R_2\left(\mathsf{b}^3t^3+(\theta-\mathfrak{s}(t))^2\right)^{-\frac16}+3\mathsf{m}R_7+R_1\\&\leqslant 3\mathsf{m}^4\left(\mathsf{b}^3t^3+(\theta-\mathfrak{s}(t))^2\right)^{-\frac16}\,,\end{aligned}$$

where we have used (5.2), and that $t\leqslant\overline{\varepsilon}$. Hence, we obtain the bound (5.143). □

5.8.9 The Behavior of $w^{(n)}$, $z^{(n)}$, and $k^{(n)}$ on the Shock Curve

Lemma 5.20 *Assume that* $(w^{(n-1)}, z^{(n-1)}, k^{(n-1)}, a^{(n-1)}) \in \mathcal{X}_{\overline{\varepsilon}}$ *and that* $w^{(n)} \in \mathcal{X}_{\overline{\varepsilon}}$. *Then for all* $t \in (0, \overline{\varepsilon}]$ *we have*

$$\left|[\![w^{(n)}(t)]\!] - [\![w^{(1)}(t)]\!]\right| \leqslant 2R_1 t\,, \qquad \left|\langle\!\langle w^{(n)}(t)\rangle\!\rangle - \langle\!\langle w^{(1)}(t)\rangle\!\rangle\right| \leqslant R_1 t\,, \tag{5.145a}$$

$$\left|\tfrac{d}{dt}[\![w^{(n)}]\!](t) - \tfrac{d}{dt}[\![w^{(1)}]\!](t)\right| \leqslant 2R_1, \qquad \left|\tfrac{d}{dt}\langle\!\langle w^{(n)}\rangle\!\rangle(t) - \tfrac{d}{dt}\langle\!\langle w^{(1)}\rangle\!\rangle(t)\right| \leqslant R_1\,, \tag{5.145b}$$

where R_1 *is as defined in (5.142). In particular, in view of (5.15a) and (5.15b), we have that*

$$\left|[\![w^{(n)}]\!](t) - 2\mathsf{b}^{\frac{3}{2}}t^{\frac{1}{2}}\right| \leqslant 3\mathsf{m}^3 t\,, \qquad \left|\langle\!\langle w^{(n)}\rangle\!\rangle(t) - \kappa\right| \leqslant \tfrac{1}{2}\mathsf{m}^4 t\,, \tag{5.146a}$$

$$\left|\tfrac{d}{dt}[\![w^{(n)}]\!](t) - \mathsf{b}^{\frac{3}{2}}t^{-\frac{1}{2}}\right| \leqslant 3\mathsf{m}^4, \qquad \left|\tfrac{d}{dt}\langle\!\langle w^{(n)}\rangle\!\rangle(t)\right| \leqslant 2\mathsf{m}^4\,, \tag{5.146b}$$

for all $t \in (0, \overline{\varepsilon}]$.

Proof of Lemma 5.20 By assumption, $w^{(n)}$ satisfies the bound (5.141a), and so the inequalities in (5.145a) follow. In order to prove (5.145b), we shall use that $\left|[\![w^{(n-1)}(t)]\!]\right| \leqslant \left|[\![w^{(1)}(t)]\!]\right| + \left|[\![w^{(n-1)}(t) - w^{(1)}(t)]\!]\right|$, and hence by (5.15a) and (5.141a),

$$\left|[\![w^{(n-1)}(t)]\!]\right| \leqslant \tfrac{21}{10}\mathsf{b}^{\frac{3}{2}}t^{\frac{1}{2}} + 2R_1 t \leqslant \tfrac{11}{5}\mathsf{b}^{\frac{3}{2}}t^{\frac{1}{2}} \tag{5.147}$$

where we have taken $\overline{\varepsilon}$ sufficiently small for the last inequality. Next, we have that from (5.144),

$$\begin{aligned}
&\tfrac{d}{dt}[\![w]\!]^{(n)} - \tfrac{d}{dt}[\![w^{(1)}]\!] \\
&\quad= [\![\partial_t w^{(n)} - \partial_t w^{(1)}]\!] + \dot{\mathfrak{s}}[\![\partial_\theta w^{(n)} - \partial_\theta w^{(1)}]\!] \\
&\quad= \big(\dot{\mathfrak{s}}(t) - w^{(1)}\big)[\![\partial_\theta w^{(n)} - \partial_\theta w^{(1)}]\!] + \big(\lambda_3^{(n-1)} - w^{(1)}\big)[\![\partial_\theta w^{(n)} - \partial_\theta w^{(1)}]\!] \\
&\qquad - [\![\lambda_3^{(n-1)}]\!](\partial_\theta w^{(n)} - \partial_\theta w^{(1)}) \\
&\qquad - (\lambda_3^{(n-1)} - w^{(1)})[\![\partial_\theta w^{(1)}]\!] - [\![\lambda_3^{(n-1)} - w^{(1)}]\!]\partial_\theta w^{(1)} \\
&\qquad - \tfrac{8}{3}a^{(n-1)}[\![w^{(n-1)}]\!] + \tfrac{1}{6}[\![(c^{(n-1)})^2\partial_\theta k^{(n-1)}]\!]\,.
\end{aligned}$$

By (5.1c), (5.13), and (5.20), we see that $w^{(1)} = w_{\mathsf{B}}$ evaluated on the shock curve, $\left|\dot{\mathfrak{s}} - w^{(1)}\right| = \mathcal{O}(t)$. Thus, using the bounds (5.141) and (5.147) shows that

$$\left|\tfrac{d}{dt}[\![w]\!]^{(n)} - \tfrac{d}{dt}[\![w^{(1)}]\!]\right| \leqslant \left|[\![\lambda_3^{(n-1)} - w^{(1)}]\!]\partial_\theta w^{(1)}\right| + Ct^{\frac{1}{2}} \leqslant 2R_1\,,$$

for $\overline{\varepsilon}$ taken sufficiently small. This proves the first bound in (5.145b), while the second follows similarly. □

Having established Lemma 5.20, the conditions of Lemmas 5.12, Lemma 5.13, and Corollary 5.14 are satisfied, which together yield

Lemma 5.21 ($z_-^{(n)}$ and $k_-^{(n)}$ on the shock curve) *Let $w^{(n)}$ be as in Lemma 5.20. Applying Lemma 5.12, on the shock curve we define $z_-^{(n)}$ and $k_-^{(n)}$ as the solutions of (5.127) with n replacing $n+1$. In particular, $z_-^{(n)}$ and $k_-^{(n)}$ are explicit functions of $[\![w^{(n)}]\!]$ and $\langle\!\langle w^{(n)}\rangle\!\rangle$ and satisfy the following bounds:*

$$\left|z_-^{(n)}(t) + \tfrac{9[\![w^{(n)}]\!](t)^3}{16\langle w^{(n)}\rangle(t)^2}\right| \leqslant C_0 t^{\frac{5}{2}}\,. \tag{5.148a}$$

$$\left|k_-^{(n)}(t) - \tfrac{4[\![w^{(n)}]\!](t)^3}{\langle w^{(n)}\rangle(t)^3}\right| \leqslant C_0 t^{\frac{5}{2}}\,, \tag{5.148b}$$

where $C_0 = C_0(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$ is an explicitly computable constant. Moreover,

$$\left|z_-^{(n)}(t)\right| \leqslant 5\mathsf{b}^{\frac{9}{2}}\kappa^{-2}t^{\frac{3}{2}}\,, \qquad \left|k_-^{(n)}(t)\right| \leqslant 40\mathsf{b}^{\frac{9}{2}}\kappa^{-3}t^{\frac{3}{2}}\,, \tag{5.149a}$$

$$\left|\tfrac{d}{dt}z_-^{(n)}(t)\right| \leqslant 8\mathsf{b}^{\frac{9}{2}}\kappa^{-2}t^{\frac{1}{2}}\,, \qquad \left|\tfrac{d}{dt}k_-^{(n)}(t)\right| \leqslant 50\mathsf{b}^{\frac{9}{2}}\kappa^{-3}t^{\frac{1}{2}}\,, \tag{5.149b}$$

for all $t \in (0, \overline{\varepsilon}]$, assuming that $\overline{\varepsilon}$ is sufficiently small.

5.8.10 Existence, Uniqueness, and Invertibility of Characteristics

The following lemma follows from (5.39)–(5.45) and Lemma 5.10.

Lemma 5.22 (Bijection set of labels) *Assume that $(w^{(n)}, z^{(n)}, k^{(n)}, a^{(n)}) \in \mathcal{X}_{\overline{\varepsilon}}$. Then, for each $t \in (0, \overline{\varepsilon}]$, there exists a* largest *$x_+^{(n)}(t) > 0$ and a* smallest *$x_-^{(n)} = x_-(t) < 0$ such that*

$$\mathfrak{s}(t) = \eta^{(n)}(x_\pm^{(n)}(t), t) \tag{5.150}$$

where

$$-\tfrac{6}{5}(\mathsf{b}t)^{\frac{3}{2}} < x_-^{(n)}(t) < -\tfrac{4}{5}(\mathsf{b}t)^{\frac{3}{2}} \quad \textit{and} \quad \tfrac{4}{5}(\mathsf{b}t)^{\frac{3}{2}} < x_+^{(n)}(t) < \tfrac{6}{5}(\mathsf{b}t)^{\frac{3}{2}}\,. \tag{5.151}$$

Furthermore, there exists a set of labels

$$\Upsilon^{(n)}(t) = \mathbb{T} \setminus [x_-^{(n)}(t), x_+^{(n)}(t)]\,,$$

such that $\eta^{(n)}(\cdot, t)\colon \Upsilon^{(n)} \to \mathbb{T}\setminus\{\mathfrak{s}(t)\}$ is a bijection, and the inverse map $\eta_{\mathrm{inv}}^{(n)}\colon \mathcal{D}_{\overline{\varepsilon}} \to \mathbb{T}\setminus\{0\}$ is continuous in spacetime.

Lemma 5.23 (Bounds for 3-characteristics) *Assume that $(w^{(n)}, z^{(n)}, k^{(n)}, a^{(n)}) \in \mathcal{X}_{\overline{\varepsilon}}$. Then, we have*

$$\tfrac{1}{2} \leqslant \partial_x \eta^{(n)}(x, t) \leqslant \tfrac{7}{4}, \qquad \textit{for all} \quad x \in \Upsilon^{(n)}\,, \tag{5.152a}$$

$$\left|\eta^{(n)}(x,t) - \eta^{(1)}(x,t)\right| \leqslant \tfrac{3}{2}\mathsf{R}_1 t^2, \qquad \textit{for all} \quad x \in \Upsilon^{(n)}, \tag{5.152b}$$

$$\left|\partial_x \eta^{(n)}(x,t) - \partial_x \eta^{(1)}(x,t)\right| \leqslant (16\mathsf{R}_1 \mathsf{b}^{-\frac{3}{2}} + 8\mathsf{R}_2) t^{\frac{1}{2}}, \qquad \textit{for all} \quad x \in \Upsilon^{(n)}, \tag{5.152c}$$

and

$$\int_0^t \left|\partial_\theta w^{(1)}(\eta^{(n)}(x,s),s)\right| ds \leqslant \tfrac{19}{40}. \tag{5.153}$$

Proof of Lemma 5.23 From Lemma 5.22, all of the conditions of Lemma 5.10 hold, so the stated inequalities are thus obtained. ◻

Lemma 5.24 *For $n \geqslant 1$, assume that $w^{(n)}$ and $z^{(n)}$ satisfy the bounds (5.141a)–(5.141d).*

Then, for every $(\theta,t) \in \mathcal{D}_{\overline{\varepsilon}}$ there exists a unique Lipschitz smooth integral curve $\psi_t^{(n)}(\theta,\cdot)\colon [0,t] \to \mathcal{D}_{\overline{\varepsilon}}$ satisfying (5.120a). There exists a unique point $\mathring{\theta}_1^{(n)} \in \mathbb{T}$ such that $\psi_{\overline{\varepsilon}}^{(n)}(\mathring{\theta}_1^{(n)},0) = 0$, which allows us to define as in Definition 5.16 the curve $\mathfrak{s}_1^{(n)}$ and the space-time region $\mathcal{D}_{\overline{\varepsilon}}^{z,(n)}$. For every $(\theta,t) \in \mathcal{D}_{\overline{\varepsilon}}^{z,(n)}$, there exists a unique shock-intersection time $0 < \mathfrak{z}^{(n)}(\theta,t) < t$ satisfying (5.94). Moreover, for $(\theta,t) \in (\overline{\mathcal{D}_{\overline{\varepsilon}}^{z,(n)}})^{\complement}$, the characteristic curve $(\psi_t^{(n)}(\theta,s),s)_{s\in[0,t]}$ does not intersect the shock curve $(\mathfrak{s}(s),s)_{s\in[0,t]}$.

Similarly, for every $(\theta,t) \in \mathcal{D}_{\overline{\varepsilon}}$ there exists a unique Lipschitz smooth integral curve $\phi_t^{(n)}(\theta,\cdot)\colon [0,t] \to \mathcal{D}_{\overline{\varepsilon}}$ satisfying (5.120b). There exists a unique point $\mathring{y}_2^{(n)} \in \mathbb{T}$ such that $\phi_{\overline{\varepsilon}}^{(n)}(\mathring{\theta}_2^{(n)},0) = 0$, which allows us to define as in Definition 5.16 the curve $\mathfrak{s}_2^{(n)}$ and the space-time region $\mathcal{D}_{\overline{\varepsilon}}^{k,(n)}$. For every $(\theta,t) \in \mathcal{D}_{\overline{\varepsilon}}^{k,(n)}$, there exists a unique shock-intersection time $0 < \mathfrak{T}^{(n)}(\theta,t) < t$ satisfying (5.94). Moreover, for $(\theta,t) \in (\overline{\mathcal{D}_{\overline{\varepsilon}}^{k,(n)}})^{\complement}$, the characteristic curve $(\phi_t^{(n)}(\theta,s),s)_{s\in[0,t]}$ does not intersect the shock curve $(\mathfrak{s}(s),s)_{s\in[0,t]}$.

Lastly, we have the estimates

$$\begin{aligned}
\psi_t^{(n)}(\theta,s) &= \tfrac{1}{3}\kappa s + (\theta - \tfrac{1}{3}kt) + \mathcal{O}(t^{\frac{4}{3}}) \\
&= \tfrac{1}{3}\kappa s + (\theta - \mathfrak{s}_1^{(n)}(t)) + \mathcal{O}(t^{\frac{4}{3}}), \quad (\theta,t) \in \mathcal{D}_{\overline{\varepsilon}}^{z,(n)}, \qquad (5.154a) \\
\phi_t^{(n)}(\theta,s) &= \tfrac{2}{3}\kappa s + (\theta - \tfrac{2}{3}kt) + \mathcal{O}(t^{\frac{4}{3}}) \\
&= \tfrac{2}{3}\kappa s + (\theta - \mathfrak{s}_2^{(n)}(t)) + \mathcal{O}(t^{\frac{4}{3}}), \quad (\theta,t) \in \mathcal{D}_{\overline{\varepsilon}}^{z,(n)}, \qquad (5.154b)
\end{aligned}$$

and

$$\sup_{s\in[0,t]} \left|\partial_\theta \phi_t^{(n)}(\theta,s) - 1\right| \leqslant C t^{\frac{1}{3}}, \qquad \sup_{s\in[0,t]} \left|\partial_\theta \psi_t^{(n)}(\theta,s) - 1\right| \leqslant C t^{\frac{1}{3}}, \qquad (\theta,t) \in \mathcal{D}_{\overline{\varepsilon}}, \tag{5.155}$$

where the constant $C > 0$ only depends on κ, b, and m.

Proof of Lemma 5.24 We prove the lemma for the 1-characteristics $\psi_t^{(n)}$, the proof for the 2-characteristics $\phi_t^{(n)}$ being exactly the same.

We begin with the existence and uniqueness of 1-characteristics passing through any point $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}$. Using the definition (5.120a), we see that

$$\begin{aligned}\partial_s \psi_t^{(n)}(\theta, s) &= \lambda_1^{(n)}(\psi_t^{(n)}(\theta, s), s) \\ &= \tfrac{1}{3} w^{(1)}(\psi_t^{(n)}(\theta, s), s) \\ &\quad + \left(\tfrac{1}{3}(w^{(n)} - w^{(1)}) + z^{(n)}\right)(\psi_t^{(n)}(\theta, s), s)\,, \end{aligned} \tag{5.156a}$$

$$\psi_t^{(n)}(\theta, t) = \theta\,, \tag{5.156b}$$

where we recall cf. (5.123) that $w^{(1)} = w_{\mathsf{B}}$ and $z^{(1)} = 0$. The bounds (5.37a), (5.141b), and (5.141d) show that $\lambda_1^{(n)}$ is Lipschitz continuous in $\mathcal{D}_{\overline{\varepsilon}}$; moreover, as long as $\psi_t^{(n)}(\theta, s) \in \mathcal{D}_{\overline{\varepsilon}}$, we have the explicit estimate

$$\begin{aligned} &\left|\partial_\theta \lambda_1^{(n)}(\psi_t^{(n)}(\theta, s), s)\right| \\ &\leqslant \tfrac{4}{15}\left((\mathsf{b}s)^3 + \left|\psi_t^{(n)}(\theta, s) - \mathfrak{s}(s)\right|^2\right)^{-\frac{1}{3}} \\ &\quad + \tfrac{1}{3} R_2 \left((\mathsf{b}s)^3 + \left|\psi_t^{(n)}(\theta, s) - \mathfrak{s}(s)\right|^2\right)^{-\frac{1}{6}} + R_4 s^{\frac{1}{2}} \\ &\leqslant \tfrac{1}{3}\mathsf{b}\left((\mathsf{b}s)^3 + \left|\psi_t^{(n)}(\theta, s) - \mathfrak{s}(s)\right|^2\right)^{-\frac{1}{3}} + 2\mathsf{m}^3\,. \end{aligned} \tag{5.157}$$

Hence, by the Cauchy-Lipschitz theorem, for each such $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}$, there is a unique local in time solution time to (5.156). Using (5.157) and the bound $|\lambda_1^{(n)}| \leqslant \frac{1}{2}\mathsf{m}$, this solution $\psi_t^{(n)}(\theta, s)$ may be *maximally extended* as a Lipschitz function of s on the time interval $[s_*, t]$, where $\psi_t(\theta, s_*) \in \partial\mathcal{D}_{\overline{\varepsilon}}$. In our case, this means that either $[s_*, t] = [0, t]$ (if $(\psi_t^{(n)}(s), s)$ does not intersect the shock curve $(\mathfrak{s}(s), s)$ for $s \in (0, t]$), or $[s_*, t] = [\jmath^{(n)}(\theta, t), t]$, where we have denoted by $\jmath^{(n)}(\theta, t) \in [0, t)$ the *largest* value of s at which $\psi_t^{(n)}(\theta, s) = \mathfrak{s}(s)$. Of course, if $t < \overline{\varepsilon}$ the solution $\psi_t(\theta, s)$ may also be similarly maximally extended to times s past t, up to the time s^* at which $\psi_t(\theta, s^*)$ reaches $\partial\mathcal{D}_{\overline{\varepsilon}}$.

In order to complete the existence and uniqueness part claimed in Lemma 5.24, we need to show that if $\jmath^{(n)}(\theta, t) \in (0, t)$, then the integral curve may be uniquely continued as a Lipschitz function of s also on the time interval $[0, \jmath^{(n)}(\theta, t)]$. We note that in this case the limit $\lim_{s \to \jmath^{(n)}(\theta, t)^+} \psi_t^{(n)}(\theta, s)$ is well-defined, and so to ensure continuity we let $\psi_t^{(n)}(\theta, \jmath^{(n)}(\theta, t))$ equal this limit. The desired claim follows once we prove the following two statements: first, that the shock surface $(\mathfrak{s}(s), s)_{s \in [0, t]}$ is a non-characteristic surface for the ODE (5.156), so that $\psi_t^{(n)}(\theta, \jmath^{(n)}(\theta, t)) = \mathfrak{s}(\jmath^{(n)}(\theta, t)$ may serve as Cauchy data for the *transversal characteristic* $\psi_t(\theta, s)$ with $s < \jmath^{(n)}(\theta, t)$; second, that the curve $\psi_t(\theta, s)$ does not intersect the shock curve for $s \in [0, \jmath^{(n)}(\theta, t))$, thereby ensuring the uniqueness/well-definedness of $\jmath^{(n)}(\theta, t)$ implicitly assumed in Definition 5.15.

The *transversality* of $\psi_t^{(n)}$ and the shock surface is established as follows. We first carefully estimate $\lambda_1^{(n)}$ in the vicinity of the shock curve. By (5.35), (5.22b), (5.141a), and (5.141c), for any $\widetilde{\theta}$ such that $|\widetilde{\theta} - \mathfrak{s}(s)| \leqslant \kappa t$ we have that

$$\left|\lambda_1^{(n)}(\widetilde{\theta}, s) - \tfrac{1}{3}\kappa\right| \leqslant \tfrac{1}{3}\left|w_0(\eta_{\mathsf{B}}{}^{-1}(\widetilde{\theta}, s)) - \kappa\right| + \tfrac{1}{3}R_1 s + R_3 s^{\frac{3}{2}} \leqslant \tfrac{3}{2}\mathsf{b}(\kappa t)^{\frac{1}{3}}, \quad (5.158)$$

since $\overline{\varepsilon}$, and hence $s \leqslant t$, are sufficiently small. Note that if $|\widetilde{\theta} - \mathfrak{s}(s)| \leqslant \kappa s$, then in the upper bound (5.158) we may replace $t^{\frac{1}{3}}$ by $s^{\frac{1}{3}}$. Next, we note that the vector normal to the shock curve is given by $\big(-1, \dot{\mathfrak{s}}(s)\big)$ while the tangent vector to the characteristic curve is given by $\big(\partial_s \psi_t^{(n)}(\theta, s), 1\big) = \big(\lambda_1^{(n)}(\psi_t^{(n)}(\theta, s), s), 1\big)$. Computing the dot-product, and appealing to (5.13) and (5.158), we obtain that

$$\big(-1, \dot{\mathfrak{s}}(s)\big) \cdot \big(\partial_s \psi_t^{(n)}(s), 1\big) = \dot{\mathfrak{s}}(s) - \lambda_1^{(n)}(\psi_t^{(n)}(\theta, s), s) = \tfrac{2}{3}\kappa + \mathcal{O}(s^{\frac{1}{3}}) \geqslant \tfrac{1}{2}\kappa\,, \quad (5.159)$$

since $\overline{\varepsilon}$ is small enough, and $s = \mathfrak{z}^{(n)}(\theta, t)$. Therefore, the characteristic curve $\psi_t^{(n)}$ intersects the shock curve transversally, and the crossing angle is bounded from below uniformly for on $[0, \overline{\varepsilon}]$. As mentioned above, this means that we can use the values of the flows $\psi_t^{(n)}$ on the shock curve as Cauchy data, and continue the solutions in a Lipschitz fashion for $s < \mathfrak{z}^{(n)}(\theta, t)$. The fact that the angle measured in (5.159) has a sign, and the smoothness of $\mathfrak{s}$, also ensures the uniqueness of the *shock-intersection time* $\mathfrak{z}(\theta, t) \in (0, t)$, so that it is a well-defined object. This concludes the proof of existence, uniqueness, and Lipschitz regularity for the characteristic curves $\psi_t^{(n)}(\theta, \cdot) \colon [0, t] \to \mathbb{T}$.

Next, we turn to the proof of the bound (5.155). Differentiating (5.156) shows that

$$\begin{aligned}\partial_\theta \psi_t^{(n)}(\theta, s) &= e^{\int_s^t (\partial_\theta \lambda_1^{(n)})(\psi_t^{(n)}(\theta, s'), s') ds'} \\ &= e^{\frac{1}{3}\int_s^t \partial_\theta w^{(1)}(\psi_t^{(n)}(\theta, s'), s') ds'} e^{\int_s^t \left(\frac{1}{3}(\partial_\theta w^{(n)} - \partial_\theta w^{(1)}) + \partial_\theta z^{(n)}\right)(\psi_t^{(n)}(\theta, s'), s') ds'}\,. \end{aligned} \quad (5.160)$$

For $s' \in [s, t]$ such that $|\psi_t^{(n)}(\theta, s') - \mathfrak{s}(s')| \geqslant \kappa t$, from (5.157) we deduce that $|\partial_\theta \lambda_1^{(n)}(\psi_t^{(n)}(\theta, s'), s')| \leqslant \frac{1}{3}\mathsf{b}(\kappa t)^{-\frac{2}{3}} + 2\mathsf{m}^3 \leqslant \frac{2}{3}\mathsf{b}(\kappa t)^{-\frac{2}{3}}$, and thus the contribution from such s' to the integral on the right side of (5.160) is bounded from above by $\exp(2\mathsf{b}\kappa^{-\frac{2}{3}} t^{\frac{1}{3}})$. On the other hand, $s' \in [s, t]$ such that $|\psi_t^{(n)}(\theta, s') - \mathfrak{s}(s')| \leqslant \kappa t$, we may appeal to (5.158), so that $\partial_s \psi_t^{(n)}(\theta, s') \leqslant \frac{1}{2}s'$; this allows us to apply Lemma 5.11 with $\gamma = \psi_t^{(n)}(\theta, \cdot)$ and $\mu = \frac{1}{2}$, for these intervals of s', and together with the bounds (5.141) we deduce that the contribution from such s' to the integral on the right side of (5.160) is bounded from above by $\exp(30\mathsf{b}\kappa^{-\frac{2}{3}} t^{\frac{1}{3}})$. Combining these estimates we deduce that for all $s \in [0, t]$ and $t \in (0, \overline{\varepsilon}]$,

$$\left|\partial_\theta \psi_t^{(n)}(\theta, s) - 1\right| \leqslant 40\mathsf{b}\kappa^{-\frac{2}{3}} t^{\frac{1}{3}}\,, \quad (5.161)$$

when $\overline{\varepsilon}$ is sufficiently small. This proves (5.155) for the flow $\psi_t^{(n)}$, which implies that $\psi_t^{(n)}$ is continuous on $\mathbb{T} \times [0, t]$, and is uniformly Lipschitz continuous both with respect to θ and with respect to s.

The bound (5.161) does not just provide regularity with respect to θ of the flow $\psi_t^{(n)}(\theta, s)$, but it also shows that it is a monotone increasing function of θ. This allows us to show the existence and uniqueness of a point $\mathring{\theta}_1^{(n)} \in \mathbb{T}$ such that $\psi_{\overline{\varepsilon}}^{(n)}(\mathring{\theta}_1^{(n)}, 0) = 0$. Existence follows by the intermediate function theorem, applied to $\psi_{\overline{\varepsilon}}^{(n)}(\theta, 0)\colon \mathbb{T} \to \mathbb{T}$: indeed, from (5.158) (applied with $t = \overline{\varepsilon}$) and (5.13), we see that for $\mathfrak{s}(\overline{\varepsilon}) < \theta$ we have $\psi_{\overline{\varepsilon}}^{(n)}(\theta, 0) \geqslant \frac{1}{4}\kappa\overline{\varepsilon} > 0$; on the other hand, for $\theta < \mathfrak{s}(\overline{\varepsilon}) - \frac{3}{4}\kappa\overline{\varepsilon}$, we have $\psi_{\overline{\varepsilon}}^{(n)}(\theta, 0) \leqslant -\frac{1}{8}\kappa\overline{\varepsilon} < 0$. The uniqueness of $\mathring{\theta}_1^{(n)}$ follows by the monotonicity in θ guaranteed by (5.161). Note that the above argument gives the rough bound $\mathfrak{s}(\overline{\varepsilon}) - \frac{3}{4}\kappa\overline{\varepsilon} \leqslant \mathring{\theta}_1^{(n)} \leqslant \mathfrak{s}(\overline{\varepsilon})$.

Thus, as in Definition 5.16 the curve $\mathfrak{s}_1^{(n)}$ and the space-time region $\mathcal{D}_{\overline{\varepsilon}}^{z,(n)}$ are now well-defined. The fact that for $(\theta, t) \in (\overline{\mathcal{D}_{\overline{\varepsilon}}^{z,(n)}})^{\complement}$ the curve $(\psi_t^{(n)}(\theta, s), s)_{s\in[0,t]}$ does not intersect the shock curve $(\mathfrak{s}(s), s)_{s\in[0,t]}$, and the fact that for $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}^{z,(n)}$ intersection does indeed occur at a unique time $\jmath^{(n)}(\theta, t)$, now follows from the monotonicity of $\psi_t^{(n)}(\theta, s)$ with respect to θ, the definition of $\mathfrak{s}_1^{(n)}$, the transversality (5.159), and its consequences discussed earlier.

In order to conclude the proof, it remains to establish (5.154). From the aforementioned rough bound on $\mathring{\theta}_1^{(n)}$, appealing to the definition $\mathfrak{s}_1(s) = \psi_{\overline{\varepsilon}}^{(n)}(\mathring{\theta}_1^{(n)}, s)$, the bound (5.13), integrating (5.158) with $\widetilde{\theta} = \mathfrak{s}_1^{(n)}(s)$, and using that $\mathfrak{s}_1^{(n)}(0) = 0 = \mathfrak{s}(0)$, we see that

$$\begin{aligned}\left|\mathfrak{s}(s) - \mathfrak{s}_1^{(n)}(s) - \tfrac{2}{3}\kappa s\right| &\leqslant \left|\mathfrak{s}_1^{(n)}(s) - \tfrac{1}{3}\kappa s\right| + \left|\mathfrak{s}(s) - \kappa s\right| \\ &\leqslant \tfrac{3}{2}\mathsf{b}\kappa^{-\frac{1}{3}} s^{\frac{4}{3}} + \mathsf{m}^4 s^2 \leqslant 2\mathsf{b}\kappa^{-\frac{1}{3}} s^{\frac{4}{3}} \qquad \text{for all} \qquad s \in [0, \overline{\varepsilon}]\,.\end{aligned} \tag{5.162}$$

More generally, for any $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}^{z,(n)}$, we may integrate (5.158) with $\widetilde{\theta} = \psi_t^{(n)}(\theta, s)$ and deduce that

$$\psi_t^{(n)}(\theta, s) = \theta - \int_s^t \lambda_1^{(n)}(\psi_t^{(n)}(\theta, s'))ds' = \theta - \frac{1}{3}\kappa(t - s) + \mathcal{O}(t^{\frac{4}{3}})\,, \tag{5.163}$$

which proves the first equality in (5.154). The second equality follows by combining (5.163) with (5.162), which in turn shows via (5.13) that $\mathfrak{s}_1^{(n)}(t) = \frac{1}{3}\kappa t + \mathcal{O}(t^{\frac{4}{3}})$.

The arguments for the 2-characteristic $\phi_t^{(n)}(\theta, s)$ are identical, except that $\frac{1}{3}\kappa t$ must be replaced with $\frac{2}{3}\kappa t$ because $\lambda_2^{(n)}$ contains $\frac{2}{3}w^{(1)}$ instead of $\frac{1}{3}w^{(1)}$. We omit these redundant details. □

5.8.11 Stability of the Iteration Space

Proposition 5.25 ($\mathcal{X}_{\overline{\varepsilon}}$ is stable under iteration) *Let $\overline{\varepsilon}$ be taken sufficiently small with respect to κ, b, c, and m. For all $n \geqslant 1$, the map*

$$(w^{(n)}, z^{(n)}, k^{(n)}, a^{(n)}) \mapsto (w^{(n+1)}, z^{(n+1)}, k^{(n+1)}, a^{(n+1)})$$

maps $\mathcal{X}_{\overline{\varepsilon}} \to \mathcal{X}_{\overline{\varepsilon}}$. In particular, the iterates $(w^{(n+1)}, z^{(n+1)}, k^{(n+1)}, a^{(n+1)})$ satisfy the bounds (5.141).

Proof of Proposition 5.25 In the course of the proof, we will repeatedly let $\overline{\varepsilon}$, and hence t, to be sufficiently small with respect to κ, b, c, m.
Estimates for $w^{(n+1)}$. By Lemma 5.22, for any $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}$, there exists a label $x \in \Upsilon^{(n)}(t)$ such that $\eta^{(n)}(x, t) = \theta$.

By the triangle inequality,

$$\begin{aligned}\left|\left(w^{(n+1)} - w^{(1)}\right) \circ \eta^{(n)}\right| &\leqslant \left|w^{(n+1)} \circ \eta^{(n)} - w_0\right| + \left|w^{(1)} \circ \eta^{(1)} - w^{(1)} \circ \eta^{(n)}\right| \\ &= \left|w^{(n+1)} \circ \eta^{(n)} - w_0\right| + \left|w_0 \circ {\eta_{\mathsf{B}}}^{-1} \circ \eta^{(n)} - w_0\right| .\end{aligned}$$

By the fundamental theorem of calculus,

$$\begin{aligned} w_0 \circ {\eta_{\mathsf{B}}}^{-1} \circ \eta^{(n)} - w_0 &= \int_0^t \frac{d}{dt}\left(w_0 \circ {\eta_{\mathsf{B}}}^{-1} \circ \eta^{(n)}\right) ds \\ &= \int_0^t w_0' \circ {\eta_{\mathsf{B}}}^{-1} \circ \eta^{(n)} \left((\partial_t {\eta_{\mathsf{B}}}^{-1}) \circ \eta^{(n)} + (\partial_\theta {\eta_{\mathsf{B}}}^{-1}) \circ \eta \partial_t \eta^{(n)} \right) ds \\ &= \int_0^t w_0' \circ {\eta_{\mathsf{B}}}^{-1} \circ \eta^{(n)} (\lambda_3^{(n)} - w^{(1)}) \circ \eta^{(n)} \left(\eta_x ({\eta_{\mathsf{B}}}^{-1} \circ \eta^{(n)})\right)^{-1} ds .\end{aligned}$$

The bounds (5.17b), (5.152a) and (5.141) show that

$$\left|w_0 \circ {\eta_{\mathsf{B}}}^{-1} \circ \eta^{(n)} - w_0\right| \leqslant \tfrac{1}{2} R_1 t . \tag{5.164}$$

Next, using the identity (5.125), we have that

$$\left|w^{(n+1)} \circ \eta^{(n)} - w_0\right| \leqslant \tfrac{8}{3} \int_0^t \left|\left(a^{(n)} w^{(n)}\right) \circ \eta^{(n)}\right| ds + \tfrac{1}{4} \int_0^t \left|c^{(n)} \circ \eta^{(n)} \tfrac{d}{ds}\left(k^{(n)} \circ \eta^{(n)}\right)\right| ds.$$

The bounds (5.141) with $\overline{\varepsilon}$ taken sufficiently small,

$$\left|w^{(n+1)} \circ \eta^{(n)} - w_0\right| \leqslant 3\mathsf{m} R_7 t .$$

Together with the bound (5.164) and the fact that $\eta^{(n)}(x, t)$ is a diffeomorphism for each label $x \in \Upsilon^{(n)}$, we have that for all $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}$,

$$\left|w^{(n+1)}(\theta, t) - w^{(1)}(\theta, t)\right| \leqslant \tfrac{3}{4} R_1 t ,$$

as long as $12\mathsf{m}R_7 \leqslant R_1$. This inequality holds due to the choices in (5.142).

Let us now show that the estimate (5.141b) holds. Following the procedure we used to obtain the identity (5.99), we differentiating (5.124), use (5.138a), and obtain that

$$\tfrac{d}{dt}\big(w_\theta^{(n+1)} \circ \eta^{(n)}\, \eta_x^{(n)}\big) = \tfrac{1}{4}\tfrac{d}{dt}\big((c^{(n)} k_\theta^{(n)}) \circ \eta^{(n)}\, \eta_x^{(n)}\big) + \mathcal{F}_{w_\theta}^{(n)} \circ \eta^{(n)}\, \eta_x^{(n)} , \tag{5.165}$$

where

$$\begin{aligned} \mathcal{F}_{w_\theta}^{(n)} = k_\theta^{(n)} \Big(&\tfrac{1}{6} c^{(n)} c_\theta^{(n)} + \tfrac{1}{6} c^{(n-2)} z_\theta^{(n)} + \tfrac{2}{3} a^{(n-2)} c^{(n-2)} \\ &+ \tfrac{1}{4} (\lambda_3^{(n-2)} - \lambda_3^{(n)}) c_\theta^{(n)} \Big) \circ \eta^{(n)}\, \eta_x^{(n)} \\ &- \tfrac{8}{3} \partial_\theta (a^{(n)} w^{(n)}) . \end{aligned} \tag{5.166}$$

An equivalent form of (5.165) is given by

$$\begin{aligned} &\partial_t w_\theta^{(n+1)} + \lambda_3^{(n)} w_{\theta\theta}^{(n+1)} + \partial_\theta \lambda_3^{(n)} w_\theta^{(n+1)} \\ &\quad = \tfrac{1}{4} \left(\tfrac{d}{dt}\big((c^{(n)} k_\theta^{(n)}) \circ \eta^{(n)}\, \eta_x^{(n)}\big) \big(\eta_x^{(n)}\big)^{-1} \right) \circ \eta_{\rm inv}^{(n)} + \mathcal{F}_{w_\theta}^{(n)} , \end{aligned} \tag{5.167}$$

Therefore,

$$\begin{aligned} &\tfrac{d}{dt} \left((w_\theta^{(n+1)} - w_\theta^{(1)}) \circ \eta^{(n)} \right) + w_\theta^{(n)} \circ \eta^{(n)} (w_\theta^{(n+1)} - w_\theta^{(1)}) \circ \eta^{(n)} \\ &\quad = -w_\theta^{(1)} \circ \eta^{(n)} (w_\theta^{(n)} - w_\theta^{(1)}) \circ \eta^{(n)} - (w^{(n)} - w^{(1)}) \circ \eta^{(n)} w_{\theta\theta}^{(1)} \circ \eta^{(n)} \\ &\qquad + \tfrac{1}{4} \tfrac{d}{dt} \Big((c^{(n)} k_\theta^{(n)}) \circ \eta^{(n)}\, \eta_x^{(n)} \Big) \big(\eta_x^{(n)}\big)^{-1} + \mathcal{F}_{w_\theta}^{(n)} \circ \eta^{(n)} . \end{aligned}$$

For $0 \leqslant s \leqslant t$, let us define the integrating factor $\mathcal{I}_{s,t} = e^{\int_s^t w_y^{(n)}(\eta^{(n)}(x,r),r)dr}$. Then, we have that

$$\begin{aligned} &(w_\theta^{(n+1)} - w_\theta^{(1)}) \circ \eta^{(n)} \\ &= \overbrace{\int_0^t -\mathcal{I}_{t,s} \Big(w_\theta^{(1)} \circ \eta^{(n)} (w_\theta^{(n)} - w_\theta^{(1)}) \circ \eta^{(n)} \Big) ds}^{\mathsf{I}_1} \\ &\quad + \overbrace{\int_0^t -\mathcal{I}_{t,s} \Big((w^{(n)} - w^{(1)}) \circ \eta^{(n)} w_{\theta\theta}^{(1)} \circ \eta^{(n)} \Big) ds}^{\mathsf{I}_2} \\ &\quad + \underbrace{\tfrac{1}{4} \int_0^t \mathcal{I}_{t,s} \left(\tfrac{d}{dt} \Big((c^{(n)} k_\theta^{(n)}) \circ \eta^{(n)}\, \eta_x^{(n)} \Big) \big(\eta_x^{(n)}\big)^{-1} \right) ds}_{\mathsf{I}_3} + \underbrace{\int_0^t \mathcal{I}_{t,s} \mathcal{F}_{w_\theta}^{(n)} \circ \eta^{(n)} ds}_{\mathsf{I}_4} . \end{aligned} \tag{5.168}$$

From (5.141b), $\left|w_\theta^{(n)} - w_\theta^{(1)}\right| \leqslant R_2 t^{-\frac{1}{2}}$ and thanks to (5.153), we have that for $\overline{\varepsilon}$ small enough,

$$\left|\mathcal{I}_{s,t}\right| = e^{\int_s^t \left|(w_y^{(n)} - w_\theta^{(1)})(\eta^{(n)}(x,r),r)\right| dr} e^{\int_s^t \left|w_\theta^{(1)}(\eta^{(n)}(x,r),r)\right| dr} \leqslant \tfrac{17}{10} .$$

Let us now estimate each integral I_1, I_2, I_3, and I_4 on the right side of (5.168). First, we have that

$$|\mathsf{I}_1| \leqslant \tfrac{17}{10} R_2 \int_0^t \left|w_\theta^{(1)} \circ \eta^{(n)}\right| \left(\mathsf{b}^3 s^3 + \left(\eta^{(n)}(x,s) - \mathfrak{s}(s)\right)^2\right)^{-\frac{1}{6}} ds . \tag{5.169}$$

The Burgers characteristic satisfies $\partial_t(\mathfrak{s}(t) - \eta_{\mathsf{B}}(x,t)) = \dot{\mathfrak{s}}(s) - w_0(x)$. Integration from s to t for $0 \leqslant s \leqslant t$ together with the inequality (5.13), the fact that $|x| \geqslant \frac{3}{4}(\mathsf{b}t)^{\frac{3}{2}}$, and taking $\overline{\varepsilon}$ sufficiently small, shows that

$$\begin{aligned}\mathfrak{s}(s) - \eta_{\mathsf{B}}(x,s) &\geqslant \mathfrak{s}(t) - \eta_{\mathsf{B}}(x,t) + (t-s)(\kappa - w_0(x) - Ct) \\ &\geqslant \mathfrak{s}(t) - \eta_{\mathsf{B}}(x,t) + (\tfrac{3}{4})^{\frac{1}{3}} \mathsf{b}^{\frac{3}{2}} t^{\frac{1}{2}} (t-s) - C(t-s)t .\end{aligned}$$

Using that that $\theta = \eta^{(n)}(x,t)$, (5.1c) and (5.152b), and taking $\overline{\varepsilon}$ even smaller if necessary, we see that

$$\mathfrak{s}(s) - \eta^{(n)}(x,s) \geqslant \mathfrak{s}(t) - \theta + \tfrac{\sqrt{3}}{2} \mathsf{b}^{\frac{3}{2}} t^{\frac{1}{2}} (t-s) ,$$

and hence

$$\mathsf{b}^3 s^3 + \left(\eta^{(n)}(x,s) - \mathfrak{s}(s)\right)^2 \geqslant \left(\theta - \mathfrak{s}(t)\right)^2 + \tfrac{3}{4}\mathsf{b}^3 t(t-s)^2 + \mathsf{b}^3 s^3 .$$

The function $\frac{3}{4}\mathsf{b}^3 t(t-s)^2 + \mathsf{b}^3 s^3$ has a minimum at $s = \frac{t}{2}$ and takes the value there of $\frac{5}{16}\mathsf{b}^3 t^3$, so that

$$\mathsf{b}^3 s^3 + \left(\eta^{(n)}(x,s) - \mathfrak{s}(s)\right)^2 \geqslant \tfrac{5}{16}\left(\left(\theta - \mathfrak{s}(t)\right)^2 + \mathsf{b}^3 t^3\right) . \tag{5.170}$$

Thus, with (5.170), the integral I_1 in (5.169) is bounded as

$$\begin{aligned}|\mathsf{I}_1| &\leqslant \tfrac{17}{10}(\tfrac{5}{16})^{-\frac{1}{6}} R_2 \left(\left(\theta - \mathfrak{s}(t)\right)^2 + \mathsf{b}^3 t^3\right)^{-\frac{1}{6}} \int_0^t \left|w_\theta^{(1)} \circ \eta^{(n)}\right| ds \\ &\leqslant \tfrac{19}{40}\tfrac{17}{10}(\tfrac{5}{16})^{-\frac{1}{6}} R_2 \left(\left(\theta - \mathfrak{s}(t)\right)^2 + \mathsf{b}^3 t^3\right)^{-\frac{1}{6}} ,\end{aligned} \tag{5.171}$$

the last inequality following from (5.153). It is important to note that $\frac{19}{40}\frac{17}{10}(\frac{5}{16})^{-\frac{1}{6}} < \frac{99}{100}$.

For the integral I_2 in (5.168), the estimate (5.37b) shows that

$$|\mathsf{I}_2| \leqslant 2\tfrac{17}{10} R_1 \mathsf{b} \int_0^t s\Big((\mathsf{b}s)^3 + |y - \mathfrak{s}(s)|^2\Big)^{-\frac{5}{6}} ds\,.$$

Using (5.170) and that $(\frac{5}{16})^{-\frac{5}{6}} \leqslant 3$, we then have that

$$|\mathsf{I}_2| \leqslant 3\tfrac{17}{10} R_1 \mathsf{b}\Big((\mathsf{b}t)^3 + |\theta - \mathfrak{s}(t)|^2\Big)^{-\frac{5}{6}} t^2 \leqslant 6R_1\mathsf{b}\Big((\mathsf{b}t)^3 + |\theta - \mathfrak{s}(t)|^2\Big)^{-\frac{1}{6}}\,.$$

Thus, $w^{(n+1)}$ satisfies (5.141b) as soon as we choose $1200R_1\mathsf{b} \leqslant R_2$. In view of (5.2), this inequality is ensured by the choice of R_2 and R_1 given in (5.142).

To bound I_3, we integrate-by-parts and find that

$$\mathsf{I}_3 = \tfrac{1}{4}\mathcal{I}_{t,s}\Big((c^{(n)}k_\theta^{(n)}) \circ \eta^{(n)}\Big) - \tfrac{1}{4}\int_0^t (c^{(n)}k_\theta^{(n)}) \circ \eta^{(n)}\; \eta_x^{(n)} \tfrac{d}{dt}\Big(\mathcal{I}_{t,s}\big(\eta_x^{(n)}\big)^{-1}\Big) ds\,.$$

Since $\partial_t \mathcal{I}_{0,t} = \mathcal{I}_{0,t} w_\theta^{(n)} \circ \eta^{(n)}$ and $\partial_t\big(\eta_x^{(n)}\big)^{-1} = -\big(\eta_x^{(n)}\big)^{-1}\partial_\theta\lambda_3^{(n)} \circ \eta^{(n)}$, using the bounds (5.141), we obtain

$$|\mathsf{I}_3| \leqslant Ct^{\frac{1}{2}}\,.$$

Finally, using the definition of $\mathcal{F}^{(n)}_{w_\theta}$ in (5.166) and the bounds (5.141), we also find that

$$|\mathsf{I}_4| \leqslant Ct^{\frac{1}{2}}\,.$$

By combining the bounds for I_1, I_2, I_3, and I_4, we taking $\bar{\varepsilon}$ sufficiently small so we have shown that

$$\Big|w_\theta^{(n+1)}(\theta,t) - w_\theta^{(1)}(\theta,t)\Big| \leqslant \tfrac{999}{1000} R_2\,,$$

for all $(\theta,t) \in \mathcal{D}_{\bar{\varepsilon}}$, thus establishing that (5.141b) holds.

Estimates for $z^{(n+1)}$. Let $(\theta,t) \in \mathcal{D}^{z,(n+1)}_{\bar{\varepsilon}}$. We integrate (5.136b) from $\mathfrak{z}^{(n)}(\theta,t)$ to t and obtain

$$z^{(n+1)}(\theta,t) = z_-^{(n+1)}(\mathfrak{s}(\mathfrak{z}^{(n)}(\theta,t))) - \int_{\mathfrak{z}^{(n)}(\theta,t)}^t \Big(\tfrac{8}{3}a^{(n)}z^{(n)} - \tfrac{1}{6}(c^{(n)})^2\partial_\theta k^{(n)}\Big) \circ \psi_t^{(n)} ds'\,, \tag{5.172}$$

Having shown that $w^{(n+1)} \in \mathcal{X}_{\bar{\varepsilon}}$ (continuity will be established below), then $w^{(n+1)}$ satisfies the criteria of Lemma 5.20 and thus we can appeal to Lemma 5.21 for the bound of $z_-^{(n+1)}(\mathfrak{s}(\mathfrak{Z}^{(n)}(\theta,t)))$. It follows from (5.141) and (5.149a) that

$$\left|z^{(n+1)}(\theta,t)\right| \leqslant (5\mathsf{b}^{\frac{9}{2}}\kappa^{-2}+\tfrac{1}{8}\kappa^2 R_6)t^{\frac{3}{2}}\,,$$

which shows that (5.141c) holds for $z^{(n+1)}$ if $5\mathsf{b}^{\frac{9}{2}}\kappa^{-2}+\frac{1}{8}\kappa^2 R_6 \leqslant R_3$. Using (5.2), this inequality holds due to the definition of R_3 and R_6 in (5.142).

Next, integrating (5.130a) from $\mathfrak{Z}^{(n)}$ to t and using the definitions of $\mathcal{F}_{z_\theta}$ and $\mathcal{H}_{z_\theta}$ given by (5.103b) and (5.115), respectively, for all $(\theta,t)\in\mathcal{D}_{\overline{\varepsilon}}^{z,(n+1)}$,

$$\partial_\theta z^{(n+1)} = \mathcal{H}_{z_\theta}(\mathcal{U}_{\mathsf{L}}^{(n)},\dot{\mathcal{U}}_{\mathsf{L}}^{(n)},\psi_t^{(n)},\mathfrak{Z}^{(n)})+\mathcal{F}_{z_\theta}(U^{(n)},\psi_t^{(n)},\mathfrak{Z}^{(n)})\,. \tag{5.173}$$

It follows from (5.57a), (5.141), (5.149b), and (5.155) that for t sufficiently small,

$$\left|\partial_\theta z^{(n+1)}(\theta,t)\right| \leqslant 2\kappa^{-3}(8\mathsf{b}^{\frac{9}{2}}+50\mathsf{b}^{\frac{9}{2}})t^{\frac{1}{2}}+\tfrac{\kappa}{2}R_6 t^{\frac{1}{2}} \leqslant R_4 t^{\frac{1}{2}}\,, \tag{5.174}$$

which proves that (5.141d) holds for $\partial_\theta z^{(n+1)}$ whenever $116\kappa^{-3}\mathsf{b}^{\frac{9}{2}}+\frac{\kappa}{2}R_6 \leqslant R_4$. Using (5.2), this inequality holds by defining R_4 and R_6 as in (5.142).

Estimates for $k^{(n+1)}$. We have shown that $w^{(n+1)}$ and $z^{(n+1)}$ satisfy the bounds (5.141), and we will prove below that both functions are continuous on $\mathcal{D}_{\overline{\varepsilon}}$ and hence are in the set $\mathcal{X}_{\overline{\varepsilon}}$. For each $(\theta,t)\in\mathcal{D}^{(n+1)}$, we then have existence of unique characteristics $\phi_t^{(n+1)}(\theta,s)$ and shock-intersection times $\mathfrak{T}^{(n+1)}(\theta,t)$ satisfying the properties in Lemma 5.24.

Let $(\theta,t)\in\mathcal{D}_{\overline{\varepsilon}}^{k,(n+1)}$. We integrate (5.133a) from $\mathfrak{T}^{(n+1)}(\theta,t)$ to t and obtain that

$$k^{(n+1)}(\theta,t) = k_-^{(n+1)}(\mathfrak{s}(\mathfrak{T}^{(n+1)}(\theta,t)))\,. \tag{5.175}$$

Again, appealing to Lemma 5.21, the bound (5.149a) then gives

$$\left|k^{(n+1)}(\theta,t)\right| \leqslant 40\mathsf{b}^{\frac{9}{2}}\kappa^{-3}t^{\frac{3}{2}}\,, \tag{5.176}$$

which shows that (5.141e) holds for $k^{(n+1)}$ if $40\mathsf{b}^{\frac{9}{2}}\kappa^{-3} \leqslant R_5$. The condition (5.2) justifies the definition of R_5 in (5.142).

In the same way that we obtained (5.106) and (5.109), we also have that

$$\partial_\theta k^{(n+1)}(\theta,t) = \frac{\dot{\mathcal{K}}^{(n+1)}(\mathfrak{T}^{(n+1)}(\theta,t)))}{\dot{\mathfrak{s}}(\mathfrak{T}^{(n+1)}(\theta,t)))-\partial_s\phi_t^{(n+1)}(\theta,\mathfrak{T}^{(n+1)}(\theta,t))} \partial_\theta\phi_t^{(n+1)}(\mathfrak{s}(\mathfrak{T}^{(n+1)}(\theta,t)),\mathfrak{T}^{(n+1)}(\theta,t))\,, \tag{5.177}$$

and thus from (5.149b), and (5.155) that for t sufficiently small,

$$\left|\partial_\theta k^{(n+1)}(\theta,t)\right| \leqslant 200\mathsf{b}^{\frac{9}{2}}\kappa^{-4}t^{\frac{1}{2}}\,, \tag{5.178}$$

which shows that (5.141f) holds for $\partial_\theta k^{(n+1)}$ if $200\mathsf{b}^{\frac{9}{2}}\kappa^{-4} \leqslant R_6$. The condition (5.2)justifies the definition of R_6 in (5.142).

Estimates for $a^{(n+1)}$. We consider any point $(\theta, t) \in \mathcal{D}^{k,(n)}_{\bar{\varepsilon}}$. By Lemma 5.24, the characteristic curve $\phi_t^{(n)}(\theta, s)$ exists for all $s \in [0, t]$. From (5.128a), we have that

$$\tfrac{d}{dt}\big(a^{(n+1)} \circ \phi_t^{(n)}\big) = \Big(-\tfrac{4}{3}(a^{(n)})^2 + \tfrac{1}{6}(w^{(n)})^2 + \tfrac{1}{6}(z^{(n)})^2 + w^{(n)}z^{(n)}\Big) \circ \phi_t^{(n)} , \tag{5.179}$$

and hence

$$\begin{aligned} a^{(n+1)}(\theta, t) &= a_0(\phi_t^{(n)}(\theta, 0)) \\ &\quad + \int_0^t \Big(-\tfrac{4}{3}(a^{(n)})^2 + \tfrac{1}{6}(w^{(n)})^2 + \tfrac{1}{6}(z^{(n)})^2 + w^{(n)}z^{(n)}\Big) \circ \phi_t^{(n)} ds . \end{aligned} \tag{5.180}$$

Using (5.1a), (5.4), and (5.141), we find that

$$\big|a^{(n+1)}(\theta, t)\big| \leqslant \mathsf{m} + Ct \leqslant 2\mathsf{m} . \tag{5.181}$$

Differentiating (5.179) gives

$$\begin{aligned} \tfrac{d}{dt}\big(\partial_\theta a^{(n+1)} \circ \phi_t^{(n)}\ \partial_\theta \phi_t^{(n)}\big) &= \partial_\theta\Big(-\tfrac{4}{3}(a^{(n)})^2 \\ &\quad + \tfrac{1}{6}(w^{(n)})^2 + \tfrac{1}{6}(z^{(n)})^2 + w^{(n)}z^{(n)}\Big) \circ \phi_t^{(n)}\ \partial_\theta \phi_t^{(n)} , \end{aligned}$$

and so

$$\begin{aligned} \partial_\theta a^{(n+1)}(\theta, t) &= a_0'(\phi_t^{(n)}(\theta, 0))\partial_\theta \phi_t^{(n)}(y, 0) \\ &\quad + \int_0^t \partial_\theta\Big(-\tfrac{4}{3}(a^{(n)})^2 + \tfrac{1}{6}(w^{(n)})^2 \\ &\quad + \tfrac{1}{6}(z^{(n)})^2 + w^{(n)}z^{(n)}\Big) \circ \phi_t^{(n)}\ \partial_\theta \phi_t^{(n)} ds . \end{aligned}$$

Employing the bounds (5.1a), (5.57a), (5.141), and (5.155), we find that

$$\big|\partial_\theta a^{(n+1)}(\theta, t)\big| \leqslant \mathsf{m} + Ct^{\frac{1}{3}} \leqslant 2\mathsf{m} ,$$

which together with (5.181) shows that (5.141g) holds for $a^{(n+1)}$ given that R_7 is defined by (5.142).

Continuity of $w^{(n+1)}$, $z^{(n+1)}$, $k^{(n+1)}$, **and** $a^{(n+1)}$. Composing (5.125) with $\eta_{\mathrm{inv}}^{(n)}$, we see that

$$w^{(n+1)}(\theta, t) = w_0(\eta_{\mathrm{inv}}^{(n)}(\theta, t)) - \tfrac{8}{3}\int_0^t \Big(a^{(n)}w^{(n)}\Big)(\eta^{(n)}(\eta_{\mathrm{inv}}^{(n)}(\theta, t), t'), t')dt'$$

$$+ \tfrac{1}{4} \int_0^t c^{(n)}(\eta^{(n)}(\eta_{\rm inv}^{(n)}(\theta,t),t'),t') \tfrac{d}{dt'} \Big(k^{(n)}(\eta^{(n)}(\eta_{\rm inv}^{(n)}(\theta,t),t'),t') \Big)\, dt' .$$

By Lemma 5.22, $\eta_{\rm inv}^{(n)}$ is continuous on $\mathcal{D}_{\bar{\varepsilon}}$, and hence by the definition of the set $\mathcal{X}_{\bar{\varepsilon}}$ given in (5.140), we see that $w^{(n+1)}$ is then continuous on $\mathcal{D}_{\bar{\varepsilon}}$.

Continuity of the shock-intersection time $\jmath(\theta,t)$ follows from the continuity of ψ_t on $\mathcal{D}_{\bar{\varepsilon}}$ and the continuity of $\mathfrak{s}(t)$. From (5.149b), we see that $z_-^{(n+1)}(t)$ is continuous. Therefore, the identity (5.149b) together with the definition of $\mathcal{X}_{\bar{\varepsilon}}$ shows that $z^{(n+1)}$ is continuous on $\mathcal{D}_{\bar{\varepsilon}}$. Continuity of $k^{(n+1)}$ follows in the same way from the identity (5.176). The identity (5.180) together with the (5.140) and the continuity of a_0 shows that $a^{(n+1)}$ is also continuous on $\mathcal{D}_{\bar{\varepsilon}}$. □

5.8.12 Contractivity of the Iteration Map

We set

$$\begin{aligned}
&\delta w^{(n)} := w^{(n)} - w^{(n-1)} , \quad \delta z^{(n)} := z^{(n)} - z^{(n-1)} , \quad \delta k^{(n)} := k^{(n)} - k^{(n-1)} , \\
&\delta c^{(n)} := c^{(n)} - c^{(n-1)} , \quad \delta \lambda_i := \lambda_i^{(n)} - \lambda_i^{(n-1)} ,
\end{aligned}$$

for $i \in \{1,2,3\}$.

Proposition 5.26 (The iteration is contractive) *The map*

$$(w^{(n)}, z^{(n)}, k^{(n)}, a^{(n)}) \mapsto (w^{(n+1)}, z^{(n+1)}, k^{(n+1)}, a^{(n+1)}) : \mathcal{X}_{\bar{\varepsilon}} \to \mathcal{X}_{\bar{\varepsilon}}$$

satisfies the contractive estimate

$$\begin{aligned}
&\max_{s\in[0,t]} \Big(\big\|\delta w^{(n+1)}(\cdot,s)\big\|_{L^\infty} + \big\|\delta z^{(n+1)}(\cdot,s)\big\|_{L^\infty} \\
&\qquad + \big\|\delta k^{(n+1)}(\cdot,s)\big\|_{L^\infty} + \big\|\delta a^{(n+1)}(\cdot,s)\big\|_{L^\infty} \Big) \\
&\quad \leqslant \tfrac{3}{4} \max_{s\in[0,t]} \Big(\big\|\delta w^{(n)}(\cdot,s)\big\|_{L^\infty} + \big\|\delta z^{(n)}(\cdot,s)\big\|_{L^\infty} \\
&\qquad + \big\|\delta k^{(n)}(\cdot,s)\big\|_{L^\infty} + \big\|\delta a^{(n)}(\cdot,s)\big\|_{L^\infty} .
\end{aligned} \tag{5.182}$$

Proof of Proposition 5.26 From (5.126a), we see that for any $(\theta,t) \in \mathcal{D}_{\bar{\varepsilon}}$,

$$\begin{aligned}
&\partial_t \delta w^{(n+1)} + \lambda_3^{(n)} \partial_\theta \delta w^{(n+1)} + \delta \lambda_3^{(n)} \partial_\theta w^{(n)} \\
&\quad = \tfrac{1}{4} c^{(n)} \Big(\partial_t \delta k^{(n)} + \lambda_3^{(n)} \partial_\theta \delta k^{(n)} \Big) + \tfrac{1}{4} \delta \lambda_3^{(n)} c^{(n)} \partial_\theta k^{(n-1)} \\
&\qquad + \tfrac{1}{4} \delta c^{(n)} \Big(\lambda_3^{(n)} - \lambda_2^{(n-1)} \Big) \partial_\theta k^{(n-1)} - \tfrac{8}{3} a^{(n)} \delta w^{(n)} - \tfrac{8}{3} \delta a^{(n)} \delta w^{(n-1)} ,
\end{aligned}$$

and thus for all $x \in \Upsilon^{(n)}(t)$,

$$\partial_t \Big(\delta w^{(n+1)} \circ \eta^{(n)} \Big)$$

$$= \tfrac{1}{4}c^{(n)} \circ \eta^{(n)} \partial_t \left(\delta k^{(n)} \circ \eta^{(n)}\right) + (\delta w^{(n)} + \tfrac{1}{3}\delta z^{(n)}) \left(\tfrac{1}{4}c^{(n)} \partial_\theta k^{(n-1)} - \partial_\theta w^{(n)}\right) \circ \eta^{(n)}$$
$$+ \tfrac{1}{8}(\delta w^{(n)} + \delta z^{(n)}) \left(\lambda_3^{(n)} - \lambda_2^{(n-1)}\right) \partial_\theta k^{(n-1)} \circ \eta^{(n)}$$
$$- \tfrac{8}{3}\delta w^{(n)} a^{(n)} \circ \eta^{(n)} - \tfrac{8}{3}\delta a^{(n)} w^{(n-1)} \circ \eta^{(n)}.$$

Using (5.137) and integrating by parts in time,

$$\tfrac{1}{4}\int_0^t c^{(n)} \circ \eta^{(n)} \partial_s \left(\delta k^{(n)} \circ \eta^{(n)}\right) ds = \tfrac{1}{4}c^{(n)} \delta k^{(n)} \circ \eta^{(n)}$$
$$+ \tfrac{1}{8}\int_0^t \left(\lambda_3^{(n-1)} w_\theta^{(n)} + \lambda_1^{(n-1)} z_\theta^{(n)} + \tfrac{64}{3} a^{(n-1)} c^{(n-1)}\right) \delta k^{(n)} \circ \eta^{(n)} ds\,,$$

and thus, we have that

$$\begin{aligned}
\delta w^{(n+1)} \circ \eta^{(n)} = &- \int_0^t w_\theta^{(1)} \delta w^{(n)} \circ \eta^{(n)} ds - \int_0^t (w_\theta^{(n)} - w_\theta^{(1)}) \delta w^{(n)} \circ \eta^{(n)} ds \\
&+ \tfrac{1}{4}\delta k^{(n)} c^{(n)} \circ \eta^{(n)} \\
&+ \tfrac{1}{8}\int_0^t \delta k^{(n)} \left(\lambda_3^{(n-1)} w_\theta^{(n)} + \lambda_1^{(n-1)} z_\theta^{(n)} + \tfrac{64}{3} a^{(n-1)} c^{(n-1)}\right) \circ \eta^{(n)} ds \\
&+ \tfrac{1}{8}\int_0^t \delta w^{(n)} \left(\left(2w^{(n)} - \tfrac{2}{3}z^{(n)} - \lambda_2^{(n-1)}\right) - \tfrac{64}{3} a^{(n)}\right) \partial_\theta k^{(n-1)} \circ \eta^{(n)} \\
&+ \tfrac{1}{24}\int_0^t \delta z^{(n)} \left((4w^{(n)} + z^{(n)} - 3\lambda_3^{(n-1)}) \partial_\theta k^{(n-1)} - 8\partial_\theta w^{(n)}\right) \circ \eta^{(n)} ds \\
&- \tfrac{8}{3}\int_0^t \delta a^{(n)} w^{(n-1)} \circ \eta^{(n)} ds\,.
\end{aligned} \tag{5.183}$$

Appealing to (5.56a) and (5.141), we find that

$$\begin{aligned}
\max_{s\in[0,t]} \left\|\delta w^{(n+1)}(\cdot, s)\right\|_{L^\infty} \leqslant &\left(\tfrac{1}{2} + Ct^{\frac{1}{2}}\right) \max_{s\in[0,t]} \left\|\delta w^{(n)}(\cdot, s)\right\|_{L^\infty} \\
&+ C \max_{s\in[0,t]} \left\|\delta k^{(n)}(\cdot, s)\right\|_{L^\infty} \\
&+ \left(\tfrac{1}{6} + Ct^{\frac{1}{2}}\right) \max_{s\in[0,t]} \left\|\delta z^{(n)}(\cdot, s)\right\|_{L^\infty} \\
&+ Ct \max_{s\in[0,t]} \left\|\delta a^{(n)}(\cdot, s)\right\|_{L^\infty}.
\end{aligned} \tag{5.184}$$

Using the evolution of $z^{(n)}$ given by (5.130a), in the same way that we obtained (5.183), we find that for any $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}^{z,(n)}$,

$$\delta z^{(n+1)}(\theta, t) = \delta z_-^{(n+1)}(\mathfrak{s}(\mathcal{J}(\theta, t))) + \tfrac{1}{4}(\delta k_-^{(n)} c^{(n)})(\mathfrak{s}(\mathcal{J}(\theta, t))) - \tfrac{1}{4}\delta k^{(n)} c^{(n)}(\theta, t)$$
$$+ \tfrac{1}{4}\int_{\mathcal{J}(\theta,t)}^t \delta k^{(n)} \left((\lambda_1^{(n)} - \lambda_1^{(n-1)}) c_\theta^{(n)} + \tfrac{2}{3} c^{(n-1)} w_\theta^{(n)}\right.$$
$$\left. - \tfrac{8}{3} a^{(n-1)} c^{(n-1)}\right) \circ \psi_t^{(n)}(\theta, s) ds$$

$$+\int_{\mathcal{J}(\theta,t)}^{t} \delta w^{(n)} \left(-\tfrac{1}{3}z_\theta^{(n)} - \tfrac{1}{12}c^{(n)}k_\theta^{(n-1)} + \tfrac{1}{3}c^{(n-1)}k_\theta^{(n-1)}\right) \circ \psi_t^{(n)}(\theta,s)ds$$
$$+\int_{\mathcal{J}(\theta,t)}^{t} \delta z^{(n)} \left(z_\theta^{(n)} - \tfrac{1}{4}c^{(n)}k_\theta^{(n)} + \tfrac{1}{3}c^{(n-1)}k_\theta^{(n-1)} - \tfrac{8}{3}a^{(n)}\right)$$
$$\circ\, \psi_t^{(n)}(\theta,s)ds - \tfrac{8}{3}\int_{\mathcal{J}(\theta,t)}^{t} \delta a^{(n)} z^{(n-1)} \circ \psi_t^{(n)}(\theta,s)ds\,.$$

Using this identity together with (5.57a), (5.141), and (5.148) shows that

$$\begin{aligned}\max_{s\in[0,t]}\left\|\delta z^{(n+1)}(\cdot,s)\right\|_{L^\infty} &\leqslant Ct\max_{s\in[0,t]}\left\|\delta w^{(n+1)}(\cdot,s)\right\|_{L^\infty} + Ct^{\frac{3}{2}}\max_{s\in[0,t]}\left\|\delta w^{(n)}(\cdot,s)\right\|_{L^\infty}\\ &\quad + C\max_{s\in[0,t]}\left\|\delta k^{(n)}(\cdot,s)\right\|_{L^\infty} + Ct\max_{s\in[0,t]}\left\|\delta z^{(n)}(\cdot,s)\right\|_{L^\infty}\\ &\quad + Ct^{\frac{5}{2}}\max_{s\in[0,t]}\left\|\delta a^{(n)}(\cdot,s)\right\|_{L^\infty}.\end{aligned} \tag{5.185}$$

Next, the identity (5.175) together with the bound (5.148) provides us with the estimates

$$\begin{aligned}\max_{s\in[0,t]}\left\|\delta k^{(n+1)}(\cdot,s)\right\|_{L^\infty} &\leqslant Ct\max_{s\in[0,t]}\left\|\delta w^{(n+1)}(\cdot,s)\right\|_{L^\infty},\\ \max_{s\in[0,t]}\left\|\delta k^{(n)}(\cdot,s)\right\|_{L^\infty} &\leqslant Ct\max_{s\in[0,t]}\left\|\delta w^{(n)}(\cdot,s)\right\|_{L^\infty}.\end{aligned} \tag{5.186}$$

Finally, using (5.128a), we find that for $(\theta,t)\in\mathcal{D}_{\bar{\varepsilon}}^{k,(n)}$,

$$\begin{aligned}\delta a^{(n+1)}(\theta,t) &= \int_0^t \delta w^{(n)}\left(\tfrac{1}{3}\delta w^{(n)} + z^{(n)} - \tfrac{2}{3}a_\theta^{(n)}\right)\circ\phi_t^{(n)}ds\\ &\quad+\int_0^t \delta z^{(n)}\left(\tfrac{1}{3}\delta z^{(n)} + w^{(n-1)} - \tfrac{2}{3}a_\theta^{(n)}\right)\circ\phi_t^{(n)}ds\\ &\quad-\int_0^t \delta a^{(n)}\delta a^{(n)}\circ\phi_t^{(n)}ds\,,\end{aligned}$$

and therefore

$$\begin{aligned}&\max_{s\in[0,t]}\left\|\delta a^{(n+1)}(\cdot,s)\right\|_{L^\infty}\\ &\quad\leqslant Ct\max_{s\in[0,t]}\Big(\left\|\delta w^{(n)}(\cdot,s)\right\|_{L^\infty} + \left\|\delta z^{(n)}(\cdot,s)\right\|_{L^\infty}\\ &\qquad + \left\|\delta k^{(n)}(\cdot,s)\right\|_{L^\infty} + \left\|\delta a^{(n)}(\cdot,s)\right\|_{L^\infty}\Big).\end{aligned} \tag{5.187}$$

Summing the inequalities (5.184)–(5.187) yields

$$\begin{aligned}&\max_{s\in[0,t]}\Big(\left\|\delta w^{(n+1)}(\cdot,s)\right\|_{L^\infty} + \left\|\delta z^{(n+1)}(\cdot,s)\right\|_{L^\infty}\\ &\quad + \left\|\delta k^{(n+1)}(\cdot,s)\right\|_{L^\infty} + \left\|\delta a^{(n+1)}(\cdot,s)\right\|_{L^\infty}\Big)\end{aligned}$$

$$\leqslant \tfrac{1}{2} \max_{s\in[0,t]} \left\| \delta w^{(n)}(\cdot, s) \right\|_{L^\infty} + \tfrac{1}{6} \max_{s\in[0,t]} \left\| \delta z^{(n)}(\cdot, s) \right\|_{L^\infty} + Ct \max_{s\in[0,t]} \left\| \delta w^{(n+1)}(\cdot, s) \right\|_{L^\infty}$$
$$+ Ct^{\frac{1}{2}} \max_{s\in[0,t]} \left(\left\| \delta w^{(n)}(\cdot, s) \right\|_{L^\infty} + \left\| \delta z^{(n)}(\cdot, s) \right\|_{L^\infty} \right)$$
$$+ Ct \max_{s\in[0,t]} \left(\left\| \delta k^{(n)}(\cdot, s) \right\|_{L^\infty} + \left\| \delta a^{(n)}(\cdot, s) \right\|_{L^\infty} \right).$$

Choosing $\overline{\varepsilon}$ sufficiently small, we obtain the bound (5.182). □

5.8.13 Convergence of the Iteration Scheme

We define $\mathsf{y} = \theta - \mathfrak{s}(t)$ and

$$\mathsf{w}(\mathsf{y}, t) = w(\theta, t), \quad \mathsf{z}(\mathsf{y}, t) = z(\theta, t), \quad \mathsf{k}(\mathsf{y}, t) = k(\theta, t), \quad \mathsf{a}(\mathsf{y}, t) = a(\theta, t).$$

The space-time gradient is denoted as $\nabla_{\mathsf{y},t}$, and it is convenient to introduce

$$\mathsf{D}_{\overline{\varepsilon}} = (\mathbb{T} \setminus \{0\}) \times (0, \overline{\varepsilon}).$$

The contractive estimate (5.182) shows that $(\mathsf{w}^{(n)}, \mathsf{z}^{(n)}, \mathsf{k}^{(n)}, \mathsf{a}^{(n)}) \to (\mathsf{w}, \mathsf{z}, \mathsf{k}, \mathsf{a})$ uniformly in $\mathsf{D}_{\overline{\varepsilon}}$, and in particular we have that

$$\lim_{n\to\infty} \left\| \mathsf{w} - \mathsf{w}^{(n)} \right\|_{L^\infty(\mathsf{D}_{\overline{\varepsilon}})} = 0. \tag{5.188}$$

Let us now describe the bounds on derivatives. According to (5.141b) and (5.143), for all $\mathsf{y} \neq 0$ and $t \in [0, \overline{\varepsilon}]$, we have that

$$\left\| \left(t^3 + \mathsf{y}^2\right)^{\frac{1}{6}} \nabla_{\mathsf{y},t} (\mathsf{w}^{(n)} - w_{\mathsf{B}}) \right\|_{\mathsf{D}_{\overline{\varepsilon}}} \leqslant C.$$

By the Banach–Alaoglu theorem, there exists a limiting function f and a subsequence such that

$$\left(t^3 + \mathsf{y}^2\right)^{\frac{1}{6}} \nabla_{\mathsf{y},t} \mathsf{w}^{(n')} \rightharpoonup \left(t^3 + \mathsf{y}^2\right)^{\frac{1}{6}} f,$$

the convergence in $L^\infty(\mathsf{D}_{\overline{\varepsilon}})$ weak-*. Let us show that $f = \nabla_{\mathsf{y},t}\mathsf{w}$, the weak derivative of the uniform limit w, and that the convergence holds for any subsequence. For test functions $\varphi \in W_0^{1,1}(\mathsf{D}_{\overline{\varepsilon}})$,

$$\lim_{n\to\infty} \int_{\mathsf{D}_{\overline{\varepsilon}}} (\mathsf{w} - \mathsf{w}^{(n)}) \partial_{\mathsf{y}} \left(\left(t^3 + \mathsf{y}^2\right)^{\frac{1}{6}} \varphi \right) d\mathsf{y}dt$$
$$= \lim_{n\to\infty} \tfrac{1}{3} \int_{\mathsf{D}_{\overline{\varepsilon}}} (\mathsf{w} - \mathsf{w}^{(n)}) \mathsf{y} \left(t^3 + \mathsf{y}^2\right)^{-\frac{5}{6}} \varphi \, d\mathsf{y}dt$$
$$+ \lim_{n\to\infty} \int_{\mathsf{D}_{\overline{\varepsilon}}} (\mathsf{w} - \mathsf{w}^{(n)}) \left(\left(t^3 + \mathsf{y}^2\right)^{\frac{1}{6}} \partial_{\mathsf{y}}\varphi \right) d\mathsf{y}dt.$$

It follows that

$$\left|\lim_{n\to\infty}\int_{\mathsf{D}_{\overline{\varepsilon}}}(\mathsf{w}-\mathsf{w}^{(n)})\partial_{\mathsf{y}}\left(\left(t^3+\mathsf{y}^2\right)^{\frac{1}{6}}\varphi\right)d\mathsf{y}dt\right|$$
$$\leqslant \lim_{n\to\infty}\|\mathsf{w}-\mathsf{w}^{(n)}\|_{L^\infty(\mathsf{D}_{\overline{\varepsilon}})}\left(\tfrac{1}{3}\|\varphi\|_{L^\infty(\mathsf{D}_{\overline{\varepsilon}})}\int_{\mathsf{D}_{\overline{\varepsilon}}}\mathsf{y}^{-\frac{2}{3}}d\mathsf{y}dt+2\int_{\mathsf{D}_{\overline{\varepsilon}}}\partial_{\mathsf{y}}\varphi\,d\mathsf{y}dt\right)=0$$

by (5.188). Similarly, if we replace ∂_{y} with ∂_t, then the integral $\frac{1}{3}\int_{\mathsf{D}_{\overline{\varepsilon}}}\mathsf{y}^{-\frac{2}{3}}d\mathsf{y}dt$ is replaced with $\frac{1}{2}\int_{\mathsf{D}_{\overline{\varepsilon}}}t^{-\frac{1}{2}}d\mathsf{y}dt$ - $\frac{1}{3}\int_{\mathsf{D}_{\overline{\varepsilon}}}\dot{\mathfrak{s}}(t)\mathsf{y}^{-\frac{2}{3}}d\mathsf{y}dt$, and the same conclusion holds, since again both integrals are bounded (using (5.13)). This shows that[8]

$$\left(t^3+\mathsf{y}^2\right)^{\frac{1}{6}}\nabla_{\mathsf{y},t}\mathsf{w}^{(n)}\rightharpoonup\left(t^3+\mathsf{y}^2\right)^{\frac{1}{6}}\nabla_{\mathsf{y},t}\mathsf{w}\qquad\text{in } L^\infty(\mathsf{D}_{\overline{\varepsilon}})\text{ weak-*},$$

and hence we have by lower semi-continuity that w satisfies (5.141a), (5.141b), and (5.143). The weak convergence for $(\partial_{\mathsf{y}}\mathsf{z}^{(n)},\partial_{\mathsf{y}}\mathsf{k}^{(n)},\partial_{\mathsf{y}}\mathsf{a}^{(n)})\rightharpoonup(\partial_{\mathsf{y}}\mathsf{z},\partial_{\mathsf{y}}\mathsf{k},\partial_{\mathsf{y}}\mathsf{a})$ in $L^\infty(\mathsf{D}_{\overline{\varepsilon}})$ weak-* is standard. We conclude that

$$(w,z,k,a)\in\mathcal{X}_{\overline{\varepsilon}}\,. \tag{5.189}$$

Let $\varphi\in C_0^\infty(\mathsf{D}_{\overline{\varepsilon}})$. Integration of (5.136a) shows that

$$\int_{\mathsf{D}_{\overline{\varepsilon}}}\Big(\partial_t\mathsf{w}^{(n+1)}+\left(\mathsf{w}^{(n)}-w_{\mathsf{B}}\right)\partial_{\mathsf{y}}\mathsf{w}^{(n+1)}$$
$$+\left(\tfrac{1}{3}\mathsf{z}^{(n)}+w_{\mathsf{B}}-\dot{\mathfrak{s}}(t)\right)\partial_{\mathsf{y}}\mathsf{w}^{(n+1)}+\tfrac{8}{3}\mathsf{a}^{(n)}\mathsf{w}^{(n)}-\tfrac{1}{6}(c^{(n)})^2\partial_{\mathsf{y}}\mathsf{k}^{(n)}\Big)\varphi\,d\mathsf{y}dt$$
$$=\overbrace{\int_{\mathsf{D}_{\overline{\varepsilon}}}\left(\partial_t\mathsf{w}^{(n+1)}+\left(\mathsf{w}^{(n)}-w_{\mathsf{B}}\right)\partial_{\mathsf{y}}\mathsf{w}^{(n+1)}\right)\varphi d\mathsf{y}dt}^{\mathcal{I}_1(\mathsf{w}^{(n)})}$$
$$+\underbrace{\int_{\mathsf{D}_{\overline{\varepsilon}}}\left(\tfrac{1}{3}\mathsf{z}^{(n)}+w_{\mathsf{B}}-\dot{\mathfrak{s}}(t)\right)\partial_{\mathsf{y}}\mathsf{w}^{(n+1)}+\tfrac{8}{3}\mathsf{a}^{(n)}\mathsf{w}^{(n)}-\tfrac{1}{6}(c^{(n)})^2\partial_{\mathsf{y}}\mathsf{k}^{(n)}\Big)\varphi\,d\mathsf{y}dt}_{\mathcal{I}_2(\mathsf{w}^{(n)},\mathsf{z}^{(n)})}\,.$$

Its clear that $\mathcal{I}_2(\mathsf{w}^{(n)},\mathsf{z}^{(n)},\mathsf{k}^{(n)},\mathsf{a}^{(n)})\to\mathcal{I}_2(\mathsf{w},\mathsf{z},\mathsf{k},\mathsf{a})$. Let us show that $\mathcal{I}_1(\mathsf{w}^{(n)})\to\mathcal{I}_1(\mathsf{w})$. We have that

$$\left|\mathcal{I}_1(\mathsf{w})-\mathcal{I}_1(\mathsf{w}^{(n)})\right|$$
$$\leqslant\left|\int_{\mathsf{D}_{\overline{\varepsilon}}}\left(t^3+\mathsf{y}^2\right)^{\frac{1}{6}}\left(\partial_t\mathsf{w}-\partial_t\mathsf{w}^{(n+1)}\right)\varphi\,\left(t^3+\mathsf{y}^2\right)^{-\frac{1}{6}}\,d\mathsf{y}dt\right|$$

8 In fact, $\left(t^3+\mathsf{y}^2\right)^{\frac{1}{6}}\mathsf{w}^{(n)}\rightharpoonup\left(t^3+\mathsf{y}^2\right)^{\frac{1}{6}}\mathsf{w}$ in $W^{1,\infty}(\mathsf{D}_{\overline{\varepsilon}})$ weak-*.

$$+\left|\int_{\mathsf{D}_{\overline{\varepsilon}}}\left(t^3+\mathsf{y}^2\right)^{\frac{1}{6}}\left(\partial_{\mathsf{y}}\mathsf{w}-\partial_{\mathsf{y}}\mathsf{w}^{(n+1)}\right)\varphi\left(\mathsf{w}-w_{\mathsf{B}}\right)\left(t^3+\mathsf{y}^2\right)^{-\frac{1}{6}}\,d\mathsf{y}dt\right|$$
$$+\left\|\mathsf{w}-\mathsf{w}^{(n)}\right\|_{L^\infty(\mathsf{D}_{\overline{\varepsilon}})}\left|\int_{\mathsf{D}_{\overline{\varepsilon}}}\left(t^3+\mathsf{y}^2\right)^{\frac{1}{6}}\partial_\theta\mathsf{w}^{(n+1)}\varphi\left(t^3+\mathsf{y}^2\right)^{-\frac{1}{6}}\,d\mathsf{y}dt\right|.$$

Since $\left(t^3+\mathsf{y}^2\right)^{-\frac{1}{6}}\in L^1(\mathsf{D}_{\overline{\varepsilon}})$, we see that the first two summands converges to 0 by weak-* convergence in $L^\infty(\mathsf{D}_{\overline{\varepsilon}})$, while the second term converges to 0 by the strong convergence (5.188). It follows that w satisfies

$$\int_{\mathsf{D}_{\overline{\varepsilon}}}\left(\partial_t\mathsf{w}+\lambda_3\partial_{\mathsf{y}}\mathsf{w}+\tfrac{8}{3}\mathsf{a}\mathsf{w}-\tfrac{1}{6}c^2\partial_{\mathsf{y}}\mathsf{k}\right)\varphi\,d\mathsf{y}dt=0\,,$$

and together with the standard weak convergence argument for the other variables, we have that $(\mathsf{w},\mathsf{z},\mathsf{k},\mathsf{a})$ are solutions to (5.91) in $\mathcal{D}_{\overline{\varepsilon}}$.

Thanks to the uniform convergence $(w^{(n)},z^{(n)},k^{(n)},a^{(n)})\to(w,z,k,a)$ in $\mathcal{D}_{\overline{\varepsilon}}$, it follows that the time derivatives $\partial_s(\eta^{(n)},\phi_t^{(n)},\psi_t^{(n)})\to\partial_s(\eta,\phi_t,\psi_t)$ uniformly, and that $\eta(\cdot,t)\colon\Upsilon(t)\to\mathbb{T}\setminus\{\mathfrak{s}(t)\}$ is a bijection, and the inverse map $\eta_{\mathrm{inv}}\colon\mathcal{D}_{\overline{\varepsilon}}\to\mathbb{T}\setminus\{0\}$ is continuous in spacetime, where the set of labels $\Upsilon^{(n)}(t)\to\Upsilon(t)$ in the sense that $\Upsilon(t)=\mathbb{T}\setminus[x_-(t),x_+(t)]$ and $x_-^{(n)}(t)\to x_-(t)$ and $x_+^{(n)}(t)\to x_+(t)$ uniformly.

Moreover, the uniform convergence $(w^{(n)},z^{(n)},k^{(n)},a^{(n)})\to(w,z,k,a)$ in $\mathcal{D}_{\overline{\varepsilon}}$, combined with the definitions (5.127) and the continuity of $\mathcal{E}_1$ and $\mathcal{E}_2$, implies that $\mathcal{E}_1(w_{\text{-}},w_+,z_-,\mathfrak{e}_-)=\mathcal{E}_2(w_{\text{-}},w_+,z_-,\mathfrak{e}_-)=0$. Thus, the equations relating z_- and k_- to w_- and w_+ hold on the given shock curve.

5.9 Proof of Proposition 5.6

The analysis given in Sects. 5.5–5.8 completes the proof of Propoisition 5.6, here we just summarize our findings. Given a regular shock curve $\mathfrak{s}$ satisfying (5.13), we have shown that there exists $\overline{\varepsilon}>0$ sufficiently small (solely in terms of κ, b, c, m) such that the iteration described in Section 5.8 produces a limit point $(w,z,k,a)\in\mathcal{X}_{\overline{\varepsilon}}$ (see (5.189)), which solves the azimuthal form of the Euler equations (5.91) in $\mathcal{D}_{\overline{\varepsilon}}$; this proves items (i), and (ii). From the last paragraph of the above section, we have that (w_-,w_+,z_-,k_-) satisfy the system of algebraic equations (3.13a)-(3.13b), arising from the Rankine–Hugoniot conditions, and by passing $n\to\infty$ in (5.146) and (5.148), we have that $[\![w]\!]$, $[\![z]\!]$, and $[\![k]\!]$ satisfy the bounds claimed in (5.63) and respectively (5.69); this proves items (iii), (v), (vi), and (vii). The stated bounds on $\mathfrak{s}_1$ and $\mathfrak{s}_2$, which are uniform limits of $\mathfrak{s}_1^{(n)}$ and $\mathfrak{s}_2^{(n)}$, follow by passing $n\to\infty$ in Lemma 5.24, proving item (iv).

5.10 Evolution of the Shock Curve

Proposition 5.6 shows that given a shock curve $(\mathfrak{s}(t),t)_{t\in[0,\overline{\varepsilon}]}$ which satisfies assumptions (5.13), we may compute a solution (w,z,k,a) of the azimuthal form of the Euler Eqs. (3.5)–(3.6) on the spacetime region $\mathcal{D}_{\overline{\varepsilon}}=(\mathbb{T}\times[0,\overline{\varepsilon}])\setminus(\mathfrak{s}(t),t)_{t\in[0,\overline{\varepsilon}]}$;

moreover, this solution exhibits a jump discontinuity from the $(w_+, 0, 0)$ state on the right of the shock curve to the state (w_-, z_-, k_-) on the left of the shock curve, and this jump is consistent with the system of algebraic Eqs. (3.13a)–(3.13b) arising from the Rankine–Hugoniot conditions. Throughout this section we shall implicitly use that we have a map

$$(\mathfrak{s}, w_0, a_0) \xrightarrow{\text{Proposition 5.6}} (w, z, k, a)\,. \tag{5.190}$$

Since at this stage of the proof uniqueness has not yet been established (this is achieved in Sect. 5.11 below), in the map (5.190) we select any one of the solutions guaranteed by Proposition 5.6.

We note that throughout the proof of Proposition 5.6, the shock curve itself is *fixed*, and does not solve an evolutionary equation. The goal of this section is to provide an iteration scheme whose fixed point $\mathfrak{s}$ is a C^2 smooth curve which solves the equation (3.12b) (recall that in view of Lemma 5.12 the jump conditions (3.12b) and (3.12a) are equivalent), which we recall is

$$\dot{\mathfrak{s}}(t) = \mathcal{F}_{\mathfrak{s}}(t)\,, \qquad \mathfrak{s}(0) = 0\,, \tag{5.191}$$

where

$$\mathcal{F}_{\mathfrak{s}}(t) = \frac{2}{3}\frac{(w_-(t) - z_-(t))^2(w_-(t) + z_-(t)) - w_+(t)^3}{(w_-(t) - z_-(t))^2 - w_+(t)^2} \tag{5.192}$$

and we have implicitly used the notation (5.7) to denote the limits from the left (indicated by a $-$ index) and the limit from the right (indicated by a $+$ index) at the shock point $(\mathfrak{s}(t), t)$ for the functions (w, z, k). We emphasize however that the (w_-, w_+, z_-, k_-) appearing in (5.192) do not just depend on $\mathfrak{s}$ because they are one sided limits of their respective functions (w, z, k) on the curve $(\mathfrak{s}(t), t)$; they also depend on $\mathfrak{s}$ because the functions (w, z, k) themselves arise from the mapping (5.190) given by Proposition 5.6; this mapping is *implicit* and *nonlinear*. Moreover, we note that due to Lemma 5.12 the z_- and k_- appearing in (5.192) are themselves smooth functions of w_- and w_+, so that $\mathcal{F}_{\mathfrak{s}}$ is truly a function that depends solely on w_- and w_+, or alternatively, $[\![w]\!]$ and $\langle\!\langle w\rangle\!\rangle$.

5.10.1 Properties of $\mathcal{F}_{\mathfrak{s}}$

Before giving the iteration scheme used to construct a solution to (5.191), we establish a few useful properties of the function $\mathcal{F}_{\mathfrak{s}}$ defined in (5.192).

Lemma 5.27 *Assume that $\mathfrak{s}$ satisfies (5.13), let (w, z, k) be defined via (5.190), and $\mathcal{F}_{\mathfrak{s}}$ be given by (5.192). We then have that*

$$|\mathcal{F}_{\mathfrak{s}}(t) - \kappa| \leqslant \tfrac{1}{2}\mathsf{m}^4 t\,, \tag{5.193a}$$

$$\left|\tfrac{d}{dt}\mathcal{F}_{\mathfrak{s}}(t)\right| \leqslant 5\mathsf{m}^4\,, \tag{5.193b}$$

for all $t \in (0, \overline{\varepsilon}]$.

Proof of Lemma 5.27 First we note that the function w satisfies (5.63) with

$$\mathsf{R}_j = 1 + 2R_1 = 1 + 100\mathsf{m}^2 \leqslant \mathsf{m}^3 \quad \text{and} \quad \mathsf{R}_m = \tfrac{1}{3}\mathsf{m}^4 + R_1 \leqslant \tfrac{1}{3}\mathsf{m}^4 + \mathsf{m}^3 . \tag{5.194}$$

This holds in view of (5.15a), (5.141a), and the definition of R_1 in (5.142).

Due to (5.68a), in order to approximate the function $\mathcal{F}_{\mathfrak{s}}$ it is natural to insert $-\frac{9}{16}[\![w]\!]^3 \langle\!\langle w \rangle\!\rangle^{-2}$ in (5.192), instead of z_-. Using the identities $w_- = \langle\!\langle w \rangle\!\rangle + \frac{1}{2}[\![w]\!]$ and $w_+ = \langle\!\langle w \rangle\!\rangle - \frac{1}{2}[\![w]\!]$, this gives us the leading order terms in $\mathcal{F}_{\mathfrak{s}}$ defined by

$$\mathcal{F}_{\mathfrak{s}}^{\text{app}} = \frac{2}{3} \frac{(w_- + \frac{9[\![w]\!]^3}{16\langle w\rangle^2})^2 (w_- - \frac{9[\![w]\!]^3}{16\langle w\rangle^2}) - w_+^3}{(w_- + \frac{9[\![w]\!]^3}{16\langle w\rangle^2})^2 - w_+^2} . \tag{5.195}$$

Furthermore, since the formula in (5.195) is explicit, using (5.63) we obtain that

$$\left| \mathcal{F}_{\mathfrak{s}}^{\text{app}} - \langle\!\langle w \rangle\!\rangle + \frac{7[\![w]\!]^2}{24\langle\!\langle w \rangle\!\rangle} \right| \leqslant C t^{\frac{3}{2}} \tag{5.196}$$

since $t \leqslant \overline{\varepsilon}$, and $\overline{\varepsilon}$ is sufficiently small; here $C = C(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$. The error we make in the approximation (5.195) may be bounded using the intermediate value theorem and the bounds (5.63), (5.68a), (5.69a) as

$$\begin{aligned} \left|\mathcal{F}_{\mathfrak{s}} - \mathcal{F}_{\mathfrak{s}}^{\text{app}}\right| &\leqslant \frac{2}{3} \left| z_- + \frac{9[\![w]\!]^3}{16\langle\!\langle w \rangle\!\rangle^2} \right| \left| 1 + \frac{[\![w]\!](\langle\!\langle w \rangle\!\rangle - \frac{1}{2}[\![w]\!])}{([\![w]\!] - z_*)^2} - \frac{\langle\!\langle w \rangle\!\rangle^2 - \frac{1}{4}[\![w]\!]^2}{(2\langle\!\langle w \rangle\!\rangle - z_*)^2} \right| \\ &\leqslant C t^{\frac{5}{2}} \left(1 + \frac{3\mathsf{b}^{\frac{3}{2}} \kappa t^{\frac{1}{2}}}{(\mathsf{b}^{\frac{3}{2}} t^{\frac{1}{2}} - 5\kappa^{-2} \mathsf{b}^{\frac{9}{2}} t^{\frac{3}{2}})^2} + \frac{\kappa^2}{4(\kappa - 5\kappa^{-2} \mathsf{b}^{\frac{9}{2}} t^{\frac{3}{2}})^2} \right) \\ &\leqslant C t^2 \end{aligned} \tag{5.197}$$

since $t \leqslant \overline{\varepsilon}$ is sufficiently small; here z_* lies in between z_- and $-\frac{9}{16}[\![w]\!]^3 \langle\!\langle w \rangle\!\rangle^{-2}$, and $C = C(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$. Combining (5.196)–(5.197) and (5.63) — with R_j and R_m as determined by (5.194), we arrive at

$$|\mathcal{F}_{\mathfrak{s}}(t) - \kappa| \leqslant \left(\tfrac{1}{3}\mathsf{m}^4 + \mathsf{m}^3 + 2\mathsf{b}^3 \kappa^{-1} \right) t \leqslant \tfrac{1}{2}\mathsf{m}^4 ,$$

thereby proving (5.193a). In this last inequality we have also appealed to (5.2).

In order to prove (5.193b), we first differentiate (5.192) with respect to t, to arrive at

$$\frac{3}{2} \frac{d}{dt} \mathcal{F}_{\mathfrak{s}} = \left(1 + \frac{[\![w]\!](\langle\!\langle w \rangle\!\rangle - \frac{1}{2}[\![w]\!])}{([\![w]\!] - z_-)^2} - \frac{\langle\!\langle w \rangle\!\rangle^2 - \frac{1}{4}[\![w]\!]^2}{(2\langle\!\langle w \rangle\!\rangle - z_-)^2} \right) \frac{d}{dt} z_-$$

$$+ \frac{[\![w]\!]^3 - 2[\![w]\!]^2 z_- + [\![w]\!] z_-^2 + z_-(2\langle\!\langle w\rangle\!\rangle - z_-)^2}{2([\![w]\!] - z_-)^2(2\langle\!\langle w\rangle\!\rangle - z_-)} \frac{d}{dt}[\![w]\!]$$
$$+ \left(1 + \frac{(\langle\!\langle w\rangle\!\rangle - 2[\![w]\!])(\langle\!\langle w\rangle\!\rangle + \frac{1}{2}[\![w]\!] - z_-)([\![w]\!] + z_-)}{2([\![w]\!] - z_-)(\langle\!\langle w\rangle\!\rangle - \frac{1}{2}z_-)^2}\right) \frac{d}{dt}\langle\!\langle w\rangle\!\rangle \tag{5.198}$$

By combining (5.198) with the bounds the derivative bounds (5.84) (which holds due to (5.145b) with constant $\mathsf{R} = R_1 \leqslant \mathsf{m}^3$ as defined in (5.142)), (5.81), and the amplitude estimates (5.63) (with (5.194)) and (5.69a), we arrive at

$$\begin{aligned}\frac{3}{2}\left|\frac{d}{dt}\mathcal{F}_{\mathfrak{s}}\right| &\leqslant \kappa \mathsf{b}^{-\frac{3}{2}} t^{-\frac{1}{2}} \left|\frac{d}{dt} z_-\right| + 2\mathsf{b}^{\frac{3}{2}}\kappa^{-1} t^{\frac{1}{2}} \left|\frac{d}{dt}[\![w]\!]\right| + \left(\frac{3}{2} + 2\mathsf{b}^3\kappa^{-2} t\right)\left|\frac{d}{dt}\langle\!\langle w\rangle\!\rangle\right| \\ &\leqslant \kappa \mathsf{b}^{-\frac{3}{2}} t^{-\frac{1}{2}}\left(8\mathsf{b}^{\frac{9}{2}}\kappa^{-2} t^{\frac{1}{2}}\right) + 2\mathsf{b}^{\frac{3}{2}}\kappa^{-1} t^{\frac{1}{2}}\left(2\mathsf{b}^{\frac{3}{2}} t^{-\frac{1}{2}}\right) + 2\left(3\mathsf{m}^4 + 3\mathsf{m}^3\right) \\ &\leqslant 7\mathsf{m}^4 .\end{aligned} \tag{5.199}$$

In the second inequality above we have used that $t \leqslant \overline{\varepsilon}$ is sufficiently small with respect to κ, b, c, and m, while in the third inequality we have used (5.2). This concludes the proof of (5.193b). □

5.10.2 The Shock Curve Iteration

In view of (5.191) and (5.193a)–(5.193b) we note that the inequalities (5.13) are stable (since $\frac{1}{2} < 1$ and $5 < 6$). Upon integrating in time, the condition $|\dot{\mathfrak{s}}(t) - \kappa| \leqslant \mathsf{m}^4 t$ present in (5.13), automatically implies $\mathfrak{s}(t) \in \Sigma(t)$.

Next, we define a sequence of curves $\mathfrak{s}^{(i)}$ for $i \geqslant 0$, as follows. For $i = 0$, we let $\mathfrak{s}^{(0)}(t) = \kappa t$. This curve trivially satisfies the conditions in (5.13). Next, given a curve $\mathfrak{s}^{(i)}$ defined on $[0, \overline{\varepsilon}]$ which satisfies (5.13), we first compute via (5.190) a tuple $(w, k, z, a)^{(i)}$ associated to $\mathfrak{s}^{(i)}$:

$$(\mathfrak{s}^{(i)}, w_0, a_0) \xrightarrow{\text{Proposition 5.6}} (w^{(i)}, z^{(i)}, k^{(i)}, a^{(i)}) . \tag{5.200}$$

Then, according to (5.192), from $(w_-^{(i)}, w_+^{(i)}, z_-^{(i)})$, which are one-sided restrictions on $\mathfrak{s}^{(i)}$, we may uniquely define a velocity field $\mathcal{F}_{\mathfrak{s}^{(i)}}(t)$, which may be in turn integrated to define

$$\mathfrak{s}^{(i+1)}(t) = \int_0^t \mathcal{F}_{\mathfrak{s}^{(i)}}(s) ds \tag{5.201}$$

for all $t \in [0, \overline{\varepsilon}]$. Since $\mathfrak{s}^{(i)}$ satisfies (5.13), by Lemma 5.27, we have that $\mathcal{F}_{\mathfrak{s}^{(i)}}$ satisfies the bounds in (5.193a)–(5.193b). Using (5.201) and Lemma 5.27, we in turn deduce that $\mathfrak{s}^{(i+1)}$ satisfies (5.13), on the same time interval $\overline{\varepsilon}$. Thus, under the above described iteration $\mathfrak{s}^{(i)} \mapsto \mathfrak{s}^{(i+1)}$, the set of inequalities (5.13) is stable.

The sequence of curves $\{\mathfrak{s}^{(i)}\}_{i\geqslant 0}$ is uniformly bounded in $W^{2,\infty}(0,\overline{\varepsilon})$, in light of the bounds (5.13), and for $i\geqslant 0$ it satisfies (5.201). From the Arzela-Ascoli theorem, we may thus deduce that there exists at least one sub-sequential uniform limit $\mathfrak{s}$, of the family $\{\mathfrak{s}^{(i)}\}_{i\geqslant 0}$, which inherits the bounds (5.13). However, in order to show that this limit point $\mathfrak{s}$ solves (5.191), we would need to show that $\mathcal{F}_{\mathfrak{s}^{(i)}}\to\mathcal{F}_{\mathfrak{s}}$ when $\mathfrak{s}^{(i)}\to\mathfrak{s}$. This continuity of $\mathcal{F}_{\mathfrak{s}}$ with respect to $\mathfrak{s}$ is addressed in the next section, where we in fact show that the sequence $\{\mathfrak{s}^{(i)}\}_{i\geqslant 0}$ is in fact Cauchy in $W^{1,\infty}(0,\overline{\varepsilon})$.

5.10.3 Contraction Mapping and Convergence of the Shock Curve Iteration

By (5.192), in order to compare $\mathcal{F}_{\mathfrak{s}^{(i+1)}}$ and $\mathcal{F}_{\mathfrak{s}^{(i)}}$, it is obviously sufficient and necessary to compare the tuples $(w_-^{(i+1)}, w_+^{(i+1)}, z_-^{(i+1)})$ and $(w_-^{(i+1)}, w_+^{(i+1)}, z_-^{(i+1)})$. Note however that these tuples represent restrictions of the functions $(w^{(i+1)}, z^{(i+1)})$ and $(w^{(i)}, z^{(i)})$, which are themselves *defined on different domains*; thus in order to compare $(w^{(i+1)}, z^{(i+1)})$ and $(w^{(i)}, z^{(i)})$, we need to re-map them of a fixed domain, by shifting $\mathsf{y}=\theta-\mathfrak{s}^{(i+1)}(t)$, respectively $\mathsf{y}=\theta-\mathfrak{s}^{(i)}(t)$.

As such, for every $i\geqslant 0$, and for $(\mathsf{y},t)\in(\mathbb{T}\setminus\{0\})\times[0,\overline{\varepsilon}]$, we define

$$(\mathsf{w}^{(i)},\mathsf{z}^{(i)},\mathsf{k}^{(i)},\mathsf{a}^{(i)})\,(\mathsf{y},t)=(w^{(i)},z^{(i)},k^{(i)},a^{(i)})\left(\mathsf{y}+\mathfrak{s}^{(i)}(t),t\right)\,, \tag{5.202}$$

where $s^{(0)}(t)=\kappa t$, and for $i\geqslant 1$ the curve $s^{(i)}$ is defined recursively via (5.201). Since Proposition 5.6 and the bound (5.189) guarantee that $(w^{(i)}, z^{(i)}, k^{(i)}, a^{(i)})\in\mathcal{X}_{\overline{\varepsilon}}$ are well-defined and differentiable on the spacetime domain $\mathbb{T}\times[0,\overline{\varepsilon}]\setminus\{(\mathfrak{s}^{(i)}(t),t)\}_{t\in[0,\overline{\varepsilon}]}$, the new unknowns $(\mathsf{w}^{(i)},\mathsf{z}^{(i)},\mathsf{k}^{(i)},\mathsf{a}^{(i)})$ are all well-defined and differentiable on the i-independent domain $(\mathbb{T}\setminus\{0\})\times[0,\overline{\varepsilon}]$ with bounds inherited from the space $\mathcal{X}_{\overline{\varepsilon}}$ defined in (5.140), allowing us to compare them to each other. Note that due to the shift (5.202), we have

$$w_-^{(i)}(t)=\lim_{\mathsf{y}\to 0^-}\mathsf{w}^{(i)}(\mathsf{y},t)\,,\ \ w_+^{(i)}(t)=\lim_{\mathsf{y}\to 0^+}\mathsf{w}^{(i)}(\mathsf{y},t)\,,$$
$$z_-^{(i)}(t)=\lim_{\mathsf{y}\to 0^-}\mathsf{z}^{(i)}(\mathsf{y},t)\,,\ \ k_-^{(i)}(t)=\lim_{\mathsf{y}\to 0^-}\mathsf{k}^{(i)}(\mathsf{y},t)\,,$$

the system of equations (5.66) (which encode the jump conditions) are satisfied for every $i\geqslant 0$, $t\in[0,\overline{\varepsilon}]$, and $\mathcal{F}_{\mathfrak{s}^{(i)}}$ may be expressed in terms of the above variables. Moreover, by (5.91) we have that for each $i\geqslant 0$ the unknowns in (5.202) solve the system of equations

$$\left(\partial_t+(\lambda_3^{(i)}-\dot{\mathfrak{s}}^{(i)})\partial_{\mathsf{y}}\right)\mathsf{w}^{(i)}=-\tfrac{8}{3}\mathsf{a}^{(i)}\mathsf{w}^{(i)}+\tfrac{1}{4}\mathsf{c}^{(i)}\left(\partial_t+(\lambda_3^{(i)}-\dot{\mathfrak{s}}^{(i)})\partial_{\mathsf{y}}\right)\mathsf{k}^{(i)}\,, \tag{5.203a}$$

$$\left(\partial_t+(\lambda_1^{(i)}-\dot{\mathfrak{s}}^{(i)})\partial_{\mathsf{y}}\right)\mathsf{z}^{(i)}=-\tfrac{8}{3}\mathsf{a}^{(i)}\mathsf{z}^{(i)}-\tfrac{1}{4}\mathsf{c}^{(i)}\left(\partial_t+(\lambda_1^{(i)}-\dot{\mathfrak{s}}^{(i)})\partial_{\mathsf{y}}\right)\mathsf{k}^{(i)}\,, \tag{5.203b}$$

$$\left(\partial_t+(\lambda_2^{(i)}-\dot{\mathfrak{s}}^{(i)})\partial_{\mathsf{y}}\right)\mathsf{k}^{(i)}=0\,, \tag{5.203c}$$

$$\left(\partial_t + (\lambda_2^{(i)} - \dot{\mathfrak{s}}^{(i)})\partial_{\mathsf{y}}\right) \mathsf{a}^{(i)} = -\tfrac{4}{3}(\mathsf{a}^{(i)})^2 + \tfrac{1}{3}(\mathsf{w}^{(i)} + \mathsf{z}^{(i)})^2 - \tfrac{1}{6}(\mathsf{w}^{(i)} - \mathsf{z}^{(i)})^2 , \tag{5.203d}$$

in the interior of $(\mathbb{T} \setminus \{0\}) \times [0, \overline{\varepsilon}]$, where we have denoted $\mathsf{c}^{(i)} = \frac{1}{2}(\mathsf{w}^{(i)} - \mathsf{z}^{(i)})$, and have use the usual notation for the three wave speeds at level i.

Since we have seen earlier that for all $i \geqslant 0$ the curves $\mathfrak{s}^{(i)}$ satisfy (5.13), by the proof of Lemma 5.27 (see the first line of estimate (5.199)) and the mean value theorem, for all $i \geqslant 0$ we have that

$$\begin{aligned}\left|\mathcal{F}_{\mathfrak{s}^{(i+1)}} - \mathcal{F}_{\mathfrak{s}^{(i)}}\right| &\leqslant \tfrac{2}{3}\kappa \mathsf{b}^{-\frac{3}{2}} t^{-\frac{1}{2}} \left|\mathsf{z}_-^{(i+1)} - \mathsf{z}_-^{(i)}\right| + \tfrac{4}{3}\mathsf{b}^{\frac{3}{2}}\kappa^{-1} t^{\frac{1}{2}} \left|[\![\mathsf{w}^{(i+1)}]\!] - [\![\mathsf{w}^{(i)}]\!]\right| \\ &\quad + \left(1 + 3\mathsf{b}^3\kappa^{-2} t\right) \left|\langle\!\langle \mathsf{w}^{(i+1)}\rangle\!\rangle - \langle\!\langle \mathsf{w}^{(i)}\rangle\!\rangle\right| ,\end{aligned} \tag{5.204}$$

holds uniformly for $t \in (0, \overline{\varepsilon}]$. Thus, it remains to estimate the right side of (5.204).

For this purpose, we fix an $i \geqslant 0$, and denote

$$\begin{aligned}(\delta\mathsf{w}, \delta\mathsf{z}, \delta\mathsf{k}, \delta\mathsf{a}, \delta\mathsf{c}, \delta\dot{\mathfrak{s}}) &= (\mathsf{w}^{(i+1)}, \mathsf{z}^{(i+1)}, \mathsf{k}^{(i+1)}, \mathsf{a}^{(i+1)}, \mathsf{c}^{(i+1)}, \dot{\mathfrak{s}}^{(i+1)}) \\ &\quad - (\mathsf{w}^{(i)}, \mathsf{z}^{(i)}, \mathsf{k}^{(i)}, \mathsf{a}^{(i)}, \mathsf{c}^{(i)}, \dot{\mathfrak{s}}^{(i)}) .\end{aligned} \tag{5.205}$$

We note that $(\delta\mathsf{w}, \delta\mathsf{z}, \delta\mathsf{k}, \delta\mathsf{a}, \delta\mathsf{c})(x, 0) = 0$. We subtract from (5.203) at level $i + 1$, the equations (5.203) at level i, in order to estimate the increments defined above, via the maximum principle, to obtain

- From (5.203c) we have that

$$\left(\partial_t + (\lambda_2^{(i+1)} - \dot{\mathfrak{s}}^{(i+1)})\partial_{\mathsf{y}}\right) \delta\mathsf{k} = -\partial_{\mathsf{y}}\mathsf{k}^{(i)} \left(\tfrac{2}{3}\delta w + \tfrac{2}{3}\delta z - \delta\dot{\mathfrak{s}}\right) .$$

  Since Proposition 5.6 guarantees that $k^{(i)} \in \mathcal{X}_{\overline{\varepsilon}}$, the function $\mathsf{k}^{(i)}$ satisfies the bound (5.141f), and so similarly to (5.186) we may obtain

$$\sup_{[0,t]} \|\delta\mathsf{k}\|_{L^\infty} \leqslant \mathsf{m}^3 t^{\frac{3}{2}} \left(\sup_{[0,t]} \|\delta\mathsf{w}\|_{L^\infty} + \sup_{[0,t]} \|\delta\mathsf{z}\|_{L^\infty} + \sup_{[0,t]} |\delta\dot{\mathfrak{s}}|\right) \tag{5.206}$$

  where the L^∞ norms are taken over the domain $\mathbb{T} \setminus \{0\}$.
- Similarly, from (5.203d) we have

$$\begin{aligned}\left(\partial_t + (\lambda_2^{(i+1)} - \dot{\mathfrak{s}}^{(i+1)})\partial_{\mathsf{y}}\right) \delta\mathsf{a} &= -\partial_{\mathsf{y}}\mathsf{a}^{(i)} \left(\tfrac{2}{3}\delta w + \tfrac{2}{3}\delta z - \delta\dot{\mathfrak{s}}\right) - \tfrac{4}{3}(\mathsf{a}^{(i+1)} + \mathsf{a}^{(i)})\delta\mathsf{a} \\ &\quad + \tfrac{1}{3}(\mathsf{w}^{(i+1)} + \mathsf{w}^{(i)} + \mathsf{z}^{(i+1)} + \mathsf{z}^{(i)})(\delta\mathsf{w} + \delta\mathsf{z}) \\ &\quad - \tfrac{1}{6}(\mathsf{w}^{(i+1)} + \mathsf{w}^{(i)} - \mathsf{z}^{(i+1)} - \mathsf{z}^{(i)})(\delta\mathsf{w} - \delta\mathsf{z}) .\end{aligned}$$

Using that $(w^{(i)}, z^{(i)}, k^{(i)}, a^{(i)}) \in \mathcal{X}_{\overline{\varepsilon}}$, and since $|w^{(1)}(\theta, t)| = |w_{\mathsf{B}}(\theta, t)| \leqslant \mathsf{m}$, similarly to (5.187) we obtain

$$\sup_{[0,t]} \|\delta \mathsf{a}\|_{L^\infty} \leqslant \mathsf{m}^3 t \left(\sup_{[0,t]} \|\delta \mathsf{w}\|_{L^\infty} + \sup_{[0,t]} \|\delta \mathsf{z}\|_{L^\infty} + \sup_{[0,t]} |\delta \dot{\mathfrak{s}}| \right) + 3\mathsf{m}^3 t \sup_{[0,t]} \|\delta \mathsf{a}\|_{L^\infty}$$
$$+ \left(\mathsf{m} t + \mathsf{m}^3 t^2 + \mathsf{m}^3 t^{\frac{5}{2}} \right) \left(\sup_{[0,t]} \|\delta \mathsf{w}\|_{L^\infty} + \sup_{[0,t]} \|\delta \mathsf{z}\|_{L^\infty} \right)$$

and thus, taking into account (5.142),

$$\sup_{[0,t]} \|\delta \mathsf{a}\|_{L^\infty} \leqslant 4\mathsf{m}^3 t \left(\sup_{[0,t]} \|\delta \mathsf{w}\|_{L^\infty} + \sup_{[0,t]} \|\delta \mathsf{z}\|_{L^\infty} + \sup_{[0,t]} |\delta \dot{\mathfrak{s}}| \right) \tag{5.207}$$

since $t \leqslant \overline{\varepsilon} \ll 1$.

- Next, we turn to (5.203a), which gives

$$\left(\partial_t + (\lambda_3^{(i+1)} - \dot{\mathfrak{s}}^{(i+1)})\partial_{\mathsf{y}}\right) \delta \mathsf{w} = -\partial_{\mathsf{y}} \mathsf{w}^{(i)} \left(\delta \mathsf{w} + \tfrac{1}{3} \delta z - \delta \dot{\mathfrak{s}}\right) - \tfrac{8}{3} a^{(i+1)} \delta \mathsf{w} - \tfrac{8}{3} \mathsf{w}^{(i)} \delta \mathsf{a}$$
$$+ \tfrac{1}{4} \mathsf{c}^{(i+1)} \left(\partial_t + (\lambda_3^{(i+1)} - \dot{\mathfrak{s}}^{(i+1)})\partial_{\mathsf{y}}\right) \delta \mathsf{k}$$
$$- \tfrac{1}{4} \mathsf{c}^{(i+1)} \left(\delta \mathsf{w} + \tfrac{1}{3} \delta \mathsf{z} - \delta \dot{\mathfrak{s}}^{(i)}\right) \partial_{\mathsf{y}} \mathsf{k}^{(i)}$$
$$+ \tfrac{1}{4} \delta \mathsf{c} \left(\lambda_3^{(i)} - \lambda_2^{(i)}\right) \partial_{\mathsf{y}} \mathsf{k}^{(i)}$$

Recalling that $c^{(i+1)}$ solves $(\partial_t + \lambda_3^{(i+1)} \partial_\theta) c^{(i+1)} = -\frac{8}{3} a^{(i+1)} c^{(i+1)} - \frac{2}{3} c^{(i+1)} \partial_\theta z^{(i+1)}$, see e.g. (5.138b), we obtain from the above that

$$\left(\partial_t + (\lambda_3^{(i+1)} - \dot{\mathfrak{s}}^{(i+1)})\partial_{\mathsf{y}}\right) \left(\delta \mathsf{w} - \tfrac{1}{4} \mathsf{c}^{(i+1)} \delta \mathsf{k}\right)$$
$$= -\partial_{\mathsf{y}} \mathsf{w}^{(i)} \left(\delta \mathsf{w} + \tfrac{1}{3} \delta \mathsf{z} - \delta \dot{\mathfrak{s}}\right) - \tfrac{8}{3} a^{(i+1)} \delta \mathsf{w} - \tfrac{8}{3} \mathsf{w}^{(i)} \delta \mathsf{a}$$
$$- \tfrac{1}{4} \delta \mathsf{k} \left(\tfrac{8}{3} \mathsf{a}^{(i+1)} \mathsf{c}^{(i+1)} + \tfrac{2}{3} \mathsf{c}^{(i+1)} \partial_{\mathsf{y}} \mathsf{z}^{(i+1)}\right)$$
$$- \tfrac{1}{4} \mathsf{c}^{(i+1)} \left(\delta \mathsf{w} + \tfrac{1}{3} \delta \mathsf{z} - \delta \dot{\mathfrak{s}}^{(i)}\right) \partial_{\mathsf{y}} \mathsf{k}^{(i)} + \tfrac{1}{6} \mathsf{c}^{(i)} \partial_{\mathsf{y}} \mathsf{k}^{(i)} \delta \mathsf{c} \,.$$

Following (5.183), the above equation is composed with the flow of $\lambda_3^{(i+1)} - \dot{\mathfrak{s}}^{(i+1)}$, which of course is just $\eta^{(i+1)} - \mathfrak{s}^{(i+1)}$, and then integrated in time. Note that $\partial_{\mathsf{y}} \mathsf{w}^{(i)} \circ (\eta^{(i+1)} - \mathfrak{s}^{(i+1)}) = (\partial_\theta w^{(i)}) \circ \eta^{(i+1)}$ and (5.56a) holds. Thus, using that $(w^{(i)}, z^{(i)}, k^{(i)}, a^{(i)}) \in \mathcal{X}_{\overline{\varepsilon}}$ and $(w^{(i+1)}, z^{(i+1)}, k^{(i+1)}, a^{(i+1)}) \in \mathcal{X}_{\overline{\varepsilon}}$ similarly to (5.184) we may deduce that

$$\sup_{[0,t]} \|\delta \mathsf{w}\|_{L^\infty} \leqslant \mathsf{m} \left(1 + 4\mathsf{m}^3 t\right) \sup_{[0,t]} \|\delta \mathsf{k}\|_{L^\infty} + \left(\tfrac{19}{40} + 4\mathsf{m}^3 t\right) \sup_{[0,t]} \|\delta \mathsf{w}\|_{L^\infty}$$

$$+\left(\tfrac{1}{6}+2\mathsf{m}^4 t^{\frac{3}{2}}\right)\sup_{[0,t]}\|\delta\mathsf{z}\|_{L^\infty}$$
$$+3\mathsf{m}t\sup_{[0,t]}\|\delta\mathsf{a}\|_{L^\infty}+\left(\tfrac{19}{40}+\mathsf{m}^4 t^{\frac{3}{2}}\right)\sup_{[0,t]}|\delta\dot{\mathfrak{s}}|\,.$$

Upon taking $\overline{\varepsilon}$ to be sufficiently small with respect to m, taking into account (5.142) we deduce

$$\sup_{[0,t]}\|\delta\mathsf{w}\|_{L^\infty}\leqslant \mathsf{m}^3\sup_{[0,t]}\|\delta\mathsf{k}\|_{L^\infty}+\tfrac{1}{2}\sup_{[0,t]}\|\delta\mathsf{z}\|_{L^\infty}+2\mathsf{m}^3 t\sup_{[0,t]}\|\delta\mathsf{a}\|_{L^\infty}$$
$$+\left(\tfrac{19}{21}+8\mathsf{m}^3 t\right)\sup_{[0,t]}|\delta\dot{\mathfrak{s}}| \tag{5.208}$$

- Lastly, from (5.203b) and (5.138c) we similarly deduce

$$\begin{aligned}&\left(\partial_t+(\lambda_1^{(i+1)}-\dot{\mathfrak{s}}^{(i+1)})\partial_{\mathsf{y}}\right)\left(\delta\mathsf{z}+\tfrac{1}{4}\mathsf{c}^{(i+1)}\delta\mathsf{k}\right)\\&=-\partial_{\mathsf{y}}\mathsf{z}^{(i)}\left(\tfrac{1}{3}\delta\mathsf{w}+\delta\mathsf{z}-\delta\dot{\mathfrak{s}}\right)-\tfrac{8}{3}a^{(i+1)}\delta\mathsf{z}-\tfrac{8}{3}\mathsf{z}^{(i)}\delta\mathsf{a}\\&\quad+\tfrac{1}{4}\delta\mathsf{k}\left(\tfrac{8}{3}\mathsf{a}^{(i+1)}\mathsf{c}^{(i+1)}-\tfrac{2}{3}\mathsf{c}^{(i+1)}\partial_{\mathsf{y}}\mathsf{w}^{(i+1)}\right)\\&\quad+\tfrac{1}{4}\mathsf{c}^{(i+1)}\left(\tfrac{1}{3}\delta\mathsf{w}+\delta\mathsf{z}-\delta\dot{\mathfrak{s}}^{(i)}\right)\partial_{\mathsf{y}}\mathsf{k}^{(i)}+\tfrac{1}{6}\mathsf{c}^{(i)}\partial_{\mathsf{y}}\mathsf{k}^{(i)}\delta\mathsf{c}\end{aligned}$$

and then similarly to (5.208) we have

$$\sup_{[0,t]}\|\delta\mathsf{z}\|_{L^\infty}\leqslant \mathsf{m}^3\sup_{[0,t]}\|\delta\mathsf{k}\|_{L^\infty}+\mathsf{m}^6 t^{\frac{3}{2}}\sup_{[0,t]}\|\delta\mathsf{w}\|_{L^\infty}$$
$$+3\mathsf{m}^3 t^{\frac{5}{2}}\sup_{[0,t]}\|\delta\mathsf{a}\|_{L^\infty}+\mathsf{m}^6 t^{\frac{3}{2}}\sup_{[0,t]}|\delta\dot{\mathfrak{s}}|\,. \tag{5.209}$$

Combining the estimates (5.206)-(5.209), and defining

$$N_i(t):=\sup_{[0,t]}\|\delta\mathsf{w}\|_{L^\infty}+t^{-\frac{3}{4}}\sup_{[0,t]}\|\delta\mathsf{z}\|_{L^\infty}+t^{-1}\sup_{[0,t]}\|\delta\mathsf{k}\|_{L^\infty}+t^{-\frac{1}{2}}\sup_{[0,t]}\|\delta\mathsf{a}\|_{L^\infty}\ , \tag{5.210}$$

where we recall the notation in (5.205), we arrive at

$$N_i(t)\leqslant 3(1+\mathsf{m}^3)t^{\frac{1}{4}}N_i(t)+(\tfrac{19}{21}+6\mathsf{m}^3 t^{\frac{1}{2}})\sup_{[0,t]}|\delta\dot{\mathfrak{s}}|$$

and thus upon taking $t\leqslant\overline{\varepsilon}$ to be sufficiently small in terms of m, we deduce

$$N_i(t)\leqslant\tfrac{20}{21}\sup_{[0,t]}|\delta\dot{\mathfrak{s}}|=\tfrac{20}{21}\sup_{[0,t]}\left|\dot{\mathfrak{s}}^{(i+1)}-\dot{\mathfrak{s}}^{(i)}\right|. \tag{5.211}$$

Recalling the definitions (5.201) and (5.210), from the bounds (5.204) and (5.211) we deduce that

$$\begin{aligned}\sup_{[0,t]}\big|\dot{\mathfrak{s}}^{(i+2)}-\dot{\mathfrak{s}}^{(i+1)}\big| &= \sup_{[0,t]}\big|\mathcal{F}_{\mathfrak{s}^{(i+1)}}-\mathcal{F}_{\mathfrak{s}^{(i)}}\big| \\ &\leqslant \tfrac{2}{3}\kappa \mathsf{b}^{-\frac{3}{2}}t^{-\frac{1}{2}}\sup_{[0,t]}\Big\|\mathsf{z}^{(i+1)}-\mathsf{z}^{(i)}\Big\|_{L^\infty} \\ &\quad+\Big(1+3\mathsf{b}^{\frac{3}{2}}\kappa^{-1}t^{\frac{1}{2}}+3\mathsf{b}^3\kappa^{-2}t\Big)\sup_{[0,t]}\Big\|\mathsf{w}^{(i+1)}-\mathsf{w}^{(i)}\Big\| \\ &\leqslant \tfrac{2}{3}\kappa \mathsf{b}^{-\frac{3}{2}}t^{\frac{1}{4}}N_i(t)+\Big(1+\tfrac{8}{3}\mathsf{b}^{\frac{3}{2}}\kappa^{-1}t^{\frac{1}{2}}+3\mathsf{b}^3\kappa^{-2}t\Big)N_i(t) \\ &\leqslant (1+t^{\frac{1}{5}})N_i(t) \\ &\leqslant \tfrac{20}{21}(1+t^{\frac{1}{5}})\sup_{[0,t]}\big|\dot{\mathfrak{s}}^{(i+1)}-\dot{\mathfrak{s}}^{(i)}\big| \\ &\leqslant \tfrac{41}{42}\sup_{[0,t]}\big|\dot{\mathfrak{s}}^{(i+1)}-\dot{\mathfrak{s}}^{(i)}\big| \end{aligned} \tag{5.212}$$

upon taking $\overline{\varepsilon}$, and hence t, sufficiently small with respect to κ, b, c, and m. Note that $\frac{41}{42}<1$, and so we have a contraction. Since $s^{(0)}=\kappa t$, and all the sequence of iterates satisfy (5.13), we deduce that

$$\sup_{[0,t]}\big|\dot{\mathfrak{s}}^{(i+1)}-\dot{\mathfrak{s}}^{(i)}\big| \leqslant \big(\tfrac{41}{42}\big)^i \sup_{[0,t]}\big|\dot{\mathfrak{s}}^{(1)}-\kappa\big| \leqslant \big(\tfrac{41}{42}\big)^i \mathsf{m}^4 t\,. \tag{5.213}$$

The bounds (5.212)–(5.213) have as consequence the fact that the sequence of shock curve iterates $\{\mathfrak{s}^{(i)}\}_{i\geqslant 0}$ defined in (5.201) is Cauchy in $W^{1,\infty}(0,\overline{\varepsilon})$, and thus has a unique limit point

$$\mathfrak{s}=\lim_{i\to\infty}\mathfrak{s}^{(i)} \qquad \text{in} \qquad W^{1,\infty}(0,\overline{\varepsilon})\,, \tag{5.214}$$

which inherits the bound (5.13). The bound (5.212) moreover shows that $\mathcal{F}_{\mathfrak{s}^{(i)}}\to\mathcal{F}_{\mathfrak{s}}$ as $i\to\infty$ in $C^0(0,\overline{\varepsilon})$, and by (5.201) we obtain that $\mathfrak{s}$ solves shock evolution equation (5.191), as desired.

Lastly, in view of (5.190), associated to this limit point $\mathfrak{s}$, which satisfies the bound (5.13), Proposition 5.6 determines a unique solution $(w,z,k,a)\in\mathcal{X}_{\overline{\varepsilon}}$ of the azimuthal form of the Euler Eqs. (3.5)–(3.6) on either side of the shock curve, which also satisfies the Rankine-Hugoniot jump conditions (3.13a)–(3.13b), and the shock speed $\dot{\mathfrak{s}}$ is given by (3.12b), as desired.

5.11 Uniqueness of Solutions

The uniqueness of solutions holds in the following sense. Consider w_0 which satisfies (5.1), and a_0 which satisfies (5.4). For $i\in\{1,2\}$, assume that $\mathfrak{s}^{(i)}$ is a C^2 smooth shock curve defined on $[0,T]$ for some $T>0$, which satisfies (5.13) on $[0,T]$. Assume that

$(w, z, k, a)^{(i)}$ are $C^1_{x,t}$ smooth solutions of the azimuthal form of the Euler Eqs. (3.5)–(3.6) on the spacetime domain $\mathcal{D}_T$, i.e., on either side of the shock curve $\mathfrak{s}$, with initial datum $(w_0, 0, 0, a_0)$. Moreover, assume that the restrictions of $(w, z, k)^{(i)}$ satisfy the Rankine-Hugoniot jump conditions (3.13a)–(3.13b), and that the shock speed $\dot{\mathfrak{s}}$ is given by (3.12b). Lastly, assume that $(w, z, k, a)^{(i)} \in \mathcal{X}_{\overline{\varepsilon}}$, as defined in (5.141)–(5.140). Then, if $\overline{\varepsilon} \leqslant T$ is sufficiently small (in terms of the constants κ, b, c, m), we have that $\mathfrak{s}^{(1)} \equiv \mathfrak{s}^{(2)}$ on $0, \overline{\varepsilon}$, and $(w, z, k, a)^{(1)} \equiv (w, z, k, a)^{(2)}$ on $\mathcal{D}_{\overline{\varepsilon}}$.

The proof of this statement is a direct consequence of the contraction mapping established in Section 5.10, and of the fact that $z^{(i)}(\cdot, t) \equiv 0$ on $\mathbb{T} \setminus [\mathfrak{s}_1^{(i)}(t), \mathfrak{s}^{(i)}(t)]$, and $k^{(i)}(\cdot, t) \equiv 0$ on $\mathbb{T}\setminus[\mathfrak{s}_2^{(i)}(t), \mathfrak{s}^{(i)}(t)]$. More precisely, for $i \in \{1, 2\}$ use the definition (5.202) to remap the two sets of solutions to the same space-time domain, and then use (5.205) (with $i = 1$) to denote their difference. As in (5.210), define

$$N(t) := \sup_{[0,t]} \|\delta \mathsf{w}\|_{L^\infty} + t^{-\frac{3}{4}} \sup_{[0,t]} \|\delta \mathsf{z}\|_{L^\infty} + t^{-1} \sup_{[0,t]} \|\delta \mathsf{k}\|_{L^\infty} + t^{-\frac{1}{2}} \sup_{[0,t]} \|\delta \mathsf{a}\|_{L^\infty} \ .$$

Then, as in (5.211) and (5.212), we may show that the bounds

$$N(t) \leqslant \tfrac{20}{21} \sup_{[0,t]} |\delta \dot{\mathfrak{s}}|$$

and

$$\sup_{[0,t]} |\delta \dot{\mathfrak{s}}| \leqslant (1 + t^{\frac{1}{5}}) N(t)$$

hold for all $t \in [0, \overline{\varepsilon}]$, whenever $\overline{\varepsilon}$ is chosen to be sufficiently small with respect to the aforementioned parameters. This shows that $N(t) = 0 = \delta\dot{\mathfrak{s}}(t)$ for all $t \in [0, \overline{\varepsilon}]$. Since $\mathfrak{s}^{(i)}(0) = 0$, it follows that $\delta\mathfrak{s} \equiv 0$, and thus also that $N \equiv 0$, thereby concluding the uniqueness proof.

5.12 Proof of Theorem 5.5

The proof of Theorem 5.5 is a direct consequence of Proposition 5.6, of the contraction mapping established in Sect. 5.10, and of the uniqueness in Sect. 5.11, as described next.

The parameter $\overline{\varepsilon} > 0$ in item (i) is chosen to be possibly smaller than what is required in Proposition 5.6, as required by the estimates in Sections 5.10 and 5.11. The *existence* of the regular shock curve $\mathfrak{s}$ and of the solution $(w, z, k, a) \in \mathcal{X}_{\overline{\varepsilon}}$ to the azimuthal form of the Euler equations (3.5), follows from the contraction mapping in Section 5.10. Note that in view of (5.191), the shock curve $\mathfrak{s}$ obeys the correct ODE, while the desired properties for (w, z, k, a) follow from Proposition 5.6 applied to this limiting shock curve. The *uniqueness* of the solution $(\mathfrak{s}, w, k, z, a)$ such that $\mathfrak{s}$ satisfies (5.13) and $(w, z, k, a) \in \mathcal{X}_{\overline{\varepsilon}}$, is established in section 5.11. Taking into account Proposition 5.6, we have thus established items (i), (ii), (iii), (iv), (vii), and along with the support properties for k and z claimed in items (v) and (vi).

In order to complete the proof of the theorem, it remains to establish the following: the precise bounds for k near $\mathfrak{s}_2$ (as claimed in item (v)), the precise bounds for z near $\mathfrak{s}_1$ (as claimed in item (vi)), the specific vorticity bounds (and its continuity across $\mathfrak{s}$) claimed in item (viii), and the continuity of a, respectively the jump for $\partial_\theta a$ across $\mathfrak{s}$, as claimed in item (ix). These properties of the solution are established in Subsections 5.12.1 and 5.12.2, below.

5.12.1 Improved Bounds for z and k near $\mathfrak{s}_1$ Respectively $\mathfrak{s}_2$

The information $(w, z, k, a) \in \mathcal{X}_{\bar\varepsilon}$ does not directly provide estimates for $z(\theta, t)$ and $k(\theta, t)$ which vanish as $\theta \to \mathfrak{s}_1(t)^+$, respectively $\theta \to \mathfrak{s}_2(t)^+$. Such bounds may however be easily obtained, as follows.

From (3.5c), the definitions of the stopping time $\mathcal{T}$ and of the flow ϕ_t, and the estimate (5.69b), we obtain

$$\big|k(\theta, t)\big| = \big|k_-(\mathfrak{s}(\mathcal{T}(\theta, t)), \mathcal{T}(\theta, t))\big| \leqslant 40\mathsf{b}^{\frac{9}{2}}\kappa^{-3}\mathcal{T}(\theta, t)^{\frac{3}{2}} \tag{5.215}$$

for all $(\theta, t) \in \mathcal{D}^k_{\bar\varepsilon}$. Similarly, from (5.110), (5.81), and (5.155) (with $n \to \infty$) we deduce that

$$\big|\partial_\theta k(\theta, t)\big| \leqslant \tfrac{4}{\kappa}\big|\tfrac{d}{dt}k_-(\mathcal{T}(\theta, t))\big| \leqslant 200\mathsf{b}^{\frac{9}{2}}\kappa^{-4}\mathcal{T}(\theta, t)^{\frac{1}{2}} \tag{5.216}$$

for all $(\theta, t) \in \mathcal{D}^k_{\bar\varepsilon}$. Since $\mathcal{T}(\theta, t) \approx \frac{3}{\kappa}(\theta - \mathfrak{s}_2(t))$, see e.g. (6.144a) below, the above two estimates give a precise order of vanishing for k and k_y as $y \to \mathfrak{s}_2(t)^+$.

Next, let us consider the behavior of z near $\mathfrak{s}_2(t)$. For $(\theta, t) \in \mathcal{D}^k_{\bar\varepsilon}$, from (3.5b) we obtain

$$\begin{aligned} z(\theta, t) &= z(\mathfrak{s}(\mathcal{J}(\theta, t)), \mathcal{J}(\theta, t))e^{-\frac{8}{3}\int_{\mathcal{J}(\theta,t)}^t a\circ\psi_t ds} \\ &\quad + \tfrac{1}{6}\int_{\mathcal{J}(\theta,t)}^t (c^2 k_\theta)\circ\psi_t e^{-\frac{8}{3}\int_s^t a\circ\psi_t ds'} ds\,. \end{aligned} \tag{5.217}$$

Using (5.69a), (5.141a), (5.141g), and (5.216), we deduce that

$$\big|z(y, t)\big| \leqslant 5\mathsf{b}^{\frac{9}{2}}\kappa^{-2}\mathcal{J}(\theta, t)^{\frac{3}{2}} + 40\mathsf{m}^2\mathsf{b}^{\frac{9}{2}}\kappa^{-4}\int_{\mathcal{J}(\theta,t)}^t \mathcal{T}(\psi_t(\theta, s), s)^{\frac{1}{2}} ds\,.$$

In order to estimate the integral term in the above estimate, we use (5.154) to bound $\frac{5}{2}\kappa^{-1}(\theta - \mathfrak{s}_2(t)) \leqslant \mathcal{T}(\theta, t) \leqslant \frac{7}{2}\kappa^{-1}(\theta - \mathfrak{s}_2(t))$ for all $\mathfrak{s}_2(t) < \theta < \mathfrak{s}(t)$, for $\bar\varepsilon$ sufficiently small. As such, it is natural to define $\gamma(s) = \psi_t(\theta, s) - \mathfrak{s}_2(s)$, and note that due to (5.158), we have $\dot\gamma(s) = \lambda_1(\psi_t(\theta, s), s) - \dot{\mathfrak{s}}_2(s) \in [-\frac{\kappa}{2}, -\frac{\kappa}{4}]$. Hence,

$$\int_{\mathcal{J}(\theta,t)}^t \mathcal{T}(\psi_t(\theta, s), s)^{\frac{1}{2}} ds \leqslant 2\kappa^{-\frac{1}{2}}\int_{\mathcal{J}(\theta,t)}^t \gamma(s)^{\frac{1}{2}} ds$$

$$\begin{aligned}
&\leqslant -8\kappa^{-\frac{3}{2}} \int_{\jmath(\theta,t)}^{t} \dot{\gamma}(s)(\gamma(s))^{\frac{1}{2}} ds \\
&= 6\kappa^{-\frac{3}{2}} \left(\gamma(\jmath(\theta,t))^{\frac{3}{2}} - \gamma(t)^{\frac{3}{2}}\right) \\
&\leqslant 6\kappa^{-\frac{3}{2}} (\mathfrak{s}(\jmath(\theta,t)) - \mathfrak{s}_2(\jmath(\theta,t)))^{\frac{3}{2}} \\
&\leqslant 2\jmath(\theta,t)^{\frac{3}{2}} .
\end{aligned}$$

Combining the above two inequalities we arrive at

$$\left|z(\theta,t)\right| \leqslant 12\mathsf{b}^{\frac{9}{2}}\kappa^{-2}\jmath(\theta,t)^{\frac{3}{2}} \tag{5.218}$$

for all $(y,t) \in \mathcal{D}^k_{\overline{\varepsilon}}$. For $(\theta,t) \in \mathcal{D}^z_{\overline{\varepsilon}} \setminus \overline{\mathcal{D}^k_{\overline{\varepsilon}}}$, the same bound as in (5.218) holds. Indeed, for $s \in [\jmath(\theta,t),t]$ such that $\psi_t(\theta,s) \notin \mathcal{D}^k_{\overline{\varepsilon}}$, we have that $k_\theta(\psi_t(\theta,s),s) = 0$, so that the integrand in the second term in (5.217) vanishes for such s. On the other hand, for $s \in [\jmath(\theta,t),t]$ such that $\psi_t(\theta,s) \in \mathcal{D}^k_{\overline{\varepsilon}}$ we again appeal to (5.216), and to the fact that $\jmath(\psi_t(\theta,s),s) = \jmath(\theta,t)$. Estimate (5.218) and the bound $\frac{5}{2}\kappa^{-1}(\theta - \mathfrak{s}_1(t)) \leqslant \jmath(\theta,t) \leqslant \frac{7}{2}\kappa^{-1}(\theta - \mathfrak{s}_2(t))$, which holds for $\mathfrak{s}_1(t) < \theta < \mathfrak{s}_2(t)$ and $\overline{\varepsilon}$ sufficiently small, gives the rate of vanishing of $z(\theta,t)$ as $\theta \to \mathfrak{s}_1(t)^+$. Moreover, since $z(\mathfrak{s}_1(t),t) = 0$ by using the definition of the derivative as the limit of finite differences, from (5.218) we immediately deduce also that

$$(\partial_\theta z)(\mathfrak{s}_1(t),t) = 0 . \tag{5.219}$$

5.12.2 Bounds for the Specific Vorticity, the Radial Velocity, and Its Derivative

The continuity of the radial velocity a on $\mathbb{T}\times[0,\varepsilon]$ is a consequence of the construction: the continuous initial data a_0 (see (5.4)) is propagated smoothly along the characteristic flow of λ_2 (which is continuous, in fact Lipschitz continuous in space and time) in the domain $(\mathcal{D}^k_{\overline{\varepsilon}})^{\complement}$, and in particular a limiting value for a from the right side of the shock curve is obtained; these values of a on the shock curve then serve as Cauchy data for the region $\mathcal{D}^k_{\overline{\varepsilon}}$, using that the flow of λ_2 is transversal to the shock curve. In detail, from (5.129), the the Lipschitz regularity of $\phi^{(n)}_t(\theta,\cdot)$ with respect to both θ and t (see Lemma 5.24 and its proof, the boundedness of $\partial_t\phi^{(n)}_t$ follows in the same way as (5.155), since $\partial_t\phi^{(n)}_t$ solves the same equation as $\partial_\theta\phi^{(n)}_t$ except with datum 0 instead of 1 at (θ,t)), the continuity of a_0, and the bounds (5.141), inductively imply that $a^{(n)}$ is continuous on $\mathbb{T}\times[0,\varepsilon]$, and thus so is its uniform limit a. In particular, $[\![a(\cdot,t)]\!] = 0$.

Concerning the specific vorticity, we note that from the uniform bound (5.140) and the lower bound on w_0 in (5.1b), we have that the sequence of specific vorticities $\{\varpi^{(n)}\}_{n\geqslant 1}$, where $\varpi^{(n)} = 4(w^{(n)} + z^{(n)} - \partial_\theta a^{(n)})(c^{(n)})^{-2}e^{k^{(n)}}$, is uniformly bounded in $L^\infty(\mathcal{D}_{\overline{\varepsilon}})$, by $300\mathsf{m}\kappa^{-2}$. Thus the weak-* limiting vorticity ϖ also lies in $L^\infty(\mathcal{D}_{\overline{\varepsilon}})$, and inherits this global bound. By repeating the argument in Section 5.8.13, since the right side of (5.139) vanishes as $n \to \infty$ (when integrated against smooth test functions), we obtain that ϖ is a $L^\infty_{x,t}$ weak solution of (3.9) in $\mathcal{D}_{\bar\varepsilon}$. Since $(w, z) \in \mathcal{X}_{\bar\varepsilon}$, we have that λ_2 is Lipschitz, giving uniqueness of weak solutions to (3.9), and thus ϖ can be computed classically by integrating along the characteristics of λ_2 (see (5.221) below).

In order to obtain a sharper estimate for the limiting specific vorticity ϖ we recall that from (5.5) that

$$10\kappa^{-1} \leqslant \varpi_0(\theta) \leqslant 28\kappa^{-1} \tag{5.220}$$

for all $\theta \in \mathbb{T}$. Integrating the evolution (3.9) along the characteristics $\phi_t(\theta, s)$, for $s \in [0, t]$, we obtain that

$$\begin{aligned}\varpi(\theta,t) &= \varpi_0(\phi_t(\theta,0)) e^{\frac{8}{3}\int_0^t a(\phi_t(\theta,s),s)ds} \\ &+ \begin{cases} \frac{4}{3}\int_{\mathcal{T}(\theta,t)}^t e^{k(\phi_t(\theta,s),s)}(\partial_\theta k)(\phi_t(\theta,s),s) e^{\frac{8}{3}\int_s^t a(\phi_t(\theta,s'),s')ds'} ds\,, & \text{for } (\theta,t)\in \mathcal{D}^k_{\bar\varepsilon} \\ 0\,, & \text{for } (\theta,t)\in (\mathcal{D}^k_{\bar\varepsilon})^{\complement} \end{cases}.\end{aligned} \tag{5.221}$$

Then, for all $(\theta, t) \in \mathcal{D}_{\bar\varepsilon}$, using the bounds (5.141g), (5.141e), and (5.141f), we deduce that

$$\begin{aligned}|\varpi(\theta,t) - \varpi_0(\phi_t(\theta,0))| &\leqslant 3R_7 t\, |\varpi_0(\phi_t(\theta,0))|\, e^{3R_7 t} \\ &\quad + R_6(t^{\frac{3}{2}} - \mathcal{T}(\theta,t)^{\frac{3}{2}}) e^{3R_7 t + R_5 t^{\frac{3}{2}}} \\ &\leqslant Ct\,.\end{aligned} \tag{5.222}$$

Since $t \leqslant \bar\varepsilon \ll 1$, it follows from the above estimate and (5.220) that

$$9\kappa^{-1} \leqslant \varpi(\theta,t) \leqslant 30\kappa^{-1}\,, \tag{5.223}$$

for all $(\theta, t) \in \mathcal{D}_{\bar\varepsilon}$.

The continuity of the specific vorticity across the shock curve $\mathfrak{s}$ follows from (5.221), the continuity of ϖ_0 (see (5.5)), the continuity of a established earlier, the Lipschitz continuity of $\phi_t(\theta, \cdot)$ in both space and time (which holds in light of the argument in Lemma 5.24 and the uniform convergence $\lambda_1^{(n)} \to \lambda_1$), the transversality of the flow $\phi_t(\theta, \cdot)$ to the shock curve, the bounds (5.141), and the fact that by definition $\mathcal{T}(\theta, t) \to t$ as $y \to \mathfrak{s}(t)^-$.

It only remains to consider the behavior of $\partial_\theta a$ near the shock curve, claimed in item (ix). From (3.8) we have that $\partial_\theta a = w + z - \frac{1}{4}\varpi c^2 e^k$ and thus, using the continuity of ϖ across the shock curve, for every $t \in (0, \bar\varepsilon]$ we deduce that

$$\begin{aligned}[\![\partial_\theta a]\!] &= [\![w]\!] + [\![z]\!] - \tfrac{1}{4}\varpi|_{(\mathfrak{s}(t),t)}[\![w]\!]\langle\!\langle c\rangle\!\rangle\langle\!\langle e^k\rangle\!\rangle + \tfrac{1}{4}\varpi|_{(\mathfrak{s}(t),t)}[\![z]\!]\langle\!\langle c\rangle\!\rangle\langle\!\langle e^k\rangle\!\rangle \\ &\quad - \tfrac{1}{4}\varpi|_{(\mathfrak{s}(t),t)}\langle\!\langle c^2\rangle\!\rangle[\![e^k]\!] \\ &= \underbrace{[\![w]\!]\left(1 - \tfrac{1}{4}\varpi|_{(\mathfrak{s}(t),t)}\langle\!\langle c\rangle\!\rangle\langle\!\langle e^k\rangle\!\rangle\right)}_{=:J_{a,1}}\end{aligned}$$

$$+ \underbrace{[\![z]\!] + \tfrac{1}{4}\varpi|_{(\mathfrak{s}(t),t)}[\![z]\!]\langle\!\langle c\rangle\!\rangle\langle\!\langle e^k\rangle\!\rangle - \tfrac{1}{4}\varpi|_{(\mathfrak{s}(t),t)}\langle\!\langle c^2\rangle\!\rangle[\![e^k]\!]}_{=:J_{a,2}} .$$

Using the fact that $(w, z, k, a) \in \mathcal{X}_{\overline{\varepsilon}}$, the precise information on w_{B} provided by Proposition 5.7, that the specific vorticity satisfies (5.223), and that the jumps in z and k (hence also the jump in e^k) satisfy (5.69), we obtain

$$\left|J_{a,2}(t)\right| \leqslant Ct^{\frac{3}{2}}$$

and that

$$J_{a,2}(t) = \left(2\mathsf{b}^{\frac{3}{2}}t^{\frac{1}{2}} + \mathcal{O}(t)\right)\left(1 - \tfrac{1}{8}\varpi|_{(\mathfrak{s}(t),t)}\kappa + \mathcal{O}(t)\right)$$

$$\xrightarrow{(5.223)} \quad -3\mathsf{b}^{\frac{3}{2}}t^{\frac{1}{2}} \leqslant -\tfrac{11}{4}\mathsf{b}^{\frac{3}{2}}t^{\frac{1}{2}} - Ct \leqslant J_{a,2}(t) \leqslant -\tfrac{1}{4}\mathsf{b}^{\frac{3}{2}}t^{\frac{1}{2}} + Ct \leqslant -\tfrac{1}{5}\mathsf{b}^{\frac{3}{2}}t^{\frac{1}{2}}$$

for all $t \in (0, \overline{\varepsilon}]$. By combining the above three displays we arrive at

$$-4\mathsf{b}^{\frac{3}{2}}t^{\frac{1}{2}} \leqslant [\![\partial_\theta a]\!](t) \leqslant -\tfrac{1}{6}\mathsf{b}^{\frac{3}{2}}t^{\frac{1}{2}}$$

since $\overline{\varepsilon}$, and hence t, is sufficiently small. The above estimate concludes the proof of Theorem 5.5.

6 A Precise Description of the Higher Order Singularities

The goal of this section is to establish:

Theorem 6.1 (Shocks, cusps, and weak discontinuities) *Let $\overline{\varepsilon} > 0$, $\mathfrak{s} \in C^2$, $\mathfrak{s}_1, \mathfrak{s}_2 \in C^1$, $(w, z, k, a) \in \mathcal{X}_{\overline{\varepsilon}}$ be as in Theorem 5.5. For $t \in (0, \overline{\varepsilon}]$, we have the following upper bounds on higher order derivatives:*

$$\left|w_{\theta\theta}(\theta,t)\right| \lesssim \begin{cases} t^{-\frac{5}{3}}, & \text{if } \theta \leqslant \mathfrak{s}_2(t) \text{ or } \theta \geqslant \mathfrak{s}(t) + \frac{\kappa t}{3} \\ t^{-\frac{5}{2}} + \Upsilon(\theta,t)^{-\frac{1}{2}} & \text{if } \mathfrak{s}_2(t) < \theta < \mathfrak{s}(t) \\ t^{-\frac{5}{2}} & \text{if } \mathfrak{s}(t) < \theta < \mathfrak{s}(t) + \frac{\kappa t}{3} \end{cases}, \tag{6.1a}$$

$$\left|z_{\theta\theta}(\theta,t)\right| \lesssim \begin{cases} \Upsilon(\theta,t)^{-\frac{1}{2}}, & \text{if } \mathfrak{s}_2(t) < \theta < \mathfrak{s}(t) \\ \daleth(\theta,t)^{-\frac{1}{2}} & \text{if } \mathfrak{s}_1(t) < \theta \leqslant \mathfrak{s}_2(t) \end{cases}, \tag{6.1b}$$

$$\left|k_{\theta\theta}(\theta,t)\right| \lesssim \Upsilon(\theta,t)^{-\frac{1}{2}} \qquad \text{if } \mathfrak{s}_2(t) < \theta < \mathfrak{s}(t), \tag{6.1c}$$

$$\left|a_{\theta\theta}(\theta,t)\right| \lesssim \begin{cases} t^{-\frac{2}{3}}, & \text{if } y \leqslant \mathfrak{s}_2(t) \text{ or } \theta \geqslant \mathfrak{s}(t) + \frac{\kappa t}{3} \\ t^{-1} + t\Upsilon(\theta,t)^{-\frac{1}{2}} & \text{if } \mathfrak{s}_2(t) < \theta < \mathfrak{s}(t) \\ t^{-1} & \text{if } \mathfrak{s}(t) < \theta < \mathfrak{s}(t) + \frac{\kappa t}{3} \end{cases}, \tag{6.1d}$$

$$\left|\varpi_\theta(\theta,t)\right| \lesssim 1 + \mathbf{1}_{(\theta,t)\in\mathcal{D}^k_{\overline{\varepsilon}}}\left(t - \Upsilon(\theta,t)\right)\Upsilon(\theta,t)^{-\frac{1}{2}}, \tag{6.1e}$$

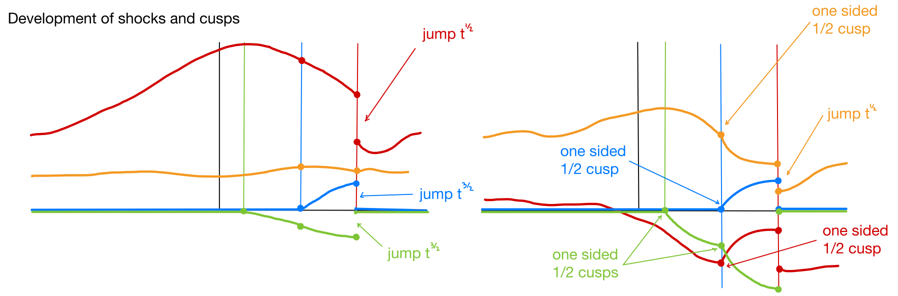

Fig. 12 Schematic of the tuple (w, z, k, a) at $t \in (0, \overline{\varepsilon}]$. On the left, we have sketched w in red, z in green, k in blue, and a in orange. On the right, we have sketched the derivatives w_θ in red, z_θ in green, k_θ in blue, and a_θ in orange

where the implicit constants in $\lesssim$ only depend on m*, cf. (6.8)–(6.13), and (6.14). In particular, for every $t > 0$, the first and second derivatives of (w, z, k, a) are bounded on both $\mathfrak{s}(t)^-$ and $\mathfrak{s}(t)^+$.*

Moreover, $\mathfrak{s}_1(t)$ and $\mathfrak{s}_2(t)$ are C^1 smooth curves of weak characteristic discontinuities in the following precise sense:

(i) The spacetime curve $\mathfrak{s}_2(t)$ is a weak contact discontinuity *with the property that second derivatives of (w, z, k, a) blow up on $\mathfrak{s}_2^+(t)$; in particular, for generic constants c and C,*

$$c(\theta - \mathfrak{s}_2(t))^{-\frac{1}{2}} \leqslant w_{\theta\theta}(\theta, t), -z_{\theta\theta}(\theta, t), k_{\theta\theta}(\theta, t), -t^{-1}a_{\theta\theta}(\theta, t) \leqslant C(\theta - \mathfrak{s}_2(t))^{-\frac{1}{2}} \tag{6.2}$$

for $\mathfrak{s}_2(t) < \theta$ and $\theta - \mathfrak{s}_2(t) \ll t$. The sum $w_{\theta\theta} + z_{\theta\theta}$ remains bounded on $\mathfrak{s}_2(t)$ and

$$\left|w_{\theta\theta}(\theta, t) + z_{\theta\theta}(\theta, t)\right| \lesssim t^{-\frac{1}{2}}, \tag{6.3}$$

for $\mathfrak{s}_2(t) < \theta < \mathfrak{s}_2(t) + \frac{\kappa t}{6}$. Lastly, the functions $(w_\theta, z_\theta, k_\theta, a_\theta)$ form $C^{\frac{1}{2}}$-cusps along $\mathfrak{s}_2(t)^+$.

(ii) The spacetime curve $\mathfrak{s}_1(t)$ is a weak discontinuity such that only $z_{\theta\theta}$ blows up on $\mathfrak{s}_1(t)^+$,

$$c(\theta - \mathfrak{s}_1(t))^{-\frac{1}{2}} \leqslant -z_{\theta\theta}(\theta, t) \leqslant C(\theta - \mathfrak{s}_1(t))^{-\frac{1}{2}}, \tag{6.4}$$

for $\mathfrak{s}_1(t) < \theta$ with $\theta - \mathfrak{s}_1(t) \ll t$, while second derivatives of (w, k, z) remain bounded in terms of inverse powers of t. The function z_θ forms a $C^{\frac{1}{2}}$-cusp along $\mathfrak{s}_1(t)^+$ (Fig. 12).

The proof of Theorem 6.1 is the subject of the remainder of this section: in Section 6.1 we give the bootstrap assumptions which yield (6.1), Sections 6.2–6.6 are dedicated to closing these bootstraps, while Sections 6.7 and 6.8 are dedicated to the analysis of the weak singularities emerging on $\mathfrak{s}_1$ and $\mathfrak{s}_2$. The summary of the proof is given in Section 6.9.

We note that the bounds for the second order derivatives of (w, z, k, a) claimed in Theorem 6.1 greatly differ according to the location of the space-time point (θ, t) where they are evaluated: while *far away* from $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}$ all information concerning w and a is propagated smoothly from the initial datum, for (θ, t) near the space-time curves $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}$, obtaining upper bounds and *matching lower bounds* for second derivatives is a delicate matter, which requires a region-by-region analysis. Accordingly, we shall consider three separate cases:

- $(\theta, t) \in \mathcal{D}^k_{\overline{\varepsilon}}$, the region between $\mathfrak{s}_2$ and $\mathfrak{s}$. Here, for all $t > 0$ the second derivatives of (w, z, k, a) are bounded as $\theta \to \mathfrak{s}(t)^-$, but they all blow up as $\theta \to \mathfrak{s}_2(t)^+$, due to the presence of the entropy.
- $(\theta, t) \in \mathcal{D}^z_{\overline{\varepsilon}} \setminus \overline{\mathcal{D}^k_{\overline{\varepsilon}}}$, the region between $\mathfrak{s}_1$ and $\mathfrak{s}_2$. In this region $k \equiv 0$, and this implies that the second derivatives of w remain bounded as $\theta \to \mathfrak{s}_2(t)^-$; nonetheless, the second derivative of a still develops a singularity here, highlighting the two-dimensional nature of Euler in azimuthal symmetry model. On the other hand, approaching $\mathfrak{s}_1(t)$ from the right side, only the second derivative of z develops a singularity.
- $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}} \setminus \overline{\mathcal{D}^z_{\overline{\varepsilon}}}$, the region which is either to the left of $\mathfrak{s}_1$ or the the right of $\mathfrak{s}$. In this region we have that $z \equiv 0$ and $k \equiv 0$, and thus the analysis reduces to the study of w and a alone. We show that for all $t > 0$, these quantities have bounded second derivatives, uniformly in this region, essentially because they are determined solely in terms of the initial data.

Remark 6.2 Naturally, the further away (θ, t) are from $\mathfrak{s}_1(t)$ (to the left) or $\mathfrak{s}(t)$ (to the right), the further away we are from any singular behavior, and so the bounds for $\partial_\theta^2 w$ and $\partial_\theta^2 a$ become better. As such, for simplicity of the presentation we only give proofs of estimates for second derivatives at points $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}} \setminus \overline{\mathcal{D}^z_{\overline{\varepsilon}}}$ which are *close* to $\mathfrak{s}_1$ or $\mathfrak{s}$: either $\mathfrak{s}_1(t) - \overline{\varepsilon}^{\frac{1}{2}} \leqslant \theta \leqslant \mathfrak{s}_1(t)$, or $\mathfrak{s}(t) < \theta < \mathfrak{s}(t) + \overline{\varepsilon}^{\frac{1}{2}}$. In particular, the closeness considered is t-independent, and thus on the complement of this region it is not hard to establish bounds for $\partial_\theta^2(\theta, t)$ and $\partial_\theta^2 a(\theta, t)$ which are uniform in time for $t \in [0, \overline{\varepsilon}]$; these bounds only depend on $\overline{\varepsilon}$, which is a fixed parameter.

Remark 6.3 By the uniform convergence of our iteration scheme and (5.154), we have that

$$\begin{aligned} \psi_t(\theta, s) &= \tfrac{1}{3}\kappa s + \left(\theta - \tfrac{1}{3}kt\right) + \mathcal{O}\left(t^{\frac{4}{3}}\right) \\ &= \tfrac{1}{3}\kappa s + (\theta - \mathfrak{s}_1(t)) + \mathcal{O}\left(t^{\frac{4}{3}}\right), \quad (\theta, t) \in \mathcal{D}^z_{\overline{\varepsilon}}, \end{aligned} \tag{6.5a}$$

$$\begin{aligned} \phi_t(\theta, s) &= \tfrac{2}{3}\kappa s + \left(\theta - \tfrac{2}{3}kt\right) + \mathcal{O}\left(t^{\frac{4}{3}}\right) \\ &= \tfrac{2}{3}\kappa s + (\theta - \mathfrak{s}_2(t)) + \mathcal{O}\left(t^{\frac{4}{3}}\right), \quad (\theta, t) \in \mathcal{D}^z_{\overline{\varepsilon}}. \end{aligned} \tag{6.5b}$$

Remark 6.4 (*Bounds on wave speeds* 1 *and* 2) Recall that ϕ_t and ψ_t are the flows of the wave speeds λ_2 and λ_1, which are the identity at time t. Throughout this section we shall use the following fact: for all $t \in [0, \overline{\varepsilon}]$, and all $y \in [\mathfrak{s}_1(t) - \overline{\varepsilon}^{\frac{1}{2}}, \mathfrak{s}(t) + \overline{\varepsilon}^{\frac{1}{2}}]$, we have

$$\left|\partial_s \phi_t(\theta, s) - \tfrac{2\kappa}{3}\right| = \left|\lambda_2(\phi_t(\theta, s), s) - \tfrac{2\kappa}{3}\right| \leqslant 4\mathsf{b}|\phi_t(\theta, s) - \mathfrak{s}(s)|^{\frac{1}{3}} + 4\mathsf{b}^{\frac{3}{2}} s^{\frac{1}{2}} \quad (6.6a)$$

$$\left|\partial_s \psi_t(\theta, s) - \tfrac{\kappa}{3}\right| = \left|\lambda_1(\psi_t(\theta, s), s) - \tfrac{\kappa}{3}\right| \leqslant 4\mathsf{b}|\psi_t(\theta, s) - \mathfrak{s}(s)|^{\frac{1}{3}} + 4\mathsf{b}^{\frac{3}{2}} s^{\frac{1}{2}} \quad (6.6b)$$

for all $s \in [0, t]$, where $C = C(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$ is a constant. The proofs of (6.6a) and (6.6b) are identical, and rely on the fact that $z(\cdot, s) = \mathcal{O}(s^{\frac{3}{2}})$, and that for $\overline{y} \in \{\phi_t(\theta, s), \psi_t(\theta, s)\}$ we have

$$\begin{aligned} |\kappa - w(\overline{\theta}, s)| &\leqslant |\kappa - w_{\mathsf{B}}(\overline{\theta}, s)| + R_1 s \\ &\leqslant |\kappa - w_0({\eta_{\mathsf{B}}}^{-1}(\overline{\theta}, s))| + R_1 s \\ &\leqslant 2\mathsf{b}|{\eta_{\mathsf{B}}}^{-1}(\overline{\theta}, s)|^{\frac{1}{3}} + R_1 s \\ &\leqslant 3\mathsf{b}^{\frac{3}{2}} s^{\frac{1}{2}} + 4\mathsf{b}|\overline{\theta} - \mathfrak{s}(s)|^{\frac{1}{3}} + R_1 s \end{aligned}$$

The aforementioned restriction on θ not being too far to the left of $\mathfrak{s}_1(t)$ or too far to the right of $\mathfrak{s}(t)$ was used in the third inequality above, because in light of (5.1c) this allows us to bound $|w_0(x) - \kappa| \leqslant 2\mathsf{b}|x|^{\frac{1}{3}}$, since $x = {\eta_{\mathsf{B}}}^{-1}(\overline{\theta}, s)$ satisfies $|x| \leqslant \overline{\varepsilon}^{\frac{1}{4}} \ll 1$. Note that a direct consequence of (6.6a)–(6.6b) and (5.13), we have that

$$\left|\mathfrak{s}(t) - \mathfrak{s}_2(t) - \tfrac{\kappa t}{3}\right| \leqslant C t^{\frac{4}{3}} \quad \text{and} \quad \left|\mathfrak{s}_2(t) - \mathfrak{s}_1(t) - \tfrac{\kappa t}{3}\right| \leqslant C t^{\frac{4}{3}} \quad (6.7)$$

holds uniformly for all $t \in [0, \overline{\varepsilon}]$, for a suitable constant $C = C(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$.

6.1 Second Derivative Bootstraps

The core of the proof of Theorem 6.1 is to obtain suitable second derivative estimates for the unknowns (w, z, k, a), and on the first derivative of ϖ, consistent with (6.1). We achieve this by postulating a number of *bootstrap bounds* — see (6.8), (6.10), (6.12) below — and then show that these same bounds hold with a constant which is better by a factor of 2. Note that the ϖ_θ and $a_{\theta\theta}$ estimates are direct consequences of these bootstrap bounds, see Lemmas 6.5 and 6.6, they are not part of the bootstraps themselves. Rigorously, the bounds (6.8), (6.10), and (6.12) need to be established iteratively for the sequence of approximations $(w^{(n)}, z^{(n)}, k^{(n)})$ which were considered in Section 5.8; then, these estimates hold for the unique limiting solution (w, z, k) by passing $n \to \infty$. When $n = 1$ the bounds (6.8), (6.10), and (6.12) are trivially seen to hold in view of the definition given in (5.123). Then, assuming the bootstraps bounds hold for $(w^{(n)}, z^{(n)}, k^{(n)})$, the analysis in Sections 6.2–6.6 below, shows that they hold for the next iterate $(w^{(n+1)}, z^{(n+1)}, k^{(n+1)})$ defined in Section 5.8, and that they in fact hold with a better constant. In the proof in this section, instead of carrying around the super-indices $\cdot^{(n)}$ and $\cdot^{(n+1)}$ (as was done in Section 5.8), we write the proof as if we had already passed $n \to \infty$, and work directly with the limiting solution. This abuse of notation is justified as described above in this paragraph.

6.1.1 Bootstraps for the Cone $\mathcal{D}^k_{\bar{\varepsilon}}$

For all $(\theta, t) \in \mathcal{D}^k_{\bar{\varepsilon}}$, we suppose that

$$\left|\partial_\theta^2 w(\theta,t) - \partial_\theta^2 w_{\mathsf{B}}(\theta,t)\right| \leqslant M_1\big(\mathcal{T}(\theta,t)^{-\frac{1}{2}} + t^{-2}\big) \tag{6.8a}$$

$$\left|\partial_\theta^2 z(\theta,t)\right| \leqslant M_2 \mathcal{T}(\theta,t)^{-\frac{1}{2}} \tag{6.8b}$$

$$\left|\partial_\theta^2 k(\theta,t)\right| \leqslant M_3 \mathcal{T}(\theta,t)^{-\frac{1}{2}}\,, \tag{6.8c}$$

where

$$M_1 = 10\mathsf{m}^4\,, \qquad M_2 = 10\mathsf{m}^3\,, \qquad M_3 = 2\mathsf{m}^2\,. \tag{6.9}$$

6.1.2 Bootstraps for the Cone $\mathcal{D}^z_{\bar{\varepsilon}} \setminus \overline{\mathcal{D}^k_{\bar{\varepsilon}}}$

For all $(\theta, t) \in \mathcal{D}^z_{\bar{\varepsilon}} \setminus \overline{\mathcal{D}^k_{\bar{\varepsilon}}}$,

$$\left|\partial_\theta^2 w(\theta,t) - \partial_\theta^2 w_{\mathsf{B}}(\theta,t)\right| \leqslant N_1 t^{-\frac{2}{3}} \tag{6.10a}$$

$$\left|\partial_\theta^2 z(\theta,t)\right| \leqslant N_2 \mathcal{J}(\theta,t)^{-\frac{1}{2}} \tag{6.10b}$$

where

$$N_1 = 5\mathsf{m}^4\,, \qquad N_2 = 8\mathsf{m}^3\,. \tag{6.11}$$

6.1.3 Bootstraps for $\mathcal{D}_{\bar{\varepsilon}} \setminus \mathcal{D}^z_{\bar{\varepsilon}}$

For all $(\theta, t) \in \mathcal{D}_{\bar{\varepsilon}} \setminus \overline{\mathcal{D}^z_{\bar{\varepsilon}}}$.

$$\left|\partial_\theta^2 w(\theta,t) - \partial_\theta^2 w_{\mathsf{B}}(\theta,t)\right| \leqslant \begin{cases} N_4 t^{-\frac{2}{3}}, & \text{if } \theta \leqslant \mathfrak{s}_1(t) \text{ or } \theta \geqslant \mathfrak{s}(t) + \frac{\kappa t}{3} \\ N_5 t^{-2}, & \text{if } \mathfrak{s}(t) < \theta < \mathfrak{s}(t) + \frac{\kappa t}{3}\,, \end{cases} \tag{6.12a}$$

where

$$N_4 = 5\mathsf{m}^4\,, \qquad N_5 = 10\mathsf{m}^4\,. \tag{6.13}$$

6.1.4 Bounds for ϖ_θ and $a_{\theta\theta}$

We first show that the bootstrap for the second derivative of k implies a good estimate for the derivative for the specific vorticity.

Lemma 6.5 *Assume that* $(w, z, k, a) \in \mathcal{X}_{\overline{\varepsilon}}$ *is such that* (6.8c) *holds. Then, for all* $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}$, *we have*

$$\left|\varpi_\theta(\theta, t)\right| \leqslant 2\mathsf{m} + \mathbf{1}_{(\theta,t)\in\mathcal{D}^k_{\overline{\varepsilon}}} 4\mathsf{m}^2 \left(t - \mathcal{T}(\theta, t)\right) \mathcal{T}(\theta, t)^{-\frac{1}{2}} . \tag{6.14}$$

Proof of Lemma 6.5 We differentiate the equation for the specific vorticity (3.9) with respect to θ and obtain

$$(\partial_t + \lambda_2\partial_\theta)\varpi_\theta + \left(\partial_\theta\lambda_2 - \tfrac{8}{3}a\right)\varpi_\theta = \tfrac{8}{3}a_\theta\varpi + \tfrac{4}{3}e^k\left(k_\theta^2 + k_{\theta\theta}\right) .$$

For any fixed $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}$, we compose the above identity with $\phi_t(\theta, s)$ and arrive at

$$\tfrac{d}{ds}\left(\varpi_\theta \circ \phi_t\right) + \left(\partial_\theta\lambda_2 \circ \phi_t - \tfrac{8}{3}a \circ \phi_t\right)\left(\varpi_\theta \circ \phi_t\right) = \left(\tfrac{8}{3}a_\theta\varpi + \tfrac{4}{3}e^k(k_\theta^2 + k_{\theta\theta})\right) \circ \phi_t .$$

Denoting the integrating factor associated to the above equation by

$$\begin{aligned} I_{\varpi_\theta} = I_{\varpi_\theta}(\theta, t; s) &= -\int_s^t \left(\partial_\theta\lambda_2(\phi_t(\theta, r), r) - \tfrac{8}{3}a(\phi_t(\theta, r), r)\right)dr \\ &= -\tfrac{2}{3}\int_s^t \left(\partial_\theta w(\phi_t(\theta, r), r) + \partial_\theta z(\phi_t(\theta, r), r)\right. \\ &\qquad \left. - 4a(\phi_t(\theta, r), r)\right)dr , \end{aligned} \tag{6.15}$$

and using that $\phi_t(\theta, t) = \theta$, we then obtain

$$\begin{aligned} \varpi_\theta(\theta, t) = \varpi_0'(\phi_t(\theta, 0))e^{I_{\varpi_\theta}(\theta,t;0)} + \int_0^t &\left(\tfrac{8}{3}a_\theta\varpi\right. \\ &\left. + \tfrac{4}{3}e^k(k_\theta^2 + k_{\theta\theta})\right)(\phi_t(\theta, s), s)e^{I_{\varpi_\theta}(\theta,t;s)}ds . \end{aligned} \tag{6.16}$$

First, we estimate the integrating factor in (6.15), for a fixed (θ, t) in the region of interest, as described in Remark 6.2. Using (6.6a) and (5.13), we have that the curve $\phi_t(\theta, s)$ is transversal to the shock curve $\mathfrak{s}$, in the sense that $\partial_s\phi_t(\theta, s) \leqslant \frac{2}{3}\kappa + \mathcal{O}(\overline{\varepsilon}^{\frac{1}{3}}) \leqslant \frac{3}{4}\kappa < \dot{\mathfrak{s}}$. Hence, we may apply Lemma 5.11 with $\gamma(s) = \phi_t(\theta, s)$, separately on the intervals $[t', t] \mapsto [\mathcal{T}(\theta, t), t]$ and $[t', t] \mapsto [s, \mathcal{T}(\theta, t)]$, with the second case being of course empty if $\mathcal{T}(\theta, t) \leqslant s$. In this way, from estimate (5.57a), (5.141b), (5.141d), and the triangle inequality, we deduce that

$$\left|I_{\varpi_\theta}(\theta, t; s)\right| \leqslant 40\mathsf{b}\kappa^{-\frac{2}{3}}t^{\frac{1}{3}} + 2R_2\mathsf{b}^{-\frac{1}{2}}t^{\frac{1}{2}} + R_4t^{\frac{3}{2}} \leqslant 50\mathsf{b}\kappa^{-\frac{2}{3}}t^{\frac{1}{3}} .$$

As such,

$$\left|e^{I_{\varpi_\theta}(\theta,t;s)} - 1\right| \leqslant 60\mathsf{b}\kappa^{-\frac{2}{3}}t^{\frac{1}{3}} \tag{6.17}$$

uniformly for $s \in [0, t]$, since $t \leqslant \overline{\varepsilon} \ll 1$.

Second, we appeal to the bounds (5.141e), (5.141f), (5.141g), and (5.223), to deduce that

$$\int_0^t \left|\tfrac{8}{3}a_\theta \varpi + \tfrac{4}{3}e^k(k_\theta^2)\right|(\phi_t(\theta,s),s)ds \leqslant 12\kappa^{-1}R_7 t + R_6^2 t^2 \leqslant Ct \tag{6.18}$$

for a suitable $C = C(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$.

Third, we use (5.141e), (6.17), the bound (6.8c), and the fact that $k \equiv 0$ on $[\mathcal{D}^k_{\bar{\varepsilon}}]^{\complement}$ to deduce that for all $(\theta, t) \in \mathcal{D}^k_{\bar{\varepsilon}}$, we have

$$\left|\int_{\mathfrak{T}(\theta,t)}^t \left(e^k k_{\theta\theta}\right)(\phi_t(\theta,s),s)e^{I_{\varpi_\theta}(\theta,t;s)}ds\right| \leqslant 4\mathsf{m}^2\left(t - \mathfrak{T}(\theta,t)\right)\mathfrak{T}(\theta,t)^{-\frac{1}{2}} \tag{6.19}$$

for a suitable $C = C(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$. Here we have implicitly used that $\mathfrak{T}((\phi_t(\theta,s),s)) = \mathfrak{T}(\theta,t)$.

Finally, by appealing to the ϖ_0' estimate in (5.5), we deduce from (6.16), (6.18), and (6.19) that

$$\begin{aligned}\left|\varpi_\theta(\theta,t)\right| &\leqslant \mathsf{m}(1 + 60\mathsf{b}\kappa^{-\frac{2}{3}}t^{\frac{1}{3}}) + Ct + \mathbf{1}_{(\theta,t)\in\mathcal{D}^k_{\bar{\varepsilon}}}4\mathsf{m}^2\left(t - \mathfrak{T}(\theta,t)\right)\mathfrak{T}(\theta,t)^{-\frac{1}{2}}\\ &\leqslant 2\mathsf{m} + \mathbf{1}_{(\theta,t)\in\mathcal{D}^k_{\bar{\varepsilon}}}4\mathsf{m}^2\left(t - \mathfrak{T}(\theta,t)\right)\mathfrak{T}(\theta,t)^{-\frac{1}{2}},\end{aligned}$$

which completes the proof of (6.14). □

The previously established estimate for the derivative of the specific vorticity, (6.14), immediately implies a bound for the second derivative of the radial velocity a:

Lemma 6.6 *Assume that $(w, z, k, a) \in \mathcal{X}_{\bar{\varepsilon}}$ is such that (6.8), (6.10), and (6.12) hold. Then, for all $(\theta, t) \in \mathcal{D}_{\bar{\varepsilon}}$ we have*

$$\left|a_{\theta\theta}(\theta,t)\right| \leqslant \begin{cases} N_3 t^{-\frac{2}{3}}, & \text{if } \theta \leqslant \mathfrak{s}_2(t) \text{ or } \theta \geqslant \mathfrak{s}(t) + \frac{\kappa t}{3}\\ M_5(t^{-1} + t\mathfrak{T}(\theta,t)^{-\frac{1}{2}}), & \text{if } \mathfrak{s}_2(t) < \theta \leqslant \mathfrak{s}(t)\\ N_7 t^{-1}, & \text{if } \mathfrak{s}(t) < \theta < \mathfrak{s}(t) + \frac{\kappa t}{3}\end{cases} \tag{6.20}$$

where the constants N_3, M_5, and N_7 are defined as as

$$N_3 = \mathsf{m}^3, \qquad M_5 = \mathsf{m}^4, \qquad N_7 = \mathsf{m}^3. \tag{6.21}$$

Proof of Lemma 6.6 The proof directly follows from the bounds on the derivative of the specific vorticity contained in the bootstrap estimate (6.14). We rewrite the definition (3.8) as $a_\theta = w + z - \frac{1}{4}c^2e^{-k}\varpi$, and upon differentiating we see that

$$\begin{aligned} a_{\theta\theta} &= w_\theta + z_\theta - \tfrac{1}{4}c(w_\theta - z_\theta)e^{-k}\varpi + \tfrac{1}{4}c^2e^{-k}k_\theta\varpi - \tfrac{1}{4}c^2e^{-k}\varpi_\theta\\ &= -\tfrac{1}{4}c^2e^{-k}\varpi_\theta + w_\theta\left(1 - \tfrac{1}{4}ce^{-k}\varpi\right) + z_\theta\left(1 + \tfrac{1}{4}ce^{-k}\varpi\right) + \tfrac{1}{4}c^2e^{-k}k_\theta\varpi\,. \end{aligned}$$

By using that $(w, z, k, a) \in \mathcal{X}_{\overline{\varepsilon}}$ and the bound (5.223), it follows that for all (θ, t) in the region of interest, we have

$$\left|a_{\theta\theta} + \tfrac{1}{4}c^2 e^{-k}\varpi_\theta - \partial_\theta w_{\mathsf{B}}\left(1 - \tfrac{1}{4}ce^{-k}\varpi\right)\right| \leqslant Ct^{-\frac{1}{2}} \tag{6.22}$$

for a suitable $C = C(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$. For $\partial_\theta w_{\mathsf{B}}$ estimates we refer to (5.37a), ϖ is bounded via (5.223), while for bounds on $\partial_\theta \varpi$ we refer to (6.14). We deduce

$$\begin{aligned}\left|a_{\theta\theta}(\theta, t)\right| \leqslant \mathsf{m}^3 + \mathbf{1}_{(\theta,t)\in\mathcal{D}^k_{\overline{\varepsilon}}}\mathsf{m}^4\left(t - \mathcal{T}(\theta, t)\right)\mathcal{T}(\theta, t)^{-\frac{1}{2}} \\ + \mathsf{m}^2\left((\mathsf{b}t)^3 + |\theta - \mathfrak{s}(t)|^2\right)^{-\frac{1}{3}}\end{aligned} \tag{6.23}$$

The bound (6.23) now directly implies (6.20), as follows.

For $\theta \leqslant \mathfrak{s}_2(t)$ or $\theta \geqslant \mathfrak{s}(t) + \frac{\kappa t}{3}$, we have that $|\theta - \mathfrak{s}(t)| \geqslant \frac{\kappa t}{3} - Ct^{\frac{4}{3}}$, and also $(\theta, t) \notin \mathcal{D}^k_{\overline{\varepsilon}}$. As such, the first bound stated in (6.20) follows from (6.23) as soon as $N_3 \geqslant 2\mathsf{m}^2(\kappa/4)^{-\frac{2}{3}}$. This condition motivates the choice of $N_3 = \mathsf{m}^3$ in (6.21). Similarly, the third bound in (6.20) follows from (6.23) as soon as $N_7 \geqslant 2\mathsf{m}^2\mathsf{b}^{-1}$; this condition holds since $N_7 = \mathsf{m}^3$ as in (6.21). Lastly, we consider the case that $\mathfrak{s}_2(t) < \theta < \mathfrak{s}(t)$, case in which (6.23) implies

$$\begin{aligned}\left|\partial^2_\theta a(\theta, t)\right| &\leqslant \mathsf{m}^3 + \mathsf{m}^4(t - \mathcal{T}(\theta, t))\mathcal{T}(\theta, t)^{-\frac{1}{2}} + \mathsf{m}^2(\mathsf{b}t)^{-1} \\ &\leqslant \mathsf{m}^4(t^{-1} + t\mathcal{T}(\theta, t)^{-\frac{1}{2}})\,.\end{aligned} \tag{6.24}$$

The bound (6.24) then clearly implies the second bound in (6.20) as soon as $M_5 \geqslant \mathsf{m}^4$; a condition which holds in view of the definition of M_5 in (6.21). ◻

6.2 Second Derivatives of the Three Wave Speeds

6.2.1 Improved Estimates for Derivatives of $\eta - \eta_{\mathsf{B}}$

Lemma 6.7 *Given $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}$, define the label $x \in \Upsilon(t)$ by $x = \eta^{-1}(\theta, t)$, where we recall that the set $\Upsilon(t)$ is defined in (5.42). Then*

$$|\partial_x \eta(x, t) - \partial_x \eta_{\mathsf{B}}(x, t)| \leqslant \begin{cases} 50\mathsf{m}t^{\frac{4}{3}}, & \text{if } \theta \notin (\mathfrak{s}_2(t), \mathfrak{s}(t) + \frac{\kappa t}{3}) \\ 10\mathsf{m}t, & \text{if } \theta \in (\mathfrak{s}_2(t), \mathfrak{s}(t) + \frac{\kappa t}{3})\,, \end{cases}, \tag{6.25a}$$

$$\left|\partial^2_x \eta(x, t) - \partial^2_x \eta_{\mathsf{B}}(x, t)\right| \leqslant \begin{cases} 10\mathsf{m}t^{\frac{1}{3}}, & \text{if } \theta \notin (\mathfrak{s}_2(t), \mathfrak{s}(t) + \frac{\kappa t}{3}) \\ 20\mathsf{m}t^{-\frac{1}{2}}, & \text{if } \theta \in (\mathfrak{s}_2(t), \mathfrak{s}(t) + \frac{\kappa t}{3}) \end{cases}. \tag{6.25b}$$

Proof of Lemma 6.7 We first record a few bounds for the derivatives of the Burgers flow map η_{B}. Using (5.17c)–(5.17d), we have that for all $\mathfrak{s}_2(t) < \theta < \mathfrak{s}(t) + \frac{\kappa t}{3}$ and with $x = \eta^{-1}(\theta, t)$

$$\left|\partial_x^2 \eta_{\mathsf{B}}(x,t)\right| \leqslant \left|t w_0''(x)\right| \leqslant \tfrac{1}{3}\mathsf{b}^{-\frac{3}{2}} t^{-\frac{3}{2}}\,, \qquad \left|\partial_x^3 \eta_{\mathsf{B}}(x,t)\right| \leqslant \left|t w_0'''(x)\right| \leqslant 2\mathsf{m}\mathsf{b}^{-4} t^{-3}\,. \tag{6.26}$$

The above estimates hold since $|x| \geqslant \frac{4}{5}(\mathsf{b}t)^{\frac{3}{2}}$, which in turn holds by the definition of $\Upsilon(t)$ and the bound (5.45). For the case that $\theta \leqslant \mathfrak{s}_2(t)$ or $\theta \geqslant \mathfrak{s}(t) + \frac{\kappa t}{3}$, similarly to (5.50) we may show that

$$|\eta(x,s) - \mathfrak{s}(s)| \geqslant |\eta(x,t) - \mathfrak{s}(t)| + \tfrac{4}{5}\mathsf{b}^{\frac{3}{2}} t^{\frac{1}{2}}(t-s) \geqslant \tfrac{\kappa t}{4} \tag{6.27}$$

and so $|x| = |\eta(x,0) - \mathfrak{s}(0)| \geqslant \frac{\kappa t}{4}$. It follows from (5.1) that for labels x such that $|x| \geqslant \frac{\kappa t}{5}$

$$\begin{aligned} \left|w_0'(x)\right| &\leqslant \mathsf{b}\kappa^{-\frac{2}{3}} t^{-\frac{2}{3}}\,,\ \left|\partial_x^2 \eta_{\mathsf{B}}(x,t)\right| \leqslant \left|t w_0''(x)\right| \\ &\leqslant 4\mathsf{b}\kappa^{-\frac{5}{3}} t^{-\frac{2}{3}}\,,\ \left|\partial_x^3 \eta_{\mathsf{B}}(x,t)\right| \leqslant \left|t w_0'''(x)\right| \leqslant 80\mathsf{m}\kappa^{-\frac{8}{3}} t^{-\frac{5}{3}}\,, \end{aligned} \tag{6.28}$$

upon taking $\overline{\varepsilon}$ small enough.

In order to prove (6.25a), we appeal to the identities

$$\eta_{\mathsf{B}x}(x,t) = 1 + \int_0^t \partial_\theta w_{\mathsf{B}} \circ \eta_{\mathsf{B}} \eta_{\mathsf{B}x} ds\,, \qquad \eta_x(x,t) = 1 + \int_0^t (w_\theta + \tfrac{1}{3} z_\theta) \circ \eta \eta_x ds\,. \tag{6.29}$$

In anticipation of subtracting the two identities above, we first derive a useful identity for $\partial_\theta w \circ \eta \eta_x$. To do so, we return to (5.98), which we rewrite as

$$\begin{aligned} &\tfrac{d}{dt}\left((w_\theta - \tfrac{1}{4}ck_\theta)\circ\eta\ \eta_x\right) + \left((\tfrac{8}{3}a - \tfrac{1}{12}ck_\theta)\circ\eta\right)\left((w_\theta - \tfrac{1}{4}ck_\theta)\circ\eta\ \eta_x\right) \\ &\quad = \left(\tfrac{1}{48}ck_\theta(ck_\theta + 4z_\theta) - \tfrac{8}{3}wa_\theta\right)\circ\eta\ \eta_x\,. \end{aligned} \tag{6.30}$$

At this stage it is convenient to introduce the w-good-unknown q^w via

$$q^w(\theta,t) = w_\theta(\theta,t) - \tfrac{1}{4}c(\theta,t)k_\theta(\theta,t)\,, \tag{6.31}$$

the integrating factor in (6.30) as

$$\mathcal{I}(x,s,t) = \int_s^t \tfrac{8}{3}a(\eta(x,s'),s') - \tfrac{1}{12}(ck_\theta)(\eta(x,s'),s')ds'\,, \tag{6.32}$$

and the forcing term in (6.30) by

$$Q^w = \tfrac{1}{48}ck_\theta(ck_\theta + 4z_\theta) - \tfrac{8}{3}wa_\theta\,. \tag{6.33}$$

With this notation, integrating (6.30) and using that $k_0 = 0$, we arrive at

$$q^w(\eta(x,t),t)\eta_x(x,t) = w_0'(x)e^{-\mathcal{I}(x,0,t)} + \int_0^t Q^w(\eta(x,s),s)\eta_x(x,s)e^{-\mathcal{I}(x,s,t)}ds \tag{6.34}$$

Upon recalling the fact that $\partial_\theta w_{\mathsf{B}} \circ \eta_{\mathsf{B}}\, \eta_{\mathsf{B}x} = w_0'$, from (6.29), (6.34), and the definition

$$Q_1 = \tfrac{1}{4}ck_\theta + \tfrac{1}{3}z_\theta\,, \tag{6.35}$$

we obtain

$$\begin{aligned}\partial_t(\eta_x - \eta_{\mathsf{B}x}) &= w_\theta \circ \eta\, \eta_x - \partial_\theta w_{\mathsf{B}} \circ \eta_{\mathsf{B}}\, \eta_{\mathsf{B}x} \\ &= w_0'(x)\left(e^{-\mathcal{I}(x,0,t)} - 1\right) + Q_1 \circ \eta\, \eta_x \\ &\quad + \int_0^t Q^w(\eta(x,s),s)\eta_x(x,s)e^{-\mathcal{I}(x,s,t)}ds \end{aligned} \tag{6.36}$$

which is the main identity relating the derivatives of η and η_{B}.

We recall that (5.1), (5.141), and (5.142) imply

$$\begin{aligned}\left|Q^w(\theta,t)\right| &\leqslant \tfrac{1}{48}(\mathsf{m}R_6t^{\frac12})(\mathsf{m}R_6t^{\frac12} + 4R_4t^{\frac12}) + \tfrac{8}{3}R_7(R_1t + \mathsf{m}) \leqslant 12\mathsf{m}^2 \\ \left|Q_1(\theta,t)\right| &\leqslant (\tfrac14\mathsf{m}R_6 + \tfrac13 R_4)t^{\frac12} \leqslant \tfrac12\mathsf{m}^{\frac32}t^{\frac12}\end{aligned}$$

while (5.54a) gives $|\eta_x(x,s)| \leqslant \frac{7}{4}$. Moreover, the integrating factor $\mathcal{I}$ defined in (6.32) satisfies

$$\left|\mathcal{I}(\cdot,s,t)\right| \leqslant \tfrac{8}{3}R_7(t-s) + \tfrac{1}{18}\mathsf{m}R_6(t^{\frac32} - s^{\frac32}) \leqslant 12\mathsf{m}(t-s)\,. \tag{6.37}$$

Estimate (6.37) will be used frequently throughout the remaining analysis.

In order to prove (6.25a), we integrate (6.36) on the interval $[0,t]$, use that $\eta_x(x,0) = 1 = \eta_{\mathsf{B}x}(x,0)$, and the fact that $(w,z,k,a) \in \mathcal{X}_\varepsilon$ (expressed through the bounds (5.141)), and obtain that

$$\begin{aligned}\left|\eta_x(x,t) - \eta_{\mathsf{B}x}(x,t)\right| &\leqslant |w_0'(x)| \int_0^t \left|e^{-\mathcal{I}(x,0,s)} - 1\right| ds + \tfrac78\mathsf{m}^{\frac32}t^{\frac32} \\ &\leqslant 8\mathsf{m}t^2|w_0'(x)| + \tfrac78\mathsf{m}^{\frac32}t^{\frac32}\,.\end{aligned} \tag{6.38}$$

In the case that $\theta = \eta(x,t) \in (\mathfrak{s}_2(t), \mathfrak{s}(t) + \frac{\kappa t}{3})$, since $|x| = |\eta^{-1}(\theta,t)| \geqslant \frac45(\mathsf{b}t)^{\frac32}$, from (5.17b) and (5.142), we obtain the second bound in (6.25a). On the other hand, for $\theta = \eta(x,t) \notin (\mathfrak{s}_2(t), \mathfrak{s}(t) + \frac{\kappa t}{3})$, from (6.27) we have $|x| \geqslant \frac{\kappa t}{4}$ and so from (6.28), (6.38), and the working assumption (5.2), we obtain that the first bound in (6.25a) holds.

We next estimate $\eta_{xx} - \eta_{\mathsf{B}xx}$. Notice that by differentiating the identity (6.36), factors of η_{xx} appear in both the integral term, which at first leads to non-optimal bounds. Instead, we twice differentiate the equations $\partial_s \eta = \lambda_3 \circ \eta$ and $\partial_s \eta_{\mathsf{B}} = w_{\mathsf{B}} \circ \eta_{\mathsf{B}}$, to find that

$$\begin{aligned}\partial_s(\eta_{xx} - \eta_{\mathsf{B}xx}) &= w_{\theta\theta} \circ \eta\, \eta_x^2 - w_{\mathsf{B}\theta\theta} \circ \eta_{\mathsf{B}}\, \eta_{\mathsf{B}x}^2 + w_\theta \circ \eta\, \eta_{xx} - w_{\mathsf{B}\theta} \circ \eta_{\mathsf{B}}\, \eta_{\mathsf{B}xx} \\ &= \underbrace{\big(w_{\mathsf{B}\theta\theta} \circ \eta_{\mathsf{B}} \circ ({\eta_{\mathsf{B}}}^{-1} \circ \eta) - w_{\mathsf{B}\theta\theta} \circ \eta_{\mathsf{B}}\big)\eta_x^2}_{\mathcal{K}_1} + \underbrace{(w_{\theta\theta} - w_{\mathsf{B}\theta\theta}) \circ \eta\, \eta_x^2}_{\mathcal{K}_2} \\ &\quad + \underbrace{w_{\mathsf{B}\theta\theta} \circ \eta_{\mathsf{B}}\, (\eta_x^2 - \eta_{\mathsf{B}x}^2)}_{\mathcal{K}_3} + \underbrace{(w_\theta \circ \eta - w_{\mathsf{B}\theta} \circ \eta_{\mathsf{B}})\eta_{\mathsf{B}xx}}_{\mathcal{K}_4} \\ &\quad + w_\theta \circ \eta\, (\eta_{xx} - \eta_{\mathsf{B}xx})\,. \end{aligned} \tag{6.39}$$

We shall first provide bounds for the terms $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, and $\mathcal{K}_4$ on the right side of (6.39) in the regions y far from $\mathfrak{s}(t)$ and y close to $\mathfrak{s}(t)$, and then apply the Grönwall inequality to estimate $\eta_{xx} - \eta_{\mathsf{B}xx}$ in these two regions. To sharpen the bounds in the region close to $\mathfrak{s}(t)$, we then return to (6.36) and differentiate it in x.

The case $\theta \leqslant \mathfrak{s}_2(t)$ or $\theta \geqslant \mathfrak{s}(t) + \frac{\kappa t}{3}$. We recall that $x = \eta^{-1}(\theta, t)$ and define the label $\overline{x} = {\eta_{\mathsf{B}}}^{-1}(\theta, t)$. As earlier, from (6.27) and (5.44) we have $|x|, |\overline{x}| \geqslant \frac{\kappa t}{5}$. Using the mean value theorem, and estimates (5.17b), (5.36b), (5.22a), (5.44), and (6.28), we obtain that

$$\left|\eta_x^{-2}\mathcal{K}_1(x, s)\right| \leqslant 2000 R_1 \mathsf{m} \kappa^{-\frac{8}{3}} s^{-\frac{2}{3}} + C s^{-\frac{1}{3}}\,,$$

so that using (5.54a), (5.142), and (5.2)

$$\left|\mathcal{K}_1(x, s)\right| \leqslant \mathsf{m}^4 s^{-\frac{2}{3}}\,. \tag{6.40}$$

Then, using (5.54a) and (6.10a) and (6.12a), we have that

$$\left|\mathcal{K}_2(x, s)\right| \leqslant 4(N_1 + N_4) s^{-\frac{2}{3}}\,. \tag{6.41}$$

In the above estimate we have implicitly used the fact that $\eta(x, s) \notin \mathcal{D}^k_{\frac{k}{\varepsilon}}$, which is a consequence of the assumption on θ being sufficiently far from $\mathfrak{s}(t)$ and of the bound (6.27). Next, by (5.36), the s-independent lower bound on x provided by (6.27), and the w_0 estimates (5.1) and (5.17b), we have

$$|\partial_\theta w_{\mathsf{B}}(\eta_{\mathsf{B}}(x, s), s)| \leqslant \tfrac{5}{3}|w_0'(x)| \leqslant 2\mathsf{b}(\kappa t)^{-\frac{2}{3}}\,, \tag{6.42a}$$

$$\left|\partial_\theta^2 w_{\mathsf{B}}(\eta_{\mathsf{B}}(x, s), s)\right| \leqslant (\tfrac{5}{3})^3|w_0''(x)| \leqslant 16\mathsf{b}(\kappa t)^{-\frac{5}{3}}\,, \tag{6.42b}$$

for all $s \in [0, t]$, and so by (5.54a) and (6.25a)

$$\left|\mathcal{K}_3(x, s)\right| \leqslant C s^{\frac{4}{3}} t^{-\frac{5}{3}} \leqslant C s^{-\frac{1}{3}}\,, \tag{6.43}$$

for a suitable $C = C(\kappa, \mathsf{b}, \mathsf{m}) > 0$. Lastly, in order to bound $\mathcal{K}_4$, we write

$$(w_\theta \circ \eta - w_{\mathsf{B}\theta} \circ \eta_\mathsf{B}) = (w_\theta \circ \eta - w_{\mathsf{B}\theta} \circ \eta_\mathsf{B})\eta_x \eta_x^{-1} ,$$

and

$$(w_\theta \circ \eta - w_{\mathsf{B}\theta} \circ \eta_\mathsf{B})\eta_x = \big(w_\theta \circ \eta\, \eta_x - \partial_\theta w_\mathsf{B} \circ \eta_\mathsf{B}\, \eta_{\mathsf{B}x}\big) - \partial_\theta w_\mathsf{B} \circ \eta_\mathsf{B}\big(\eta_x - \eta_{\mathsf{B}x}\big) .$$

Using the second equality in (6.36), similarly to (6.38) but with t replaced by s, we have that

$$\big|w_\theta \circ \eta\, \eta_x - w_{\mathsf{B}\theta} \circ \eta_\mathsf{B}\, \eta_{\mathsf{B}x}\big| \leqslant 4R_7 s|w_0'(x)| + Cs^{\frac{1}{2}} \leqslant 20\mathsf{b}\kappa^{-\frac{2}{3}}\mathsf{m}st^{-\frac{2}{3}} \leqslant Cs^{\frac{1}{3}} ,$$

where in the second inequality we have also appealed to (6.28). Hence, by combining the above three displays with (5.54a), (6.25a), (6.28), and (6.42), we have that

$$\big|\mathcal{K}_4(x, s)\big| \leqslant Ct^{-\frac{2}{3}} \left(s^{\frac{1}{3}} + t^{-\frac{2}{3}}s^{\frac{4}{3}}\right) \leqslant Cs^{-\frac{1}{3}} , \tag{6.44}$$

for a suitable $C = C(\kappa, \mathsf{b}, \mathsf{m}) > 0$.

Finally, using the bounds (6.40)–(6.44), and the estimates (5.56a) and (5.141a) we apply Grönwall to (6.39) and find that

$$\begin{aligned}\big|\eta_{xx}(x, t) - \eta_{\mathsf{B}xx}(x, t)\big| &\leqslant e^{\frac{1}{2}+2R_2\mathsf{b}^{-\frac{1}{2}}t^{\frac{1}{2}}}16(\mathsf{m}^4 + N_1 + N_4)t^{\frac{1}{3}} \\ &\leqslant 30(\mathsf{m}^4 + N_1 + N_4)t^{\frac{1}{3}} ,\end{aligned} \tag{6.45}$$

in the case that $y \notin (\mathfrak{s}_2(t), \mathfrak{s}(t) + \frac{\kappa t}{3})$.

The case $\mathfrak{s}_2(t) < \theta < \mathfrak{s}(t) + \frac{\kappa t}{3}$. We shall first use (6.39) to provide a (non optimal) bound for the difference $\eta_{xx} - \eta_{\mathsf{B}xx}$. Once we have such a bound, we will then return to the differentiated form of (6.36) to obtain the optimal bound.

Recall the definitions of the labels $\overline{x} = \eta_\mathsf{B}{}^{-1}(\eta(x, t), t)$ and $x = \eta^{-1}(\theta, t)$. At this stage it is convenient to introduce $s = \nu^\sharp(x, t) \in [0, t)$, the *largest time* at which *either* $\eta(x, s) = \mathfrak{s}_2(s) = \mathfrak{s}(s) - \frac{\kappa s}{3} + \mathcal{O}(s^{\frac{4}{3}})$ *or* $\eta(x, s) = \mathfrak{s}(s) + \frac{\kappa s}{3}$. This time $\nu(x, t)$ exists in view of the intermediate function theorem since, $|\eta(x, 0) - \mathfrak{s}(0)| = |x| \geqslant \frac{4}{5}(\mathsf{b}t)^{\frac{3}{2}} > 0$, and is unique since as in (5.50) and in Lemma (5.24), we have that the flow η is transversal to both $\mathfrak{s}_2$ and to $\mathfrak{s}$. In fact, we recall from (5.50) that

$$|\eta(x, s) - \mathfrak{s}(s)| \geqslant |y - \mathfrak{s}(t)| + \tfrac{4}{5}\mathsf{b}^{\frac{3}{2}}t^{\frac{1}{2}}(t - s) \tag{6.46}$$

and therefore, by also taking into account (6.7), we have that

$$\nu^\sharp(x, t) \geqslant \mathsf{b}^{\frac{3}{2}}\kappa^{-1}t^{\frac{3}{2}} \tag{6.47}$$

uniformly for all $x = \eta^{-1}(\theta, t)$, and $\theta \in (\mathfrak{s}_2(t), \mathfrak{s}(t) + \frac{\kappa t}{3})$.

Next, we return to bounding the terms on the right side of (6.39). Then by (5.21a), (5.36b), the mean value theorem, (5.44), and using (5.17b), (5.17c), (6.26) we obtain

$$
\begin{aligned}
\left|\mathcal{K}_1(x,s)\right| &\leqslant 4\left|{\eta_{\mathsf{B}}}^{-1}(\eta(x,s),s)-x\right|\left|\frac{(1+sw_0'(\widetilde{x}))w_0'''(\widetilde{x})-3s(w_0''(\widetilde{x}))^2}{(1+sw_0'(\widetilde{x}))^4}\right| \\
&\leqslant 4R_1s^2(\tfrac{5}{3})^4\left(4\mathsf{m}\mathsf{b}^{-4}t^{-4}+16\mathsf{b}^{-3}st^{-5}\right) \\
&\leqslant \mathsf{m}^4s^2t^{-4}\,.
\end{aligned}
\tag{6.48}
$$

Here we have use that $\widetilde{x}$ lies in between x and ${\eta_{\mathsf{B}}}^{-1}(\eta(x,s),s)$, and thus satisfies $|\widetilde{x}|\geqslant\frac{4}{5}(\mathsf{b}t)^{\frac{3}{2}}$. Next, by (5.54a), (6.8a), (6.10a), and (6.12a),

$$
\left|\mathcal{K}_2(x,s)\right|\leqslant 4(M_1+N_5)\left(s^{-2}+\mathbf{1}_{\mathfrak{s}_2(s)<\eta(x,s)<\mathfrak{s}(s)}\mathcal{T}(\eta(x,s),s)^{-\frac{1}{2}}\right). \tag{6.49}
$$

Next, using (5.36b) and the fact that $|x|\geqslant\frac{4}{5}(\mathsf{b}t)^{\frac{3}{2}}$, combined with the estimates (5.17b) and (5.17c) we obtain that $|w_{\mathsf{B}\theta\theta}(\eta_{\mathsf{B}}(x,s),s)|\leqslant 3\mathsf{b}^{-\frac{3}{2}}t^{-\frac{5}{2}}$. Hence, by also appealing to (5.54a) and (6.25a), we deduce

$$
\left|\mathcal{K}_3(x,s)\right|\leqslant Cst^{-\frac{5}{2}}\,. \tag{6.50}
$$

Finally, by (5.52), (5.141b), (5.142), and (6.26),

$$
\begin{aligned}
\left|\mathcal{K}_4(x,s)\right| &\leqslant \left(4R_1\mathsf{b}^{-\frac{3}{2}}s^{-\frac{1}{2}}+R_2(\mathsf{b}s)^{-\frac{1}{2}}\right)s|w_0''(x)| \\
&\leqslant \left(4R_1\mathsf{b}^{-\frac{3}{2}}s^{-\frac{1}{2}}+R_2(\mathsf{b}s)^{-\frac{1}{2}}\right)s\tfrac{1}{3}\mathsf{b}^{-\frac{3}{2}}t^{-\frac{5}{2}}\leqslant\mathsf{m}^4s^{\frac{1}{2}}t^{-\frac{5}{2}}\,.
\end{aligned}
\tag{6.51}
$$

Summing up the estimates (6.48)–(6.51), we obtain

$$
\begin{aligned}
&\left|\mathcal{K}_1(x,s)\right|+\left|\mathcal{K}_2(x,s)\right|+\left|\mathcal{K}_3(x,s)\right|+\left|\mathcal{K}_4(x,s)\right| \\
&\quad\leqslant\left(4(M_1+N_5)+2\mathsf{m}^4\right)s^{-2}+4M_1\mathbf{1}_{\mathfrak{s}_2(s)<\eta(x,s)\leqslant\mathfrak{s}(s)}\mathcal{T}(\eta(x,s),s)^{-\frac{1}{2}}\,.
\end{aligned}
\tag{6.52}
$$

Let $\widetilde{\nu}(t)=\mathsf{b}^{\frac{3}{2}}\kappa^{-1}t^{\frac{3}{2}}$ be the lower bound in (6.47). With (6.52) in hand we apply the Grönwall inequality to (6.39) on the time interval $[\widetilde{\nu}(t),t]$, which in view of (6.47) is slightly larger than $[\nu^\sharp(x,t),t]$. The point here is that due to (6.47) we know that either $\eta(x,\widetilde{\nu})<\mathfrak{s}_2(\widetilde{\nu})$, or $\eta(x,\widetilde{\nu})>\mathfrak{s}(\widetilde{\nu})+\frac{\kappa\widetilde{\nu}}{3}$, and thus (6.45) holds at the time $\widetilde{\nu}$. We thus deduce that

$$
\begin{aligned}
&\left|\eta_{xx}(x,t)-\eta_{\mathsf{B}xx}(x,t)\right| \\
&\quad\leqslant 30(\mathsf{m}^4+N_1+N_4)\widetilde{\nu}^{\frac{1}{3}}+(4(M_1+N_5)+2\mathsf{m}^4)\int_{\widetilde{\nu}}^t s^{-2}ds \\
&\qquad+4M_1\int_{\widetilde{\nu}}^t\mathbf{1}_{\mathfrak{s}_2(s)<\eta(x,s)\leqslant\mathfrak{s}(s)}\mathcal{T}(\eta(x,s),s)^{-\frac{1}{2}}ds
\end{aligned}
$$

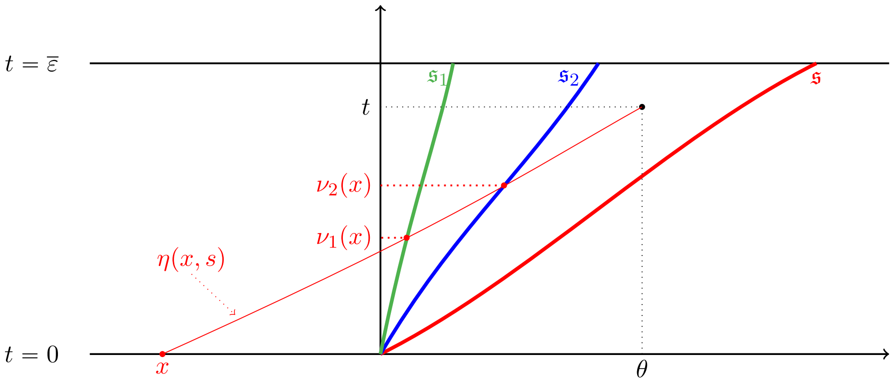

Fig. 13 Fix a point (θ, t) which lies in between $\mathfrak{s}_1$ and $\mathfrak{s}_2$, and let x be the label such that $\eta(x, t) = \theta$. The intersection time of $\eta(x, s)$ with $\mathfrak{s}_2$ is denoted by $\nu_2(x)$, while the intersection time of $\eta(x, s)$ with $\mathfrak{s}_1$ is denoted by $\nu_1(x)$

$$\begin{aligned} &\leqslant Ct^{\frac{1}{2}} + (4(M_1 + N_5) + 2\mathsf{m}^4)\kappa \mathsf{b}^{-\frac{3}{2}} t^{-\frac{3}{2}} \\ &\quad + 4M_1 \int_{\widetilde{\nu}}^{t} \mathbf{1}_{\mathfrak{s}_2(s) < \eta(x,s) \leqslant \mathfrak{s}(s)} \mathcal{T}(\eta(x, s), s)^{-\frac{1}{2}} ds \,. \end{aligned} \tag{6.53}$$

Note that if $\theta > \mathfrak{s}(t)$, then $\{\eta(x, s)\}_{s\in[0,t]}$ does not intersect $\mathcal{D}^k_{\overline{\varepsilon}}$, and so the integral term in the above is vacuous. We thus are left to consider the case $\theta \in (\mathfrak{s}_2(t), \mathfrak{s}(t))$.

In order to bound the integral term on the right side of (6.53), for every $x \in [\eta^{-1}(\mathfrak{s}_2(t), t), \eta^{-1}(\mathfrak{s}(t), t))$ we define the intersection time $s = \nu_2(x)$ at which the 3-characteristic $\eta(x, s)$ intersects the curve $\mathfrak{s}_2(s)$. Just as we showed that $\phi_t(\theta, s)$ is transverse to the shock curve in the proof of Lemma 5.24, by the same argument, for all labels $x \in \Upsilon(t)$, the curve $\eta(x, s)$ is transverse to the characteristic curve $(\mathfrak{s}_2(t), t)$, and so there exists an $\mathfrak{s}_2(t)$-intersection time $\nu_2(x)$ such that

$$\eta(x, \nu_2(x)) = \mathfrak{s}_2(\nu_2(x)) \,. \tag{6.54}$$

Note that for these values of x, we have that $\nu_2(x) = \nu^\sharp(x, t)$, as was previously defined above (6.46). When $x \notin [\eta^{-1}(\mathfrak{s}_2(t), t), \eta^{-1}(\mathfrak{s}(t), t))$ we overload notation, and define $\nu_2(x) = \overline{\varepsilon}$, to signify that $\eta(x, \cdot)$ does not intersect $\mathfrak{s}_2$.

For future purposes, for every $x \in [\eta^{-1}(\mathfrak{s}_1(t), t), \eta^{-1}(\mathfrak{s}(t), t))$ we define the intersection time $s = \nu_1(x)$ at which the 3-characteristic $\eta(x, s)$ intersects the curve $\mathfrak{s}_1(s)$, i.e.

$$\eta(x, \nu_1(x)) = \mathfrak{s}_1(\nu_1(x)) \,. \tag{6.55}$$

The existence and uniqueness of $\nu_1(x)$ is again justified by the transversality of the 3-characteristic and the 1-characteristic. Again, for $x \notin [\eta^{-1}(\mathfrak{s}_1(t), t), \eta^{-1}(\mathfrak{s}(t), t))$, we set $\nu_1(x) = \overline{\varepsilon}$ (Fig. 13).

With this notation, we return to the integral term in (6.53), and recall that $2\kappa^{-1}(\theta - \mathfrak{s}_2(s)) \leqslant \mathcal{T}(\theta, s) \leqslant 4\kappa^{-1}(\theta - \mathfrak{s}_2(s))$. This justifies defining the curve $\gamma(s) = \eta(x, s) -$

$\mathfrak{s}_2(s)$. Note that in view of Remark 6.3 and 6.4, we have that $\dot{\gamma}(s) = \lambda_3(\eta(x,s),s) - \dot{\mathfrak{s}}_2(s) \geqslant \frac{1}{4}\kappa$. Hence,

$$
\begin{aligned}
\int_{\widetilde{\nu}}^{t} \mathbf{1}_{\mathfrak{s}_2(s)<\eta(x,s)\leqslant\mathfrak{s}(s)} \mathcal{T}(\eta(x,s),s)^{-\frac{1}{2}} ds &\leqslant \int_{\nu_2(x)}^{t} \mathcal{T}(\eta(x,s))^{-\frac{1}{2}} ds \\
&\leqslant \kappa^{\frac{1}{2}} \int_{\nu_2(x)}^{t} (\eta(x,s) - \mathfrak{s}_2(s))^{-\frac{1}{2}} ds \\
&\leqslant 4\kappa^{-\frac{1}{2}} \int_{\nu_2(x)}^{t} \dot{\gamma}(s)(\gamma(s))^{-\frac{1}{2}} ds \\
&\leqslant 8\kappa^{-\frac{1}{2}} \gamma(t)^{\frac{1}{2}} = 8\kappa^{-\frac{1}{2}}(\theta - \mathfrak{s}_2(t))^{\frac{1}{2}} \\
&\leqslant 8t^{\frac{1}{2}} .
\end{aligned} \tag{6.56}
$$

In the last inequality above we have used that $|\theta - \mathfrak{s}_2(t)| \leqslant \mathfrak{s}(t) - \mathfrak{s}_2(t) \leqslant \frac{\kappa t}{2}$. From (6.53) and (6.56), we deduce the non-sharp upper bound

$$
\left|\eta_{xx}(x,t) - \eta_{\mathsf{B}\,xx}(x,t)\right| \leqslant 4(M_1 + N_5 + \mathsf{m}^4)\kappa \mathsf{b}^{-\frac{3}{2}} t^{-\frac{3}{2}} , \tag{6.57}
$$

for $x = \eta^{-1}(\theta, t)$, when $y \in (\mathfrak{s}_2(t), \mathfrak{s}(t) + \frac{\kappa t}{3})$.

Note that (6.57) is weaker than the bound claimed in the second line of (6.25b). This rough bound (6.57) may now be used to establish an optimal bound for $\eta_{xx} - \eta_{\mathsf{B}\,xx}$ as follows. Estimate (6.57) is combined with (6.26) and (6.28), together with the bound (6.45), to show that for $\overline{\varepsilon}$ taken sufficiently small we have

$$
\left|\eta_{xx}(x,t)\right| \leqslant \begin{cases} 12\mathsf{b}\kappa^{-\frac{5}{3}} t^{-\frac{2}{3}}, & \text{if } \theta \notin (\mathfrak{s}_2(t), \mathfrak{s}(t) + \frac{\kappa t}{3}) \\ 8(M_1 + N_5 + \mathsf{m}^4)\kappa \mathsf{b}^{-\frac{3}{2}} t^{-\frac{3}{2}} & \text{if } \theta \in (\mathfrak{s}_2(t), \mathfrak{s}(t) + \frac{\kappa t}{3}) \end{cases} , \tag{6.58}
$$

for $t \in [0, \overline{\varepsilon}]$ where $x = \eta^{-1}(\theta, t) \in \Upsilon(t)$. Moreover, by the definition of the time $\nu^\sharp(x,t)$ appearing in (6.47), upon letting $(\theta, t) \mapsto (\eta(x,s), s)$ in (6.58), we obtain that

$$
\left|\eta_{xx}(x,s)\right| \leqslant \begin{cases} 12\mathsf{b}\kappa^{-\frac{5}{3}} s^{-\frac{2}{3}}, & \text{if } s \leqslant \nu^\sharp(x,t) \\ 8(M_1 + N_5 + \mathsf{m}^4)\kappa \mathsf{b}^{-\frac{3}{2}} s^{-\frac{3}{2}} & \text{if } \nu^\sharp(x,t) < s \leqslant t \end{cases} , \tag{6.59}
$$

where we have overloaded notation and have defined $\nu^\sharp(x,t) := t$ whenever $\eta(x,s) < \mathfrak{s}_2(s)$ or $\eta(x,s) > \mathfrak{s}(s) + \frac{\kappa s}{3}$ for all $s \in [0,t]$.

Next, differentiating (6.36), we arrive at

$$
\begin{aligned}
\partial_s(\eta_{xx} - \eta_{\mathsf{B}\,xx}) = w_0''\left(e^{-\mathcal{I}(\cdot,0,s)} - 1\right) - w_0' e^{-\mathcal{I}(\cdot,0,s)} \partial_x \mathcal{I}(\cdot,0,s) \\
+ \partial_\theta Q_1 \circ \eta \, (\eta_x)^2 + Q_1 \circ \eta \, \eta_{xx} \\
+ \int_0^s \Big(\partial_\theta Q^w \circ \eta \, \eta_x^2 + Q^w \circ \eta \, \eta_{xx}
\end{aligned}
$$

$$-Q^w \circ \eta\, \eta_x \partial_x \mathcal{I}(\cdot, s', s)\big)\, e^{-\mathcal{I}(\cdot, s', s)} ds' \tag{6.60}$$

where we recall that $\mathcal{I}$, Q^w and Q^1 are defined as in (6.32), (6.33), and (6.35). From (5.56a), (5.141), (6.8c), and (6.56) we deduce

$$\big|\partial_x \mathcal{I}(\cdot, s', s)\big| \leqslant 24\mathsf{m}s + \mathbf{1}_{s>\nu_2(x)}(\mathsf{m}^{\frac12} + \mathsf{m}M_3)s^{\frac12} \tag{6.61a}$$

$$\big|Q_1(\cdot, s)\big| \leqslant \mathsf{m}^2 s^{\frac12} \tag{6.61b}$$

$$\big|Q^w(\cdot, s)\big| \leqslant 12\mathsf{m}\,. \tag{6.61c}$$

Moreover, differentiating (6.33) and (6.35), using (5.141) we also obtain

$$\big|\partial_\theta Q_1(\cdot, s)\big| \leqslant (\mathsf{m}s)^{\frac12}\big|w_\theta(\cdot, s)\big| + \tfrac{\mathsf{m}}{4}\big|k_{\theta\theta}(\cdot, s)\big| + \tfrac13\big|z_{\theta\theta}(\cdot, s)\big| + Cs \tag{6.62a}$$

$$\begin{aligned}\big|\partial_\theta Q^w(\cdot, s)\big| &\leqslant \mathsf{m}^3 s^{\frac12}\big|k_{\theta\theta}(\cdot, s)\big| + \mathsf{m}^2 s^{\frac12}\big|z_{\theta\theta}(\cdot, s)\big| \\ &\quad + 3\mathsf{m}\big|a_{\theta\theta}(\cdot, s)\big| + 12\mathsf{m}\big|w_\theta(\cdot, s)\big| + C\,.\end{aligned} \tag{6.62b}$$

These bounds are used to estimate the three lines on the right side of (6.60) as follows. Using (6.37) and (6.61a), we obtain

$$\text{first line on RHS of (6.60)} \leqslant 24\mathsf{m}s|w_0''(x)| + 2(\mathsf{m}^{\frac12} + \mathsf{m}M_3)s^{\frac12}|w_0'(x)|\,. \tag{6.63}$$

Next, using (6.61b) and (6.62a), combined with (5.54a), (6.8b), (6.8c), (6.10b), and (6.59), we estimate

$$\begin{aligned}&\text{second line on RHS of (6.60)} \\ &\leqslant \mathsf{m}^2 s^{\frac12}\Big(12\mathsf{b}\kappa^{-\frac53}s^{-\frac23}\mathbf{1}_{s\leqslant\nu^\sharp(x,t)} + 8(M_1 + N_5 + \mathsf{m}^4)\kappa\mathsf{b}^{-\frac32}s^{-\frac32}\mathbf{1}_{s>\nu^\sharp(x,t)}\Big) \\ &\quad + 4(\mathsf{m}s)^{\frac12}\big|w_\theta(\cdot, s)\big| \\ &\quad + (\mathsf{m}M_2 + 2M_3)\mathcal{T}(\eta(x, s), s)^{-\frac12}\mathbf{1}_{s>\nu_2(x)} \\ &\quad + 2N_2\mathcal{J}(\eta(x, s), s)^{-\frac12}\mathbf{1}_{\nu_1(x)<s<\nu_2(x)} + Cs\,.\end{aligned} \tag{6.64}$$

The estimate for the third line of (6.60) is more delicate, and proceeds in several steps. By using (6.59), (6.61a), (6.61c), and (6.62b), combined with (5.54a), (5.56a), (5.141b), (6.8), (6.10), and (6.12), we have

$$\begin{aligned}&\text{third line on RHS of (6.60)} \\ &\leqslant C + C\int_0^{\min\{s,\nu^\sharp(x,t)\}}(s')^{-\frac23}ds' + \mathbf{1}_{s>\nu^\sharp(x,t)}96(M_1 + N_5 + \mathsf{m}^4)\kappa\mathsf{b}^{-\frac32}\mathsf{m} \\ &\quad \int_{\nu^\sharp(x,t)}^s (s')^{-\frac32}ds' \\ &\quad + 4(\mathsf{m}^3 M_3 + \mathsf{m}^2 M_2 + s^{\frac12}M_5)\mathbf{1}_{s>\nu_2(x)}s^{\frac12}\int_{\nu_2(x)}^s \mathcal{T}(\eta(x, s'), s')^{-\frac12}ds'\end{aligned}$$

$$+ 4\mathsf{m}^2 N_2 \mathbf{1}_{s>\nu_1(x)} s^{\frac{1}{2}} \int_{\nu_1(x)}^{\min\{s,\nu_2(x)\}} \jmath(\eta(x,s'),s')^{-\frac{1}{2}} ds'$$
$$+ 12\mathsf{m}\left((M_5 + N_7)\mathbf{1}_{s>\nu_2(x)} \int_{\nu_2(x)}^{s} (s')^{-1} ds' + N_3 \int_{\nu_1(x)}^{\min\{s,\nu^\sharp(x,t)\}} (s')^{-\frac{2}{3}} ds' \right). \tag{6.65}$$

Next, by using (6.56), and the fact that in view of the relations $\jmath(\theta, s) \approx \kappa^{-1}(\theta - \mathfrak{s}_1(s))$ and $\lambda_3(\eta(x,s),s) - \dot{\mathfrak{s}}_1(s) \geqslant \frac{1}{2}\kappa$ the same argument used to prove (6.56) also establishes

$$\int_{\nu_1(x)}^{t} \jmath(\eta(x,s))^{-\frac{1}{2}} ds \leqslant C t^{\frac{1}{2}}, \tag{6.66}$$

and so from (6.65), (6.47), (6.56), and (6.66) we obtain that

$$\begin{aligned}\text{third line on RHS of (6.60)} &\leqslant C + \mathbf{1}_{s>\nu^\sharp(x,t)} 200(M_1 + N_5 + \mathsf{m}^4)\kappa \mathsf{b}^{-\frac{3}{2}} \mathsf{m} (\nu^\sharp(x,t))^{-\frac{1}{2}} \\ &\leqslant C + \mathbf{1}_{s>\mathsf{b}^{\frac{3}{2}}\kappa^{-1}t^{\frac{3}{2}}} 200(M_1 + N_5 + \mathsf{m}^4)\kappa^{\frac{3}{2}} \mathsf{b}^{-\frac{9}{4}} \mathsf{m} t^{-\frac{3}{4}}.\end{aligned} \tag{6.67}$$

Finally, using the bounds (6.63), (6.64) (which needs to be combined with (5.56a), (5.141b), (6.47), (6.56), (6.66)), and (6.67), we integrate (6.60) on $[0, t]$, use (6.26), and arrive at

$$\begin{aligned}(\eta_{xx} - \eta_{\mathsf{B}xx})(x,t) &\leqslant 12\mathsf{m}t^2 |w_0''(x)| + 2(\mathsf{m}^{\frac{1}{2}} + \mathsf{m}M_3) t^{\frac{3}{2}} |w_0'(x)| \\ &\quad + C \log \frac{t}{\nu^\sharp(x,t)} + C t^{\frac{1}{2}} \\ &\quad + 200(M_1 + N_5 + \mathsf{m}^4)\kappa^{\frac{3}{2}} \mathsf{b}^{-\frac{9}{4}} \mathsf{m} t^{\frac{1}{4}} \\ &\leqslant 12\mathsf{m}t^2 |w_0''(x)| + C \log t \\ &\leqslant 5\mathsf{m}\mathsf{b}^{-\frac{3}{2}} t^{-\frac{1}{2}}\end{aligned} \tag{6.68}$$

for $x = \eta^{-1}(\theta, t)$ with $\theta \in (\mathfrak{s}_2(t), \mathfrak{s}(t) + \frac{\kappa t}{3})$. This concludes the proof of the second inequality in (6.25b).

The case $\theta \leqslant \mathfrak{s}_2(t)$ or $\theta \geqslant \mathfrak{s}(t) + \frac{\kappa t}{3}$ revisited. In order to prove the Lemma, we note that the constant claimed in the first inequality in (6.25b) is different than the one previously established in (6.45); this issue plays an important role proof of Lemma 6.12.

For this purpose we combine (6.60) with the bounds (6.63), (6.64), (6.67) (the first line of this inequality is used here), and use the fact that for y as above we have that $\nu^\sharp(x,t), \nu_2(x) \geqslant t > s$, to arrive at

$$\left|\partial_s(\eta_{xx} - \eta_{\mathsf{B}xx})\right| \leqslant 24\mathsf{m}s|w_0''(x)| + 2(\mathsf{m}^{\frac{1}{2}} + \mathsf{m}M_3) s^{\frac{1}{2}} |w_0'(x)|$$

$$+ 24\mathsf{b}\kappa^{-\frac{5}{3}}\mathsf{m}^2 s^{-\frac{1}{6}} + 4\mathsf{m}^{\frac{1}{2}} s^{-\frac{1}{2}}$$
$$+ 2N_2 \mathfrak{J}(\eta(x,s),s)^{-\frac{1}{2}} \mathbf{1}_{\nu_1(x)<s<\nu_2(x)} + C\,.$$

Integrating the above estimate on $[0, t]$ and appealing to (6.28) and (6.66) we obtain

$$\begin{aligned}\left|(\eta_{xx} - \eta_{\mathsf{B}\,xx})(x,t)\right| &\leqslant 12\mathsf{m}t^2|w_0''(x)| + 2(\mathsf{m}^{\frac{1}{2}} + \mathsf{m}M_3)t^{\frac{3}{2}}|w_0'(x)| + Ct^{\frac{1}{2}} \\ &\leqslant 48\mathsf{m}\mathsf{b}\kappa^{-\frac{5}{3}}t^{\frac{1}{3}} + Ct^{\frac{1}{2}}\,.\end{aligned}$$

Taking into account (5.2) and the fact that t is sufficiently small with respect to κ, b, m, the above estimate proves the first inequality in (6.25b). □

6.2.2 Derivatives of the 1- and 2-Characteristics

Lemma 6.8 *For any $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}$,*

$$\sup_{s\in[0,t]} |\partial_\theta \phi_t(\theta,s) - 1| \leqslant 60\mathsf{b}\kappa^{-\frac{2}{3}}t^{\frac{1}{3}}\,, \qquad \sup_{s\in[0,t]} |\partial_\theta \psi_t(\theta,s) - 1| \leqslant 30\mathsf{b}\kappa^{-\frac{2}{3}}t^{\frac{1}{3}}\,. \tag{6.69}$$

Proof of Lemma 6.8 For any $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}$ and $s \in [0, t]$, $\partial_s \partial_\theta \phi_t = \partial_\theta \lambda_2 \circ \phi_t\ \partial_\theta \phi_t$, and since $\partial_\theta \phi_t(\theta, t) = 1$, we see that

$$\partial_\theta \phi_t(\theta,s) = e^{-\int_s^t \partial_\theta \lambda_2 \circ \phi_t dr} = e^{-\frac{2}{3}\int_s^t \partial_\theta w_{\mathsf{B}} \circ \phi_t dr} e^{-\frac{2}{3}\int_s^t \partial_\theta (w - w_{\mathsf{B}} + z)\circ \phi_t dr}\,.$$

Similarly, for $s \in [0, t]$, $\partial_s \partial_\theta \psi_t = \partial_\theta \lambda_1 \circ \psi_t\ \partial_\theta \psi_t$, and since $\partial_\theta \psi_t(\theta, t) = 1$, so that

$$\partial_\theta \psi_t(\theta,s) = e^{-\int_s^t \partial_\theta \lambda_1 \circ \psi_t dr} = e^{-\frac{1}{3}\int_s^t \partial_\theta w_{\mathsf{B}} \circ \psi_t dr} e^{-\int_s^t \left(\frac{1}{3}\partial_\theta (w - w_{\mathsf{B}}) + \partial_\theta z\right)\circ \psi_t dr}\,.$$

By combining the above two identities with the bounds (5.2), (5.141), and (5.57a) (with $\mu = \frac{3}{4}$ for ϕ_t and $\mu = \frac{1}{2}$ for ψ_t), and using that $\overline{\varepsilon}$ is sufficiently small, the bound (6.69) follows. □

We next derive second derivative identities and bounds for these characteristics. As we noted above, the bounds differ, depending on the spacetime region. In order to state these bounds, we first define the 2-characteristic $\mathfrak{s}_1(t)$-intersection time. Just as we showed that $\phi_t(\theta, s)$ is transverse to the shock curve in the proof of Lemma 5.24, by the same argument, the curve $\phi_t(\theta, s)$ is transverse to the characteristic curve $(\mathfrak{s}_1(t), t)$, and there exists an $\mathfrak{s}_1(t)$-intersection time $\mathcal{T}_1(\theta, t)$ such that (Fig. 14)

$$\phi_t(\theta, \mathcal{T}_1(\theta,t)) = \mathfrak{s}_1(\mathcal{T}_1(\theta,t))\,.$$

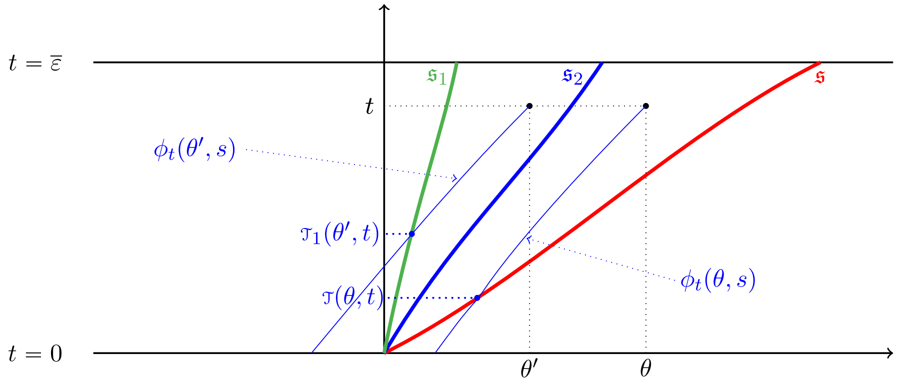

Fig. 14 Fix points: (θ', t) which lies in between $\mathfrak{s}_1$ and $\mathfrak{s}_2$, and (θ, t) which lies in between $\mathfrak{s}_1$ and $\mathfrak{s}$. The intersection time of $\phi_t(\theta', s)$ with $\mathfrak{s}_1$ is denoted by $\mathcal{T}_1(\theta', t)$, while the intersection time of $\phi_t(\theta, s)$ with $\mathfrak{s}$ is denoted as usual by $\mathcal{T}(\theta, t)$

Lemma 6.9 *Let $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}$. Then, for all $(\theta, t) \in \mathcal{D}^k_{\overline{\varepsilon}}$ we have*

$$\sup_{s\in[\mathcal{T}(\theta,t),t]} s\left|\partial_\theta^2\phi_t(\theta,s)\right| \leqslant 3\mathsf{m}^2\kappa^{-3}, \qquad \sup_{s\in[\mathcal{J}(\theta,t),t]} s\left|\partial_\theta^2\psi_t(\theta,s)\right| \leqslant \mathsf{m}^{\frac12}\kappa^{-\frac32}, \tag{6.70}$$

while for all $(\theta, t) \in \mathcal{D}^z_{\overline{\varepsilon}} \setminus \overline{\mathcal{D}^k_{\overline{\varepsilon}}}$ it holds that

$$\begin{aligned} \sup_{s\in[\mathcal{T}_1(\theta,t),t]} s^{\frac23}\left|\partial_\theta^2\phi_t(\theta,s)\right| &\leqslant 4\mathsf{b}\mathsf{m}^2\kappa^{-3}, \\ \sup_{s\in[\mathcal{J}(\theta,t),t]} \left|\partial_\theta^2\psi_t(\theta,s)\right| &\leqslant \mathsf{m}^{\frac12}\kappa^{-\frac32}\mathcal{J}(\theta,t)^{-1}. \end{aligned} \tag{6.71}$$

Lastly, for $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}} \setminus \overline{\mathcal{D}^z_{\overline{\varepsilon}}}$ we have

$$\begin{aligned} \sup_{s\in[0,t]} \left|\partial_\theta^2\phi_t(\theta,s)\right| &\leqslant 3\mathsf{m}^2\kappa^{-3}t^{-1}, \\ \sup_{s\in[0,t]} \left|\partial_\theta^2\psi_t(\theta,s)\right| &\leqslant \mathsf{m}^{\frac12}\kappa^{-\frac32}t^{-1}, \quad \mathfrak{s}(t) \leqslant \theta \leqslant \pi, \end{aligned} \tag{6.72}$$

$$\begin{aligned} \sup_{s\in[0,t]} s^{\frac23}\left|\partial_\theta^2\phi_t(\theta,s)\right| &\leqslant 3\mathsf{m}^2\kappa^{-3}, \\ \sup_{s\in[0,t]} \left|\partial_\theta^2\psi_t(\theta,s)\right| &\leqslant \mathsf{m}^{\frac12}\kappa^{-\frac32}t^{-\frac23}, \quad -\pi \leqslant \theta \leqslant \mathfrak{s}_1(t). \end{aligned} \tag{6.73}$$

Proof of Lemma 6.9 It is convenient to introduce the (temporary) variables $C = c \circ \phi_t$, $B = \partial_s\phi_t = \lambda_2 \circ \phi_t$ and $A = a \circ \phi_t$ so that using the chain-rule, the equation for c

given by (3.7) can be written as

$$\partial_s C + \tfrac{1}{2} C (\partial_\theta \phi_t)^{-1} \partial_\theta B = -\tfrac{8}{3} AC\,.$$

It follows that

$$(\partial_\theta \phi_t)^{\frac{1}{2}} \partial_s C + \tfrac{1}{2} C (\partial_\theta \phi_t)^{-\frac{1}{2}} \partial_\theta B = -\tfrac{8}{3} (\partial_\theta \phi_t)^{\frac{1}{2}} AC\,,$$

and hence

$$\partial_s \left((\partial_\theta \phi_t)^{\frac{1}{2}} C \right) + \tfrac{8}{3} A (\partial_\theta \phi_t)^{\frac{1}{2}} C = 0\,.$$

For $(\theta, t) \in \mathcal{D}^k_{\bar{\varepsilon}}$, and letting $s \in [\Upsilon(\theta, t), t)$, we integrate this equation from s to t and find that

$$\partial_\theta \phi_t(\theta, s) = e^{\frac{16}{3} \int_s^t (a \circ \phi_t)(y, s') ds'} \frac{c^2(\theta, t)}{c^2(\phi_t(\theta, s), s)}\,. \tag{6.74}$$

Differentiating (6.74), we find that

$$\begin{aligned} \partial^2_\theta \phi_t(\theta, s) &= 2 e^{\frac{16}{3} \int_s^t a \circ \phi_t ds'} \frac{c(\theta, t)}{c^3(\phi_t(\theta, s), s)} \Big(\tfrac{8}{3} c(\theta, t) c(\phi_t(\theta, s), s) \int_s^t (a_\theta \circ \phi_t \; \partial_\theta \phi_t) ds' \\ &\quad + c(\phi_t(\theta, s), s) c_\theta(\theta, t) - c(\theta, t) c_\theta(\phi_t(\theta, s), s) \partial_\theta \phi_t(\theta, s) \Big)\,. \end{aligned} \tag{6.75}$$

In essence, the two worst terms in the above identity are $c_\theta(\theta, t)$ and $c_\theta(\phi_t(\theta, s), s)$, so that in view of (5.37) and (5.141) the bounds will be determined by how close y is to $\mathfrak{s}(t)$, respectively $\phi_t(\theta, s)$ to $\mathfrak{s}(s)$.

A similar argument can be used to obtain a formula for $\partial_\theta \psi_t$. To do so, we make the observation (see also (5.138c)) that (3.7) can be written using λ_1 as the transport velocity in the special form

$$\partial_t c + \lambda_1 \partial_\theta c + 2c \partial_\theta \lambda_1 = 2c \partial_\theta z - \tfrac{8}{3} ac\,.$$

We again introduce temporary variables $C = c \circ \psi_t$ and $B = \lambda_1 \circ \psi_t = \partial_s \psi_t$, so that

$$\partial_s C + 2C (\partial_\theta \psi_t)^{-1} \partial_\theta B = \left(2\partial_\theta z - \tfrac{8}{3} a \right) \circ \psi_t \, C\,.$$

Then,

$$\partial_s \left((\partial_\theta \psi_t)^2 C \right) - \left(2\partial_\theta z - \tfrac{8}{3} a \right) \circ \psi_t \; (\partial_\theta \psi_t)^2 C = 0\,,$$

and for any $(\theta, t) \in \mathcal{D}^{z}_{\bar\varepsilon}$ and $s \in [\mathfrak{T}(\theta, t), t)$, we integrate this equation from s to t and find that

$$\partial_\theta \psi_t(\theta, s) = e^{\int_s^t \left(\frac{4}{3}a(\psi_t(\theta,s'),s') - z_\theta(\psi_t(\theta,s'),s')\right)ds'} \frac{c^{\frac12}(\theta, t)}{c^{\frac12}(\psi_t(\theta, s), s)} . \tag{6.76}$$

Differentiating (6.76) once more yields

$$\begin{aligned}\partial^2_\theta \psi_t(\theta, s) = \tfrac12 e^{\int_s^t (\frac43 a - z_\theta)\circ\psi_t ds'} & \frac{c^{\frac12}(\theta, t)}{c^{\frac12}(\psi_t(\theta, s), s)} \\ & \times \left(\int_s^t \left(\tfrac83 a_\theta - 2z_{\theta\theta}\right) \circ \psi_t\, \partial_\theta \psi_t ds' + \frac{\partial_\theta c(\theta, t)}{c(\theta, t)} \right. \\ & \left. - \frac{\partial_\theta c(\psi_t(\theta, s), s)}{c(\psi_t(\theta, s), s)} \partial_\theta \psi_t(\theta, s) \right) . \end{aligned} \tag{6.77}$$

As before, the worst terms in the above identity are $c_\theta(\theta, t)$ and $c_\theta(\phi_t(\theta, s), s)$, but in order to justify this heuristic we need to estimate the time integral of $z_{\theta\theta} \circ \psi_t$.

For $(\theta, t) \in \mathcal{D}^{k}_{\bar\varepsilon}$, we shall need a good bound for $\int_{\mathfrak{Z}(\theta,t)}^t z_{\theta\theta}(\psi_t(\theta, s), s)ds$, and to this end, we employ an argument which is very similar to the one we used to obtain (6.56). Let us define $\gamma(s) = \psi_t(\theta, s) - \mathfrak{s}_2(s)$. Since $\lambda_1(\psi_t(\theta, s), s) - \dot{\mathfrak{s}}_2(s) \leqslant -\frac{3}{10}\kappa$, we obtain $\dot\gamma(s) \leqslant -\frac{3}{10}\kappa$. Moreover, using (6.5) we have that for $\bar\varepsilon$ sufficiently small, $\mathfrak{T}(\theta, t) \geqslant \frac52 \kappa^{-1}(\theta - \mathfrak{s}_2(t))$ for all $\mathfrak{s}_2(t) \leqslant \theta \leqslant \mathfrak{s}(t)$. Hence,

$$\begin{aligned}\int_{\mathfrak{Z}(\theta,t)}^t \mathfrak{T}(\psi_t(\theta, s), s)^{-\frac12} ds &\leqslant \tfrac35 \kappa^{\frac12} \int_{\mathfrak{Z}(\theta,t)}^t (\psi_t(\theta, s) - \mathfrak{s}_2(s))^{-\frac12} ds \\ &\leqslant -2\kappa^{-\frac12} \int_{\mathfrak{Z}(\theta,t)}^t \dot\gamma(s)(\gamma(s))^{-\frac12} ds \\ &= 4\kappa^{-\frac12} \left(\gamma(\mathfrak{Z}(\theta, t))^{\frac12} - \gamma(t)^{\frac12}\right) \\ &\leqslant 4\kappa^{-\frac12} (\mathfrak{s}(\mathfrak{Z}(\theta, t)) - \mathfrak{s}_2(\mathfrak{Z}(\theta, t)))^{\frac12} \\ &\leqslant \tfrac52 \mathfrak{Z}(\theta, t)^{\frac12} \end{aligned} \tag{6.78}$$

From (6.78) and the bootstrap assumption (6.8b), we get

$$\int_{\mathfrak{Z}(\theta,t)}^t |z_{\theta\theta} \circ \psi_t|\, ds \leqslant 3M_2 \mathfrak{Z}(\theta, t)^{\frac12} . \tag{6.79}$$

First consider $(\theta, t) \in \mathcal{D}^{k}_{\bar\varepsilon}$. Combining (6.75) and (6.77), with the bounds (5.37), (5.141), (6.69), (6.79), and taking $\bar\varepsilon$ sufficiently small, we see that $\left|\partial^2_\theta \phi_t(\theta, s)\right| \leqslant 3\mathsf{m}^2\kappa^{-3}s^{-1}$ and $\left|\partial^2_\theta \psi_t(\theta, s)\right| \leqslant \mathsf{m}^{\frac12}\kappa^{-\frac32}s^{-1}$, which are the bounds stated in (6.70). Here we use that $\mathfrak{T}(\theta, t)$ and $\mathfrak{Z}(\theta, t)$ are the shock intersection times for trajectories $\phi_t(\theta, s)$ and $\psi_t(\theta, s)$.

We next consider the case $(\theta, t) \in \mathcal{D}^z_{\bar{\varepsilon}} \setminus \overline{\mathcal{D}^k_{\bar{\varepsilon}}}$. From (6.75), by using (5.141), (5.37a), and (6.69), we obtain

$$\left|\partial^2_\theta \phi_t(\theta, s)\right| \leqslant 4\mathsf{b}\mathsf{m}^2 \kappa^{-3} s^{-\frac{2}{3}} ,$$

for all $s \in [\mathcal{T}_1(\theta, t), t]$, which establishes the first bound in (6.71). Using the bootstrap assumption (6.10b) and the bound (6.79) for s such that $\psi_t(\theta, s) \in \mathcal{D}^k_{\bar{\varepsilon}}$, respectively (6.10b) and the fact that $\mathfrak{J}(\psi_t(\theta, s), s) = \mathfrak{J}(\theta, t)$ for $\psi_t(\theta, s) \in \mathcal{D}^z_{\bar{\varepsilon}} \setminus \overline{\mathcal{D}^k_{\bar{\varepsilon}}}$, we obtain

$$\int_{\mathfrak{J}(\theta,t)}^t \left|z_{\theta\theta}(\psi_t(\theta, s), s)\right| ds \leqslant t N_2 \mathfrak{J}(\theta, t)^{-\frac{1}{2}} + 3M_2 \mathfrak{J}(\theta, t)^{\frac{1}{2}} .$$

Therefore, the identity (6.77) together with (5.37), (5.141), (6.69), (6.75), (6.77), and the above estimate, show that

$$\sup_{s\in[\mathfrak{J}(\theta,t),t]} \left|\partial^2_\theta \psi_t(\theta, s)\right| \leqslant \mathsf{m}^{\frac{1}{2}} \kappa^{-\frac{3}{2}} \mathfrak{J}(\theta, t)^{-1} ,$$

for all $(\theta, t) \in \mathcal{D}^z_{\bar{\varepsilon}} \setminus \overline{\mathcal{D}^k_{\bar{\varepsilon}}}$, which establishes the second bound in (6.71). Note that this bound is only sharp when s is very close to $\mathfrak{J}(\theta, t)$.

For the case that $(\theta, t) \in \mathcal{D}_{\bar{\varepsilon}}$ such that $\theta > \mathfrak{s}(t)$, we have that $z = 0$, and so the identities (6.75) and (6.77) show that second derivatives of these characteristics are largest at points (θ, t) which are very close to $\mathfrak{s}(t)$. Using that $|\phi_t(\theta, s) - \mathfrak{s}(s)|$, $|\psi_t(\theta, s) - \mathfrak{s}(s)| \gtrsim \kappa t$ for $s \in [0, t/2]$, using (5.37) and (5.141) it follows from (6.75) and respectively (6.77) that

$$\sup_{s\in[0,t]} \left|\partial^2_\theta \phi_t(\theta, s)\right| \leqslant 3\mathsf{m}^2 \kappa^{-3} t^{-1} , \qquad \sup_{s\in[0,t]} \left|\partial^2_\theta \psi_t(\theta, s)\right| \leqslant \mathsf{m}^{\frac{1}{2}} \kappa^{-\frac{3}{2}} t^{-1} ,$$

which establishes (6.72) for $\mathfrak{s}(t) \leqslant \theta \leqslant \pi$.

For the case that $(\theta, t) \in \mathcal{D}_{\bar{\varepsilon}}$ such that $-\pi \leqslant \theta \leqslant \mathfrak{s}_1(t)$ we again have that $z = 0$. Using (5.37), (5.141), (6.69), it similarly follows from (6.75) and (6.77) that

$$\sup_{s\in[0,t]} s^{\frac{2}{3}} \left|\partial^2_\theta \phi_t(\theta, s)\right| \leqslant 3\mathsf{m}^2 \kappa^{-3} , \qquad \sup_{s\in[0,t]} \left|\partial^2_\theta \psi_t(\theta, s)\right| \leqslant \mathsf{m}^{\frac{1}{2}} \kappa^{-\frac{3}{2}} t^{-\frac{2}{3}} , \tag{6.80}$$

which is the stated bound (6.73). This improved growth rate of second derivatives makes use of the fact that for $-\pi \leqslant \theta \leqslant \mathfrak{s}_1(t)$, one the one hand we have $|\psi_t(\theta, s) - \mathfrak{s}(s)| \geqslant |\psi_t(\theta, s) - \mathfrak{s}_2(s)| \approx |\theta - \mathfrak{s}_2(t)| \gtrsim \kappa t$ for all $s \in [0, t]$, while on the other hand $|\phi_t(\theta, s) - \mathfrak{s}(s)| \geqslant |\mathfrak{s}_1(s) - \mathfrak{s}(s)| \approx \kappa s$ for all $s \in [0, t]$. □

6.3 Second Derivatives for *w* Along the Shock Curve

Lemma 6.10 *Assume that the shock curve $\mathfrak{s}$ satisfies (5.13), that $(w, z, k, a) \in \mathcal{X}_{\bar{\varepsilon}}$ (as defined in (5.141)–(5.142)), and that the second derivative bootstraps (6.8)–(6.12)* *hold. Then we have that*

$$\left|\tfrac{d^2}{dt^2}w(\mathfrak{s}(t)^\pm,t)-\tfrac{d^2}{dt^2}w_{\mathsf{B}}(\mathfrak{s}(t)^\pm,t)\right|\leqslant(4\mathsf{b}^3M_1+\mathsf{m}^5)t^{-1} \tag{6.81}$$

where $M_1=M_1(\kappa,\mathsf{b},\mathsf{c},\mathsf{m})>0$ *is the constant from (6.8a). In particular, the bound (5.82) holds with the constant* $\mathsf{R}^*=4\mathsf{b}^3M_1+\mathsf{m}^5$*, which in turn implies (5.83).*

Proof of Lemma 6.10 First, we note that from (3.5), Lemma 5.8, and the fact that $(w,z,k,a)\in\mathcal{X}_{\overline{\varepsilon}}$ cf. (5.141)–(5.142), we have that

$$|\partial_\theta w(\theta,t)|\leqslant|\partial_\theta w_{\mathsf{B}}(\theta,t)|+R_2(\mathsf{b}t)^{-\frac12}\leqslant\tfrac{9}{11}t^{-1}+R_2(\mathsf{b}t)^{-\frac12}\leqslant t^{-1} \tag{6.82a}$$

$$\begin{aligned}|\partial_t w(\theta,t)|&\leqslant(\mathsf{m}+R_1t+\tfrac13R_3t^{\frac32})t^{-1}+\tfrac{8R_7}{3}(\mathsf{m}+R_1t)+\tfrac{R_6}{24}t^{\frac12}(\mathsf{m}+R_1t+R_3t^{\frac32})^2\\&\leqslant 2\mathsf{m}t^{-1}\end{aligned} \tag{6.82b}$$

$$\begin{aligned}|\partial_t z(\theta,t)|&\leqslant(\tfrac13(\mathsf{m}+R_1t)+R_3t^{\frac32})R_4t^{\frac12}+\tfrac83R_7R_3t^{\frac32}+\tfrac{1}{24}(\mathsf{m}+R_1t+R_3t^{\frac32})^2R_6t^{\frac12}\\&\leqslant\mathsf{m}^3t^{\frac12}\end{aligned} \tag{6.82c}$$

$$|\partial_t a(\theta,t)|\leqslant\tfrac12(\mathsf{m}+R_1t+R_3t^{\frac32})R_7+\tfrac12(\mathsf{m}+R_1t+R_3t^{\frac32})^2\leqslant\mathsf{m}^3 \tag{6.82d}$$

for all $(\theta,t)\in\mathcal{D}_{\overline{\varepsilon}}$, and in particular as $\theta\to\mathfrak{s}(t)^\pm$.

From the chain rule, we obtain that

$$\begin{aligned}\tfrac{d^2}{dt^2}w(\mathfrak{s}(t)^\pm,t)&=\ddot{\mathfrak{s}}(t)(w_\theta)(\mathfrak{s}(t)^\pm,t)+(\dot{\mathfrak{s}}(t))^2(w_{\theta\theta})(\mathfrak{s}(t)^\pm,t)\\&\quad+2\dot{\mathfrak{s}}(t)(w_{t\theta})(\mathfrak{s}(t)^\pm,t)+(w_{tt})(\mathfrak{s}(t)^\pm,t)\,.\end{aligned} \tag{6.83}$$

From the evolution equations (3.5) and the definition of the wave speeds in (3.6) we have the identities

$$\begin{aligned}w_{t\theta}&=-\left(w+\tfrac13z\right)w_{\theta\theta}-\left(w_\theta+\tfrac13z_\theta\right)w_\theta-\tfrac83(aw)_\theta\\&\quad+\tfrac{1}{12}(w-z)\,(w_\theta-z_\theta)\,k_\theta+\tfrac{1}{24}(w-z)^2k_{\theta\theta}\end{aligned} \tag{6.84a}$$

$$\begin{aligned}w_{tt}&=-\left(w+\tfrac13z\right)w_{t\theta}-\left(w_t+\tfrac13z_t\right)w_\theta-\tfrac83\partial_t(aw)+\tfrac{1}{12}(w-z)\,(w_t-z_t)\,k_\theta\\&\quad+\tfrac{1}{24}(w-z)^2k_{t\theta}\\&=\left(w+\tfrac13z\right)\Big(\left(w+\tfrac13z\right)w_{\theta\theta}+\left(w_\theta+\tfrac13z_\theta\right)w_\theta+\tfrac83(aw)_\theta\\&\quad-\tfrac{1}{12}(w-z)\,(w_\theta-z_\theta)\,k_\theta-\tfrac{1}{24}(w-z)^2k_{\theta\theta}\Big)\\&\quad+\Big((w+\tfrac13z)w_\theta+\tfrac83aw+\tfrac13(\tfrac13w+z)z_\theta+\tfrac89az-\tfrac{1}{18}(w-z)^2k_\theta\Big)\,w_\theta\\&\quad-\tfrac83\partial_t(aw)\\&\quad+\tfrac{1}{12}(w-z)(w_t-z_t)k_\theta-\tfrac{1}{36}(w-z)^2\left((w+z)k_{\theta\theta}+(w_\theta+z_\theta)k_\theta\right)\end{aligned} \tag{6.84b}$$

pointwise for $(\theta,t)\in\mathcal{D}_{\overline{\varepsilon}}$. We shall in fact use (6.84) only for $\theta\to\mathfrak{s}(t)^\pm$, so that the relevant bounds on second derivatives of w are given by (6.8a), the second branch in (6.12a), and from the estimate $|\partial_\theta^2 w_{\mathsf{B}}(\theta, t)| \leqslant \frac{11}{4}\mathsf{b}^{-\frac{3}{2}} t^{-\frac{5}{2}}$, which follows from Lemma 5.8 and (5.36b); together, these bounds and the fact $\mathcal{T}(\mathfrak{s}(t)^-, t) = t$, imply that

$$|\partial_\theta^2 w(\mathfrak{s}(t)^\pm, t)| \leqslant \tfrac{11}{4}\mathsf{b}^{-\frac{3}{2}} t^{-\frac{5}{2}} + Ct^{-2} \leqslant 3\mathsf{b}^{-\frac{3}{2}} t^{-\frac{5}{2}} .$$

Similarly, for the second derivative of k we appeal to (6.8c), which gives

$$|\partial_\theta^2 k(\mathfrak{s}(t)^-, t)| \leqslant M_3 t^{-\frac{1}{2}} .$$

From the above two estimates, the bounds (6.84), the fact that $(w, z, k, a) \in \mathcal{X}_{\overline{\varepsilon}}$ cf. (5.141)–(5.142), we deduce that at $(\mathfrak{s}(t)^\pm, t)$:

$$\begin{aligned} \left| w_{t\theta} + w w_{\theta\theta} + (w_\theta)^2 \right| &\leqslant \tfrac{1}{3}|z w_{\theta\theta}| + \tfrac{8}{3}|a w_\theta| + Ct^{-\frac{1}{2}} \\ &\leqslant (R_3 \mathsf{b}^{-\frac{3}{2}} + 3R_7)t^{-1} + Ct^{-\frac{1}{2}} \\ &\leqslant \tfrac{1}{2}\mathsf{m}^3 t^{-1} \qquad (6.85a) \\ \left| w_{tt} - w^2 w_{\theta\theta} - 2w(w_\theta)^2 \right| &\leqslant |w| \left| w_{t\theta} + w w_{\theta\theta} + (w_\theta)^2 \right| + \tfrac{1}{3}|z w_{t\theta}| + \tfrac{8}{3}|a w w_\theta| \\ &\quad + \tfrac{8}{3}|a \partial_t w| + Ct^{-\frac{1}{2}} \\ &\leqslant \tfrac{1}{2}\mathsf{m}^4 t^{-1} + 3R_3 \mathsf{m} \mathsf{b}^{-\frac{3}{2}} t^{-1} + 10\mathsf{m} R_7 t^{-1} + Ct^{-\frac{1}{2}} \\ &\leqslant \mathsf{m}^4 t^{-1} \qquad (6.85b) \end{aligned}$$

upon taking $\overline{\varepsilon}$, and hence t, to be sufficiently small, and using (5.2). Combining the $\mathfrak{s}$ bounds in (5.13) with (6.83) and (6.85), we thus deduce that

$$\begin{aligned} &\left| \tfrac{d^2}{dt^2} w(\mathfrak{s}(t)^\pm, t) - (\dot{\mathfrak{s}} - w(\mathfrak{s}(t)^\pm, t))^2 w_{\theta\theta}(\mathfrak{s}(t)^\pm, t) + 2(\dot{\mathfrak{s}} - w(\mathfrak{s}(t)^\pm, t))(w_\theta(\mathfrak{s}(t)^\pm, t))^2 \right| \\ &\quad \leqslant \tfrac{1}{2}\mathsf{m}^5 t^{-1} . \qquad (6.86) \end{aligned}$$

In a similar fashion, we may show from (5.14) that $\partial_{t\theta} w_{\mathsf{B}} = -w_{\mathsf{B}} \partial_\theta^2 w_{\mathsf{B}} - (\partial_\theta w_{\mathsf{B}})^2$ and that $\partial_{tt} w_{\mathsf{B}} = {w_{\mathsf{B}}}^2 \partial_\theta^2 w_{\mathsf{B}} + 2w_{\mathsf{B}}(\partial_\theta w_{\mathsf{B}})^2$, and thus, as in (6.83), we have that

$$\begin{aligned} &\tfrac{d^2}{dt^2} w_{\mathsf{B}}(\mathfrak{s}(t)^\pm, t) - (\dot{\mathfrak{s}} - w_{\mathsf{B}}(\mathfrak{s}(t)^\pm, t))^2 w_{\mathsf{B}\theta\theta}(\mathfrak{s}(t)^\pm, t) \\ &\quad + 2(\dot{\mathfrak{s}} - w_{\mathsf{B}}(\mathfrak{s}(t)^\pm, t))(w_{\mathsf{B}\theta}(\mathfrak{s}(t)^\pm, t))^2 = 0 . \qquad (6.87) \end{aligned}$$

That is, for the Burgers solution we have (6.86) without the $\mathcal{O}(t^{-1})$ error term. In order to prove (6.81) it remains to subtract (6.86) and (6.87). We obtain that

$$\begin{aligned} &\tfrac{d^2}{dt^2} \left(w(\mathfrak{s}(t)^\pm, t) - w_{\mathsf{B}}(\mathfrak{s}(t)^\pm, t) \right) \\ &= \tfrac{1}{2} \Big((\dot{\mathfrak{s}}(t) - w(\mathfrak{s}(t)^\pm, t))^2 + (\dot{\mathfrak{s}}(t) - w_{\mathsf{B}}(\mathfrak{s}(t)^\pm, t))^2 \Big) \partial_\theta^2 (w - w_{\mathsf{B}})(\mathfrak{s}(t)^\pm, t)) \\ &\quad + (w - w_{\mathsf{B}})(\mathfrak{s}(t)^\pm, t)) \left(\dot{\mathfrak{s}}(t) - \tfrac{1}{2}(w + w_{\mathsf{B}})(\mathfrak{s}(t)^\pm, t)) \right) \partial_\theta^2 (w + w_{\mathsf{B}})(\mathfrak{s}(t)^\pm, t)) \end{aligned}$$

$$
\begin{aligned}
&- 2\left(\dot{\mathfrak{s}}(t) - \tfrac{1}{2}(w + w_{\mathsf{B}})(\mathfrak{s}(t)^{\pm}, t)\right)\right) \partial_\theta (w - w_{\mathsf{B}})(\mathfrak{s}(t)^{\pm}, t))\partial_\theta (w + w_{\mathsf{B}})(\mathfrak{s}(t)^{\pm}, t)) \\
&- (w - w_{\mathsf{B}})(\mathfrak{s}(t)^{\pm}, t)) \left((w_\theta(\mathfrak{s}(t)^{\pm}, t)))^2 + (w_{\mathsf{B}\theta}(\mathfrak{s}(t)^{\pm}, t)))^2\right) + \mathcal{O}(t^{-1})
\end{aligned}
\tag{6.88}
$$

where the $\mathcal{O}(t^{-1})$ term is bounded by the right side of (6.86). The estimate (6.88) is now combined with the working assumption (5.2), the $\dot{\mathfrak{s}}(t) - \kappa$ bound in (5.13), the w_{B} estimates established in the proof of Proposition 5.7, the estimates (5.141a)–(5.141b), and the bootstrap assumption (6.8a), to arrive at

$$
\begin{aligned}
&\left|\tfrac{d^2}{dt^2}\left(w(\mathfrak{s}(t)^{\pm}, t) - w_{\mathsf{B}}(\mathfrak{s}(t)^{\pm}, t)\right)\right| \\
&\quad\leqslant (\mathsf{b}^{\frac{3}{2}} t^{\frac{1}{2}} + (2\mathsf{m}^4 + R_1)t)^2(2M_1 t^{-2}) + R_1 t(\mathsf{b}^{\frac{3}{2}} t^{\frac{1}{2}} + (2\mathsf{m}^4 + R_1)t)\left(6\mathsf{b}^{-\frac{3}{2}} t^{-\frac{5}{2}}\right) \\
&\qquad + 2(\mathsf{b}^{\frac{3}{2}} t^{\frac{1}{2}} + (2\mathsf{m}^4 + R_1)t)R_2(\mathsf{b}t)^{-\frac{1}{2}}(2t^{-1}) + R_1 t\left(2t^{-2}\right) + \mathsf{m}^5\mathsf{b}^{-\frac{3}{2}} t^{-1} \\
&\quad\leqslant \left(4\mathsf{b}^3 M_1 + 9\mathsf{m}^3 + \tfrac{1}{2}\mathsf{m}^5\right) t^{-1}\,.
\end{aligned}
\tag{6.89}
$$

This completes the proof of the lemma, upon appealing to (5.2). □

6.4 Improving the Bootstrap Bounds for $k_{\theta\theta}$

Lemma 6.11 *For all $(\theta, t) \in \mathcal{D}^k_{\varepsilon}$ we have that*

$$
\left|\partial_\theta^2 k(\theta, t)\right| \leqslant \mathsf{m}^2 \mathcal{T}(\theta, t)^{-\frac{1}{2}}\,. \tag{6.90}
$$

This justifies the choice of the constant M_3 in (6.9) and improves the bootstrap assumption (6.8c).

Proof of Lemma 6.11 Differentiating (5.105), we have that

$$
\tfrac{d}{ds}\left(\partial_\theta^2 k \circ \phi_t \ (\partial_\theta \phi_t)^2 + \partial_\theta k \circ \phi_t \ \partial_\theta^2 \phi_t\right) = 0\,, \tag{6.91}
$$

and integrating in time from $\mathcal{T}(\theta, t)$ to t, we have that for each $(\theta, t) \in \mathcal{D}^k_{\varepsilon}$,

$$
\partial_\theta^2 k(\theta, t) = \partial_\theta^2 k(\mathfrak{s}(\mathcal{T}), \mathcal{T})\left(\partial_\theta \phi_t(\theta, \mathcal{T})\right)^2 + \partial_\theta k(\mathfrak{s}(\mathcal{T}), \mathcal{T})\partial_\theta^2 \phi_t(\theta, \mathcal{T})\,, \quad \mathcal{T} = \mathcal{T}(\theta, t)\,. \tag{6.92}
$$

It follows from (5.109) that

$$
\partial_\theta^2 k(\theta, t) = \partial_\theta^2 k(\mathfrak{s}(\mathcal{T}), \mathcal{T})\left(\partial_\theta \phi_t(\theta, \mathcal{T})\right)^2 + \tfrac{\dot{k_-(\mathcal{T}))}}{\dot{\mathfrak{s}}(\mathcal{T})) - \partial_s \phi_t(y, \mathcal{T})}\partial_\theta^2 \phi_t(y, \mathcal{T})\,, \tag{6.93}
$$

where $\mathcal{T} = \mathcal{T}(\theta, t)$. Next, by differentiating the system (5.107), a lengthy computation reveals that

$$\partial_\theta^2 k(\mathfrak{s}(t), t) = \frac{\ddot{k}_-(t)}{(\dot{\mathfrak{s}}(t)-\lambda_2(\mathfrak{s}(t),t))^2} - \Big(\ddot{\mathfrak{s}}(t) - \big(\partial_t\lambda_2(\mathfrak{s}(t), t) + (2\dot{\mathfrak{s}}(t) - \lambda_2(\mathfrak{s}(t), t))\partial_\theta\lambda_2(\mathfrak{s}(t), t)\big)\Big) \frac{\dot{k}_-}{(\dot{\mathfrak{s}}(t)-\lambda_2(\mathfrak{s}(t),t))^3} . \tag{6.94}$$

Substitution of (6.94) into (6.93) shows that for all $(\theta, t) \in \mathcal{D}^k_{\bar{\varepsilon}}$,

$$\partial_\theta^2 k(\theta, t) = \Big(\frac{\ddot{k}_-}{(\dot{\mathfrak{s}}-\lambda_2)^2} - \Big(\ddot{\mathfrak{s}} - \big(\partial_t\lambda_2 + (2\dot{\mathfrak{s}} - \lambda_2)\partial_\theta\lambda_2\big)\Big)\frac{\dot{k}_-}{(\dot{\mathfrak{s}}-\lambda_2)^3}\Big)\Big|_{(\mathfrak{s}(\mathcal{T}(\theta,t)),\mathcal{T}(\theta,t))} \big(\partial_\theta\phi_t(y, \mathcal{T}(\theta, t))\big)^2 + \frac{\dot{k}_-(\mathcal{T}(\theta,t))}{\dot{\mathfrak{s}}(\mathcal{T}(\theta,t)) - \partial_s\phi_t(y,\mathcal{T}(\theta,t))}\partial_\theta^2\phi_t(y, \mathcal{T}(\theta, t)) . \tag{6.95}$$

Given the bounds (5.141) together with (5.2), (5.13), (5.37), (5.81), (5.83), (6.6), (6.7), (6.82), (6.69), and (6.70) we find that

$$\begin{aligned}\left|\partial_\theta^2 k(\theta, t)\right| &\leqslant \left(1 + Ct^{\frac{1}{3}}\right)^2 \left(\tfrac{16}{\kappa^2}\left|\ddot{k}_-(\mathcal{T}(\theta, t))\right| + \tfrac{64}{\kappa^3}\left(6\mathsf{m}^4 + \tfrac{\kappa}{3}\mathcal{T}(\theta, t)^{-1}\right)\left|\dot{k}_-(\mathcal{T}(\theta, t))\right|\right) \\ &\quad + \tfrac{9\mathsf{m}^2}{\kappa^4}\left|\dot{k}_-(\mathcal{T}(\theta, t))\right|\mathcal{T}(\theta, t)^{-1} + C \\ &\leqslant 50\mathsf{b}^{\frac{9}{2}}\kappa^{-5}(1 + 10\mathsf{m}^2\kappa^{-2})\mathcal{T}(\theta, t)^{-\frac{1}{2}} \\ &\leqslant \mathsf{m}^2\mathcal{T}(\theta, t)^{-\frac{1}{2}}\end{aligned}$$

for all $(\theta, t) \in \mathcal{D}^k_{\bar{\varepsilon}}$. See the details in the proof of (6.152) below for a sharper bound than the one given above. The estimate (6.90) thus holds, concluding the proof. □

6.5 Improving the Bootstrap Bounds for $w_{\theta\theta}$

Lemma 6.12 *For all $(\theta, t) \in \mathcal{D}_{\bar{\varepsilon}}$, we have that*

$$\left|w_{\theta\theta}(\theta, t) - w_{\mathsf{B}\theta\theta}(\theta, t)\right| \leqslant \begin{cases} \frac{1}{2}\min\{N_1, N_4\}t^{-\frac{2}{3}}, & \theta \leqslant \mathfrak{s}_2(t) \text{ or } \theta \geqslant \mathfrak{s}(t) + \frac{\kappa t}{3} \\ \frac{1}{2}M_1(t^{-2} + \mathcal{T}(\theta, t)^{-\frac{1}{2}}) & \text{if } \mathfrak{s}_2(t) < \theta < \mathfrak{s}(t) \\ \frac{1}{2}N_5 t^{-2} & \text{if } \mathfrak{s}(t) \leqslant \theta < \mathfrak{s}(t) + \frac{\kappa t}{3} \end{cases}, \tag{6.96}$$

where M_1 is as defined as in (6.9), N_1 is given by (6.11), while N_4 and N_5 are defined in (6.13). In particular, we have improved the bootstrap bounds (6.8a), (6.10a), and (6.12a). Moreover, we have

$$\left|w_{\theta\theta}(\theta, t)\right| \leqslant \begin{cases} 15\mathsf{b}\kappa^{-\frac{5}{3}}t^{-\frac{5}{3}}, & \text{if } \theta \leqslant \mathfrak{s}_2(t) \text{ or } \theta \geqslant \mathfrak{s}(t) + \frac{\kappa t}{3} \\ 3\mathsf{b}^{-\frac{3}{2}}t^{-\frac{5}{2}} + 5\mathsf{m}^4\mathcal{T}(\theta, t)^{-\frac{1}{2}} & \text{if } \mathfrak{s}_2(t) < \theta < \mathfrak{s}(t) \\ 3\mathsf{b}^{-\frac{3}{2}}t^{-\frac{5}{2}} & \text{if } \mathfrak{s}(t) \leqslant \theta < \mathfrak{s}(t) + \frac{\kappa t}{3} \end{cases} . \tag{6.97}$$

Proof of Lemma 6.12 Throughout this proof we will take $\overline{\varepsilon}$, and hence t, to be sufficiently small with respect to κ, b, c and m. For any $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}$, we define $x \in \Upsilon(t)$ by $x = \eta^{-1}(\theta, t)$.

Recall that the good unknown q^w is defined in (6.31), and it satisfies (6.34). Differentiating (6.34) with respect to the label x, we obtain the identity

$$\partial_\theta q^w(\eta(x,t),t)\,\eta_x^2(x,t) = -q^w(\eta(x,t))\eta_{xx}(x,t) \\ + \partial_x \left(w_0'(x) e^{-\mathcal{I}(x,0,t)} + \int_0^t Q^w(\eta(x,s),s)\eta_x(x,s) e^{-\mathcal{I}(x,s,t)} ds \right). \tag{6.98}$$

Taking into account the definition of q^w in (6.31) and the identity $\partial_\theta^2 w_{\mathsf{B}} \circ \eta_{\mathsf{B}}\, {\eta_{\mathsf{B}}}_x^2 + \partial_\theta w_{\mathsf{B}} \circ \eta_{\mathsf{B}}\, {\eta_{\mathsf{B}}}_{xx} = w_0''(x)$, we thus obtain that

$$\partial_\theta q^w \circ \eta\, \eta_x^2 - \partial_\theta^2 w_{\mathsf{B}} \circ \eta_{\mathsf{B}}\, {\eta_{\mathsf{B}}}_x^2 \\ = w_0'(x) {\eta_{\mathsf{B}}}_x^{-1} \left({\eta_{\mathsf{B}}}_{xx} - \eta_{xx} \right) + \left(\partial_\theta w_{\mathsf{B}} \circ \eta_{\mathsf{B}} - \partial_\theta w \circ \eta + \tfrac{1}{4}(ck_\theta)\circ\eta \right) \eta_{xx} \\ + w_0'' \left(e^{-\mathcal{I}(\cdot,0,s)} - 1 \right) - w_0' e^{-\mathcal{I}(\cdot,0,s)} \partial_x \mathcal{I}(\cdot,0,s) \\ + \int_0^s \left(\partial_\theta Q^w \circ \eta\, \eta_x^2 + Q^w \circ \eta\, \eta_{xx} - Q^w \circ \eta\, \eta_x \partial_x \mathcal{I}(\cdot,s',s) \right) e^{-\mathcal{I}(\cdot,s',s)} ds' . \tag{6.99}$$

The key observation is that second line in (6.99) is precisely the first line in (6.60), while the third line in (6.99) is precisely the third line in (6.60); we will use this fact to avoid redundant bounds.

Bounds in the region $\mathfrak{s}_2(t) < \theta < \mathfrak{s}(t) + \frac{\kappa t}{3}$. By taking into account (5.54a), (6.25b), (5.52), (6.26), (5.141) and (5.142), we obtain that

$$\text{the first line on RHS of (6.99)} \\ \leqslant 40\mathsf{m}t^{-\frac{3}{2}} + \left(8R_1 \mathsf{b}^{-\frac{3}{2}} t^{-\frac{1}{2}} + R_2(\mathsf{b}t)^{-\frac{1}{2}} + \tfrac{1}{4}\mathsf{m}R_6 t^{\frac{1}{2}} \right) \left(\tfrac{1}{3}\mathsf{b}^{-\frac{3}{2}} t^{-\frac{3}{2}} + 20\mathsf{m}t^{-\frac{1}{2}} \right) \\ \leqslant \mathsf{m}^3 (\mathsf{b}t)^{-2} , \tag{6.100}$$

since t is sufficiently small. Next, since second line in (6.99) equals the first line in (6.60), from (6.63), (6.26), and the fact that $t > \nu^\sharp(x,t)$, we obtain

$$\text{the second line on RHS of (6.99)} \leqslant 24\mathsf{m}t|w_0''(x)| + 2(\mathsf{m}^{\frac{1}{2}} + \mathsf{m}M_3)t^{\frac{1}{2}}|w_0'(x)| \\ \leqslant 10\mathsf{m}(\mathsf{b}t)^{-\frac{3}{2}} . \tag{6.101}$$

Similarly, since third line in (6.99) equals the third line in (6.60), from (6.67), (6.26), and the fact that $t > \nu^\sharp(x,t)$, we obtain

$$\text{the third line on RHS of (6.99)} \leqslant Ct^{-\frac{3}{4}} . \tag{6.102}$$

By adding (6.100), (6.101), and (6.102), since t is sufficiently small we deduce that

$$\left|\partial_\theta q^w \circ \eta\, \eta_x^2 - \partial_\theta^2 w_{\mathsf{B}} \circ \eta_{\mathsf{B}}\, \eta_{\mathsf{B}x}^2\right| \leqslant 2\mathsf{m}^3 (\mathsf{b}t)^{-2} . \tag{6.103}$$

Next, by recalling the definition of q^w in (6.31), and appealing to (5.141), (5.142), and (6.90), we deduce

$$\left|w_{\theta\theta} \circ \eta (\eta_x)^2 - w_{\mathsf{B}\theta\theta} \circ \eta_{\mathsf{B}} (\eta_{\mathsf{B}x})^2\right| \leqslant 3\mathsf{m}^3 (\mathsf{b}t)^{-2} + \mathsf{m}^3 \mathcal{T}(\theta, t)^{-\frac{1}{2}} . \tag{6.104}$$

With (6.104) in hand, we use the notation introduced in (6.39) to rewrite

$$\begin{aligned} w_{\theta\theta}(\theta, t) - w_{\mathsf{B}\theta\theta}(\theta, t) &= \eta_x^{-2} \left(w_{\theta\theta} \circ \eta (\eta_x)^2 - w_{\mathsf{B}\theta\theta} \circ \eta_{\mathsf{B}} (\eta_{\mathsf{B}x})^2 \right) \\ &\quad - \eta_x^{-2} \mathcal{K}_3 - \eta_x^{-2} \mathcal{K}_1 , \end{aligned} \tag{6.105}$$

and thus we may combine (5.54a), (6.48), and (6.50), to arrive at

$$\left|w_{\theta\theta}(\theta, t) - w_{\mathsf{B}\theta\theta}(\theta, t)\right| \leqslant 5\mathsf{m}^4 t^{-2} + 2\mathsf{m}^3 \mathcal{T}(\theta, t)^{-\frac{1}{2}} \tag{6.106}$$

since m is large compared to b. The above estimate proves the second and third bounds in (6.96) once we ensure that $\frac{1}{2} M_1 \geqslant 5\mathsf{m}^4$ and $\frac{1}{2} N_5 \geqslant 5\mathsf{m}^4$. These conditions hold in view of the definitions (6.9) and (6.13).

Bounds in the region $\theta \leqslant \mathfrak{s}_2(t)$ **or** $\theta \geqslant \mathfrak{s}(t) + \frac{\kappa t}{3}$. In order to estimate the first line on the right side of (6.99), we rewrite

$$w_\theta \circ \eta - w_{\mathsf{B}\theta} \circ \eta = \eta_x^{-1} \left(w_\theta \circ \eta\, \eta_x - w_{\mathsf{B}\theta} \circ \eta_{\mathsf{B}}\, \eta_{\mathsf{B}x} \right) - \eta_x^{-1} w_{\mathsf{B}\theta} \circ \eta_{\mathsf{B}} \left(\eta_x - \eta_{\mathsf{B}x} \right) \tag{6.107}$$

so that from the second equality in (6.36), (5.54a), (6.37), (6.61b), (6.61c), and (6.25a), we have

$$\begin{aligned} \left|(w_\theta \circ \eta - w_{\mathsf{B}\theta} \circ \eta)(x, t)\right| &\leqslant 40\mathsf{m}t |w_0'(x)| + \mathsf{m}^2 t^{\frac{1}{2}} + Ct + 200\mathsf{m} |w_0'(x)| t^{\frac{4}{3}} \\ &\leqslant 50\mathsf{m}t |w_0'(x)| + 2\mathsf{m}^2 t^{\frac{1}{2}} . \end{aligned} \tag{6.108}$$

Thus, analogously to (6.100), using (5.54a), (6.25b), (6.28), and the fact that $k(\theta, t) = 0$, we have

$$\begin{aligned} \text{the first line on RHS of (6.99)} &\leqslant 20|w_0'(x)|\mathsf{m}t^{\frac{1}{3}} + \left(50\mathsf{m}t|w_0'(x)| + 2\mathsf{m}^2 t^{\frac{1}{2}}\right) \\ &\qquad \left(4\mathsf{b}\kappa^{-\frac{5}{3}} t^{-\frac{2}{3}} + 10\mathsf{m}t^{\frac{1}{3}}\right) \\ &\leqslant Ct^{-\frac{1}{3}} . \end{aligned} \tag{6.109}$$

Next, similarly to (6.101) we have that

$$\text{the second line on RHS of (6.99)} \leqslant 24\mathsf{m}t|w_0''(x)| + 2(\mathsf{m}^{\frac{1}{2}} + \mathsf{m}M_3)t^{\frac{1}{2}}|w_0'(x)|$$

$$\leqslant 96\mathsf{mb}\kappa^{-\frac{5}{3}} t^{-\frac{2}{3}} + C t^{-\frac{1}{6}} . \tag{6.110}$$

As in (6.102), but this time using that $\nu^\sharp(x,t) \geqslant t$, we obtain from the first line in (6.67) that

$$\text{the third line on RHS of (6.99)} \leqslant C . \tag{6.111}$$

By adding (6.109), (6.110), and (6.111), using that $k(\eta(x,s),s) = 0$, since t is sufficiently small we deduce

$$\left| w_{\theta\theta} \circ \eta (\eta_x)^2 - w_{\mathsf{B}\theta\theta} \circ \eta_{\mathsf{B}} (\eta_{\mathsf{B}x})^2 \right| \leqslant 20\mathsf{m} t^{-\frac{2}{3}} . \tag{6.112}$$

Here we have also used (5.2). Finally, using the decomposition (6.105), and appealing to the bounds (6.40) and (6.43) we deduce that

$$\left| w_{\theta\theta}(\theta,t) - w_{\mathsf{B}\theta\theta}(\theta,t) \right| \leqslant 2\mathsf{m}^4 t^{-\frac{2}{3}} . \tag{6.113}$$

The above estimate proves the first bound in (6.96) once we ensure that $\frac{1}{2}\min\{N_1, N_4\} \geqslant 2\mathsf{m}^4$. This condition holds in view of the definitions (6.11) and (6.13).

In order to complete the proof of the lemma, we note that (6.97) follows from (6.96), the triangle inequality, and (5.37b). □

Lemma 6.13 *Recall the definition of q^w in (6.31). For all $(\theta,t) \in \mathcal{D}^k_{\varepsilon}$ such that $\mathfrak{s}_2(t) < \theta < \mathfrak{s}_2(t) + \frac{\kappa t}{6}$, we have that*

$$\left| q^w_\theta(\theta,t) \right| \leqslant 3\mathsf{b}(\kappa t)^{-\frac{5}{3}} . \tag{6.114}$$

Proof of Lemma 6.13 Combining (6.103) with (5.1), (5.36b), (5.54a), (6.46) (with $s = 0$), and (6.7) we deduce that $|\eta^{-1}(\theta,t)| \geqslant \frac{\kappa t}{7}$ and thus

$$\begin{aligned} \left| q^w_\theta(\theta,t) \right| &\leqslant 2 \left| w_{\mathsf{B}\theta\theta}(\eta_{\mathsf{B}}(\eta^{-1}(\theta,t),t),t) \right| + 2\mathsf{m}^3(\mathsf{b}t)^{-2} \\ &\leqslant 10 \left| w_0''(\eta^{-1}(\theta,t)) \right| + 2\mathsf{m}^3(\mathsf{b}t)^{-2} \\ &\leqslant 3\mathsf{b}(\kappa t)^{-\frac{5}{3}} . \end{aligned} \tag{6.115}$$

The bound (6.114) is thus proven. □

6.6 Improving the Bootstrap Bounds for $z_{\theta\theta}$

Just as we defined the function $q^w(\theta,t)$ in (6.31), we introduce the function

$$q^z(\theta,t) = z_\theta(\theta,t) + \tfrac{1}{4} c(\theta,t) k_\theta(\theta,t) . \tag{6.116}$$

Using this unknown, we rewrite the equation (5.102) as

$$\tfrac{d}{ds}(q^z \circ \psi_t \partial_\theta \psi_t) = -Q^z \circ \psi_t \; \partial_\theta \psi_t \,, \tag{6.117}$$

where

$$Q^z = ck_\theta(\tfrac{1}{12} w_\theta + \tfrac{1}{12} z_\theta + \tfrac{2}{3} a) + \tfrac{8}{3}\partial_\theta(az)\,. \tag{6.118}$$

Differentiating (6.117), we have that

$$\tfrac{d}{ds}(q^z_\theta \circ \psi_t (\partial_\theta \psi_t)^2 + q^z \circ \psi_t \partial^2_\theta \psi_t) = -\partial_\theta Q^z \circ \psi_t \; (\partial_\theta \psi_t)^2 - Q^z \circ \psi_t \; \partial^2_\theta \psi_t \,, \tag{6.119}$$

which may be integrated on $[\jmath(\theta, t), t]$ to obtain that

$$\begin{aligned} q^z_\theta(\theta, t) = &\left(q^z_\theta (\partial_\theta \psi_t)^2 + q^z \partial^2_\theta \psi_t\right)\Big|_{(\mathfrak{s}(\jmath), \jmath)} \\ &- \int_{\jmath(\theta,t)}^t \left(\partial_\theta Q^z \circ \psi_t \; (\partial_\theta \psi_t)^2 + Q^z \circ \psi_t \; \partial^2_\theta \psi_t\right) ds\,. \end{aligned} \tag{6.120}$$

for all $(\theta, t) \in \mathcal{D}^z_{\bar{\varepsilon}}$. Here we have used that $\psi_t(\theta, \jmath(\theta, t)) = \mathfrak{s}(\jmath(\theta, t))$ and the fact that $\psi_t(\theta, t) = \theta$, which implies $\partial_\theta \psi_t(\theta, t) = 1$ and $\partial^2_\theta \psi_t(\theta, t) = 0$. In order to estimate the right side of (6.120), we first establish:

Lemma 6.14 *For* $(\theta, t) \in \mathcal{D}^z_{\bar{\varepsilon}}$,

$$\int_{\jmath(\theta,t)}^t \left|\partial_\theta Q^z \circ \psi_t \; (\partial_\theta \psi_t)^2 + Q^z \circ \psi_t \; \partial^2_\theta \psi_t\right| ds \leqslant 3\mathsf{m}^3 \jmath(\theta, t)^{-\frac{1}{2}}\,. \tag{6.121}$$

Proof of Lemma 6.14 We decompose $\partial_\theta Q^z = \mathcal{Q}_1 + \mathcal{Q}_2$, where

$$\begin{aligned} \mathcal{Q}_1 &= \overbrace{\tfrac{1}{12} ck_\theta w_{\theta\theta}}^{\mathcal{Q}_{1a}} + \overbrace{\tfrac{1}{24} k_\theta w_\theta w_\theta}^{\mathcal{Q}_{1b}} + \overbrace{ck_{\theta\theta}(\tfrac{1}{12} w_\theta + \tfrac{1}{12} z_\theta + \tfrac{2}{3} a)}^{\mathcal{Q}_{1c}}\,, \\ \mathcal{Q}_2 &= ck_\theta(\tfrac{1}{12} z_{\theta\theta} + \tfrac{2}{3} a_\theta) + \tfrac{8}{3}(az)_{\theta\theta} + \tfrac{1}{24} k_\theta z_\theta w_\theta\,. \end{aligned}$$

For $(\theta, t) \in \mathcal{D}^z_{\bar{\varepsilon}} \setminus \overline{\mathcal{D}^k_{\bar{\varepsilon}}}$, is convenient to introduce a time $\jmath_1(\theta, t)$, which is defined as the time at which the curve $\psi_t(\theta, \cdot)$ intersects the curve $\mathfrak{s}_2$; recall that $\jmath(\theta, t)$ is the time at which $\psi_t(\theta, \cdot)$ intersects the shock curve $\mathfrak{s}$. From (6.7), (6.69), and the definitions of $\jmath$ and $\jmath_1$, we note that

$$\jmath_1(\theta, t) = 2\jmath(\theta, t) + \mathcal{O}(\jmath(\theta, t)^{\frac{4}{3}})\,. \tag{6.122}$$

When $(\theta, t) \in \mathcal{D}^k_{\bar{\varepsilon}}$, we abuse notation and write $\jmath_1(\theta, t) = t$, emphasizing that $\psi_t(\theta, \cdot)$ does not intersect $\mathfrak{s}_2$. By definition, note that for $s \in (\jmath_1(\theta, t), t]$, all the terms in $\mathcal{Q}_1 \circ \psi_t$ and $\mathcal{Q}_2 \circ \psi_t$ vanish.

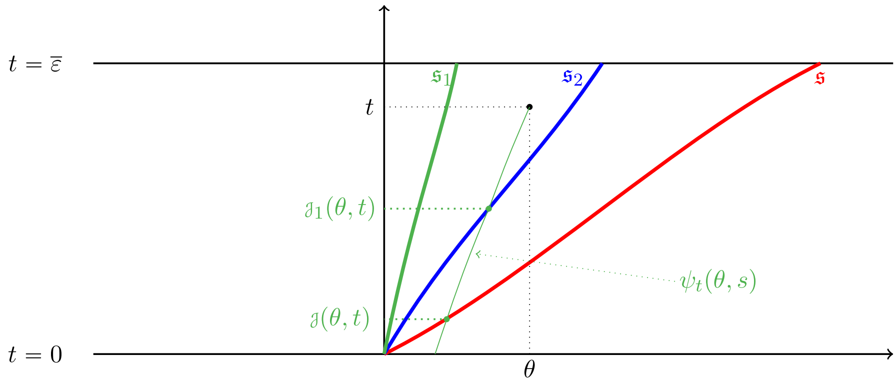

Fig. 15 Fix a point (θ, t) which lies in between $\mathfrak{s}_1$ and $\mathfrak{s}_2$. The intersection time of $\psi_t(\theta, s)$ with $\mathfrak{s}_2$ is denoted by $\jmath_1(\theta, t)$, while the intersection time with $\mathfrak{s}$ is denoted as usual by $\jmath(\theta, t)$

Let us thus consider first the case $(\theta, t) \in \mathcal{D}^k_{\overline{\varepsilon}}$. From (5.216) we have that

$$\left|k_\theta(\psi_t(\theta, s), s)\right| \leqslant 200\mathsf{b}^{\frac{9}{2}}\kappa^{-4}\daleth(\psi_t(\theta, s), s)^{\frac{1}{2}} . \tag{6.123}$$

for all $s \in [\jmath(\theta, t), t]$. Thus, using (6.123) together with (5.57c), (6.8a), and the fact that $\daleth(\psi_t(\theta, s), s) \leqslant \jmath(\psi_t(\theta, s), s) = \jmath(\theta, t)$, we have that

$$\begin{aligned}\int_{\jmath(\theta,t)}^t \left|\mathcal{Q}_{1a} \circ \psi_t\right| (\partial_\theta \psi_t)^2 ds &\leqslant \tfrac{1}{4}(1 + Ct^{\frac{1}{3}})\mathsf{m}\left(\int_{\jmath(\theta,t)}^t \left|k_\theta(w_{\theta\theta} - w_{\mathsf{B}\theta\theta}) \circ \psi_t\right| ds\right.\\ &\quad \left.+ \int_{\jmath(\theta,t)}^t \left|k_\theta w_{\mathsf{B}\theta\theta} \circ \psi_t\right| ds\right)\\ &\leqslant 60\mathsf{m}\mathsf{b}^{\frac{9}{2}}\kappa^{-4}\left(\int_{\jmath(\theta,t)}^t \left(M_1 + \jmath(\theta, t)^{\frac{1}{2}} s^{-2}\right) ds\right.\\ &\quad \left.+ 20\kappa^{-1}\jmath(\theta, t)^{-\frac{1}{2}}\right)\\ &\leqslant 40\mathsf{m}\jmath(\theta, t)^{-\frac{1}{2}} .\end{aligned} \tag{6.124}$$

In the last inequality we have taken t to be sufficiently small, and have used (5.2).

Next, using (6.82a), (5.141b), (6.69), (5.57a), and (6.123), we have that

$$\begin{aligned}\int_{\jmath(\theta,t)}^t \left|\mathcal{Q}_{1b} \circ \psi_t\right| (\partial_\theta \psi_t)^2 ds &\leqslant C\jmath(\theta, t)^{-\frac{1}{2}} \int_{\jmath(\theta,t)}^t \left(\left|(w_\theta - w_{\mathsf{B}\theta}) \circ \psi_t\right| + \left|w_{\mathsf{B}\theta} \circ \psi_t\right|\right) ds\\ &\leqslant Ct^{\frac{1}{3}}\jmath(\theta, t)^{-\frac{1}{2}}\end{aligned} \tag{6.125}$$

and with (5.141), (6.8c), and (6.78),

$$\int_{\jmath(\theta,t)}^t \left|\mathcal{Q}_{1c} \circ \psi_t\right| (\partial_\theta \psi_t)^2 ds \leqslant 2\mathsf{m}\left(\tfrac{1}{12}\jmath(\theta, t)^{-1} + \tfrac{1}{12} R_4 t^{\frac{1}{2}} + \tfrac{2}{3} R_7\right)$$

$$
\begin{aligned}
&\int_{\jmath(\theta,t)}^{t} \left|k_{\theta\theta}\circ\psi_t\right| ds \\
&\leqslant \tfrac{1}{3}\mathsf{m} M_3 \jmath(\theta,t)^{-1} \int_{\jmath(\theta,t)}^{t} \Im(\psi_t(\theta,s),s)^{-\frac{1}{2}} ds \\
&\leqslant 2\mathsf{m}^3 \jmath(\theta,t)^{-\frac{1}{2}} . \qquad (6.126)
\end{aligned}
$$

In the last inequality we have taken into account the definition of M_3 in (6.9),

Note that if $\theta \in (\mathfrak{s}_1(t), \mathfrak{s}_2(t))$ then the integrals in (6.124), (6.125), and (6.126) range from $\jmath(\theta,t)$ up to $\jmath_1(\theta,t) < t$, but this has no effect on the bounds established in (6.124), (6.125), and (6.126).

Returning to our decomposition of $\partial_\theta Q^z$ as $\mathcal{Q}_1 + \mathcal{Q}_2$, we note that by the same bounds and arguments as above, and by appealing also to (5.218), we also have that

$$
\begin{aligned}
\int_{\jmath(\theta,t)}^{t} \left|\mathcal{Q}_2\circ\psi_t\right|(\partial_\theta\psi_t)^2 ds &\leqslant 2\int_{\jmath(\theta,t)}^{t} \Big(3R_7\left|z_{\theta\theta}\circ\psi_t\right| + C\jmath(\theta,t)^{\frac{3}{2}}\left|a_{\theta\theta}\circ\psi_t\right| + C\Big) ds \\
&\leqslant C\jmath(\theta,t)^{\frac{1}{2}} + Ct\jmath(\theta,t)^{-\frac{1}{2}} \\
&\quad + C\jmath(\theta,t)^{\frac{3}{2}} \log\tfrac{t}{\jmath(\theta,t)} + Ct \\
&\leqslant Ct\jmath(\theta,t)^{-\frac{1}{2}} . \qquad (6.127)
\end{aligned}
$$

We note that for the bounds (6.124)–(6.127), we have taken $\overline{\varepsilon}$ sufficiently small.

Lastly, from (6.70) we have that for $(\theta,t) \in \mathcal{D}^k_{\overline{\varepsilon}}$ we have $\left|\partial_\theta^2\psi_t(\theta,s)\right| \leqslant Cs^{-1}$ so with the definition of Q^z in (6.118) and the bounds (5.57a), (5.141), and (6.123),

$$
\int_{\jmath(\theta,t)}^{t} \left|Q^z\circ\psi_t\ \partial_\theta^2\psi_t\right| ds \leqslant Ct^{\frac{1}{3}}\jmath(\theta,t)^{-\frac{1}{2}} . \qquad (6.128)
$$

On the other hand, for $(\theta,t) \in \mathcal{D}^z_{\overline{\varepsilon}} \setminus \overline{\mathcal{D}^k_{\overline{\varepsilon}}}$, we have that $\left|\partial_\theta^2\psi_t(\theta,s)\right| \leqslant C\jmath(\theta,t)^{-1}$ for $s \in [\jmath(\theta,t), \jmath_1(\theta,t)]$ and hence using (6.122)

$$
\begin{aligned}
\int_{\jmath(\theta,t)}^{t} \left|Q^z\circ\psi_t\ \partial_\theta^2\psi_t\right| ds &\leqslant \int_{\jmath(\theta,t)}^{\jmath_1(\theta,t)} \left|Q^z\circ\psi_t\ \partial_\theta^2\psi_t\right| ds + \int_{\jmath_1(\theta,t)}^{t} \left|Q^z\circ\psi_t\ \partial_\theta^2\psi_t\right| ds \\
&\leqslant C\jmath(\theta,t)^{-\frac{1}{6}} + Ct^{\frac{1}{3}}\jmath(\theta,t)^{-\frac{1}{2}} \\
&\leqslant Ct^{\frac{1}{3}}\jmath(\theta,t)^{-\frac{1}{2}} . \qquad (6.129)
\end{aligned}
$$

Combining the bounds (6.124)–(6.128), and taking $\overline{\varepsilon}$ sufficiently small, we obtain the inequality (6.121). □

Lemma 6.15 *For all $(\theta,t) \in \mathcal{D}^z_{\overline{\varepsilon}}$ we have the bounds*

$$
\left|z_{\theta\theta}(\theta,t)\right| \leqslant \begin{cases} \frac{1}{2}M_2\Im(\theta,t)^{-\frac{1}{2}}, & \text{if } (\theta,t) \in \mathcal{D}^k_{\overline{\varepsilon}} \\ \frac{1}{2}N_2\jmath(\theta,t)^{-\frac{1}{2}} & \text{if } (\theta,t) \in \mathcal{D}^z_{\overline{\varepsilon}} \setminus \overline{\mathcal{D}^k_{\overline{\varepsilon}}} \end{cases} , \qquad (6.130)
$$

where M_2 and N_2 are defined in (6.9), respectively in (6.11). Thus, the bootstrap assumptions (6.8b) and (6.10b) are improved. Moreover, the quantity q^z defined in (6.116) satisfies the bound

$$\left|q^z_{\theta}(\theta,t)\right| \leqslant 4\mathsf{m}^3 \jmath(\theta,t)^{-\frac{1}{2}} \tag{6.131}$$

for all $\mathfrak{s}_2(t) < \theta < \mathfrak{s}_2(t) + \frac{\kappa t}{6}$.

Proof of Lemma 6.15 Using (6.120) and the definition of q^z in (6.116), we see that for all $(\theta,t) \in \mathcal{D}^z_{\frac{\varepsilon}{}}$ and with $\jmath = \jmath(\theta,t)$, we have

$$\begin{aligned} z_{\theta\theta}(\theta,t) &= \overbrace{\left(z_{\theta\theta}(\partial_\theta\psi_t)^2 + z_\theta\circ\psi_t\partial_\theta^2\psi_t\right)\Big|_{(\mathfrak{s}(\jmath),\jmath)}}^{\mathcal{H}_1} \\ &\quad + \overbrace{\tfrac{1}{4}\left(ck_{\theta\theta}(\partial_\theta\psi_t)^2 + c_\theta k_\theta(\partial_\theta\psi_t)^2 + ck_\theta\partial_\theta^2\psi_t\right)\Big|_{(\mathfrak{s}(\jmath),\jmath)}}^{\mathcal{H}_2} \\ &\quad \underbrace{- \tfrac{1}{4}(ck_{\theta\theta} + c_\theta k_\theta)(\theta,t)}_{\mathcal{H}_3} - \int_{\jmath(\theta,t)}^{t}\left(\partial_\theta Q^z\circ\psi_t\ (\partial_\theta\psi_t)^2 + Q^z\circ\psi_t\ \partial_\theta^2\psi_t\right)ds\,. \end{aligned} \tag{6.132}$$

In order to get a good bound for the term $\mathcal{H}_1$ in (6.132) on the shock curve, it remains for us to express $z_{\theta\theta}(\mathfrak{s}(\jmath(\theta,t)),\jmath(\theta,t))$ in terms of derivatives of functions along the shock curve. Differentiating the system (5.111), taking into account the identity $\frac{d}{dt}(f(\mathfrak{s}(t),t)) = ((\partial_t + \dot{\mathfrak{s}}\partial_\theta)f)(\mathfrak{s}(t),t)$, and the formulas

$$\begin{aligned} \partial_t(c^2k_\theta) + \lambda_2\partial_\theta(c^2k_\theta) &= -2(\partial_\theta\lambda_2 + \tfrac{8}{3}a)(c^2k_\theta)\,, \\ \partial_{t\theta}z &= -\lambda_1 z_{\theta\theta} - \partial_\theta\lambda_1 z_\theta - \tfrac{8}{3}(az)_\theta + \tfrac{1}{6}\partial_\theta(c^2k_\theta) \end{aligned}$$

which are direct consequences of (3.5b), (3.5c), and (3.7), after a straightforward but lengthy computation we arrive at

$$\begin{aligned} \ddot{z_-} &= (\partial_t + \dot{\mathfrak{s}}\partial_\theta)^2 z - (\partial_t + \dot{\mathfrak{s}}\partial_\theta)(\partial_t + \lambda_1\partial_\theta)z - (\partial_t + \dot{\mathfrak{s}}\partial_\theta)\left(\tfrac{1}{6}c^2k_\theta - \tfrac{8}{3}az\right) \\ &= (\dot{\mathfrak{s}} - \lambda_1)z_{t\theta} + \dot{\mathfrak{s}}(\dot{\mathfrak{s}} - \lambda_1)z_{\theta\theta} + z_\theta(\ddot{\mathfrak{s}} - \partial_t\lambda_1 - \dot{\mathfrak{s}}\partial_\theta\lambda_1) + (\partial_t + \dot{\mathfrak{s}}\partial_\theta)\left(\tfrac{1}{6}c^2k_\theta - \tfrac{8}{3}az\right) \\ &= (\dot{\mathfrak{s}} - \lambda_1)^2 z_{\theta\theta} - (\dot{\mathfrak{s}} - \lambda_1)\left(\partial_\theta\lambda_1 z_\theta + \tfrac{8}{3}(az)_\theta - \tfrac{1}{6}\partial_\theta(c^2k_\theta)\right) \\ &\quad + z_\theta\left(\ddot{\mathfrak{s}} + \tfrac{1}{3}\lambda_3 w_\theta + \lambda_1 z_\theta - \tfrac{2}{9}c^2k_\theta + \tfrac{8}{9}aw + \tfrac{8}{3}az - \dot{\mathfrak{s}}\left(\tfrac{1}{3}w_\theta + z_\theta\right)\right) \\ &\quad - \tfrac{1}{3}\left(\partial_\theta\lambda_2 + \tfrac{8}{3}a\right)(c^2k_\theta) + \tfrac{1}{6}(\dot{\mathfrak{s}} - \lambda_2)\partial_\theta(c^2k_\theta) \\ &\quad - \tfrac{8}{3}a\left((\dot{\mathfrak{s}} - \lambda_1)z_\theta - \tfrac{8}{3}az + \tfrac{1}{6}c^2k_\theta\right) - \tfrac{8}{3}z\left((\dot{\mathfrak{s}} - \lambda_2)a_\theta - \tfrac{4}{3}a^2 + \tfrac{1}{6}(w^2 + z^2) + wz\right) \\ &= (\dot{\mathfrak{s}} - \lambda_1)^2 z_{\theta\theta} + \tfrac{1}{3}\left(\dot{\mathfrak{s}} - \tfrac{1}{2}w - \tfrac{5}{6}z\right)c^2k_{\theta\theta} \end{aligned}$$

$$+\tfrac{1}{3}\Big(\big(\dot{\mathfrak{s}}-\tfrac{5}{6}w-\tfrac{1}{2}z\big)ck_\theta-\big(2\dot{\mathfrak{s}}-\tfrac{4}{3}w-\tfrac{4}{3}z\big)z_\theta\Big)w_\theta+\mathcal{R}_{z_{\theta\theta}} \tag{6.133}$$

where we have denoted the remainder term $\mathcal{R}_{z_{\theta\theta}}$ by

$$\mathcal{R}_{z_{\theta\theta}}:=z_\theta\left(\ddot{\mathfrak{s}}+2\left(\lambda_1-\dot{\mathfrak{s}}\right)z_\theta-\tfrac{1}{6}(2\dot{\mathfrak{s}}+\tfrac{1}{3}w-3z)ck_\theta-\tfrac{8}{3}a(2\dot{\mathfrak{s}}-3\lambda_1)\right)$$
$$-\tfrac{8}{3}a\left(\tfrac{1}{2}c^2k_\theta-\tfrac{8}{3}az\right)-\tfrac{8}{3}z\left((2\dot{\mathfrak{s}}-\lambda_2-\lambda_1)a_\theta-\tfrac{4}{3}a^2+\tfrac{1}{6}(w^2+z^2)+wz\right) \tag{6.134}$$

At this stage we note that the reason we call the term $\mathcal{R}_{z_{\theta\theta}}$ a remainder term is as follows; from (5.13), (5.141), and the properties of w_{B}, we may directly show that

$$\left|\mathcal{R}_{z_{\theta\theta}}(\mathfrak{s}(t),t)\right|\leqslant Ct^{\frac{1}{2}} \tag{6.135}$$

for a suitable constant $C=C(\kappa,\mathsf{b},\mathsf{c},\mathsf{m})>0$. In comparison, the remaining terms in (6.133) will be shown to be $\mathcal{O}(t^{-\frac{1}{2}})$, so that $\mathcal{R}_{z_{\theta\theta}}$ is negligible.

The identities (6.133) and (6.134) are valid at any point $(\mathfrak{s}(t),t)$ on the shock curve, so in particular at $(\mathfrak{s}(\jmath),\jmath)$. Hence, we see that

$$z_{\theta\theta}(\mathfrak{s}(\jmath),\jmath)$$
$$=\frac{\ddot{z_-}-\frac{1}{3}\left(\dot{\mathfrak{s}}-\frac{1}{2}w-\frac{5}{6}z\right)(c^2k_{\theta\theta})-\frac{1}{3}\left(\left(\dot{\mathfrak{s}}-\frac{5}{6}w-\frac{1}{2}z\right)ck_\theta-\left(2\dot{\mathfrak{s}}-\frac{4}{3}w-\frac{4}{3}z\right)z_\theta\right)w_\theta}{(\dot{\mathfrak{s}}-\lambda_1)^2}\bigg|_{(\mathfrak{s}(\jmath),\jmath)}$$
$$-\frac{\mathcal{R}_{z_{\theta\theta}}}{(\dot{\mathfrak{s}}-\lambda_1)^2}\bigg|_{(\mathfrak{s}(\jmath),\jmath)}. \tag{6.136}$$

By combining (6.136) with (5.109) (in which we replace $\mathcal{T}$ with $\jmath$), (5.113), (6.94) (with t replaced by $\jmath$), and the estimates (5.69), (5.81), (5.83), (5.15), (5.141), (6.69), (6.90), and taking $\overline{\varepsilon}$ sufficiently small, we find that

$$\left|z_{\theta\theta}(\mathfrak{s}(\jmath),\jmath)\partial_\theta\psi_t(\mathfrak{s}(\jmath),\jmath)^2\right|\leqslant 3\kappa^{-2}(4\mathsf{b}^{\frac{9}{2}}\kappa^{-2}+\kappa^3\mathsf{m}^2+(\kappa^2R_6+\kappa R_4))\jmath^{-\frac{1}{2}}$$
$$+C\jmath^{\frac{1}{2}}\leqslant 6\kappa\mathsf{m}^2\jmath^{-\frac{1}{2}}\,. \tag{6.137}$$

On the other hand, from (5.141) and (6.70),

$$\left|z_\theta(\mathfrak{s}(\jmath),\jmath)\partial_\theta^2\psi_t(\mathfrak{s}(\jmath),\jmath)\right|\leqslant R_4\jmath^{\frac{1}{2}}\mathsf{m}^{\frac{1}{2}}\kappa^{-\frac{3}{2}}\jmath^{-1}\leqslant\mathsf{m}^2\jmath^{-\frac{1}{2}}\,. \tag{6.138}$$

Combining (6.137) and (6.138), we have thus bounded the first term $\mathcal{H}_1$ on the right side of (6.132) as

$$\left|\mathcal{H}_1\right|\leqslant 7\kappa\mathsf{m}^2\jmath(\theta,t)^{-\frac{1}{2}}\,. \tag{6.139}$$

Next, we turn our attention to the second term, $\mathcal{H}_2$, in (6.132). Using (5.141), (6.69), (6.70), (6.71), (6.90), and the fact that $\mathcal{T}(\mathfrak{s}(\jmath(\theta,t)),\jmath(\theta,t))=\jmath(\theta,t)$, we similarly obtain that

$$\begin{aligned}\left|\mathcal{H}_2\right| &\leqslant \kappa \mathsf{m}^2 \jmath(\theta,t)^{-\frac12} + R_6 \jmath(\theta,t)^{-\frac12} + \kappa^{-\frac12} R_6 \mathsf{m}^{\frac12} \jmath(\theta,t)^{-\frac12} + C \\ &\leqslant 2\kappa \mathsf{m}^2 \jmath(\theta,t)^{-\frac12}\,. \end{aligned} \tag{6.140}$$

Since the integral term in (6.132) was previously estimated in Lemma 6.14, it thus remains to bound the term $\mathcal{H}_3$ on the right side of (6.132). Note that if $(\theta,t) \in \mathcal{D}^z_{\bar\varepsilon} \setminus \overline{\mathcal{D}}^k_{\bar\varepsilon}$, then k vanishes, and so $\mathcal{H}_3 = 0$. In the case that $(\theta,t) \in \mathcal{D}^k_{\bar\varepsilon}$, by appealing to (5.141), the bound $\frac{\kappa}{5} \leqslant c(\theta,t) \leqslant \mathsf{m}$, and (6.90), we obtain

$$\begin{aligned}\left|\mathcal{H}_3\right| &\leqslant \tfrac14 \mathsf{m}^3 \mathcal{T}(\theta,t)^{-\frac12} + \tfrac18 (t^{-1} + R_4 t^{\frac12}) R_6 t^{-\frac12} \\ &\leqslant \tfrac14 \mathsf{m}^3 \mathcal{T}(\theta,t)^{-\frac12} + \mathsf{m} t^{-\frac12} \\ &\leqslant \mathsf{m}^3 \mathcal{T}(\theta,t)^{-\frac12}\,. \end{aligned} \tag{6.141}$$

In the last inequality of (6.141) we have used that $\mathcal{T}(\theta,t) \leqslant t$.

By combining the identity (6.132) with the bounds (6.139), (6.140), (6.141), (6.121), we have that

$$\left|z_{\theta\theta}(\theta,t)\right| \leqslant 9\kappa \mathsf{m}^2 \jmath(\theta,t)^{-\frac12} + 3\mathsf{m}^3 \jmath(\theta,t)^{-\frac12} + \begin{cases} \mathsf{m}^3 \mathcal{T}(\theta,t)^{-\frac12}\,, & \text{for} \quad \theta \in \mathcal{D}^k_{\bar\varepsilon} \\ 0\,, & \text{for} \quad \theta \in \mathcal{D}^z_{\bar\varepsilon} \setminus \overline{\mathcal{D}}^k_{\bar\varepsilon} \end{cases}. \tag{6.142}$$

Taking into account that for $(\theta,t) \in \mathcal{D}^k_{\bar\varepsilon}$ by (6.5) we have that $\mathcal{T}(\theta,t) \leqslant \jmath(\theta,t)$, and we have $9\kappa \leqslant \mathsf{m}$, the above bound completes the proof of (6.130), once we ensure that $\frac12 M_2 \geqslant 5\mathsf{m}^3$ and $\frac12 N_2 \geqslant 4\mathsf{m}^3$. This justifies the choices of M_2 and N_2 are defined in (6.9), respectively in (6.11).

In order to complete the proof of the Lemma, we need to establish the bound (6.131), which is useful later in the proof. For this purpose, note that in view of (6.120), (6.132), the fact that $q^z_\theta(\theta,t) = z_{\theta\theta}(\theta,t) + \mathcal{H}_3$, and of the bounds bounds (6.139), (6.140), (6.121), we have that

$$\left|q^z_\theta(\theta,t)\right| \leqslant 9\kappa \mathsf{m}^2 \jmath(\theta,t)^{-\frac12} + 3\mathsf{m}^3 \jmath(\theta,t)^{-\frac12} \tag{6.143}$$

which thus concludes the proof of (6.131), and of the lemma. □

6.7 Lower Bounds for Second Derivatives

In this section we prove that various second derivatives of the solution blow up as we approach the curves $\mathfrak{s}_1$ and $\mathfrak{s}_2$ from the right side. Throughout this section we fix $t \in (0, \bar\varepsilon]$ and shall make reference to the following asymptotic descriptions:

$$\lim_{\theta \to \mathfrak{s}_2(t)^+} \frac{\theta - \mathfrak{s}_2(t)}{\mathcal{T}(\theta,t)} = \frac{\kappa}{3} \tag{6.144a}$$

$$\lim_{\theta \to \mathfrak{s}_2(t)} \mathfrak{Z}(\theta, t) \geqslant \frac{t}{3} \tag{6.144b}$$

$$\lim_{\theta \to \mathfrak{s}_1(t)^+} \frac{\theta - \mathfrak{s}_1(t)}{\mathfrak{Z}(\theta, t)} = \frac{\kappa}{3}. \tag{6.144c}$$

Here we have implicitly used that $\phi_t(\mathfrak{s}_2(t), s) = \mathfrak{s}_2(s)$, and $\psi_t(\mathfrak{s}_1(t), s) = \mathfrak{s}_1(s)$. The bounds are a consequence of (6.5), (6.69), and the definitions of $\mathfrak{s}_1, \mathfrak{s}_2, \phi_t$, and ψ_t. For example, in order to prove (6.144a), note that by the mean value theorem we have

$$\begin{aligned}\mathfrak{s}(\mathfrak{T}(\theta, t)) - \mathfrak{s}_2(\mathfrak{T}(\theta, t)) &= \phi_t(\theta, \mathfrak{T}(\theta, t)) - \phi_t(\mathfrak{s}_2(t), \mathfrak{T}(\theta, t)) \\ &= (\theta - \mathfrak{s}_2(t)) \underbrace{\partial_\theta \phi_t(\overline{y}, \mathfrak{T}(\theta, t))}_{=1+\mathcal{O}(\mathfrak{T}^{\frac{1}{3}})}\end{aligned}$$

while by (6.7) we have

$$\mathfrak{s}(\mathfrak{T}(\theta, t)) - \mathfrak{s}_2(\mathfrak{T}(\theta, t)) = \tfrac{\kappa}{3}\mathfrak{T}(\theta, t) + \mathcal{O}(\mathfrak{T}(\theta, t)^{\frac{4}{3}}).$$

The proof of (6.144c) is similar. Lastly, in order to prove (6.144b), we use that one the hand

$$\mathfrak{s}(\mathfrak{Z}(\theta, t)) - \mathfrak{s}_2(\mathfrak{Z}(\theta, t)) = \tfrac{\kappa}{3}\mathfrak{Z}(\theta, t) + \mathcal{O}(\mathfrak{Z}(\theta, t)^{\frac{4}{3}}),$$

while on the other hand

$$\begin{aligned}\mathfrak{s}(\mathfrak{Z}(\theta, t)) - \mathfrak{s}_2(\mathfrak{Z}(\theta, t)) &= \psi_t(\mathfrak{s}_2(t), \mathfrak{Z}(\theta, t)) - \phi_t(\mathfrak{s}_2(t), \mathfrak{Z}(\theta, t)) \\ &= \int_{\mathfrak{Z}(\theta,t)}^{t} \underbrace{(\partial_s \phi_t - \partial_s \psi_t)}_{=\frac{\kappa}{3}+\mathcal{O}(r^{\frac{1}{3}})} ds \\ &= (t - \mathfrak{Z}(\theta, t)) \left(\tfrac{\kappa}{3} + \mathcal{O}(\overline{\varepsilon}^{\frac{1}{3}})\right).\end{aligned}$$

By combining the above two estimates, it follows that $\mathfrak{Z}(\mathfrak{s}_2(t), t) \geqslant t(\frac{1}{2} - \mathcal{O}(\overline{\varepsilon}^{\frac{1}{3}})) \geqslant \frac{t}{3}$, proving (6.144b).

6.7.1 Singularities on $\mathfrak{s}_2$, from the Right Side

Note that the second derivative upper bounds established in (6.8) blow up as $\theta \to \mathfrak{s}_2(t)^+$; the purpose of this subsection is to obtain lower bounds which are within a constant factor of these upper bounds, and thus also diverge as $\theta \to \mathfrak{s}_2(t)^+$.

In this proof we shall frequently use the following facts. First, that $\frac{\kappa}{5} \leqslant c(\theta, t) \leqslant \mathsf{m}$ for all $(\theta, t) \in \mathcal{D}_{\overline{\varepsilon}}$. This follows from the identity $c(\theta, t) = \frac{1}{2} w_{\mathsf{B}}(\theta, t) + \frac{1}{2}(w - w_{\mathsf{B}} - z)$, which in view of (5.35), and (5.141) implies $c(\theta, t) = \frac{1}{2} w_0(\eta_{\mathsf{B}}{}^{-1}(\theta, t)) + \mathcal{O}(t)$; the desired bound now follows from (5.1a) and (5.1b). Second, we note that a slightly sharper bound is required for $\partial_\theta w_{\mathsf{B}}$ on the shock curve (when compared to (5.37a)).

From (5.34) we note that $\eta_{\mathsf{B}}{}^{-1}(\mathfrak{s}(t)^-, t) = -(\mathsf{b}t)^{\frac{3}{2}} + \mathcal{O}(t^2)$. By appealing to (5.1d) we then obtain that $w_0'(\eta_{\mathsf{B}}{}^{-1}(\mathfrak{s}(t)^-, t)) = -\frac{1}{3}t^{-1} + \mathcal{O}(t^{-\frac{1}{2}})$ as $t \to 0$. We then conclude from (5.36a) that

$$\partial_\theta w_{\mathsf{B}}(\mathfrak{s}(t), t) = \frac{-\frac{1}{3}t^{-1} + \mathcal{O}\left(t^{-\frac{1}{2}}\right)}{1 + t\left(-\frac{1}{3}t^{-1} + \mathcal{O}\left(t^{-\frac{1}{2}}\right)\right)} = -\frac{1}{2t} + \mathcal{O}(t^{-\frac{1}{2}}) \tag{6.145}$$

as for $0 < t \leqslant \overline{\varepsilon}$.

Lower bound for $|k_{\theta\theta}|$ **on** $\mathfrak{s}_2^+$. The desired lower bound turns out to be a consequence of (6.95).

We first consider the second line of (6.95). Let $\theta > \mathfrak{s}_2(t)$ with $\theta - \mathfrak{s}_2(t) \leqslant \frac{\kappa t}{6}$. Note in this range of θ, due to (6.144a) and the fact that $t \leqslant \overline{\varepsilon}$, we have $\Upsilon(\theta, t) \leqslant \frac{t}{3} \leqslant \overline{\varepsilon}$. We claim that for a constant $C = C(\kappa, \mathsf{m}, \mathsf{b}, \mathsf{c}) > 0$ we have

$$\partial_\theta^2 \phi_t(\theta, \Upsilon(\theta, t)) \geqslant \frac{\kappa^2}{100\mathsf{m}^3}\Upsilon(\theta, t)^{-1} - Ct^{-\frac{2}{3}} \geqslant \frac{\kappa^2}{100\mathsf{m}^3}\Upsilon(\theta, t)^{-1} - C\Upsilon(\theta, t)^{-\frac{2}{3}} \geqslant 0 \tag{6.146}$$

for $\mathfrak{s}_2(t) < \theta < \mathfrak{s}_2(t) + \frac{\kappa t}{6}$, once $\overline{\varepsilon}$ is sufficiently small. In order to prove (6.146), we consider the formula (6.75) with $s = \Upsilon(\theta, t)$. We note that the largest term in (6.75), the one containing $c(\theta, t)c_\theta(\mathfrak{s}(\Upsilon), \Upsilon)\partial_\theta\phi_t(\theta, \Upsilon)$, is positive. Indeed, from the bounds $\frac{\kappa}{5} \leqslant c(\theta, t) \leqslant \mathsf{m}$, (6.145), the bound (6.69), (5.141b), and (5.141d), we obtain that

$$\begin{aligned}
&c(\theta, t)c_\theta(\mathfrak{s}(\Upsilon(\theta, t))^+, \Upsilon(\theta, t))\partial_\theta\phi_t(\theta, \Upsilon(\theta, t)) \\
&= \tfrac{1}{2}c(\theta, t)\partial_\theta\phi_t(\theta, \Upsilon)\left(\partial_\theta w_{\mathsf{B}}(\mathfrak{s}(\Upsilon)^+, \Upsilon) + \partial_\theta(w - w_{\mathsf{B}} - z)(\mathfrak{s}(\Upsilon)^+, \Upsilon)\right) \\
&= \tfrac{1}{2}c(\theta, t)(1 + \mathcal{O}(t^{\frac{1}{3}}))\left(-\tfrac{1}{2}\Upsilon^{-1} + \mathcal{O}(\Upsilon^{-\frac{1}{2}})\right) \\
&\leqslant -\tfrac{\kappa}{40}\Upsilon(\theta, t)^{-1}
\end{aligned}$$

since $\Upsilon(\theta, t) \leqslant \frac{t}{3} \ll 1$. The remaining terms in (6.75) may be estimated from above by

$$2e^{16\mathsf{m}t}\left(\frac{80\mathsf{m}^2}{\kappa^2}R_7 t + \frac{25\mathsf{m}}{\kappa^2}\left(\frac{4\mathsf{b}}{5}(\tfrac{\kappa t}{6})^{-\frac{2}{3}} + R_2(\tfrac{\kappa t}{6})^{-\frac{1}{2}} + R_4 t^{\frac{1}{2}}\right)\right) \leqslant Ct^{-\frac{2}{3}}$$

for a constant $C = C(\kappa, \mathsf{m}, \mathsf{b}, \mathsf{c}) > 0$. The above two estimates then imply

$$\partial_\theta^2\phi_t(\theta, \Upsilon(\theta, t)) \geqslant 2e^{-16\mathsf{m}t}\frac{\kappa}{5\mathsf{m}^3}\frac{\kappa}{40}\Upsilon(\theta, t)^{-1} - Ct^{-\frac{2}{3}},$$

and (6.146) follows.

Next, we return to the second line of (6.95), from (5.81) we have

$$\dot{k_-}(\Upsilon(\theta, t)) = \frac{48\mathsf{b}^{\frac{9}{2}}}{\kappa^3}\Upsilon(\theta, t)^{\frac{1}{2}} + \mathcal{O}(\Upsilon(\theta, t)) \geqslant 0 \tag{6.147}$$

since $\mathcal{T}(\theta,t) \leqslant t$ is small. Moreover, from (6.6a) and (5.13) we have $\frac{\kappa}{4} \leqslant (\dot{\mathfrak{s}} - \partial_s \phi_t)(\theta, \mathcal{T}(\theta,t)) \leqslant \frac{\kappa}{2}$. As a consequence, from (6.146) and (6.147), we obtain

$$
\begin{aligned}
\text{second line of (6.95)} \geqslant \tfrac{2}{\kappa} &\left(\tfrac{48\mathsf{b}^{\frac{9}{2}}}{\kappa^3} \mathcal{T}(\theta,t)^{\frac{1}{2}} - C\mathcal{T}(\theta,t) \right) \\
&\left(\tfrac{\kappa^2}{100\mathsf{m}^3} \mathcal{T}(\theta,t)^{-1} - C\mathcal{T}(\theta,t)^{-\frac{2}{3}} \right) \\
&\geqslant \tfrac{48\mathsf{b}^{\frac{9}{2}}}{50\kappa^2\mathsf{m}^2} \mathcal{T}(\theta,t)^{-\frac{1}{2}} - C\mathcal{T}(\theta,t)^{-\frac{1}{6}} \\
&\geqslant \tfrac{\mathsf{b}^{\frac{9}{2}}}{2\kappa^2\mathsf{m}^2} \mathcal{T}(\theta,t)^{-\frac{1}{2}} \qquad \text{as} \qquad \theta \to \mathfrak{s}_2(t)^+ \,.
\end{aligned} \tag{6.148}
$$

Next, we consider the terms on the first line of (6.95). From the definition of λ_2 in (3.6) and the evolution equation (3.5a), we obtain that

$$
\begin{aligned}
(\partial_t \lambda_2 + (2\dot{\mathfrak{s}} - \lambda_2)\partial_\theta \lambda_2) - \ddot{\mathfrak{s}} = \tfrac{2}{3}&\partial_\theta w(\dot{\mathfrak{s}} - \lambda_2 + \dot{\mathfrak{s}} - w - \tfrac{1}{3}z) \\
&+ \tfrac{2}{3}(2\dot{\mathfrak{s}} - \lambda_2)\partial_\theta z + \tfrac{1}{2}\partial_t z \\
&+ \tfrac{2}{3}\left(\tfrac{1}{24}(w-z)^2 \partial_\theta k - \tfrac{8}{2} a w \right) - \ddot{\mathfrak{s}} \,.
\end{aligned}
$$

Taking into account the bound (5.13), Proposition 5.7, and the fact that $(w, z, k, a) \in \mathcal{X}_{\overline{\varepsilon}}$ (in particular, that (6.82) holds), we obtain

$$
\begin{aligned}
&(\partial_t \lambda_2 + (2\dot{\mathfrak{s}} - \lambda_2)\partial_\theta \lambda_2)\,(\mathfrak{s}(\mathcal{T}(\theta,t)), \mathcal{T}(\theta,t)) - \ddot{\mathfrak{s}}(\mathcal{T}(\theta,t)) \\
&\quad = \tfrac{2}{3}\partial_\theta w_{\mathsf{B}}(\mathfrak{s}(\mathcal{T}(\theta,t), \mathcal{T}(\theta,t)) \left(2\dot{\mathfrak{s}}(\mathcal{T}(\theta,t)) - \tfrac{5}{3} w_{\mathsf{B}}(\mathfrak{s}(\mathcal{T}(\theta,t)), \mathcal{T}(\theta,t)) \right) \\
&\qquad + \mathcal{O}(\mathcal{T}(\theta,t)^{-\frac{1}{2}})
\end{aligned} \tag{6.149}
$$

as $\theta \to \mathfrak{s}_2(t)^+$, or equivalently, as $\mathcal{T}(\theta,t) \to 0^+$. Next, from (5.81) and (5.83) we note that

$$
\ddot{k_-}(\mathcal{T}(\theta,t))) = \tfrac{1}{2\mathcal{T}(\theta,t)} \dot{k_-}(\mathcal{T}(\theta,t))) + \mathcal{O}(1) \tag{6.150}
$$

as $\mathcal{T}(\theta,t) \to 0^+$. By combining (6.149), (6.150), the bound $\frac{\kappa}{2} \geqslant \dot{\mathfrak{s}}(\mathcal{T}(\theta,t)) - \lambda_2(\mathfrak{s}(\mathcal{T}(\theta,t)), \mathcal{T}(\theta,t)) \geqslant \frac{\kappa}{4}$, and (5.81), we deduce

$$
\begin{aligned}
&\text{first line of (6.95)} \\
&\quad = \left(\tfrac{\partial_\theta \phi_t(\theta, \mathcal{T}(\theta,t))}{\dot{\mathfrak{s}}(\mathcal{T}(\theta,t)) - \lambda_2(\mathfrak{s}(\mathcal{T}(\theta,t)), \mathcal{T}(\theta,t))} \right)^2 \dot{k_-}(\mathcal{T}(\theta,t)) \\
&\qquad \left(\tfrac{1}{2\mathcal{T}(\theta,t)} + \tfrac{2}{3}\partial_\theta w_{\mathsf{B}}(\mathfrak{s}(\mathcal{T}(\theta,t), \mathcal{T}(\theta,t)) \right) + \mathcal{O}(1)
\end{aligned} \tag{6.151}
$$

as $\mathcal{T}(\theta,t) \to 0^+$. At this stage we appeal to (6.145) with t replaced by $\mathcal{T} = \mathcal{T}(\theta,t) \to 0^+$, which is the relevant regime for $\theta \to \mathfrak{s}_2(t)^+$. From (6.151), (6.145), (6.69), and

(5.81) we finally conclude that

$$\begin{aligned}\text{first line of (6.95)} &= \left(\tfrac{\partial_\theta \phi_t(\theta,\mathcal{T}(\theta,t))}{\dot{\mathfrak{s}}(\mathcal{T}(\theta,t))-\lambda_2(\mathfrak{s}(\mathcal{T}(\theta,t)),\mathcal{T}(\theta,t))}\right)^2 \dot{k}_-(\mathcal{T}(\theta,t))\tfrac{1}{6\mathcal{T}(\theta,t)} + \mathcal{O}(1)\\ &\geqslant \tfrac{3}{\kappa^2}\tfrac{48\mathsf{b}^{\frac{9}{2}}\mathcal{T}(\theta,t)^{\frac{1}{2}}}{\kappa^3}\tfrac{1}{6\mathcal{T}(\theta,t)} - C\\ &\geqslant \tfrac{24\mathsf{b}^{\frac{9}{2}}}{\kappa^4}\mathcal{T}(\theta,t)^{-\frac{1}{2}}\end{aligned} \tag{6.152}$$

as $\mathcal{T}(\theta,t)\to 0^+$.

Lastly, by combining (6.148) with (6.152), we obtain that

$$\lim_{\theta\to\mathfrak{s}_2(t)^+} \partial_\theta^2 k(\theta,t)\mathcal{T}(\theta,t)^{\frac{1}{2}} \geqslant \tfrac{24\mathsf{b}^{\frac{9}{2}}}{\kappa^4}\,. \tag{6.153}$$

In view of (6.144a), the above estimate and (6.8c) thus precisely determines the blowup rate of $\partial_\theta^2 k(\theta,t)$ as $\theta\to\mathfrak{s}_2(t)^+$: this rate lies within two constants of $(\theta-\mathfrak{s}_2(t))^{-\frac{1}{2}}$.

Lower bound for $|z_{\theta\theta}|$ **on** $\mathfrak{s}_2^+$. Next, we show that the upper bound (6.8b) also has a corresponding lower bound which blows up as $\theta\to\mathfrak{s}_2(t)^+$. We start by recalling the function q^z defined in (6.117), and the formula for its derivative in (6.120). As above, we let $\mathfrak{s}_2(t)<\theta<\mathfrak{s}_2(t)+\frac{\kappa t}{6}$ and denote $\mathcal{J}=\mathcal{J}(\theta,t)$. From estimate (6.131), and by appealing to (6.144a) which yields $\mathcal{J}(\theta,t)\geqslant\frac{1}{4}t$ in the range of θ considered here, we arrive at

$$\left|z_{\theta\theta}+\tfrac{1}{4}ck_{\theta\theta}+\tfrac{1}{4}c_\theta k_\theta\right|(\theta,t) = \left|\partial_\theta q^z(\theta,t)\right| \leqslant C\mathcal{J}(\theta,t)^{-\frac{1}{2}} \leqslant Ct^{-\frac{1}{2}}\,,$$

for all $\theta>\mathfrak{s}_2(t)$ which is close to $\mathfrak{s}_2(t)$. Furthermore, since (5.141) and (6.82a) imply that $|c_\theta k_\theta|(\theta,t)\leqslant\frac{1}{2}(t^{-1}+R_4t^{\frac{1}{2}})R_6t^{\frac{1}{2}}\leqslant Ct^{-\frac{1}{2}}$, the above estimate implies

$$\left|z_{\theta\theta}+\tfrac{1}{4}ck_{\theta\theta}\right|(\theta,t)\leqslant Ct^{-\frac{1}{2}}\,, \tag{6.154}$$

for a suitable constant $C=C(\kappa,\mathsf{b},\mathsf{c},\mathsf{m})>0$.

Lastly, since $\frac{\kappa}{5}\leqslant c(\theta,t)\leqslant\mathsf{m}$, we see that the blowup rate for $k_{\theta\theta}$ as $\theta\to\mathfrak{s}_2^+(t)$, given by (6.153), is immediately transferred to $z_{\theta\theta}$, and we have

$$\lim_{\theta\to\mathfrak{s}_2(t)^+}\partial_\theta^2 z(\theta,t)\mathcal{T}(\theta,t)^{\frac{1}{2}} \leqslant -\tfrac{1}{4}\lim_{\theta\to\mathfrak{s}_2(t)^+} c(\theta,t)\partial_\theta^2 k(\theta,t)\mathcal{T}(\theta,t)^{\frac{1}{2}} \leqslant -\tfrac{\mathsf{b}^{\frac{9}{2}}}{\kappa^3}\,. \tag{6.155}$$

Here we have used the fact that $\lim_{\theta\to\mathfrak{s}(t)^+}\mathcal{T}(\theta,t)t^{-\frac{1}{2}}=0$. The estimate (6.155), and the upper bound (6.8b), show that $\partial_\theta^2 z(\theta,t)\to-\infty$ as $\theta\to\mathfrak{s}_2(t)^+$, at a rate which is proportional to $-(\theta-\mathfrak{s}_2(t))^{-\frac{1}{2}}$.

Lower bound for $|w_{\theta\theta}|$ **on** $\mathfrak{s}_2^+$. The argument is nearly identical to the one for the second derivative of z. We recall that the variable q^w defined in (6.31) satisfies the derivative bound (6.114). By appealing to the fact that $(w, z, k, a) \in \mathcal{X}_{\bar{\varepsilon}}$, the estimate (5.37b) for the second derivative of the Burgers solution, and to (6.114), we arrive at

$$\begin{aligned}\left|w_{\theta\theta} - \tfrac{1}{4} c k_{\theta\theta}\right|(\theta, t) &\leqslant \tfrac{1}{4}\left|c_\theta k_\theta\right|(\theta, t) + \left|q^w_\theta(\theta, t)\right| \\ &\leqslant \tfrac{1}{8}(t^{-1} + R_4 t^{\frac{1}{2}}) R_6 t^{\frac{1}{2}} + 3\mathsf{b}(\kappa t)^{-\frac{5}{3}} \\ &\leqslant C t^{-\frac{5}{3}} \end{aligned} \tag{6.156}$$

for all $\theta \in (\mathfrak{s}_2(t), \mathfrak{s}_2(t) + \frac{\kappa t}{6})$, for a suitable constant $C = C(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$. This estimate is the parallel bound to (6.154) for the second derivative of z. It implies, in a similar fashion to (6.156), that

$$\lim_{\theta \to \mathfrak{s}_2(t)^+} \partial^2_\theta w(\theta, t) \mathcal{T}(\theta, t)^{\frac{1}{2}} \geqslant \frac{\mathsf{b}^{\frac{9}{2}}}{\kappa^3}\,. \tag{6.157}$$

The estimate (6.157), and the upper bound (6.8a), show that $\partial^2_\theta w(\theta, t) \to +\infty$ as $\theta \to \mathfrak{s}_2(t)^+$, at a rate which is proportional to $(\theta - \mathfrak{s}_2(t))^{-\frac{1}{2}}$.

Lower bound for $|a_{\theta\theta}|$ on $\mathfrak{s}_2^+$. As before, consider $\theta \in (\mathfrak{s}_2(t), \mathfrak{s}_2(t) + \frac{\kappa t}{6})$. By combining (5.37a), (5.141e), (5.223), and (6.22), we arrive at the bound

$$\left|a_{\theta\theta} + \tfrac{1}{4} c^2 e^{-k} \varpi_\theta\right| \leqslant C t^{-\frac{2}{3}} + C t^{-\frac{1}{2}} \leqslant C t^{-\frac{2}{3}}\,. \tag{6.158}$$

The desired lower bound on $a_{\theta\theta}$ is thus inherited from ϖ_θ, which we recall is given by (6.16). The principal contribution is due to the term containing the time integral of $k_{\theta\theta}$. Indeed, using the same argument used to prove (6.14), we have that

$$\left|\varpi_\theta(\theta, t) - \tfrac{4}{3} \int_{\mathcal{T}(\theta,t)}^t \left(e^k k_{\theta\theta}\right)(\phi_t(\theta, s), s) e^{I_{\varpi_\theta}(\theta,t;s)} ds\right| \leqslant C\,. \tag{6.159}$$

The analysis reduces to establishing a lower bound which is commensurate with the upper bound (6.19). The main idea here is as follows. From (6.148) and (6.152), as in (6.153) we have that $\partial^2_\theta k(\theta, t) \geqslant 24\mathsf{b}^{\frac{9}{2}} \kappa^{-4} \mathcal{T}(\theta, t)^{-\frac{1}{2}}$, for all θ sufficiently close to $\mathfrak{s}_2(t)$, i.e. $\mathfrak{s}_2(t) < \theta < \mathfrak{s}_2(t) + \frac{\kappa t}{6}$. Therefore, if the point (θ, t) is replaced by the point $(\phi_t(\theta, s), s)$, which in view of Remark 6.5 and estimate (6.7) is such that $\phi_t(\theta, s)$ is sufficiently close to $\mathfrak{s}_2(s)$, we have that

$$\partial^2_\theta k(\phi_t(\theta, s), s) \geqslant 24\mathsf{b}^{\frac{9}{2}} \kappa^{-4} \mathcal{T}(\phi_t(\theta, s), s)^{-\frac{1}{2}} = 24\mathsf{b}^{\frac{9}{2}} \kappa^{-4} \mathcal{T}(\theta, t)^{-\frac{1}{2}}$$

uniformly for all $s \in [\mathcal{T}(\theta, t), t]$. In particular, $\partial^2_\theta k \circ \phi_t > 0$, and so by combining (6.158)–(6.159), with (5.141f), (6.17), and with the estimate $\frac{\kappa}{5} \leqslant c \leqslant \mathsf{m}$, we arrive at

$$\begin{aligned} a_{\theta\theta}(\theta, t) &\leqslant -\tfrac{1}{4} c^2 e^{-k} \varpi_\theta + C t^{-\frac{2}{3}} \\ &\leqslant -\tfrac{\kappa^2}{75} \int_{\mathcal{T}(\theta,t)}^t k_{\theta\theta}(\phi_t(\theta, s), s) ds + C t^{-\frac{2}{3}} \end{aligned}$$

$$\leqslant -\frac{\mathsf{b}^{\frac{9}{2}}}{4\kappa^2}(t - \mathcal{T}(\theta,t))\mathcal{T}(\theta,t)^{-\frac{1}{2}} . \tag{6.160}$$

The above estimate implies

$$\lim_{\theta\to\mathfrak{s}_2(t)^+} \mathcal{T}(\theta,t)^{\frac{1}{2}} a_{\theta\theta}(\theta,t) \leqslant -\frac{\mathsf{b}^{\frac{9}{2}}}{4\kappa^2} t , \tag{6.161}$$

which may be combined with the upper bound (6.20) show that $a_{\theta\theta}(\theta,t) \to -\infty$ as $\theta \to \mathfrak{s}_2(t)^+$, at a rate which is proportional to $t(\theta - \mathfrak{s}_2(t))^{-\frac{1}{2}}$.

6.7.2 Singularities on $\mathfrak{s}_1$, from the Right Side

Passing to the limit $\theta \to \mathfrak{s}_1(t)^+$ in the estimates (6.10), we obtain that

$$\lim_{\theta\to\mathfrak{s}_1(t)^+} |w_{\theta\theta}(\theta,t)| \leqslant Ct^{-\frac{5}{3}} , \qquad \text{and} \qquad \lim_{\theta\to\mathfrak{s}_1(t)^+} |a_{\theta\theta}(\theta,t)| \leqslant Ct^{-\frac{2}{3}} ,$$

for a suitable constant $C = C(\kappa, \mathsf{b}, \mathsf{c}, \mathsf{m}) > 0$, which shows that these quantities do not blow up as θ approaches $\mathfrak{s}_1$ from the right side. The only quantity that does indeed blow up is the second derivative of z.

Here we establish a lower bound for $|\partial_\theta^2 z(\theta,t)|$ which is commensurate with (6.10b) as $\theta \to \mathfrak{s}_1(t)^+$; more precisely we claim that

$$\lim_{\theta\to\mathfrak{s}_1(t)^+} \mathfrak{Z}(\theta,t)^{\frac{1}{2}} z_{\theta\theta}(\theta,t) \leqslant -\tfrac{1}{4}\mathsf{b}^{\frac{9}{2}}\kappa^{-4} , \tag{6.162}$$

which shows the precise rate of divergence of $\partial_\theta^2 z$ towards $-\infty$ as θ approaches $\mathfrak{s}_1$ from the right side. The proof of (6.162) is quite involved, and will be broken up into several parts, which correspond to estimating the various terms in (6.120). We rewrite this identity as

$$z_{\theta\theta}(\theta,t) = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 , \tag{6.163}$$

where we define

$$\begin{aligned}
\mathcal{B}_1 &:= q_\theta^z(\mathfrak{s}(\mathfrak{Z}(\theta,t)), \mathfrak{Z}(\theta,t))(\partial_\theta\psi_t)^2(\theta, \mathfrak{Z}(\theta,t)) \\
&\quad + q^z(\mathfrak{s}(\mathfrak{Z}(\theta,t)), \mathfrak{Z}(\theta,t))\partial_\theta^2\psi_t(\theta, \mathfrak{Z}(\theta,t)) \\
&= \mathcal{B}_{11} + \mathcal{B}_{12} &(6.164)\\
\mathcal{B}_2 &:= -\int_{\mathfrak{Z}_1(\theta,t)}^{t} \Big(\partial_\theta Q^z\circ\psi_t\ (\partial_\theta\psi_t)^2 + Q^z\circ\psi_t\ \partial_\theta^2\psi_t\Big)ds &(6.165)\\
\mathcal{B}_3 &:= -\int_{\mathfrak{Z}(\theta,t)}^{\mathfrak{Z}_1(\theta,t)} \Big(\partial_\theta Q^z\circ\psi_t\ (\partial_\theta\psi_t)^2 + Q^z\circ\psi_t\ \partial_\theta^2\psi_t\Big)ds &(6.166)
\end{aligned}$$

and $\mathfrak{Z}_1$ is the time at which $\psi_t(\theta,\cdot)$ intersects the curve $\mathfrak{s}_2$; as given by (6.122), see also Figure 15. Since $\theta \to \mathfrak{s}_1(t)^+$ is equivalent in view of (6.144c) to $\mathfrak{Z}(\theta,t) \to 0$, our goal is to extract the leading order term in $\mathcal{B}_1$ with respect to $\jmath \ll 1$, and then to obtain sharp estimates for $\mathcal{B}_2$ and $\mathcal{B}_3$ with respect to $\jmath$. In this direction we claim:

Lemma 6.16 *Fix $t \in (0, \overline{\varepsilon}]$ and $\mathfrak{s}_1(t) < \theta < \mathfrak{s}_1(t) + \frac{\kappa t}{6}$. Then we have that*

$$-\tfrac{86}{16}\mathsf{b}^{\frac{9}{2}}\kappa^{-4}\jmath(\theta,t)^{-\frac{1}{2}} \leqslant \mathcal{B}_1 \leqslant -\tfrac{85}{16}\mathsf{b}^{\frac{9}{2}}\kappa^{-4}\jmath(\theta,t)^{-\frac{1}{2}}\,, \tag{6.167}$$

where the term $\mathcal{B}$ is as defined in (6.164).

Lemma 6.17 *Fix $t \in (0, \overline{\varepsilon}]$ and $\mathfrak{s}_1(t) < \theta < \mathfrak{s}_1(t) + \frac{\kappa t}{6}$. Then we have that*

$$\mathcal{B}_2 \leqslant t^{\frac{1}{4}}\mathsf{b}^{\frac{9}{2}}\kappa^{-4}\jmath(\theta,t)^{-\frac{1}{2}}\,, \tag{6.168a}$$

$$\mathcal{B}_3 \leqslant \tfrac{9}{2}\mathsf{b}^{\frac{9}{2}}\kappa^{-4}\jmath(\theta,t)^{-\frac{1}{2}} + C\jmath(\theta,t)^{-\frac{1}{6}}\,, \tag{6.168b}$$

where the terms $\mathcal{B}_2$ and $\mathcal{B}_3$ defined in (6.165) and respectively in (6.166). Note that the sum of the estimates in (6.168) gives an improvement over (6.121), in the sense that the constant is sharper.

Proof of (6.162) We note that the bound (6.162) follows from (6.163), (6.167), (6.168a), (6.168b), and the inequality

$$\tfrac{9}{2} + Ct^{\frac{1}{4}} + C\jmath^{\frac{1}{3}} - \tfrac{85}{16} \leqslant -\tfrac{1}{4}\,,$$

for $\overline{\varepsilon}$, and hence t and $\jmath$, sufficiently small. Thus, in order to complete the proof of the (6.162), it only remains to prove Lemmas 6.16 and 6.17. These proofs occupy the remainder of this subsection.

Proof of Lemma 6.16 We recall that q^z is defined in (6.116) as $z_\theta + \frac{1}{4}ck_\theta$. The easiest term is the sound speed. From (5.1c), (5.20), (5.141a), and (5.141c) we note that

$$\begin{aligned} c(\mathfrak{s}(\jmath),\jmath) &= \tfrac{1}{2}w_{\mathsf{B}}(\mathfrak{s}(\jmath),\jmath) + \tfrac{1}{2}(w - w_{\mathsf{B}} + z)(\mathfrak{s}(\jmath),\jmath) \\ &= \tfrac{1}{2}w_0(x_{\mathsf{B},-}(\jmath)) + \mathcal{O}(\jmath) \\ &= \tfrac{\kappa}{2} - \tfrac{1}{2}\mathsf{b}^{\frac{3}{2}}\jmath^{\frac{1}{2}} + \mathcal{O}(\jmath)\,, \end{aligned} \tag{6.169}$$

as $\jmath \to 0$. The next term we consider is the y derivative of k, restricted to the shock curve. This term is given by (5.108), with t replaced by $\jmath$. The denominator of this fraction is given by $\dot{\mathfrak{s}}(\jmath) - \lambda_2(\mathfrak{s}(\jmath),\jmath) = \dot{\mathfrak{s}}(\jmath) - \frac{4}{3}c(\mathfrak{s}(\jmath),\jmath) - \frac{4}{3}z(\mathfrak{s}(\jmath),\jmath) = \frac{1}{3}\kappa + \mathcal{O}(\jmath^{\frac{1}{2}})$, by appealing to (5.141c) and (6.169). By combining the above estimate with the identity (5.81), we arrive at

$$k_\theta(\mathfrak{s}(\jmath),\jmath) = \frac{\frac{48\mathsf{b}^{\frac{9}{2}}}{\kappa^3}\jmath^{\frac{1}{2}} + \mathcal{O}(\jmath)}{\frac{1}{3}\kappa + \mathcal{O}(\jmath^{\frac{1}{2}})} = 144\mathsf{b}^{\frac{9}{2}}\kappa^{-4}\jmath^{\frac{1}{2}} + \mathcal{O}(\jmath)\,, \tag{6.170}$$

as $\jmath \to 0$. The last ingredient needed to compute q^z on the shock curve is to obtain a leading order term for the derivative of z. For this term we appeal to identity (5.112) with t replaced by $\jmath$. As above, we may show that $\dot{\mathfrak{s}}(\jmath) - \lambda_1(\mathfrak{s}(\jmath), \jmath) = \frac{2}{3}\kappa + \mathcal{O}(\jmath^{\frac{1}{2}})$, and we may appeal to the estimate (5.81) and the already established (6.169) and (6.170), to deduce

$$z_\theta(\mathfrak{s}(\jmath), \jmath) = \frac{\left(-\frac{27\mathsf{b}^{\frac{9}{2}}}{4\kappa^2}\jmath^{\frac{1}{2}} + \mathcal{O}(\jmath)\right) - \frac{1}{6}\left((\frac{\kappa}{2} + \mathcal{O}(\jmath^{\frac{1}{2}}))^2 \frac{144\mathsf{b}^{\frac{9}{2}}}{\kappa^4}\jmath^{\frac{1}{2}} + \mathcal{O}(\jmath)\right)}{\frac{2}{3}\kappa + \mathcal{O}(\jmath^{\frac{1}{2}})}$$
$$= -\frac{153}{8}\mathsf{b}^{\frac{9}{2}}\kappa^{-3}\jmath^{\frac{1}{2}} + \mathcal{O}(\jmath)\,. \tag{6.171}$$

We then combine the definition of q^z in (6.116) with (6.169)–(6.171) and arrive at

$$q^z(\mathfrak{s}(\jmath), \jmath) = -\frac{9}{8}\mathsf{b}^{\frac{9}{2}}\kappa^{-3}\jmath^{\frac{1}{2}} + \mathcal{O}(\jmath)\,, \tag{6.172}$$

as $\jmath \to 0$. In order to have a complete asymptotic description of the second term on the right side of (6.164), we need to determine $\partial_\theta^2\psi_t(\theta, \jmath)$. For this purpose, we use (6.77) with s replaced by $\jmath = \jmath(\theta, t)$, and we recall that we are interested in the region $\mathfrak{s}_1(t) < \theta < \mathfrak{s}_1(t) + \frac{\kappa t}{6}$. By using (5.37a), (5.141), (6.69), (6.79), (6.145), (6.169)

$$\partial_\theta^2\psi_t(\theta, \jmath) = \frac{1}{2}e^{\int_\jmath^t(\frac{4}{3}a - z_\theta)\circ\psi_t ds'}\frac{c^{\frac{1}{2}}(\theta,t)}{c^{\frac{1}{2}}(\mathfrak{s}(\jmath),\jmath)}\Big(\int_\jmath^t\left(\tfrac{8}{3}a_\theta - 2z_{\theta\theta}\right)\circ\psi_t\,\partial_\theta\psi_t ds'$$
$$+ \frac{\partial_\theta c(\theta,t)}{c(\theta,t)} - \frac{\partial_\theta c(\mathfrak{s}(\jmath),\jmath)}{c(\mathfrak{s}(\jmath),\jmath)}\partial_\theta\psi_t(\theta, \jmath)\Big)$$
$$= \frac{1}{2}e^{\mathcal{O}(t-\jmath)}\frac{(\frac{\kappa}{2}+\mathcal{O}(t^{\frac{1}{3}}))^{\frac{1}{2}}}{(\frac{\kappa}{2}+\mathcal{O}(\jmath^{\frac{1}{2}}))^{\frac{1}{2}}}\Big(\mathcal{O}(t-\jmath) + \mathcal{O}(\jmath^{\frac{1}{2}})$$
$$- \frac{\mathcal{O}(t^{-\frac{2}{3}})}{\frac{\kappa}{2}+\mathcal{O}(t^{\frac{1}{3}})} - \frac{-\frac{1}{4}\jmath^{-1}+\mathcal{O}(\jmath^{-\frac{1}{2}})}{\frac{\kappa}{2}+\mathcal{O}(\jmath^{\frac{1}{2}})}(1 + \mathcal{O}(\jmath^{\frac{1}{3}}))\Big)$$
$$= \frac{1}{4\kappa}\jmath^{-1} + \mathcal{O}(t^{-\frac{2}{3}}) + \mathcal{O}(\jmath^{-\frac{1}{2}}) \tag{6.173}$$

for $\jmath < t \ll 1$. From (6.172) and (6.173), and using that $\jmath \leqslant t$, we finally obtain that the second term in (6.164) is given by

$$\mathcal{B}_{12} = -\left(\tfrac{9}{8}\mathsf{b}^{\frac{9}{2}}\kappa^{-3}\jmath^{\frac{1}{2}} - \mathcal{O}(\jmath)\right)\left(\tfrac{1}{4\kappa}\jmath^{-1} + \mathcal{O}(\jmath^{-\frac{2}{3}})\right)$$
$$= -\frac{9}{32}\mathsf{b}^{\frac{9}{2}}\kappa^{-4}\jmath^{-\frac{1}{2}} + \mathcal{O}(\jmath^{-\frac{1}{6}})\,. \tag{6.174}$$

It remains to consider the first term on the right side of (6.164). We recall that $q^z_\theta = z_{\theta\theta} + \frac{1}{4}c_\theta k_\theta + \frac{1}{4}ck_{\theta\theta}$. Thus, in view of (6.169) and (6.170), we need to estimate separately three terms on the shock curve: c_θ, $k_{\theta\theta}$, and $z_{\theta\theta}$. First, similarly to (6.169), we have from (6.145) and (5.141) that

$$c_\theta(\mathfrak{s}(\jmath), \jmath) = \tfrac{1}{2}(\partial_\theta w_{\mathsf{B}})(\mathfrak{s}(\jmath), \jmath) + \tfrac{1}{2}\partial_\theta(w - w_{\mathsf{B}} + z)(\mathfrak{s}(\jmath), \jmath)$$

$$= -\tfrac{1}{4}\jmath^{-1} + \mathcal{O}(\jmath^{-\frac{1}{2}})\,, \tag{6.175}$$

as $\jmath \to 0$. Next, we turn to $\partial_\theta^2 k$, which is given by (6.94). By appealing to (6.149), (6.145), (5.81), (5.83), and (5.141), we obtain

$$\begin{aligned}
k_{\theta\theta}(\mathfrak{s}(\jmath),\jmath) &= \tfrac{\ddot{k_-}(\jmath)}{(\dot{\mathfrak{s}}(\jmath)-\lambda_2(\mathfrak{s}(\jmath),\jmath))^2} \\
&\quad + \Big(\big(\partial_t\lambda_2(\mathfrak{s}(\jmath),\jmath) + (2\dot{\mathfrak{s}}(\jmath) - \lambda_2(\mathfrak{s}(\jmath),\jmath))\partial_\theta\lambda_2(\mathfrak{s}(\jmath),\jmath)\big) \\
&\quad - \ddot{\mathfrak{s}}(\jmath)\Big)\tfrac{\dot{k_-}}{(\dot{\mathfrak{s}}(\jmath)-\lambda_2(\mathfrak{s}(\jmath),\jmath))^3} \\
&= \frac{\frac{24\mathsf{b}^{\frac{9}{2}}}{\kappa^3}\jmath^{-\frac{1}{2}}+\mathcal{O}(1)}{(\frac{\kappa}{3}+\mathcal{O}(\jmath^{\frac{1}{2}}))^2} + \Big(-\tfrac{\kappa}{9}\jmath^{-1} + \mathcal{O}(\jmath^{-\frac{1}{2}})\Big)\frac{\frac{48\mathsf{b}^{\frac{9}{2}}}{\kappa^3}\jmath^{\frac{1}{2}}+\mathcal{O}(\jmath)}{(\frac{\kappa}{3}+\mathcal{O}(\jmath^{\frac{1}{2}}))^3} \\
&= 72\mathsf{b}^{\frac{9}{2}}\kappa^{-5}\jmath^{-\frac{1}{2}} + \mathcal{O}(1)
\end{aligned} \tag{6.176}$$

as $\jmath \to 0$. Lastly, we turn to $\partial_\theta^2 z$, which is given by the expression (6.136). By using (5.13), (5.141), and (6.134), we first rewrite

$$\begin{aligned}
z_{\theta\theta}(\mathfrak{s}(\jmath),\jmath) &= \frac{\ddot{z_-} - \frac{1}{3}(\kappa - \frac{1}{2}w)(c^2k_{\theta\theta}) - \frac{1}{3}\Big((\kappa - \frac{5}{6}w)ck_\theta - (2\kappa - \frac{4}{3}w)z_\theta\Big)w_\theta}{(\dot{\mathfrak{s}} - \lambda_1)^2}\Big|_{(\mathfrak{s}(\jmath),\jmath)} \\
&\quad + \mathcal{O}(\jmath^{\frac{1}{2}})\,.
\end{aligned} \tag{6.177}$$

Then, by appealing to (5.83), (5.141), (6.145), (6.169), (6.170), (6.171), (6.175), and (6.176), from the above formula we obtain

$$\begin{aligned}
z_{\theta\theta}(\mathfrak{s}(\jmath),\jmath) &= \frac{\ddot{z_-} - \frac{1}{3}\left(\kappa - \frac{1}{2}w_{\mathsf{B}}\right)(c^2k_{\theta\theta}) - \frac{1}{3}\left(\left(\kappa - \frac{5}{6}w_{\mathsf{B}}\right)ck_\theta - \left(2\kappa - \frac{4}{3}w_{\mathsf{B}}\right)z_\theta\right)\partial_\theta w_{\mathsf{B}}}{\left(\frac{2\kappa}{3} + \mathcal{O}\left(\jmath^{\frac{1}{2}}\right)\right)^2}\Big|_{(\mathfrak{s}(\jmath),\jmath)} + \mathcal{O}(1) \\
&= \frac{-\frac{27\mathsf{b}^{\frac{9}{2}}}{8\kappa^2}\jmath^{-\frac{1}{2}} - \frac{1}{3}\frac{\kappa}{2}\frac{\kappa^2}{4}\frac{72\mathsf{b}^{\frac{9}{2}}}{\kappa^5}\jmath^{-\frac{1}{2}} - \frac{1}{3}\left(\frac{\kappa}{6}\frac{\kappa}{2}\frac{144\mathsf{b}^{\frac{9}{2}}}{\kappa^4}\jmath^{\frac{1}{2}} + \frac{2\kappa}{3}\frac{153\mathsf{b}^{\frac{9}{2}}}{8\kappa^3}\jmath^{\frac{1}{2}}\right)\left(-\frac{1}{2\jmath}\right)}{\left(\frac{2\kappa}{3} + \mathcal{O}\left(\jmath^{\frac{1}{2}}\right)\right)^2} \\
&\quad + \mathcal{O}(1) \\
&= -\tfrac{81}{16}\mathsf{b}^{\frac{9}{2}}\kappa^{-4}\jmath^{-\frac{1}{2}} + \mathcal{O}(1)\,,
\end{aligned} \tag{6.178}$$

as $\jmath \to 0$. Using the definition of q_y^z, upon combining (6.169), (6.170), (6.175), (6.176), and (6.178) we obtain

$$\begin{aligned}
q_\theta^z(\mathfrak{s}(\jmath),\jmath) &= -\tfrac{81}{16}\mathsf{b}^{\frac{9}{2}}\kappa^{-4}\jmath^{-\frac{1}{2}} + \mathcal{O}(1) + \tfrac{1}{4}\left(-\tfrac{1}{4}\jmath^{-1} + \mathcal{O}(\jmath^{-\frac{1}{2}})\right) \\
&\quad \left(144\mathsf{b}^{\frac{9}{2}}\kappa^{-4}\jmath^{\frac{1}{2}} + \mathcal{O}(\jmath)\right) \\
&\quad + \tfrac{1}{4}\left(\tfrac{\kappa}{2} + \mathcal{O}(\jmath^{\frac{1}{2}})\right)\left(72\mathsf{b}^{\frac{9}{2}}\kappa^{-5}\jmath^{-\frac{1}{2}} + \mathcal{O}(1)\right) \\
&= -\tfrac{81}{16}\mathsf{b}^{\frac{9}{2}}\kappa^{-4}\jmath^{-\frac{1}{2}} + \mathcal{O}(1)\,.
\end{aligned} \tag{6.179}$$

Lastly, by combining (6.69) with (6.179), and using that $\jmath \leqslant t$, we obtain that the first term in (6.164) is given by

$$\begin{aligned}\mathcal{B}_{11} &= \left(-\tfrac{81}{16}\mathsf{b}^{\frac{9}{2}}\kappa^{-4}\jmath^{-\frac{1}{2}} + \mathcal{O}(1)\right)\left(1+\mathcal{O}(t^{\frac{1}{3}})\right)^2 \\ &= -\left(\tfrac{81}{16}\mathsf{b}^{\frac{9}{2}}\kappa^{-4} + \mathcal{O}(t^{\frac{1}{3}})\right)\jmath^{-\frac{1}{2}} + \mathcal{O}(1)\,. \qquad (6.180)\end{aligned}$$

Adding the bounds (6.174) and (6.180) completes the proof of the lemma. □

Proof of Lemma 6.17 Recall from (6.118) that $Q^z = ck_\theta(\frac{1}{12}w_\theta + \frac{1}{12}z_\theta + \frac{2}{3}a) + \frac{8}{3}\partial_\theta(az)$. As in the proof of Lemma 6.14, we write $\partial_\theta Q^z = \mathcal{Q}_1 + \mathcal{Q}_2$, where

$$\begin{aligned}\mathcal{Q}_1 &= \overbrace{\tfrac{1}{12}ck_\theta w_{\theta\theta}}^{\mathcal{Q}_{1a}} + \overbrace{\tfrac{1}{24}k_\theta w_\theta w_\theta}^{\mathcal{Q}_{1b}} + \overbrace{ck_{\theta\theta}(\tfrac{1}{12}w_\theta + \tfrac{1}{12}z_\theta + \tfrac{2}{3}a)}^{\mathcal{Q}_{1c}}\,,\\ \mathcal{Q}_2 &= ck_\theta(\tfrac{1}{12}z_{\theta\theta} + \tfrac{2}{3}a_\theta) + \tfrac{8}{3}(az)_{\theta\theta} + \tfrac{1}{24}k_\theta z_\theta w_\theta\,.\end{aligned}$$

We first give the proof of the more difficult bound, (6.168b). Several times in this proof we require a bound on $\int_\jmath^{\jmath_1}|k_{\theta\theta}w_\theta|\circ\psi_t$. In order to obtain a suitable estimate, we recall the bound of $\jmath_1$ in (6.122), and introduce the time which lies half way in between $\jmath$ and $\jmath_1$, namely $\jmath_2 = \jmath + \frac{1}{2}(\jmath_1 - \jmath) = \frac{3}{2}\jmath + \mathcal{O}(\jmath^{\frac{4}{3}})$. The reason is as follows. For $s \in [\jmath, \jmath_2]$, from Remark 6.3 we may deduce that $\mathcal{T}(\psi_t(\theta, s), s) \geqslant \frac{1}{5}\jmath(\theta, t)$; this lower bound is useful when combined with (6.8c), (6.69), (5.141b), and (5.57a) for $\gamma(s) = \psi_t(\theta, s)$:

$$\int_\jmath^{\jmath_2}|k_{\theta\theta}w_\theta|\circ\psi_t\,(\partial_\theta\psi_t)^2 ds \lesssim \jmath^{-\frac{1}{2}}\int_\jmath^{\jmath_2}(|\partial_\theta w_{\mathsf{B}}\circ\psi_t| + s^{-\frac{1}{2}})ds \lesssim \jmath^{-\frac{1}{6}}\,. \qquad (6.181)$$

On the other hand, for the contribution coming from $s \in [\jmath_2, \jmath_1]$, the trick is to use that $|\psi_t(\theta, s) - \mathfrak{s}(s)| \geqslant \frac{\kappa s}{8}$. Then, we may appeal to the bound (6.78), to (5.141b), and to the estimate (5.37a), which in this region gives that $|\partial_\theta w_{\mathsf{B}}(\psi_t(\theta, s), s)| \leqslant \frac{4}{5}\mathsf{b}|\psi_t(\theta, s) - \mathfrak{s}(s)|^{-\frac{2}{3}} \lesssim s^{-\frac{2}{3}} \lesssim \jmath^{-\frac{2}{3}}$, concluding in

$$\int_{\jmath_2}^{\jmath_1}|k_{\theta\theta}w_\theta|\circ\psi_t\,(\partial_\theta\psi_t)^2 ds \lesssim \jmath^{-\frac{2}{3}}\int_{\jmath_2}^{\jmath_1}\mathcal{T}(\psi_t(\theta, s), s)^{-\frac{1}{2}}ds \lesssim \jmath^{-\frac{1}{6}}\,. \qquad (6.182)$$

Combining the above two bounds, and the fact that $\frac{\kappa}{5} \leqslant c \leqslant \mathsf{m}$, we conclude that

$$\int_\jmath^{\jmath_1}|ck_{\theta\theta}w_\theta|\circ\psi_t\,(\partial_\theta\psi_t)^2 ds \lesssim \jmath^{-\frac{1}{6}}\,. \qquad (6.183)$$

The remaining contribution to $\mathcal{Q}_{1c}$ is bounded as

$$\int_\jmath^{\jmath_1}\left|ck_{\theta\theta}(\tfrac{1}{12}z_\theta + \tfrac{2}{3}a)\right|\circ\psi_t(\partial_\theta\psi_t)^2 ds \lesssim \jmath^{\frac{1}{2}}\,. \qquad (6.184)$$

Next, let us estimate $-\int_{\mathfrak{z}(\theta,t)}^{\mathfrak{z}_1(\theta,t)} \mathcal{Q}_{1b} \circ \psi_t \, (\partial_\theta \psi_t)^2 ds$. From (6.122), we see that $\mathfrak{z}_1(\theta,t) = 2\mathfrak{z}(\theta,t) + \mathcal{O}(t^{\frac{4}{3}})$. Hence, using the bounds (5.57a), (5.141b), (6.123), and (6.145), we have that

$$\int_{\mathfrak{z}}^{\mathfrak{z}_1} \big(k_\theta (w_\theta)^2\big) \circ \psi_t (\partial_\theta \psi_t)^2 ds \lesssim \mathfrak{z}^{-\frac{1}{2}} \int_{\mathfrak{z}}^{\mathfrak{z}_1} \big(\big|w_\theta - w_{\mathsf{B}\theta}\big| \circ \psi_t + \big|w_{\mathsf{B}\theta}\big| \circ \psi_t\big)\, ds$$
$$\lesssim \mathfrak{z}^{-\frac{1}{6}} . \tag{6.185}$$

Next, in order to bound the contribution from $\mathcal{Q}_{1a}$, we define

$$\mathcal{A} = -\tfrac{1}{12}\big(ck_\theta w_{\theta\theta} + cw_\theta k_{\theta\theta}\big)$$
$$\mathcal{G} = \tfrac{2}{3} cw_\theta k_{\theta\theta} + \tfrac{1}{3} cc_\theta (k_\theta)^2 + \tfrac{1}{6} c^2 k_{\theta\theta} k_\theta - \tfrac{8}{3}(aw)_\theta k_\theta - k_\theta (w_\theta)^2 + k_\theta w_\theta z_\theta .$$

A straightforward computation shows that the product $k_\theta w_y$ solves the equation

$$\partial_s (k_\theta w_\theta) + \lambda_1 \partial_\theta (k_\theta w_\theta) + 2(k_\theta w_\theta)\partial_\theta \lambda_1 = \tfrac{48}{3}\mathcal{A} + \mathcal{G} . \tag{6.186}$$

We now obtain an explicit solution to (6.186). In order to solve (6.186), we set

$$\chi = (k_\theta w_\theta) \circ \psi_t , \qquad \mathcal{F} = (\tfrac{48}{3}\mathcal{A} + \mathcal{G}) \circ \psi_t ,$$

and by employing the chain-rule, we write (6.186) as

$$\partial_s \chi + 2\chi (\partial_\theta \psi_t)^{-1} \partial_s (\partial_\theta \psi_t) = \mathcal{F} .$$

It follows that

$$\tfrac{d}{ds}\Big((\partial_\theta \psi_t)^2 \chi\Big) = (\partial_\theta \psi_t)^2 \mathcal{F} ,$$

and integration from $\mathfrak{z}$ to $\mathfrak{z}_1$ yields the identity

$$(k_\theta w_\theta)(\mathfrak{s}_2(\mathfrak{z}_1), \mathfrak{z}_1)(\partial_\theta \psi_t(\theta,\mathfrak{z}_1))^2 - (k_\theta w_\theta)(\mathfrak{s}(\mathfrak{z}), \mathfrak{z}) w_\theta(\mathfrak{s}(\mathfrak{z}), \mathfrak{z})(\partial_\theta \psi_t(\theta,\mathfrak{z}))^2$$
$$= \int_{\mathfrak{z}}^{\mathfrak{z}_1} (\tfrac{48}{3}\mathcal{A} + \mathcal{G}) \circ \psi_t \, (\partial_\theta \psi_t)^2 ds$$
$$= \int_{\mathfrak{z}}^{\mathfrak{z}_1} \big(\mathcal{G} - \tfrac{4}{3} cw_\theta k_{\theta\theta}\big) \circ \psi_t \, (\partial_\theta \psi_t)^2 ds - \tfrac{48}{3} \int_{\mathfrak{z}}^{\mathfrak{z}_1} \big(\tfrac{1}{12} ck_\theta w_{\theta\theta}\big) \circ \psi_t \, (\partial_\theta \psi_t)^2 ds . \tag{6.187}$$

First, we note that since $\mathcal{T}(\mathfrak{s}_2(\mathfrak{z}_1), \mathfrak{z}_1) = 0$, the estimate (5.216) implies that $k_\theta(\mathfrak{s}_2(\mathfrak{z}_1), \mathfrak{z}_1) = 0$, and so the first term on the left side of (6.187) vanishes. The first term on the right side of (6.187) is estimated using (5.37a), (5.141), (6.8c), (6.69), (6.78), (6.123), (6.183), and (6.185) as

$$\int_{\jmath}^{\jmath_1} \left|\mathcal{G} - \tfrac{4}{3} c w_\theta k_{\theta\theta}\right| \circ \psi_t \, (\partial_\theta \psi_t)^2 ds \lesssim \jmath^{-\frac{1}{6}} . \tag{6.188}$$

Moreover, the estimates (5.141b), (6.69), (6.145), and (6.170) show that

$$\begin{aligned} -(k_\theta w_\theta)(\mathfrak{s}(\jmath), \jmath)(\partial_\theta \psi_t(\theta, \jmath))^2 &= \left(\tfrac{144 \mathsf{b}^{\frac{9}{2}}}{\kappa^4} \jmath^{\frac{1}{2}} + \mathcal{O}(\jmath)\right) \left(\tfrac{1}{2} \jmath^{-1} + \mathcal{O}(\jmath^{-\frac{1}{2}})\right) \\ &\quad \left(1 + \mathcal{O}(\jmath^{\frac{1}{3}})\right)^2 \\ &= \tfrac{72 \mathsf{b}^{\frac{9}{2}}}{\kappa^4} \jmath^{-\frac{1}{2}} + \mathcal{O}(\jmath^{-\frac{1}{6}}) . \end{aligned} \tag{6.189}$$

By using (6.187), the observation $k_\theta(\mathfrak{s}_2(\jmath_1), \jmath_1) = 0$, and the bounds (6.188) and (6.189) we obtain that

$$-\int_{\jmath}^{\jmath_1} \left(\tfrac{1}{12} c k_\theta w_{\theta\theta}\right) \circ \psi_t \, (\partial_\theta \psi_t)^2 ds = \tfrac{9 \mathsf{b}^{\frac{9}{2}}}{2\kappa^4} \jmath(\theta, t)^{-\frac{1}{2}} + \mathcal{O}(\jmath^{-\frac{1}{6}}) . \tag{6.190}$$

Combining (6.183), (6.184), (6.185), (6.188), and (6.190), we have proven that for $\overline{\varepsilon}$ small enough,

$$-\int_{\jmath(\theta,t)}^{\jmath_1(\theta,t)} \mathcal{Q}_1 \circ \psi_t \, (\partial_\theta \psi_t)^2 ds \leqslant \tfrac{9 \mathsf{b}^{\frac{9}{2}}}{2\kappa^4} \jmath(\theta, t)^{-\frac{1}{2}} + C \jmath(\theta, t)^{-\frac{1}{6}} . \tag{6.191}$$

In addition to the bounds (5.37a), (5.141), (6.8c), (6.69), (6.78), (6.123), by also appealing to (6.8b) and (6.79), we deduce that

$$-\int_{\jmath(\theta,t)}^{\jmath_1(\theta,t)} \mathcal{Q}_2 \circ \psi_t \, (\partial_\theta \psi_t)^2 ds \lesssim \jmath(\theta, t)^{\frac{1}{2}} . \tag{6.192}$$

Moreover, by using the identity (6.77) for $\partial_\theta^2 \psi_t$, we see that the integrand $Q^z \circ \psi_t \, \partial_\theta^2 \psi_t$ is estimated in the identical fashion as the term $\mathcal{Q}_{1b}$ in (6.185), and hence we have that

$$-\int_{\jmath(\theta,t)}^{\jmath_1(\theta,t)} Q^z \circ \psi_t \, \partial_\theta^2 \psi_t ds \lesssim \jmath^{-\frac{1}{6}} . \tag{6.193}$$

Together, the bounds (6.191), (6.192), and (6.193) establish the desired inequality (6.168b), for $\jmath(\theta, t) \ll 1$.

The proof of the lemma is completed once we establish (6.168a). These estimates are however simpler because by the definition of the time $\jmath_1(\theta, t)$, for all $\mathfrak{s}_1(t) < \theta < \mathfrak{s}_1(t) + \frac{\kappa t}{6}$, and for all $s \in (\jmath_1(\theta, t), t)$, we have that $(\psi_t(\theta, s), s) \in \mathcal{D}^z_{\frac{\varepsilon}{2}} \setminus \overline{\mathcal{D}^k_{\frac{\varepsilon}{2}}}$, and $k \equiv 0$ in this region. In particular, this means that in this region we have that $Q^z = \frac{8}{3} \partial_\theta(az)$, and $\partial_\theta Q^z = \mathcal{Q}_2 = \frac{8}{3}\partial_\theta^2(az)$; there are no dangerous k terms. As such the bounds we seek directly follow from (6.127) and (6.128):

$$\mathcal{B}_2 \leqslant \int_{\mathfrak{J}_1(\theta,t)}^t \left|\partial_\theta Q^z \circ \psi_t\right|(\partial_\theta \psi_t)^2 ds + \int_{\mathfrak{J}_1(\theta,t)}^t \left|Q^z \circ \psi_t \; \partial_\theta^2 \psi_t\right| ds \leqslant C t^{\frac{1}{3}} \mathfrak{J}(\theta,t)^{-\frac{1}{2}} \tag{6.194}$$

for a suitable constant C. The bound (6.168a) follows since $t \leqslant \overline{\varepsilon} \ll 1$. This completes the proof of Lemma 6.17. □

6.8 Precise Hölder Estimates for Derivatives

Here we combine the upper bounds established in Section 6.1, with the lower bounds proven in Section 6.7, to precisely characterize the behavior of $(w_\theta, z_\theta, k_\theta, a_\theta)$ as $\theta \to \mathfrak{s}_1(t)^+$ and $\theta \to \mathfrak{s}_2(t)^+$.

We first consider the behavior of these derivatives on $\mathfrak{s}_1(t)$. Note that on the left side of $\mathfrak{s}_1(t)$, by (6.12) and (6.97) we have that the order second derivatives of w and a are finite for every $t \in (0, \overline{\varepsilon}]$, but that the bounds are not uniform in t as $t \to 0^+$ (as should be expected, since $w_0, a_0 \notin C^2$). On this left side of $\mathfrak{s}_1(t)$, we moreover have that $k \equiv z \equiv 0$. Similarly, on the right side of $\mathfrak{s}_1(t)$, the second derivative of w is bounded due to (6.97), the second derivative of a is bounded in light of (6.20), these bounds not being uniform as $t \to 0^+$, while $k \equiv 0$. It remains to consider the behavior of $z_\theta(\theta, t)$ as $\theta \to \mathfrak{s}_1(t)^+$. From (5.219) we know that $z_\theta(\mathfrak{s}_1(t), t) = 0$, so that using (6.10b) and (6.144c)

$$\begin{aligned}
\sup_{0<h<\frac{\kappa t}{6}} \frac{\left|z_\theta(\mathfrak{s}_1(t)+h,t) - z_\theta(\mathfrak{s}_1(t),t)\right|}{h^\alpha} &\leqslant \sup_{0<h<\frac{\kappa t}{6}} h^{1-\alpha} \int_0^1 \left|z_{\theta\theta}(\mathfrak{s}_1(t)+\lambda h, t)\right| d\lambda \\
&\leqslant N_2 \sup_{0<h<\frac{\kappa t}{6}} h^{1-\alpha} \int_0^1 \mathfrak{J}(\mathfrak{s}_1(t)+\lambda h, t)^{-\frac{1}{2}} d\lambda \\
&\leqslant 2N_2 \kappa^{-\frac{1}{2}} \sup_{0<h<\frac{\kappa t}{6}} h^{1-\alpha} \int_0^1 \left|\lambda h\right|^{-\frac{1}{2}} d\lambda \\
&= 32\mathsf{m}^3 \kappa^{-\frac{1}{2}} \sup_{0<h<\frac{\kappa t}{6}} h^{\frac{1}{2}-\alpha}.
\end{aligned} \tag{6.195}$$

The right side of (6.195) is finite whenever $\alpha \leqslant \frac{1}{2}$. Thus, from (6.10b), (6.144b), and (6.195), we deduce that $z \in C^{1,\frac{1}{2}}$ in $\mathcal{D}^z_{\overline{\varepsilon}} \setminus \overline{\mathcal{D}^k_{\overline{\varepsilon}}}$. The remarkable fact is that due to (6.162), this upper bound is sharp: for any $\alpha > \frac{1}{2}$, $z \notin C^{1,\alpha}$ near $\mathfrak{s}_1$. Indeed, by (6.162), we have that for h sufficiently small but positive,

$$\frac{z_\theta(\mathfrak{s}_1(t)+h,t) - z_\theta(\mathfrak{s}_1(t),t)}{h^\alpha} = h^{1-\alpha} \int_0^1 z_{\theta\theta}(\mathfrak{s}_1(t)+\lambda h, t) d\lambda$$

$$\leqslant -\tfrac{1}{8}\mathsf{b}^{\frac{9}{2}}\kappa^{-4}h^{1-\alpha}\int_0^1 \mathcal{J}(\mathfrak{s}_1(t)+\lambda h,t)^{-\frac{1}{2}}d\lambda$$
$$\leqslant -\tfrac{1}{16}\mathsf{b}^{\frac{9}{2}}\kappa^{-\frac{9}{2}}h^{1-\alpha}\int_0^1 (\lambda h)^{-\frac{1}{2}}d\lambda$$
$$\leqslant -\tfrac{1}{8}\mathsf{b}^{\frac{9}{2}}\kappa^{-\frac{9}{2}}h^{\frac{1}{2}-\alpha}\,. \tag{6.196}$$

For $\alpha > \frac{1}{2}$, the right side of (6.196) converges to $-\infty$ as $h \to 0^+$, proving that $z_\theta \notin C^\alpha$ in the vicinity of $\mathfrak{s}_1$.

Next, we consider the behavior of derivatives on $\mathfrak{s}_2(t)$. On the left side of $\mathfrak{s}_2(t)$ we have that $k \equiv 0$, while the second derivatives of w, z, and a are bounded in terms of inverse powers of t in view of (6.10b), (6.144b), (6.20), and (6.97). On the right side of $\mathfrak{s}_2$, the situation is different. Similarly to (6.195), we may use (6.97), (6.8b), (6.20), and (6.144a) to show that $w_\theta, z_\theta, k_\theta, a_\theta \in C^\alpha$ near $\mathfrak{s}_2$, for any $\alpha \leqslant \frac{1}{2}$. Indeed, the only difference to (6.195) is that $\mathcal{J}(\mathfrak{s}_1(t)+\lambda h,t)^{-\frac{1}{2}}$ is replaced by $\mathcal{T}(\mathfrak{s}_1(t)+\lambda h,t)^{-\frac{1}{2}} \leqslant 2\kappa^{-\frac{1}{2}}(\lambda h)^{-\frac{1}{2}}$. Moreover, for any $\alpha > \frac{1}{2}$, similarly to (6.196), we may use (6.153), (6.155), (6.157), and (6.161) to prove

$$\frac{k_\theta(\mathfrak{s}_2(t)+h,t)-k_\theta(\mathfrak{s}_1(t),t)}{h^\alpha} \geqslant 12\mathsf{b}^{\frac{9}{2}}\kappa^{-\frac{9}{2}}h^{\frac{1}{2}-\alpha} \tag{6.197a}$$
$$\frac{z_\theta(\mathfrak{s}_2(t)+h,t)-z_\theta(\mathfrak{s}_1(t),t)}{h^\alpha} \leqslant -\tfrac{1}{2}\mathsf{b}^{\frac{9}{2}}\kappa^{-\frac{7}{2}}h^{\frac{1}{2}-\alpha} \tag{6.197b}$$
$$\frac{w_\theta(\mathfrak{s}_2(t)+h,t)-w_\theta(\mathfrak{s}_1(t),t)}{h^\alpha} \geqslant \tfrac{1}{2}\mathsf{b}^{\frac{9}{2}}\kappa^{-\frac{7}{2}}h^{\frac{1}{2}-\alpha} \tag{6.197c}$$
$$\frac{a_\theta(\mathfrak{s}_2(t)+h,t)-a_\theta(\mathfrak{s}_1(t),t)}{h^\alpha} \leqslant -\tfrac{1}{8}\mathsf{b}^{\frac{9}{2}}\kappa^{-\frac{5}{2}}t h^{\frac{1}{2}-\alpha}\,, \tag{6.197d}$$

for $h > 0$ sufficiently small. The estimates in (6.197) show that $k_\theta, z_\theta, w_\theta, a_\theta \notin C^\alpha$ for any $\alpha > \frac{1}{2}$.

6.9 Proof of Theorem 6.1

The bounds in (6.1) are merely a restatement of the bootstrap bounds stated in (6.1) for $(w_{\theta\theta}, z_{\theta\theta}, k_{\theta\theta})$. The bounds for $a_{\theta\theta}$ and ϖ_θ follow as shown in Lemmas 6.5 and 6.6. These bootstrap estimates were closed (i.e., improved by a factor of 2) by the analysis in Sections 6.2–6.6. As discussed in the first paragraph of Section 6.1, this analysis should formally be carried out at the level of the approximating sequence $(w^{(n)}, z^{(n)}, k^{(n)})$, but we have not chosen to do so for simplicity of the presentation. One remark is in order at this point: when dealing with the approximating sequence $(w^{(n)}, z^{(n)}, k^{(n)}, a^{(n)})$ the identities (6.75) and (6.77) for the second derivatives of ϕ_t and ψ_t are not available; this is because the structure of the equation for the sound speed at $c^{(n+1)}$, given in (5.138a)–(5.138c), lacks a necessary $n \to n+1$ symmetry; in this case, estimates for $\partial_\theta^2\phi_t^{(n)}$ and $\partial_\theta^2\psi_t^{(n)}$ are obtained simply by differentiating (5.120a) and (5.120b) twice with respect to y and appealing to the bootstrap bounds for $\partial_\theta^2 w^{(n)}$ and $\partial_\theta^2 z^{(n)}$; the resulting bounds are however exactly the same as the ones given in Lemma 6.9.

The bounds in (6.2) follow from (6.1) on the one hand, and (6.144), (6.153), (6.155), (6.157), (6.161), on the other hand. The estimate (6.3) follows by adding the bounds in (6.154) and (6.156), observing that the terms $\frac{1}{4}ck_{\theta\theta}$ cancel. The characterization of the singularity formed by $(w_\theta, z_\theta, k_\theta, a_\theta)$ as $\theta \to \mathfrak{s}_2(t)^+$ as being precisely a $C^{\frac{1}{2}}$ cusp is given by Section 6.8, estimate (6.197). The estimate (6.4) is implied by bounds (6.1) and (6.162). The characterization of the singularity formed by z_θ as $\theta \to \mathfrak{s}_1(t)^+$ as being precisely a $C^{\frac{1}{2}}$ cusp is given by Section 6.8, estimate (6.196). This concludes the proof of Theorem 6.1.

7 Shock Development for 2D Euler

In view of the transformations $(u_\theta, u_r, \sigma, S) \mapsto (b, c, k, a) \mapsto (w, z, k, a)$ described in (3.2) and (3.4), the results obtained in Sections 4–6 for the azimuthal variables (w, z, k, a, ϖ) imply the following results for the usual hydrodynamic variables (u, ρ, E, p). First, from Theorem 4.1 we deduce:

Theorem 7.1 (Shock formation for 2D Euler with azimuthal symmetry) *There exists $\kappa_0 > 1$ sufficiently large, and $\varepsilon > 0$ sufficiently small, such that the following holds. Consider initial data at time $-\varepsilon$ given by*

$$(u_r, u_\theta, \sigma, S)\,(r, \theta, -\varepsilon) := \left(ra(\theta, -\varepsilon), \tfrac{r}{2}w(\theta, -\varepsilon), \tfrac{r}{2}w(\theta, -\varepsilon), 0\right) ,$$

with $(w, a)(\cdot, -\varepsilon) \in C^5(\mathbb{T})$ satisfying conditions (4.17)–(4.26). In particular, the initial data is smooth and has azimuthal symmetry. Then, there exists $T_* > -\varepsilon$ (explicitly computable), and a unique solution $(u, \sigma, S) \in C^0([-\varepsilon, T_*); C^4(\mathbb{R}^2 \setminus \{0\}))$ of the Euler equations (2.33), which has the azimuthal symmetry (3.2). The associated density is $\rho = \frac{1}{4}\sigma^2 e^{-S} = \frac{1}{16}r^2 w^2$, and the total energy is $E = \frac{1}{2}\rho\,|u|^2 + \frac{1}{2}\rho^2 e^S = \frac{1}{32}r^4 w^2(a^2 + \frac{5}{16}w^2)$. Moreover, at time blowup time T_* we have $S(\theta, T_*) = 0$, and there exists a unique angle $\xi_* \in \mathbb{T}$ (explicitly computable) such that an azimuthal pre-shock *forms on the half-infinite ray $\{(r, \xi_*, T_*)\}_{r \in \mathbb{R}_+}$. The azimuthal pre-shock is described by the fact that for $|\theta - \xi_*| \ll_\varepsilon 1$ we have*

$$u_\theta(r, \theta, T_*) = \tfrac{1}{2}r\left(\kappa_* + \mathsf{a}_1(\theta - \xi_*)^{\frac{1}{3}} + \mathsf{a}_2(\theta - \xi_*)^{\frac{2}{3}} + \mathsf{a}_3(\theta - \xi_*) + \mathcal{O}((\theta - \xi_*)^{\frac{4}{3}})\right)$$
$$u_r(r, \theta, T_*) = r\left(\mathsf{a}_0' + \mathsf{a}_1'(\theta - \xi_*) + \mathsf{a}_2'(\theta - \xi_*)^{\frac{4}{3}} + \mathcal{O}((\theta - \xi_*)^{\frac{5}{3}})\right)$$
$$\rho(r, \theta, T_*) = \tfrac{1}{16}r^2\left(\kappa_*^2 + 2\mathsf{a}_1\kappa_*(\theta - \xi_*)^{\frac{1}{3}} + (\mathsf{a}_1^2 + 2\mathsf{a}_2\kappa_*)(\theta - \xi_*)^{\frac{2}{3}} + \mathcal{O}(\theta - \xi_*)\right)$$
$$E(r, \theta, T_*) = \tfrac{1}{32}r^4\left(\kappa_*^2(a_0'^2 + \tfrac{5\kappa_*^2}{16}) + \mathsf{a}_1\kappa_*(2\mathsf{a}_0'^2 + \tfrac{5\kappa_*^2}{4})(\theta - \xi_*)^{\frac{1}{3}} + \mathcal{O}((\theta - \xi_*)^{\frac{2}{3}})\right)$$

where $\kappa_*, \mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \mathsf{a}_0', \mathsf{a}_1', \mathsf{a}_2'$ are suitable constants which may be computed in terms of the data. Moreover, in view of (4.4) we have that these asymptotic descriptions *are valid (to leading order), for the first three derivatives of the solution, and for* $|\theta-\xi_*| \ll_\varepsilon 1$. *For angles* θ *which are at any fixed distance away from* ξ_*, *the functions* $(u, \rho, E)(r, \theta, T_*)$ *are* C^4 *smooth. Lastly, the specific vorticity and its derivatives remain uniformly bounded up to* T_*.

The above result, which establishes the formation of the pre-shock and gives its detailed description, is nothing but a rewriting of Theorem 4.1 in terms of the usual fluid variables. This is possible in view of the mapping $(u_r, u_\theta, \sigma, S) = (ra, \frac{1}{2}rw, \frac{1}{2}rw, 0)$, valid on $[-\varepsilon, T_*]$, and the above mentioned formulas for the density and energy. The series expansion for the radial velocity $ra(\theta, T_*)$ is not explicitly stated in Theorem 4.1, but it immediately follows from the fact that a has regularity precisely $C^{1,1/3}$ and no better, and from the bounds on $a \circ \eta$ obtained in Section 4.

For the development part of our result, for simplicity of notation it is convenient to re-label the pre-shock location $(r, \xi_*, T_*) \mapsto (r, 0, 0)$. Moreover, the fields at which we arrive at the end of the formation part, namely $(u, \sigma, S)(\cdot, T_*)$, are re-labeled as (u_0, σ_0, S_0). Then, from Theorems 5.5 and 6.1 we obtain:

Theorem 7.2 (Shock development for 2D Euler with azimuthal symmetry) *Given pre-shock initial data*

$$(u_r, u_\theta, \sigma, S)|_{t=0} := (ra_0, \tfrac{r}{2}(w_0 + z_0), \tfrac{r}{2}(w_0 - z_0), k_0),$$

with (w_0, z_0, a_0, k_0) *satisfying conditions (5.1)–(5.5), there exist:*

(i) $\overline{\varepsilon} > 0$ *sufficiently small;*
(ii) a shock surface $\mathcal{S} := \{(r, \theta, t) \in \mathbb{R}^2 \times [0, \overline{\varepsilon}] : \theta = \mathfrak{s}(t)\}$ *with* $\mathfrak{s} \in C^2([0, \overline{\varepsilon}])$*;*
(iii) fields (u, ρ, E) *with* $\rho = \frac{1}{4}\sigma^2 e^{-S}$ *and* $E = \frac{1}{2}\rho\,|u|^2 + \frac{1}{2}\rho^2 e^S$, *such that the* $(u, \rho, E, \mathcal{S})$ *is a* regular shock solution of the compressible Euler equations *(1.1) on the time interval* $[0, \overline{\varepsilon}]$, *in the sense of Definition 1.1;*
(iv) two C^1 *smooth functions* $\mathfrak{s}_1, \mathfrak{s}_2 : [0, \overline{\varepsilon}] \to \mathbb{T}$, *with* $\mathfrak{s}_1(0) = \mathfrak{s}_2(0) = 0$ *and* $\mathfrak{s}_1(t) < \mathfrak{s}_2(t) < \mathfrak{s}(t)$ *for* $t \in (0, \overline{\varepsilon}]$, *such that* $\mathcal{S}_i := \{(r, \theta, t) \in \mathbb{R}^2 \times [0, \overline{\varepsilon}] : \theta = \mathfrak{s}_i(t)\}$ *is a characteristic surface for the* λ_i *wave-speed, where* $\lambda_1 = u_\theta - \frac{1}{2}\sigma$ *and* $\lambda_2 = u_\theta$*;*

such that for any $t \in (0, \overline{\varepsilon}]$ *all fields are twice differentiable at points* (r, θ) *with* $\theta \notin \{\mathfrak{s}_1(t), \mathfrak{s}_2(t), \mathfrak{s}(t)\}$, *and the following hold:*

(v) letting $\mathcal{D}^{(2)}_{\overline{\varepsilon}} = \{(r, \theta, t) \in \mathbb{R}^2 \times (0, \overline{\varepsilon}] : \mathfrak{s}_2(t) < \theta < \mathfrak{s}(t)\}$ *we have that*

- $S \in C^{1,1/2}(\mathcal{D}^{(2)}_{\overline{\varepsilon}})$, $S \equiv 0$ *on* $(\mathcal{D}^{(2)}_{\overline{\varepsilon}})^{\complement}$, *and* $\frac{1}{C} \leqslant (\theta - \mathfrak{s}_2(t))^{\frac{1}{2}} \partial_\theta^2 S(r, \theta, t) \leqslant C$ *as* $\theta \to \mathfrak{s}_2(t)^+$,
- $p, u_\theta \in C^2(\mathcal{D}^{(2)}_{\overline{\varepsilon}})$, $|\partial_\theta^2 u_\theta(r, \theta, t)| \leqslant Crt^{-\frac{1}{2}}$ *and* $|\partial_\theta^2 p(r, \theta, t)| \leqslant Cr^4 t^{-2}$ *as* $\theta \to \mathfrak{s}_2(t)^+$,
- $u_r \in C^{1,1/2}(\mathcal{D}^{(2)}_{\overline{\varepsilon}})$ *and* $-rtC \leqslant (\theta - \mathfrak{s}_2(t))^{\frac{1}{2}} \partial_\theta^2 u_r(r, \theta, t) \leqslant -\frac{1}{C} rt$ *as* $\theta \to \mathfrak{s}_2(t)^+$,
- $\rho \in C^{1,1/2}(\mathcal{D}^{(2)}_{\overline{\varepsilon}})$ *and* $-r^2 C \leqslant (\theta - \mathfrak{s}_2(t))^{\frac{1}{2}} \partial_\theta^2 \rho(r, \theta, t) \leqslant -\frac{1}{C} r^2$ *as* $\theta \to \mathfrak{s}_2(t)^+$,

for a suitable constant $C > 0$*;*

(vi) letting $\mathcal{D}^{(1)}_{\overline{\varepsilon}} = \{(r,\theta,t) \in \mathbb{R}^2 \times (0,\overline{\varepsilon}]\colon \mathfrak{s}_1(t) < \theta < \mathfrak{s}_2(t)\}$, *we have*

- $S(r,\theta,t) = 0$ *on* $\mathcal{D}^{(1)}_{\overline{\varepsilon}}$,
- $u_\theta \in C^{1,1/2}(\mathcal{D}^{(1)}_{\overline{\varepsilon}})$ *and* $\frac{1}{C} r \leqslant (\theta - \mathfrak{s}_1(t))^{\frac{1}{2}} \partial^2_\theta u_\theta(r,\theta,t) \leqslant Cr$ *as* $\theta \to \mathfrak{s}_1(t)^+$,
- $u_r \in C^2(\mathcal{D}^{(1)}_{\overline{\varepsilon}})$ *and* $|\partial^2_\theta u_r(r,\theta,t)| \leqslant Crt^{-1}$ *as* $\theta \to \mathfrak{s}_1(t)^+$,
- $\rho \in C^{1,1/2}(\mathcal{D}^{(1)}_{\overline{\varepsilon}})$ *and* $-r^2 C \leqslant (\theta - \mathfrak{s}_1(t))^{\frac{1}{2}} \partial^2_\theta \rho(r,\theta,t) \leqslant -\frac{1}{C} r^2$ *as* $\theta \to \mathfrak{s}_1(t)^+$,

for a suitable constant $C > 0$;

(vii) on $\mathcal{S}$, *the functions* $u_\theta(r,\cdot,t)$ *and* $\partial_\theta u_r(r,\cdot,t)$ *exhibit* $\mathcal{O}(rt^{\frac{1}{2}})$ *jumps, the density* $\rho(r,\cdot,t)$ *exhibits an* $\mathcal{O}(r^2 t^{\frac{1}{2}})$ *jump, the entropy* $S(r,\cdot,t)$ *exhibits an* $\mathcal{O}(t^{\frac{3}{2}})$ *jump, the total energy* $E(r,\cdot,t)$ *exhibits an* $\mathcal{O}(r^4 t^{\frac{1}{2}})$ *jump (cf. (5.63) and (5.69)), while* $u_r(r,\cdot,t)$ *does not jump.*

Moreover, this solution is unique in the class of entropy producing regular shock solutions (cf. Definition 1.1) with azimuthal symmetry, such that the corresponding azimuthal variables (w,z,k,a) *belong to the space* $\mathcal{X}_{\overline{\varepsilon}}$ *(cf. Definition 5.3).*

The above theorem directly follows from our previous two Theorems 5.5 and 6.1, by taking into account the relation between the fluid variables and the azimuthal variables in (3.2), and in turn to the Riemann variables in (3.4). The bounds on second derivatives are all a consequence of Theorem 6.1. In the region $\mathcal{D}^{(2)}_{\overline{\varepsilon}}$, the bounds for the entropy S and radial velocity u_r follow from (6.2). Since $u_\theta = \frac{r}{2}(w+z)$, the bound for the second derivative of u_θ in the region $\mathcal{D}^{(2)}_{\overline{\varepsilon}}$, which does not blow up as $\theta \to \mathfrak{s}_2(t)^+$ in positive time, follows from (6.3). Since $\rho = \frac{r^2}{4} c^2 e^{-k}$, the claimed bound for the second derivative of the density follows from (5.11), (6.2), (6.114), and (6.131) since we may write

$$
\begin{aligned}
\tfrac{16}{r^2}\partial^2_\theta \rho &= ce^{-k}\,(2c_{\theta\theta} - ck_{\theta\theta}) + (\text{terms which are bounded as } \theta \to \mathfrak{s}_2(t)^+ \\
&\qquad \text{in terms of powers of } t^{-1}) \\
&= ce^{-k}\left(q^w_\theta - q^z_\theta - \tfrac{1}{2}ck_{\theta\theta}\right) + (\text{terms which are bounded as } \theta \to \mathfrak{s}_2(t)^+ \\
&\qquad \text{in terms of powers of } t^{-1}) \\
&= -\tfrac{1}{2}c^2 e^{-k} k_{\theta\theta} + (\text{terms which are bounded as } \theta \to \mathfrak{s}_2(t)^+ \\
&\qquad \text{in terms of powers of } t^{-1})\,.
\end{aligned}
$$

and so the singularity of $k_{\theta\theta}$ on $\mathfrak{s}_2$ carries over to ρ. Lastly, the claimed estimate for the second derivative of pressure, which does not blow up as $\theta \to \mathfrak{s}_2(t)^+$ in positive time, follows from the identity $p = \frac{1}{32} r^4 c^4 e^{-k}$ and a similar computation as above

$$
\begin{aligned}
\tfrac{32}{r^4}\partial^2_\theta p &= c^3 e^{-k}\,(4c_{\theta\theta} - ck_{\theta\theta}) + (\text{terms which are bounded as } \theta \to \mathfrak{s}_2(t)^+ \\
&\qquad \text{in terms of powers of } t^{-1}) \\
&= 2c^3 e^{-k}\left(q^w_\theta - q^z_\theta\right) + (\text{terms which are bounded as } \theta \to \mathfrak{s}_2(t)^+
\end{aligned}
$$

$$\text{in terms of powers of } t^{-1})$$
$$= (\text{terms which are bounded as } \theta \to \mathfrak{s}_2(t)^+ \text{ in terms of powers of } t^{-1}) \,. \tag{7.1}$$

The dependence of the bound on t^{-1} follows from (5.11), (6.114), and (6.131).

In the region $\mathcal{D}^{(1)}_{\bar{\varepsilon}}$, we have that $w_{\theta\theta}$ is bounded in terms of inverse powers of t and $z_{\theta\theta}$ satisfies (6.4), which gives the bounds on u_θ and ρ. The bound for the radial velocity appears in (6.1a).

The size of the jumps along the shock curve, and the uniqueness statement, follow directly from Theorem 5.5. To avoid redundancy we omit further details.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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