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The role of the 1.5 order formalism and the gauging of spacetime groups in the development of gravity and supergravity theories

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The 1.5 formalism played a key role in the discovery of supergravity and it has been used to prove the invariance of essentially all supergravity theories under local supersymmetry. It emerged from the gauging of the super Poincaré group to find supergravity. We review both of these developments as well as the auxiliary fields for simple supergravity and its most general coupling to matter using the tensor calculus.

Keywords:

A theory of supergravity was first proposed by Ferrara *et al.*[1] entitled "Progress towards a theory of supergravity" which contained the vierbein $e_\mu{}^a$ and the gravitino $\psi_{\mu\alpha}$. They proposed the action

$$A = \int d^4x \left\{ \frac{e}{2\kappa^2} R - \frac{1}{2} \bar{\psi}_\mu R^\mu \right\} \tag{1}$$

and the local supersymmetry transformations

$$\delta e_\mu{}^a = \kappa \bar{\varepsilon} \gamma^a \psi_\mu, \quad \delta \psi_\mu = 2\kappa^{-1} D_\mu(w(e,\psi))\varepsilon. \tag{2}$$

In these equations

$$\begin{gathered} R = R_{\mu\nu}{}^{ab} e_a{}^\mu e_b{}^\nu, \quad R^\mu = \varepsilon^{\mu\nu\rho\kappa} i\gamma_5 \gamma_\nu D_\rho(w(e,\psi))\psi_\kappa, \\ R_{\mu\nu}{}^{ab} \frac{\sigma_{ab}}{4} = [D_\mu, D_\nu], \quad \gamma_\mu = e_\mu{}^a \gamma_a, \end{gathered} \tag{3}$$

where the Lorentz covariant derivative is given by

$$D_\mu(w(e,\psi)) = \partial_\mu + w_{\mu ab} \frac{\sigma^{ab}}{4}, \tag{4}$$

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with

$$w_{\mu ab} = \frac{1}{2} e^{\nu}{}_a (\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}) - \frac{1}{2} e_b{}^{\nu} (\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu})$$
$$- \frac{1}{2} e_a{}^{\rho} e_b{}^{\sigma} (\partial_\rho e_{\sigma c} - \partial_\sigma a_{\rho c}) e_\mu{}^c + \frac{\kappa^2}{4} (\bar{\psi}_\mu \gamma_a \psi_b + \bar{\psi}_a \gamma_\mu \psi_b - \bar{\psi}_\mu \gamma_b \psi_a). \tag{5}$$

They showed that the action was invariant up to, and including, cubic terms in the gravitino and stated that the quintic terms vanished using a computer programme. They also showed that the supersymmetry transformations closed up to terms cubic in the gravitino if one uses the equations of motion. This theory was in second-order formalism as it contained the vierbein and gravitino but not the spin connection as an independent field. This supergravity is often referred to as $N = 1$, $D = 4$ supergravity, or simple supergravity.

A bit later, a theory of supergravity involving the vierbein $e_\mu{}^a$, the gravitino $\psi_{\mu\alpha}$ and a spin connection $\omega_\mu{}^{ab}$ was proposed.[2] This paper proposed an action which was in first-order formalism, that is, the spin connection was an independent field and had an independent supersymmetry transformation. These authors showed that their theory did not have anomalous characteristics of its surfaces of propagation. In other words, it has a consistent propagation. It was known that the propagation of a spin 3/2 particle coupled to a spin 1 particle was not consistent and the same was suspected to be the case for generic higher spin theories. Reference 2 contains a two-sentence discussion of the invariance of the action under the supersymmetry transformations that uses Eq. (11) which is, in effect, an equation of motion. The paper also does not discuss the closure of the supersymmetry transformations. There has subsequently been almost no work on supergravity in first-order formalism and it remains an interesting open problem to develop it further.

A different approach to supergravity was taken some months later in Ref. 3. This paper considered the gauge theory of the super Poincaré group, which has the generators P_a, Q_α and J_{ab} and the algebra

$$\begin{gathered}
[P_a, P_b] = J_{ab}, \quad [P_a, J_{bc}] = (\eta_{ab} P_c - \eta_{ac} P_b), \\
[J_{ab}, J_{cd}] = (\eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{bc} J_{ad}), \\
\{Q_\alpha, Q_\beta\} = -2(\gamma^a C^{-1})_{\alpha\beta} P_a, \\
[J_{ab}, Q_\alpha] = -\frac{1}{2} (\gamma_{ab} Q)_\alpha, \quad [P_a, Q_\alpha] = 0.
\end{gathered} \tag{6}$$

As such they introduced the connection

$$A_\mu = e_\mu{}^a P_a - \frac{1}{2} \omega_\mu{}^{ab} J_{ab} + \frac{1}{2} \bar{\psi}^\alpha Q_\alpha \tag{7}$$

and the corresponding field strengths defined by $[\hat{D}_\mu, \hat{D}_\nu] = -R_{\mu\nu}{}^a P_a + \frac{1}{2} R_{\mu\nu}{}^{ab} J_{ab} + \frac{1}{2} \bar{\Psi}_{\mu\nu} Q$ where $D_\mu = \partial_\mu - A_\mu$. The field strengths are

$$
\begin{aligned}
R_{\mu\nu}{}^{a} &= \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega_\mu{}^a{}_c e^c_\nu - \omega_\nu{}^a{}_c e^c_\mu + \tfrac{1}{2}\bar{\psi}_\mu \gamma^a \psi_\nu, \\
R_{\mu\nu}{}^{ab} &= \partial_\mu \omega_\nu{}^{ab} + \omega_\mu{}^{ac}\omega_{\nu c}{}^{b} - (\mu \leftrightarrow \nu), \\
\Psi_{\mu\nu} &= \left(\partial_\mu - \frac{1}{4}\gamma_{cd}\omega_\mu{}^{cd}\right)\psi_\nu - (\mu \leftrightarrow \nu) \equiv D_\mu \psi_\nu - (\mu \leftrightarrow \nu).
\end{aligned} \tag{8}
$$

The variations of the fields under gauge transformations of the form $\Lambda = v^a P_a - \frac{1}{2}\omega^{ab}J_{ab} + \frac{1}{2}\bar{\epsilon}^\alpha Q_\alpha$ are given by $\delta A_\mu = \partial_\mu \Lambda - [A_\mu, \Lambda]$ and so the individual fields transform as

$$
\begin{aligned}
\delta e^a_\mu &= \partial_\mu v^a - \omega^a{}_c e^c_\mu + \omega_\mu{}^{ac} v_c + \frac{1}{2}\bar{\epsilon}\gamma^a \psi_\mu, \\
\delta \omega_\mu{}^{ab} &= \partial_\mu \omega^{ab} - (\omega^{ac}\omega_{\mu c}{}^{b} - \omega^{bc}\omega_{\mu c}{}^{a}), \\
\delta &= 2\left(\partial_\mu - \frac{1}{4}\gamma_{cd}\omega_\mu{}^{cd}\right)\epsilon + \frac{1}{4}\gamma_{cd}\omega^{cd}\psi_\mu \equiv D_\mu \epsilon + \frac{1}{4}\gamma_{cd}\omega^{cd}\psi_\mu.
\end{aligned} \tag{9}
$$

In Ref. 3, the action was taken to be linear in the field strengths and the unique such action which is invariant under local Lorentz transformations is of the form

$$
-\frac{1}{8}\int d^4x\, \epsilon^{\mu\nu\rho\lambda}(\epsilon_{abcd}e^a_\mu e^b_\nu R_{\rho\lambda}{}^{cd} - 2if\bar{\psi}_\mu \gamma_5 \gamma_\nu \Psi_{\rho\lambda}), \tag{10}
$$

where f is a constant. Since the action is not of the form of the squares of the field strengths, it cannot be invariant under the above gauge transformations. However, the authors of Ref. 3 only demanded invariance up to the condition

$$
R^a_{\mu\nu} = 0. \tag{11}
$$

We will now vary the action of Eq. (10) under the transformations of Eq. (9) subject to the condition of Eq. (11). The argument follows the steps of Ref. 3 except that, for simplicity, we will take $f = -1$, which is the value determined from the variation. The variation of the Einstein part is given by

$$
\delta \int \frac{e}{2\kappa^2}(e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{ab})d^4x = \int d^4x \left\{\frac{1}{\kappa}\{\bar{\varepsilon}\gamma^\mu \psi_a\}\left\{-R_\mu{}^a + \frac{1}{2}e_\mu{}^a R\right\}\right\} \tag{12}
$$

while the variations of the Rarita–Schwinger part of the action give the following three terms:

$$
\begin{aligned}
\delta \int \left(-\frac{i}{2}\bar{\psi}_\mu \gamma_5 e_\nu{}^a \gamma_a D_\rho \psi_\kappa \varepsilon^{\mu\nu\rho\kappa}\right)d^4x &= \int d^4x \left\{-\frac{i}{\kappa}\bar{\varepsilon}\vec{D}_\mu \gamma_5 \gamma_\nu D_\rho \psi_\kappa \varepsilon^{\mu\nu\rho\kappa}\right\} \\
&\times \left\{-\frac{i}{\kappa}\bar{\psi}_\mu \gamma_5 \gamma_\nu \vec{D}_\rho D_\kappa \varepsilon \varepsilon^{\mu\nu\rho\kappa} - \frac{\kappa}{2}i\bar{\varepsilon}\gamma^a \psi_\nu \bar{\psi}_\mu \gamma_5 \gamma_a D_\rho \psi_\kappa \varepsilon^{\mu\nu\rho\kappa}\right\}.
\end{aligned} \tag{13}
$$

Flipping the spinors using their Majorana property, we find that the second term of the above equation takes the form

$$
-\frac{i}{8\kappa}\bar{\psi}_\mu \gamma_5 \gamma_\nu R_{\rho\kappa}{}^{cd}\sigma_{cd}\varepsilon \varepsilon^{\mu\nu\rho\kappa} = -\frac{i}{8\kappa}\bar{\varepsilon}\sigma_{cd}\gamma_\nu \gamma_5 \psi_\mu R_{\rho\kappa}{}^{cd}\varepsilon^{\mu\nu\rho\kappa}. \tag{14}
$$

Integrating the first term of Eq. (13) by parts and neglecting surface terms, we find that it is given by

$$\frac{i}{\kappa}\bar{\varepsilon}\gamma_5[D_\mu, \gamma_\nu]D_\rho\psi_\kappa\varepsilon^{\mu\nu\rho\kappa} + \frac{i}{\kappa}\bar{\varepsilon}\gamma_5\gamma_\nu D_\mu D_\rho\psi_\kappa\varepsilon^{\mu\nu\rho\kappa}. \tag{15}$$

Using Eq. (3) the second of these terms is given by

$$\frac{i}{8\kappa}\bar{\varepsilon}\gamma_5\gamma_\nu R_{\rho\kappa}{}^{cd}\sigma_{cd}\psi_\mu\varepsilon^{\mu\nu\rho\kappa}. \tag{16}$$

The term given in Eq. (16) and that in Eq. (14) add together to give the result

$$\begin{aligned} &+\frac{i}{2\cdot 4\kappa}\bar{\varepsilon}\gamma_5(\gamma_\nu\sigma_{cd} + \sigma_{cd}\gamma_\nu)\psi_\mu R_{\rho\kappa}{}^{cd}\varepsilon^{\mu\nu\rho\kappa} \\ &= \frac{1}{4\kappa}\bar{\varepsilon}\gamma_f\psi_\mu\varepsilon_{f\nu cd}\varepsilon^{\mu\nu\rho\kappa}R_{\rho\kappa}{}^{cd} \\ &= -\frac{1}{2\kappa}\bar{\varepsilon}\gamma^a\psi_\mu\{e_a{}^\mu R - 2R_a{}^\mu\}e \end{aligned} \tag{17}$$

which exactly cancels the variation of the Einstein action given in Eq. (12).

Consequently, we are just left with the first term of Eq. (15) and the last term of Eq. (13). Performing a Fierz transformation (see, for example, the Appendix of Ref. 5 for details) on the latter term, it becomes

$$-\frac{\kappa}{2\cdot 4}i\bar{\varepsilon}\gamma^a\gamma_R\gamma_a\gamma_5 D_\rho\psi_\kappa\varepsilon^{\mu\nu\rho\kappa}\bar{\psi}_\mu\gamma_R\psi_\nu = +\frac{\kappa}{4}i\bar{\varepsilon}\gamma_c\gamma_5 D_\rho\psi_\kappa\varepsilon^{\mu\nu\rho\kappa}\bar{\psi}_\mu\gamma^c\psi_\nu. \tag{18}$$

The first term in Eq. (15) is most easily evaluated by going to inertial coordinates, that is, we set $\partial_\mu e_\nu{}^a = 0$; it becomes

$$\begin{aligned} &\frac{i}{4\kappa}\bar{\varepsilon}\gamma_5[\sigma^{cd}, \gamma_\nu]w_{\mu cd}D_\rho\psi_\kappa\varepsilon^{\mu\nu\rho\kappa} \\ &= \frac{i}{\kappa}\bar{\varepsilon}\gamma_5\gamma^c D_\rho\psi_\kappa w_{\mu c\nu}\varepsilon^{\mu\nu\rho\kappa} \\ &= \frac{\kappa}{4}i\bar{\varepsilon}\gamma_5\gamma^c D_\rho\psi_\kappa\bar{\psi}_\mu\gamma_c\psi_\nu\varepsilon^{\mu\nu\rho\kappa}. \end{aligned} \tag{19}$$

This term cancels with that of Eq. (18). This completes the proof of invariance.

Adopting the constraint of Eq. (11) was somewhat unconventional and we now discuss it in more detail. Equation (11) allows one to express the spin connection in terms of the vierbein and gravitino, indeed this is all the information it contains. The resulting expression is nothing but the equation of motion of the spin connection of the action of Eq. (10) with $f = -1$. As such, the constraint of Eq. (11) takes the theory from the first- to second-order formalism. Indeed, adopting this value for the spin connection, Eq. (11) is identically true. Enforcing the condition of Eq. (11), the action of Eq. (10) and the transformations of the veirbein and gravitino of Eq. (9) are just those found in Ref. 1. Thus gauging the super Poincaré group leads to the supergravity theory discovered in Ref. 1, that is, the same action and transformation laws, however, it had the great advantage which also showed that it was invariant under the local supersymmetry transformations.

It remains to comment on the fact that in the steps above we essentially did not vary the spin connection in the action. Varying the action of Eq. (1), or equivalently Eq. (10) with $f = -1$, we have

$$\delta A = \int d^4x \left(\frac{\delta A}{\delta e_\mu{}^a} \delta e_\mu{}^a + \frac{\delta A}{\delta \psi_{\mu\alpha}} \delta \psi_{\mu\alpha} + \frac{\delta A}{\delta \omega_\mu{}^{ab}} \delta \omega_\mu{}^{ab} \right). \tag{20}$$

Since we are in second-order formalism, the variation of $\omega_\mu{}^{ab}$ is just that found by varying the vierbein and graviton upon which it depends. Just to be completely clear

$$\delta \omega_\mu{}^{ab} = \frac{\delta \omega_\mu{}^{ab}}{\delta e_\mu{}^c} \delta e_\mu{}^c + \frac{\delta \omega_\mu{}^{ab}}{\delta \psi_{\mu\alpha}} \delta \psi_{\mu\alpha}, \tag{21}$$

where the variations of the vierbein and graviton are those of Eq. (2). However, the last term of the variation of the action, given in Eq. (20), vanishes

$$\frac{\delta A}{\delta \omega_\mu{}^{ab}} = \frac{e}{2} R_{\kappa\lambda}{}^c (e_c^\lambda (e_a^\mu e_b^\kappa - e_b^\mu e_a^\kappa) + e_c^\mu e_a^\kappa e_b^\lambda) = 0 \tag{22}$$

as a consequence of the constraint Eq. (11). Thus in effect, one does not have to vary the spin connection in the action. In second-order formalism $\omega_\mu{}^{ab}$ is not an independent field but is given in terms of the vierbein and graviton. Indeed, it is the equation of motion of $\omega_\mu{}^{ab}$ that determines the spin connection in this way. As such, Eq. (22) is not an equation of motion but an identity. A straightforward account of the invariance of the action was given in Ref. 6. This paper adopted the steps in Ref. 3, that is Eqs. (12)–(19), but also implemented Eq. (22).

With the above steps, the discovery of supergravity was complete, the transformations rules of Ref. 1 were shown to be an invariance of the action of the seminal Ref. 1 using the usual analytic methods given in Ref. 3. The advantage of this was that any reader could verify that the action was invariant so opening up the way to further discoveries. The method of Ref. 3 has been used to show the invariance of all supergravity actions in all dimensions.

At some point, the above procedure was given the name the 1.5 order formalism, a name by which it is now known. However, the supergravity of theory of Ref. 3 and indeed the discussion of Eqs. (12)–(19) are in second-order formalism as we have implemented the constraint of Eq. (11). The proof of invariance presented in Ref. 3 is really a method and not a formalism. This aspect has mislead some authors such as in Ref. 7.

The method of gauging the supersymmetry algebra to derive supergravity presented in Ref. 3 was simultaneously also presented in an alternative form based on gauging the Orthosymplectic algebra $\mathrm{OSP}(1,4)$ which has the generators P_a, J_{ab} and Q_α.[7] In this work it was shown how to obtain the $N = 1$ supergravity fields e_μ^a, ω_μ^{ab} and ψ_μ as gauge fields of $\mathrm{OSP}(1,4)$. The transformations of the fields were calculated and it was shown how to recover the above results based on the super Poincaré algebra of Ref. 3 by rescaling $P_a \to RP_a$, $Q_\alpha \to \sqrt{R} Q_\alpha$, $J_{ab} \to J_{ab}$ then taking the infinite radius limit $R \to \infty$. This was taken up in the work of MacDowell and Mansouri who completed the calculation using the Orthosymplectic algebra and constructed an action based on field strengths squared.[8] Despite the aesthetic appearance of the action one still has to impose Eq. (11). A similar calculation was also presented in Ref. 9.

A restriction of the results of Ref. 3 can also be used to find Einstein's theory of general relativity in a very simple way. We begin by gauging just the Poincaré group and so set to zero the gravitino in the above equations. The resulting theory has just local translations and rotations. We also adopt the constraint of Eq. (11) with the gravitino set to zero. The remarks above about the spin connection and the proof of invariance apply in the same way to the case of pure gravity. A review of the above gauging of the Poincaré group can be found in Sec. 13.1.3 in the book of Ref. 4.

Adopting the condition of Eq. (11) is rather unconventional as it breaks by hand the gauge symmetry and in particular the local translations. However, one can write the gauge transformation of the vierbein of Eq. (8) as

$$\delta e_\mu{}^a = \partial_\mu \xi^\lambda e_\lambda{}^a + \xi^\lambda \partial_\lambda e_\mu{}^a - (\xi^\lambda w_\lambda{}^a{}_b) e_\mu{}^b - \frac{1}{2}(\xi^\lambda \psi_\lambda)\gamma^a \psi_\mu + \xi^\lambda R_{\mu\lambda}, \tag{23}$$

where $\xi^\mu = e^\mu{}_b v^b$. We recognize this transformation as a diffeomorphism, a local Lorentz transformation and a local supersymmetry transformation provided the constraint of Eq. (11) holds. Thus, we have the paradoxical result that imposing the condition of Eq. (11) we find that the local translations become a combination of a diffeomorphism, a local Lorentz and a local supersymmetry transformation which are symmetries of the final theory. This feature appears in the other applications of Ref. 3 to the gauging other spacetime groups. A way to proceed without taking the constraint of Eq. (11), and so not breaking the gauge symmetry by hand, was to introduce some more fields which are constrainted.[14]

It will be instructive to recall previous developments on the connection between the Poincaré group and general relativity. The vierbein was introduced into general relativity by Herman Weyl in 1929.[15] In Refs. 16–18, it was shown that the spin connection of general relativity in first-order form could be thought of as the gauge field for the Lorentz group, indeed the Riemann curvature was just the corresponding field strength. The authors of Refs. 17 and 18 also considered what they called the gauge theory of the Poincaré group. In this approach, they took the well-known coordinate transformations of the Poincaré group on Minkowski spacetime

$$x^\mu \to x^\mu + \omega^\mu{}_\nu x^\nu + a^\mu \tag{24}$$

and let the constant parameters $\omega^\mu{}_\nu$ and a^μ be local, that is, depend on spacetime. As they pointed out, in this way, one introduces a diffeomorphism. This approach has been extensively pursued and there is now a substantial literature, see, for example, Ref. 19. This literature is not the same as taking the gauge theory of the Poincaré group in the sense of Yang–Mills which was the approach of Ref. 3. As is well known, the unique action of a Yang–Mills theory consists of its field strength squared and this is not the case of gravity. This fact had, perhaps, put off researchers from carrying out a direct gauging of the Poincaré, In this context, it is amusing to read the first sentence of the introduction of Ref. 3; the authors were just PhD students!

The advantage of the gauge approach of Ref. 3 was that it provided a simple way to construct Einstein's theory of general relativity if one gauge the Poincaré group, and supergravity if one gauged the super Poincaré group. Indeed this gauging approach was used to construct the theories of super conformal gravity,[20] the super conformal tensor calculus as well as gravity and supergravity in three dimensions,[30] ... It also underlies the construction of higher spin theories[21] where one gauges an infinite-dimensional gauge group rather than the Poincaré, or super Poincaré group.

One drawback of the original formulation of supergravity[1,3] was that the local supersymmetry algebra only closed when one used the equations of motion. This meant that the coupling of supergravity to any super matter, that is, any combination of the super Yang–Mills and Wess–Zumino models, was a formidable task. Indeed, the task had to be repeated for each new matter model as the equations of motion were different and, as a consequence, so were the local supersymmetry transformations. This changed with the discovery of the auxiliary fields M, N and b_μ for the simplest supergravity in four dimensions.[10,11] The action was given by

$$A = \int d^4x \left\{ \frac{e}{2\kappa^2} R - \frac{1}{2}\bar{\psi}_\mu R^\mu - \frac{1}{3} e (M^2 + N^2 - b_\mu b^\mu) \right\} \tag{25}$$

and the transformations by

$$\begin{aligned}
\delta e_\mu{}^a &= \kappa \bar{\varepsilon}\gamma^a \psi_\mu, \\
\delta\psi_\mu &= 2\kappa^{-1} D_\mu(w(e,\psi))\varepsilon + i\gamma_5 \left(b_\mu - \frac{1}{3}\gamma_\mu \not{b} \right)\varepsilon - \frac{1}{3}\gamma_\mu (M + i\gamma_5 N)\varepsilon, \\
\delta M &= -\frac{1}{2} e^{-1}\bar{\varepsilon}\gamma_\mu R^\mu - \frac{\kappa}{2} i\bar{\varepsilon}\gamma_5 \psi_\nu b^\nu - \kappa\bar{\varepsilon}\gamma^\nu \psi_\nu M + \frac{\kappa}{2}\bar{\varepsilon}(M + i\gamma_5 N)\gamma^\mu \psi_\mu, \\
\delta N &= -\frac{e^{-1}}{2} i\bar{\varepsilon}\gamma_5\gamma_\mu R^\mu + \frac{\kappa}{2}\bar{\varepsilon}\psi_\nu b^\nu - \kappa\bar{\varepsilon}\gamma^\nu\psi_\nu N - \frac{\kappa}{2} i\bar{\varepsilon}\gamma_5 (M + i\gamma_5 N)\gamma^\mu \psi_\mu \\
\delta b_\mu &= \frac{3i}{2} e^{-1}\bar{\varepsilon}\gamma_5 \left(g_{\mu\nu} - \frac{1}{3}\gamma_\mu\gamma_\nu \right) R^\nu + \kappa\bar{\varepsilon}\gamma^\nu b_\nu \psi_\mu - \frac{\kappa}{2}\bar{\varepsilon}\gamma^\nu\psi_\nu b_\mu \\
&\quad - \frac{\kappa}{2} i\bar{\psi}_\mu \gamma_5 (M + i\gamma_5 N)\varepsilon - \frac{i\kappa}{4}\varepsilon_\mu{}^{bcd} b_b \bar{\varepsilon}\gamma_5\gamma_c\psi_d.
\end{aligned} \tag{26}$$

These transformations closed without the use of equations of motion, namely

$$\begin{aligned}
[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] &= \delta_{\text{supersymmetry}}(-\kappa\xi^\nu\psi_\nu) + \delta_{\text{general coordinate}}(2\xi_\mu) \\
&\quad + \delta_{\text{Local Lorentz}}\left(-\frac{2\kappa}{3}\varepsilon_{ab\lambda\rho} b^\lambda \xi^\rho - \frac{2\kappa}{3}\bar{\varepsilon}_2 \sigma_{ab}(M + i\gamma_5 N)\varepsilon_1 + 2\xi^d w_d{}^{ab} \right),
\end{aligned} \tag{27}$$

where $\xi_\mu = \bar{\varepsilon}_2 \gamma_\mu \varepsilon_1$.

It was straightforward to extend the proof of invariance of the action without auxiliary fields[3] to include them. With this last step we now have a supergravity theory which possess local supersymmetry transformations that satisfy a closing algebra which is independent of any specific dynamics and leave the action invariant. Of course this is the usual situation with symmetries before that of supersymmetry. It was straightforward to quantize the simple supergravity theory using the usual BRST techniques.[12] This contrasts with the statements given in Ref. 13 which finds not only this result, but the 1.5 formalism itself to be troublesome.

The discovery of the auxiliary fields allowed the construction of a tensor calculus for supergravity which made it easy to compute the most general coupling of $D = 4$, $N = 1$ supergravity to the most general matter, which, in turn, paved the way to construct a realistic spontaneously broken supersymmetric model. We will now explain how the tensor calculus was constructed.[22,23] Matter consists of chiral multiplets (Wess–Zumino) Σ^a and vector multiplets V. The chiral multiplets have the field content

$$\Sigma^a = (z^a, \chi_L^a, h^a), \tag{28}$$

where $z^a = A^a + iB^a$ are complex scalar fields, χ_L^a are left-handed Weyl spinors and $h^a = F^a + iG^a$ are complex auxiliary fields. Taking the complex conjugate of the above chiral super multiplet, we find it contains a spin zero field z_a, which is the complex conjugate of z^a, and also a spinor of the opposite chirality. The index a on Σ^a is an internal symmetry index which corresponds to fact that Σ^a can belong to a representation of a gauge group G.

The vector multiplet V is real and has the components

$$V = (C, \zeta, H, K, v_\mu, \lambda, D), \tag{29}$$

which belong to the adjoint representation of the gauge group G. In this equation ζ, λ are Majorana spinors and C, H, K are scalars while D, which is also a scalar, is an auxiliary field. These super multiplets have been used to construct realistic models of nature that have rigid supersymmetry. The quarks, leptons and Higgs are expected to be contained in chiral super multiplets while the vector super multiplets contain the spin one gauge particles. It is far from clear how to break supersymmetry in the context of rigid supersymmetry.

As we have mentioned the introduction of the auxiliary fields leads to a theory of simple supergravity whose fields possessed transformations that closed without the use of the equations of motion. This is the supergravity analogue of the closure of two general coordinate transformations in general relativity. With this result, the chiral and vector multiplets of rigid supersymmetry could then be generalized to be multiplets of this local supersymmetry, that is carry a representation of this local algebra. In particular, their supersymmetry transformations should have a local spinor parameter and they must have a closing algebra that is the same as that for the supergravity fields, in other words that of Eq. (26). To achieve this, their transformations must be extended to include terms involving the supergravity fields.

Given these local chiral and vector super multiplets we can, as in general relativity, construct a tensor calculus. In other words, we can construct local chiral and vector multiplets out of products of such multiplets. The precise formulae can be found in Refs. 22 and 23, or the review of Chap. 13 in the book of Ref. 5.

The final step in the construction of the tensor calculus is to construct the supersymmetric invariants for the chiral and vector multiplets. For rigid supersymmetry, these are just given by the integrals over spacetime of the two auxiliary fields F and D, respectively. These are called the F and D terms. Their generalization to be invariant under local supersymmetry are easy to find given the local supersymmetry transformations of the fields.[22,23] The invariant for the chiral super multiplet, the F term, is given by

$$A_F = \int d^4x\, e\left(F - (MA + NB) + \frac{1}{2}\bar{\psi}_\mu\gamma^\mu\chi + \frac{1}{4}\bar{\psi}_\mu\gamma^{\mu\nu}(A + i\gamma_5 B)\psi_\nu\right). \quad (30)$$

While the invariant for the vector super multiplet, the D term, is given by

$$\begin{aligned} A_D = \int d^4x\, e\bigg\{ & D - \frac{i\kappa}{2}\bar{\psi}_\mu\gamma^\mu\gamma_5\lambda + \frac{2}{3}(MK - NH) \\ & - \frac{2\kappa}{3}A_\mu\left(b^\mu + \frac{3\kappa e^{-1}}{8}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\nu\gamma_\rho\psi_\sigma\right) \\ & - \frac{\kappa}{3}\bar{\zeta}(i\gamma_5\gamma_\mu R^\mu + \frac{3\kappa}{8}\epsilon^{\mu\nu\rho\sigma}\psi_\mu\bar{\psi}_\nu\gamma_\rho\psi_\sigma\bigg) - \frac{2\kappa^2}{3}e^{-1}L_{\mathrm{SG}}\bigg\}, \end{aligned} \quad (31)$$

where L_{SG} is the Lagrangian of simple supergravity which can be read off from Eq. (25). We observe that all the fields of the relevant super multiplet occur in these local F and D terms, as do all the supergravity fields. A complete discussion of the tensor calculus can be found in Chap. 13 of the book of Ref. 5.

Using the tensor calculus, it is easy to find the most general coupling of super matter to simple supergravity; one just has to apply the formulae for the composition of the super multiplets and the above density formulae of Eqs. (30) and (31). The resulting action can be expressed in terms of three functions $g(z^a)$, $\phi(z^a, z_a)$ and $f_{\alpha\beta}(z^a)$. The super potential function $g(z^a)$ is the lowest element of the most general gauge singlet chiral multiplet formed out of the chiral multiplets Σ^a. We can write it as

$$g(z^a) = A_{a_1a_2\cdots a_m}z^{a_1}z^{a_2}\cdots z^{a_m}. \quad (32)$$

The function $\phi(z^a, z_a)$ represents the most general gauge singlet vector multiplet formed out of the chiral multiplet Σ^a and its hermitian conjugate Σ_a whose first components are z^a and its complex conjugate z_a, respectively. It can be written as

$$\phi(z^a, z_a) = B_{a_1\cdots a_m}{}^{b_1\cdots b_n}z^{a_1}\cdots z^{a_m}z_{b_1}\cdots z_{b_n}. \quad (33)$$

The coefficients $A_{a_1a_2\cdots a_m}$ and $B_{a_1\cdots a_m}{}^{b_1\cdots b_n}$ are arbitrary parameters except that they are chosen to maintain invariance under the gauge group G. The function $f_{\alpha\beta}(z^a)$ is the lowest component of a chiral function transforming as the symmetric product of the adjoint representation of G. In most models $f_{\alpha\beta}(z^a)$ is taken to be $\delta_{\alpha\beta}$.

The final action resulting from the tensor calculus will contain the auxiliary fields for supergravity and matter multiplets. They can be eliminated using their equations of motion, although once this step is taken the resultant Lagrangian will be invariant under supersymmetry transformation only after using the equations of motion. The full Lagrangian is too long to list here, so we will only write the bosonic part which can be transformed to be of the form[24–26]

$$\int d^4x\, e\bigg\{\frac{1}{2\kappa^2}R - \frac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu\alpha} - \frac{1}{\kappa^2}\mathcal{G}_{,a}{}^{b}D_\mu z^a D^\mu z_b$$

$$- \frac{1}{\kappa^4}e^{-\mathcal{G}}(3 + (\mathcal{G}^{-1})_a{}^b\mathcal{G}_,{}^a\mathcal{G}_{,b}\,) - \frac{1}{8\kappa^4}|g_\alpha\mathcal{G}_{,a}(T^\alpha z)^a|^2\bigg\}, \tag{34}$$

where $F^\alpha_{\mu\nu}$ is the Yang–Mills field strength and the function $\mathcal{G}$ is defined by

$$\mathcal{G} = 3\ln\left(-\frac{\kappa^2}{3}\phi(z^a, z_a)\right) - \ln\left(\frac{\kappa^6}{4}|g(z^a)|^2\right). \tag{35}$$

We have defined $D_\mu z^a$ as covariant derivative with respect to gauge group G, $\mathcal{G}_,^a = \frac{\partial\mathcal{G}}{\partial z_a}$, $\mathcal{G}_{,a} = \frac{\partial\mathcal{G}}{\partial z^a}$, $\mathcal{G}_{,a}{}^b = \frac{\partial^2\mathcal{G}}{\partial z^a\partial z_b}$, T^α and the g_α are the matrices and gauge couplings associated with the representation carried by z^a.

Unlike the case for rigid supersymmetry, the potential in Eq. (34) is no longer positive definite because of the corrections from supergravity. This fact is already apparent from the way the auxiliary fields occur in the supergravity action of Eq. (26) and the tensor calculus density formulae of Eqs. (30) and (31). Realistic supergravity models can be constructed by considering a set of fields $z^A = (z, z^a)$ where z is a field belonging to the super-Higgs sector, which is the sector responsible for supersymmetry breaking, and z^a are the remaining matter fields. This can be achieved by considering the super potential[24,28]

$$g(z^A) = g_1(z^a) + g_2(z), \tag{36}$$

where in the limit $\kappa \to 0$, there is no interaction between the fields z^a and z.

For nonzero κ, these fields do have super-gravitational interactions. The most dramatic effect occurs in the z^a sector, which due to influence of the field z, has soft breaking terms due to the super-Higgs effect. The superpotential $g_2(z)$ is taken to be of the form[28]

$$g_2(z) = \kappa^{-1}m^2 f(\kappa z) \tag{37}$$

so that the expectation value of z at the minimum of the potential is such that $\kappa\langle z\rangle = O(1)$ and $\langle g_2\rangle = O(\kappa^{-1}m^2)$. Such supersymmetry breaking leads to a gravitino mass and low-energy supersymmetric particles of size $m_s = \kappa^2\langle g_2\rangle = \kappa m^2$. If we choose $m \sim 10^{10}$ GeV, then $m_s = O$ is of the order of (Tev).

The supersymmetric partner χ of the field z is the Goldstino, that is, the Goldstone fermion arising from the supersymmetry breaking. It gets absorbed by the gravitino making it massive with mass $m_{\frac{3}{2}} = \kappa^{-1}\langle e^{\frac{1}{2}\mathcal{G}}\rangle$. Taking the unification group G to be SU(5), or SO(10), the gauge coupling constants unify at a scale $M_{\mathrm{G}} \simeq 2\times 10^{16}$ GeV and the group breaks to $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. It was shown in Ref. 28 that, for the superpotential of the form given in Eqs. (36) and (37), the gauge hierarchy is preserved for both M_{Pl} and M_{G}. For the case $f_{\alpha\beta}(z^A) = \delta_{\alpha\beta}$ and a flavor blind Kähler potential $\phi(z^A, z_A)$ the effective potential takes the simple form[27]

$$
\begin{aligned}
V_{\mathrm{eff}} = \left|\frac{\partial\tilde{g}}{\partial z^\alpha}\right|^2 &+ m_0^2 z^\alpha z_\alpha + (B_0\tilde{g})^{(2)} + A_0\tilde{g}^{(3)} + \mathrm{h.c.}) \\
&+ \frac{1}{2\kappa^4}|g_\sigma G_\alpha (T^\sigma z)^\alpha|^2,
\end{aligned} \tag{38}
$$

where $\tilde{g}$ is the superpotential containing only the quadratic and cubic functions of the light fields z^α, i.e. $\tilde{g}(z^\alpha) = \tilde{g}^{(2)}(z^\alpha) + \tilde{g}^{(3)}(z^\alpha)$, m_0, A_0, B_0 are soft breaking parameters of size m_s and $G_\alpha = \tilde{g}_{,\alpha} + \frac{\kappa^2}{2} z_\alpha \tilde{g}$. The most remarkable feature, however, is that the breaking of supergravity in the hidden sector induces the breaking of $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$.

The fact that the super-Higgs mass scale m_s of the soft breaking parameters and the scale of SU(2) × U(1) breaking are comparable, i.e. both lie in the TeV region, is a natural consequence of the heavy top quark. The two Higgs doublets have an effective coupling in the superpotential in the form $\mu_0 H_1 H_2$ with μ_0 is of size m_s. Thus, one is led to a simple model with five universal parameters at the GUT scale: m_0, $m_{\frac{1}{2}}$, A_0, B, μ_0 where $m_{\frac{1}{2}}$ is the mass of the gauginos. These parameters characterize the way the super-Higgs field interacts with the matter fields.

While global (rigid) supersymmetry models can accommodate over 134 soft breaking parameters, the supergravity models, called variously SUGRA GUT model, minimal supergravity model, CMSS or mSUGRA allows one to build simple models that are relatively natural and with a significantly reduced number of soft terms. However, experimental results over the last few years have restricted the five parameter space of the models discussed above to a rather small volume and it would seem that one has to consider more complicated models in order to remain consistent with experimental results. For a review of the construction of realistic models of supersymmetry, see the review of Ref. 29.

In this review, we have explained the formulation of supergravity[3] that results from gauging the super Poincaré group and how it contained an analytic proof of the invariance of $D = 4$, $N = 1$ (simple) supergravity under local supersymmetry transformations. Indeed this method has been used to prove the invariance of essentially all supergravity theories including those in ten and eleven dimensions. These results allowed for the systematic development of supergravity theories and in particular the discovery of the auxiliary, fields, that is, a formulation of simple supergravity whose transformations closed without the use of the equations of motion.[10,11] The resulting algebra allowed the construction of the analogue of the tensor calculus for general relativity for simple supergravity.[22,23] This in turn allowed for the construction of the most general matter coupling to supergravity and as a result the realistic models making it possible to test low energy supersymmetry experimentally.

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