

Discretely shrinking targets in moduli space

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Abstract

We consider the discrete shrinking target problem for Teichmüller geodesic flow on the moduli space of abelian or quadratic differentials and prove that the discrete geodesic trajectory of almost every differential will hit a shrinking family of targets infinitely often provided the measures of the targets are not summable. This result applies to any ergodic $\mathrm{SL}(2, \mathbb{R})$ -invariant measure and any nested family of spherical targets. Under stronger conditions on the targets, we moreover prove that almost every differential will eventually always hit the targets. As an application, we obtain a logarithm law describing the rate at which generic discrete trajectories accumulate on a given point in moduli space. These results build on work of Kelmer [Kel] and generalize theorems of Aimino, Nicol, and Todd [ANT].

1 Introduction

In a finite measure space (X, μ) with a measure preserving transformation $T: X \rightarrow X$, the *discrete shrinking target problem* considers the question of which trajectories $\{T^n(x)\}_{n \in \mathbb{N}}$ will hit a shrinking family of nested measurable subsets $B_1 \supset B_2 \supset \cdots$ infinitely often. More precisely, defining the *hitting set* for the family $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ to be

$$\mathcal{E}_T(\mathcal{B}) = \left\{ x \in X \mid T^n(x) \in B_n \text{ for infinitely many } n \in \mathbb{N} \right\},$$

the goal of the shrinking target problem is to study $\mathcal{E}_T(\mathcal{B})$ and characterize, for example, when it has full or null measure. In the case of an ergodic transformation, it is easy to see that $\mathcal{E}_T(\mathcal{B})$ will have full measure whenever $\mu(B_n)$ does not converge to 0. On the other hand, the Borel–Cantelli Lemma implies that $\mu(\mathcal{E}_T(\mathcal{B})) = 0$ whenever $\sum_n \mu(B_n) < \infty$. In the opposite case $\sum_n \mu(B_n) = \infty$, the converse to the Borel–Cantelli Lemma implies that $\mathcal{E}_T(\mathcal{B})$ has full measure provided the events $\{x \mid T^n(x) \in B_n\}$ are independent. There are nevertheless many naturally occurring settings in which this independence condition fails but the hitting set $\mathcal{E}_T(\mathcal{B})$ does have full measure.

Moduli space of differentials. This paper considers the discrete shrinking target problem for the Teichmüller geodesic flow on the moduli spaces of unit area abelian or quadratic differentials. Fix a finite type surface Σ (not the thrice-punctured sphere) with $\chi(\Sigma) < 0$

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and let $\mathcal{M} = \mathcal{M}(\Sigma)$ denote the moduli space of Riemann surface structures on Σ up to biholomorphism. An abelian differential on a Riemann surface $X \in \mathcal{M}$ is a holomorphic section ω of the holomorphic cotangent bundle, and a quadratic differential is a holomorphic section q of its symmetric square. We write $\mathcal{D}(X)$ for the disjoint union of the vector spaces $\mathcal{A}(X)$ and $\mathcal{Q}(X)$ of abelian and quadratic differentials on X , and write $\mathcal{D}^1(X) = \mathcal{A}^1(X) \sqcup \mathcal{Q}^1(X)$ for those differentials that have unit area with respect to the piecewise Euclidean structure they induce on X ; see §2.4 for details. The space $\mathcal{D} = \mathcal{D}(\Sigma)$ of all abelian or quadratic differentials forms a bundle $\mathcal{D} \rightarrow \mathcal{M}$ over moduli space. The subbundle $\mathcal{Q} \rightarrow \mathcal{M}$ of quadratic differentials is canonically identified with the cotangent bundle $T^*(\mathcal{M})$, and the restriction $\mathcal{Q}^1 \rightarrow \mathcal{M}$ to unit area quadratic differentials is the unit cotangent bundle for the Finsler Teichmüller metric on \mathcal{M} . The group $G = \mathrm{SL}(2, \mathbb{R})$ acts on \mathcal{D} and \mathcal{D}^1 via affine adjustment of the piecewise Euclidean structure; see §2.4. The action of the diagonal subgroup $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ then gives the *Teichmüller geodesic flow* on $\mathcal{Q}^1 = T^*(\mathcal{M})$: unit-speed Teichmüller geodesics in \mathcal{M} are precisely given by projections of the paths $t \mapsto g_t q$ for $q \in \mathcal{Q}^1$.

Shrinking targets. Our main result characterizes which shrinking targets in moduli space, or more generally abelian or quadratic differential space, are hit by almost every discrete geodesic trajectory. For the statement, a *spherical* subset of \mathcal{D} is one that is invariant under the action of the orthogonal subgroup $K = \mathrm{SO}(2) \leq G$. Since K preserves each fiber of $\pi: \mathcal{D}^1 \rightarrow \mathcal{M}$, preimage sets $\pi^{-1}(B) \subset \mathcal{D}^1$ are automatically spherical.

Theorem 1.1. *Let μ be any ergodic, $G = \mathrm{SL}(2, \mathbb{R})$ -invariant, probability measure on \mathcal{D}^1 . Let $B_1 \supset B_2 \supset \dots$ be a nested family \mathcal{B} of μ -measurable spherical subsets $B_n \subset \mathcal{D}^1$, and consider the hitting set $\mathcal{E}_g(\mathcal{B}) = \left\{ \xi \in \mathcal{D}^1 \mid \{n \in \mathbb{N} \mid g_n \xi \in B_n\} \text{ is infinite} \right\}$.*

1. *If $\sum_{n=1}^{\infty} \mu(B_n) < \infty$, then $\mu(\mathcal{E}_g(\mathcal{B})) = 0$.*
2. *If $\sum_{n=1}^{\infty} \mu(B_n) = \infty$, then $\mathcal{E}_g(\mathcal{B})$ has full μ -measure.*
3. *If, moreover, $\{n\mu(B_n)\}_{n \in \mathbb{N}}$ is unbounded, then there is a subsequence n_j such that*

$$\lim_{j \rightarrow \infty} \frac{\#\{1 \leq i \leq n_j \mid g_i \xi \in B_{n_j}\}}{n_j \mu(B_{n_j})} = 1 \quad \text{for } \mu\text{-almost every } \xi \in \mathcal{D}^1.$$

Remark 1.2. There are numerous measures—such as the extensively studied Masur–Veech measures (see §2.5) or measures coming from arbitrary orbit closures—to which Theorem 1.1 applies. Any ergodic measure μ is in fact supported on one of the subsets \mathcal{A}^1 or \mathcal{Q}^1 , which are both preserved by G . We have opted to work with their disjoint union \mathcal{D}^1 simply to economize on notation.

Remark 1.3. A random sequence ξ_1, \dots, ξ_n would land in a measurable subset B approximately $n\mu(B)$ times; thus the conclusion of Theorem 1.1.3 says there is a subsequence of targets that are hit asymptotically the correct number of times.

In [ANT, Theorem 6.7], Aimino, Nicol, and Todd consider the *continuous* shrinking target problem for the Teichmüller geodesic and prove the continuous-time analog of Theorem 1.1.1–2 for the Masur–Veech measure on the principle stratum (see §2.4) in the case of targets that are preimages $\pi^{-1}(B_t)$ of shrinking metric balls $B_t \subset \mathcal{M}$. As requiring a trajectory $\{g_t \xi\}_{t \in \mathbb{R}_+}$ to hit the targets at infinitely many *integer* times is more subtle and restrictive than requiring $\{t \in \mathbb{R}_+ \mid \pi(g_t \xi) \in B_t\}$ to be unbounded, Theorem 1.1 thus

strengthens the results in [ANT] while also extending them to arbitrary measures and more general families of targets.

We also consider the finer question of which differentials eventually always hit the targets. Given a shrinking family $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ in \mathcal{D}^1 , say that $\xi \in \mathcal{D}^1$ *hits* the target B_n if $g_i \xi \in B_n$ for some $1 \leq i \leq n$. We then let $\mathcal{E}_g^a(\mathcal{B})$ denote the set of differentials that hit B_n for all sufficiently large n . If $\xi \in \mathcal{E}_g^a(\mathcal{B})$, then for all large n we may choose $1 \leq j_n \leq n$ so that $g_{j_n} \xi \in B_n \subset B_{j_n}$. Clearly this sequence $\{j_n\}$ is unbounded unless ξ lies in $g_{-k}(\cap_n B_n)$ for some $k \in \mathbb{N}$. As $\cup_k g_k(\cap_n B_n)$ has measure zero when $\mu(B_n) \rightarrow 0$ and otherwise $\mathcal{E}_g(\mathcal{B})$ has full measure (Theorem 1.1.2), we conclude that $\mathcal{E}_g^a(\mathcal{B}) \subset \mathcal{E}_g(\mathcal{B})$ up to a set of measure zero. The next theorem gives conditions ensuring this smaller set has full measure and, moreover, that generic differentials hit *all* small targets approximately the expected number of times.

Theorem 1.4. *Let μ and $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ be as in Theorem 1.1, and set*

$$\mathcal{E}_g^a(\mathcal{B}) = \left\{ \xi \in \mathcal{D}^1 \mid \{g_1 \xi, \dots, g_n \xi\} \cap B_n \neq \emptyset \text{ for all sufficiently large } n \right\}.$$

1. *If there is an increasing sequence n_1, n_2, \dots in \mathbb{N} with $\sum_j (n_j \mu(B_{n_{j+1}}))^{-1}$ finite, then $\mathcal{E}_g^a(\mathcal{B})$ has full μ -measure.*
2. *If there exists $\lambda > 1$ and an increasing sequence n_1, n_2, \dots in \mathbb{N} with $\sum_j (n_j \mu(B_{n_{j+1}}))^{-1}$ finite and $n_{j+1} \mu(B_{n_j}) \leq \lambda n_j \mu(B_{n_{j+1}})$ for all j , then μ -almost every $\xi \in \mathcal{D}^1$ satisfies*

$$\frac{1}{2\lambda} \leq \frac{\#\{1 \leq i \leq n \mid g_i \xi \in B_n\}}{n \mu(B_n)} \leq 2\lambda \quad \text{for all sufficiently large } n.$$

Remark 1.5. While the hypotheses of Theorem 1.4.1–2 may seem restrictive, they in fact hold in many natural situations. For example, they are automatically satisfied (with $n_j = 2^j$) provided the targets B_n decay at the rate $\mu(B_n) \asymp n^{-\kappa}$ for some $0 < \kappa < 1$. We will use these always-hitting results to obtain the strong logarithm laws in Theorem 1.6 below.

Theorems 1.1 and 1.4 extend Kelmer’s work on shrinking targets for discrete time flows on hyperbolic manifolds to the context of Teichmüller geodesic flow and may be viewed as analogues of Theorems 1 and 2, respectively, in [Kel].

Logarithm laws. Since the geodesic flow is ergodic, the orbit closure $\overline{\{g_n \xi \mid n \in \mathbb{N}\}}$ of almost every differential $\xi \in \mathcal{D}^1$ will equal the support of the measure. That is, the sequence $\{g_n \xi\}$ ultimately accumulates on almost every point ξ_0 . It is natural to quantify this phenomenon by asking how long it takes for $g_n \xi$ to land within some fixed distance of ξ_0 or how close the orbit gets to ξ_0 in its first n steps. Our shrinking target results imply *logarithm laws* describing these quantities.

More precisely, consider the path metric $d_{\mathcal{M}}$ that Moduli space inherits from the Teichmüller metric on \mathcal{T} ; see §2.2. For any Riemann surface $X \in \mathcal{M}$ and differential $\xi \in \mathcal{D}^1$, we then let

$$d_n(\xi, X) = \min_{0 \leq j \leq n} d_{\mathcal{M}}(\pi(g_j \xi), X)$$

denote the closest that the projected orbit comes to X in its first $n \geq 0$ steps and dually let

$$\tau_r(\xi, X) = \inf\{n \in \mathbb{N} \mid d_{\mathcal{M}}(\pi(g_n \xi), X) \leq r\}$$

denote the first time the orbit is within $r > 0$ of X . Calculating the decay/growth rate of d_n and τ_r requires fine estimates for the measure of small metric balls in \mathcal{M} . Such estimates

are most readily obtained ([Lemma 2.5](#)) in the case of the Masur–Veech measure λ_β^1 on the principle stratum $\mathcal{Q}^1(\alpha)$ where $\alpha = (-1, \dots, -1, 1, \dots, 1; -1)$; see [§2.4](#) and [§2.6](#) for details.

Theorem 1.6. *Fix a Riemann surface $X \in \mathcal{M}$ and let λ_α^1 denote the Masur–Veech measure on the principle stratum $\mathcal{Q}^1(\alpha)$ of unit-area quadratic differentials. Then for λ_α^1 -almost every $q \in \mathcal{Q}^1$ one has*

$$\lim_{n \rightarrow \infty} \frac{\log(d_n(q, X))}{\log(1/n)} = \frac{1}{\dim_{\mathbb{R}}(\mathcal{M})} \quad \text{and} \quad \lim_{r \rightarrow 0} \frac{\log(\tau_r(q, X))}{\log(1/r)} = \dim_{\mathbb{R}}(\mathcal{M}).$$

We emphasize that [Theorem 1.6](#) provides actual limits describing the decay and growth of d_n and τ_r , rather than merely limits superior as is typically the strongest consequence of shrinking target results as in [Theorem 1.1](#). Here we obtain precise limits by employing the stronger results from [Theorem 1.4](#).

Remark 1.7. Applying dynamical shrinking target results to geometric questions about logarithm laws requires using measures that are related to the geometry of moduli space, since to quantitatively use [Theorem 1.1](#) one must know how the measure of a metric ball decays with its radius. This is the reason we obtain [Theorem 1.6](#) only for the Masur–Veech measure λ_α^1 and not for general ergodic $\mathrm{SL}(2, \mathbb{R})$ -invariant measures; see [§2.6](#).

Historical Context. This work builds on a long history of shrinking target problems and dynamical Borel–Cantelli lemmas in various settings. Sullivan’s 1982 paper [\[Sul\]](#) established a logarithm law for cusp excursions of the geodesic flow on finite-volume hyperbolic manifolds; this was later extended to the general setting of diagonal flows on homogeneous spaces by Kleinbock and Margulis [\[KM\]](#). These results take horospherical cusp neighborhoods as the shrinking targets; turning instead to precompact targets, Maucourant [\[Mau\]](#) solved the continuous shrinking target problem for metric balls in hyperbolic manifolds and obtained a logarithm law describing the rate that typical geodesic trajectories approach a given point. These results were later refined and extended to discrete geodesic flows by Kleinbock and Zhao [\[KZ\]](#). In the setting of the Teichmüller geodesic flow, Masur [\[Mas2\]](#) proved a logarithm law for cusp excursions of typical geodesics in a Teichmüller disk, and as mentioned above, Aimino, Nicol, and Todd [\[ANT\]](#) recently solved a continuous-time shrinking target problem for the case of nested metric balls. Their approach is combinatorial in nature and is based on Rauzy–Veech–Zorich renormalization on the space of interval exchange maps and the application of this framework to Teichmüller geodesic flow on the space of translation surfaces. For more related result, see also [\[Dol\]](#) [\[Gad\]](#) [\[Gal\]](#) [\[GS\]](#) [\[GNO\]](#) [\[HNPV\]](#) [\[Phi\]](#) and [\[Ath\]](#) for a nice survey of this area.

Many of these results utilize exponential decay of correlations and require some sort of regularity on the targets considered. In our case of the Teichmüller geodesic flow on the moduli space of quadratic differentials, the shrinking target results in [\[ANT\]](#) rely on the exponential mixing for Hölder observables proven by Avila–Gouëzel–Yoccoz [\[AGY\]](#). Indeed, the restriction to metric balls here is due in part to the necessity of using Hölder observables.

Kelmer [\[Kel\]](#) recently introduced a new spectral theory approach that utilizes effective mean ergodic theorems (see also [\[GK\]](#)) to conclude strong shrinking target results—which apply to very general targets—for both the discrete geodesic and discrete horocyclic flows on hyperbolic manifolds. These ideas were subsequently extended to homogeneous spaces by Kelmer and Yu [\[KY\]](#). We follow Kelmer’s approach and adapt it to the Teichmüller flow setting by using Eskin and Mirzakhani’s result [\[EM\]](#) that every ergodic $\mathrm{SL}(2, \mathbb{R})$ -invariant

probability measure μ on \mathcal{D}^1 is algebraic and Avila and Gouëzel's result [AG] that the Laplacian operator on $L^2(\mathcal{D}^1, \mu)$ has a spectral gap for any such measure μ .

Outline. In §2 we establish notation and review the necessary background material, including the unitary representation theory of $\mathrm{SL}(2, \mathbb{R})$ (§2.1), and the moduli space of differentials (§2.4) along with its period coordinates and invariant measures (§2.5). Additionally, in §2.6 we derive estimates for the measures of balls in \mathcal{M} . Our main results are proven in §3, with §3.3 and §3.4 respectively devoted to Theorem 1.1 and Theorem 1.4. The key ingredients for these proofs are established in §3.1, where we follow Kelmer's [Kel] approach in deriving an effective mean ergodic theorem for spherical functions in $L^2(\mathcal{D}^1, \mu)$, and in §3.2, where we formulate the quasi-independence needed for the converse to the Borel–Cantelli lemma. Finally, the logarithm laws of Theorem 1.6 are proven in §4.

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2 Preliminaries

Throughout, G will denote the Lie group $G = \mathrm{SL}(2, \mathbb{R})$ of real 2×2 matrices with determinant one. For $t \in \mathbb{R}$ and $\theta \in \mathbb{R}$ consider the elements

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \text{and} \quad r_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

Varying t, θ , these respectively comprise the *diagonal* $A = \{g_t \mid t \in \mathbb{R}\}$ and *rotation* subgroups $K = \{r_\theta \mid \theta \in \mathbb{R}\} = \mathrm{SO}(2)$. The Lie algebra of G is the vector space \mathfrak{g} of 2×2 real matrices with trace 0; it has a basis given by

$$W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.1)$$

2.1 Unitary representation theory of $\mathrm{SL}(2, \mathbb{R})$

We briefly review the necessary representation theory of the Lie group $G = \mathrm{SL}(2, \mathbb{R})$. For details, we direct the reader to [Kna]. A *unitary representation* of G is a homomorphism $\pi: G \rightarrow \mathrm{U}(\mathcal{H})$ of G into the unitary group $\mathrm{U}(\mathcal{H})$ of a complex Hilbert space \mathcal{H} such that the induced action map $G \times \mathcal{H} \rightarrow \mathcal{H}$ is continuous. Two such representations $\pi_i: G \rightarrow \mathrm{U}(\mathcal{H}_i)$ are *unitarily equivalent* if there is a unitary isomorphism $\varphi: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ so that $\varphi\pi_1(g) = \pi_2(g)\varphi$ for all $g \in G$. We refer to a representation simply by the name of its Hilbert space \mathcal{H} and use the notation $g \cdot v$, where $g \in G$ and $v \in \mathcal{H}$, as shorthand for $\pi(g)(v)$. A subspace V of a unitary representation \mathcal{H} is *invariant* if $g \cdot V \subset V$ for all $g \in G$; the representation is *irreducible* if the only closed invariant subspaces are $\{0\}$ and \mathcal{H} .

A vector v in a unitary representation \mathcal{H} is a C^k -vector, where $k \in \mathbb{N} \cup \{\infty\}$, if the assignment $g \mapsto g \cdot v$ defines a C^k map $G \rightarrow \mathcal{H}$. The set of C^∞ -vectors is dense in \mathcal{H} . A vector $v \in \mathcal{H}$ is *spherical* if $k \cdot v = v$ for all $k \in K$ and is *K-finite* if $K \cdot v$ is contained in a finite-dimensional closed subspace of \mathcal{H} .

Each Lie vector $X \in \mathfrak{g}$ determines an unbounded operator L_X on any unitary representation \mathcal{H} defined on the subspace of C^1 -vectors by the rule

$$L_X(v) = \lim_{t \rightarrow 0} \frac{\exp(tX) \cdot v - v}{t}.$$

The *Casimir operator* of \mathcal{H} is then defined on the space of C^2 -vectors as

$$\Omega = (L_W^2 - L_Q^2 - L_V^2)/4,$$

where $\{W, Q, V\}$ is the basis of \mathfrak{g} specified in (2.1). From the elementary calculation $\langle L_X(v), w \rangle = -\langle v, L_X(w) \rangle$ we see that Ω is symmetric: $\langle \Omega(v), w \rangle = \langle v, \Omega(w) \rangle$ for all C^2 -vectors $v, w \in \mathcal{H}$. It is known that closure of Ω is self-adjoint and that Ω commutes with L_X for every $X \in \mathfrak{g}$ and with $\pi(g)$ for each $g \in G$: $\Omega(g \cdot v) = g \cdot \Omega(v)$. When \mathcal{H} is irreducible, Schur's Lemma therefore provides a scalar $\lambda = \lambda(\mathcal{H})$ such that $\Omega(v) = \lambda v$ for all C^2 -vectors $v \in \mathcal{H}$; the fact that Ω is symmetric assures that $\lambda(\mathcal{H}) \in \mathbb{R}$. The *spectrum* of the unitary representation \mathcal{H} is

$$\Lambda(\mathcal{H}) = \{\lambda \in \mathbb{C} \mid \Omega - \lambda I \text{ does not have a bounded inverse}\} \subset \mathbb{R}.$$

A function $G \rightarrow \mathbb{C}$ of the form $g \mapsto \langle g \cdot v, w \rangle$, where $v, w \in \mathcal{H}$, is called a *matrix coefficient* of the unitary representation \mathcal{H} . Much is known about the asymptotic behavior of matrix coefficients along the diagonal subgroup $A = \{g_t \mid t \in \mathbb{R}\}$, especially when v and w are K -finite; see for example [Kna, VIII.§13]. We shall only require the following basic lemma which is derived from Ratner's work [Rat].

Lemma 2.2. *There exists a universal constant C_0 such that the following holds. Fix $0 < \delta < 1$ and let \mathcal{H} be a unitary representation of G without nonzero invariant vectors and such that $\Lambda(\mathcal{H}) \cap (0, \frac{1-\delta^2}{4}) = \emptyset$. Then for any spherical vectors $v, w \in \mathcal{H}$ and $t \geq 1$ we have*

$$|\langle g_t \cdot v, w \rangle| \leq C_0 \|v\| \|w\| t e^{-t(1-\delta)}.$$

Proof. The lemma follows immediately from the quantitative version of [Rat, Theorem 3] that Matheus gives in [Mat]. For this, set $\lambda = \beta(\mathcal{H}) = \inf(\Lambda(\mathcal{H}) \cap (0, \frac{1}{4}))$ (note that our Casimir operator Ω is the negative of that used in [Mat]). The relevant terms from [Mat] are then the function

$$b_{\mathcal{H}}(t) = b_{\lambda}(t) = \begin{cases} t e^{-t}, & \lambda \geq \frac{1}{4} \\ t e^{-(1-\sqrt{1-4\lambda})t}, & 0 < \lambda < \frac{1}{4} \\ t e^{-2t}, & \lambda \leq 0 \end{cases}$$

and the constant

$$\tilde{K}_{\mathcal{H}} = \tilde{K}_{\lambda} \leq \max \left\{ (1 + 2\sqrt{2})e + \frac{32+\sqrt{2}}{3e^3(1-e^{-4})^2}, 3e + e^2 + \frac{4}{9e^3(1-e^{-4})}, e^2 \right\} =: C_0.$$

Our hypothesis on $\Lambda(\mathcal{H})$ implies $\lambda \geq \frac{1-\delta^2}{4}$ and accordingly $-(1 - \sqrt{1-4\lambda}) \leq -(1 - \delta)$. Since $-2 \leq -1 \leq -(1 - \delta)$ as well, in any case we have $b_{\mathcal{H}}(t) \leq t e^{-t(1-\delta)}$.

We may now apply [Mat, Theorem 2] to estimate $|\langle g_t \cdot v, w \rangle|$. Since our vectors v, w are spherical, they satisfy $L_W(v) = L_W(w) = 0$ and therefore the bound reduces to

$$|\langle g_t \cdot v, w \rangle| \leq \tilde{K}_{\mathcal{H}} \|v\| \|w\| b_{\mathcal{H}}(t) \leq C_0 \|v\| \|w\| t e^{-t(1-\delta)}. \quad \square$$

Remark 2.3. Related estimates on $|\langle g_t \cdot v, w \rangle|$ hold more generally but are more involved when v, w are not spherical. It is likely that our arguments could be carried out, albeit with more technical considerations, in the case of shrinking targets that are merely K -finite.

2.2 Teichmüller space and its metric

Throughout we let Σ denote a fixed orientable closed surface Σ of genus $g \geq 0$ equipped with a (potentially empty) finite set $P \subset \Sigma$ of distinguished points that we term *punctures*. Writing $p = |P| \geq 0$, we assume that $\chi(\Sigma \setminus P) < 0$ and that $(g, p) \neq (0, 3)$, which is equivalent to $3g - 3 + p > 0$. We write $\mathcal{T} = \mathcal{T}(\Sigma)$ for the *Teichmüller space* of Riemann surface structures on Σ up to isotopy fixing P . The *mapping class group* $\text{Mod}(\Sigma)$ is the quotient of the group $\text{Homeo}^+(\Sigma, P)$ of orientation-preserving homeomorphisms of Σ that setwise preserve P by the path component $\text{Homeo}_0(\Sigma, P)$ of the identity. The group of homeomorphisms naturally acts on the set of Riemann surface structures—a homeomorphism ϕ sends a Riemann surface structure with atlas $\{z_\alpha: \Sigma \supset U_\alpha \rightarrow \mathbb{C}\}$ to the structure defined by the atlas $\{z_\alpha \circ \phi^{-1}\}$ —and this descends to an action of $\text{Mod}(\Sigma)$ on \mathcal{T} . The quotient is the *moduli space* $\mathcal{M} = \mathcal{M}(\Sigma)$ of Riemann surfaces of genus g with p distinguished points.

The *Teichmüller distance* $d_{\mathcal{T}}$ between two Riemann surface structures X and Y on Σ is the infimum of $\frac{1}{2} \log(K_\phi(X, Y))$ over all maps $\phi \in \text{Homeo}_0(\Sigma, P)$ that are quasiconformal with respect to these structures, where $K_\phi(X, Y)$ denotes the quasiconformal constant of ϕ . Topologically, \mathcal{T} is a smooth manifold homeomorphic to $\mathbb{R}^{6g-6+2p}$, and there is a Finsler metric on \mathcal{T} such that the Teichmüller distance $d_{\mathcal{T}}(X, Y)$ is realized as the shortest Finsler length of a path from X to Y . This metric is invariant under the $\text{Mod}(\Sigma)$ action and thus descends to a Finsler metric on \mathcal{M} with associated path metric $d_{\mathcal{M}}$.

2.3 Notation

Given some parameter δ (such as a point in space or a measure on that space) we use the notation $A \ll_\delta B$ or $B \gg_\delta A$ to mean that there is a constant $c > 0$, depending only on our fixed surface Σ and the parameter δ , such that $A \leq cB$. We write $A \asymp_\delta B$ to mean $A \ll_\delta B$ and $B \ll_\delta A$. We similarly use \ll and \asymp when the constant c depends only on Σ .

2.4 Moduli space of differentials

An *abelian differential* on a Riemann surface structure $X \in \mathcal{T}(\Sigma)$ is a holomorphic 1-form, that is, a section of the holomorphic cotangent bundle of X . A *quadratic differential* is a section of the symmetric square of the meromorphic cotangent bundle whose poles are all simple and occur in the distinguished set P . Abelian and quadratic differentials may, respectively, be concretely represented in local coordinates $\{z_\alpha: U_\alpha \rightarrow \mathbb{C}\}$ by families of functions $h_\alpha, \phi_\alpha: z_\alpha(U_\alpha) \rightarrow \mathbb{C}$ such that on any two overlapping charts z_α and z_β one has

$$h_\beta(z_\beta) \left(\frac{dz_\beta}{dz_\alpha} \right) = h_\alpha(z_\alpha) \quad \text{or, respectively,} \quad \phi_\beta(z_\beta) \left(\frac{dz_\beta}{dz_\alpha} \right)^2 = \phi_\alpha(z_\alpha)$$

and, furthermore, the functions h_α are holomorphic while the functions ϕ_α have only simple poles and are holomorphic on $\Sigma \setminus P$. In particular, the location and multiplicities of the zeros and poles of an abelian/quadratic differential on X are well-defined. From this it is clear that each abelian differential ω or quadratic differential q gives rise to a natural coordinate atlas $\{z_\alpha: U_\alpha \rightarrow \mathbb{C}\}$ for X such that each U_α contains at most one zero and the above functions take the form $h_\alpha(z_\alpha) = z_\alpha^k$ or $\phi_\alpha(z_\alpha) = z_\alpha^k$ when U_α contains a zero of order $k \geq -1$ and otherwise have the form $h_\alpha = 1$ or $\phi_\alpha = 1$. We call these the *translation charts* of ω or q since they define a translation surface structure on $X \setminus \{\text{zeros of } \omega\}$ in the abelian case, or a half-translation surface structure on $X \setminus \{\text{zeros of } q\}$ in the quadratic case. In

this translation structure, an order- k zero of ω acquires a cone angle of $2\pi(k+1)$ whereas an order- k zero of q acquires a cone angle of $\pi(k+2)$.

We write $\mathcal{A}(X)$ and $\mathcal{Q}(X)$ for the \mathbb{C} -vector spaces of abelian and quadratic differentials, respectively, on a Riemann surface $X \in \mathcal{T}$. For a tuple $\beta = (\beta_1, \dots, \beta_n; 0)$ where $n \geq p$ and $\beta_i \geq 0$ are integers such that $\sum_i \beta_i = 2g - 2$ and $\beta_i \geq 1$ for $i > p$, we let $\mathcal{A}(X, \beta)$ denote the subset of abelian differentials on X that have zeros of orders $\{\beta_1, \dots, \beta_p\}$ at the punctures and $n - p$ other zeros of orders $\{\beta_{p+1}, \dots, \beta_n\}$. Similarly for $\kappa = (\kappa_1, \dots, \kappa_n; \epsilon)$, where $n \geq p$, $\epsilon \in \{\pm 1\}$, and $\kappa_i \geq -1$ are integers with $\sum_i \kappa_i = 4g - 4$ and $\kappa_i \geq 1$ for $i > p$, we write $\mathcal{Q}(X, \kappa)$ for the subset of quadratic differentials with zeros of orders $\{\kappa_1, \dots, \kappa_p\}$ at the punctures and $n - p$ other zeros or orders $\{\kappa_{p+1}, \dots, \kappa_n\}$ that are, if $\epsilon = 1$, or are not, if $\epsilon = -1$, the square of an abelian differential.

To economize on notation, we let $\mathcal{D}(X)$ be the disjoint $\mathcal{A}(X) \sqcup \mathcal{Q}(X)$. Further, for any integer vector $\alpha = (\alpha_1, \dots, \alpha_n; \epsilon)$ we respectively define $\mathcal{D}(X, \alpha)$ to mean $\mathcal{A}(X, \alpha)$ or $\mathcal{Q}(X, \alpha)$ in the cases that α satisfies the conditions on β or κ above. For any such space, the notation \mathcal{A}^1 , \mathcal{Q}^1 , or \mathcal{D}^1 will indicate the corresponding subset of differentials that induce (half) translation structures with total area 1. As X varies in \mathcal{T} we obtain vector bundles \mathcal{AT} and \mathcal{QT} over Teichmüller space which, under the quotient by the action of the mapping class group, descend to bundles $\mathcal{A}, \mathcal{Q} \rightarrow \mathcal{M}$ whose disjoint union $\mathcal{D} = \mathcal{D}(\Sigma)$ is stratified by the associated subsets $\mathcal{D}(\alpha)$.

The bundles \mathcal{AT} and \mathcal{QT} are endowed with an action of the group $G = \mathrm{SL}(2, \mathbb{R})$ of real 2×2 matrices as follows: Given a differential ξ with translation charts $\{z_\alpha: U_\alpha \rightarrow \mathbb{C}\}$, postcomposing by $A \in G$ defines translations charts $\{A \circ z_\alpha\}$ for a new differential $A\xi$ with the same total area and orders of zeros, but over a potentially different Riemann surface. This commutes with the action of $\mathrm{Mod}(\Sigma)$ by precomposition and so descends to an action $G \curvearrowright \mathcal{D}$ that preserves all the strata $\mathcal{D}^1(\alpha)$ introduced above.

2.5 Invariant measures

A (quadratic or abelian) differential ξ allows one to develop any path γ in Σ in the translation charts and measure its “total displacement” in \mathbb{C} . In this way the differential ξ defines a relative cohomology class $\Phi(\xi) \in H^1(\Sigma, Z(\xi); \mathbb{C})$, where $Z(\xi) \subset \Sigma$ denotes the union of P with the zeros of ξ . Choosing a symplectic basis $\{\gamma_1, \dots, \gamma_m\}$ allows us to view this class in coordinates as

$$\Phi(\xi) \in H^1(\Sigma, Z(\xi); \mathbb{C}) = \mathrm{Hom}(H_1(\Sigma, Z(\xi); \mathbb{C}), \mathbb{C}) \cong \mathbb{C}^m.$$

By consistently choosing symplectic bases of $H_1(\Sigma, Z(\xi'); \mathbb{C})$ for all ξ' in some neighborhood of ξ , in this way we obtain locally defined maps

$$\Phi_\alpha: \mathcal{D}(\alpha) \rightarrow \mathbb{C}^{m_\alpha}$$

on each stratum. These *period coordinates* are local diffeomorphisms and endow each stratum with a canonical complex affine structure. Pulling back Lebesgue measure from \mathbb{C}^{m_α} defines a canonical *Masur–Veech* measure λ_α on each stratum $\mathcal{D}(\alpha)$ that is independent of the choice of basis. Furthermore, λ_α induces a measure λ_α^1 on the subset $\mathcal{D}^1(\alpha)$ of unit area differentials by the rule $\lambda_\alpha^1(U) = \lambda_\alpha(\{tU : t \in (0, 1]\})$. Since $\mathrm{SL}(2, \mathbb{R})$ acts via area-preserving transformations of the translation charts, the Masur–Veech measures λ_α are invariant under the $\mathrm{SL}(2, \mathbb{R})$ action.

It is a fundamental result of Masur [Mas1] and Veech [Vee] that each measure λ_α^1 has finite total mass and is ergodic for the $\mathrm{SL}(2, \mathbb{R})$ action on \mathcal{D} . While there are many other

ergodic measures on \mathcal{D} (for example, coming from orbit closures $\overline{G\xi}$), the celebrated work of Eskin and Mirzakhani [EM] shows that every such measure is *affine* in the sense that it essentially arises from the Lebesgue measure of an immersed submanifold $Y \looparrowright \mathcal{D}(\alpha)$ that is affine in the period coordinates of some stratum.

Given any probability measure μ on the space \mathcal{D}^1 of unit-area (abelian or quadratic) differentials, we may consider the Hilbert space $L^2(\mathcal{D}^1, \mu)$ of square-integrable functions. The $\mathrm{SL}(2, \mathbb{R})$ action makes this into a unitary representation of G in which $A \in G$ sends a function $f \in L^2(\mathcal{D}^1, \mu)$ to the new function $\xi \mapsto (A \cdot f)(\xi) = f(A^{-1}\xi)$. We rely crucially on the following result of Avila and Gouëzel, which, in this formulation, itself depends on the affine structure provided by Eskin and Mirzakhani [EM]:

Theorem 2.4 (Avila–Gouëzel [AG]). *Let μ be any $\mathrm{SL}(2, \mathbb{R})$ -invariant, ergodic probability measure on \mathcal{D}^1 . Then the Casimir operator for the unitary representation $L^2(\mathcal{D}^1, \mu)$ has a spectral gap. More precisely, $\Lambda(L^2(\mathcal{D}^1, \mu)) \cap (0, \frac{1-\delta^2}{4}) = \emptyset$ for some $0 < \delta < 1$.*

2.6 The measures of balls

In order to derive logarithm laws for the geodesic flow, we need to know the measure of certain geometrically significant subsets of \mathcal{D}^1 . For this, we consider the principle stratum $\mathcal{D}(\alpha) = \mathcal{Q}(\alpha)$ where $\alpha = (-1, \dots, -1, 1, \dots, 1; -1)$. The associated measure λ_α^1 pushes forward, under the bundle map $\pi: \mathcal{Q} \rightarrow \mathcal{M}$, to a measure $\mathfrak{n} = \pi_*(\lambda_\alpha^1)$ on \mathcal{M} . The Finsler metric on \mathcal{M} induces a *Buseman volume form* that in turn gives rise to an associated measure ν on \mathcal{M} ; in fact ν is the Hausdorff measure for the Teichmüller metric [APT, Bus]. Using Masur’s result that the Teichmüller geodesic flow is Hamiltonian [Mas3], in [DDM, Corollary 4.7] it was shown that the measures ν and \mathfrak{n} are absolutely continuous and in fact related by inequalities $c_1 \mathfrak{n} \leq \nu \leq c_2 \mathfrak{n}$ for some scalar $c_2 > c_1 > 0$. This gives rise to the following estimate:

Lemma 2.5. *Fix $X \in \mathcal{M}$. For all sufficiently small $r > 0$ the \mathfrak{n} -measure of the Teichmüller metric ball $B_X(r) \subset \mathcal{M}$ of radius r satisfies*

$$\mathfrak{n}(B_X(r)) \asymp_X r^{6g-6+2n}$$

Proof. Choose a coordinate chart $\varphi: V \rightarrow \mathbb{R}^{6g-6+2n}$ on some compact set $V \subset \mathcal{M}$ whose interior contains X . Let ν^φ and $B_X^\varphi(r)$ denote the Riemannian volume and metric balls associated to the standard Riemannian metric in φ -coordinates. Since all norms on \mathbb{R}^k are equivalent and V is compact, there is a constant $c > 1$ so that the Euclidean norm in φ -coordinates agrees with the Teichmüller Finsler norm up to a multiple of c . It follows that the measures ν^φ and ν agree up to a multiple of $c^{6g-6+2p}$ and that for all sufficiently small $r > 0$ we have $B_X(r) \leq B_X^\varphi(rc)$ and $B_X^\varphi(r) \subset B_X(rc)$. The lemma now follows by noting

$$\mathfrak{n}(B_X(r)) \asymp \nu(B_X(r)) \asymp_c \nu^\varphi(B_X^\varphi(r)) \asymp r^{6g-6+2n}. \quad \square$$

3 Shrinking targets

For the entirety of this section, we fix an ergodic, G -invariant probability measure μ on the moduli space \mathcal{D}^1 of unit area abelian or quadratic differentials on Σ . The Hilbert space $\mathcal{L} = L^2(\mathcal{D}^1, \mu)$ of square-integrable complex-valued functions is then a unitary representation of G on which $A \in G$ sends $f \in \mathcal{L}$ to the function $A \cdot f$ given by $\xi \mapsto (A \cdot f)(\xi) = f(A^{-1}\xi)$.

For $f \in \mathcal{L}$ we write $\|f\|_2 = (\int f \bar{f} d\mu)^{1/2}$ for the norm of f in \mathcal{L} . Let $\mathcal{L}_0 \leq \mathcal{L}$ be the closed subspace of G -invariant vectors, and let \mathcal{L}_0^\perp denote its orthogonal complement. For any $f \in \mathcal{L}$, we let $f^0 \in \mathcal{L}_0$ denote its orthogonal projection to \mathcal{L}_0 and set $f' = f - f^0 \in \mathcal{L}_0^\perp$. Since the action $G \curvearrowright \mathcal{D}^1$ is ergodic, every G -invariant function is μ -almost everywhere constant. Thus $\{1\}$ is a basis for \mathcal{L}_0 and f^0 is equal in \mathcal{L} to the constant function $\langle f, 1 \rangle$. Observe that f' is spherical if and only if f is spherical.

By [Theorem 2.4](#) there exists $0 < \delta < 1$ so that the $\Lambda(\mathcal{L})$ and hence also $\Lambda(\mathcal{L}_0^\perp)$ is disjoint from $(0, \frac{1-\delta^2}{4})$. Since \mathcal{L}_0^\perp does not have nonzero invariant vectors, it follows from [Lemma 2.2](#) that there exists a constant $\sigma > 0$ depending only on μ such that

$$|\langle g_t \cdot f', h' \rangle| \ll \|f'\|_2 \|h'\|_2 t e^{-t\sigma} \quad \text{for all } t \geq 1 \text{ and all spherical } f, h \in \mathcal{L}. \quad (3.1)$$

3.1 Effective mean ergodic theorem

Consider the average of a function $f \in \mathcal{L}$ over the first $n \geq 1$ iterates of the geodesic flow:

$$\beta_n(f) = \frac{1}{n} \sum_{j=1}^n g_{-j} \cdot f, \quad \text{so} \quad (\beta_n(f))(\xi) = \frac{f(g_1\xi) + \cdots + f(g_n\xi)}{n}. \quad (3.2)$$

The Birkhoff ergodic theorem implies that the images $\beta_n(f)$ of f under the operators $\beta_n: \mathcal{L} \rightarrow \mathcal{L}$ pointwise almost everywhere converge to the constant function $\int f d\mu = \langle f, 1 \rangle$ as $n \rightarrow \infty$. When f is spherical, $\{\beta_n(f)\}_n$ in fact converges in \mathcal{L} at a definite rate depending on $\|f\|_2$ as described by the following effective mean ergodic theorem:

Theorem 3.3. *For any spherical $f \in \mathcal{L} = L^2(\mathcal{D}^1, \mu)$ and $n \geq 1$, we have*

$$\|\beta_n(f) - f^0\|_2 \ll_\mu \frac{\|f\|_2}{n^{1/2}}$$

Proof. Since $f^0 \in \mathcal{L}_0$ is invariant under β_n , we first observe that

$$\beta_n(f) = \beta_n(f^0 + f') = \beta_n(f^0) + \beta_n(f') = f^0 + \beta_n(f').$$

Therefore $\|\beta_n(f) - f^0\|_2 = \|\beta_n(f')\|_2$. To bound latter quantity, we write

$$\|\beta_n(f')\|_2^2 = \frac{1}{n^2} \sum_{j=1}^n \sum_{k=1}^n \langle g_{-j} \cdot f', g_{-k} \cdot f' \rangle = \frac{1}{n^2} \sum_{j=1}^n \sum_{k=1}^n \langle g_{k-j} \cdot f', f' \rangle \quad (3.4)$$

Noting that the quantity $C_i = \#\{(j, k) \mid j, k = 1, \dots, n \text{ with } k - j = i\}$ is at most n for every i , and using the fact $\langle g_{-i} \cdot f', f' \rangle = \langle g_i \cdot f', f' \rangle$, equation (3.4) gives

$$\|\beta_n(f')\|_2^2 = \left| \frac{1}{n^2} \sum_{j=1}^n \sum_{k=1}^n \langle g_{k-j} \cdot f', f' \rangle \right| = \frac{1}{n^2} \left| \sum_{i=1-n}^{n-1} C_i \langle g_i \cdot f', f' \rangle \right| \leq \frac{2}{n} \sum_{i=0}^{n-1} |\langle g_i \cdot f', f' \rangle|.$$

Invoking equation (3.1) and using $\sum_{i=1}^\infty i e^{-i\sigma} < \infty$, we now conclude the required estimate

$$\|\beta_n(f) - f^0\|_2^2 = \|\beta_n(f')\|_2^2 \ll \frac{2}{n} \|f'\|_2^2 \left(1 + \sum_{i=1}^{n-1} i e^{-i\sigma} \right) \ll_\mu \frac{\|f\|_2^2}{n}. \quad \square$$

As we will see in [§3.3](#) below, in the case of spherical shrinking targets $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ with $\{n\mu(B_n)\}$ unbounded, the effective mean ergodic theorem provides strong results about typical hitting frequencies and the hitting sets $\mathcal{E}_g(\mathcal{B})$ and $\mathcal{E}_g^a(\mathcal{B})$. In the absence of this strong condition on $\{n\mu(B_n)\}$, we must take a different approach based on independence:

3.2 Quasi-independence

Consider a family \mathcal{B} of spherical, μ -measurable shrinking targets $B_1 \supset B_2 \supset \dots$ in \mathcal{D}^1 . We are interested in the associated sets

$$E_n = g_n^{-1}(B_n) = \{\xi \in \mathcal{D}^1 \mid g_n \xi \in B_n\}.$$

Notice that $\mu(E_n) = \mu(B_n)$. As mentioned in the introduction, if the events E_n were independent in the sense that $\mu(E_{n_1} \cap \dots \cap E_{n_k}) = \mu(E_{n_1}) \dots \mu(E_{n_k})$ for all distinct indices n_1, \dots, n_k , the converse to the Borel–Cantelli lemma would imply $\mu(\mathcal{E}_g(\mathcal{B})) = 1$. In general, this independence condition need not hold for the family \mathcal{B} , but the spectral theory of $\mathcal{L} = L^2(\mathcal{D}^1, \mu)$ guarantees that a weaker form of *quasi-independence* does:

Proposition 3.5. *Let $B_1 \supset B_2 \supset \dots$ be a sequence of μ -measurable spherical subsets $B_n \subset \mathcal{D}^1$. For $n \geq 1$ set $E_n = g_n^{-1}(B_n)$. Then for all $N > M \geq 1$ we have*

$$\sum_{m,n=M}^N \left(\mu(E_m \cap E_n) - \mu(E_m)\mu(E_n) \right) \ll_\mu \sum_{n=M}^N \mu(B_n).$$

Proof. For $n \geq 1$, let $f_n = \chi_{B_n}$ be the characteristic function for the n th target, and note that f_n is spherical. Consider $f_n^0 = \langle f_n, 1 \rangle = \mu(B_n)$ and $f'_n = f_n - f_n^0 \in \mathcal{L}_0^\perp$. We have

$$\begin{aligned} \|f'_n\|_2^2 &= \langle f_n - f_n^0, f_n - f_n^0 \rangle = \langle f_n, f_n \rangle - \mu(B_n)\langle 1, f_n \rangle - \mu(B_n)\langle f_n, 1 \rangle + \langle f_n^0, f_n^0 \rangle \\ &= \mu(B_n) - \mu(B_n)^2 \leq \mu(B_n). \end{aligned}$$

Observe that $(g_{-n} \cdot f_n)(\xi) = \chi_{B_n}(g_n \xi)$ for all $\xi \in \mathcal{D}^1$, showing that $g_{-n} \cdot f_n = \chi_{E_n}$. Hence

$$\begin{aligned} \mu(E_m \cap E_n) &= \langle g_{-m} \cdot f_m, g_{-n} \cdot f_n \rangle = \langle f_m^0, f_n^0 \rangle + \langle g_{n-m} f'_m, f'_n \rangle \\ &= \mu(B_m)\mu(B_n) + \langle g_{n-m} f'_m, f'_n \rangle \end{aligned}$$

for all $m, n \geq 1$. Using equation (3.1) and the fact $\mu(E_n) = \mu(B_n)$, it follows that

$$R_{m,n} := \left(\mu(E_m \cap E_n) - \mu(E_m)\mu(E_n) \right) \ll \sqrt{\mu(B_m)\mu(B_n)} |n-m| e^{-|n-m|\sigma}$$

whenever $|n-m| \geq 1$. On the other hand, $R_{n,n} = \mu(E_n)(1 - \mu(E_n)) \leq \mu(E_n)$ for all n . Since the targets are shrinking, we also have $\mu(B_m) \leq \mu(B_n)$ whenever $n \leq m$. Combining these observations, we find that

$$\begin{aligned} \sum_{m,n=M}^N R_{m,n} &= \sum_{n=M}^N R_{n,n} + 2 \sum_{n=M}^N \sum_{i=1}^{N-n} R_{n,n+i} \\ &\ll \sum_{n=M}^N \mu(E_n) + 2 \sum_{n=M}^N \sum_{i=1}^{N-n} \sqrt{\mu(B_n)\mu(B_{n+i})} i e^{-i\sigma} \\ &\leq \sum_{n=M}^N \mu(B_n) + 2 \sum_{n=M}^N \mu(B_n) \sum_{i=1}^{N-n} i e^{-i\sigma} \\ &\leq \sum_{n=M}^N \mu(B_n) \left(1 + 2 \sum_{i=1}^{\infty} i e^{-i\sigma} \right). \end{aligned} \quad \square$$

3.3 Hitting the targets

We now use the estimates from §§3.1–3.2 to obtain our main result on shrinking targets. Let \mathcal{B} be any family of spherical μ -measurable shrinking targets $B_1 \supset B_2 \supset \dots$ in \mathcal{D}^1 , where μ remains the arbitrary G -invariant measure specified at the start of §3. Set $E_n = g_n^{-1}(B_n)$ and consider the associated hitting set for \mathcal{B} :

$$\mathcal{E}_g(\mathcal{B}) = \{\xi \in \mathcal{D}^1 \mid \{n \in \mathbb{N} \mid g_n \xi \in B_n\} \text{ is infinite}\} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n.$$

Proof of Theorem 1.1.1. Suppose $\sum_{n=1}^{\infty} \mu(B_n) < \infty$. Then we have

$$\mu(\mathcal{E}_g(\mathcal{B})) = \mu\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n\right) \leq \inf_{k \geq 1} \mu\left(\bigcup_{n=k}^{\infty} E_n\right) \leq \inf_{k \geq 1} \sum_{n=k}^{\infty} \mu(B_n) = 0. \quad \square$$

Proof of Theorem 1.1.2. Suppose that $\sum_{n=1}^{\infty} \mu(B_n) = \infty$. It has long been known, going back to the work of Schmidt, that the conclusion $\mu(\mathcal{E}_g(\mathcal{B})) = 1$ may be deduced from this hypothesis and Proposition 3.5; for example, [Spr, Chapter 1, Lemma 10] gives an even stronger conclusion. We include a brief proof here for completeness. Let $\Phi_N(x) = \sum_{n=1}^N \chi_{E_n}$ and $S_N = \int \Phi_N d\mu = \sum_{n=1}^N \mu(B_n)$. Hence $\xi \in \mathcal{E}_g(\mathcal{B})$ iff $\{\Phi_N(\xi)\}$ is unbounded. Using Proposition 3.5, we see that

$$\begin{aligned} \int (\Phi_N(\xi) - S_N)^2 d\mu(\xi) &= \int \left(\sum_{m,n=1}^N \chi_{E_m} \chi_{E_n} - 2\mu(E_m) \chi_{E_n} + \mu(E_m) \mu(E_n) \right) d\mu \\ &= \sum_{m,n=1}^N \left(\mu(E_m \cap E_n) - \mu(E_m) \mu(E_n) \right) \ll_{\mu} \sum_{n=1}^N \mu(B_n) = S_N. \end{aligned}$$

Therefore $\|\Phi_N - S_N\|_2 \ll_{\mu} \sqrt{S_N}$. Setting $\Psi_N = \frac{1}{S_N} \Phi_N$, it follows that

$$\|\Psi_N - 1\|_2 \ll_{\mu} \frac{1}{\sqrt{S_N}}.$$

Since $S_N \rightarrow \infty$ by assumption, we conclude that Ψ_N converges in \mathcal{L} to the constant function 1. After passing to a subsequence N_i , it follows that $\Psi_{N_i}(\xi) \rightarrow 1$ for μ -almost-every $\xi \in \mathcal{D}^1$ (see, for example, [Fol, Corollary 2.32]). But by definition this implies $\Phi_{N_i}(\xi) \rightarrow \infty$ and hence $\xi \in \mathcal{E}_g(\mathcal{B})$. Therefore $\mathcal{E}_g(\mathcal{B})$ has full measure, as claimed. \square

Proof of Theorem 1.1.3. Suppose the sequence $\{n\mu(B_n)\}_{n \in \mathbb{N}}$ is unbounded. Let f_n be the normalized characteristic function $f_n = \frac{1}{\mu(B_n)} \chi_{B_n}$. For the averaging operator β_n from (3.2), we have that

$$\beta_n(f_n)(\xi) = \frac{1}{n} \sum_{i=1}^n f_n(g_i \xi) = \frac{\#\{1 \leq i \leq n \mid g_i \xi \in B_n\}}{n\mu(B_n)}.$$

Since $f_n^0 = \langle f_n, 1 \rangle = 1$ and $\|f_n\|_2 = \frac{1}{\mu(B_n)} \|\chi_{B_n}\|_2 = 1/\sqrt{\mu(B_n)}$, Theorem 3.3 implies that

$$\|\beta_n(f_n) - 1\|_2 \ll_{\mu} \frac{\|f_n\|_2}{n^{1/2}} = \frac{1}{\sqrt{n\mu(B_n)}}.$$

Choosing a subsequence n_j so that $n_j \mu(B_{n_j}) \rightarrow \infty$, it follows that $\beta_{n_j}(f_{n_j})$ converges to the constant function 1 in \mathcal{L} . Passing to a further subsequence, for μ -almost-every $\xi \in \mathcal{D}^1$ we conclude that

$$\lim_{j \rightarrow \infty} \frac{\#\{1 \leq i \leq n_j \mid g_i \xi \in B_{n_j}\}}{n_j \mu(B_{n_j})} = \lim_{j \rightarrow \infty} \beta_{n_j}(f_{n_j})(\xi) = 1. \quad \square$$

3.4 Always hitting the targets

We next turn to [Theorem 1.4](#) and the question of when differentials will eventually always hit the targets. Again, fix a shrinking family $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ of μ -measurable spherical targets. Let $f_n = \chi_{B_n}$ be the associated characteristic functions. For $n, m \in \mathbb{N}$, let

$$W_{m,n} = \{\xi \in \mathcal{D}^1 \mid \{g_1 \xi, \dots, g_m \xi\} \cap B_n = \emptyset\}$$

be the set of differentials whose first m iterates miss the target B_n . Then

$$\mathcal{E}_g^a(\mathcal{B}) = \{\xi \in \mathcal{D}^1 \mid \xi \notin W_{n,n} \text{ for all sufficiently large } n\} = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (\mathcal{D}^1 - W_{n,n}). \quad (3.6)$$

We know from [Theorem 3.3](#) that $\beta_m(f_n)$ converges to the constant function $\mu(B_n) = \langle f_n, 1 \rangle$ as $m \rightarrow \infty$. To analyze those differentials for which this convergence is poor, for each $\kappa > 1$ and $m, n \in \mathbb{N}$ let us set

$$Y_{m,n}^{\kappa} = \left\{ \xi \in \mathcal{D}^1 \mid \frac{\#\{1 \leq i \leq m \mid g_i \xi \in B_n\}}{m} = \beta_m(f_n)(\xi) \notin [\kappa^{-1} \mu(B_n), \kappa \mu(B_n)] \right\}.$$

Now, for each $\xi \in W_{m,n}$ we have $\beta_m(f_n)(\xi) = 0$ by definition of β_m and $W_{m,n}$. Therefore

$$\|\beta_m(f_n) - \mu(B_n)\|_2^2 \geq \int_{W_{m,n}} |\beta_m(f_n) - \mu(B_n)|^2 d\mu = \mu(W_{m,n}) \mu(B_n)^2.$$

For each $\xi \in Y_{m,n}^{\kappa}$ we similarly have $|\beta_m(f_n)(\xi) - \mu(B_n)| \geq (\kappa - 1) \kappa^{-1} \mu(B_n)$ and hence

$$\|\beta_m(f_n) - \mu(B_n)\|_2^2 \geq \int_{Y_{m,n}^{\kappa}} |\beta_m(f_n) - \mu(B_n)|^2 d\mu \geq \mu(Y_{m,n}^{\kappa}) \frac{(\kappa - 1)^2}{\kappa^2} \mu(B_n)^2.$$

Since $\langle f_n, 1 \rangle = \mu(B_n) = \|f_n\|_2^2$, the effective mean ergodic theorem ([Theorem 3.3](#)) now gives

$$\begin{aligned} \mu(W_{m,n}) &\leq \frac{\|\beta_m(f_n) - \mu(B_n)\|_2^2}{\mu(B_n)^2} \ll_{\mu} \frac{\|f_n\|_2^2}{m \mu(B_n)^2} = \frac{1}{m \mu(B_n)}, \quad \text{and} \\ \mu(Y_{m,n}^{\kappa}) &\leq \frac{\kappa^2 \|\beta_m(f_n) - \mu(B_n)\|_2^2}{(\kappa - 1)^2 \mu(B_n)^2} \ll_{\mu} \frac{\kappa^2}{m (\kappa - 1)^2 \mu(B_n)}. \end{aligned} \quad (3.7)$$

Proof of [Theorem 1.4.1](#). Suppose n_j is an increasing sequence in \mathbb{N} with $\{(n_j \mu(B_{n_{j+1}}))^{-1}\}$ summable. Notice that whenever $n_j \leq n \leq n_{j+1}$, we have $W_{n,n} \subset W_{n_j, n_{j+1}}$ by definition of $W_{n,m}$ and the fact $B_{n_{j+1}} \subset B_n$. Consequently, using [\(3.6\)](#) we may write

$$\mathcal{D}^1 - \mathcal{E}_g^a(\mathcal{B}) = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} W_{n,n} \subset \bigcap_{k=1}^{\infty} \bigcup_{n_j \geq k} W_{n_j, n_{j+1}}.$$

The always hitting set $\mathcal{E}_g^a(\mathcal{B})$ therefore has full measure since by (3.7) its complement has

$$\mu(\mathcal{D}^1 - \mathcal{E}_g^a(\mathcal{B})) \leq \inf_{k \geq 1} \left(\sum_{n_j \geq k} \mu(W_{n_j, n_{j+1}}) \right) \leq \inf_{k \geq 1} \left(\sum_{n_j \geq k} \frac{1}{n_j \mu(B_{n_{j+1}})} \right) = 0. \quad \square$$

Proof of Theorem 1.4.2. Suppose an increasing sequence n_j in \mathbb{N} with $\{(n_j \mu(B_{n_{j+1}}))^{-1}\}$ summable satisfies $n_{j+1} \mu(B_{n_j}) \leq \lambda n_j \mu(B_{n_{j+1}})$ for some $\lambda > 1$. For any $\kappa > 1$, let

$$Y^\kappa = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \left\{ \xi \in \mathcal{D}^1 \mid \beta_n(f_n)(\xi) \notin [\kappa^{-1} \mu(B_n), \kappa \mu(B_n)] \right\} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} Y_{n,n}^\kappa$$

denote the set of differentials ξ such that for each $k \in \mathbb{N}$ there is some $n \geq k$ so that $\beta_n(f_n)(\xi)$ lies outside $[\kappa^{-1} \mu(B_n), \kappa \mu(B_n)]$. Then for each $\xi \notin Y^\kappa$ there exists some $K = K(\xi)$ such that for all $n \geq K$ we have

$$\frac{\#\{1 \leq i \leq n \mid g_i \xi \in B_n\}}{n \mu(B_n)} = \frac{\beta_n(f_n)(\xi)}{\mu(B_n)} \in [\kappa^{-1}, \kappa].$$

Hence to prove the theorem it suffices to show $\mu(Y^{2\lambda}) = 0$.

For this, observe that for each $n_j \leq n \leq n_{j+1}$, the averaged function $\beta_n(f_n)$ satisfies

$$\frac{n_j}{n_{j+1}} \beta_{n_j}(f_{n_{j+1}}) \leq \frac{n_j}{n} \beta_{n_j}(f_n) \leq \beta_n(f_n) \leq \frac{n_{j+1}}{n} \beta_{n_{j+1}}(f_n) \leq \frac{n_{j+1}}{n_j} \beta_{n_{j+1}}(f_{n_j})$$

by the definition of β_m and the fact that $0 \leq f_{n_{j+1}} \leq f_n \leq f_{n_j}$. Thus when $\beta_n(f_n)(\xi) > \kappa \mu(B_n)$ we find that

$$\beta_{n_{j+1}}(f_{n_j})(\xi) \geq \frac{n_j}{n_{j+1}} \beta_n(f_n)(\xi) > \frac{n_j}{n_{j+1}} \kappa \mu(B_n) \geq \frac{\kappa}{\lambda} \mu(B_{n_j})$$

(here we have used our hypothesis $\mu(B_n) \geq \mu(B_{n_{j+1}}) \geq \frac{n_{j+1}}{\lambda n_j} \mu(B_{n_j})$). On the other hand, when $\beta_n(f_n)(\xi) < \kappa^{-1} \mu(B_n)$ we have

$$\beta_{n_j}(f_{n_{j+1}})(\xi) \leq \frac{n_{j+1}}{n_j} \beta_n(f_n)(\xi) < \frac{n_{j+1}}{n_j \kappa} \mu(B_n) \leq \frac{\lambda}{\kappa} \mu(B_{n_{j+1}}).$$

Setting $\kappa = 2\lambda > 1$, it follows that $Y_{n,n}^{2\lambda} \subset Y_{n_j, n_{j+1}}^2 \cup Y_{n_{j+1}, n_j}^2$ whenever $n_j \leq n \leq n_{j+1}$. Since the numbers n_j are increasing and the targets B_{n_j} are shrinking, we have that $n_j \mu(B_{n_{j+1}}) \leq n_{j+1} \mu(B_{n_j})$ for all $j \in \mathbb{N}$. Therefore our hypothesis implies the sequences $\{(n_j \mu(B_{n_{j+1}}))^{-1}\}_{j \in \mathbb{N}}$ and $\{(n_{j+1} \mu(B_{n_j}))^{-1}\}_{j \in \mathbb{N}}$ are *both* summable. Using these observations together with (3.7), we now conclude the desired estimate

$$\begin{aligned} \mu(Y^{2\lambda}) &= \mu \left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} Y_{n,n}^{2\lambda} \right) \leq \mu \left(\bigcap_{k=1}^{\infty} \bigcup_{n_j \geq k} Y_{n_j, n_{j+1}}^2 \cup Y_{n_{j+1}, n_j}^2 \right) \\ &\leq \inf_{k \geq 1} \sum_{n_j \geq k} \left(\frac{4}{n_j \mu(B_{n_{j+1}})} + \frac{4}{n_{j+1} \mu(B_{n_j})} \right) = 0. \quad \square \end{aligned}$$

4 Logarithm Laws

Here we explicitly consider the Masur–Veech measure λ_α^1 on the principle stratum $\mathcal{Q}^1(\alpha) = \mathcal{D}^1(\alpha)$ where $\alpha = (-1, \dots, -1, 1, \dots, 1; -1)$. Recall from §2.6 that we have used $\mathbf{n} = \pi_*(\lambda_\alpha^1)$ to denote the push-forward of measure under $\pi: \mathcal{Q}^1 \rightarrow \mathcal{M}$. Recall also that $d_{\mathcal{M}}(X, Y)$ denotes the distance between points X and Y in the moduli space \mathcal{M} and that for $X \in \mathcal{M}$ and $\xi \in \mathcal{D}^1$ we have defined

$$d_n(\xi, X) = \min_{0 \leq j \leq n} d_{\mathcal{M}}(\pi(g_j \xi), X) \quad \text{and} \quad \tau_r(\xi, X) = \inf\{n \in \mathbb{N} \mid d_{\mathcal{M}}(\pi(g_n \xi), X) < r\}.$$

Proof of Theorem 1.6. For $\epsilon > 0$ fixed, set $r_m^\pm = m^{\frac{-(1 \pm \epsilon)}{6g-6+2p}}$ and let $B_m^\pm = \pi^{-1}(B_X(r_m^\pm)) \subset \mathcal{Q}^1$ be the preimage of the radius r_m^\pm metric ball about X . Consider the shrinking families $\mathcal{B}^\pm = \{B_m^\pm\}_{m \in \mathbb{N}}$. By Lemma 2.5 we have

$$\lambda_\alpha^1(B_m^\pm) = \mathbf{n}(B_X(r_m^\pm)) \asymp_X (r_m^\pm)^{6g-6+2p} = \frac{1}{m^{1 \pm \epsilon}}.$$

Therefore $\sum_{m=1}^\infty \lambda_\alpha^1(B_m^+) < \infty$ and Theorem 1.1.1 implies that $\lambda_\alpha^1(\mathcal{E}_g(\mathcal{B}^+)) = 0$. The set Z of $\xi \in \mathcal{Q}^1$ such that $\pi(g_k \xi) = X$ of some $k \in \mathbb{N}$ also has measure zero. We claim that each $\xi \notin \mathcal{E}_g(\mathcal{B}^+) \cup Z$ satisfies

$$\frac{\log(d_m(\xi, X))}{\log(m)} \geq \frac{-1 - \epsilon}{6g - 6 + 2p} \quad \text{for all sufficiently large } m. \quad (4.1)$$

Indeed if this is not the case we may choose indices m_j tending to infinity and associated indices $k_j \leq m_j$ such that

$$0 < d_{\mathcal{M}}(\pi(g_{k_j} \xi), X) = \min_{k \leq m_j} d(\pi(g_k \xi), X) = d_{m_j}(\xi, X) < m_j^{\frac{-1 - \epsilon}{6g-6+2p}} = r_{m_j}^+ \leq r_{k_j}^+.$$

These inequalities imply that $g_{k_j} \xi \in B_{k_j}^+$ for each j while also forcing (since $r_{m_j}^+ \rightarrow 0$) the set $\{k_j \mid j \in \mathbb{N}\}$ to be infinite. However, this contradicts $\xi \notin \mathcal{E}_g(\mathcal{B}^+)$, proving the claim.

For \mathcal{B}^- , on the other hand, Theorem 1.4.1 implies that $\mathcal{E}_g^a(\mathcal{B}^-)$ has full measure since

$$\sum_{j=1}^\infty (2^j \lambda_\alpha^1(B_{2^{j+1}}^-))^{-1} \asymp_X \sum_{j=1}^\infty \frac{(2^{j+1})^{1-\epsilon}}{2^j} = 2 \sum_{j=1}^\infty (2^{-\epsilon})^{j+1} < \infty.$$

For each $\xi \in \mathcal{E}_g^a(\mathcal{B}^-)$ we know that $\{g_1 \xi, \dots, g_m \xi\} \cap B_m^-$ is nonempty for all sufficiently large m . But this is equivalent to saying

$$d_m(\xi, X) = \min_{k \leq m} d_{\mathcal{M}}(\pi(g_k \xi), X) < r_m^- = m^{\frac{-1 + \epsilon}{6g-6+2p}} \quad \text{for all sufficiently large } m. \quad (4.2)$$

Since (4.1) and (4.2) each hold for a full measure set, we conclude that

$$\frac{-1 - \epsilon}{6g - 6 + 2p} \leq \liminf_{m \rightarrow \infty} \frac{\log(d_m(\xi, X))}{\log(m)} \leq \limsup_{m \rightarrow \infty} \frac{\log(d_m(\xi, X))}{\log(m)} \leq \frac{-1 + \epsilon}{6g - 6 + 2p}$$

for λ_α^1 -almost every $\xi \in \mathcal{Q}^1$. Since $\epsilon > 0$ was arbitrary, the first claim follows.

For notational convenience, for $y \in \mathbb{R}_+$, let us now define

$$d'_y(\xi, X) = d_{\lfloor y \rfloor}(\xi, X) = \min\{d_{\mathcal{M}}(\pi(g_i \xi), X) \mid i \in \mathbb{Z} \text{ with } 0 \leq i \leq y\}.$$

Notice that $\lim_{y \rightarrow \infty} \frac{\log(d'_y(\xi, X))}{\log(y)} = \lim_{m \rightarrow \infty} \frac{\log(d_m(\xi, X))}{\log(m)}$ since $\frac{\log y}{\log \lfloor y \rfloor} \rightarrow 1$. Also observe from the definition of $\tau_r(\xi, X)$ that

$$\tau_r(\xi, X) \leq y \iff d'_y(\xi, X) < r \quad \text{and} \quad \tau_r(\xi, X) > y \iff d'_y(\xi, X) \geq r. \quad (4.3)$$

By the first result, there is a λ_α^1 -full measure set of $\xi \in \mathcal{Q}^1$ such that for any $\epsilon > 0$ we have

$$\frac{-(1+\epsilon)}{6g-6+2p} \leq \frac{\log(d'_y(\xi, X))}{\log(y)} < \frac{-(1-\epsilon)}{6g-6+2p} \quad \text{for all sufficiently large } y.$$

Applying this to $y = r^{\frac{6g-6+2p}{-(1\pm\epsilon)}}$ (so that $r = y^{\frac{-(1\pm\epsilon)}{6g-6+2p}}$) and invoking (4.3), it follows that

$$r^{\frac{6g-6+2p}{-(1+\epsilon)}} < \tau_r(\xi, X) \leq r^{\frac{6g-6+2p}{-(1-\epsilon)}} \quad \text{for all sufficiently small } r > 0.$$

Rearranging, we conclude that there is a λ_α^1 -full measure set of $\xi \in \mathcal{Q}^1$ for which

$$\frac{6g-6+2p}{1+\epsilon} < \frac{\log(\tau_r(\xi, X))}{\log(1/r)} \leq \frac{6g-6+2p}{1-\epsilon} \quad \text{holds for all sufficiently small } r > 0.$$

Since $\epsilon > 0$ is arbitrary, the result follows. \square

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