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Summing Inflationary Logarithms in Nonlinear Sigma Models

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ABSTRACT

We consider two nonlinear sigma models on de Sitter background which involve the same derivative interactions as quantum gravity but without the gauge issue. The first model contains only a single field, which can be reduced to a free theory by a local field redefinition; the second contains two fields and cannot be so reduced. Loop corrections in both models produce large temporal and spatial logarithms which cause perturbation theory to break down at late times and large distances. Many of these logarithms derive from the "tail" part of the propagator and can be summed using a variant of Starobinsky's stochastic formalism involving a curvature-dependent effective potential. The remaining logarithms derive from the ultraviolet and can be summed using a variant of the renormalization group based on a special class of curvature-dependent renormalizations. Explicit results are derived at 1-loop and 2-loop orders.

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1 Introduction

In their pioneering work on perturbation theory in nontrivial geometries, DeWitt and Brehme devoted special attention to the behavior of propagators and Green's functions near coincidence [1]. They noted that while the leading singularity is a universal function of the invariant separation, there is a less singular part which depends on the geometry and on the properties of the field. They called this sub-dominant singularity the "tail" term.

The tail terms of certain fields become maximally strong during the accelerated expansion of inflation. For many purposes the background geometry of inflation can be taken to be de Sitter,

$$ds^2 = a^2(\eta)\Big[-d\eta^2 + d\vec{x}\cdot d\vec{x}\Big] = -dt^2 + a^2 d\vec{x}\cdot d\vec{x} \quad , \quad a(\eta) = -\frac{1}{H\eta} = e^{Ht} \; . \tag{1}$$

On $D = 4$ dimensional de Sitter background the propagator of a massless, minimally coupled scalar, in Bunch-Davies vacuum, is,

$$i\Delta(x;x')\Big\vert_{D=4} = \frac{1}{4\pi^2}\Bigg\{\frac{1}{aa'\Delta x^2} - \frac{H^2}{2}\ln\Big[\frac{1}{4}H^2\Delta x^2\Big]\Bigg\} \; , \tag{2}$$

where the Poincaré interval is,

$$\Delta x^2(x;x') \equiv \Big\Vert\vec{x}-\vec{x}'\Big\Vert^2 - \Big(|\eta-\eta'|-i\epsilon\Big)^2 \; . \tag{3}$$

The tail term of expression (2) is the part involving the logarithm.

A curious feature of the massless, minimally coupled scalar tail is that its coincidence limit grows with time [2–4],

$$i\Delta(x;x) = \Big(\text{Divergent constant}\Big) + \frac{H^2}{4\pi^2}\ln(a) \; . \tag{4}$$

Expression (4) was the first example of a general sort of secular effect encountered in loop corrections involving interactions between nearly massless and minimally coupled scalars [5–9]. These secular logarithms attracted much attention during the opening decade of the 21st century because they have the potential to enhance loop corrections to the power spectrum [10–24].

A fascinating aspect of the secular logarithms encountered in loop corrections to scalar potential models,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi g^{\mu\nu}\sqrt{-g} - V(\Phi)\sqrt{-g} \; , \tag{5}$$

is that the steady growth of $\ln(a) = Ht$ must eventually overwhelm even the smallest loop-counting parameter. One cannot conclude from this that loop corrections ever become large, just that the standard loop expansion breaks down. Some sort of nonperturbative resummation is required to determine what actually happens.

Starobinsky quite early developed a stochastic formalism which not only predicts the leading logarithms of scalar potential models at each order in perturbation theory [25], but also gives the late time form in those cases for which a static limit is approached [26]. Starobinsky's formalism is based on replacing the full field operator $\Phi(t, \vec{x})$ with a stochastic field $\varphi(t, \vec{x})$ which commutes with itself $[\varphi(t, \vec{x}), \varphi(t', \vec{x}')] = 0$, and whose correlators are completely free of ultraviolet divergences. This stochastic field $\varphi(t, \vec{x})$ is constructed from the same free creation and annihilation operators that appear in $\Phi(t, \vec{x})$ in such a way that the two fields produce the same leading logarithms at each order in perturbation theory. The Heisenberg field equation for Φ gives rise to a Langevin equation for φ (which we express in co-moving coordinates),

$$\frac{\delta S[\Phi]}{\delta \Phi(x)} = \partial_\mu \Big[\sqrt{-g}\, g^{\mu\nu} \partial_\nu \Phi\Big] - V'(\Phi)\sqrt{-g} \longrightarrow 3Ha^3 \Big[\dot{\varphi} - \dot{\varphi}_0\Big] - V'(\varphi) a^3 \; . \quad (6)$$

Here $\varphi_0(t, \vec{x})$ is a truncation of the Yang-Feldman free field with the ultraviolet excised and the mode function taken to its limiting infrared form,

$$\varphi_0(t, \vec{x}) \equiv \int \frac{d^3k}{(2\pi)^3} \theta\Big(aH - k\Big) \frac{\theta(k - H)H}{\sqrt{2k^3}} \Big\{ \alpha_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + \alpha^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} \Big\} \; , \qquad (7)$$

One derives (6) by first integrating the exact field equation to reach the Yang-Feldman form. One then notes that reaching leading logarithm order requires each free field to contribute an infrared logarithm, so there will be no change to correlators, at leading logarithm order, if the full free field mode sum is replaced by (7). Differentiating this truncated Yang-Feldman equation gives Starobinsky's Langevin equation [27].

The problem of summing up large logarithms in flat space scattering amplitudes seems similar, and that has prompted particle theorists to try applying renormalization group methods to understanding the evolution of cosmological correlators [28–39]. However, the problems with this approach become obvious upon closer examination of the analogy on which it is based. The renormalization group of flat space describes how correlators change when the positions of field operators are adiabatically expanded (or compressed) by some constant:

$$\text{Renormalization Group}: \qquad x^\mu \longrightarrow A \times x^\mu \; . \tag{8}$$

What we really want to know in cosmology is how correlators change when *infinitesimal intervals* are expanded by the *time-dependent* scale factor:

$$\text{Cosmological Evolution}: \qquad dx^\mu \longrightarrow a(\eta) \times dx^\mu \; . \tag{9}$$

It is not clear how to relate the two processes, and simple correspondences such as $A \longrightarrow a(\eta)$, or the renormalization scale $\mu \longrightarrow H$, can easily be shown to fail by direct computation [40]. Another crucial obstacle is that the leading logarithms of scalar potential models arise entirely from the infrared, without regard to renormalization. And the fact is that, despite years of heroic effort by talented physicists [41,42], no one has yet been able to devise a version of the renormalization group which gives complete agreement for the leading logarithms of scalar potential models [43].

Massless, minimally coupled scalars also engender large logarithms when they interact with fermions [44] and with photons [45–48]. In both cases the other fields do not themselves generate large logarithms, but their dynamics modify the ways in which these logarithms manifest. Such modifications derive as much from the ultraviolet as from the infrared, so no simple truncation procedure captures the correct result. However, integrating out the other fields produces a scalar potential model whose large logarithms are correctly captured by the stochastic formalism [44, 49].

On a general cosmological background it turns out that dynamical gravitons obey the same equation as the massless, minimally coupled [50], so loop corrections from inflationary gravitons should also induce large logarithms. Of course the computations are much more difficult but a number of 1PI (one-particle-irreducible) 1-point and 2-point functions have been evaluated at 1-loop and 2-loop orders in pure gravity [51–54] and in gravity plus various matter theories [55–62]. When the 1PI 2-point functions are used to quantum-correct the linearized effective field equations one often (but not always) finds large logarithmic corrections to mode functions and to exchange potentials [63–69]. These are *very* challenging calculations, and it has been suggested that some of them may be gauge artifacts [70–77]. A procedure has been developed to purge gauge dependence from 1PI 2-point functions [78, 79], and its implementation is being undertaken as rapidly as the formidable computational challenges permit [80].

Assuming the large logarithms of inflationary gravitons are real, the question is how they can be re-summed. The derivative couplings of gravity pose an obstacle to a completely stochastic explanation of these results because derivatives preclude every free field from inducing a large logarithm, which was an essential part of the proof that the stochastic formalism works for scalar potential models [27]. Further, direct studies have shown that some of the logarithms cannot be explained using the stochastic formalism [81], nor are all of the logarithms due to the tail part of the graviton propagator [82]. What we need is a simple format in which the complications of derivative interactions can be sorted out without intricate computations which require a year or more to complete.

Nonlinear sigma models would seem to provide a natural paradigm for derivative interactions. These models consist of normal scalar kinetic terms which are multiplied by functions of undifferentiated scalars, giving rise to the same sort of derivative interactions as quantum gravity but without the distractions of tensor indices and gauge fixing. Early work focused on deriving a completely stochastic representation of the large logarithms induced by these models [27], and that approach has been extensively pursued by Kitamoto and Kitazawa [83–85]. We have thought it good to revisit this problem after the realization that no completely stochastic approach can capture all the large logarithms induced inflationary gravitons [81,82]. The point of this paper is to demonstrate that the large logarithmics of nonlinear sigma models on de Sitter can be explained by combining a variant of Starobinsky's stochastic formalism with a variant of the renormalization group.

This paper consists of six sections, of which the first is nearly done. In section 2 we introduce the two nonlinear sigma models that will be studied. Section 3 works out 1-loop corrections to the mode functions and exchange potentials of the first model, as well as to 1-loop and 2-loop expectation values of the field and its square. The same things are computed for the second model in section 4. Section 5 collects the various large logarithms exposed by all this work. We then demonstrate that many of these large logarithms arise from stochastic effects associated with a curvature-dependent effective potential induced by the kinetic terms. The remaining large logarithms follow from employing the Callan-Symanzik equation to a special class of counterterms that can be viewed as curvature-dependent renormalizations of the bare theories. Our conclusions comprise section 6, particularly the lessons for quantum gravity.

2 Two Nonlinear Sigma Models

This section introduces the two models upon which this study is based. The first is a single field model which gives a free theory by a local field redefinition; the second is a model based on two fields which is fundamentally interacting. For each model we give the bare Lagrangian and the first two variations of the action. We also present the Feynman rules and some important identities for the coincidence limits of the propagator.

2.1 Single Field Model

The Lagrangian of the first model we will study is,

$$\mathcal{L} = -\frac{1}{2} f^2(\Phi) \partial_\mu \Phi \partial_\nu \Phi g^{\mu\nu} \sqrt{-g} \; . \tag{10}$$

A nonlinear sigma model based on a single field can be reduced to free theories by local field redefinitions. For the case of (10) the free field $\Psi(x)$ obeys,

$$d\Psi \equiv f(\Phi) d\Phi \qquad \Longrightarrow \qquad \mathcal{L} = -\frac{1}{2} \partial_\mu \Psi \partial_\nu \Psi g^{\mu\nu} \sqrt{-g} \; . \tag{11}$$

Of course the existence of such a local field redefinition means that the flat space S-matrix is unity but interactions can still cause interesting changes to the kinematics of free fields, and to the evolution of the Φ background. Quantifying these changes at 1-loop and 2-loop orders will teach us much.

We must select the function $f(\Phi)$ in order to define a specific model. The simplest choice involves a single, dimensionful coupling constant λ,

$$f(\Phi) = 1 + \frac{\lambda}{2} \Phi \quad \Longrightarrow \quad \Psi[\Phi] = \Phi + \frac{\lambda}{4} \Phi^2 \quad \Longleftrightarrow \quad \Phi[\Psi] = \frac{2}{\lambda} \Big[\sqrt{1 + \lambda \Psi} - 1 \Big] \; . \tag{12}$$

With this choice of $f(\Phi)$ the Heisenberg field equation is,

$$\frac{\delta S[\Phi]}{\delta \Phi(x)} = \Big(1 + \frac{1}{2} \lambda \Phi \Big) \partial_\mu \Big[(1 + \frac{1}{2} \lambda \Phi \Big) \sqrt{-g} g^{\mu\nu} \partial_\nu \Phi \Big] \; . \tag{13}$$

We will also sometimes need the second variation,

$$\frac{\delta^2 S[\Phi]}{\delta \Phi(x) \delta \Phi(x')} = \lambda \delta^D(x - x') \partial_\mu \Big[\Big(1 + \frac{1}{2} \lambda \Phi \Big) \sqrt{-g} g^{\mu\nu} \partial_\nu \Phi \Big]$$
$$- \frac{1}{4} \lambda^2 \delta^D(x - x') \sqrt{-g} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \partial_\mu \Big[\Big(1 + \frac{1}{2} \lambda \Phi \Big)^2 \sqrt{-g} g^{\mu\nu} \partial_\nu \delta^D(x - x') \Big] . \tag{14}$$

In D spacetime dimensions the propagators of both the Φ and the Ψ fields obey the same equation,

$$\partial^\mu \Big[a^{D-2} \partial_\mu i\Delta(x;x')\Big] \equiv \mathcal{D} i\Delta(x;x') = i\delta^D(x-x') \; . \tag{15}$$

The solution is [5, 6],

$$i\Delta(x;x') = F\Big(y(x;x')\Big) + k\ln(aa') \qquad , \qquad k \equiv \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \; , \tag{16}$$

where the de Sitter length function is $y(x;x') \equiv aa'H^2\Delta x^2(x;x')$ and the first derivative of the function $F(y)$ is,

$$F'(y) = -\frac{H^{D-2}}{4(4\pi)^{\frac{D}{2}}} \Bigg\{ \Gamma\Big(\frac{D}{2}\Big)\Big(\frac{4}{y}\Big)^{\frac{D}{2}} + \Gamma\Big(\frac{D}{2}+1\Big)\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} + \sum_{n=0}^{\infty} \Bigg[\frac{\Gamma(n+\frac{D}{2}+2)}{\Gamma(n+3)}\Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2} - \frac{\Gamma(n+D)}{\Gamma(n+\frac{D}{2}+1)}\Big(\frac{y}{4}\Big)^{n}\Bigg]\Bigg\}. \tag{17}$$

The coincidence limits of the propagator and its first two derivatives are,

$$i\Delta(x;x) = k\Big[-\pi\cot\Big(\frac{D\pi}{2}\Big) + 2\ln(a)\Big] \quad , \quad \partial_\mu i\Delta(x;x')\Big|_{x'=x} = kHa\delta^0_{\ \mu} \; , \tag{18}$$

$$\partial_\mu \partial'_\nu i\Delta(x;x')\Big|_{x'=x} = -\Big(\frac{D-1}{D}\Big)kH^2 g_{\mu\nu} \quad , \quad \partial_\mu i\Delta(x;x) = 2kHa\delta^0_{\ \mu} \; . \tag{19}$$

2.2 Two Field Model

The simplest truly interacting nonlinear sigma model would seem to be,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A \partial_\nu A g^{\mu\nu}\sqrt{-g} - \frac{1}{2}\Big(1+\frac{1}{2}\lambda A\Big)^2 \partial_\mu B \partial_\nu B g^{\mu\nu}\sqrt{-g} \; . \tag{20}$$

The first variations of its action are,

$$\frac{\delta S[A,B]}{\delta A(x)} = \partial_\mu\Big[\sqrt{-g}g^{\mu\nu}\partial_\nu A\Big] - \frac{1}{2}\lambda\Big(1+\frac{1}{2}\lambda A\Big)\partial_\mu B\partial_\nu B g^{\mu\nu}\sqrt{-g} \; , \tag{21}$$

$$\frac{\delta S[A,B]}{\delta B(x)} = \partial_\mu\Big[\Big(1+\frac{1}{2}\lambda A\Big)^2\sqrt{-g}g^{\mu\nu}\partial_\nu B\Big] \; . \tag{22}$$

And the second variations work out to be,

$$\frac{\delta^2 S[A,B]}{\delta A(x)\delta A(x')} = \partial_\mu \Bigl[\sqrt{-g}g^{\mu\nu}\partial_\nu \delta^D(x-x')\Bigr] \\ -\frac{1}{4}\lambda^2 \delta^D(x-x')\partial_\mu B \partial_\nu B g^{\mu\nu}\sqrt{-g} \; , \tag{23}$$

$$\frac{\delta^2 S[A,B]}{\delta B(x)\delta B(x')} = \partial_\mu \Bigl[\Bigl(1+\frac{1}{2}\lambda A\Bigr)^2 \sqrt{-g}g^{\mu\nu}\partial_\nu \delta^D(x-x')\Bigr] \; . \tag{24}$$

The propagators of both A and B are the same as $i\Delta(x;x')$ given in expression (16). The other Feynman rules for the bare action are the $\lambda A \partial B \partial B$ and $\lambda^2 A^2 \partial B \partial B$ vertices. All are depicted in Figure 1.

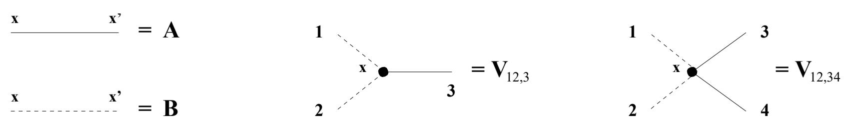

Figure 1: The Feynman rules of the A-B model. A lines are solid whereas B lines are dashed, and both propagators have the functional form (16).

3 Large Logarithms in the Single Field Model

In this section we calculate 1-loop and 2-loop corrections to a variety of quantities in the single field model. We begin with the 1-loop self-mass, which is then used to compute 1-loop corrections to the plane wave mode function and the response to a point source. The section closes with a evaluation of the expectation values of the field and its square at 1-loop and 2-loop orders.

3.1 The Self-Mass

The 1PI 2-point function can be expressed as,

$$-iM^2(x;x') \equiv \Bigl\langle \Omega \Bigl| \frac{i\delta S[\Phi]}{\delta \Phi(x)} \frac{i\delta S[\Phi]}{\delta \Phi(x')} + \frac{i\delta^2 S[\Phi]}{\delta \Phi(x)\delta \Phi(x')} \Bigr| \Omega \Bigr\rangle \; . \tag{25}$$

Substituting (13) into the first term and using (12) gives,

$$-iM^2_{\Phi 3}(x;x') = \Bigl(\frac{i\lambda}{2}\Bigr)^2 \Bigl\{ \frac{1}{2}\mathcal{D}\mathcal{D}' \Bigl[i\Delta(x;x')\Bigr]^2$$

$$-\mathcal{D}\Big[a'^{D-2}\partial'^{\rho} i\Delta(x;x')\partial'_{\rho} i\Delta(x;x')\Big] - \mathcal{D}'\Big[a^{D-2}\partial^{\mu} i\Delta(x;x')\partial_{\mu} i\Delta(x;x')\Big]$$
$$+2(aa')^{D-2}\partial^{\mu}\partial'^{\rho} i\Delta(x;x')\partial_{\mu}\partial'_{\rho} i\Delta(x;x')\Big\}. \quad (26)$$

The second term in (25) comes from the second variation (14),

$$-iM^2_{\Phi 4}(x;x') = \frac{i\lambda^2}{4}\Big\{-\delta^D(x-x')\, a^{D-2}\partial'_{\mu}\partial^{\mu} i\Delta(x;x')$$
$$+\partial^{\mu}\Big[i\Delta(x;x)a^{D-2}\partial_{\mu}\delta^D(x-x')\Big]\Big\}. \quad (27)$$

The 3-point contribution (26) can be reduced by a series of partial integrations whose general form will occur repeatedly. We will present them this once in detail and not again,

$$a^{D-2}\partial^{\mu} i\Delta(x;x')\partial_{\mu} i\Delta(x;x') = \partial^{\mu}\Big[a^{D-2} i\Delta(x;x')\partial_{\mu} i\Delta(x;x')\Big]$$
$$-i\Delta(x;x')\mathcal{D} i\Delta(x;x') = \frac{1}{2}\mathcal{D}\Big[i\Delta(x;x')\Big]^2 - i\Delta(x;x) i\delta^D(x-x')\ , \quad (28)$$
$$2(aa')^{D-2}\partial^{\mu}\partial'^{\rho} i\Delta(x;x')\partial_{\mu}\partial'_{\rho} i\Delta(x;x') = 2\partial^{\mu}\Big[(aa')^{D-2}\partial'^{\rho} i\Delta(x;x')$$
$$\times\partial_{\mu}\partial'_{\rho} i\Delta(x;x')\Big] - 2a'^{D-2}\partial'^{\rho} i\Delta(x;x')\partial'_{\rho}\mathcal{D} i\Delta(x;x')\ , \quad (29)$$
$$= \mathcal{D}\Big[a'^{D-2}\partial'^{\rho} i\Delta(x;x')\partial'_{\rho} i\Delta(x;x')\Big] - 2a'^{D-2}\partial'^{\rho} i\Delta(x;x')\partial'_{\rho} i\delta^D(x-x')\ , \quad (30)$$
$$= \frac{1}{2}\mathcal{D}\mathcal{D}'\Big[i\Delta(x;x')\Big]^2 - \mathcal{D}\Big[i\Delta(x;x) i\delta^D(x-x')\Big] - 2kHa^{D-1}\partial_0 i\delta^D(x-x')\ . \quad (31)$$

Reducing the four parts of (26) gives,

$$-iM^2_{\Phi 3}(x;x') = \frac{\lambda^2}{4}\Big\{-i\Delta(x;x)\mathcal{D}\Big[i\delta^D(x-x')\Big] + 2kHa^{D-1}\partial_0 i\delta^D(x-x')\Big\}. \quad (32)$$

Applying the similar reductions to (27) allows the 4-point contribution to be expressed as,

$$-iM^2_{\Phi 4}(x;x') = \frac{\lambda^2}{4}\Big\{i\Delta(x;x)\mathcal{D}\Big[i\delta^D(x-x')\Big]$$
$$-2kHa^{D-1}\partial_0 i\delta^D(x-x') + (D-1)kH^2 a^D i\delta^D(x-x')\Big\}. \quad (33)$$

When (32) and (33) are added, all the divergences cancel and we can take the unregulated limit for the final result,

$$-iM^2_\Phi(x;x') = \frac{3\lambda^2 H^4 a^4}{32\pi^2}\, i\delta^4(x-x') \; . \tag{34}$$

Equation (34) corresponds to a tachyonic mass of $m^2_\Phi = -3\lambda^2 H^4/32\pi^2$. Note that the unit S-matrix implied by (12) does not preclude interactions from changing the free field kinematics. We will see that evolution can also occur, and that composite operators still require field strength renormalization.

3.2 1-Loop Mode Function and Exchange Potential

The self-mass supplies the quantum correction to the linearized effective field equation for $\Phi(x)$,

$$\mathcal{D}\Phi(x) - \int d^4x' M^2(x;x')\Phi(x') = \Bigl[\mathcal{D} + \frac{3\lambda^2 H^4 a^4}{32\pi^2} + O(\lambda^4)\Bigr]\Phi(x) = J(x) \; , \tag{35}$$

where $\mathcal{D} \equiv \partial^\mu a^2 \partial_\mu$ is the kinetic operator which was introduced in equation (15). We will study 1-loop corrections to the kinematics of free scalar fields (with $J(x) = 0$) and to the response to a point source (with $J(x) = K a \delta^3(\vec{x})$). It will be useful to consider the scale factor a as the time variable,

$$\mathcal{D} \equiv a^2\Bigl[-\partial_0^2 - 2aH\partial_0 + \nabla^2\Bigr] = a^4 H^2 \Bigl[-a^2\partial_a^2 - 4a\partial_a + \frac{\nabla^2}{a^2H^2}\Bigr] \; . \tag{36}$$

3.2.1 Mode Function

Scalar radiation takes the form,

$$J(x) = 0 \qquad \Longrightarrow \qquad \Phi(x) = u_\Phi(\eta,k) e^{i\vec{k}\cdot\vec{x}} \; , \tag{37}$$

where the mode function $u_\Phi(\eta,k)$ obeys,

$$\Bigl[a^2\partial_a^2 + 4a\partial_a + \frac{k^2}{a^2H^2} - \frac{3\lambda^2 H^2}{32\pi^2} + O(\lambda^4)\Bigr] u_\Phi(\eta,k) = 0 \; . \tag{38}$$

The canonically normalized solution for Bunch-Davies vacuum is,

$$u_\Phi(\eta,k) = i\sqrt{\frac{\pi}{4Ha^3}}\, H^{(1)}_\nu\Bigl(\frac{k}{aH}\Bigr) \qquad , \qquad \nu \equiv \sqrt{\frac{9}{4} - \frac{m^2_\Phi}{H^2}} \; . \tag{39}$$

The form at late times is,

$$u_\Phi(\eta,k) \longrightarrow \frac{\Gamma(\nu)}{\sqrt{4\pi H a^3}} \Big(\frac{2aH}{k}\Big)^{\nu} \Big\{1 + \frac{1}{\nu{-}1} \Big(\frac{k}{2aH}\Big)^2 + \dots\Big\} \; . \qquad (40)$$

Because $\nu = \frac{3}{2} - \frac{m^2}{3H^2} + O(\lambda^4)$ is greater than $\frac{3}{2}$ for tachyonic masses, we see that the mode function (40) experiences slow growth at late times.

3.2.2 Exchange Potential

The exchange potential is the response to a point source,

$$J(x) = K a \delta^3(\vec{x}) \qquad \Longrightarrow \qquad \Phi(x) = P_\Phi(\eta,r) \; , \qquad (41)$$

where $r \equiv \|\vec{x}\|$ and the potential obeys,

$$a^4 H^2 \Big[-a^2 \partial_a^2 - 4a\partial_a + \frac{\nabla^2}{a^2 H^2} + \frac{3\lambda^2 H^2}{32\pi^2} + O(\lambda^4)\Big] P_\Phi(\eta,r) = K a \delta^3(\vec{x}) \; . \quad (42)$$

The order λ^0 solution and its late time limit are [86],

$$P_0(\eta,r) = \frac{KH}{4\pi} \Big\{ \ln(Hr) + \ln\Big(1{+}\frac{1}{aHr}\Big) - \frac{1}{aHr}\Big\} \; , \qquad (43)$$

$$\longrightarrow \frac{KH}{4\pi} \Big\{ \ln(Hr) - \frac{1}{2(aHr)^2} + \dots \Big\} \; . \qquad (44)$$

The simplest way of solving equation (42) is by using the retarded Green's function for a massive, minimally coupled scalar,

$$m^2 \neq 0 \Longrightarrow G_{\rm ret}(x;x') = -\frac{1}{4\pi} \Big\{ \frac{\delta(\Delta\eta{-}\Delta r)}{a a' \Delta r} - \frac{H^2 \theta(\Delta\eta{-}\Delta r)}{2\Gamma(\frac{1}{2}{+}\nu)\Gamma(\frac{1}{2}{-}\nu)} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{3}{2}{+}\nu{+}n)\Gamma(\frac{3}{2}{-}\nu{+}n)}{n!(n{+}1)!} \Big(\frac{y}{4}\Big)^n \Big\}, \quad (45)$$

where $\nu^2 \equiv \frac{9}{4} - \frac{m^2}{H^2}$ and $y \equiv a a' H^2 \Delta x^2$ is the de Sitter length function. (Note that expression (45) takes the form predicted by DeWitt and Brehme [1] with a universal light-cone singularity plus a mass-dependent tail term.) For small m^2/H^2 we can expand $G_{\rm ret}(x;x')$,

$$G_{\rm ret}(x;x') = -\frac{1}{4\pi} \Big\{ \frac{\delta(\Delta\eta{-}\Delta r)}{a a' \Delta r} + H^2 \theta(\Delta\eta{-}\Delta r) - m^2 \theta(\Delta\eta{-}\Delta r) \Big[\frac{1}{3} \ln\Big(1{-}\frac{y}{4}\Big) {+} \frac{1}{2} {+} \frac{\frac{y}{6}}{y{-}4}\Big] + O(m^4) \Big\}. \quad (46)$$

Integrating against the source and taking the late time limit gives,

$$P_\Phi(\eta, r) = \int d^4x' G_{\rm ret}(x;x') \times K a' \delta^3(\vec{x}') \; , \tag{47}$$

$$\longrightarrow \frac{KH}{4\pi} \Bigl\{ \ln(Hr) + \frac{\lambda^2 H^2}{32\pi^2} \ln(a) \ln(Hr) + O(\lambda^4) \Bigr\} \; . \tag{48}$$

3.3 The Expectation Values of $\Phi(x)$ and $\Phi^2(x)$

The field definition (12) makes it simple to evaluate expectation values of $\Phi(x)$ and its square. The first power is,

$$\Phi[\Psi] = \Psi - \frac{1}{4}\lambda \Psi^2 + \frac{1}{8}\lambda^2 \Psi^3 - \frac{5}{24}\lambda^3 \Psi^4 + O\Bigl(\lambda^4 \Psi^5\Bigr) \; . \tag{49}$$

Taking the expectation value gives,

$$\Bigl\langle \Omega \Bigl| \Phi(x) \Bigr| \Omega \Bigr\rangle = -\frac{1}{4}\lambda i\Delta(x;x) - \frac{15}{64}\lambda^3 \Bigl[i\Delta(x;x) \Bigr]^2 + O\Bigl(\lambda^5 \Bigl[i\Delta(x;x) \Bigr]^3 \Bigr) \; . \tag{50}$$

Expression (18) shows that the coincident propagator is time dependent, so we see that $\langle \Omega | \Phi(x) | \Omega \rangle$ evolves, in spite of vanishing flat space scattering amplitudes.

The expansion of Φ^2 is,

$$\Phi^2(x) = \Psi^2(x) - \frac{1}{2}\lambda \Psi^3(x) + \frac{5}{16}\lambda^2 \Psi^4(x) + O\Bigl(\lambda^3 \Psi^5\Bigr) \; . \tag{51}$$

Taking its expectation value gives,

$$\Bigl\langle \Omega \Bigl| \Phi^2(x) \Bigr| \Omega \Bigr\rangle = i\Delta(x;x) + \frac{15}{16}\lambda^2 \Bigl[i\Delta(x;x) \Bigr]^2 + O\Bigl(\lambda^4 \Bigl[i\Delta(x;x) \Bigr]^3 \Bigr) \; . \tag{52}$$

$\Phi^2(x)$ is a composite operator and requires renormalization with counterterms of the form,

$$\delta\Phi^2 = K_{\Phi 1} R + K_{\Phi 2} R \Phi^2 + K_{\Phi 3} R^2 + O(\lambda^4) \; . \tag{53}$$

Comparison with the primitive expression (52) implies,

$$K_{\Phi 1} = \frac{\mu^{D-4}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \frac{\pi \cot(\frac{D\pi}{2})}{D(D-1)} \; , \tag{54}$$

$$K_{\Phi 2} = \frac{15\lambda^2 \mu^{D-4}}{8(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \frac{\pi \cot(\frac{D\pi}{2})}{D(D-1)} \; , \tag{55}$$

$$K_{\Phi 3} = \frac{15\lambda^2 \mu^{D-4}}{16(4\pi)^D} \Bigl[\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \frac{\pi \cot(\frac{D\pi}{2})}{D(D-1)} \Bigr]^2 \; . \tag{56}$$

Using these values in (53), adding $\delta\Phi^2$ to (52), and taking the unregulated limit gives the fully renormalized result,

$$\Big\langle \Omega \Big| \Phi^2(x) \Big| \Omega \Big\rangle_{\rm ren} = \frac{H^2}{4\pi^2} \ln\Big(\frac{\mu a}{H}\Big) + \frac{15\lambda^2 H^4}{256\pi^4} \ln^2\Big(\frac{\mu a}{H}\Big) + O(\lambda^4) \; . \qquad (57)$$

4 Large Logarithms in the Two Field Model

The task of this section is computing the same things for the two field model (20) that we previously did for the single field model (10). The order of presentation is the same as in the previous section, although our labor is complicated by the inability to remove interactions by a local field redefinition. We must also digress to explain the Schwinger-Keldysh formalism when solving the effective field equations.

4.1 1-Loop Self-Masses for A and B

The four counterterms we require for renormalizing the self-masses at 1-loop are,

$$\Delta\mathcal{L} = -\frac{1}{2} C_{A1} \Box A \Box A \sqrt{-g} - \frac{1}{2} C_{A2}\, R \partial_\mu A \partial_\nu A g^{\mu\nu} \sqrt{-g} \\ -\frac{1}{2} C_{B1} \Box B \Box B \sqrt{-g} - \frac{1}{2} C_{B2}\, R \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g} \; . \qquad (58)$$

The three diagrams which contribute to the A self-mass at 1-loop are shown in Figure 2.

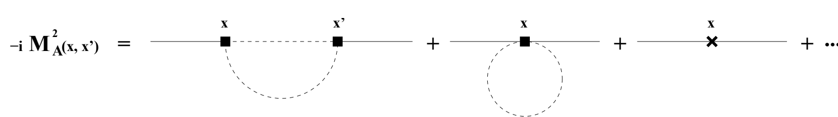

Figure 2: Diagrams which represent the 1-loop contributions to the A-field self-mass $-iM^2_A(x;x')$. Recall that A lines are solid, whereas B lines are dashed.

From left to right, their analytic expressions are,

$$-iM^2_{A3}(x;x') = \frac{(-i\lambda)^2}{2} (aa')^{D-2} \partial^\mu \partial'^\rho i\Delta(x;x') \partial_\mu \partial'_\rho i\Delta(x;x') \; , \qquad (59)$$

$$-iM^2_{A4}(x;x') = -\frac{\lambda^2}{4} i\delta^D(x-x')\, a^{D-2} \partial^\mu \partial'_\mu i\Delta(x;x') \; , \qquad (60)$$

$$-iM^2_{Ac}(x;x') = -C_{A1}\mathcal{D}\mathcal{D}'\Big[\frac{i\delta^D(x-x')}{(aa')^{\frac{D}{2}}}\Big] + C_{A2}\partial^\mu\Big[Ra^{D-2}\partial_\mu i\delta^D(x-x')\Big] \; . \quad (61)$$

After the same sort of reductions employed in the single-field model, the two primitive diagrams take the forms,

$$-iM^2_{A3}(x;x') = \frac{(-i\lambda)^2}{2}\Bigg\{\frac14\mathcal{D}\mathcal{D}'\Big[i\Delta(x;x')\Big]^2 - \frac12\mathcal{D}\Big[i\Delta(x;x)i\delta^D(x-x')\Big] -kHa^{D-1}\partial_0 i\delta^D(x-x')\Bigg\}, \quad (62)$$

$$-iM^2_{A4}(x;x') = -\frac{i\lambda^2}{4}\delta^D(x-x')\times-(D-1)kH^2a^D \; . \quad (63)$$

The square of the propagator in expression (62) is logarithmically divergent so we need only retain dimensional regularization for the leading term and can take $D=4$ for the rest,

$$\Big[i\Delta(x;x')\Big]^2 = \frac{\Gamma^2(\frac{D}{2}-1)}{16\pi^D}\frac{1}{(aa'\Delta x^2)^{D-2}} - \frac{H^2}{16\pi^4}\frac{\ln(\frac14 H^2\Delta x^2)}{aa'\Delta x^2} + \frac{H^4}{64\pi^4}\ln^2\Big(\frac14 H^2\Delta x^2\Big) + O(D-4) \; . \quad (64)$$

The fundamental logarithmic divergence is $1/\Delta x^{2D-4}$. We localize this by first extracting a d'Alembertian, then adding zero in the form of the massless propagator equation in flat space [5,6],

$$\frac{1}{\Delta x^{2D-4}} = \frac{\partial^2}{2(D-3)(D-4)}\Big[\frac{1}{\Delta x^{2D-6}}\Big] = \frac{\mu^{D-4}}{2(D-3)(D-4)}\frac{4\pi^{\frac{D}{2}}i\delta^D(x-x')}{\Gamma(\frac{D}{2}-1)} + \frac{\partial^2}{2(D-3)(D-4)}\Big[\frac{1}{\Delta x^{2D-6}} - \frac{\mu^{D-4}}{\Delta x^{D-2}}\Big] \; , \quad (65)$$

$$= \frac{\mu^{D-4}}{2(D-3)(D-4)}\frac{4\pi^{\frac{D}{2}}i\delta^D(x-x')}{\Gamma(\frac{D}{2}-1)} - \frac{\partial^2}{4}\Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big] + O(D-4) \; . \quad (66)$$

Here μ is the mass scale of dimensional regularization.

Comparison with expression (61) gives the two A-type counterterms,

$$C_{A1} = -\frac{\lambda^2\mu^{D-4}}{32\pi^{\frac{D}{2}}}\frac{\Gamma(\frac{D}{2}-1)}{2(D-3)(D-4)} \quad , \quad C_{A2} = \frac{\lambda^2\mu^{D-4}}{4(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\frac{\pi\cot(\frac{D\pi}{2})}{D(D-1)} \; . \quad (67)$$

Combining $-iM^2_{A3}(x;x')$ and $-iM^2_{A4}(x;x')$ with $-iM^2_{3c}(x;x')$ and taking the unregulated limit gives the renormalized self-mass at one loop,

$$-iM^2_A(x;x') = -\frac{3\lambda^2 H^4 a^4}{32\pi^2}\, i\delta^4(x-x') + \frac{\lambda^2 H^2 \partial^\mu}{16\pi^2}\Bigl[\ln\Bigl(\frac{\mu a}{H}\Bigr) a^2 \partial_\mu i\delta^4(x-x')\Bigr] + \frac{\lambda^2 \mathcal{D}\mathcal{D}'}{512\pi^4}\Biggl\{\frac{\ln(aa')4\pi^2 i\delta^4(x-x')}{(aa')^2} + \frac{\partial^2}{(aa')^2}\Bigl[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Bigr] + \frac{4H^2}{aa'}\frac{\ln(\frac{1}{4}H^2\Delta x^2)}{\Delta x^2} - H^4\ln^2\Bigl(\frac{1}{4}H^2\Delta x^2\Bigr)\Biggr\}. \quad (68)$$

Note that the first term represents a positive mass-squared of the same magnitude as the tachyonic mass we saw in expression (34). It is accompanied by many other contributions which signal that this system is not reducible to a free field.

Figure 3 depicts the three diagrams which contribute to the self-mass of B at 1-loop.

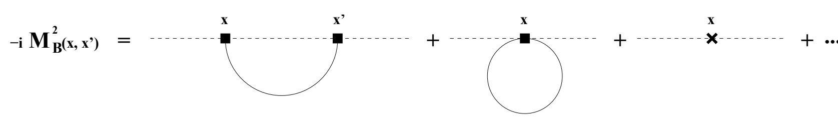

Figure 3: Diagrams which represent the 1-loop contributions to the B-field self-mass $-iM^2_B(x;x')$. Recall that A lines are solid, whereas B lines are dashed.

The analytic expressions for these three diagrams are, from left to right,

$$-iM^2_{B3}(x;x') = (i\lambda)^2 \partial^\mu \partial'^\rho\Bigl[i\Delta(x;x')(aa')^{D-2}\partial_\mu \partial'_\rho i\Delta(x;x')\Bigr] , \quad (69)$$

$$-iM^2_{B4}(x;x') = \frac{\lambda^2}{4}\partial^\mu\Bigl[i\Delta(x;x)a^{D-2}\partial_\mu i\delta^D(x-x')\Bigr] , \quad (70)$$

$$-iM^2_{Bc}(x;x') = -C_{B1}\mathcal{D}\mathcal{D}'\Bigl[\frac{i\delta^D(x-x')}{(aa')^{\frac{D}{2}}}\Bigr] + C_{B2}\partial^\mu\Bigl[R a^{D-2}\partial_\mu i\delta^D(x-x')\Bigr] . \quad (71)$$

After some familiar reductions the two primitive diagrams take the form,

$$-iM^2_{B3}(x;x') = \frac{(i\lambda)^2}{2}\mathcal{D}\mathcal{D}'\Bigl[i\Delta(x;x')\Bigr]^2 - (i\lambda)^2\partial^\mu\partial'^\rho\Bigl[(aa')^{D-2}\partial_\mu i\Delta(x;x')\partial'_\rho i\Delta(x;x')\Bigr] , \quad (72)$$

$$-iM^2_{B4}(x;x') = -\frac{\lambda^2}{4}k\pi\cot\Bigl(\frac{D\pi}{2}\Bigr)\mathcal{D}\Bigl[i\delta^D(x-x')\Bigr]$$

$$+\frac{\lambda^2 H^2 \partial^\mu}{16\pi^2}\Big[\ln(a) a^2 \partial_\mu i\delta^4(x-x')\Big] + O(D-4)\ . \quad (73)$$

Expression (73) is already fully reduced, and the reduction of the first term of (72) is identical to that of (64), but the second term requires new analysis. Because this contribution is quadratically divergent we must retain dimensional regularization for the first two terms in the power series of the propagator,

$$\partial_\mu i\Delta(x;x') = -\frac{\Gamma(\frac{D}{2})}{2\pi^{\frac{D}{2}}(aa')^{\frac{D}{2}-1}}\Bigg\{\frac{\Delta x_\mu}{\Delta x^D} + \frac{[\frac{1}{2}aH\delta^0{}_\mu + \frac{D}{8}aa'H^2\Delta x_\mu]}{\Delta x^{D-2}} + \dots\Bigg\}. \quad (74)$$

Taking the product of two such differentiated propagators, extracting derivatives and taking $D=4$ in integrable terms gives,

$$-(i\lambda)^2\partial^\mu\partial'^\rho\Big[(aa')^{D-2}\partial_\mu i\Delta(x;x')\partial'_\rho i\Delta(x;x')\Big]$$
$$= -\frac{(i\lambda)^2\Gamma^2(\frac{D}{2})}{4\pi^D}\Bigg\{\frac{\mathcal{D}\mathcal{D}'}{4(D-2)^2}\Big[\frac{1}{(aa'\Delta x^2)^{D-2}}\Big] - \frac{DH^2\partial\cdot\partial'}{8(D-2)}\Big[\frac{aa'}{\Delta x^{2D-4}}\Big]$$
$$+\frac{1}{8}\mathcal{D}\mathcal{D}'\Big[\frac{H^2}{aa'\Delta x^2}\Big] + \frac{a^2a'^2H^4\nabla^2}{8}\Big[\frac{1}{\Delta x^2}\Big] - \frac{H^4}{16}\mathcal{D}\mathcal{D}'\ln(H^2\Delta x^2)\Bigg\}. \quad (75)$$

Comparison with expression (71) gives the two B-type counterterms,

$$C_{B1} = -\frac{\lambda^2\mu^{D-4}}{16\pi^{\frac{D}{2}}}\frac{\Gamma(\frac{D}{2}-1)}{2(D-3)(D-4)}\ , \quad (76)$$

$$C_{B2} = \frac{\lambda^2\mu^{D-4}}{4(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\frac{\pi\cot(\frac{D\pi}{2})}{D(D-1)} - \frac{\lambda^2\mu^{D-4}}{32\pi^{\frac{D}{2}}}\frac{\Gamma(\frac{D}{2}-1)}{2(D-3)(D-4)}\Big(\frac{D-2}{D-1}\Big)\ . \quad (77)$$

Note that C_{B2} cancels divergences in $-iM^2_{B4}(x;x')$ — the left hand contribution to (77) — and in $-iM^2_{B3}(x;x')$ — the right hand term of (77). Combining the two primitive diagrams with the counterterm and taking the unregulated limit gives,

$$-iM^2_B(x;x') = \frac{\lambda^2\mathcal{D}\mathcal{D}'}{64\pi^2}\Big[\frac{\ln(aa')i\delta^4(x-x')}{(aa')^2}\Big] - \frac{\lambda^2H^2\partial^\mu}{16\pi^2}\Big[\ln\Big(\frac{Ha}{\mu}\Big)a^2\partial_\mu i\delta^4(x-x')\Big]$$
$$+\frac{\lambda^2\mathcal{D}\mathcal{D}'}{256\pi^4}\Bigg\{\frac{\partial^2}{(aa')^2}\Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big] + \frac{H^2\partial^2}{aa'}\Big[\ln^2\Big(\frac{H^2\Delta x^2}{4}\Big)\Big] - 2H^4\Big[\ln^2\Big(\frac{H^2\Delta x^2}{4}\Big)$$
$$+2\ln\Big(\frac{H^2\Delta x^2}{4}\Big)\Big]\Bigg\} + \frac{\lambda^2H^2\partial\cdot\partial'}{64\pi^4}\Bigg\{aa'\partial^2\Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big]\Bigg\} + \frac{\lambda^2H^4(aa')^2\nabla^2}{32\pi^4}\frac{1}{\Delta x^2}. \quad (78)$$

4.2 1-Loop Mode Functions & Exchange Potentials

4.2.1 Schwinger-Keldysh Effective Field Equations

The linearized effective field equations for A reads,

$$\mathcal{D}A(x) = J(x) + \int d^4x' \, M_A^2(x;x') A(x') \; . \tag{79}$$

The B equation is the same with $A(x)$ replaced by $B(x)$ and $M_A^2(x;x')$ replaced by $M_B^2(x;x')$. Setting the source $J(x) = 0$ describes the propagation of scalar radiation, while the choice $J(x) = a(\eta)\delta^3(\vec{x})$ gives the scalar exchange potential.

Using our in-out results (68) and (78) in equation (79) would result in two problems:

- The fact that the self-masses are complex precludes real solutions; and
- The fact that the self-masses $M_{A,B}^2(x;x')$ are nonzero for x'^{μ} outside the past light-cone of x^{μ} leads to a response *before* its cause.

Both of these embarrassments can be avoided by employing the self-masses of the Schwinger-Keldysh formalism [87–90]. There are many fine references on the Schwinger-Keldysh effective field equations [91–93] but we only need the simple rules for converting the in-out self-masses (68) and (78) to in-in self-masses [94]:

- The Dirac delta function terms are not changed;
- For every term involving the Poincaré interval $\Delta x^2(x;x')$, defined in expression (3), one must subtract the very same function of,

$$\Delta x_{+-}^2(x;x') \equiv \Big\Vert \vec{x} - \vec{x}' \Big\Vert^2 - \Big(\eta - \eta' + i\epsilon \Big)^2 \; . \tag{80}$$

In converting the in-out self-mass to its in-in cognate it is desirable to extract d'Alembertians from inverse powers of $1/\Delta x^2$ to reach powers of logarithms,

$$\frac{1}{\Delta x^2} = \frac{\partial^2}{4} \Big[\ln(\mu^2 \Delta x^2) \Big] \quad , \quad \frac{\ln(\mu^2 \Delta x^2)}{\Delta x^2} = \frac{\partial^2}{8} \Big[\ln^2(\mu^2 \Delta x^2) - 2\ln(\mu^2 \Delta x^2) \Big] \; . \tag{81}$$

Differences of powers of logarithms of Δx^2 and Δx^2_{+-} give a form that makes the reality and causality of $-iM^2_{A,B}(x;x')$ manifest,

$$\ln(\mu^2\Delta x^2) - \ln(\mu^2\Delta x^2_{+-}) = 2\pi i\theta(\Delta\eta{-}\Delta r) \; , \tag{82}$$

$$\ln^2(\mu^2\Delta x^2) - \ln^2(\mu^2\Delta x^2_{+-}) = 4\pi i\theta(\Delta\eta{-}\Delta r)\ln[\mu^2(\Delta\eta^2{-}\Delta r^2)] \; . \tag{83}$$

where $\Delta\eta \equiv \eta - \eta'$ and $\Delta r \equiv \|\vec{x} - \vec{x}'\|$.

Finally, we must adapt (79) to the fact that only one loop results for the self-masses are available. This means we can only solve the equation perturbatively in powers of λ^2,

$$A(x) \equiv A_0(x){+}\lambda^2 A_1(x){+}O(\lambda^4) \quad , \quad B(x) \equiv B_0(x){+}\lambda^2 B_1(x){+}O(\lambda^4) \; . \tag{84}$$

The zeroth order solutions obey,

$$\mathcal{D}A_0(x) = J_A(x) \quad , \quad \mathcal{D}B_0(x) = J_B(x) \; . \tag{85}$$

At 1-loop order we have,

$$\begin{aligned}
\mathcal{D}A_1(x) &= \frac{3H^4a^4A_0}{32\pi^2} - \frac{H^2\partial^\mu}{16\pi^2}\Big[\ln\Big(\frac{\mu a}{H}\Big)a^2\partial_\mu A_0\Big] - \frac{\mathcal{D}}{64\pi^2}\Big[\frac{\ln(a)\mathcal{D}A_0}{a^4}\Big] \\
&-\frac{\mathcal{D}}{512\pi^3}\int\! d^4x'\Big\{\frac{\partial^4}{2a^2a'^2}\Big[\theta(\Delta\eta{-}\Delta r)\Big(\ln[\mu^2(\Delta\eta^2{-}\Delta r^2)]{-}1\Big)\Big] \\
&\qquad +\frac{2H^2\partial^2}{aa'}\Big[\theta(\Delta\eta{-}\Delta r)\Big(\ln\Big[\frac14 H^2(\Delta\eta^2{-}\Delta r^2)\Big]{-}1\Big)\Big] \\
&\qquad\qquad -4H^4\theta(\Delta\eta{-}\Delta r)\ln\Big[\frac14 H^2(\Delta\eta^2{-}\Delta r^2)\Big]\Big\}\mathcal{D}'A_0(x') \; ,
\end{aligned} \tag{86}$$

$$\begin{aligned}
\mathcal{D}B_1(x) &= -\frac{\mathcal{D}}{32\pi^2}\Big[\frac{\ln(a)\mathcal{D}B_0}{a^4}\Big] + \frac{H^2\partial^\mu}{16\pi^2}\Big[\ln\Big(\frac{Ha}{\mu}\Big)a^2\partial_\mu B_0\Big] \\
&-\frac{\mathcal{D}}{256\pi^3}\int\! d^4x'\Big\{\frac{\partial^4}{2a^2a'^2}\Big[\theta(\Delta\eta{-}\Delta r)\Big(\ln[\mu^2(\Delta\eta^2{-}\Delta r^2)]{-}1\Big)\Big] \\
&\qquad +\frac{4H^2\partial^2}{aa'}\Big(\theta(\Delta\eta{-}\Delta r)\ln\Big[\frac14 H^2(\Delta\eta^2{-}\Delta r^2)\Big]\Big) \\
&\qquad\qquad -8H^4\theta(\Delta\eta{-}\Delta r)\Big(\ln\Big[\frac14 H^2(\Delta\eta^2{-}\Delta r^2)\Big]{+}1\Big)\Big\}\mathcal{D}'B_0(x') \\
&-\frac{H^2}{128\pi^3}\int\! d^4x'\partial{\cdot}\partial'\Big\{aa'\partial^4\Big[\theta(\Delta\eta{-}\Delta r)\Big[\ln[\mu^2(\Delta\eta^2{-}\Delta x^2)]{-}1\Big]\Big]\Big\}B_0(x') \\
&\qquad\qquad -\frac{H^4}{64\pi^3}\int\! d^4x' a^2a'^2\nabla^2\partial^2\Big[\theta(\Delta\eta{-}\Delta r)\Big]B_0(x') \; .
\end{aligned} \tag{87}$$

4.2.2 1-Loop Corrected Mode Functions

Scalar radiation corresponds to $J_{A,B}(x)=0$ and has zeroth order solution,

$$A_0(x) = B_0(x) = u_0(\eta,k) e^{i\vec{k}\cdot\vec{x}} \qquad \Longrightarrow \qquad u_0(\eta,k) = \frac{H}{\sqrt{2k^3}} \Big[1 + ik\eta\Big] e^{-ik\eta} \; . \tag{88}$$

Because $\mathcal{D}A_0(x) = 0 = \mathcal{D}B_0(x)$, very few terms of the 1-loop field equations (86-87) survive,

$$\mathcal{D}A_1 \;=\; \frac{3H^4 a^4 A_0}{32\pi^2} + \frac{H^3 a^3 \partial_0 A_0}{16\pi^2} \; , \tag{89}$$

$$\mathcal{D}B_1 \;=\; -\frac{H^3 a^3 \partial_0 B_0}{16\pi^2} + \frac{H^4 k^2 a^2}{64\pi^3} \int \! d^4x' \, \theta(\Delta\eta - \Delta r) \partial'^2 \Big[a'^2 B_0(x')\Big] \tag{90}$$

$$-\frac{H^2 \partial^\mu}{128\pi^3} \Big\{ a\partial^2 \!\! \int \! d^4x' \, \theta(\Delta\eta - \Delta r) \Big[\ln[\mu^2(\Delta\eta^2 - \Delta r^2)] - 1\Big] \partial'^2 \Big[a' \partial'_\mu B_0(x')\Big]\Big\} . \tag{91}$$

1-loop corrections to the A mode function are dominated by the first term on the right hand side of expression (89) which corresponds to a mass,

$$m_A^2 = \frac{3\lambda^2 H^4}{32\pi^2} + O(\lambda^4) \quad \Longrightarrow \quad \nu \equiv \sqrt{\frac{9}{4} - \frac{m_A^2}{H^2}} = \frac{3}{2} - \frac{\lambda^2 H^2}{32\pi^2} + O(\lambda^4) \; . \tag{92}$$

Substituting (92) into our previous result (40) for the late time form of a massive scalar mode function, and expanding in powers of λ^2 gives,

$$i\sqrt{\frac{\pi}{4Ha^3}} \, H^{(1)}_\nu\Big(\frac{k}{aH}\Big) \longrightarrow \frac{H}{\sqrt{2k^3}} \Big\{ 1 + \frac{k^2}{2a^2 H^2} + O\Big(\frac{k^4}{a^4 H^4}\Big) - \frac{\lambda^2 H^2}{32\pi^2} \Big[\ln\Big(\frac{2aH}{k}\Big) + \psi\Big(\frac{3}{2}\Big) + \Big[\ln\Big(\frac{2aH}{k}\Big) + \psi\Big(\frac{3}{2}\Big) - 2\Big] \frac{k^2}{2a^2 H^2} + O\Big(\frac{k^4}{a^4 H^4}\Big)\Big] + O(\lambda^4)\Big\} . \tag{93}$$

In contrast, the final term in (89) is down by two factors of a,

$$\partial_0 u_0(\eta,k) = \frac{H}{\sqrt{2k^3}} \times - \frac{k^2}{aH} \exp\Big[\frac{ik}{aH}\Big] \; , \tag{94}$$

and corrects the A mode function by terms which fall off like $1/a^2$.

The first term on the right hand side of (91) is opposite to the final term of (89), and is similarly irrelevant. To evaluate the nonlocal contribution to (91) we first note,

$$\partial'^2 \Big[a'^2 B_0(x')\Big] = 12 a'^4 H^2 B_0(x') + 2a'^3 H \partial'_0 B_0(x') \longrightarrow 12 a'^4 H^2 \times \frac{H}{\sqrt{2k^3}} \; . \tag{95}$$

Hence the leading late time form of the right hand side of (91) is,

$$\frac{H^4k^2a^2}{64\pi^3}\int d^4x'\theta(\Delta\eta-\Delta r)\partial'^2\Big[a'^2B_0(x')\Big]$$
$$\longrightarrow \frac{H^4k^2a^2}{64\pi^3}\int d^4x'\theta(\Delta\eta-\Delta r)\times 12a'^4H^2u_0(0,k)\;, \tag{96}$$
$$= \frac{H^6k^2a^2}{4\pi^2}\,u_0(0,k)\int_{\eta_i}^{\eta}d\eta' a'^4\Delta\eta^3 \longrightarrow \frac{H^2k^2a^2\ln(a)}{4\pi^2}\,u_0(0,k)\;. \tag{97}$$

The nonlocal source in (91) is therefore only enhanced over the minuscule local contribution by a factor of $\ln(a)$. The net effect is no large logarithms in 1-loop corrections to $u_B(\eta,k)$, just a slightly slower approach to the constant late time limit of the tree order result,

$$u_B(\eta,k)\longrightarrow\Big\{1+\frac{\lambda^2H^2}{8\pi^2}\Big(\frac{k}{aH}\Big)^2\ln(a)+O(\lambda^4)\Big\}u_0(\eta,k)\;. \tag{98}$$

4.2.3 1-Loop Corrected Exchange Potentials

We define the exchange potential as the response to a point source $J(\eta,\vec{x})=Ka\delta^3(\vec{x})$. These potentials are functions of η and $r\equiv\|\vec{x}\|$. The order λ^0 solutions for A and B are the same [86],

$$\mathcal{D}P_0(\eta,r)=Ka\delta^3(\vec{x})\Longrightarrow P_0(\eta,r)=\frac{KH}{4\pi}\Big\{\ln\Big(Hr+\frac{1}{a}\Big)-\frac{1}{aHr}\Big\}\;. \tag{99}$$

Other derivatives of $P_0(\eta,r)$ are,

$$\partial_0P_0(\eta,r)=\frac{KH^2}{4\pi}\Big[\frac{1}{Hr}-\frac{1}{Hr+\frac{1}{a}}\Big]\;\;,\;\;\partial_0^2P_0(\eta,r)=-\frac{KH^3}{4\pi}\frac{1}{(Hr+\frac{1}{a})^2}\;, \tag{100}$$
$$\nabla^2P_0(\eta,r)=\frac{K\delta^3(\vec{x})}{a}+\frac{KH^3}{4\pi}\Big[\frac{2a}{Hr}-\frac{2a}{Hr+\frac{1}{a}}-\frac{1}{(Hr+\frac{1}{a})^2}\Big]\;. \tag{101}$$

Recall also that the late time limit of $Hr\gg\frac{1}{a}$ is constant in time but not in space,

$$P_0(\eta,r)\longrightarrow\frac{KH}{4\pi}\Big\{\ln(Hr)+\frac{1}{2a^2H^2r^2}+\dots\Big\}\;. \tag{102}$$

The exchange potential for A takes the form,

$$P_A(\eta,r)=P_0(\eta,r)+P_{A1}(\eta,r)+O(\lambda^4)\;. \tag{103}$$

From equations (86) and (99) we see that $\mathcal{D}P_{A1}(\eta,r)$ is,

$$a^4H^2m_A^2P_0-\frac{\lambda^2H^2\ln(\frac{\mu a}{H})Ka\delta^3(\vec{x})}{16\pi^2}+\frac{\lambda^2H^3a^3\partial_0P_0}{16\pi^2}-\frac{\lambda^2\mathcal{D}}{64\pi^2}\Big[\frac{\ln(a)K\delta^3(\vec{x})}{a^3}\Big]$$
$$-\frac{\lambda^2K\mathcal{D}}{512\pi^3}\!\int\! d^4\eta'\Big\{\frac{\partial^4}{2a^2a'}\Big[\theta(\Delta\eta-r)\Big(\ln[\mu^2(\Delta\eta^2-r^2)]-1\Big)\Big]+\frac{2H^2\partial^2}{a}\Big[\theta(\Delta\eta-r)$$
$$\times\Big(\ln\Big[\frac{1}{4}H^2(\Delta\eta^2-r^2)\Big]-1\Big)\Big]-4a'H^4\theta(\Delta\eta-r)\ln\Big[\frac{1}{4}H^2(\Delta\eta^2-r^2)\Big]\Big\}. \tag{104}$$

It turns out that only the first two contributions to (104) make significant contributions to $P_{A1}(\eta,r)$ at late times. Of course the term proportional to $m_A^2=\frac{3\lambda^2H^2}{32\pi^2}=-m_\Phi^2$ makes the opposite contribution from the tachyonic mass of Φ that we worked out in expression (48),

$$\mathcal{D}\Delta P_{A1}=a^4H^2m_A^2P_0\Longrightarrow\Delta P_{A1}(\eta,r)\longrightarrow-\frac{\lambda^2H^2}{32\pi^2}\ln(a)\times\frac{KH}{4\pi}\ln(Hr)\;. \tag{105}$$

To work out the result from the second term in (104) we simply integrate against the λ^0 retarded Green's function (46),

$$\mathcal{D}\Delta P_{A1}=-\frac{\lambda^2H^2\ln(\frac{\mu a}{H})Ka\delta^3(\vec{x})}{16\pi^2}$$
$$\Longrightarrow\Delta P_{A1}(\eta,r)=\int_{\eta_i}^{0}d\eta'\Big\{\frac{\delta(\Delta\eta-r)}{aa'r}+H^2\theta(\Delta\eta-r)\Big\}\frac{\lambda^2KH^2a'\ln(\frac{\mu a'}{H})}{64\pi^3}, \tag{106}$$
$$=\frac{\lambda^2KH^3}{64\pi^3}\Big\{\frac{1}{aHr}\ln\Big(\frac{\mu a(\eta-r)}{H}\Big)+\frac{1}{2}\ln^2\Big(\frac{\mu a(\eta-r)}{H}\Big)\Big\}, \tag{107}$$
$$\longrightarrow\frac{\lambda^2H^2}{32\pi^2}\ln(Hr)\times\frac{KH}{4\pi}\ln(Hr)\;. \tag{108}$$

The final step is facilitated by noting that $a(\eta-r)=1/(Hr+\frac{1}{a})$. Combining (105) and (108) gives the leading late time correction at 1-loop order,

$$P_{A1}(\eta,r)\longrightarrow\Big\{-\frac{\lambda^2H^2}{32\pi^2}\ln(a)+\frac{\lambda^2H^2}{32\pi^2}\ln(Hr)\Big\}\times\frac{KH}{4\pi}\ln(Hr)\;. \tag{109}$$

The B exchange potential can be written as,

$$P_B(\eta,r)=P_0(\eta,r)+P_{B1}(\eta,r)+O(\lambda^4)\;. \tag{110}$$

Substituting the tree order solution (99) in the generic 1-loop equation (87) implies that the 1-loop correction obeys,

$$
\begin{aligned}
\mathcal{D}P_{B1}(x) = {} & \frac{\lambda^2 H^2 \ln(\frac{\mu a}{H}) K a \delta^3(\vec{x})}{16\pi^2} - \frac{\lambda^2 H^3 a^3 \partial_0 P_0}{16\pi^2} - \frac{\lambda^2 \mathcal{D}}{32\pi^2} \Big[\frac{\ln(a) K \delta^3(\vec{x})}{a^3}\Big] \\
& - \frac{\lambda^2 K \mathcal{D}}{256\pi^3} \int d\eta' \Big\{ \frac{\partial^4}{2a^2 a'} \Big[\theta(\Delta\eta - r)\Big(\ln[\mu^2(\Delta\eta^2 - r^2)] - 1\Big)\Big] + \frac{4H^2\partial^2}{a}\Big(\theta(\Delta\eta - r) \\
& \qquad \times \ln\Big[\frac{1}{4}H^2(\Delta\eta^2 - r^2)\Big]\Big) - 8a' H^4 \theta(\Delta\eta - r)\Big(\ln\Big[\frac{1}{4}H^2(\Delta\eta^2 - r^2)\Big] + 1\Big)\Big\} \\
& + \frac{\lambda^2 H^2 \partial^\mu}{128\pi^3} \Big\{ a\partial^4 \int d^4x' a' \theta(\Delta\eta - \Delta r) \Big[\ln[\mu^2(\Delta\eta^2 - \Delta x^2)] - 1\Big] \partial'_\mu P_0(x') \Big\} \\
& - \frac{\lambda^2 a^2 H^4 \partial^2}{64\pi^3} \int d^4x' a'^2 \theta(\Delta\eta - \Delta r) \nabla^2 P_0(x') \; . \qquad (111)
\end{aligned}
$$

Many of the contributions in (111) are similar to those of (104), and it turns out that only the first one induces large logarithms at late times. Because the sign of this first term is opposite to its cousin in (104) we need only reverse the sign of (108) to obtain the leading late time contribution,

$$
P_{B1}(\eta, r) \longrightarrow -\frac{\lambda^2 H^2}{32\pi^2} \ln(Hr) \times \frac{KH}{4\pi} \ln(Hr) \; . \qquad (112)
$$

4.3 Expectation Values at 1 and 2 Loops

Computing expectation values is *much* more difficult without a local field redefinition like (12) which expresses the full field in terms of a free field. However, we struggle through most of the same computations for the two field model (20) that we did for the single field model (10). We first evaluate primitive results in dimensional regularization, then renormalize and take the unregulated limit.

4.3.1 Primitive Result for $\langle A(x) \rangle$

Figure 4 shows the 1-loop contributions to the expectation value of $A(x)$.

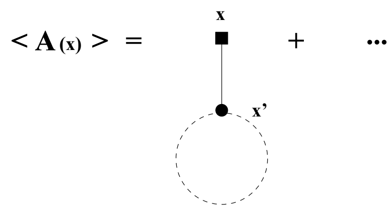

Figure 4: Diagram which represents the 1-loop contribution to the expectation value of $A(x)$. Recall that A lines are solid, whereas B lines are dashed. Square vertices are fixed, whereas circular vertices are integrated.

The initial expression of this diagram is,

$$A_1 = -\frac{i}{2}\lambda \int d^D x' \, i\Delta(x;x') \sqrt{-g(x')} g^{\mu\nu}(x') \times \partial'_\mu \partial''_\nu i\Delta(x';x'') \Big|_{x''=x'} . \qquad (113)$$

Employing relation (19) to evaluate the doubly differentiated, coincident propagator, and then interpreting the diagram in the Schwinger-Keldysh sense gives,

$$A_1 = \frac{i}{2}\lambda (D-1) k H^2 \int d^D x' \sqrt{-g(x')} \Big[i\Delta_{++}(x;x') - i\Delta_{+-}(x;x') \Big] . \qquad (114)$$

Expression (114) is ultraviolet finite so we can set $D = 4$ and use expressions (2) and (81) to conclude,

$$\Big\langle \Omega \Big| A(x) \Big| \Omega \Big\rangle = \frac{\lambda H^2}{16\pi^2} \Big[\ln(a) - \frac{1}{3} + \frac{1}{3a^3} \Big] + O(\lambda^3) . \qquad (115)$$

4.3.2 Primitive Results for $\langle [A(x) - \langle A(x) \rangle]^2 \rangle$

The expectation value of $A^2(x)$ contains a disconnected part that is the square of (115),

$$\Big\langle \Omega \Big| A^2(x) \Big| \Omega \Big\rangle = \Big\langle \Omega \Big| A(x) \Big| \Omega \Big\rangle^2 + \Big\langle \Omega \Big| \Big[A(x) - \langle \Omega | A(x) | \Omega \rangle \Big]^2 \Big| \Omega \Big\rangle . \qquad (116)$$

Figure 5 shows the 1-loop and 2-loop contributions to the connected part.

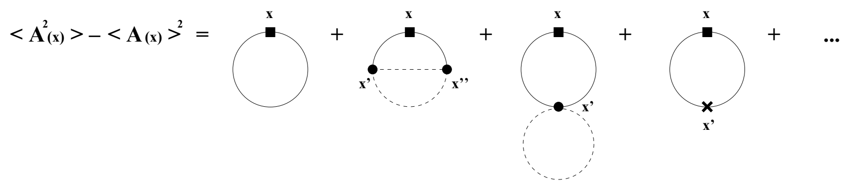

Figure 5: Diagrams representing those 1-loop and 2-loop contributions to the expectation value of $A^2(x)$ which are not already included in the square of the expectation value of $A(x)$. Recall that A lines are solid, whereas B lines are dashed. Square vertices are fixed, whereas circular vertices are integrated.

Of course the 1-loop part is just the coincident propagator (18). Our initial expressions for the three 2-loop contributions are,

$$\begin{aligned}A^2_{2a} &= \frac{(-i\lambda)^2}{2}\int d^Dx' \sqrt{-g(x')} g^{\mu\nu}(x')\, i\Delta(x;x') \\ &\quad\times\int d^Dx'' \sqrt{-g(x'')} g^{\rho\sigma}(x'')\, i\Delta(x;x'')\, \partial'_\mu \partial''_\rho i\Delta(x';x'') \partial'_\nu \partial''_\sigma i\Delta(x';x'')\ , \quad (117)\\ A^2_{2b} &= -\frac{i\lambda^2}{4}\int d^Dx' \sqrt{-g(x')} g^{\mu\nu}(x') \Big[i\Delta(x;x')\Big]^2 \partial'_\mu \partial''_\nu i\Delta(x';x'')\Big\vert_{x''=x'}\ , \quad (118)\\ A^2_{2c} &= \int d^Dx' \sqrt{-g(x')} \Big\{ -iC_{A1} \square' i\Delta(x;x') \square' i\Delta(x;x') \\ &\qquad\qquad - iC_{A2}\, R g^{\mu\nu}(x') \partial'_\mu i\Delta(x;x') \partial'_\nu i\Delta(x;x') \Big\} . \quad (119)\end{aligned}$$

We reduce expression (117) with (31) and an invocation of the Schwinger-Keldysh formalism,

$$\begin{aligned}A^2_{2a} &= -\frac{\lambda^2}{8}\Big[i\Delta(x;x)\Big]^2 \\ &\quad -\frac{i\lambda^2 (D\!-\!1)kH^2}{4}\int d^Dx' \sqrt{-g(x')} \Big\{ \Big[i\Delta(x;x')\Big]^2 - \Big[i\Delta_{+-}(x;x')\Big]^2 \Big\} . \quad (120)\end{aligned}$$

The 4-point contribution (118) follows from (19) and another application of the Schwinger-Keldysh formalsim,

$$A^2_{2b} = \frac{i\lambda^2 (D\!-\!1)kH^2}{4}\int d^Dx' \sqrt{-g(x')} \Big\{ \Big[i\Delta(x;x')\Big]^2 - \Big[i\Delta_{+-}(x;x')\Big]^2 \Big\} . \quad (121)$$

And the counterterm insertion (119) follows from the propagator equation and an application of (28),

$$A^2_{2c} = -C_{A2}\, R\, i\Delta(x;x)\ . \quad (122)$$

4.3.3 Primitive Results for $\langle B^2(x)\rangle$

The expectation value of $B(x)$ vanishes to all orders by virtue of the shift symmetry of (20). Hence there is no disconnected part to the expectation value of $B^2(x)$. The diagrams which contribute to it are depicted in Figure 6

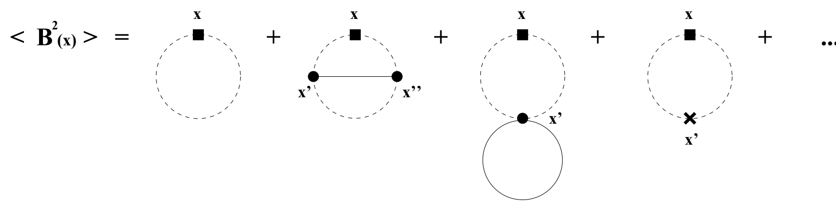

Figure 6: Diagrams which represent the 1-loop and 2-loop contributions to the expectation value of $B^2(x)$. Recall that A lines are solid, whereas B lines are dashed. Square vertices are fixed, whereas circular vertices are integrated.

The associated analytic expressions are,

$$\begin{aligned} B^2_{2a} &= (-i\lambda)^2 \int d^D x' \sqrt{-g(x')} g^{\mu\nu}(x')\, \partial'_\mu i\Delta(x;x') \\ &\quad \times \int d^D x'' \sqrt{-g(x'')} g^{\rho\sigma}(x'')\, \partial''_\rho i\Delta(x;x'')\, i\Delta(x';x'') \partial'_\nu \partial''_\sigma i\Delta(x';x'') \;, \qquad (123) \\ B^2_{2b} &= -\frac{i\lambda^2}{4} \int d^D x' \sqrt{-g(x')} g^{\mu\nu}(x') \partial'_\mu i\Delta(x;x') \partial'_\nu i\Delta(x;x')\, i\Delta(x';x') \;, \qquad (124) \\ B^2_{2c} &= \int d^D x' \sqrt{-g(x')} \Big\{ -iC_{B1} \square' i\Delta(x;x') \square' i\Delta(x;x') \\ &\qquad\qquad -iC_{B2}\, R g^{\mu\nu}(x') \partial'_\mu i\Delta(x;x') \partial'_\nu i\Delta(x;x') \Big\} . \qquad (125) \end{aligned}$$

Familiar partial integration procedures reduce (123-125) to the form,

$$\begin{aligned} B^2_{2a} &= \frac{3\lambda^2}{4} \Big[i\Delta(x;x) \Big]^2 \\ &\quad - \frac{i\lambda^2 (D-1) k H^2}{2} \int d^D x' \sqrt{-g(x')} \Big\{ \Big[i\Delta(x;x') \Big]^2 - \Big[i\Delta_{+-}(x;x') \Big]^2 \Big\}, \qquad (126) \\ B^2_{2b} &= -\frac{\lambda^2}{4} \Big[i\Delta(x;x) \Big]^2 \\ &\quad + \frac{i\lambda^2 (D-1) k H^2}{4} \int d^D x' \sqrt{-g(x')} \Big\{ \Big[i\Delta(x;x') \Big]^2 - \Big[i\Delta_{+-}(x;x') \Big]^2 \Big\}, \qquad (127) \\ B^2_{2c} &= -C_{B2}\, R\, i\Delta(x;x) \;. \qquad (128) \end{aligned}$$

The coincident propagator was given in (18), so we need only employ the Schwinger-Keldysh formalism to show that,

$$-\frac{i\lambda^2(D-1)kH^2}{4}\int d^Dx'\sqrt{-g(x')}\Bigl\{\Bigl[i\Delta(x;x')\Bigr]^2-\Bigl[i\Delta_{+-}(x;x')\Bigr]^2\Bigr\}$$
$$=\frac{\lambda^2H^{2D-4}}{(4\pi)^D}\frac{\Gamma(D)}{\Gamma(\frac{D}{2})}\Bigl\{\frac{\Gamma(\frac{D}{2}-1)}{2(D-3)(D-4)}-\frac{\ln^2(a)}{3}-\frac{2}{3}\ln(a)+\frac{79}{54}+O(a^{-1})\Bigr\}. \quad (129)$$

Note that we have simplified ultraviolet finite contributions to (129) by taking $D=4$.

4.3.4 Renormalized Results

The expectation value of $A(x)$ is ultraviolet finite and requires no renormalization. The square of $A(x)$ is a composite operator and renormalizing it at 1-loop and 2-loop orders requires three counterterms,

$$\delta A^2 = K_{A1}R + K_{A2}RA^2 + K_{A3}R^2 + \dots \quad (130)$$

Comparison with expressions (120-122) reveals that the three constants are,

$$K_{A1} = \frac{\mu^{D-4}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\frac{\pi\cot(\frac{D\pi}{2})}{D(D-1)}\,, \quad (131)$$
$$K_{A2} = 0\,, \quad (132)$$
$$K_{A3} = -\frac{\lambda^2\mu^{2D-8}}{8(4\pi)^D}\frac{\Gamma^2(D-1)}{\Gamma^2(\frac{D}{2})}\frac{\pi^2\cot^2(\frac{D\pi}{2})}{D^2(D-1)^2}\,. \quad (133)$$

Putting everything together gives the final renormalized result,

$$\Bigl\langle\Omega\Bigl|A^2(x)\Bigr|\Omega\Bigr\rangle_{\rm ren} = \frac{H^2}{4\pi^2}\ln\Bigl(\frac{\mu a}{H}\Bigr)-\frac{\lambda^2H^4}{128\pi^4}\ln^2\Bigl(\frac{\mu a}{H}\Bigr)$$
$$\frac{\lambda^2H^4}{256\pi^4}\Bigl[\ln(a)-\frac{1}{3}+\frac{1}{3a^3}+O(\lambda^2)\Bigr]^2+O(\lambda^4)\,. \quad (134)$$

The square of $B(x)$ is also a composite operator and requires similar counterterms,

$$\delta B^2 = K_{B1}R + K_{B2}RB^2 + K_{B3}R^2 + \dots \quad (135)$$

Comparison with expressions (126-128) determines the constants to be,

$$K_{B1} = \frac{\mu^{D-4}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \frac{\pi\cot(\frac{D\pi}{2})}{D(D-1)} , \tag{136}$$

$$K_{B2} = \frac{5\lambda^2\mu^{D-4}}{4(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \frac{\pi\cot(\frac{D\pi}{2})}{D(D-1)} - \frac{\lambda^2\mu^{D-4}}{32\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}-1)}{2(D-3)(D-4)} \Big(\frac{D-2}{D-1}\Big), \tag{137}$$

$$K_{B3} = \frac{\lambda^2\mu^{2D-8}}{2(4\pi)^D} \frac{\Gamma^2(D-1)}{\Gamma^2(\frac{D}{2})} \frac{\pi^2\cot^2(\frac{D\pi}{2})}{D^2(D-1)^2}$$
$$- \frac{\lambda^2\mu^{2D-8}}{(4\pi)^D} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \frac{\Gamma(\frac{D}{2}-1)}{2(D-3)(D-4)} \frac{1}{D^2(D-1)} - \frac{\lambda^2}{128\pi^4} \frac{79}{2592} \tag{138}$$

Substituting these values in (135), adding that to the sum of (126-128), and taking the unregulated limit gives,

$$\Big\langle \Omega \Big| B^2(x) \Big| \Omega \Big\rangle_{\rm ren} = \frac{H^2}{4\pi^2} \ln\Big(\frac{\mu a}{H}\Big) + \frac{\lambda^2 H^4}{128\pi^4} \Big\{ 3\ln^2(a) - 2\ln(a)$$
$$+ 8\ln\Big(\frac{\mu}{H}\Big)\ln(a) + 4\ln^2\Big(\frac{\mu}{H}\Big) - 3\ln\Big(\frac{\mu}{H}\Big) \Big\} + O(\lambda^4) \; . \tag{139}$$

5 Describing the Large Logarithms

The purpose of this section is to explain the various large logarithms derived in the previous two sections. We begin by summarizing them. We then show how to infer stochastic effects based on effective potentials for $\Phi(x)$ and $A(x)$. The remaining large logarithms can be explained using a variant of the renormalization group which is based on a special class of counterterms that can be viewed as curvature-dependent renormalizations of parameters in the bare theories. The section closes with a color-coded summary of the various logarithms, in which stochastic effects are red and renormalization group effects are green.

5.1 Summary

The large logarithms of the single scalar model (10) reside in expressions (40), (48), (50) and (57). Table 1 summarizes these results.

Quantity	Leading Logarithms
$u_\Phi(\eta,k)$	$\Big\{1+\frac{\lambda^2 H^2}{32\pi^2}\ln(a)+O(\lambda^4)\Big\}\times\frac{H}{\sqrt{2k^3}}$
$P_\Phi(\eta,r)$	$\Big\{1+\frac{\lambda^2 H^2}{32\pi^2}\ln(a)+O(\lambda^4)\Big\}\times\frac{KH}{4\pi}\ln(Hr)$
$\langle\Omega\vert\Phi(x)\vert\Omega\rangle$	$-\Big\{1+\frac{15\lambda^2 H^2}{64\pi^2}\ln(a)+O(\lambda^4)\Big\}\times\frac{\lambda H^2}{16\pi^2}\ln(a)$
$\langle\Omega\vert\Phi^2(x)\vert\Omega\rangle_{\rm ren}$	$\Big\{1+\frac{15\lambda^2 H^2}{64\pi^2}\ln(a)+O(\lambda^4)\Big\}\times\frac{H^2}{4\pi^2}\ln(a)$

Table 1: Leading logarithms in the single scalar model (10).

The large logarithms of the two scalar model (20) reside in expressions (93), (98), (109), (112), (115), (134) and (139). Of course the expectation value of $B(x)$ vanishes to all orders. Table 2 summarizes these results.

Quantity	Leading Logarithms
$u_A(\eta,k)$	$\Big\{1-\frac{\lambda^2 H^2}{32\pi^2}\ln(a)+O(\lambda^4)\Big\}\times\frac{H}{\sqrt{2k^3}}$
$u_B(\eta,k)$	$\Big\{1+0+O(\lambda^4)\Big\}\times\frac{H}{\sqrt{2k^3}}$
$P_A(\eta,r)$	$\Big\{1-\frac{\lambda^2 H^2}{32\pi^2}\ln(a)+\frac{\lambda^2 H^2}{32\pi^2}\ln(Hr)+O(\lambda^4)\Big\}\times\frac{KH}{4\pi}\ln(Hr)$
$P_B(\eta,r)$	$\Big\{1-\frac{\lambda^2 H^2}{32\pi^2}\ln(Hr)+O(\lambda^4)\Big\}\times\frac{KH}{4\pi}\ln(Hr)$
$\langle\Omega\vert A(x)\vert\Omega\rangle$	$\Big\{1+O(\lambda^2)\Big\}\times\frac{\lambda H^2}{16\pi^2}\ln(a)$
$\langle\Omega\vert A^2(x)\vert\Omega\rangle_{\rm ren}$	$\Big\{1-\frac{\lambda^2 H^2}{64\pi^2}\ln(a)+O(\lambda^4)\Big\}\times\frac{H^2}{4\pi^2}\ln(a)$
$\langle\Omega\vert B(x)\vert\Omega\rangle$	0
$\langle\Omega\vert B^2(x)\vert\Omega\rangle_{\rm ren}$	$\Big\{1+\frac{3\lambda^2 H^2}{32\pi^2}\ln(a)+O(\lambda^4)\Big\}\times\frac{H^2}{4\pi^2}\ln(a)$

Table 2: Leading logarithms in the two scalar model (20).

5.2 1-Loop Effective Potentials

Stochastic effects in both models can understood using the effective potential. One derives this by setting the undifferentiated fields equal to a constant and then integrating the differentiated fields out of the field equations.

5.2.1 Single Field Model

The key to evaluating $V_{\rm eff}(\Phi)$ is that the Φ propagator in the presence of a constant field configuration $\Phi(x) = \Phi_0$ is a field strength renormalization of the free propagator,

$$\Big\langle \Omega \Big| T\Big[\Phi(x)\Phi(x')\Big]\Big|\Omega\Big\rangle_{\Phi_0} = \frac{i\Delta(x;x')}{(1{+}\frac12\lambda\Phi_0)^2} \; . \tag{140}$$

The 1-loop effective potential follows from taking the expectation value of the action's first variation (13) in the presence of constant $\Phi(x) = \Phi_0$,

$$\begin{aligned} -V_{\rm eff}'(\Phi_0)a^D &= \Big(1{+}\frac12\lambda\Phi_0\Big)\partial^\mu\Big[\frac14\lambda a^{D-2}\partial_\mu\Big\langle\Omega\Big|\Phi^2(x)\Big|\Omega\Big\rangle_{\Phi_0}\Big] \; , & (141) \\ &= -\frac{\frac12\lambda(D{-}1)kH^2a^D}{1{+}\frac12\lambda\Phi_0} \; . & (142) \end{aligned}$$

Expression (142) is ultraviolet finite and corresponds to a 1-loop effective potential of,

$$V_{\rm eff}(\Phi) = \frac{3H^4}{8\pi^2}\ln\Big|1 + \frac12\lambda\Phi\Big| \; . \tag{143}$$

The effective potential (143) explains the tachyonic mass (34) we found after the lengthy computation of the 1-loop self-mass,

$$m_\Phi^2 \equiv \frac{\partial^2 V_{\rm eff}(\Phi)}{\partial\Phi^2}\Big|_{\Phi=0} = -\frac{3\lambda^2H^4}{32\pi^2} \; . \tag{144}$$

Recall expression (40) for the late time limit of the massive mode function,

$$u_\Phi(\eta,k) \longrightarrow \frac{\Gamma(\nu)}{\sqrt{4\pi Ha^3}}\Big(\frac{2aH}{k}\Big)^\nu \qquad , \qquad \nu = \sqrt{\frac94 - \frac{m_\Phi^2}{H^2}} = \frac32 - \frac{m_\Phi^2}{3H^2} + \dots \tag{145}$$

Substituting (144) into (145) and expanding for small λ gives quantitative agreement with the entry for the mode function in Table 1,

$$u_\Phi(\eta,k) \longrightarrow \Big\{1 + \frac{\lambda^2H^2}{32\pi^2}\Big[\ln\Big(\frac{2aH}{k}\Big) + \psi\Big(\frac32\Big)\Big] + O(\lambda^4)\Big\}\frac{H}{\sqrt{2k^3}} \; . \tag{146}$$

From expression (48) we see that the stochastically generated mass (144) also explains the entry for the exchange potential.

The effective potential also explains the tendency for the expectation value of $\Phi(x)$ to become more and more negative as per expression (50). To see this we write the homogeneous evolution equation (in co-moving time) that follows from adding the variation of the effective potential to the classical variation (13),

$$-\Big(1+\frac{\lambda}{2}\Phi\Big)\frac{d}{dt}\Big[\Big(1+\frac{\lambda}{2}\Phi\Big)a^3\dot{\Phi}\Big] - V'_{\rm eff}(\Phi)a^3 = 0 \; . \tag{147}$$

Because the evolution of Φ is much slower than that of the scale factor $a = e^{Ht}$, the largest contribution to the first term of (147) is from the external derivative acting on the factor of a^3. At this point the equation can be integrated,

$$3H\Big(1+\frac{\lambda}{2}\Phi\Big)^2\dot{\Phi} \simeq -\frac{3\lambda H^4}{16\pi^2}\frac{1}{1+\frac{1}{2}\lambda\Phi} \Longrightarrow \frac{1}{2\lambda}\Big[\Big(1+\frac{\lambda}{2}\Phi\Big)^4 - 1\Big] \simeq -\frac{\lambda H^2}{16\pi^2}\ln(a) \; . \tag{148}$$

Inverting to solve for Φ gives,

$$\Phi = \frac{2}{\lambda}\Big\{\Big[1 - \frac{\lambda^2 H^2}{8\pi^2}\ln(a)\Big]^{\frac{1}{4}} - 1\Big\} = -\frac{\lambda H^2}{16\pi^2}\ln(a)\Big\{1 + \frac{3\lambda^2 H^2}{64\pi^2}\ln(a) + O(\lambda^4)\Big\}. \tag{149}$$

The fact that the order λ^3 contribution (149) disagrees with Table 1 is due to not having included fluctuations around the homogeneous solution driven by the stochastically truncated free field,

$$\varphi_0(t,\vec{x}) \equiv \int\!\frac{d^3k}{(2\pi)^3}\theta\Big(aH - k\Big)\frac{\theta(k-H)H}{\sqrt{2k^3}}\Big\{\alpha_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} + \alpha^\dagger_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}}\Big\} \; , \tag{150}$$

where $\alpha^\dagger_{\vec{k}}$ and $\alpha_{\vec{k}}$ are canonically normalized creation and annihilation operators,

$$\Big[\alpha_{\vec{k}}, \alpha^\dagger_{\vec{p}}\Big] = (2\pi)^3\delta^3(\vec{k}-\vec{p}) \; . \tag{151}$$

If we use the symbol $\varphi(t,\vec{x})$ to distinguish the full ultraviolet finite stochastic field from Φ, then the Langevin equation associated with (148) is,

$$\dot{\varphi} = \dot{\varphi}_0 - \frac{\lambda H^3}{16\pi^2}\frac{1}{(1+\frac{1}{2}\lambda\Psi)^3} \; . \tag{152}$$

It is simple to generate a perturbative solution which includes stochastic fluctuations around (149),

$$\varphi = \varphi_0 - \frac{\lambda H^2}{16\pi^2} \ln(a) + \frac{3\lambda^2 H^3}{32\pi^2} \int_0^t dt' \varphi_0 \\ - \frac{3\lambda^3 H^4}{1024\pi^4} \ln^2(a) - \frac{3\lambda^3 H^3}{32\pi^2} \int_0^t dt' \varphi_0^2 + O(\lambda^4) \; . \tag{153}$$

The expectation value of (153) reproduces the entry for $\langle\Omega|\Phi(x)|\Omega\rangle$ in Table 1,

$$\Bigl\langle \Omega \Bigl| \varphi(t, \vec{x}) \Bigr| \Omega \Bigr\rangle = 0 - \frac{\lambda H^2}{16\pi^2} \ln(a) + 0 \\ - \frac{3\lambda^3 H^4}{1024\pi^4} \ln^2(a) - \frac{3\lambda^3 H^3}{32\pi^2} \int_0^t dt' \frac{H^2}{4\pi^2} \ln(a') + O(\lambda^5) \; , \tag{154}$$

$$= -\frac{\lambda H^2}{16\pi^2} \ln(a) \Biggl\{ 1 + \frac{15\lambda^2 H^2}{64\pi^4} \ln^2(a) + O(\lambda^4) \Biggr\} . \tag{155}$$

Note that stochastic fluctuations cause the expectation value of the field to roll down its potential more rapidly than the result (149) because a downward fluctuation is more probable than an upward one.

5.2.2 Two Field Model

The same techniques can be applied to the two scalar model (20). The exact shift symmetry of B precludes there being any effective potential for the field B, but A has one. We can compute it by noting that the expectation value of B in the presence of constant $A(x) = A_0$ is a field strength renormalization,

$$\Bigl\langle \Omega \Bigl| T\Bigl[B(x) B(x') \Bigr] \Bigr| \Omega \Bigr\rangle_{A_0} = \frac{i\Delta(x; x')}{(1 + \frac{1}{2}\lambda A_0)^2} \; . \tag{156}$$

The 1-loop effective potential follows from taking the expectation value of the action's first A variation (21),

$$-V'_{\rm eff}(A_0) a^D = -\frac{1}{2}\lambda \Bigl(1 + \frac{1}{2}\lambda A_0\Bigr) a^{D-2} \Bigl\langle \Omega \Bigl| \partial^\mu B(x) \partial_\mu B(x) \Bigr| \Omega \Bigr\rangle_{A_0} \; , \tag{157}$$

$$= + \frac{\frac{1}{2}\lambda (D-1) k H^2 a^D}{1 + \frac{1}{2}\lambda A_0} \; . \tag{158}$$

Taking the unregulated limit and integrating gives the 1-loop effective potential,

$$V_{\rm eff}(A) = -\frac{3H^4}{8\pi^2} \ln\Bigl|1 + \frac{1}{2}\lambda A\Bigr| \; . \qquad (159)$$

The A effective potential (159) explains the positive mass-squared we found in expression (68) after a lengthy computation,

$$m_A^2 = \frac{\partial^2 V_{\rm eff}(A)}{\partial A^2}\Bigr|_{A=0} = \frac{3\lambda^2 H^4}{32\pi^2} \; . \qquad (160)$$

This is opposite of the tachyonic mass for Φ (that is, $m_A^2 = -m_\Phi^2$), which explains the factors of $-\frac{\lambda^2 H^2}{32\pi^2}\ln(a)$ in the entries for $u_A(\eta,k)$ and $P_A(\eta,r)$ in Table 2. The effective potential for A also explains the tendency for $\langle\Omega|A(x)|\Omega\rangle$ to grow without bound. Specializing the A field equation (21) to homogeneous evolution in co-moving coordinates, adding the effective potential, and neglecting derivatives of A with respect to derivatives of the scale factor a gives,

$$-\frac{d}{dt}\Bigl(a^3 \dot{A}\Bigr) - V'_{\rm eff}(A) a^3 = 0 \qquad \Longrightarrow \qquad 3H\dot{A} \simeq \frac{3\lambda H^4}{16\pi^2}\frac{1}{1+\frac{1}{2}\lambda A} \; . \qquad (161)$$

Equation (161) can be solved exactly,

$$A \simeq \frac{2}{\lambda}\Biggl[\sqrt{1 + \frac{\lambda^2 H^2}{16\pi^2}\ln(a)} - 1\Biggr] = \frac{\lambda H^2}{16\pi^2}\ln(a)\Biggl\{1 - \frac{\lambda^2 H^2}{64\pi^2}\ln(a) + O(\lambda^4)\Biggr\}. \qquad (162)$$

Gaining quantitative agreement with $\langle\Omega|A(x)|\Omega\rangle$ requires the inclusion of stochastic jitter from the truncated free field $\mathcal{A}_0(t,\vec{x})$,

$$\mathcal{A}_0(t,\vec{x}) \equiv \int\!\frac{d^3k}{(2\pi)^3}\theta\Bigl(aH - k\Bigr)\frac{\theta(k-H)H}{\sqrt{2k^3}}\Bigl\{\alpha_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + \alpha^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}}\Bigr\} \; . \qquad (163)$$

The Langevin equation associated with (161) is,

$$\dot{\mathcal{A}} = \dot{\mathcal{A}}_0 + \frac{\lambda H^3}{16\pi^2}\frac{1}{1+\frac{1}{2}\lambda\mathcal{A}} \; . \qquad (164)$$

Iteration of (164) generates a solution which includes the $\mathbb{C}$-number solution (162) plus stochastic jitter involving $\mathcal{A}_0$,

$$\mathcal{A} = \mathcal{A}_0 + \frac{\lambda H^2}{16\pi^2}\ln(a) - \frac{\lambda^2 H^3}{32\pi^2}\int_0^t\! dt' \mathcal{A}_0 - \frac{\lambda^3 H^4}{1024\pi^4}\ln^2(a) + \frac{\lambda^3 H^3}{64\pi^2}\int_0^t\! dt' \mathcal{A}_0^2 + O(\lambda^4) \; . \qquad (165)$$

The expectation value of (165) agrees exactly with the entry for $\langle\Omega|A(x)|\Omega\rangle$ in Table 2,

$$\begin{aligned}\Big\langle\Omega\Big|\mathcal{A}(t,\vec{x})\Big|\Omega\Big\rangle &= 0+\frac{\lambda H^2}{16\pi^2}\ln(a)+0 \\ &\quad -\frac{\lambda^3H^4}{1024\pi^4}\ln^2(a)+\frac{\lambda^3H^3}{64\pi^2}\int_0^t dt'\frac{H^2}{4\pi^2}\ln(a')+O(\lambda^5)\ , \qquad (166)\\ &= \frac{\lambda H^2}{16\pi^2}\ln(a)\Big\{1+\frac{\lambda^2H^2}{64\pi^2}\ln(a)+O(\lambda^4)\Big\}. \qquad (167)\end{aligned}$$

Note again that stochastic jitter again increases the rate at which the field rolls down its potential.

The expectation value of $\mathcal{A}^2(t,\vec{x})$ is also straightforward,

$$\begin{aligned}\Big\langle\Omega\Big|\mathcal{A}^2(t,\vec{x})\Big|\Omega\Big\rangle &= \Big\langle\Omega\Big|\mathcal{A}_0^2+\frac{\lambda H^2}{8\pi^2}\ln(a)\mathcal{A}_0 \\ &\quad +\frac{\lambda^2H^4}{256\pi^4}\ln^2(a)-\frac{\lambda^2H^3}{16\pi^2}\mathcal{A}_0\int_0^t dt'\mathcal{A}_0+O(\lambda^3)\Big\rangle, \qquad (168)\\ &= \frac{H^2}{4\pi^2}\ln(a)\Big\{1-\frac{\lambda^2H^2}{64\pi^2}\ln(a)+O(\lambda^4)\Big\}. \qquad (169)\end{aligned}$$

Expression (169) is in perfect agreement with the entry for $\langle\Omega|A^2(x)|\Omega\rangle$ in Table 2. This means that the leading logarithms of the field $A(x)$ and its square are purely stochastic — at least to this order.

5.3 Curvature-Dependent Renormalizations

Let us start with the renormalized expectation of $\Phi^2(x)$. Recall from relation (53) that the renormalized composite operator is,

$$\Phi^2_{\rm ren}\equiv\Phi^2+K_{\Phi1}R+K_{\Phi2}R\Phi^2+K_{\Phi3}R^2+O(\lambda^4)\ . \qquad (170)$$

Some of the 1-loop and 2-loop counterterms in expression (170) have no flat space analogs, but the $K_{\Phi2}R\Phi^2$ counterterm can be regarded as part of a curvature-dependent field strength renormalization, $\Phi^2=\sqrt{Z_{\Phi^2}}\times\Phi^2_{\rm ren}$ with,

$$Z_{\Phi^2}=1-2K_{\Phi2}\times R+O(\lambda^4)\ . \qquad (171)$$

The associated γ function is,

$$\gamma_{\Phi^2}\equiv\frac{\partial\ln(Z_{\Phi^2})}{\partial\ln(\mu^2)}=-\frac{15\lambda^2H^2}{32\pi^2}+O(\lambda^4)\ . \qquad (172)$$

Expression (57) shows that the renormalized expectation value of $\Phi^2(x)$ actually depends on the product $\mu a/H$. Hence we can replace the $\mu\frac{\partial}{\partial\mu}$ term in the Callan-Symanzik equation with $a\frac{\partial}{\partial a}$,

$$\Big[a\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial\lambda} + \gamma_{\Phi^2}\Big]\Big\langle\Omega\Big|\Phi^2(x)\Big|\Omega\Big\rangle_{\rm ren} = 0\ . \tag{173}$$

The β function for this model is of order λ^3, so we see that equation (173) perfectly explains the order λ^2 (2-loop) contribution to $\langle\Omega|\Phi^2(x)|\Omega\rangle_{\rm ren}$

$$a\frac{\partial}{\partial a}\Big\{\frac{15\lambda^2H^4}{256\pi^4}\ln^2(a)\Big\} - \frac{15\lambda^2H^2}{32\pi^2}\times\frac{H^2}{4\pi^2}\ln(a) = 0 \tag{174}$$

Note that the order λ^0 (1-loop) contribution does not obey the Callan-Symanzik equation (173); the $\frac{H^2}{4\pi^2}\ln(a)$ contribution is a stochastic effect which is not explained by the renormalization group.

Expressions (130) and (132) show that the composite operator $A^2(x)$ does not require a curvature-dependent field strength renormalization at this order,

$$Z_{A^2} = 1 - 2K_{A2}\times R + O(\lambda^4) = 1 + O(\lambda^4) \quad\Longrightarrow\quad \gamma_{A^2} = 0 + O(\lambda^4)\ . \tag{175}$$

Hence the Callan-Symanzik equation does not constrain $\langle\Omega|A^2(x)|\Omega\rangle_{\rm ren}$ at this order, and we saw from expression (169) that stochastic effects completely explain the 1-loop and 2-loop contributions for this entry in Table 2.

The composite operator $B^2(x)$ experiences a curvature-dependent field strength renormalization we can read off from expression (135) and (137),

$$Z_{B^2} = 1 - 2K_{B2}\times R + O(\lambda^4) \qquad\Longrightarrow\qquad \gamma_{B^2} = -\frac{3\lambda^2H^2}{16\pi^2} + O(\lambda^4H^4)\ . \tag{176}$$

The Callan-Symanzik equation for $\langle\Omega|B^2(x)|\Omega\rangle_{\rm ren}$ is,

$$\Big[a\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial\lambda} + \gamma_{B^2}\Big]\Big\langle\Omega\Big|B^2(x)\Big|\Omega\Big\rangle_{\rm ren} = 0\ . \tag{177}$$

Because the β function of the two field model is of order λ^3, the equation precisely predicts the $\frac{3\lambda^2H^2}{32\pi^2}\times\frac{H^2}{4\pi^2}\ln(a)$ contribution reported in Table 2. Note again that the Callan-Symanzik equation does not predict the logarithm at order λ^0 which is a stochastic effect.

Expression (58) shows that we can also think of the 1PI 2-point functions for A and B experiencing curvature-dependent field strength renormalizations whose associated γ functions can be computed from (67) and (77),

$$Z_A = 1 + C_{A2}\times R + O(\lambda^4) \quad , \quad Z_B = 1 + C_{B2}\times R + O(\lambda^4) \; , \tag{178}$$

$$\gamma_A = +\frac{\lambda^2 H^2}{32\pi^2} + O(\lambda^4) \quad , \quad \gamma_B = -\frac{\lambda^2 H^2}{32\pi^2} + O(\lambda^4) \; . \tag{179}$$

The tree order mode functions both approach constants at late times, so the 1-loop corrections are unconstrained by the Callan-Symanzik equation. However, the tree order exchange potentials approach $\frac{KH}{4\pi}\ln(Hr)$ at late times. We must therefore interpret the $\mu\frac{\partial}{\partial\mu}$ term as $r\frac{\partial}{\partial r}$. The 1-loop corrections are integrals of the 1PI 2-point functions, so the Callan-Symanzik equations read,

$$\Big[r\frac{\partial}{\partial r} + \beta\frac{\partial}{\partial\lambda} - 2\gamma_A\Big] P_A(\eta,r) = 0 = \Big[r\frac{\partial}{\partial r} + \beta\frac{\partial}{\partial\lambda} - 2\gamma_B\Big] P_B(\eta,r) \; . \tag{180}$$

With the γ functions (179), these equations predict the $\pm\frac{\lambda^2 H^2}{32\pi^2}\ln(Hr) \times \frac{KH}{4\pi}\ln(Hr)$ contributions for $P_A(\eta,r)$ and $P_B(\eta,r)$ in Table 2. Note that they do not predict the $-\frac{\lambda^2 H^2}{32\pi^2}\ln(a) \times \frac{KH}{4\pi}\ln(Hr)$ contribution to $P_A(\eta,r)$. This is a stochastic effect from the mass generated by the effective potential $V_{\rm eff}(A)$ of expression (159).

5.4 Color-Coded Tables

So many different logarithms occurred that we have thought it good to provide color-coded versions of Tables 1 and 2 to distinguish stochastic effects (in red) from those explained by the renormalization group (in green).

Quantity	Leading Logarithms
$u_\Phi(\eta,k)$	$\Big\{1+\frac{\lambda^2 H^2}{32\pi^2}\ln(a) + O(\lambda^4)\Big\} \times \frac{H}{\sqrt{2k^3}}$
$P_\Phi(\eta,r)$	$\Big\{1+\frac{\lambda^2 H^2}{32\pi^2}\ln(a) + O(\lambda^4)\Big\} \times \frac{KH}{4\pi}\ln(Hr)$
$\langle\Omega\vert\Phi(x)\vert\Omega\rangle$	$-\Big\{1+\frac{15\lambda^2 H^2}{64\pi^2}\ln(a) + O(\lambda^4)\Big\} \times \frac{\lambda H^2}{16\pi^2}\ln(a)$
$\langle\Omega\vert\Phi^2(x)\vert\Omega\rangle_{\rm ren}$	$\Big\{1+\frac{15\lambda^2 H^2}{64\pi^2}\ln(a) + O(\lambda^4)\Big\} \times \frac{H^2}{4\pi^2}\ln(a)$

Table 3: Color-coded explanations of single scalar logarithms from Table 1. Red denotes stochastic logarithms and green those explained by the renormalization group.

Note that mass effects are considered stochastic because m_Φ^2 and m_A^2 were induced by the effective potentials $V_{\rm eff}(\Phi)$ and $V_{\rm eff}(A)$ which give rise to the Langevin equations (152) and (164).

Quantity	Leading Logarithms
$u_A(\eta, k)$	$\Big\{1 - \frac{\lambda^2 H^2}{32\pi^2} \ln(a) + O(\lambda^4)\Big\} \times \frac{H}{\sqrt{2k^3}}$
$u_B(\eta, k)$	$\Big\{1 + 0 + O(\lambda^4)\Big\} \times \frac{H}{\sqrt{2k^3}}$
$P_A(\eta, r)$	$\Big\{1 - \frac{\lambda^2 H^2}{32\pi^2} \ln(a) + \frac{\lambda^2 H^2}{32\pi^2} \ln(Hr) + O(\lambda^4)\Big\} \times \frac{KH}{4\pi} \ln(Hr)$
$P_B(\eta, r)$	$\Big\{1 - \frac{\lambda^2 H^2}{32\pi^2} \ln(Hr) + O(\lambda^4)\Big\} \times \frac{KH}{4\pi} \ln(Hr)$
$\langle\Omega\vert A(x)\vert\Omega\rangle$	$\Big\{1 + O(\lambda^2)\Big\} \times \frac{\lambda H^2}{16\pi^2} \ln(a)$
$\langle\Omega\vert A^2(x)\vert\Omega\rangle_{\rm ren}$	$\Big\{1 - \frac{\lambda^2 H^2}{64\pi^2} \ln(a) + O(\lambda^4)\Big\} \times \frac{H^2}{4\pi^2} \ln(a)$
$\langle\Omega\vert B(x)\vert\Omega\rangle$	0
$\langle\Omega\vert B^2(x)\vert\Omega\rangle_{\rm ren}$	$\Big\{1 + \frac{3\lambda^2 H^2}{32\pi^2} \ln(a) + O(\lambda^4)\Big\} \times \frac{H^2}{4\pi^2} \ln(a)$

Table 4: Color-coded explanations of two scalar logarithms from Table 2. Red denotes stochastic logarithms and green those explained by the renormalization group.

6 Epilogue

Proponents of the renormalization group have long contended with supporters of the stochastic formalism in attempting to explain and re-sum the large logarithms which arise when making perturbative computations during inflation. Although the stochastic formalism provides a complete description for scalar potential models, the outcome for nonlinear sigma models is more nuanced. Many of their logarithms can be explained by a variant of the stochastic formalism which is based on using the effective potential to infer a scalar potential model. Unlike the cases of Yukawa theory [44] and Scalar Quantum Electrodynamics [49], the effective potentials of nonlinear sigma models derive from kinetic terms and would vanish in flat space background. The remaining logarithms can be explained by a variant of the renormalization group based on regarding certain higher-derivative counterterms as curvature-dependent renormalizations of couplings in the bare theory.

We considered a single field model (10), which can be reduced to a free theory by a field redefinition,[1] and a two field model (20) which cannot be. For the mode functions and exchange potentials of each model we derived tree order and 1-loop results; for the expectation values of the fields and the squares we derived 1-loop and 2-loop results. Tables 3 and 4 give a color-coded summary of which logarithms have a stochastic explanation and which ones derive from the renormalization group. Note that many of the entries derive partially from one technique and partly from the other. This is particularly evident for 1-loop corrections to the exchange potential for A in the two field model (20).

The need to combine ultraviolet and stochastic techniques can be seen in the passage from the exact field equation (21) for $A(x)$ to its stochastic realization (164). The exact Heisenberg operator equation is,

$$\partial_\mu \Bigl(\sqrt{-g}\, g^{\mu\nu} \partial_\nu A \Bigr) - \frac{\lambda}{2} \Bigl(1 + \frac{\lambda}{2} A \Bigr) \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g} = 0 \; . \qquad (181)$$

The stochastic realization of the first term in (181) is straightforward [27],

$$\partial_\mu \Bigl(\sqrt{-g}\, g^{\mu\nu} \partial_\nu A \Bigr) \longrightarrow -3H \Bigl(\dot{\mathcal{A}} - \dot{\mathcal{A}}_0 \Bigr) a^3 \; . \qquad (182)$$

However, there is no completely stochastic derivation of the stochastic realization of the second term,

$$-\frac{\lambda}{2} \Bigl(1 + \frac{\lambda}{2} A \Bigr) \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g} \longrightarrow + \frac{3 \lambda H^4}{16 \pi^2} \frac{a^3}{1 + \frac{1}{2} \lambda \mathcal{A}} \; , \qquad (183)$$

because it depends on the ultraviolet sector of $B(x)$ to produce the correct stochastic result,

$$\partial_\mu B(x) \partial_\nu B(x) \longrightarrow -\frac{(\frac{D-1}{D}) k H^2 g_{\mu\nu}(x)}{[1 + \frac{1}{2} \lambda \mathcal{A}(x)]^2} \; . \qquad (184)$$

Any stochastic truncation of $B(x)$ would result in ultraviolet finite fields whose expectation values could never reproduce the indefinite signature so evident in expression (184). Note also that it can be *the same field* whose ultraviolet must be integrated out in derivative terms to give the appropriate

[1] The absence of flat space scattering in no way precludes interactions from changing the kinematics of free fields or the evolution of the background.

Langevin equation. This is evident for the single field model (10) in the passage from the exact field equation (13),

$$\Bigl(1+\frac{\lambda}{2}\Phi\Bigr)^2 \partial_\mu \Bigl(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \Phi\Bigr) + \frac{\lambda}{2}\Bigl(1+\frac{\lambda}{2}\Phi\Bigr) \partial_\mu \Phi \partial_\nu \Phi g^{\mu\nu} \sqrt{-g} = 0 \; , \qquad (185)$$

to its stochastic realization (152),

$$-\Bigl(1+\frac{\lambda}{2}\varphi\Bigr)^2 \times 3 H a^3 \Bigl(\dot{\varphi} - \dot{\varphi}_0\Bigr) - \frac{3\lambda H^4}{16\pi^2} \frac{a^3}{1+\frac{1}{2}\lambda\varphi} = 0 \; . \qquad (186)$$

Quantum field theories in an expanding universe have instantaneous energy eigenstates, but the expansion of the universe prevents these eigenstates from evolving onto one another. So what was the minimum energy state at one instant is not generally minimum energy later on. Bunch-Davies vacuum corresponds to the state that was minimum energy in the distant past. Although Starobinsky's stochastic formalism, and hence also our variant of it, was derived assuming quantum fields in Bunch-Davies vacuum, it ought to apply broadly to states which are perturbatively nearby. On the other hand, this is not true for states which are highly excited from Bunch-Davies vacuum. Indeed, by making suitable Bogoliubov transformations one can change the scalar and tensor power spectra by potentially momentum-dependent factors which range from zero to infinity! That same ambiguity must also afflict the stochastic formalism, as it does all the other predictions of inflationary cosmology. We suspect that the process by which the states of originally trans-Planckian wave numbers are red-shifted to the point where quantum general relativity can be used as an effective field theory leaves these states near Bunch-Davies vacuum. However, that is a conjecture which can and should be studied.

We should comment on the peculiar notion of applying the renormalization group to nonrenormalizable theories such as (10) and (20). At the order we have worked, no renormalization was required for the 1PI 2-point functions of the single field model (10), but we did need four counterterms (58) to renormalize the self-masses for A and B,

$$\Delta\mathcal{L} = -\frac{1}{2} C_{A1} \square A \square A \sqrt{-g} - \frac{1}{2} C_{A2}\, R \partial_\mu A \partial_\nu A g^{\mu\nu} \sqrt{-g} \\ -\frac{1}{2} C_{B1} \square B \square B \sqrt{-g} - \frac{1}{2} C_{B2}\, R \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g} \; . \qquad (187)$$

Each of these counterterms involves higher derivatives, but there is an important distinction between when those derivatives act on A and B and when they act on the metric. The terms proportional to C_{A1} and C_{B1} involve higher derivatives of the fields A and B and play no role in the generation of large inflationary logarithms. However, the terms proportional to C_{A2} and C_{B2} can be viewed as curvature-dependent field strength renormalizations of A and B, respectively. It is the flow of these couplings which serves to capture the green-colored logarithms in Tables 3 and 4. It is also worth noting that the basis for stochastic effects, the effective potentials (143) and (159), are also curvature-dependent and would vanish in the flat space limit.

This project suggests a number of extensions. The most urgent of these is working out the curvature-dependent coupling constant renormalizations that would allow us to determine the renormalization group flows. We would like to determine the late time behavior and also whether or not renormalization group improvement of the effective potentials matters at leading logarithm order. It would also be interesting to compute the order λ^3 (two loop) contribution to the expectation value of $A(x)$ to see if it agrees with the stochastic prediction in expression (167). And we would like to know whether or not the renormalization group can be used to explain the subdominant logarithms one sometimes encounters in the rate at which the mode functions freeze in,

$$u_0(\eta,k) \longrightarrow \frac{H}{\sqrt{2k^3}}\Bigl\{1+\frac{k^2}{2a^2H^2}+\dots\Bigr\}\;, \tag{188}$$

$$u_1(\eta,k) \longrightarrow \frac{H}{\sqrt{2k^3}}\Bigl\{0+\frac{\#\lambda^2k^2\ln(a)}{a^2}+\dots\Bigr\}\;. \tag{189}$$

A final spin-off is understanding the painfully accumulated collection of large logarithms induced by inflationary gravitons in the mode functions and exchange potentials of various matter theories [63, 65, 66, 95] and gravity itself [69, 96].

Of course the primary motivation for studying nonlinear sigma models was to understand the derivative interactions of quantum gravity, without the plethora of indices and the miasma of confusion associated with gauge fixing. We particularly wish to understand the viability of back-reaction to slow the expansion rate in Λ-driven inflation [51, 97]. It is therefore worth summarizing what this project suggests for quantum gravity:

- The large logarithms induced by inflationary gravitons can likely be explained using a combination of curvature-dependent effective potentials and curvature-dependent renormalization group flows;

- There seems to be no obstacle to inferring an effective potential by integrating the ultraviolet out of the invariant Lagrangian [98],

$$\mathcal{L}_{\rm inv} = a^{D-2}\sqrt{-\widetilde{g}}\,\widetilde{g}^{\alpha\beta}\widetilde{g}^{\gamma\delta}\widetilde{g}^{\epsilon\zeta}\Big[\frac{1}{2}h_{\alpha\gamma,\epsilon}h_{\zeta\delta,\beta}-\frac{1}{2}h_{\alpha\beta,\gamma}h_{\delta\epsilon,\zeta}+\frac{1}{4}h_{\alpha\beta,\gamma}h_{\epsilon\zeta,\delta} -\frac{1}{4}h_{\alpha\gamma,\epsilon}h_{\beta\delta,\zeta}\Big] + \frac{1}{2}(D-2)a^{D-1}H\sqrt{-\widetilde{g}}\,\widetilde{g}^{\alpha\beta}\widetilde{g}^{\gamma\delta}h_{\alpha\beta,\gamma}h_{\delta 0}\ , \quad (190)$$

  where $\widetilde{g}_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu}$ is considered to be constant ($\kappa^2 \equiv 16\pi G$), at which point we can follow back-reaction by solving for the homogeneous evolution with the effective potential, the same way we did with equations (148-149) for the single field model and with equations (161-162) for the two field model;

- Of the two invariant 1-loop counterterms,

$$\Delta\mathcal{L} = \alpha_1 R^2\sqrt{-g} + \alpha_2 C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}\sqrt{-g}\ , \quad (191)$$

  the one proportional to α_2 likely plays no role in producing large logarithms while the one proportional to α_1 can be viewed as a curvature-dependent renormalization of Newton's constant, and its flow has the potential to explain the unnaturally large value of α_1 in Starobinsky's original model of inflation [99];

- As long as the curvature remains nonzero there is no reason to assume that evolution approaches a static limit, and the two field model provides an explicit example of significant evolution persisting to arbitrarily late times, cf. expression (162);

- It seems inevitable that significant modifications to the expansion rate and to the force of gravity will persist to late times; and

- Curvature-dependent effective potentials and renormalization group flows pose a challenge when back-reaction causes the curvature to change, but they also provide a natural mechanism through which the effects of inflationary gravitons can become dormant during radiation domination (with $R = 0$) and then reassert themselves at late times, after the transition to matter domination.

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