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How Inflationary Gravitons Affect the Force of Gravity

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ABSTRACT

We employ an unregulated computation the graviton self-energy from gravitons on de Sitter background to infer the renormalized result. This is used to quantum-correct the linearized Einstein equation. We solve this equation for the potentials which represent the gravitational response to static, point mass. We find large spatial and temporal logarithmic corrections to the Newtonian potential and to the gravitational shift. Although suppressed by a minuscule loop-counting parameter, these corrections cause perturbation theory to break down at large distances and late times. Another interesting fact is that gravitons induce up to three large logarithms whereas a loop of massless, minimally coupled scalars produces only a single large logarithm. This is in line with corrections to the graviton mode function: a loop of gravitons induces two large logarithms whereas a scalar loop gives none.

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1 Introduction

A key prediction of primordial inflation is that virtual gravitons of cosmological scale are ripped out of the vacuum [1,2]. The occupation number for each wave vector $\vec{k}$ is staggering,

$$N(\eta,k) = \frac{\pi \Delta^2_h(k)}{64 G k^2} \times a^2(\eta) \; , \tag{1}$$

where $\Delta^2_h(k)$ is the tensor power spectrum, G is Newton's constant and $a(\eta)$ is the scale factor at conformal time η. Our goal is to study how these gravitons change the force of gravity.

We can describe the background geometry of cosmology in conformal coordinates,

$$ds^2 = a^2(\eta) \Bigl[-d\eta^2 + d\vec{x} \cdot d\vec{x} \Bigr] \qquad \Longrightarrow \qquad H \equiv \frac{a'}{a^2} \quad , \quad \epsilon \equiv -\frac{H'}{aH^2} \; , \tag{2}$$

where $H(\eta)$ is the Hubble parameter and $\epsilon(\eta)$ is the first slow roll parameter. A reasonable paradigm for inflation is provided by the special case of de Sitter ($\epsilon = 0$, constant H and $a(\eta) = -1/H\eta$), which is tempting because there are analytic expressions for the graviton propagator [3,4] and because there is no mixing between gravitons and the matter fields that drive inflation [5,6]. One quantum-corrects the linearized Einstein equation using the graviton self-energy $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ which is the 1PI (one particle irreducible) 2-graviton function,

$$\mathcal{D}^{\mu\nu\rho\sigma} h_{\rho\sigma}(x) - \int d^4x' \Bigl[^{\mu\nu}\Sigma^{\rho\sigma}\Bigr](x;x') h_{\rho\sigma}(x') = \frac{1}{2} \kappa T^{\mu\nu}_{\rm lin}(x) \; . \tag{3}$$

Here $\kappa^2 \equiv 16\pi G$ is the loop-counting parameter, $h_{\mu\nu} \equiv (g_{\mu\nu} - a^2 \eta_{\mu\nu})/\kappa$ is the graviton field, $T^{\mu\nu}_{\rm lin}(x)$ is the linearized stress tensor and $\mathcal{D}^{\mu\nu\rho\sigma}$ is the graviton kinetic operator in the same gauge that was used to compute $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$. Our two aims in this work are (1) to infer a fully renormalized result for $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ at one loop from an old computation [7] that was made without regularization, and (2) to work out one loop corrections to the gravitational response to a point mass.

There are four sections to this paper, of which this Introduction is the first. Section 2 describes our procedure for extracting the renormalized self-energy from the unregulated result, with technical details consigned to an Appendix. Section 3 solves (3) for one loop corrections to the gravitational potentials induced by a point mass. Our conclusions comprise section 4.

2 Quantum Linearized Einstein Equation

This section derives an explicit expression for the quantum-corrected Einstein equation (3). Our first tasks are specifying the gauge-fixed kinetic operator $\mathcal{D}^{\mu\nu\rho\sigma}$, explaining how we represent the tensor structure of the graviton self-energy, and giving 3+1 decompositions of both. The main part of this section is describing the process through which we infer most of the renormalized, Schwinger-Keldysh result for the graviton self-energy from an unregulated, noncoincident computation [7]. At the section's end we give a direct, dimensionally regulated computation of the local 4-point contribution, and we discuss the need for a fully dimensionally regulated calculation.

2.1 $3+1$ Decomposition

In the simplest gauge and $D = 3+1$ dimensions, the gauge-fixed kinetic operator takes the form [3, 4],

$$\mathcal{D}^{\mu\nu\rho\sigma} = \frac{1}{2}\eta^{\mu(\rho}\eta^{\sigma)\nu}D_A - \frac{1}{4}\eta^{\mu\nu}\eta^{\rho\sigma}D_A + 2a^4H^2\delta^{(\mu}{}_0\eta^{\nu)(\rho}\delta^{\sigma)}{}_0 \; . \tag{4}$$

Here D_A is the massless, minimally coupled scalar kinetic operator,

$$D_A = -a^2\Bigl[\partial_0^2 + 2aH\partial_0 - \nabla^2\Bigr] = \partial^\mu a^2 \partial_\mu \; . \tag{5}$$

The $3+1$ decomposition of $\mathcal{D}^{\mu\nu\rho\sigma}h_{\rho\sigma}$ is,

$$\mathcal{D}^{00\rho\sigma}h_{\rho\sigma} = \frac{1}{4}D_A(h_{00}+h_{kk}) - 2a^4H^2h_{00} \; , \tag{6}$$

$$\mathcal{D}^{0i\rho\sigma}h_{\rho\sigma} = -\frac{1}{2}D_B h_{0i} \; , \tag{7}$$

$$\mathcal{D}^{ij\rho\sigma}h_{\rho\sigma} = \frac{1}{2}D_A\Bigl[h_{ij} + \frac{1}{2}\delta_{ij}(h_{00}-h_{kk})\Bigr] \; , \tag{8}$$

where D_B stands for the kinetic operator of a massless, conformally coupled scalar,

$$D_B = -a^2\Bigl[\partial_0^2 + 2aH\partial_0 - \nabla^2 + 2a^2H^2\Bigr] = a\partial^2 a \; . \tag{9}$$

Note that adding (6) and the trace of (8) gives a relation for h_{00},

$$\Bigl(\mathcal{D}^{00\rho\sigma} + \mathcal{D}^{kk\rho\sigma}\Bigr)h_{\rho\sigma} = D_B h_{00} \; . \tag{10}$$

Using general tensor analysis on a general cosmological background (2), we can represent the graviton self-energy as a sum of 21 tensor differential operators $[^{\mu\nu}\mathcal{D}^{\rho\sigma}]$ acting on scalar functions of η, η' and $\|\vec{x}-\vec{x}'\|$ [8],

$$-i\Big[^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x') = \sum_{i=1}^{21}\Big[^{\mu\nu}\mathcal{D}_i^{\rho\sigma}\Big]\times T^i(x;x') \; . \tag{11}$$

The 21 basis tensors are constructed from $\delta^{\mu}_{\ 0}$, the spatial part of the Minkowski metric $\overline{\eta}^{\mu\nu} \equiv \eta^{\mu\nu} + \delta^{\mu}_{\ 0}\delta^{\nu}_{\ 0}$ and the spatial derivative operator $\overline{\partial}^{\mu} \equiv \partial^{\mu} + \delta^{\mu}_{\ 0}\partial_0$. These 21 tensors are listed in Table 1.

i	$[^{\mu\nu}\mathcal{D}_i^{\rho\sigma}]$	i	$[^{\mu\nu}\mathcal{D}_i^{\rho\sigma}]$	i	$[^{\mu\nu}\mathcal{D}_i^{\rho\sigma}]$
1	$\overline{\eta}^{\mu\nu}\overline{\eta}^{\rho\sigma}$	8	$\overline{\partial}^{\mu}\overline{\partial}^{\nu}\overline{\eta}^{\rho\sigma}$	15	$\delta^{(\mu}_{\ 0}\overline{\partial}^{\nu)}\delta^{\rho}_{\ 0}\delta^{\sigma}_{\ 0}$
2	$\overline{\eta}^{\mu(\rho}\overline{\eta}^{\sigma)\nu}$	9	$\delta^{(\mu}_{\ 0}\overline{\eta}^{\nu)(\rho}\delta^{\sigma)}_{\ 0}$	16	$\delta^{\mu}_{\ 0}\delta^{\nu}_{\ 0}\overline{\partial}^{\rho}\overline{\partial}^{\sigma}$
3	$\overline{\eta}^{\mu\nu}\delta^{\rho}_{\ 0}\delta^{\sigma}_{\ 0}$	10	$\delta^{(\mu}_{\ 0}\overline{\eta}^{\nu)(\rho}\overline{\partial}^{\sigma)}$	17	$\overline{\partial}^{\mu}\overline{\partial}^{\nu}\delta^{\rho}_{\ 0}\delta^{\sigma}_{\ 0}$
4	$\delta^{\mu}_{\ 0}\delta^{\nu}_{\ 0}\overline{\eta}^{\rho\sigma}$	11	$\overline{\partial}^{(\mu}\overline{\eta}^{\nu)(\rho}\delta^{\sigma)}_{\ 0}$	18	$\delta^{(\mu}_{\ 0}\overline{\partial}^{\nu)}\delta^{(\rho}_{\ 0}\overline{\partial}^{\sigma)}$
5	$\overline{\eta}^{\mu\nu}\delta^{(\rho}_{\ 0}\overline{\partial}^{\sigma)}$	12	$\overline{\partial}^{(\mu}\overline{\eta}^{\nu)(\rho}\overline{\partial}^{\sigma)}$	19	$\delta^{(\mu}_{\ 0}\overline{\partial}^{\nu)}\overline{\partial}^{\rho}\overline{\partial}^{\sigma}$
6	$\delta^{(\mu}_{\ 0}\overline{\partial}^{\nu)}\overline{\eta}^{\rho\sigma}$	13	$\delta^{\mu}_{\ 0}\delta^{\nu}_{\ 0}\delta^{\rho}_{\ 0}\delta^{\sigma}_{\ 0}$	20	$\overline{\partial}^{\mu}\overline{\partial}^{\nu}\delta^{(\rho}_{\ 0}\overline{\partial}^{\sigma)}$
7	$\overline{\eta}^{\mu\nu}\overline{\partial}^{\rho}\overline{\partial}^{\sigma}$	14	$\delta^{\mu}_{\ 0}\delta^{\nu}_{\ 0}\delta^{(\rho}_{\ 0}\overline{\partial}^{\sigma)}$	21	$\overline{\partial}^{\mu}\overline{\partial}^{\nu}\overline{\partial}^{\rho}\overline{\partial}^{\sigma}$

Table 1: The 21 basis tensors used in expression (11). The pairs $(3,4)$, $(5,6)$, $(7,8)$, $(10,11)$, $(14,15)$, $(16,17)$ and $(19,20)$ are related by reflection.

Table 2 gives the 7 pairs of the $T^i(x;x')$ which are related by reflection invariance, $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x') = -i[^{\rho\sigma}\Sigma^{\mu\nu}](x';x)$.

i	Relation	i	Relation
$3,4$	$T^4(x;x') = +T^3(x';x)$	$14,15$	$T^{15}(x;x') = -T^{14}(x';x)$
$5,6$	$T^6(x;x') = -T^5(x';x)$	$16,17$	$T^{17}(x;x') = +T^{16}(x';x)$
$7,8$	$T^8(x;x') = +T^7(x';x)$	$19,20$	$T^{20}(x;x') = -T^{19}(x';x)$
$10,11$	$T^{11}(x;x') = -T^{10}(x';x)$		

Table 2: Scalar coefficient functions in expression (11) which are related by reflection.

The $3+1$ decomposition of $[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')h_{\rho\sigma}(x')$ is,

$$\Big[^{00}\Sigma^{\rho\sigma}\Big]h_{\rho\sigma} \longrightarrow iT^4 h_{kk} + iT^{13}h_{00} + iT^{14}h_{0k,k} + iT^{16}h_{k\ell,k\ell} \; , \tag{12}$$

$$\Big[^{0i}\Sigma^{\rho\sigma}\Big]h_{\rho\sigma} \longrightarrow \frac{i}{2}\partial_i\Big[T^6 h_{kk} + T^{15}h_{00} + T^{18}h_{0k,k} + T^{19}h_{k\ell,k\ell}\Big] + \frac{i}{2}T^9 h_{0i} + \frac{i}{2}T^{10}h_{ik,k} \; , \tag{13}$$

$$\Big[^{ij}\Sigma^{\rho\sigma}\Big]h_{\rho\sigma} \longrightarrow i\delta_{ij}\Big[T^1 h_{kk} + T^3 h_{00} + T^5 h_{0k,k} + T^7 h_{k\ell,k\ell}\Big] + iT^2 h_{ij} + i\partial_{(i}\Big[T^{11}h_{j)0} + T^{12}h_{j)k,k}\Big] + i\partial_i\partial_j\Big[T^8 h_{kk} + T^{17}h_{00} + T^{20}h_{0k,k} + T^{21}h_{k\ell,k\ell}\Big]. \tag{14}$$

Some of these relations were simplified using transition invariance to partially integrate spatial derivatives from the coefficient functions $T^i(x;x')$ onto the graviton field.

2.2 The Quantum Correction

Suppose that $S[g]$ stands for the classical action, with ghost and gauge fixing action $S_g[h,\overline{\theta},\theta]$, and counterterms $\Delta S[g]$. We can give an analytic expression for the one loop graviton self-energy using an expectation value of variations of these actions,

$$-i\Big[^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x') = \Big\langle\Omega\Big|T^*\Big[\Big[\frac{i\delta S[g]}{\delta h_{\mu\nu}(x)}\Big]_{hh}\Big[\frac{i\delta S[g]}{\delta h_{\rho\sigma}(x')}\Big]_{hh} + \Big[\frac{i\delta S[g]}{\delta h_{\mu\nu}(x)}\Big]_{\overline{\theta}\theta} \times\Big[\frac{i\delta S[g]}{\delta h_{\rho\sigma}(x')}\Big]_{\overline{\theta}\theta} + \Big[\frac{i\delta^2 S[g]}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Big]_{hh} + \Big[\frac{i\delta^2\Delta S[g]}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Big]_1\Big]\Big|\Omega\Big\rangle. \tag{15}$$

The T^*-ordering symbol indicates that derivatives are taken outside the time ordering symbol, and the various subscripts give the number of weak fields which contribute. The analogous Feynman diagrams are shown in Figure 1.

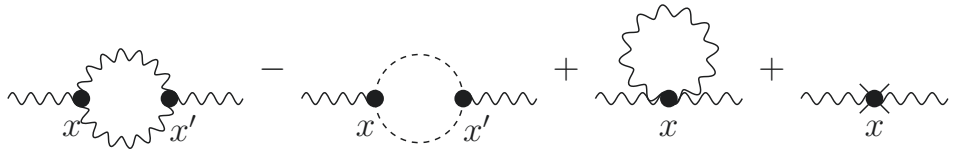

Figure 1: Diagrams contributing to the one loop graviton self-energy, shown in the same order, left to right, as the four contributions to (15). Graviton lines are wavy and ghost lines are dashed.

2.2.1 The $D = 4$ Result

The unregulated result [7] can best be understood by considering how a dimensionally regulated computation of $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ would look. The general forms of the 3-graviton and 4-graviton vertices are [3, 9],[1]

$$\kappa a^{D-2} h \partial h \partial h \quad , \quad \kappa H a^{D-1} h h \partial h \; , \tag{16}$$

$$\kappa^2 a^{D-2} h h \partial h \partial h \quad , \quad \kappa^2 H a^{D-1} h h h \partial h \; . \tag{17}$$

There are a plethora of different index contractions, but contributions to the first two (nonlocal) diagrams of Figure 1 take the general form,

$$\kappa a^{D-2} \times \partial\partial' i\Delta(x;x') \times \partial\partial' i\Delta(x;x') \times \kappa a'^{D-2} \; , \tag{18}$$

with $i\Delta(x;x')$ standing for a ghost or graviton propagator, and the understanding that one derivative at each vertex could be replaced by a factor of H times the appropriate scale factor. Note also that, when an external leg happens to be differentiated, then minus the derivative acts on everything. On the other hand, the third (4-point) diagram of Figure 1 is local,

$$\kappa^2 a^{D-2} \times \partial\partial' i\Delta(x;x') \times i\delta^D(x-x') \; , \tag{19}$$

with the same understanding concerning derivatives. The last (counterterm) diagram of Figure 1 is also local,

$$\frac{\kappa^2 a^{D-4}}{D-4} \times \partial^2 \partial'^2 \times i\delta^D(x-x') \; , \tag{20}$$

with the stipulation that any number of the four derivatives could each be replaced by a factor of Ha.

The gauge for this computation was fixed by adding [3, 4],

$$\mathcal{L}_{GF} = -\frac{a^{D-2}}{2} \eta^{\mu\nu} F_\mu F_\nu \;\; , \;\; F_\mu = \eta^{\rho\sigma}\Big(h_{\mu\rho,\sigma} - \frac{1}{2} h_{\rho\sigma,\mu} + (D\text{–}2) a H h_{\mu\rho} \delta^0{}_\sigma\Big) \; . \tag{21}$$

In this gauge the ghost and graviton propagators become sums of constant tensor factors multiplied by simple scalar propagators,

$$i\Big[{}_\mu\Delta_\rho\Big](x;x') = \overline{\eta}_{\mu\rho} \times i\Delta_A(x;x') - \delta^0{}_\mu \delta^0{}_\nu \times i\Delta_B(x;x') \; , \tag{22}$$

$$i\Big[{}_{\mu\nu}\Delta_{\rho\sigma}\Big](x;x') = \sum_{I=A,B,C} \Big[{}_{\mu\nu}T^I_{\rho\sigma}\Big] \times i\Delta_I(x;x') \; . \tag{23}$$

[1] Vertices involving ghosts take the same form as (16).

The various $[{}_{\mu\nu}T^I_{\rho\sigma}]$ are,

$$\Big[{}_{\mu\nu}T^A_{\rho\sigma}\Big] = 2\overline{\eta}_{\mu(\rho}\overline{\eta}_{\sigma)\nu} - \frac{2}{D-3}\overline{\eta}_{\mu\nu}\overline{\eta}_{\rho\sigma} \quad , \quad \Big[{}_{\mu\nu}T^B_{\rho\sigma}\Big] = -4\delta^0{}_{(\mu}\overline{\eta}_{\nu)(\rho}\delta^0{}_{\sigma)} \; , \tag{24}$$

$$\Big[{}_{\mu\nu}T^C_{\rho\sigma}\Big] = \frac{2E_{\mu\nu}E_{\rho\sigma}}{(D-2)(D-3)} \quad , \quad E_{\mu\nu} \equiv (D-3)\delta^0{}_\mu\delta^0{}_\nu + \overline{\eta}_{\mu\nu} \; . \tag{25}$$

Most of the scalar propagators can be expressed using a function $A(y)$ of the de Sitter length function $y(x;x') \equiv aa' H^2 \Delta x^2$,

$$i\Delta_A(x;x') = A(y) + k\ln(aa') \qquad k \equiv \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \; , \tag{26}$$

$$i\Delta_B(x;x') = B(y) \equiv -\frac{[(4y-y^2)A'(y)+(2-y)k]}{2(D-2)} \; , \tag{27}$$

$$i\Delta_C(x;x') = C(y) \equiv \frac{1}{2}(2-y)B(y) + \frac{k}{D-3} \; . \tag{28}$$

The first derivative of $A(y)$ is [10, 11],

$$A'(y) = -\frac{H^{D-2}}{4(4\pi)^{\frac{D}{2}}}\Bigg\{\Gamma\Big(\frac{D}{2}\Big)\Big(\frac{4}{y}\Big)^{\frac{D}{2}} + \Gamma\Big(\frac{D}{2}+1\Big)\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} + \sum_{n=0}^{\infty}\Bigg[\frac{\Gamma(n+\frac{D}{2}+2)}{\Gamma(n+3)}\Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2} - \frac{\Gamma(n+D)}{\Gamma(n+\frac{D}{2}+1)}\Big(\frac{y}{4}\Big)^n\Bigg]\Bigg\}. \tag{29}$$

Note that the y^n and $y^{n-\frac{D}{2}-2}$ terms cancel for $D = 4$, so they only contribute when multiplied by a sufficiently singular term.

Divergences occur in the effective field equation (3) when the integration over x'^μ carries it to coincidence, $x'^\mu = x^\mu$. Hence the first two (nonlocal) diagrams of Figure 1 can be taken to $D = 4$ away from coincidence, which also makes the two local diagrams vanish. This was done for the unregulated computation [7]. That computation was tractable because taking $D = 4$ simplifies the propagators,

$$i\Big[{}_\mu\Delta^{D=4}_\rho\Big](x;x') = \frac{1}{4\pi^2}\Bigg\{\frac{\eta_{\mu\rho}}{aa'\Delta x^2} - \frac{1}{2}H^2\ln(H^2\Delta x^2)\overline{\eta}_{\mu\rho}\Bigg\}, \tag{30}$$

$$i\Big[{}_{\mu\nu}\Delta^{D=4}_{\rho\sigma}\Big](x;x') = \frac{1}{4\pi^2}\Bigg\{\frac{(2\eta_{\mu(\rho}\eta_{\sigma)\nu} - \eta_{\mu\nu}\eta_{\rho\sigma})}{aa'\Delta x^2} - H^2\ln(H^2\Delta x^2)\Big(\overline{\eta}_{\mu(\rho}\overline{\eta}_{\sigma)\nu} - \overline{\eta}_{\mu\nu}\overline{\eta}_{\rho\sigma}\Big)\Bigg\}. \tag{31}$$

i	Coefficient Functions $T^i_L(x;x')$ in expression (32)
1	$8a^2a'^2H^4 \times [\frac{4\Delta\eta^2}{\Delta x^6} + \frac{1}{\Delta x^4}] + 4a^3a'^3H^6 \times [\frac{4\Delta\eta^4}{\Delta x^6} - \frac{\Delta\eta^2}{\Delta x^4} + \frac{3}{\Delta x^2}]$
2	$-16a^2a'^2H^4 \times [\frac{4\Delta\eta^2}{\Delta x^6} + \frac{1}{\Delta x^4}] - 4a^3a'^3H^6 \times [\frac{8\Delta\eta^4}{\Delta x^6} + \frac{1}{\Delta x^2}]$
3	$8a^3a'^3H^6 \times [\frac{\Delta\eta^2}{\Delta x^4} - \frac{2}{\Delta x^2}] - 4a^3a'^2H^5 \times [\frac{4\Delta\eta^3}{\Delta x^6} - \frac{\Delta\eta}{\Delta x^4}]$
5	$16a^2a'^2H^4 \times \frac{\Delta\eta}{\Delta x^4} + 4a^3a'^2H^5 \times [\frac{2\Delta\eta^2}{\Delta x^4} + \frac{3}{\Delta x^2}]$
7	$-8a^2a'^2H^4 \times \frac{1}{\Delta x^2} - 2a^3a'^3H^6 \times \frac{\Delta\eta^2}{\Delta x^2} - 2a^3a'^2H^5 \times \frac{\Delta\eta}{\Delta x^2}$
9	$-96aa'H^2 \times [\frac{16\Delta\eta^4}{\Delta x^{10}} + \frac{12\Delta\eta^2}{\Delta x^8} + \frac{1}{\Delta x^6}] - 4a^2a'^2H^4 \times [\frac{24\Delta\eta^4}{\Delta x^8} + \frac{8\Delta\eta^2}{\Delta x^6} - \frac{1}{\Delta x^4}]$
10	$96aa'H^2 \times [\frac{2\Delta\eta^3}{\Delta x^8} + \frac{\Delta\eta}{\Delta x^6}] + 12a^2a'^2H^4 \times [\frac{4\Delta\eta^3}{\Delta x^6} + \frac{\Delta\eta}{\Delta x^4}] + a^3a'^3H^6 \times [\frac{8\Delta\eta^3}{\Delta x^4} - \frac{4\Delta\eta}{\Delta x^2}]$ $-8a^2a'H^3 \times [\frac{4\Delta\eta^2}{\Delta x^6} + \frac{1}{\Delta x^4}] - 4a^3a'^2H^5 \times [\frac{2\Delta\eta^2}{\Delta x^4} - \frac{1}{\Delta x^2}]$
12	$-8aa'H^2 \times [\frac{4\Delta\eta^2}{\Delta x^6} + \frac{1}{\Delta x^4}] - 2a^2a'^2H^4 \times [\frac{6\Delta\eta^2}{\Delta x^4} - \frac{9}{\Delta x^2}] + 4a^3a'^3H^6 \times \frac{\Delta\eta^2}{\Delta x^2}$
13	$-96aa'H^2 \times [\frac{16\Delta\eta^4}{\Delta x^{10}} + \frac{12\Delta\eta^2}{\Delta x^8} + \frac{1}{\Delta x^6}] + 4a^2a'^2H^4 \times [\frac{24\Delta\eta^4}{\Delta x^8} + \frac{56\Delta\eta^2}{\Delta x^6} + \frac{11}{\Delta x^4}]$ $+8a^3a'^3H^6 \times [\frac{4\Delta\eta^4}{\Delta x^6} - \frac{2\Delta\eta^2}{\Delta x^4} + \frac{3}{\Delta x^2}]$
14	$192aa'H^2 \times [\frac{2\Delta\eta^3}{\Delta x^8} + \frac{\Delta\eta}{\Delta x^6}] + 8a^2a'^2H^4 \times [\frac{4\Delta\eta^3}{\Delta x^6} - \frac{3\Delta\eta}{\Delta x^4}]$ $-16a^2a'H^3 \times [\frac{4\Delta\eta^2}{\Delta x^6} + \frac{1}{\Delta x^4}] - 16a^3a'^2H^5 \times [\frac{\Delta\eta^2}{\Delta x^4} + \frac{1}{\Delta x^2}]$
16	$-8aa'H^2 \times [\frac{4\Delta\eta^2}{\Delta x^6} + \frac{1}{\Delta x^4}] + 2a^2a'^2H^4 \times [-\frac{6\Delta\eta^2}{\Delta x^4} + \frac{1}{\Delta x^2}] - 2a^3a'^3H^6 \times \frac{\Delta\eta^2}{\Delta x^2}$ $+16a^2a'H^3 \times \frac{\Delta\eta}{\Delta x^4} + 6a^3a'^2H^5 \times \frac{\Delta\eta}{\Delta x^2}$
18	$-24aa'H^2 \times [\frac{4\Delta\eta^2}{\Delta x^6} + \frac{1}{\Delta x^4}] - 2a^2a'^2H^4 \times [\frac{6\Delta\eta^2}{\Delta x^4} + \frac{5}{\Delta x^2}]$
19	$8aa'H^2 \times \frac{\Delta\eta}{\Delta x^4} + 6a^2a'^2H^4 \times \frac{\Delta\eta}{\Delta x^2} - 4a^2a'H^3 \times \frac{1}{\Delta x^2}$

Table 3: Each tabulated term must be multiplied by $-\frac{\kappa^2}{64\pi^4}$.

Because one of the propagators in the nonlocal diagrams (18) might not carry any derivatives, the coefficient functions $T^i(x;x')$ in our representation (11) of the graviton self-energy take the form,

$$T^i(x;x') \equiv T^i_N(x;x') + T^i_L(x;x') \times \ln(H^2\Delta x^2) \; . \tag{32}$$

The coefficient functions $T^i_L(x;x')$ are given in Table 3, and the $T^i_N(x;x')$ are given in Table 4. Both are functions of a, a', $\Delta\eta \equiv \eta - \eta'$ and inverse powers of the Poincaré interval $\Delta x^2 \equiv \|\vec{x} - \vec{x}'\|^2 - (|\eta - \eta'| - i\varepsilon)^2$.

i	Coefficient Functions $T^i_N(x;x')$ in expression (32)
1	$\frac{\frac{736}{5}}{\Delta x^8} - aa'H^2[\frac{616\Delta\eta^2}{\Delta x^8} + \frac{\frac{220}{3}}{\Delta x^6}] - a^2a'^2H^4[\frac{96\Delta\eta^4}{\Delta x^8} + \frac{\frac{812}{3}\Delta\eta^2}{\Delta x^6} + \frac{19}{\Delta x^4}] - a^3a'^3H^6[\frac{64\Delta\eta^4}{\Delta x^6} + \frac{22\Delta\eta^2}{\Delta x^4}]$
2	$\frac{\frac{1952}{5}}{\Delta x^8} - aa'H^2[\frac{416\Delta\eta^2}{\Delta x^8} + \frac{\frac{128}{3}}{\Delta x^6}] - a^2a'^2H^4[\frac{\frac{112}{3}\Delta\eta^2}{\Delta x^6} - \frac{56}{\Delta x^4}] + a^3a'^3H^6[\frac{32\Delta\eta^4}{\Delta x^6} + \frac{12\Delta\eta^2}{\Delta x^4}]$
3	$-184[\frac{\frac{8}{5}\Delta\eta^2}{\Delta x^{10}} + \frac{1}{\Delta x^8}] + 32aa'H^2[\frac{36\Delta\eta^4}{\Delta x^{10}} + \frac{14\Delta\eta^2}{\Delta x^8} - \frac{\frac{43}{3}}{\Delta x^6}]$ $-a^2a'^2H^4[\frac{288\Delta\eta^4}{\Delta x^8} - \frac{\frac{1228}{3}\Delta\eta^2}{\Delta x^6} + \frac{145}{\Delta x^4}] + 4a^3a'^3H^6[\frac{16\Delta\eta^4}{\Delta x^6} + \frac{5\Delta\eta^2}{\Delta x^4}]$ $-8aH[\frac{232\Delta\eta^3}{\Delta x^{10}} + \frac{203\Delta\eta}{\Delta x^8}] + 4a^2a'H^3[\frac{74\Delta\eta^3}{\Delta x^8} - \frac{\frac{319}{3}\Delta\eta}{\Delta x^6}] - a^3a'^2H^5 \times \frac{38\Delta\eta}{\Delta x^4}$
5	$\frac{\frac{368}{5}\Delta\eta}{\Delta x^8} - 32aa'H^2[\frac{9\Delta\eta^3}{\Delta x^8} - \frac{\frac{4}{3}\Delta\eta}{\Delta x^6}] + 4a^2a'^2H^4[\frac{8\Delta\eta^3}{\Delta x^6} - \frac{\Delta\eta}{\Delta x^4}]$ $+232aH[\frac{2\Delta\eta^2}{\Delta x^8} + \frac{1}{\Delta x^6}] - 2a^2a'H^3[\frac{\frac{28}{3}\Delta\eta^2}{\Delta x^6} - \frac{23}{\Delta x^4}] - a^3a'^2H^5 \times \frac{12\Delta\eta^2}{\Delta x^4}$
7	$-\frac{\frac{92}{15}}{\Delta x^6} + aa'H^2[\frac{24\Delta\eta^2}{\Delta x^6} + \frac{\frac{7}{3}}{\Delta x^4}] + a^2a'^2H^4[\frac{8\Delta\eta^2}{\Delta x^4} - \frac{3}{\Delta x^2}] + a^3a'^3H^6 \times \frac{\Delta\eta^2}{\Delta x^2}$ $-aH \times \frac{\frac{116}{3}\Delta\eta}{\Delta x^6} - a^2a'H^3 \times \frac{\frac{23}{3}\Delta\eta}{\Delta x^4} + a^3a'^2H^5 \times \frac{\Delta\eta}{\Delta x^2}$
9	$-16[\frac{\frac{488}{5}\Delta\eta^2}{\Delta x^{10}} + \frac{61}{\Delta x^8}] + 16aa'H^2[\frac{88\Delta\eta^4}{\Delta x^{10}} - \frac{10\Delta\eta^2}{\Delta x^8} - \frac{35}{\Delta x^6}] + 4a^2a'^2H^4[\frac{8\Delta\eta^4}{\Delta x^8} + \frac{2\Delta\eta^2}{\Delta x^6} - \frac{3}{\Delta x^4}]$
10	$\frac{\frac{976}{5}\Delta\eta}{\Delta x^8} - 16aa'H^2[\frac{8\Delta\eta^3}{\Delta x^8} - \frac{\frac{23}{3}\Delta\eta}{\Delta x^6}] - 2a^2a'^2H^4[\frac{\frac{16}{3}\Delta\eta^3}{\Delta x^6} - \frac{23\Delta\eta}{\Delta x^4}]$ $+a^3a'^3H^6 \times \frac{4\Delta\eta^3}{\Delta x^4} + aH \times \frac{\frac{64}{3}}{\Delta x^6} + 8a^2a'H^3[\frac{\frac{4}{3}\Delta\eta^2}{\Delta x^6} - \frac{5}{\Delta x^4}] - a^3a'^2H^5 \times \frac{4\Delta\eta^2}{\Delta x^4}$
12	$-\frac{\frac{488}{15}}{\Delta x^6} + \frac{32}{3}aa'H^2[\frac{\Delta\eta^2}{\Delta x^6} - \frac{1}{\Delta x^4}] - a^2a'^2H^4[\frac{\frac{10}{3}\Delta\eta^2}{\Delta x^4} - \frac{8}{\Delta x^2}] - a^3a'^3H^6 \times \frac{2\Delta\eta^2}{\Delta x^2}$
13	$16[\frac{336\Delta\eta^4}{\Delta x^{12}} + \frac{336\Delta\eta^2}{\Delta x^{10}} + \frac{63}{\Delta x^8}] + 4aa'H^2[\frac{336\Delta\eta^4}{\Delta x^{10}} + \frac{868\Delta\eta^2}{\Delta x^8} + \frac{409}{\Delta x^6}]$ $+a^2a'^2H^4[\frac{424\Delta\eta^4}{\Delta x^8} + \frac{144\Delta\eta^2}{\Delta x^6} + \frac{557}{\Delta x^4}] - 24a^3a'^3H^6[\frac{4\Delta\eta^4}{\Delta x^6} - \frac{\Delta\eta^2}{\Delta x^4}]$
14	$-672[\frac{\frac{8}{5}\Delta\eta^3}{\Delta x^{10}} + \frac{\Delta\eta}{\Delta x^8}] - 8aa'H^2[\frac{30\Delta\eta^3}{\Delta x^8} + \frac{\frac{107}{3}\Delta\eta}{\Delta x^6}] - 4a^2a'^2H^4[\frac{16\Delta\eta^3}{\Delta x^6} - \frac{7\Delta\eta}{\Delta x^4}]$ $-16aH[\frac{4\Delta\eta^2}{\Delta x^8} + \frac{\frac{35}{3}}{\Delta x^6}] - 2a^2a'H^3[\frac{\frac{68}{3}\Delta\eta^2}{\Delta x^6} + \frac{137}{\Delta x^4}] + a^3a'^2H^5 \times \frac{16\Delta\eta^2}{\Delta x^4}$
16	$4[\frac{\frac{84}{5}\Delta\eta^2}{\Delta x^8} + \frac{\frac{13}{3}}{\Delta x^6}] + aa'H^2[\frac{24\Delta\eta^2}{\Delta x^6} + \frac{\frac{127}{3}}{\Delta x^4}] + a^2a'^2H^4[\frac{6\Delta\eta^2}{\Delta x^4} + \frac{5}{\Delta x^2}]$ $-a^3a'^3H^6 \times \frac{3\Delta\eta^2}{\Delta x^2} - aH \times \frac{28\Delta\eta}{\Delta x^6} + a^2a'H^3 \times \frac{\frac{49}{3}\Delta\eta}{\Delta x^4} + a^3a'^2H^5 \times \frac{\Delta\eta}{\Delta x^2}$
18	$8[\frac{\frac{168}{5}\Delta\eta^2}{\Delta x^8} + \frac{\frac{29}{3}}{\Delta x^6}] + 4aa'H^2[\frac{12\Delta\eta^2}{\Delta x^6} - \frac{23}{\Delta x^4}] + a^2a'^2H^4 \times \frac{10\Delta\eta^2}{\Delta x^4}$
19	$-\frac{\frac{112}{5}\Delta\eta}{\Delta x^6} - 8aa'H^2 \times \frac{\Delta\eta}{\Delta x^4} - a^2a'^2H^4 \times \frac{\Delta\eta}{\Delta x^2} + aH \times \frac{\frac{50}{3}}{\Delta x^4}$
21	$\frac{\frac{14}{5}}{\Delta x^4}$

Table 4: Each of the tabulated terms must be multiplied by $-\frac{\kappa^2}{64\pi^4}$.

2.2.2 Recovering the Renormalized Result

In [12] we presented a 4-step procedure for reconstructing the dimensionally regulated result for the first two diagrams of Figure 1:

1. Express each $T^i_L(x;x')$ as a sum of derivatives acting on three integrable functions,
$$\frac{1}{\Delta x^2} \quad , \quad \frac{\Delta\eta}{\Delta x^2} \quad , \quad \frac{\Delta\eta^2}{\Delta x^2} \; ; \tag{33}$$

2. Commute the various derivatives to the left of the multiplicative factor of $\ln(H^2\Delta x^2)$;

3. Write the sum of the remainder $\Delta T^i_L(x;x')$ from step 2, and $T^i_N(x;x')$, as a sum of derivatives acting on the same integrable functions (33) and $1/\Delta x^4$; and

4. Recognize the factors of $1/\Delta x^4$ from step 3 as the $D = 4$ limit of $1/\Delta x^{2D-4}$, and isolate the ultraviolet divergences on delta functions which can be absorbed into counterterms.

Below we explain the rationale for each step and provide details. We also implement the various steps on $T^{12}(x;x')$,

$$T^{12}_L(x;x') = -\frac{\kappa^2 \ln(H^2\Delta x^2)}{64\pi^4}\Biggl\{aa'H^2\Bigl[-\frac{32\Delta\eta^2}{\Delta x^6} - \frac{8}{\Delta x^4}\Bigr] + a^2a'^2H^4\Bigl[-\frac{12\Delta\eta^2}{\Delta x^4} + \frac{18}{\Delta x^2}\Bigr] + a^3a'^3H^6\Bigl[\frac{4\Delta\eta^2}{\Delta x^2}\Bigr]\Biggr\}, \tag{34}$$

$$T^{12}_N(x;x') = -\frac{\kappa^2}{64\pi^4}\Biggl\{-\frac{\frac{488}{15}}{\Delta x^6} + aa'H\Bigl[\frac{\frac{32}{3}\Delta\eta^2}{\Delta x^6} - \frac{\frac{32}{3}}{\Delta x^6}\Bigr] + a^2a'^2H^4\Bigl[-\frac{\frac{10}{3}\Delta\eta^2}{\Delta x^4} + \frac{8}{\Delta x^2}\Bigr] + a^3a'^3H^6\Bigl[-\frac{2\Delta\eta^4}{\Delta x^2}\Bigr]\Biggr\}. \tag{35}$$

To understand the rationale behind **Step 1**, note that a single factor of $\ln(H^2\Delta x^2)$ from the propagators (30-31) can only contribute to one of the $T^i_L(x;x')$ if no derivatives act on one of the two propagators in (18). In that case all of the derivatives must act on the other propagator, and it is this differentiated propagator, multiplied by the scale factors from the vertices, which appear in $T^i_L(x;x')$. It follows that we can express $T^i_L(x;x')$ as a sum

of products of scale factors multiplied by derivatives of the three integrable functions (33). For example, $T^{12}_L(x;x')$ in expression (34) can be written as,

$$T^{12}_L(x;x') = -\frac{\kappa^2 \ln(H^2\Delta x^2)}{64\pi^4}\Big\{aa'H^2\times -4\partial_0^2\Big(\frac{1}{\Delta x^2}\Big) \\ +a^2a'^2H^4\Big[-6\partial_0\Big(\frac{\Delta\eta}{\Delta x^2}\Big)+\frac{24}{\Delta x^2}\Big]+a^3a'^3H^6\Big[\frac{4\Delta\eta^2}{\Delta x^2}\Big]\Big\}. \quad (36)$$

The Appendix contains a number of useful identities (109-118) for extracting derivatives.

Step 2 consists of commuting the multiplicative factor of $\ln(H^2\Delta x^2)$ through the derivatives to multiply the three integrable functions (33). Of course this produces a "remainder" $\Delta T^i_L(x;x')$ in which derivatives act on the logarithm to produce a term like those in $T^i_N(x;x')$. For example, carrying out **Step 2** on expression (36) for $T^{12}_L(x;x')$ gives,

$$T^{12}_L(x;x') = -\frac{\kappa^2}{64\pi^4}\Big\{aa'H^2\times -4\partial_0^2\Big(\frac{\ln(H^2\Delta x^2)}{\Delta x^2}\Big)+a^2a'^2H^4\Big[-6\partial_0 \\ \times\Big(\frac{\Delta\eta\ln(H^2\Delta x^2)}{\Delta x^2}\Big)+\frac{24\ln(H^2\Delta x^2)}{\Delta x^2}\Big]+a^3a'^3H^6\times\frac{4\Delta\eta^2\ln(H^2\Delta x^2)}{\Delta x^2}\Big\} \\ -\frac{\kappa^2}{64\pi^4}\Big\{aa'H^2\Big[-\frac{48\Delta\eta^2}{\Delta x^6}-\frac{8}{\Delta x^4}\Big]+a^2a'^2H^4\times-\frac{12\Delta\eta^4}{\Delta x^6}\Big\}. \quad (37)$$

Identities (119-127) in the Appendix facilitate these reductions. It is useful at this stage to identify six integrable functions, with a factor of $2\pi i$ extracted for future convenience,

$$2\pi iA_1 \equiv \frac{\ln(H^2\Delta x^2)}{\Delta x^2} \quad , \quad 2\pi iA_2 \equiv \frac{1}{\Delta x^2}\ , \quad (38)$$

$$2\pi iB_1 \equiv \frac{\Delta\eta\ln(H^2\Delta x^2)}{\Delta x^2} \quad , \quad 2\pi iB_2 \equiv \frac{\Delta\eta}{\Delta x^2}\ , \quad (39)$$

$$2\pi iC_1 \equiv \frac{\Delta\eta^2\ln(H^2\Delta x^2)}{\Delta x^2} \quad , \quad 2\pi iC_2 \equiv \frac{\Delta\eta^2}{\Delta x^2}\ . \quad (40)$$

Hence we can write,

$$T^{12}_L = -\frac{i\kappa^2}{32\pi^3}\Big\{-4aa'H^2\partial_0^2A_1-6a^2a'^2H^4\Big[\partial_0B_1-4A_1\Big]+4a^3a'^3H^6C_1\Big\}+\Delta T^2_L\ , \quad (41)$$

where the remainder term is,

$$\Delta T^{12}_L(x;x') = -\frac{\kappa^2}{64\pi^4}\Bigg\{aa'H^2\Big[-\frac{48\Delta\eta^2}{\Delta x^6}-\frac{8}{\Delta x^4}\Big]+a^2a'^2H^4\times-\frac{12\Delta\eta^4}{\Delta x^6}\Bigg\}. \quad (42)$$

The terms involving A_1, B_1 and C_1 would be ultraviolet finite in dimensional regularization so it is perfectly valid to leave then in $D = 4$. Results for all the algebraically independent coefficient functions are given in Table 5.

i	Nonlocal Contributions to $iT^i_{SK}(x;x')$ which involve A_1, B_1 and C_1
1	$4a^2a'^2H^4 \times \partial_0^2 A_1 + 2a^3a'^3H^6 \times [\partial_0^2 C_1 - 6\partial_0 B_1 + 10A_1]$
2	$-8a^2a'^2H^4 \times \partial_0^2 A_1 - 4a^3a'^3H^6 \times [\partial_0^2 C_1 - 5\partial_0 B_1 + 4A_1]$
3	$4a^3a'^3H^6 \times [\partial_0 B_1 - 5A_1] - 2a^3a'^2H^5 \times [\partial_0^2 B_1 - 4\partial_0 A_1]$
5	$8a^2a'^2H^4 \times \partial_0 A_1 + 4a^3a'^2H^5 \times [\partial_0 B_1 + 2A_1]$
7	$-8a^2a'^2H^4 \times A_1 - 2a^3a'^3H^6 \times C_1 - 2a^3a'^2H^5 \times B_1$
9	$-4aa'H^2 \times \partial_0^4 A_1 - 2a^2a'^2H^4 \times [\partial_0^3 B_1 - 4\partial_0^2 A_1]$
10	$4aa'H^2 \times \partial_0^3 A_1 + 6a^2a'^2H^4 \times [\partial_0^2 B_1 - 2\partial_0 A_1]$ $+a^3a'^3H^6 \times [4\partial_0 C_1 - 12B_1] - 4a^2a'H^3 \times \partial_0^2 A_1 - a^3a'^2H^5 \times [4\partial_0 B_1 - 8A_1]$
12	$-4aa'H^2 \times \partial_0^2 A_1 - 6a^2a'^2H^4 \times [\partial_0 B_1 - 4A_1] + 4a^3a'^3H^6 \times C_1$
13	$-4aa'H^2 \times \partial_0^4 A_1 + 2a^2a'^2H^4 \times [\partial_0^3 B_1 + 8\partial_0^2 A_1]$ $+4a^3a'^3H^6 \times [\partial_0^2 C_1 - 7\partial_0 B_1 + 11A_1]$
14	$8aa'H^2 \times \partial_0^3 A_1 + 4a^2a'^2H^4 \times [\partial_0^2 B_1 - 6\partial_0 A_1]$ $-8a^2a'H^3 \times \partial_0^2 A_1 - 8a^3a'^2H^5 \times [\partial_0 B_1 + A_1]$
16	$-4aa'H^2 \times \partial_0^2 A_1 - a^2a'^2H^4 \times [6\partial_0 B_1 - 8A_1] - 2a^3a'^3H^6 \times C_1$ $+8a^2a'H^3 \times \partial_0 A_1 + 6a^3a'^2H^5 \times B_1$
18	$-12aa'H^2 \times \partial_0^2 A_1 - a^2a'^2H^4 \times [6\partial_0 B_1 + 4A_1]$
19	$4aa'H^2 \times \partial_0 A_1 + 6a^2a'^2H^4 \times B_1 - 4a^2a'H^3 \times A_1$

Table 5: Each tabulated term must be multiplied by $\frac{\kappa^2}{32\pi^3}$.

i	Nonlocal Contributions to $iT^i_{SK}(x;x')$ which involve A_2, B_2 and C_2
1	$-a^2a'^2H^4[2\partial_0^3B_2+\frac{95}{6}\partial_0^2A_2]-a^3a'^3H^6[5\partial_0^2C_2-12\partial_0B_2+2A_2]$
2	$-\frac{50}{3}a^2a'^2H^4\partial_0^2A_2-a^3a'^3H^6[2\partial_0^2C_2-16\partial_0B_2+12A_2]$
3	$3aa'H^2\partial_0^4A_2-a^2a'^2H^4[6\partial_0^3B_2-\frac{523}{6}\partial_0^2A_2]+a^3a'^3H^6[8\partial_0^2C_2$ $-26\partial_0B_2+10A_2]+\frac{37}{6}a^2a'H^3\partial_0^3A_2-a^3a'^2H^5[3\partial_0^2B_2+8\partial_0A_2]$
5	$-6aa'H^2\partial_0^3A_2+a^2a'^2H^4[4\partial_0^2B_2-6\partial_0A_2]$ $-\frac{7}{3}a^2a'H^3\partial_0^2A_2-2a^3a'^2H^5[\partial_0B_2-A_2]$
7	$3aa'H^2\partial_0^2A_2+a^2a'^2H^4[4\partial_0B_2-7A_2]+a^3a'^3H^6C_2$ $-\frac{23}{6}a^2a'H^3\partial_0A_2+a^3a'^2H^5B_2$
9	$-\frac{14}{3}aa'H^2\partial_0^4A_2-a^2a'^2H^4[3\partial_0^3B_2-13\partial_0^2A_2]$
10	$\frac{14}{3}aa'H^2\partial_0^3A_2+a^2a'^2H^4[\frac{23}{3}\partial_0^2C_2+6\partial_0A_2]+a^3a'^3H^6[6\partial_0C_2-12B_2]$ $-\frac{14}{3}a^2a'H^3\partial_0^2A_2-6a^3a'^2H^5[\partial_0B_2-A_2]$
12	$-\frac{14}{3}aa'H^2\partial_0^2A_2-a^2a'^2H^4[\frac{23}{3}\partial_0B_2-\frac{47}{3}A_2]-2a^3a'^3H^6C_2$
13	$-\frac{29}{6}aa'H^2\partial_0^4A_2+a^2a'^2H^4[\frac{25}{2}\partial_0^3B_2-15\partial_0^2A_2]$ $-a^3a'^3H^6[6\partial_0^2C_2-34\partial_0B_2+22A_2]$
14	$\frac{29}{3}aa'H^2\partial_0^3A_2-a^2a'^2H^4[2\partial_0^2B_2-8\partial_0A_2]-\frac{53}{3}a^2a'H^3\partial_0^2A_2$
16	$-3aa'H^2\partial_0^2A_2-a^2a'^2H^4[3\partial_0B_2-8A_2]-3a^3a'^3H^6C_2$ $+\frac{97}{6}a^2a'H^3\partial_0A_2+a^3a'^2H^5B_2$
18	$-12aa'H^2\partial_0^2A_2-a^2a'^2H^4[\partial_0B_2-A_2]$
19	$-a^2a'^2H^4B_2$

Table 6: Each of the tabulated terms must be multiplied by $\frac{\kappa^2}{32\pi^3}$.

In **Step 3** we first combine $T^i_N(x;x')$ with the remainder $\Delta T^i_L(x;x')$. For our example of $T^{12}(x;x')$ we add (35) and (42),

$$T^{12}_N(x;x')+\Delta T^{12}_L(x;x')=-\frac{\kappa^2}{64\pi^4}\Bigg\{-\frac{\frac{488}{15}}{\Delta x^6}+aa'H\Big[-\frac{\frac{112}{3}\Delta\eta^2}{\Delta x^6}-\frac{\frac{56}{3}}{\Delta x^4}\Big]$$

$$+a^2a'^2H^4\Big[-\frac{\frac{46}{3}\Delta\eta^2}{\Delta x^4}+\frac{8}{\Delta x^2}\Big]+a^3a'^3H^6\times-\frac{2\Delta\eta^2}{\Delta x^2}\Big\}. \quad (43)$$

These sums typically contain ultraviolet divergences. If we again employ the Appendix identities (109-118) to extract derivatives the result involves factors of $1/\Delta x^4$ in addition to the three integrable functions (33). For example, expression (43) gives,

$$\Big[T^{12}_N+\Delta T^{12}_L\Big](x;x') = -\frac{\kappa^2}{64\pi^4}\Big\{-\partial^2\Big(\frac{\frac{61}{15}}{\Delta x^4}\Big)+aa'H^2\Big[-\partial_0^2\Big(\frac{\frac{14}{3}}{\Delta x^2}\Big)-\frac{\frac{28}{3}}{\Delta x^4}\Big]$$
$$+a^2a'^2H^4\Big[-\partial_0\Big(\frac{\frac{23}{3}\Delta\eta}{\Delta x^2}\Big)+\frac{\frac{47}{3}}{\Delta x^2}\Big]+a^3a'^3H^6\times-\frac{2\Delta\eta^2}{\Delta x^2}\Big\}. \quad (44)$$

The ultraviolet finite factors of A_2, B_2 and C_2 are reported in Table 6, whereas we retain the factors of $1/\Delta x^4$ for further analysis,

$$T^{12}_N+\Delta T^{12}_L = -\frac{i\kappa^2}{32\pi^3}\Big\{-\frac{14}{3}aa'H^2\partial_0^2A_2+a^2a'^2H^4\Big[-\frac{23}{3}\partial_0B_2+\frac{47}{3}A_2\Big]$$
$$-2a^3a'^3H^6C_2\Big\}-\frac{\kappa^2}{64\pi^4}\Big\{-\partial^2\Big(\frac{\frac{61}{15}}{\Delta x^4}\Big)-aa'H^2\frac{\frac{28}{3}}{\Delta x^4}\Big\}. \quad (45)$$

In **Step 4** we isolate the logarithmic ultraviolet divergence implicit in the factors of $1/\Delta x^4$ produced by **Step 3**. We first note that factors of $1/\Delta x^4$ would appear as $1/\Delta x^{2D-4}$ had dimensional regularization been retained. Extracting a d'Alembertian from this uncovers an explicit factor of $1/(D-4)$,

$$\frac{1}{\Delta x^4}\longrightarrow\frac{1}{\Delta x^{2D-4}}=\frac{\partial^2}{2(D-3)(D-4)}\Big[\frac{1}{\Delta x^{2D-6}}\Big]. \quad (46)$$

The ultraviolet divergence is localized by adding a term proportional to the flat space background massless propagator equation [10, 11],

$$\frac{1}{\Delta x^4}\longrightarrow\frac{\partial^2}{2(D-3)(D-4)}\Big[\frac{1}{\Delta x^{2D-6}}\Big]$$
$$=\frac{\partial^2}{2(D-3)(D-4)}\Big[\frac{1}{\Delta x^{2D-6}}-\frac{\mu^{D-4}}{\Delta x^{D-2}}\Big]+\frac{\mu^{D-4}4\pi^{\frac{D}{2}}i\delta^D(x-x')}{2(D-3)(D-4)\Gamma(\frac{D}{2}-1)}. \quad (47)$$

The nonlocal part of (47) is both integrable and finite for $D=4$. We can take the unregulated limit of the nonlocal part of (47),

$$\frac{\partial^2}{2(D-3)(D-4)}\Big[\frac{1}{\Delta x^{2D-6}}-\frac{\mu^{D-4}}{\Delta x^{D-2}}\Big]\longrightarrow-\frac{\partial^2}{4}\Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big]\equiv-\frac{\partial^2}{4}\Big[2\pi iA_3\Big]. \quad (48)$$

These ultraviolet finite terms are given in Table 7.

i	Nonlocal Contributions to $iT^i_{SK}(x;x')$ which involve A_3
1	$-\frac{23}{120}\partial^6 A_3 + aa'H^2[\frac{77}{12}\partial_0^2 - \frac{11}{12}\partial^2]\partial^2 A_3 - \frac{49}{6}a^2a'^2H^4\partial^2 A_3$
2	$-\frac{61}{120}\partial^6 A_3 + aa'H^2[\frac{13}{3}\partial_0^2 - \frac{5}{6}\partial^2]\partial^2 A_3 - \frac{55}{3}a^2a'^2H^4\partial^2 A_3$
3	$\frac{23}{120}\nabla^2\partial^4 A_3 + aa'H^2[\frac{52}{12}\partial_0^2 + \frac{173}{12}\partial^2]\partial^2 A_3 + \frac{425}{6}a^2a'^2H^4\partial^2 A_3$ $+aH[\frac{29}{12}\partial_0^3 + \frac{29}{6}\partial_0\partial^2]\partial^2 A_3 + \frac{215}{6}a^2a'H^3\partial_0\partial^2 A_3$
5	$-\frac{23}{60}\partial_0\partial^4 A_3 - \frac{35}{3}aa'H^2\partial_0\partial^2 A_3 - \frac{29}{6}aH[\partial_0^2 + \partial^2]\partial^2 A_3 - \frac{38}{3}a^2a'H^3\partial^2 A_3$
7	$\frac{23}{120}\partial^4 A_3 + \frac{11}{12}aa'H^2\partial^2 A_3 + \frac{29}{12}aH\partial_0\partial^2 A_3$
9	$\frac{61}{60}\nabla^2\partial^4 A_3 + aa'H^2[\frac{29}{3}\partial_0^2 + \frac{41}{3}\partial^2]\partial^2 A_3 + 4a^2a'^2H^4\partial^2 A_3$
10	$-\frac{61}{60}\partial_0\partial^4 A_3 - \frac{29}{3}aa'H^2\partial_0\partial^2 A_3 - \frac{2}{3}aH\partial^4 A_3 + \frac{29}{3}a^2a'H^3\partial^2 A_3$
12	$\frac{61}{60}\partial^4 A_3 + \frac{7}{3}aa'H^2\partial^2 A_3$
13	$-\frac{7}{10}\nabla^4\partial^2 A_3 - aa'H^2[\frac{86}{3}\partial_0^2 + \frac{431}{12}\partial^2]\partial^2 A_3 - 139a^2a'^2H^4\partial^2 A_3$
14	$\frac{7}{5}\nabla^2\partial_0\partial^2 A_3 + \frac{43}{3}aa'H^2\partial_0\partial^2 A_3 + aH[\frac{2}{3}\partial_0^2 + \frac{11}{2}\partial^2]\partial^2 A_3 + \frac{191}{3}a^2a'H^3\partial^2 A_3$
16	$-[\frac{7}{10}\partial_0^2 + \frac{23}{120}\partial^2]\partial^2 A_3 - \frac{121}{12}aa'H^2\partial^2 A_3 + \frac{7}{4}aH\partial_0\partial^2 A_3$
18	$-[\frac{14}{5}\partial_0^2 + \frac{61}{60}\partial^2]\partial^2 A_3 + 23aa'H^2\partial^2 A_3$
19	$\frac{7}{5}\partial_0\partial^2 A_3 - \frac{25}{6}aH\partial^2 A_3$
21	$-\frac{7}{10}\partial^2 A_3$

Table 7: Each of the tabulated terms must be multiplied by $\frac{\kappa^2}{32\pi^3}$.

It remains to renormalize the local divergence in expression (47). This turns out to always produce a finite local term proportional to $\ln(a)$. It arises from the incomplete cancellation between primitive divergences like (47) and counterterms, which contain an extra factor of a^{D-4} from the measure,

$$\frac{\mu^{D-4}4\pi^{\frac{D}{2}}i\delta^D(x-x')}{2(D-3)(D-4)\Gamma(\frac{D}{2}-1)} - \frac{a^{D-4}\mu^{D-4}4\pi^{\frac{D}{2}}i\delta^D(x-x')}{2(D-3)(D-4)\Gamma(\frac{D}{2}-1)} \longrightarrow -2\pi^2 i \times \ln(a)\delta^4(x-x') \; . \quad (49)$$

These local terms are reported in Table 8.

i	Local Contributions to $iT^i_{SK}(x;x')$
1	$-\frac{23}{30}\partial^4\delta^4(x-x')+aa'H^2[\frac{77}{3}\partial_0^2-\frac{11}{3}\partial^2]\delta^4(x-x')-\frac{98}{3}a^2a'^2H^4\delta^4(x-x')$
2	$-\frac{61}{30}\partial^4\delta^4(x-x')+aa'H^2[\frac{52}{3}\partial_0^2-\frac{10}{3}\partial^2]\delta^4(x-x')-\frac{220}{3}a^2a'^2H^4\delta^4(x-x')$
3	$\frac{23}{30}\nabla^2\partial^2\delta^4(x-x')+aa'H^2[\frac{52}{3}\partial_0^2+\frac{173}{3}\partial^2]\delta^4(x-x')+\frac{850}{3}a^2a'^2H^4$ $\times\delta^4(x-x')+aH[\frac{29}{3}\partial_0^3+\frac{58}{3}\partial_0\partial^2]\delta^4(x-x')+\frac{430}{3}a^2a'H^3\partial_0\delta^4(x-x')$
5	$-\frac{23}{15}\partial_0\partial^2\delta^4(x-x')-\frac{140}{3}aa'H^2\partial_0\delta^4(x-x')$ $-\frac{58}{3}aH\nabla^2\delta^4(x-x')-\frac{152}{3}a^2a'H^3\delta^4(x-x')$
7	$\frac{23}{30}\partial^2\delta^4(x-x')+\frac{11}{3}aa'H^2\delta^4(x-x')+\frac{29}{3}aH\partial_0\delta^4(x-x')$
9	$\frac{61}{15}\nabla^2\partial^2\delta^4(x-x')+aa'H^2[\frac{116}{3}\partial_0^2+\frac{164}{3}\partial^2]\delta^4(x-x')+16a^2a'^2H^4\delta^4(x-x')$
10	$-\frac{61}{15}\partial_0\partial^2\delta^4(x-x')-\frac{116}{3}aa'H^2\partial_0\delta^4(x-x')-\frac{8}{3}aH\partial^2\delta^4(x-x')$ $+\frac{116}{3}a^2a'H^3\delta^4(x-x')$
12	$\frac{61}{15}\partial^2\delta^4(x-x')+\frac{28}{3}aa'H^2\delta^4(x-x')$
13	$-\frac{14}{5}\nabla^4\delta^4(x-x')-aa'H^2[\frac{344}{3}\partial_0^2+\frac{431}{3}\partial^2]\delta^4(x-x')-556a^2a'^2H^4\delta^4(x-x')$
14	$\frac{28}{5}\nabla^2\partial_0\delta^4(x-x')+\frac{172}{3}aa'H^2\partial_0\delta^4(x-x')$ $+aH[\frac{8}{3}\partial_0^2+22\partial^2]\delta^4(x-x')+\frac{764}{3}a^2a'H^3\delta^4(x-x')$
16	$-[\frac{14}{5}\partial_0^2+\frac{23}{30}\partial^2]\delta^4(x-x')-\frac{121}{3}aa'H^2\delta^4(x-x')+7aH\partial_0\delta^4(x-x')$
18	$-[\frac{56}{5}\partial_0^2+\frac{61}{15}\partial^2]\delta^4(x-x')+92aa'H^2\delta^4(x-x')$
19	$\frac{28}{5}\partial_0\delta^4(x-x')-\frac{50}{3}aH\delta^4(x-x')$
21	$-\frac{14}{5}\delta^4(x-x')$

Table 8: Each of the tabulated terms must be multiplied by $\frac{\kappa^2\ln(a)}{32\pi^2}$.

To see that primitive divergences are free of D-dependent scale factors, note first that the two nonlocal diagrams of Figure 1, corresponding to the generic expression (18), acquire a factor of $(aa')^{D-2}$ from the two 3-point vertices. The D-dependence of these vertex scale factors is cancelled by scale factors

from the two propagators. The most singular part of each propagator is,

$$\frac{H^{D-2}\Gamma(\frac{D}{2}-1)}{(4\pi)^{\frac{D}{2}}}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} = \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}}\Bigl(\frac{1}{aa'\Delta x^2}\Bigr)^{\frac{D}{2}-1} . \tag{50}$$

Less singular terms differ among the various propagators, but their scale factors all have the form $(aa')^{1-\frac{D}{2}} \times (aa')^N$ necessary to cancel the D-dependence of the vertex scale factors.

2.2.3 The Schwinger-Keldysh Result

Even though the graviton field is Hermitian, the nonlocal factors (48) and (38-40) are neither real nor causal because the Feynman diagrams from which they derive are in-out matrix elements rather than expectation values. We can derive true expectation values using the Schwinger-Keldysh formalism [13–17] which is a diagrammatic technique that is almost as simple as the Feynman rules. These expectation values obey effective field equations that are real and causal, albeit nonlocal [18–20].

There is no point to deriving the rules for converting the 1PI N-point functions such as $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ from in-out amplitudes to the Schwinger-Keldysh formalism. We merely list the rules [21]:

- Spacetime points carry a $\pm$ polarity.
- Because propagators have two points, each with two polarities, there are four Schwinger-Keldysh propagators $i\Delta_{\pm\pm}(x;x')$. The $++$ case is just the Feynman propagator, whereas the $--$ case is its conjugate. The $-+$ propagator is the free expectation value of the field at x^μ times the field at x'^μ, and the $+-$ propagator is the free expectation value of the reverse-ordered product.
- Each vertex has a $\pm$ polarity. The $+$ vertices are the same as those of the in-out formalism while the $-$ vertices are complex conjugates.
- Every in-out 1PI N-point function gives rise to 2^N N-point functions in the Schwinger-Keldysh formalism.
- The factor of $[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ in the linearized quantum Einstein equation (3) is replaced by the sum of $[^{\mu\nu}\Sigma^{\rho\sigma}_{++}](x;x')$, which is the same as the in-out result, and $[^{\mu\nu}\Sigma^{\rho\sigma}_{+-}](x;x')$.

- On our simple background (2), one can infer the result for $[^{\mu\nu}\Sigma^{\rho\sigma}_{+-}](x;x')$ from that for $[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ by dropping all the local contributions of Table 8, multiplying the nonlocal terms by -1, and converting the coordinate interval Δx^2 from

$$\Delta x^2_{++}(x;x') \equiv \Bigl\Vert \vec{x} - \vec{x}' \Bigr\Vert^2 - \Bigl(|\eta - \eta'| - i\varepsilon \Bigr)^2 \; , \tag{51}$$

to

$$\Delta x^2_{+-}(x;x') \equiv \Bigl\Vert \vec{x} - \vec{x}' \Bigr\Vert^2 - \Bigl(\eta - \eta' + i\varepsilon \Bigr)^2 \; . \tag{52}$$

Implementing these rules is straightforward. First, recall that the only dependence on the coordinate interval Δx^2 in the nonlocal results of Tables 5, 6 and 7 comes through the integrable functions A_{1-3}, B_{12} and C_{1-2}, which were defined in expressions (38-40) and (48). We can eliminate the factors of $1/\Delta x^2$ using identities (128-136) of the Appendix. For example, the $++$ and $+-$ versions of $2\pi i \times A_1$ are,

$$2\pi i \times A_1 = \frac{\ln(H^2 \Delta x^2_{+\pm})}{\Delta x^2_{+\pm}} = \frac{\partial^2}{8} \Bigl[\ln^2(H^2 \Delta x^2_{+\pm}) - 2 \ln(H^2 \Delta x^2_{+\pm}) \Bigr] \; . \tag{53}$$

Because the scale factors and derivatives are identical in the $++$ and $+-$ contributions, we just need to consider differences of logarithms,

$$\ln(H^2 \Delta x^2_{++}) - \ln(H^2 \Delta x^2_{+-}) = 2\pi i \times \theta(\Delta\eta - r) \; , \tag{54}$$

$$\ln^2(H^2 \Delta x^2_{++}) - \ln^2(H^2 \Delta x^2_{+-}) = 4\pi i \times \theta(\Delta\eta - r) \ln[H^2(\Delta\eta^2 - r^2)] \; , \tag{55}$$

where $r \equiv \Vert \vec{x} - \vec{x}' \Vert$. For example, the factors of A_1 on Table 5 have the Schwinger-Keldysh correspondence,

$$A_1 \longrightarrow + \frac{\partial^2}{4} \Biggl\{ \theta(\Delta\eta - r) \Bigl[\ln[H^2(\Delta\eta^2 - r^2)] - 1 \Bigr] \Biggr\} \; . \tag{56}$$

Identities (137-143) in the Appendix give the reductions needed for any of the integrable functions A_{1-3}, B_{1-2} and C_{1-2}.

2.3 The 4-Point Contribution

The previous discussion concerned the two nonlocal diagrams of Figure 1, and the local counterterms needed to renormalize them. There are also finite

local contributions from the 3rd diagram. It derives from the 42 4-graviton interactions given in equation (4.1) of [9]. One connects two of the graviton fields to the external legs and then replaces the remaining two fields by graviton propagator. The procedure is tedious and we shall content ourselves with simply sketching it and giving the final result.

As an example we reduce the first of the 42 interactions,

$$S_1 \equiv \frac{\kappa^2}{32} \int d^D x \, a^{D-2} h^2 h_{,\theta} h^{,\theta} \; , \tag{57}$$

where a comma denotes differentiation and the trace of the graviton field is $h \equiv h^{\alpha}_{\;\alpha} \equiv \eta^{\alpha\beta} h_{\alpha\beta}$. We first take variational derivatives of the action integral with respect to $h_{\mu\nu}(x)$ and $h_{\rho\sigma}(x')$ as in expression (15),

$$\frac{i\delta^2 S_1}{\delta h_{\mu\nu}(x) \delta h_{\rho\sigma}(x')} = \frac{\kappa^2}{32} \eta^{\mu\nu} \eta^{\rho\sigma} \Big\{ -\partial_\theta \Big[2a^{D-2} h^{\alpha}_{\;\;\alpha}(x) h^{\beta}_{\;\;\beta}(x) \partial^\theta i\delta^D(x-x') \Big]$$
$$+4a^{D-2} h^{\alpha}_{\;\;\alpha}(x) h^{\beta}_{\;\;\beta,\theta}(x) \partial^\theta i\delta^D(x-x') - 4\partial^\theta \Big[a^{D-2} h^{\alpha}_{\;\;\alpha}(x) h^{\beta}_{\;\;\beta,\theta}(x) i\delta^D(x-x') \Big]$$
$$+2a^{D-2} h^{\alpha}_{\;\;\alpha,\theta}(x) h^{\beta}_{\;\;\beta,\theta}(x) \Big] i\delta^D(x-x') \Big\} . \tag{58}$$

Now compute the expectation value of the T^*-ordered product, which amounts to replacing the remaining two graviton fields of each term by the appropriate coincident (and sometimes differentiated) propagator,

$$\Big\langle \Omega \Big| T^* \Big[\frac{i\delta^2 S_1}{\delta h_{\mu\nu}(x) \delta h_{\rho\sigma}(x')} \Big] \Big| \Omega \Big\rangle = \frac{\kappa^2}{32} \eta^{\mu\nu} \eta^{\rho\sigma} \Big\{ -\partial_\theta \Big[2a^{D-2} \times i \Big[{}^{\alpha}_{\alpha} \Delta^{\beta}_{\;\beta} \Big](x;x)$$
$$\times \partial^\theta i\delta^D(x-x') \Big] + 4a^{D-2} \times \partial'_\theta i \Big[{}^{\alpha}_{\alpha} \Delta^{\beta}_{\;\beta} \Big](x;x') \Big|_{x'=x} \times \partial^\theta i\delta^D(x-x') - 4\partial^\theta \Big[a^{D-2}$$
$$\times i\delta^D(x-x') \times \partial'_\theta i \Big[{}^{\alpha}_{\alpha} \Delta^{\beta}_{\;\beta} \Big](x;x') \Big] + 2a^{D-2} i\delta^D(x-x') \times \partial_\theta \partial'^\theta i \Big[{}^{\alpha}_{\alpha} \Delta^{\beta}_{\;\beta} \Big](x;x') \Big\} . \tag{59}$$

Finally, we express the tensor structure using the 21 basis tensors of Table 1,

$$\eta^{\mu\nu} \eta^{\rho\sigma} = \Big(\overline{\eta}^{\mu\nu} - \delta^{\mu}_{\;0} \delta^{\nu}_{\;0} \Big) \Big(\overline{\eta}^{\rho\sigma} - \delta^{\rho}_{\;0} \delta^{\sigma}_{\;0} \Big) \; , \tag{60}$$
$$= \Big[{}^{\mu\nu}\mathcal{D}^{\rho\sigma}_{1} \Big] - \Big[{}^{\mu\nu}\mathcal{D}^{\rho\sigma}_{3} \Big] - \Big[{}^{\mu\nu}\mathcal{D}^{\rho\sigma}_{4} \Big] + \Big[{}^{\mu\nu}\mathcal{D}^{\rho\sigma}_{13} \Big] \; . \tag{61}$$

The coincidence limits of the three propagators which appear in the graviton propagator (23) are,

$$i\Delta_A(x;x) = k \Big[-\pi \cot\Big(\frac{\pi D}{2} \Big) + 2 \ln(a) \Big] \qquad , \qquad i\Delta_B(x;x) = -\frac{k}{D-2} \; , \tag{62}$$

$$i\Delta_C(x;x) = \frac{k}{(D-2)(D-3)} \quad , \quad k \equiv \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \; . \tag{63}$$

Note that only the undifferentiated A-type propagator is ultraviolet divergent in dimensional regularization. The undifferentiated A-type propagator is also the only way to get a factor of $\ln(a)$. First derivatives of coincident propagators are all finite,

$$\partial_\alpha i\Delta_A(x;x')\Big|_{x'=x} = aHk\delta^0{}_\alpha \;\; , \;\; \partial_\alpha i\Delta_B(x;x')\Big|_{x'=x} = 0 = \partial_\alpha i\Delta_C(x;x')\Big|_{x'=x} \; . \tag{64}$$

Mixed second derivatives are also finite,

$$\partial_\alpha \partial'_\beta i\Delta_A(x;x')\Big|_{x'=x} = -\Big(\frac{D-1}{D}\Big) kH^2 g_{\alpha\beta} \; , \tag{65}$$

$$\partial_\alpha \partial'_\beta i\Delta_B(x;x')\Big|_{x'=x} = \frac{1}{D} kH^2 g_{\alpha\beta} \; , \tag{66}$$

$$\partial_\alpha \partial'_\beta i\Delta_C(x;x')\Big|_{x'=x} = -\frac{2}{D(D-2)} kH^2 g_{\alpha\beta} \; . \tag{67}$$

i	Nonzero contributions to $iT^i_{SK}(x;x')$ from the 4-point diagram
1	$-8a^2H^2(\partial_0+2aH)\partial_0\delta^4(x-x')$
2	$8a^2H^2(\partial_0+2aH)\partial_0\delta^4(x-x')$
3	$-8a^2H^2[\nabla^2+aH\partial_0+3a^2H^2]\delta^4(x-x')$
5	$16a^2H^2(\partial_0+2aH)\delta^4(x-x')$
9	$16a^2H^2\nabla^2\delta^4(x-x')$
10	$-16a^2H^2\partial_0\delta^4(x-x')$
13	$-72a^4H^4\delta^4(x-x')$
14	$-16a^3H^3\delta^4(x-x')$
16	$8a^2H^2\delta^4(x-x')$
18	$-16a^2H^2\delta^4(x-x')$

Table 9: Each of the tabulated terms must be multiplied by $\frac{\kappa^2\ln(a)}{32\pi^2}$.

Note that all primitive contributions have factors of a^{D-2}, a^{D-1} or a^D. The counterterms which absorb ultraviolet divergences possess the very same dependence on a so renormalization engenders no finite factors of $\ln(a)$ the way it did for the nonlocal diagrams of expression (49). It does produce factors of $\ln(H/\mu)$ but we report only the $\ln(a)$ contributions in Table 9.

2.4 Anomalous Local Contributions

Our result for the renormalized self-energy consists of the local contributions, collected in Tables 8 and 9, plus the nonlocal contributions of Tables 5, 6 and 7. The nonlocal contributions obey the Ward identity, just as did the noncoincident, $D = 4$ result [7] from which they were inferred. However, it turns out that the local contributions do not. It is possible that the missing terms are associated with contributions from the first two (nonlocal) diagrams of Figure 1 in which an A-type propagator is undifferentiated and the derivatives on the other propagator are contracted into one another,

$$\kappa a^{D-2} \times i\Delta_A(x;x') \times \partial^\mu \partial'_\mu i\Delta(x;x') \times \kappa a'^{D-2} \; . \tag{68}$$

In that case the contracted derivatives would produce a delta function not recovered by the noncoincident, $D = 4$ result [7],

$$\partial^\mu \partial'_\mu i\Delta(x;x') = -\frac{i\delta^D(x-x')}{a^{D-2}} + O\Bigl(\frac{1}{\Delta x^{D-2}}\Bigr) \; . \tag{69}$$

It is also possible that the Feynman rules need to include contributions from the functional measure factor. A fully dimensionally regulated calculation would seem to be necessary to resolve this. One should also re-examine the contribution from a loop of massless, minimally coupled scalars [22] to see if it shows similar anomalous local contributions. In the meantime, we can proceed with the nonlocal contributions because it turns out that the local contributions do not affect the potentials as strongly at late times and large distances.

3 The Effect on the Force of Gravity

In this section we solve the effective field equations (3) to find one loop corrections to the gravitational response to a point mass. Our first step is to

specialize the general equation (3) appropriately for a perturbative determination of the potentials. We next compute the source terms induced by integrating the one loop self-energy against the classical potentials. We close the section by solving for the leading one loop corrections at late times and large distances.

3.1 Equations for the Potentials

The linearized stress-energy for a static point mass is,

$$8\pi G T^{\mu\nu}_{\rm lin}(x) = 8\pi G M a \delta^3(\vec{x}) \; . \tag{70}$$

The gravitational response to such a source is given by four scalar potentials,

$$\kappa h_{00} \equiv -2\Psi \quad , \quad \kappa h_{0i} \equiv -\partial_i \Omega \quad , \quad \kappa h_{ij} \equiv -2\delta_{ij}\Phi - 2\partial_i\partial_j \chi \; . \tag{71}$$

We can derive an equation for Ψ from the sum of the $\mu = 0 = \nu$ and the spatial trace,

$$\Big[\mathcal{D}^{00\rho\sigma} + \mathcal{D}^{kk\rho\sigma}\Big] \kappa h_{\rho\sigma}(x) = -2D_B \Psi(x) = 8\pi G M a \delta^3(\vec{x}) \\ + \int d^4x' \Big\{ \Big[{}^{00}\Sigma^{\rho\sigma}\Big](x;x') + \Big[{}^{kk}\Sigma^{\rho\sigma}\Big](x;x') \Big\} \kappa h_{\rho\sigma}(x') \; , \tag{72}$$

where D_B is the kinetic operator of a conformally coupled scalar (9). The $\mu = 0, \nu = i$ components give an equation for Ω,

$$\mathcal{D}^{0i\rho\sigma} \kappa h_{\rho\sigma} = \frac{\partial_i}{2} D_B \Omega(x) = 0 + \int d^4x' \Big[{}^{0i}\Sigma^{\rho\sigma}\Big](x;x') \kappa h_{\rho\sigma}(x') \; . \tag{73}$$

And the equation for χ and $\Psi - \Phi$ is,

$$\Big[\mathcal{D}^{ij\rho\sigma} - \delta^{ij}\mathcal{D}^{kk\rho\sigma}\Big] \kappa h_{\rho\sigma} = -\partial^i\partial^j D_A \chi + \delta^{ij} D_A(\Psi - \Phi) = 0 \\ + \int d^4x' \Big\{ \Big[{}^{ij}\Sigma^{\rho\sigma}\Big](x;x') - \delta^{ij}\Big[{}^{kk}\Sigma^{\rho\sigma}\Big](x;x') \Big\} \kappa h_{\rho\sigma}(x') \; , \tag{74}$$

where D_A is the kinetic operator of a massless, minimally coupled scalar (5).

Although equations (72-74) are correct, they cannot be solved exactly because we only possess one loop results for the graviton self-energy. This means we must develop perturbative solutions,

$$\Psi = \Psi_0 + \kappa^2 \Psi_1 + \kappa^4 \Psi_2 + \ldots \; , \tag{75}$$

and so on for the other potentials. The zeroth order solutions are,

$$\Psi_0(x) = \Phi_0(x) = \frac{GM}{ar} \qquad , \qquad \Omega_0(x) = \chi_0(x) = 0 \; . \tag{76}$$

It is only these zeroth order potentials that appear on the right hand side of equations (72-74). If we use the symbol $T^i(x;x')$ to stand for just the one loop contribution to the graviton self-energy then the one loop correction to Ψ is given by,

$$\begin{aligned} -2D_B\kappa^2\Psi_1(x) = \int d^4x' \Big\{ \Big[9iT^1 + 3iT^2 + 3(iT^3 + iT^4) + iT^{13} \Big] \\ + \Big[3(iT^7 + iT^8) + iT^{12} + (iT^{16} + iT^{17}) \Big] \nabla'^2 + iT^{21} \nabla'^4 \Big\} \times -2\Psi_0(x') \; . \end{aligned} \tag{77}$$

The equations for Ω_1 and χ_1 are,

$$D_B\kappa^2\Omega_1(x) = \int d^4x' \Big\{ \Big[3iT^6 + iT^{10} + iT^{15} \Big] + iT^{19} \nabla'^2 \Big\} \times -2\Psi_0(x') \; , \tag{78}$$

$$D_A\kappa^2\chi_1(x) = -\int d^4x' \Big\{ \Big[3iT^8 + iT^{12} + iT^{17} \Big] + iT^{21} \nabla'^2 \Big\} \times -2\Psi_0(x') \; . \tag{79}$$

And the equation for the gravitational slip is,

$$\begin{aligned} D_A\kappa^2 \Big[\Psi_1(x) - \Phi_1(x) \Big] = -\int d^4x' \Big\{ \Big[6iT^1 + 2iT^2 + 2iT^3 \Big] \\ + \Big[2iT^7 + 3iT^8 + iT^{12} + iT^{17} \Big] \nabla'^2 + iT^{21} \nabla'^4 \Big\} \times -2\Psi_0(x') \; . \end{aligned} \tag{80}$$

3.2 Performing the Source Integrations

From equation (77) we see that Ψ_1 is sourced by various combinations of the Schwinger-Keldysh coefficient functions multiplied by zero, one or two powers of ∇'^2 acing on $-2\Psi_0(x')$. Having a factor of ∇'^2 simplifies the source integration enormously because,

$$\nabla'^2 \times -2\Psi_0(x') = \frac{8\pi GM \delta^3(\vec{x'})}{a'} \tag{81}$$

The ∇'^4 source comes entirely from $iT^{21}_{SK} = \frac{\kappa^2}{32\pi^3} \times A_3$. Table 10 gives the combination which multiplies ∇'^2.

Operator	Factor
$-12aa'H^2\partial_0^2 + 8aa'(a-a')H^3\partial_0 - 8a^2a'^2H^4$	A_1
$-18a^2a'^2H^4\partial_0$	B_1
$-12a^3a'^3H^6$	C_1
$\frac{22}{3}aa'H^2\partial_0^2+\frac{14}{3}aa'(a-a')H^3\partial_0-\frac{31}{3}a^2a'^2H^4$	A_2
$\frac{31}{3}a^2a'^2H^4\partial_0+4a^2a'^2(a-a')H^5$	B_2
$-2a^3a'^3H^6$	C_2
$-(\frac{7}{5}\partial_0^2-\frac{107}{60}\partial^2)\partial^2+9(a-a')H\partial_0\partial^2-\frac{37}{3}aa'H^2\partial^2$	A_3
$-\ln(a)(\frac{28}{5}\partial_0^2-\frac{107}{15}\partial^2)-\frac{100}{3}\ln(a)a^2H^2$	$\delta^4(\Delta x)$

Table 10: Contributions for $3(iT^7+iT^8)+iT^{12}+(iT^{16}+iT^{17})$. Each tabulated term must be multiplied by $\frac{\kappa^2}{32\pi^3}$.

These ∇'^2 source integrations can be evaluated exactly, for example,

$$\frac{\kappa^2}{32\pi^3}\int d^4x'[-8a^2a'^2H^4]A_1(x;x')\times\frac{8\pi GM\delta^3(\vec{x}')}{a'}$$
$$= -\frac{GM\kappa^2H^4a^2\partial^2}{2\pi^2}\int_{\eta_i}^{\eta-r}d\eta' a'\Big\{\ln\Big[H^2(\Delta\eta^2-r^2)\Big]-1\Big\}\;, \tag{82}$$
$$= \frac{GM\kappa^2H^3a^2\partial^2}{2\pi^2}\Big\{\ln^2\Big(Hr+\frac{1}{a}\Big)-\ln\Big(Hr+\frac{1}{a}\Big)+\sum_{n=1}^{\infty}\frac{1}{n^2}\Big[1-\Big(Hr+\frac{1}{a}\Big)^n\Big]$$
$$+\sum_{n=1}^{\infty}\frac{1}{n^2}\Big[\Big(\frac{Hr-\frac{1}{a}}{Hr+\frac{1}{a}}\Big)^n-\Big(Hr-\frac{1}{a}\Big)^n\Big]\Big\}. \tag{83}$$

However, all that really matters for us is the limiting form for $aHr\gg 1$ with $Hr\ll 1$,

$$\frac{\kappa^2}{32\pi^3}\int d^4x'[-8a^2a'^2H^4]A_1(x;x')\times\frac{8\pi GM\delta^3(\vec{x}')}{a'}$$
$$\longrightarrow\frac{2GM\kappa^2H^3a^2\ln(Hr)}{\pi^2r^2}\;. \tag{84}$$

Operator	Factor
$-4aa'H^2\partial_0^4+28a^2a'^2H^4\partial_0^2+24a^2a'^2(a-a')H^5\partial_0+56a^3a'^3H^4$	A_1
$2a^2a'^2H^4\partial_0^3-6a^2a'^2(a-a')H^5\partial_0^2-52a^3a'^3H^6\partial_0$	B_1
$10a^3a'^3H^6\partial_0^2$	C_1
$\frac{79}{6}aa'H^2\partial_0^4+\frac{37}{2}aa'(a-a')H^3\partial_0^3+\frac{631}{2}a^2a'^2H^4\partial_0^2$ $-24a^2a'^2(a-a')H^5\partial_0-16a^3a'^3H^6$	A_2
$-\frac{83}{2}a^2a'^2H^4\partial_0^3-9a^2a'^2(a-a')H^5\partial_0^2+34a^3a'^3H^6\partial_0$	B_2
$-9a^3a'^3H^6\partial_0^2$	C_2
$-(\frac{7}{10}\partial_0^4+\frac{1}{4}\partial_0^2\partial^2+\frac{14}{5}\partial^4)\partial^2+\frac{29}{4}(a-a')H(\partial_0^2+2\partial^2)\partial_0\partial^2$ $+aa'H^2(\frac{817}{12}\partial_0^2+\frac{478}{12}\partial^2)\partial^2+\frac{215}{2}aa'(a-a')H^3\partial_0\partial^2+\frac{315}{2}a^2a'^2H^4\partial^2$	A_3
$-\ln(a)(\frac{14}{5}\partial_0^4+\partial_0^2\partial^2+\frac{56}{5}\partial^4)+\ln(a)a^2H^2(\frac{529}{3}\partial_0^2+\frac{334}{3}\partial^2)$ $-96\ln(a)a^3H^3\partial_0+486\ln(a)a^4H^4$	$\delta^4(\Delta x)$

Table 11: Contributions for $9iT^1+3i^2+3(iT^3+iT^4)+iT^{13}$. Each tabulated term must be multiplied by $\frac{\kappa^2}{32\pi^3}$.

Table 11 gives the combination of coefficient functions contributing to $\Psi_1(\eta,r)$ which carry no factors of ∇'^2. These terms cannot be evaluated exactly, but there is no problem getting them in the limit $aHr \gg 1$ and $Hr \ll 1$. Consider the example,

$$\frac{\kappa^2}{32\pi^3}\int d^4x'[56a^3a'^3H^4]A_1(x;x')\times-\frac{2GM}{a'r'}$$

$$=-\frac{7GM\kappa^2H^4a^3\partial^2}{8\pi^3}\int d^4x'\frac{a'^2\theta(\Delta\eta-r')}{\|\vec{x}+\vec{x}'\|}\Big\{\ln\Big[H^2(\Delta\eta^2-r'^2)\Big]-1\Big\}\,, \quad (85)$$

$$=-\frac{7GM\kappa^2H^4a^3\partial^2}{2\pi^2}\int_{\eta_i}^{\eta}d\eta'a'^2\int_0^{\Delta\eta}dr'r'^2\Big[\frac{\theta(r-r')}{r}+\frac{\theta(r'-r)}{r'}\Big]$$
$$\times\Big\{\ln\Big[H^2(\Delta\eta^2-r'^2)\Big]-1\Big\}\,, \quad (86)$$

$$\longrightarrow\frac{7GM\kappa^2H^4a^3\partial_0^2}{2\pi^2r}\int_{\eta_i}^{\eta}d\eta'a'^2\int_0^{\Delta\eta}dr'r'^2\Big\{\ln\Big[H^2(\Delta\eta^2-r'^2)\Big]-1\Big\}\,, \quad (87)$$

$$\longrightarrow -\frac{7GM\kappa^2 H^2 a^3 \ln^2(a)}{\pi^2 r} \; . \tag{88}$$

When all the Ψ_1 source contributions are included, the leading late time result is,

$$-2D_B\kappa^2\Psi_1 \longrightarrow -\frac{3GM\kappa^2 H^4 a^4[\ln^2(a) - \ln(Hr)]}{\pi^2 ar} + O(a^2) \; . \tag{89}$$

Operator	Factor
$12aa'H^2\partial_0^3 - 4aa'(a-2a')H^3\partial_0^2 - 12a^2a'^2H^4\partial_0 + 8a^2a'^2(a-2a')H^5$	A_1
$10a^2a'^2H^4\partial_0^2 - 4a^2a'^2(a+a')\partial_0 - 12a^3a'^3H^6$	B_1
$4a^3a'^3H^6\partial_0$	C_1
$-\frac{11}{3}aa'H^2\partial_0^3 - aa'(\frac{14}{3}a - \frac{74}{3}a')H^3\partial_0^2 - 4a^2a'^2H^4\partial_0 + 6a^2a'^2(a-a')H^5$	A_2
$\frac{53}{3}a^2a'^2H^4\partial_0^2 - 6a^2a'^2(a-a')H^5\partial_0 - 12a^3a'^3H^6$	B_2
$6a^3a'^3H^6\partial_0$	C_2
$(\frac{7}{5}\partial_0^2 - \frac{23}{30}\partial^2)\partial_0\partial^2 + \frac{1}{6}\partial^2(83a'\partial_0^2 - 4a\partial^2 + 54a'\partial^2)H$ $-\frac{91}{3}aa'H^2\partial_0\partial^2 + aa'(\frac{29}{3}a - \frac{77}{3}a')H^3\partial^2$	A_3
$\ln(a)[(\frac{28}{5}\partial_0^2 - \frac{46}{15}\partial^2)\partial_0 + aH(\frac{166}{3}\partial_0^2 + \frac{100}{3}\partial^2) - \frac{268}{3}a^2H^2\partial_0 - 48a^3H^3]$	$\delta^4(\Delta x)$

Table 12: Contributions for $3iT^6 + iT^{10} + iT^{15}$. Each tabulated term must be multiplied by $\frac{\kappa^2}{32\pi^3}$.

Equation (89) shows a source term for Ψ_1 which grows like a^3; we ignore sources with fewer factors of a. Table 12 gives the combinations of coefficient function which contribute to Ω_1 and involve no factors of ∇'^2. There is an additional source involving $iT^{19} \times \nabla'^2$. When the various source integrations are evaluated, and the late time form taken, the result is no contributions of order a^3,

$$D_B\kappa^2\Omega_1 \longrightarrow 0 + O(a^2) \; .. \tag{90}$$

Table 13 gives the Ω_1 source contributions which contain no factors of ∇'^2. There is an additional contribution involving $iT^{21}\nabla'^2$. When the source

integrations are performed the result is,

$$D_A\kappa^2\chi_1 = -\frac{GM\kappa^2H^2a^4[5-\ln(16)]}{8\pi^2ar} + O(a^2) \; . \tag{91}$$

Operator	Factor
$-8aa'H^2\partial_0^2 - 8aa'^2H^3\partial_0 + 8a^2a'^2H^4$	A_1
$-12a^2a'^2H^4\partial_0$	B_1
$-4a^3a'^3H^6$	C_1
$\frac{4}{3}aa'H^2\partial_0^2 - \frac{14}{3}aa'^2H^3\partial_0 + \frac{8}{3}a^2a'^2H^4$	A_2
$\frac{4}{3}a^2a'^2H^4\partial_0 - 4a^2a'^3H^5$	B_2
$-2a^3a'^3H^6$	C_2
$-(\frac{7}{10}\partial_0^2 - \frac{7}{5}\partial^2)\partial^2 - 9a'H\partial_0\partial^2 - 5aa'H^2\partial^2$	A_3
$-\ln(a)(\frac{14}{5}\partial_0^2 - \frac{28}{5}\partial^2) - 36\ln(a)aH\partial_0 - 12\ln(a)a^2H^2$	$\delta^4(\Delta x)$

Table 13: Contributions for $3iT^8 + iT^{12} + iT^{17}$. Each tabulated term must be multiplied by $\frac{\kappa^2}{32\pi^3}$.

Tables 14 and 15 give the source combinations for the gravitational slip which contain no factor of ∇'^2 and one factor of it, respectively. When the $iT^{21} \times \nabla'^4$ contribution is added, the leading late time result is,

$$D_A\kappa^2\Big[\Psi_1 - \Phi_1\Big] = \frac{GM\kappa^2H^4a^4[4\ln^2(a) - 3\ln(Hr)]}{\pi^2ar} + O(a^2) \; . \tag{92}$$

3.3 Solving for the Potentials

Equations (89), (90), (91) and (92) determine 1-loop corrections to the various potentials. It would be straightforward to express the potentials as integrals over the sources because we possess the exact Green's functions for D_A and D_B,

$$G_A(x;x') = -\frac{1}{4\pi}\Bigg\{\frac{\delta(\Delta\eta - \Delta r)}{aa'\Delta r} + H^2\theta(\Delta\eta - \Delta r)\Bigg\} \; , \tag{93}$$

$$G_B(x;x') = -\frac{1}{4\pi}\frac{\delta(\Delta\eta - \Delta r)}{aa'\Delta r} \; . \tag{94}$$

Operator	Factor
$8a^2a'^2H^4\partial_0^2+16a^3a'^2H^5\partial_0+48a^3a'^3H^6$	A_1
$-4a^3a'^2H^5\partial_0^2-24a^3a'^3H^6\partial_0$	B_1
$4a^3a'^3H^6\partial_0^2$	C_1
$6aa'H^2\partial_0^4+\frac{37}{3}a^2a'H^3\partial_0^3+46a^2a'^2H^4\partial_0^2-16a^3a'^2H^5\partial_0-16a^3a'^3H^6$	A_2
$-24a^2a'^2H^4\partial_0^3-6a^3a'^2H^5\partial_0^2+52a^3a'^3H^6\partial_0$	B_2
$-18a^3a'^3H^6\partial_0^2$	C_2
$(\frac{23}{60}\partial_0^2-\frac{107}{60}\partial^2)\partial^4+\frac{29}{6}aH(\partial_0^2+2\partial^2)\partial_0\partial^2+aa'H^2(\frac{335}{6}\partial_0^2+\frac{65}{3}\partial^2)\partial^2$ $+\frac{215}{3}a^2a'H^3\partial_0\partial^2+56a^2a'^2H^4\partial^2$	A_3
$\ln(a)[(\frac{23}{15}\partial_0^2-\frac{107}{15}\partial^2)\partial^2+\frac{58}{3}aH(\partial_0^2+2\partial^2)\partial_0+a^2H^2(\frac{526}{3}\partial_0^2+\frac{212}{3}\partial^2)$ $+\frac{620}{3}a^3H^3\partial_0+176a^4H^4]$	$\delta^4(\Delta x)$

Table 14: Contributions for $6iT^1+2iT^2+2iT^3$. Each tabulated term must be multiplied by $\frac{\kappa^2}{32\pi^3}$.

Operator	Factor
$-8aa'H^2\partial_0^2-8aa'^2H^3\partial_0-8a^2a'^2H^4$	A_1
$-12a^2a'^2H^4\partial_0-4a^3a'^2H^5$	B_1
$-8a^3a'^3H^6$	C_1
$\frac{22}{3}aa'H^2\partial_0^2-aa'(\frac{23}{3}a+\frac{14}{3}a')H^3\partial_0-\frac{34}{3}a^2a'^2H^4$	A_2
$\frac{28}{3}a^2a'^2H^4\partial_0+2a^2a'^2(a-2a')H^5$	B_2
0	C_2
$-(\frac{7}{10}\partial_0^2-\frac{107}{60}\partial^2)\partial^2+(\frac{29}{6}a-9a')H\partial_0\partial^2-\frac{19}{6}aa'H^2\partial^2$	A_3
$-\ln(a)(\frac{14}{5}\partial_0^2-\frac{107}{15}\partial^2)-\frac{50}{3}\ln(a)aH\partial_0-\frac{14}{3}\ln(a)a^2H^2$	$\delta^4(\Delta x)$

Table 15: Contributions for $2iT^7+3iT^8+iT^{12}+iT^{17}$. Each tabulated term must be multiplied by $\frac{\kappa^2}{32\pi^3}$.

However, this would be overkill because the various sources are only known for late times. It is better instead to change the temporal variable from η to the scale factor a, and then extract a factor of $-a^4H^2$ from the two differential operators,

$$D_A = -a^4H^2\Big[a^2\frac{\partial^2}{\partial a^2} + 4a\frac{\partial}{\partial a} - \frac{\nabla^2}{a^2H^2}\Big] , \tag{95}$$

$$D_B = -a^4H^2\Big[a^2\frac{\partial^2}{\partial a^2} + 4a\frac{\partial}{\partial a} + 2 - \frac{\nabla^2}{a^2H^2}\Big] . \tag{96}$$

The advantage of this form is that the temporal differential operators inside the brackets neither increase nor decrease number of scale factors, while the effect of the spatial derivatives is sub-dominant at late times. It is therefore trivial to invert D_A and D_B to the leading late time form for the relevant sources,

$$D_Af(a) = -a^4H^2 \times \frac{[\alpha\ln^2(a)+\beta\ln(Hr)]}{ar} , \tag{97}$$

$$\Longrightarrow f(a) \longrightarrow -\frac{[\alpha\ln^2(a)+\beta\ln(Hr)]}{2ar} , \tag{98}$$

$$D_Bg(a) = -a^4H^2 \times \frac{[\gamma\ln^2(a)+\delta\ln(Hr)]}{ar} , \tag{99}$$

$$\Longrightarrow g(a) \longrightarrow \frac{[\frac{1}{3}\gamma\ln^3(a) + \delta\ln(a)\ln(Hr)]}{ar} . \tag{100}$$

Applying expression (100) to equations (89) and (90) gives,

$$\kappa^2\Psi_1(\eta,r) \longrightarrow \frac{2GM}{ar}\Big\{-\frac{4GH^2\ln^3(a)}{\pi} + \frac{12GH^2\ln(a)\ln(Hr)}{\pi}\Big\} , \tag{101}$$

$$\kappa^2\Omega_1(\eta,r) \longrightarrow 0 . \tag{102}$$

And the last two potentials come from using expression (98) to invert D_A in equations (91) and (92),

$$\kappa^2\chi_1(\eta,r) \longrightarrow \frac{2GM}{ar}\Big\{\frac{[-5+\ln(16)]G}{2\pi}\Big\} , \tag{103}$$

$$\kappa^2(\Psi_1-\Phi_1) \longrightarrow \frac{2GM}{ar}\Big\{\frac{16GH^2\ln^2(a)}{\pi} - \frac{12GH^2\ln(Hr)}{\pi}\Big\} . \tag{104}$$

4 Epilogue

As long as the two points do not coincide, $x^\mu \neq x'^\mu$, no regularization is needed for the 1-loop graviton self-energy $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$. In section 2 of this paper we exploited an old, unregulated computation of the graviton contribution to the graviton self-energy [7] to infer the fully renormalized result. Our answer is expressed as a sum (11) of 21 coefficient functions $T^i(x;x')$, multiplied by basis tensors listed in Table 1. Our results for the renormalized coefficient functions are expressed in Tables 5, 6, 7, 8 and 9, as derivative operators and functions of the two scale factors, acting on $\delta^4(x-x')$ and seven nonlocal functions $A_{1,2,3}(x;x')$, $B_{1,2}(x;x')$ and $C_{1,2}(x;x')$, which are defined in expressions (137-143).

Although the nonlocal contributions obey the Ward Identity away from coincidence, there is a local obstacle proportional to $\ln(a)\delta^4(x-x')$. This obstacle might originate from anomalous contributions (68) to the first two diagrams of Figure 1. Such diagrams would contribute $\ln(a)\delta^4(x-x')$ terms which we would not be able to recognize from the unregulated, noncoincident result. It is also possible that we have missed some exotic, local contributions to the Feynman rules associated with the functional measure factor or time-ordering. More work is required to resolve this issue, and we believe a good venue for this study is the much simpler contribution to $-i[^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ arising from a loop of massless, minimally coupled scalars [22]. Fortunately, the missing local terms do not make leading order contributions to the gravitational potentials.

In section 3 we applied our result to work out the gravitational response to a static point mass (70) at 1-loop. Because the graviton self-energy was computed in a fixed gauge, we had to solve the effective field equations using the same gauge fixing functional [3,4]. This resulted in there being four scalar potentials (71), instead of the usual two. Our final results for the leading late time forms of the four potentials were given in equations (101), (102), (103) and (104). Of particular interest are the Newtonian potential and the gravitational slip,

$$\Psi \longrightarrow \frac{GM}{ar}\Biggl\{1 + \frac{8GH^2}{\pi}\Bigl[-\ln^3(a) + 3\ln(a)\ln(Hr)\Bigr] + \ldots\Biggr\}, \quad (105)$$

$$\Psi - \Phi \longrightarrow \frac{GM}{ar}\Biggl\{0 + \frac{8GH^2}{\pi}\Bigl[4\ln^2(a) - 3\ln(Hr)\Bigr] + \ldots\Biggr\}. \quad (106)$$

It is interesting to compare the effect of graviton contributions to the

Newtonian potential (105) with that from a loop of massless, minimally coupled scalars [23],

$$\Psi_{MMCS} \longrightarrow \frac{GM}{ar}\Biggl\{1 - \frac{GH^2}{10\pi}\Bigl[\frac{1}{3}\ln(a) + 3\ln(aHr)\Bigr] + \ldots\Biggr\} . \tag{107}$$

In both cases the 1-loop correction reduces the gravitational potential, but gravitons induce two additional factors of $\ln(a)$. The same pattern is evident for the gravitational slip, which gets two factors of $\ln(a)$ from gravitons but none at all from scalars [23]. Similarly, the 1-loop correction to the graviton mode function is enhanced by $\ln^2(a)$ [12], but is not affected at all by scalars [24]. We therefore conclude that loops of inflationary gravitons contribute more strongly than matter loops by two large logarithms. It is also noteworthy that graviton loop corrections to gravity are much strong than graviton loop corrections to fermions [25–27], to electrodynamics [28–32], and to massless, minimally coupled scalars [33–35]. The key difference seems to be that graviton loop corrections to gravity can involve two graviton propagators whereas graviton corrections to other fields involve only one.

The appearance of very large logarithms in graviton loop corrections implies the breakdown of perturbation at late times and large distances. It has been difficult to devise a resummation procedure because these logarithms derive from two sources: the "tail" part of the graviton propagator and logarithmic ultraviolet divergences of the form (49) [36]. This led to the speculation that resummation might be accomplished by combining a variant of Starobinsky's stochastic formalism [37, 38] with a variant of the renormalization group. This speculation was recently confirmed in the context of nonlinear sigma models on a nondynamical de Sitter background [39], which possess the same kinds of derivative interactions as quantum gravity and exhibit the same mixture of "tail" and ultraviolet logarithms. The technique has been applied to explain graviton loop corrections to the exchange potential of a massless, minimally coupled scalar [35], and strenuous efforts are underway to devise similar explanations for the collection of large graviton logarithms that have been patiently accumulated by direct computation over the course of two decades.

It is well known that classical modified gravity models also correct the force of gravity [40], and can induce nonzero gravitational slip [41, 42]. One is therefore led to wonder if our results (105) and (106) could be reproduced by some local, metric-based model. The answer seems to be no because the only stable, local, invariant and metric-based modification of gravity is $f(R)$

gravity [43]. However, the modified force induced by these models on de Sitter background depends only on the combination aHr [40], and cannot reproduce the distinct $\ln^3(a)$ and $\ln(a)\ln(Hr)$ terms of our result (105). It should also be noted that neither the scalar nor the tensor amplitudes in these models experience secular growth after horizon crossing [44], unlike the $\ln^2(a)$ dependence we found previously [12].

We close by commenting on the gauge issue. On flat space background the graviton self-energy is known to be highly gauge dependent [45]. Because the $H \to 0$ limit of our result agrees with the flat space limit, our de Sitter graviton self-energy must inherit this gauge dependence. The large logarithms we have found all derive from terms which carry factors of H^2, and their gauge dependence is not known, although indications from gravity plus electromagnetism suggest that there is some [32]. A procedure has been developed for removing this gauge dependence [46], which has been successfully applied on flat space background to graviton loop corrections to the massless, minimally coupled scalar [46], and to electromagnetism [47]. The massless, minimally coupled scalar exchange potential ha been identified as the simplest venue for generalizing this technique to de Sitter background [35], and it is hoped that a result will be available later this year. Based on flat space background experience [46, 47], we expect that the elimination of gauge dependence will not eliminate large graviton logarithms but might change their numerical coefficients.

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5 Appendix: Derivative Identities

This Appendix summarizes the various derivative identities we employ to convert the unregulated results of Tables 3 and 4 to the renormalized Schwinger-Keldysh results of Tables 5, 6, 7 and 8.

5.1 Extracting Derivatives

We begin with the relations needed to write each term as derivatives acting on the four fundamental expressions,

$$\frac{1}{\Delta x^4} \;,\; \frac{1}{\Delta x^2} \;,\; \frac{\Delta\eta}{\Delta x^2} \;,\; \frac{\Delta\eta^2}{\Delta x^2} \,. \tag{108}$$

Terms with large inverse powers of Δx^2 all reach $\frac{1}{\Delta x^4}$,

$$\frac{\Delta\eta^4}{\Delta x^{12}} = \Big[\frac{\partial_0^4}{1920} - \frac{\partial_0^2\partial^2}{640} + \frac{\partial^4}{5120}\Big]\frac{1}{\Delta x^4} \quad , \quad \frac{\Delta\eta^2}{\Delta x^{10}} = \Big[\frac{\partial_0^2\partial^2}{384} - \frac{\partial^4}{1536}\Big]\frac{1}{\Delta x^4}, \tag{109}$$

$$\frac{\Delta\eta^3}{\Delta x^{10}} = \Big[\frac{\partial_0^3}{192} - \frac{\partial_0\partial^2}{128}\Big]\frac{1}{\Delta x^4} \quad , \quad \frac{\Delta\eta}{\Delta x^8} = \frac{\partial_0\partial^2}{48}\Big(\frac{1}{\Delta x^4}\Big), \tag{110}$$

$$\frac{\Delta\eta^2}{\Delta x^8} = \Big[\frac{\partial_0^2}{24} - \frac{\partial^2}{48}\Big]\frac{1}{\Delta x^4} \quad , \quad \frac{1}{\Delta x^8} = \frac{\partial^4}{192}\Big(\frac{1}{\Delta x^4}\Big), \tag{111}$$

$$\frac{\Delta\eta}{\Delta x^6} = \frac{\partial_0}{4}\Big(\frac{1}{\Delta x^4}\Big) \quad , \quad \frac{1}{\Delta x^6} = \frac{\partial^2}{8}\Big(\frac{1}{\Delta x^4}\Big). \tag{112}$$

Terms with $\Delta\eta^4$ divided fewer than six powers of Δx^2 involve all four of the fundamental expressions (108),

$$\frac{\Delta\eta^4}{\Delta x^{10}} = \frac{\partial_0^4}{384}\Big(\frac{1}{\Delta x^2}\Big) - \Big[\frac{\partial_0^2}{32} - \frac{\partial^2}{128}\Big]\Big(\frac{1}{\Delta x^4}\Big) \,, \tag{113}$$

$$\frac{\Delta\eta^4}{\Delta x^8} = \frac{\partial_0^3}{48}\Big(\frac{\Delta\eta}{\Delta x^2}\Big) - \frac{\partial_0^2}{8}\Big(\frac{1}{\Delta x^2}\Big) + \frac{1}{8}\Big(\frac{1}{\Delta x^4}\Big) \,, \tag{114}$$

$$\frac{\Delta\eta^4}{\Delta x^6} = \frac{\partial_0^2}{8}\Big(\frac{\Delta\eta^2}{\Delta x^2}\Big) - \frac{5}{8}\partial_0\Big(\frac{\Delta\eta}{\Delta x^2}\Big) + \frac{3}{8}\Big(\frac{1}{\Delta x^2}\Big) \,. \tag{115}$$

The last relations we require involve fewer powers of both $\Delta\eta$ and Δx^2,

$$\frac{\Delta\eta^3}{\Delta x^8} = \frac{\partial_0^3}{48}\Big(\frac{1}{\Delta x^2}\Big) - \frac{\partial_0}{8}\Big(\frac{1}{\Delta x^4}\Big) \quad , \quad \frac{\Delta\eta^3}{\Delta x^6} = \frac{\partial_0^2}{8}\Big(\frac{\Delta\eta}{\Delta x^2}\Big) - \frac{3}{8}\partial_0\Big(\frac{1}{\Delta x^2}\Big), \tag{116}$$

$$\frac{\Delta\eta^2}{\Delta x^6} = \frac{\partial_0^2}{8}\Big(\frac{1}{\Delta x^2}\Big) - \frac{1}{4}\Big(\frac{1}{\Delta x^4}\Big) \quad , \quad \frac{\Delta\eta^2}{\Delta x^4} = \frac{\partial_0}{2}\Big(\frac{\Delta\eta}{\Delta x^2}\Big) - \frac{1}{2}\Big(\frac{1}{\Delta x^2}\Big), \tag{117}$$

$$\frac{\Delta\eta}{\Delta x^4} = \frac{\partial_0}{2}\Big(\frac{1}{\Delta x^2}\Big) \quad , \quad \frac{\Delta\eta^3}{\Delta x^4} = \frac{\partial_0}{2}\Big(\frac{\Delta\eta^2}{\Delta x^2}\Big) - \Big(\frac{\Delta\eta}{\Delta x^2}\Big). \tag{118}$$

5.2 Absorbing the Factor of $\ln(H^2\Delta x^2)$

The next step is passing derivatives through the factor of $\ln(H^2\Delta x^2)$ that multiplies all terms in Table 3. This is facilitated by the identities,

$$\partial_0\Big(\frac{1}{\Delta x^2}\Big)\times\ln(H^2\Delta x^2) = \partial_0\Big[\frac{\ln(H^2\Delta x^2)}{\Delta x^2}\Big] + \frac{2\Delta\eta}{\Delta x^4} \; , \tag{119}$$

$$\partial_0^2\Big(\frac{1}{\Delta x^2}\Big)\times\ln(H^2\Delta x^2) = \partial_0^2\Big[\frac{\ln(H^2\Delta x^2)}{\Delta x^2}\Big] + \frac{2}{\Delta x^4} + \frac{12\Delta\eta^2}{\Delta x^6} \; , \tag{120}$$

$$\partial_0^3\Big(\frac{1}{\Delta x^2}\Big)\times\ln(H^2\Delta x^2) = \partial_0^3\Big[\frac{\ln(H^2\Delta x^2)}{\Delta x^2}\Big] + \frac{36\Delta\eta}{\Delta x^6} + \frac{88\Delta\eta^3}{\Delta x^8} \; , \tag{121}$$

$$\partial_0^4\Big(\frac{1}{\Delta x^2}\Big)\times\ln(H^2\Delta x^2) = \partial_0^4\Big[\frac{\ln(H^2\Delta x^2)}{\Delta x^2}\Big] + \frac{36}{\Delta x^6} + \frac{528\Delta\eta^2}{\Delta x^8} + \frac{800\Delta\eta^4}{\Delta x^{10}} \; , \tag{122}$$

$$\partial_0\Big(\frac{\Delta\eta}{\Delta x^2}\Big)\times\ln(H^2\Delta x^2) = \partial_0\Big[\frac{\Delta\eta\ln(H^2\Delta x^2)}{\Delta x^2}\Big] + \frac{2\Delta\eta^2}{\Delta x^4} \; , \tag{123}$$

$$\partial_0^2\Big(\frac{\Delta\eta}{\Delta x^2}\Big)\times\ln(H^2\Delta x^2) = \partial_0^2\Big[\frac{\Delta\eta\ln(H^2\Delta x^2)}{\Delta x^2}\Big] + \frac{6\Delta\eta}{\Delta x^4} + \frac{12\Delta\eta^3}{\Delta x^6} \; , \tag{124}$$

$$\partial_0^3\Big(\frac{\Delta\eta}{\Delta x^2}\Big)\times\ln(H^2\Delta x^2) = \partial_0^3\Big[\frac{\Delta\eta\ln(H^2\Delta x^2)}{\Delta x^2}\Big] + \frac{6}{\Delta x^4} + \frac{72\Delta\eta^2}{\Delta x^6} + \frac{88\Delta\eta^4}{\Delta x^8} \; , \tag{125}$$

$$\partial_0\Big(\frac{\Delta\eta^2}{\Delta x^2}\Big)\times\ln(H^2\Delta x^2) = \partial_0\Big[\frac{\Delta\eta^2\ln(H^2\Delta x^2)}{\Delta x^2}\Big] + \frac{2\Delta\eta^3}{\Delta x^4} \; , \tag{126}$$

$$\partial_0^2\Big(\frac{\Delta\eta^2}{\Delta x^2}\Big)\times\ln(H^2\Delta x^2) = \partial_0^2\Big[\frac{\Delta\eta^2\ln(H^2\Delta x^2)}{\Delta x^2}\Big] + \frac{10\Delta\eta^2}{\Delta x^4} + \frac{12\Delta\eta^4}{\Delta x^6} \; . \tag{127}$$

The "remainder" terms, which carry no logarithms, are combined with the appropriate entries in Table 4, and then reduced to derivatives acting on the fundamental expressions (108) using relations (109-118).

5.3 Eliminating Inverse Powers

Reducing Table 3 according to this scheme results in a series of derivatives acting on the product of a single factor of $\ln(H^2\Delta x^2)$ times the last three terms in expression (108). The inverse powers can be eliminated using,

$$\frac{\ln(H^2\Delta x^2)}{\Delta x^2} \equiv 2\pi i\times A_1 = +\frac{\partial^2}{8}\Big[\ln^2(H^2\Delta x^2) - 2\ln(H^2\Delta x^2)\Big] \; , \tag{128}$$

$$\frac{\Delta\eta\ln(H^2\Delta x^2)}{\Delta x^2} \equiv 2\pi i\times B_1 = -\frac{\partial_0}{4}\Big[\ln^2(H^2\Delta x^2)\Big] \; , \tag{129}$$

$$\frac{\Delta\eta^2\ln(H^2\Delta x^2)}{\Delta x^2} \equiv 2\pi i\times C_1 = +\frac{\partial_0^2}{8}\Big[\Delta x^2\Big(\ln^2(H^2\Delta x^2) - 2\ln(H^2\Delta x^2) + 2\Big)\Big]$$

$$+\frac{1}{4}\ln^2(H^2\Delta x^2)\ . \quad (130)$$

The terms of Table 4 produce a series of derivatives acting on the four fundamental expressions (108). We eliminate the last three terms using,

$$\frac{1}{\Delta x^2} \equiv 2\pi i \times A_2 = +\frac{\partial^2}{4}\Big[\ln(H^2\Delta x^2)\Big]\ , \quad (131)$$

$$\frac{\Delta\eta}{\Delta x^2} \equiv 2\pi i \times B_2 = -\frac{\partial_0}{2}\Big[\ln(H^2\Delta x^2)\Big]\ , \quad (132)$$

$$\frac{\Delta\eta^2}{\Delta x^2} \equiv 2\pi i \times C_2 = +\frac{\partial_0^2}{4}\Big[\Delta x^2\Big(\ln(H^2\Delta x^2)-1\Big)\Big]+\frac{1}{2}\ln(H^2\Delta x^2)\ . \quad (133)$$

The factor of $\frac{1}{\Delta x^4}$ is divergent. When combined with the appropriate counterterm it gives,

$$\frac{1}{\Delta x^4} \longrightarrow -\frac{\partial^4}{32}\Big[\ln^2(\mu^2\Delta x^2)-2\ln(\mu^2\Delta x^2)\Big] - \ln(a)\times 2\pi^2 i\delta^4(x-x')\ , \quad (134)$$

$$\equiv 2\pi i\times\Big[-\frac{\partial^2}{4}A_3\Big] - \ln(a)\times 2\pi^2 i\delta^4(x-x')\ . \quad (135)$$

Note that any derivatives that act on expression (135) occur to the right of the factor of $\ln(a)$, for example,

$$\partial^2\Big[\frac{1}{\Delta x^4}\Big] \longrightarrow 2\pi i\times\Big[-\frac{\partial^4}{4}A_3\Big] - \ln(a)\times 2\pi^2 i\partial^2\delta^4(x-x')\ . \quad (136)$$

5.4 Schwinger-Keldysh Reductions

Each of the in-out logarithms in (128-134) gives rise in the Schwinger-Keldysh formalism to real and causal expressions for $A_{1,2,3}$, $B_{1,2}$ and $C_{1,2}$,

$$A_1 \longrightarrow +\frac{\partial^2}{4}\Big\{\theta(\Delta\eta-\Delta r)\Big[\ln[H^2(\Delta\eta^2-\Delta r^2)]-1\Big]\Big\}\ , \quad (137)$$

$$B_1 \longrightarrow -\frac{\partial_0}{2}\Big\{\theta(\Delta\eta-\Delta r)\ln[H^2(\Delta\eta^2-\Delta r^2)]\Big\}\ , \quad (138)$$

$$C_1 \longrightarrow +\frac{\partial_0^2}{4}\Big\{\theta(\Delta\eta-\Delta r)(\Delta r^2-\Delta\eta^2)\Big[\ln[H^2(\Delta\eta^2-\Delta r^2)]-1\Big]\Big\}$$
$$+\frac{1}{2}\theta(\Delta\eta-\Delta r)\ln[H^2(\Delta\eta^2-\Delta r^2)]\ , \quad (139)$$

$$A_2 \longrightarrow +\frac{\partial^2}{4}\Big\{\theta(\Delta\eta-\Delta r)\Big\}\,, \tag{140}$$

$$B_2 \longrightarrow -\frac{\partial_0}{2}\Big\{\theta(\Delta\eta-\Delta r)\Big\}\,, \tag{141}$$

$$C_2 \longrightarrow +\frac{\partial_0^2}{4}\Big\{\theta(\Delta\eta-\Delta r)(\Delta r^2-\Delta\eta^2)\Big\}+\frac{1}{2}\theta(\Delta\eta-\Delta r)\,, \tag{142}$$

$$A_3 \longrightarrow +\frac{\partial^2}{4}\Big\{\theta(\Delta\eta-\Delta r)\Big[\ln[\mu^2(\Delta\eta^2-\Delta r^2)]-1\Big]\Big\}\,. \tag{143}$$

$$\tag{144}$$

Here $\Delta r \equiv \|\vec{x}-\vec{x}'\|$

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