Dynamics of a Binary Option Market with Exogenous Information and Price Sensitivity

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In this paper, we derive and analyze a continuous binary option market with exogenous information. The resulting non-linear system has a discontinuous right hand side, which can be analyzed using zero-dimensional Filippov surfaces. Under general assumptions on purchasing rules, we show that when exogenous information is constant in the binary asset market, the price always converges. We then investigate market prices in the case of changing information, showing empirically that price sensitivity has a strong effect on price lag vs. information. We conclude with open questions on general M-ary option markets.

I. INTRODUCTION

Market models have been studied extensively both through simulation and in theory in the econophysics literature [1–10]. In contrast to traditional stock or bond markets, prediction markets have assets corresponding to future events (e.g., elections [11], sports outcomes [12] etc.) that can be bought and sold causing changes in the underlying asset prices. Asset prices may be interpreted as probabilities [13, 14].

Binary option markets were first documented on Wall Street in the nineteenth century [15]. When used with a logarithmic scoring rule [16], they are a common market structure for both prediction markets [13, 14, 17–27] and artificial prediction markets [28–30, 30]. In the finance literature, binary options are usually used as a theoretical construct since they have a particularly simple Black-Scholes formulation [31].

In this paper, we study the dynamics of binary and M-ary $(M > 2)$ option markets with logarithmic scoring rule under continuum limits. Agents in the market receive an exogenous information signal and use it along with the current market price to make decisions on option purchases. The model of the agents in the market are inspired by [32], which discusses a synthetic prediction market. In this paper we do not consider the machine learning problem associated with artificial prediction markets. Instead we are solely concerned with the market as a dynamical system.

The main results of this paper are as follows:

1. We show that under a constant information assumption, all asset prices converge. The result follows from an argument on Filippov surfaces.
2. We then study the impact of dynamic information on market dynamics and numerically quantify the effect of price sensitivity and on lag between the market price and the information signal.
3. We conjecture that in arbitrary M-ary option markets all asset prices converge assuming constant information and we show numerical evidence supporting this conjecture.

The remainder of the paper is organized as follows: We show the model derivation in Section II. In Section III we show that with constant information, all binary prediction markets converge to a constant price. In Section IV, we study the numerical results of a non-constant information. We present a general model for an M-ary option market and explore this model numerically in Section V. We conclude and present future directions in Section VI.

II. MODEL

Assume $\mathbf{q}_t = \langle q^0(t), q^1(t)\rangle$ units of Asset 0 and Asset 1 at time t have been sold. We assume the market is composed of a collection of agents with infinite cash and that the market price is fixed by a logarithmic market scoring rule

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(LMSR) [16] so that the spot prices are given by

$$p^0(t) = \frac{\exp\left[\beta q^0(t)\right]}{\exp\left[\beta q^0(t)\right] + \exp\left[\beta q^1(t)\right]} \tag{1}$$

$$p^1(t) = \frac{\exp\left[\beta q^1(t)\right]}{\exp\left[\beta q^0(t)\right] + \exp\left[\beta q^1(t)\right]}. \tag{2}$$

The Boltzmann distribution used in the spot price definition ensures $p^0(t) + p^1(t) = 1$ for all time. Here, β is a liquidity factor [33] that adjusts the amount the price will increase or decrease given a change in the asset quantities. We denote the spot-price vector $\mathbf{p}(t) = \langle p^0(t), p^1(t)\rangle$. Assume that $\mathbf{p}(0) = \langle p_0^0, p_0^1\rangle$.

Trade costs are not given by $\mathbf{p}_t$, since LMSR incorporates a market maker cost. The trade costs are given by

$$\kappa_t^0(\Delta q^0) = \frac{1}{\beta}\log\left\{\frac{\exp[\beta(q^0(t)+\Delta q^0)] + \exp[\beta q^1(t)]}{\exp[\beta q^0(t)] + \exp[\beta q^1(t)]}\right\}$$

$$\kappa_t^1(\Delta q^1) = \frac{1}{\beta}\log\left\{\frac{\exp[\beta q^0(t)] + \exp[\beta(q^1(t)+\Delta q^1)]}{\exp[\beta q^1(t)] + \exp[\beta q^0(t)]}\right\},$$

where Δq^i is the change in the quantify of Asset i as a result of purchases by an agent. For mathematical simplicity we will assume agents purchase one share at a time so that $\Delta q^i \in \{0, 1\}$. Let

$$\kappa_t^i = \kappa_t^i(1)$$

be the price of a single share of Asset i.

A. Agent Purchase Logic and Information

We modify the agent model in [32] to operate in continuous time. We are not interested in the machine learning problem posed in [32]. Instead we study the resulting continuous time market with exogenous information signals as a nonlinear dynamical system. The machine learning approach discussed in [32] assumes discrete time with agents with finite cash and is focused on solving classification problems through the market mechanism, which we do not consider here.

Assume agents are indexed in $\{1, \dots, N\}$ and suppose that agent i purchases only assets in class $y_i \in \{0, 1\}$ as a result of a time-varying signal $\mathbf{x}(t) \in \mathbb{R}^n$. That is, we assume agents specialize in the purchase of a specific asset class (either 0 or 1) and that there is some (possibly time-varying) external information $\mathbf{x}(t)$ that will inform this purchase decision.

We assume agents have an infinite cash reserve and do not sell assets back to the market maker. Each agent uses a characteristic function $\psi_i : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}$ to reason about information $\mathbf{x}$ and its decision to buy an asset in class y_i is governed by

$$\Delta q_i^{y_i} = H\left\{\sigma[\psi_i(\mathbf{x}, \mathbf{p})] - \kappa^{y_i}\right\}. \tag{3}$$

Here $\sigma : \mathbb{R} \to [0, 1]$ and $H(x)$ is the unit step function defined as 0 at $x = 0$. The expression $\sigma[\psi_i(\mathbf{x}, \mathbf{p})]$ defines the value Agent i places on Asset y_i as a function of the current spot price(s) and the information in the signal $\mathbf{x}$. If Agent i places more value on Asset y_i than its present price κ^{y_i}, then $H\left\{\sigma[\psi_i(\mathbf{x}, \mathbf{p})] - \kappa^{y_i}\right\} = 1$ and $\Delta q_i^{y_i} = 1$, meaning Agent i purchases a share of Asset i. Otherwise $\Delta q_i^{y_i} = 0$ and Agent i does nothing.

We note that any mapping $\sigma : \mathbb{R} \to [0, 1]$ can be used. It will have not impact on the asymptotic properties of the model. However, for convenience we assume

$$\sigma(x) = \frac{1}{1 + e^{-x}}.$$

in our numerical simulations, as this is a familiar and well-behaved function.

We now derive a continuous time version of this purchase rule by assuming that agent purchase decisions are made continuously and by letting β become small so that each continuous agent action does not affect the market price substantially.

B. Continuous Model

Consider $\kappa_t^1(\Delta q^1)$. We can rewrite this as

$$\kappa_t^1(\Delta q^1) = \frac{1}{\beta}\log\left\{\frac{\exp[\beta q^0(t)] + \exp[\beta(q^1(t)+\Delta q^1(t))]}{\exp[\beta q^1(t)] + \exp[\beta q^0(t)]}\right\} = \frac{1}{\beta}\log\left\{\exp\left(\beta\Delta q^1(t)\right)\frac{\exp[\beta q^1(t)]}{\exp[\beta q^1(t)]+\exp[\beta q^0(t)]} + \frac{\exp[\beta q^0(t)]}{\exp[\beta q^1(t)]+\exp[\beta q^0(t)]}\right\} = \frac{1}{\beta}\log\left[\exp\left[\beta\Delta q^1(t)\right]p^1(t) + p^0(t)\right].$$

We know that $p^0(t) + p^1(t) = 1$, so we have

$$\kappa_t^1(\Delta q^1) = \frac{1}{\beta}\log\left\{[\exp\left(\beta\Delta q^1\right) - 1]p^1 + 1\right\}.$$

For $\beta \ll 1$, we can approximate the exponential and logarithmic functions with a Taylor series: $e^x \approx 1 + x + O(x^2)$ and $\log(1+x) \approx x$ to see

$$\kappa_t^1(\Delta q^1) = \frac{1}{\beta}\log\left[\beta\Delta q^1 p^1(t) + 1\right] = \frac{1}{\beta}\left[\beta\Delta q^1 p^1(t)\right] = \Delta q^1 p^1(t).$$

Since we assume $\Delta q^1 \in \{0, 1\}$, we may assume $\kappa_t^1 \approx p^1(t)$.

Let $C_1 \subset \{1, \ldots, N\}$ be the set of agents that buy only Asset 1 and let $C_0 \subset \{1, \ldots, N\}$ be the set of agents that buy only Assert 0. Using our approximation in Eq. (3) we have

$$q^1(t+1) \approx q^1(t) + \sum_{i \in C_1} H\left\{\sigma[\psi_i(\mathbf{x}, \mathbf{p})] - p^1(t)\right\}$$
$$q^0(t+1) \approx q^0(t) + \sum_{i \in C_1} H\left\{\sigma[\psi_i(\mathbf{x}, \mathbf{p})] - p^0(t)]\right\}.$$

If we assume that transaction quantities are proportional to the size of a time slice so that

$$q^1(t+\epsilon) \approx q^1(t) + \epsilon\sum_{i \in C_1} H\left\{\sigma[\psi_i(\mathbf{x}, \mathbf{p})] - p_t^1]\right\}, \tag{4}$$

then, taking the limit as $\epsilon \to 0$, we obtain the non-linear differential equation

$$\dot{q}^1 = \sum_{i \in C_1} H\left\{\sigma[\psi_i(\mathbf{x}, \mathbf{p})] - p_t^1]\right\} \tag{5}$$
$$\dot{q}^0 = \sum_{i \in C_0} H\left\{\sigma[\psi_i(\mathbf{x}, \mathbf{p})] - p_t^0]\right\}. \tag{6}$$

Using Eqs. (1) and (2) and the quotient rule we can compute

$$\dot{p}^1 = \beta p^1 p^0\left(\dot{q}^1 - \dot{q}^0\right) \tag{7}$$
$$\dot{p}^0 = \beta p^0 p^1\left(\dot{q}^0 - \dot{q}^1\right). \tag{8}$$

These results are consistent with [34, 35] where a multi-agent learning systems using a Boltzmann (soft-max) learning rule is studied.

Using the fact that $p^0(t) + p^1(t) = 1$ we may write $p^0 = 1 - p^1$. For simplicity we can remove the superscript from p^1 and denote it as p. Therefore, $p^0 = 1 - p$. Making these substitutions in Eqs. (5) to (7) and combining them gives us a single non-linear ODE describing the dynamics of the the price of Asset 1:

$$\dot{p} = \beta p(1-p)\left(\underbrace{\sum_{i \in C_1} H\left\{\sigma\left[\psi_i(\mathbf{x}, p)\right] - p\right\}}_{\dot{q}^1} - \underbrace{\sum_{i \in C_0} H\left\{\sigma\left[\psi_i(\mathbf{x}, p)\right] - (1-p)\right\}}_{\dot{q}^0}\right). \tag{9}$$

Unlike the work in [34, 35], this ODE has a piecewise smooth and bounded right-hand-side with jump discontinuities. Therefore it is a Filippov system [36, 37].

III. ASYMPTOTIC MARKET BEHAVIOR FOR CONSTANT INFORMATION

Assume $\mathbf{x}(t)$ is constant. Then we can express $\psi_i(\mathbf{x}, \mathbf{p})$ as $\psi_i[\mathbf{x}(t), \mathbf{p}] = L_i + f_i(p)$ where $L_i \in \mathbb{R}$ describes the constant exogenous information and $f_i(p)$ describes the price sensitivity of Agent i when making asset purchase decisions. The dynamics of $p(t)$ are then

$$\dot{p} = \beta p(1-p) \left(\sum_{i \in C_1} H \left\{ \sigma \left[L_i + f_i(p) \right] - p \right\} - \sum_{i \in C_0} H \left\{ \sigma \left[L_i + f_i(p) \right] - (1-p) \right\} \right). \quad (10)$$

The right-hand-side is piecewise smooth and bounded and therefore a solution to this ODE exists [37]. It is clear that $p^* = 0$ and $p^* = 1$ are fixed points of Eq. (10). Therefore, for $p_0 \in [0,1]$, in Eq. (9), if $p(t)$ is a solution, then $p(t) \in [0,1]$ for all time.

We have just shown that the dynamics of $p(t)$ are contained entirely in the smooth manifold $M = [0,1]$. Thus the flow at any point M is defined by a scalar (since the dimension of M is 1). By convention positive flow moves toward $p^* = 1$. A *sliding mode* fixed point or fixed point on a Filippov surface is a point $p^* \in \mathrm{int}(M)$ (a zero dimensional manifold) where the flow to the left of p^* is positive and the flow to the right of p^* is negative and the flow at p^* may take either a positive or negative or zero value [37]. That is, the flow around this point is moving in opposite directions. Naturally, this definition can be extended to higher dimensional flows (see [37]) but we do not require this level of complexity.

Let

$$N(p) = \sum_{i \in C_1} H \left\{ \sigma \left[L_i + f_i(p) \right] - p \right\} - \sum_{i \in C_0} H \left\{ \sigma \left[L_i + f_i(p) \right] - (1-p) \right\}.$$

This is a discontinuous integer valued function with maximum value $|C_1|$ and minimum value $-|C_0|$. We can re-write Eq. (9) as

$$\dot{p} = \beta N(p) p(1-p).$$

From this, we see that for any fixed integer $N(p)$, Eq. (9) is just a logistic differential equation, which has a global solution. We deduce at once that Eq. (9) has a global continuous solution composed of piecewise smooth solutions to the individual logistic differential equations.

We now show that these solutions are monotonic and either asymptotically approach a value as time goes to infinity or reach a fixed point in finite time.

If $N(p_0) = 0$, then $p = p_0$ is the solution for all time. On the other hand, suppose $N(p_0) < 0$, then $p(t)$ is initially decreasing and we have three possibilities.

Case I: If $N[p(t)]$ remains negative as $p(t)$ decreases, then $p(t)$ decreases for all time, asymptotically approaching the fixed point $p^* = 0$.

Case II: If $N[p(t)]$ increases as $p(t)$ decreases and there is a finite $t^* > 0$ such that $N[p(t^*)] = 0$, then for all $t \geq t^*$, $p(t) = p(t^*)$.

Case III: If $N[p(t)]$ increases as $p(t)$ decreases and there is a smallest time $t^* > 0$ so that for all $\epsilon > 0$, $N[p(t^* + \epsilon)] > 0$ and $N[p(t^* - \epsilon)] < 0$, then $p(t^*)$ is a sliding mode fixed point. Consequently, the value of $p(t)$ becomes fixed at the location of the jump discontinuity in $N(p)$. This is illustrated in Fig. 1.

The argument for the case when $N(p_0) > 0$ is symmetric. Thus, we have shown that: (i) Every instance of Eq. (9) has a global continuous and piecewise smooth solution. (ii) Solutions are monotonic. (iii) Every solution either asymptotically approaches $p^* = 0$ or $p^* = 1$ or reaches a constant value in finite time.

IV. NUMERICAL RESULTS WITH NON-CONSTANT INFORMATION

When $\mathbf{x}(t)$ is non-constant, the market will track the function $\psi(\mathbf{x}, p)$ through the function

$$N(\mathbf{x}, p) = \left(\sum_{i \in C_1} H \left\{ \sigma \left[\psi_i(\mathbf{x}, p) \right] - p \right\} - \sum_{i \in C_0} H \left\{ \sigma \left[\psi_i(\mathbf{x}, p) \right] - (1-p) \right\} \right)$$

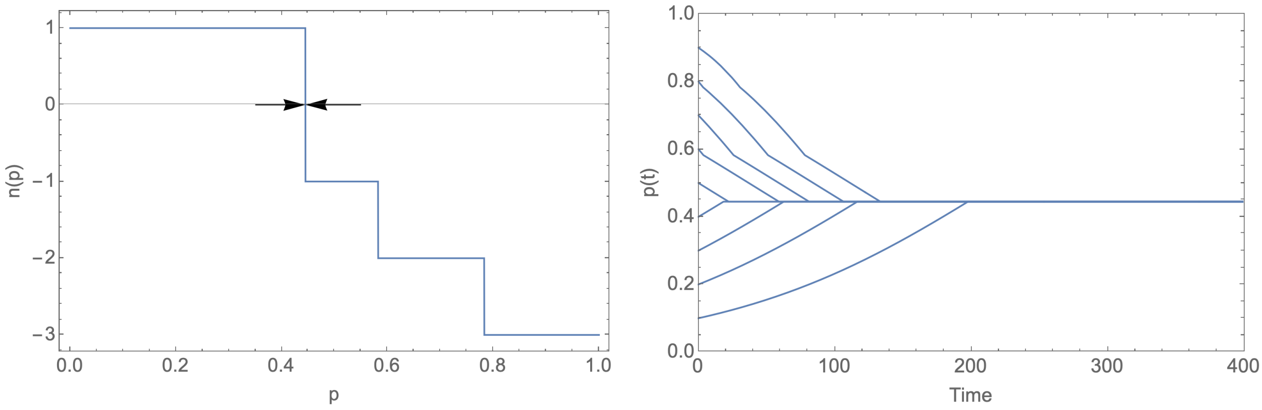

FIG. 1: Illustration of a function $n(p)$ with a jump discontinuity and the resulting convergence to the sliding mode manifold.

If there is some time $T \geq 0$ so that for all $t \geq T$ we have $N[\mathbf{x}(t), p(t)] \geq 0$ ($N[\mathbf{x}(t), p(t)] \leq 0$, respectively) then clearly $p(t)$ must converge.

When $N(\mathbf{x}, p)$ remains above (or below) 0, the market will lag the information content in $\mathbf{x}(t)$ while simultaneously smoothing the information in $\mathbf{x}(t)$. By way of example, let $\mathbf{x}(t)$ be a solution to the Lorenz system and define

$$\psi_1(\mathbf{x}, p) = \mathbf{x}_1(t)$$
$$\psi_0(\mathbf{x}, p) = -\mathbf{x}_1(t).$$

The dynamics are shown in Fig. 2 for two different initial conditions. This shows that the market preserves the

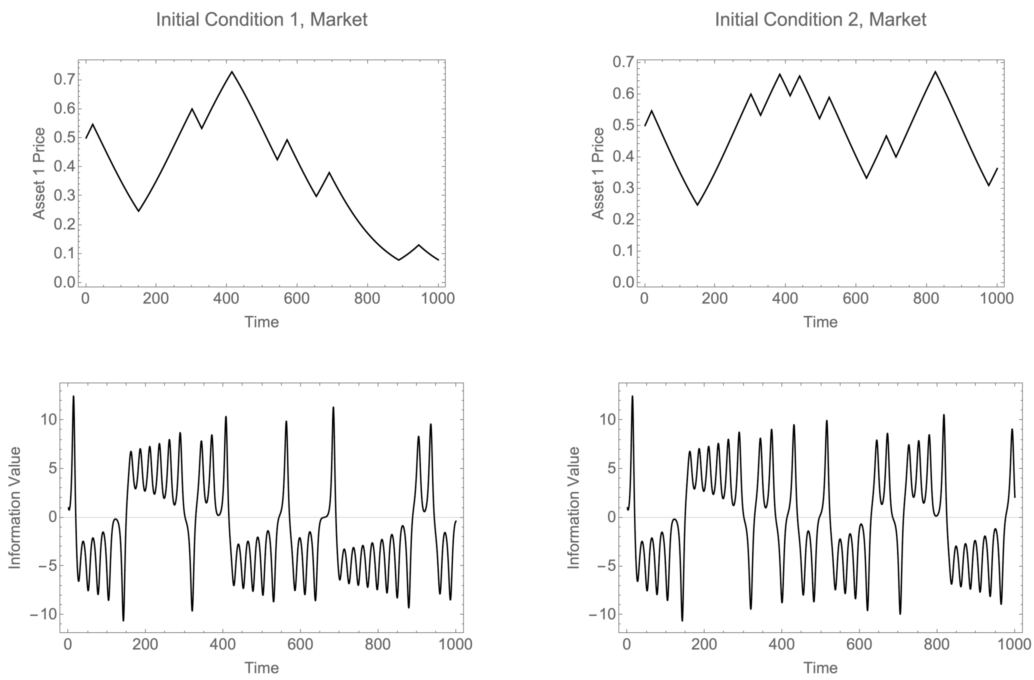

FIG. 2: Two separate Lorenz solutions (first dimension only) are shown one with initial condition $(1, -1, 1)$ and $(1.01, -1, 1)$. The market smoothes the dynamics of the Lorenz system. Here, $\beta = 0.01$.

sensitive dependence on initial conditions (as expected) and smoothes the dynamics of the Lorenz system. Numerical solution of the differential equation systems is described in B.

There is a lag in the market price as new information is assimilated. To study this lag and the effect of various market conditions, we assume that $x = A\sin(\omega t)$. For our experiments, we set $A = 1$ and $\omega = 2\pi/25000$. Define a set of agents by random intervals $\{(a_i, b_i)\}_{i=1}^N$ where if $i \in C_1$, then $a_i \geq 0$ and $b_i \leq 1$ and if $i \in C_0$, then $b_i \leq 0$ and $a_i \geq -1$. When N is large, with high probability these intervals cover the interval $[-1, 1]$. Assume

$$\psi_i(x) = \begin{cases} 1 & \text{if } a_i < x < b_i \\ -1 & \text{otherwise.} \end{cases}$$

Then the market will classify $x(t)$ as it moves from $+1$ to -1 over time. This is illustrated in Fig. 3 for two different agent sizes. The market with smaller numbers of agents tracks the information but there are times when $x(t)$ is not in any interval defining an agent and so buying stops. This leads to plateaus in the data.

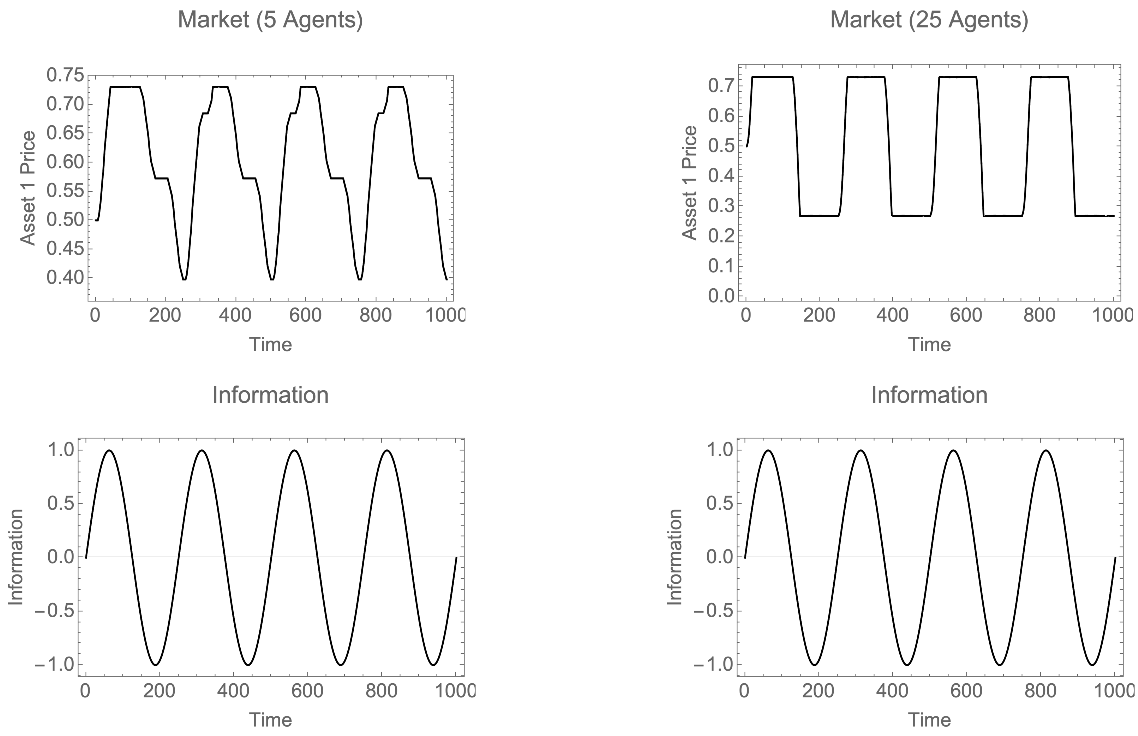

FIG. 3: (Left) A market with ten agents (five agents per class) shows tracking of the information $x(t)$. (Right) A market with fifty agents (25 agents per class) tracks that information $x(t)$ but the market price shows saturation.

Introduce two parameters $\alpha \geq 1$ and $\nu \in \{-1, 0, 1\}$ and modify Eqs. (5) and (6) to be:

$$\dot{q}_1 = \sum_{i \in C_1} H\left\{\sigma\left[\alpha\left(\psi_i(x) + \nu\left(p - \tfrac{1}{2}\right)\right)\right] - p\right\}$$

$$\dot{q}_0 = \sum_{i \in C_0} H\left\{\sigma\left[\alpha\left(\psi_0(x) + \nu\left(\tfrac{1}{2} - p\right)\right)\right] - (1 - p)\right\}.$$

This makes Agent i's purchase logic sensitive to the value of the information x relative to the interval $[a_i, b_i]$ defining the agent and also to the relative spot price p with respect to the initial price $p(0) = \frac{1}{2}$. Substituting these terms into Eq. (9) gives

$$\dot{p} = \beta p(1-p)\left(\sum_{i \in C_1} H\left\{\sigma\left[\alpha\left(\psi_i(x) + \nu\left(p - \tfrac{1}{2}\right)\right)\right] - p\right\} - \sum_{i \in C_0} H\left\{\sigma\left[\alpha\left(\psi_0(x) + \nu\left(\tfrac{1}{2} - p\right)\right)\right] - (1-p)\right\}\right)$$

We use these dynamics to numerically examine the effect asset price sensitivity (ν) and input gain (α) have on the lag in information uptake in the market. To do this, we use the measured phase difference between $p(t)$ and $x(t)$

using the argument of the Fourier transform of the solutions at the fundamental frequency ω. The ratio of the two phases provides a measure of the speed with which information is taken up by the market. We used markets with 250 agents (125 agents purchasing Asset 0 and 125 agents purchasing Asset 1). We allowed α to vary from 1 to 15 and ν to vary in the set $\{-1, 0, 1\}$. For all experiments $\beta = 0.01$. When $\nu = 0$, there is no price sensitivity. When $\nu = -1$, agents are more sensitive to price and higher prices make them value the asset less. Effectively, this creates additional friction in purchasing. When $\nu = 1$, agents are price sensitive and higher prices causes them to value an asset more. This creates additional force in the the market to purchase even at higher prices. We used 5 replications of each condition to construct confidence intervals on the mean of the phase ratios. Agent intervals were randomized in each replication. Throughout all experiments, we kept $A = 1$ and $\omega = 2\pi/25000$. Results are shown in Fig. 4. We

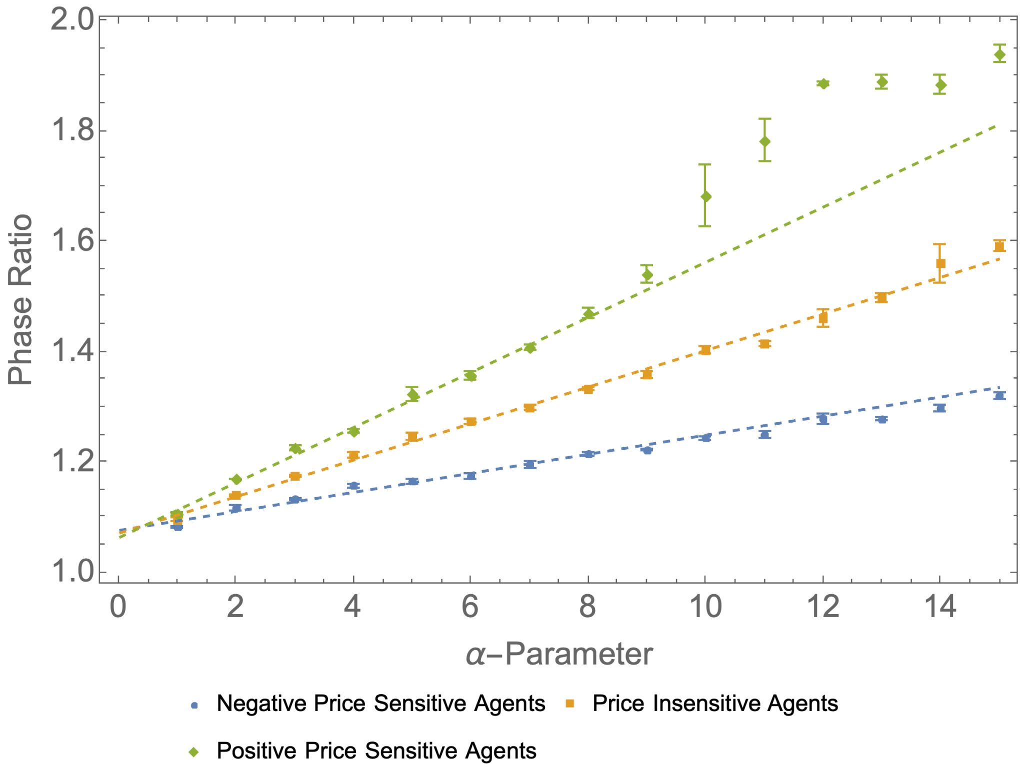

FIG. 4: The ratio between the phase angle at the fundamental frequency for the price and information dynamics are shown. The Line of fit it computed from the first five data points to show non-linear growth.

conclude that agents with positive price sensitivity resulted in a higher phase ratio while agents with a negative price sensitivity had a lower phase ratio for varying input gain (α). Price insensitive agents' phase ratio fell between the positive and negative price sensitive agents. Hence, when $\nu = -1$, information is taken up by the market faster than when $\nu = 0$ and when $\nu = 1$. We also note that the phase ratio appears to exhibit saturation in the positively price sensitive agents, which should be expected.

V. GENERALIZATION AND CONJECTURE

Generalizing the dynamics of Eq. (9) to $M > 2$ assets is straightforward. We obtain

$$\dot{p}_i = \sum_{j=1}^{M} \beta p_i p_j \left(\dot{q}_i - \dot{q}_j\right), \tag{11}$$

where

$$\dot{q}_j = \sum_{k \in C_j} H\left\{\sigma\left[\psi_k(\mathbf{x}, \mathbf{p})\right] - p_j\right\}.$$

Here C_j is the group of agents who specialize in purchasing assets in class j and p_j is the price of asset class j. This equation is consistent with [34, 35]. We conjecture that with constant information, the resulting dynamics always converge, fully generalizing the result in Section III. We illustrate this conjecture in Fig. 5 for a three asset market simulated with over 3000 variations on the dynamics assuming

$$\psi_k(\mathbf{x}, \mathbf{p}) = L_k,$$

where L_k is randomly drawn from either $[-2, 2]$ or $[-5, 5]$. See Fig. 5(left) or Fig. 5(right), respectively. Both figures show the ternary transform of price trajectories $\mathbf{p} = \langle p^1, p^2, p^3 \rangle$ for randomly chosen values of L_k in a market model with three asset classes. The cardinality $|C_j|$ is randomly drawn from 1 to 10, simulating a small market. The figures

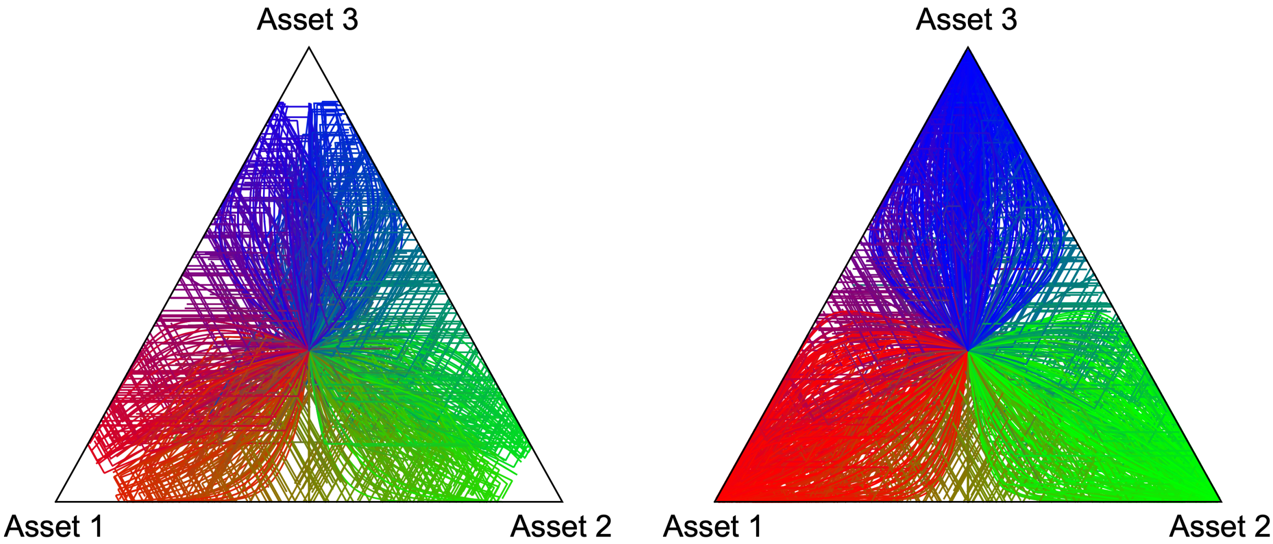

FIG. 5: The ternary transform of the trajectories $\mathbf{p} = \langle p^1, p^2, p^3 \rangle$ are shown for randomly chosen values of L_k in a market with three asset classes. (Left) L_k randomly drawn from $[-2, 2]$. (Right) L_k randomly drawn from $[-5, 5]$. In both cases the dynamics converge or asymptotically approach the extreme point of the unit simplex. No oscillations are observed.

show that oscillation is not present when information is held constant. In particular, our conjecture implies that the dynamics quickly reach a $k < M$ dimensional Filippov surface and then slide along it until they reach a fixed point (a zero dimensional subset of the Filippov surface) thus ensuring convergence of the system.

VI. CONCLUSION AND FUTURE DIRECTIONS

In this paper, we developed a continuous model of a binary option market with traders who act on an exogenous information signal. Our model is given by a nonlinear ordinary differential equation with discontinuous right-hand-side. We proved the spot prices converge in this market model under the assumption of constant external information by showing that the right-hand-side of the dynamics are piecewise smooth and bounded allowing us to reason about the Filippov surface on which the dynamics come to rest. In the presence of non-constant external information, we show empirically that a positive price sensitivity (i.e., purchases increase in response to higher prices) causes larger market lags to external information change while negative price sensitivity (i.e., purchases decrease in response to higher prices) decrease market lag to external information.

Results discussed in this paper can be extended in several ways. We have left an open conjecture on the generalization of the market convergence result to markets with $M > 2$ asset classes. We assert that under constant external information these markets also converge in spot prices. Additionally, further analysis in the case when the exogenous signal $\mathbf{x}(t)$ is non-constant may provide further insight into the behavior of these simple options markets.

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Appendix A: The Lorenz Attractor

For the example in Section IV, we use the standard Lorenz attractor:

$$\begin{aligned}\dot{x} &= s \cdot \sigma(x-y)\\ \dot{y} &= s \cdot x\,(\rho - z - y)\\ \dot{z} &= s \cdot (xy - \alpha z)\,.\end{aligned}$$

To better illustrate the market effect, we slow the dynamics down by a factor of $s = 0.05$ and set $\alpha = 1$, $\sigma = -3$, $\rho = 26.5$.

Appendix B: Numerical Solution of Large Differential Equations

When $\mathbf{x}(t)$ is complex or there are more than a few agents, numerical solution of the differential equations can become unstable. To compensate for this, we can solve Eqs. (5) and (6) using a simple Euler forward step method using a small ϵ in Eq. (4) (see below):

$$q^1(t+\epsilon) \approx q^1(t) + \epsilon \sum_{i \in C_1} H\left\{\sigma[\psi_i(\mathbf{x}, p)] - p(t)]\right\}.$$

We then compute $p(t)$ using information for $q_1(t)$ and $q_0(t)$ up to time t via Eq. (2). For small $\epsilon \approx 0.01$, the results match the output of an off-the-shelf ODE solver and are more numerically stable.

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