Critical Field Theories with OSp(1|2M) Symmetry

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In the paper [L. Fei *et al.*, J. High Energy Phys. 09 (2015) 076] a cubic field theory of a scalar field σ and two anticommuting scalar fields, θ and $\bar{\theta}$, was formulated. In $6-\epsilon$ dimensions it has a weakly coupled fixed point with imaginary cubic couplings where the symmetry is enhanced to the supergroup OSp(1|2). This theory may be viewed as a "UV completion" in $2<d<6$ of the nonlinear sigma model with hyperbolic target space $H^{0|2}$ described by a pair of intrinsic anticommuting coordinates. It also describes the $q\to 0$ limit of the critical q-state Potts model, which is equivalent to the statistical mechanics of spanning forests on a graph. In this Letter we generalize these results to a class of OSp(1|2M) symmetric field theories whose upper critical dimensions are $d_c(M)=2[(2M+1)/(2M-1)]$. They contain $2M$ anticommuting scalar fields, θ^i, $\bar{\theta}^i$, and one commuting one, with interaction $g(\sigma^2+2\theta^i\bar{\theta}^i)^{(2M+1)/2}$. In $d_c(M)-\epsilon$ dimensions, we find a weakly coupled IR fixed point at an imaginary value of g. We propose that these critical theories are the UV completions of the sigma models with fermionic hyperbolic target spaces $H^{0|2M}$. Of particular interest is the quintic field theory with OSp(1|4) symmetry, whose upper critical dimension is 10/3. Using this theory, we make a prediction for the critical behavior of the OSp(1|4) lattice system in three dimensions.

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Introduction.—This Letter builds on the paper [1] where the field theory was studied with the Euclidean action

$$S=\int d^dx\left(\partial_\mu\theta\partial^\mu\bar{\theta}+\frac{1}{2}(\partial_\mu\sigma)^2+g_1\sigma\theta\bar{\theta}+\frac{1}{6}g_2\sigma^3\right) \quad (1)$$

for two anticommuting scalar fields, θ and $\bar{\theta}$, and one commuting one, σ. The global Sp(2) symmetry of this model becomes enhanced to the supergroup OSp(1|2) because at the IR fixed point in $6-\epsilon$ dimensions the two coupling constants are *imaginary* and related by $g_2^*=2g_1^*$. At this weakly coupled fixed point, which is of the Wilson-Fisher type [2], the interaction becomes proportional to the manifestly OSp(1|2) invariant form $(\sigma^2+2\theta\bar{\theta})^{3/2}$. Using three-loop calculations, the scaling dimensions of σ and θ in $6-\epsilon$ dimensions were indeed found to be equal [1]. Since the coupling is imaginary, the Euclidean path integral associated with (1) does not suffer from the instability encountered for the real σ^3 interaction. Indeed, the cubic theory of a single scalar field with an imaginary coupling constant is known [3] to describe the Lee-Yang edge singularity.

Earlier examples of models with OSp(1|2) symmetry are the lattice models and sigma models describing the spanning forests, or equivalently the $q\to 0$ limit of the q-state Potts model [4–6]. In [1] we showed that the $6-\epsilon$ expansions of the scaling dimensions in our OSp(1|2) symmetric theory are the same as in the $q\to 0$ limit of the q-state Potts model [7]. Thus, (1) provides an explicit formulation of the field theory that governs this formal limit and is super-renormalizable in $d<6$.

Besides the $6-\epsilon$ expansion, it is interesting to develop the $2+\epsilon$ expansion for the critical theory of spanning forests. In [4,8] it was argued that it is provided by the OSp(1|2) sigma model [9,10] with the action

$$S=\frac{1}{2g^2}\int d^dx((\partial_\mu\sigma)^2+2\partial_\mu\theta\partial^\mu\bar{\theta}), \quad (2)$$

where the constraint $\sigma^2+2\theta\bar{\theta}=1$ is imposed. The constraint has two solutions, $\sigma=\pm(1-\theta\bar{\theta})$, and choosing one of these solutions breaks the Z_2 symmetry but preserves the OSp(1|2); then the global symmetries in the sigma model are the same as in the cubic theory (1). In the recent work [11,12] the interpretation of the target space was changed from (half of) the sphere $S^{0|2}$ to the space $H^{0|2}$, which is a fermionic version of the hyperboloid.

Substituting $\sigma=1-\theta\bar{\theta}$ into (2), we find the sigma model with the hyperbolic target space $H^{0|2}$. It has the following classical action in terms of the two anticommuting scalar fields:

$$S = \frac{1}{g^2}\int d^dx(\partial_\mu\theta\partial^\mu\bar\theta - \theta\bar\theta\partial_\mu\theta\partial^\mu\bar\theta), \tag{3}$$

and it is important to take $g^2 < 0$ so that the model is asymptotically free in $d = 2$ [4,11–13]. Indeed, the beta function of this theory is the same as for the O(N). nonlinear sigma model continued to $N = -1$, and the theory is asymptotically free for negative g^2. Thus, in $d = 2 + \epsilon$ this theory has a weakly coupled UV fixed point. The fact that near the upper critical dimension 6 the model has another weakly coupled description, involving an extra canonical field σ, is analogous to a similar phenomenon in the Gross-Neveu model [15] near 4 dimensions [16,17] (for a more recent discussion, see [18]).

We may interpret the cubic theory (1) as a "UV completion" of the $H^{0|2}$ sigma model in $2 < d < 6$: the IR fixed point of the cubic theory presumably describes the same physics as the UV fixed point of the sigma model. The sense of this is similar to how the super-renormalizable Gross-Neveu-Yukawa model [16,17] provides a UV completion in $2 < d < 4$ of the nonrenormalizable Gross-Neveu model. Similarly, the $O(N)$ symmetric field theory of $N > 2$ scalar fields with the quartic interaction $\lambda(\phi^i\phi^i)^2$ provides a UV completion in $2 < d < 4$ of the nonrenormalizable $O(N)$ nonlinear sigma model.

The OSp(1|2) lattice system [4–6,11,12], which underlies the continuum descriptions reviewed above, involves introducing on each lattice site x the vector $u_x = (\theta_x, \bar\theta_x, \sigma_x)$. The constraint that it belong to $H^{0|2}$ implies $\sigma_x = 1 - \theta_x\bar\theta_x$. Then for each pair of nearest neighbor lattice sites, x and y, the factor

$$e^{\beta(u_x\cdot u_y+1)/2}, \qquad u_x\cdot u_y = -\sigma_x\sigma_y - \theta_x\bar\theta_y - \theta_y\bar\theta_x, \tag{4}$$

is included in the integrand of the partition function. Since all the integrations are over the Grassmann variables $\theta_x, \bar\theta_x$, they can be performed exactly on a finite lattice. This lattice system has a particular simplicity and is found to be equivalent to the statistical mechanics of spanning forests, or alternatively the $q \to 0$ Potts model [4–6]. The Monte Carlo simulations for it were carried out in [19], indicating the second-order phase transitions in $d = 3, 4, 5$. We will show that the estimates of critical exponents based on the two-sided Padé extrapolations, using the $6 - \epsilon$ and $2 + \epsilon$ expansions, are in good agreement with the Monte Carlo results.

The OSp($1|2M$) lattice systems may be constructed analogously: in this case there are M pairs of Grassmann variables on each lattice site, $\theta^i_x, \bar\theta^i_x$, $i = 1, \ldots, M$, and

$$\sigma_x = \left(1 - 2\sum_{i=1}^M \theta^i_x\bar\theta^i_x\right)^{1/2},$$
$$u_x\cdot u_y = -\sigma_x\sigma_y - \sum_{i=1}^M(\theta^i_x\bar\theta^i_y + \theta^i_y\bar\theta^i_x). \tag{5}$$

It is of obvious interest to study possible critical behavior in such lattice systems for $M > 1$. In this Letter, we propose an extension of the field theoretic approach (1) to these systems. We are led to consider theories with interactions of order $2M + 1$, i.e., σ^{2M+1} plus terms involving the anticommuting fields [20]. Such theories have the upper critical dimensions

$$d_c(M) = 2\frac{2M+1}{2M-1}. \tag{6}$$

We use the results [22] for the O(N)-invariant field theories with interactions of order $2M + 1$, which are renormalizable at the upper critical dimensions $(10/3)$, $(14/5)$, etc., and substitute $N = -2M$ to account for the anticommuting nature of the N scalar fields. Then we find OSp($1|2M$) invariant IR fixed points where the interaction term is proportional to $(\sigma^2 + 2\theta^i\bar\theta^i)^{(2M+1)/2}$ with an imaginary coefficient. These critical theories appear to be nonperturbatively well defined, and it would be very interesting to compare the continuum results with those in the OSp($1|2M$) lattice systems.

Scaling dimensions for the OSp (1|2) model.—The one-loop beta functions and anomalous dimensions for the theory (22) are [1]

$$\beta_1 = -\frac{g_1\epsilon}{2} - \frac{1}{12(4\pi)^3}g_1(10g_1^2 + 12g_1g_2 - g_2^2),$$
$$\beta_2 = -\frac{g_2\epsilon}{2} + \frac{1}{4(4\pi)^3}(8g_1^3 - 2g_1^2g_2 - 3g_2^3),$$
$$\gamma_\theta = \frac{g_1^2}{6(4\pi)^3}, \qquad \gamma_\sigma = \frac{g_2^2 - g_1^2}{12(4\pi)^3}. \tag{7}$$

There is an OSp(1|2) invariant IR fixed point where [23]

$$g_1 = i\sqrt{\frac{(4\pi)^3\epsilon}{5}}, \qquad g_2 = 2g_1, \qquad \gamma_\theta = \gamma_\sigma = -\frac{\epsilon}{30}. \tag{8}$$

These results can be extended to the four-loop order using the formulae from [24] for the $O(N)$ invariant cubic model [25,26], and then setting $N = -2$ [27]. Then we find

$$\Delta_\theta = 2 - \frac{8}{15}\epsilon - \frac{7}{450}\epsilon^2 - \frac{269 - 702\zeta(3)}{33750}\epsilon^3 - \frac{207313 - 4212\pi^4 + 936\zeta(3) + 907200\zeta(5)}{24300000}\epsilon^4 + \mathcal{O}(\epsilon^5). \tag{9}$$

This expansion coincides with the corresponding one in the formal $q \to 0$ limit of the q-state Potts model.

The $2 + \epsilon$ expansion for the $O(N)$ sigma model is [30]

$$\Delta_\theta = \frac{1}{2}\epsilon + \frac{1}{2(N-2)}\epsilon - \frac{N-1}{2(N-2)^2}\epsilon^2 + \mathcal{O}(\epsilon^3), \tag{10}$$

which for $N=-1$ becomes

$$\Delta_\theta = \frac{1}{3}\epsilon + \frac{1}{9}\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{11}$$

Using the "two-sided" (4,3) and (6,1) Padé approximations, which utilize both the $6-\epsilon$ and $2+\epsilon$ expansions, we obtain the estimates $\Delta_\theta \approx 1.46$ in $d=5$; $\Delta_\theta \approx 0.92$ in $d=4$ and $\Delta_\theta \approx 0.415$ in $d=3$. They are in very good agreement with the Monte Carlo simulations of the $q=0$ Potts model. Using the values for γ/ν given in Table I of [19], we find that

$$\Delta_\theta = \frac{d}{2} - \frac{\gamma}{2\nu} \tag{12}$$

are very close to our Padé estimates in $d=3$, 4, 5.

The next important operator in the sigma model has dimension [30]

$$\Delta_+ = d - \nu^{-1} = 2 - \frac{1}{N-2}\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{13}$$

For $N=-1$ this becomes

$$\Delta_+ = 2 + \frac{1}{3}\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{14}$$

The expansion of Δ_+ in $d=6-\epsilon$ may be found using the theory (1), where it corresponds to the OSp(1|2) invariant primary operator $\sigma^2 + 2\theta\bar{\theta}$. Using the four-loop results [24], we find

$$\begin{aligned}\Delta_+ = 4 &- \frac{2}{3}\epsilon + \frac{1}{30}\epsilon^2 + \frac{173 - 864\zeta(3)}{27000}\epsilon^3 \\ &+ \frac{51683 - 1296\pi^4 + 140400\zeta(3) + 272160\zeta(5)}{4860000}\epsilon^4 \\ &+ \mathcal{O}(\epsilon^5).\end{aligned} \tag{15}$$

Performing the two-sided Padé approximations, we find $\Delta_+ \approx 3.36$ in $d=5$; $\Delta_+ \approx 2.8$ in $d=4$ and $\Delta_+ \approx 2.2$ in $d=3$. They are in good agreement with the Monte Carlo results for ν given in Table I of [19] for $q=0$.

Field theory for $M>1$.—The 2D sigma model with target space $\mathrm{H}^{0|2M}$ may be defined by picking one of the two solutions of the constraint

$$\sigma^2 + 2\sum_{i=1}^{M} \theta^i\bar{\theta}^i = 1. \tag{16}$$

The sigma model classical action is

$$S = \frac{1}{2g^2}\int d^2x \left((\partial_\mu\sigma)^2 + 2\sum_{i=1}^{M} \partial_\mu\theta^i\partial^\mu\bar{\theta}^i \right), \tag{17}$$

with the substitution of

$$\sigma = \left(1 - 2\sum_{i=1}^{M}\theta^i\bar{\theta}^i\right)^{1/2}. \tag{18}$$

For example, for $M=2$

$$\sigma = 1 - \theta^1\bar{\theta}^1 - \theta^2\bar{\theta}^2 - \theta^1\bar{\theta}^1\theta^2\bar{\theta}^2. \tag{19}$$

In general, the expansion of the square root in (18) terminates with the term of order $2M$ proportional to $\prod_{i=1}^{M}\theta^i\bar{\theta}^i$. Related to this fact, we will propose a critical field theory with interactions of order $2M+1$.

The $\mathrm{H}^{0|2M}$ sigma model may be thought of as the O(N) sigma model with $N = 1$–$2M$. For $g^2 < 0$ it is asymptotically free in $d=2$ and therefore has a UV fixed point in $d=2+\epsilon$. An interesting question is how to continue this theory to the dimension slightly below the upper critical one. For $M=1$ its upper critical dimension is 6, and in $d<6$ we can view it as Euclidean field theory (1) with interaction $(\sigma^2 + 2\theta^i\bar{\theta}^i)^{3/2}$. However, such a description cannot be applicable to $M>1$. Indeed, already for $M=2$ the potential would contain the term $\sim\theta^1\bar{\theta}^1\theta^2\bar{\theta}^2/\sigma$ which is not admissible in renormalizable field theory. We propose that the proper generalization of our $M=1$ construction to higher M involves higher powers in the OSp(1|2M) invariant potential, so that its expansion in the anticommuting variables does not contain any terms with negative powers of σ; namely,

$$\begin{aligned}(\sigma^2 + 2\theta^i\bar{\theta}^i)^{(2M+1)/2} = \sigma^{2M+1} &+ (2M+1)\sigma^{2M-1}\sum_{i=1}^{M}\theta^i\bar{\theta}^i \\ &+ \cdots + (2M+1)!!\sigma\prod_{i=1}^{M}\theta^i\bar{\theta}^i.\end{aligned} \tag{20}$$

In $d_c(M) - \epsilon$ there is a weakly coupled IR fixed point of the interacting field theory

$$S = \int d^dx \left(\partial_\mu\theta^i\partial^\mu\bar{\theta}^i + \frac{1}{2}(\partial_\mu\sigma)^2 + g(\sigma^2 + 2\theta^i\bar{\theta}^i)^{(2M+1)/2} \right), \tag{21}$$

where at the fixed point g is imaginary and $\sim\sqrt{\epsilon}$.

The $M=2$ model is particularly interesting, since its critical dimension $(10/3)$ is above 3. In the $(10/3)-\epsilon$ expansion, we expect good results since $\epsilon = 1/3$ is small. This was indeed the case for the model which is described by the $i\sigma^5$ theory. The $(10/3)-\epsilon$ expansion for this model was obtained in [31], where it was called the Blume-Capel or the tricritical Lee-Yang universality class [32].

The quintic model with four anticommuting scalars and $Sp(4)$ symmetry has the Euclidean action

$$S=\int d^dx\Big(\partial_\mu\theta^i\partial^\mu\bar\theta^i+\frac{1}{2}(\partial_\mu\sigma)^2+\frac{g_1}{6}\sigma(\theta^i\bar\theta^i)^2 + \frac{g_2}{6}\sigma^3\theta^i\bar\theta^i+\frac{g_3}{120}\sigma^5\Big). \tag{22}$$

Since the one-loop β and γ functions were calculated in [22] for the quintic model with N additional scalar fields and O(N) symmetry, we may set $N=-4$ to obtain the corresponding results for the theory (22). After performing a suitable multiplicative redefinition of the couplings, $g_j\to Ag_j$, which simplifies the formulae, we find the one-loop beta functions for the theory (22):

$$\begin{aligned}\beta_1&=-\frac{3g_1\epsilon}{2}-\frac{1}{480}(1224g_1^3+4320g_1^2g_2+17\,304g_1g_2^2\\&\quad+19\,440g_2^3+1440g_1g_2g_3+1620g_2^2g_3-3g_1g_3^2),\\\beta_2&=-\frac{3g_2\epsilon}{2}+\frac{1}{480}(480g_1^3+5768g_1^2g_2+19\,440g_1g_2^2\\&\quad-7032g_2^3+240g_1^2g_3+1080g_1g_2g_3\\&\quad-12\,960g_2^2g_3-2691g_2g_3^2),\\\beta_3&=-\frac{3g_3\epsilon}{2}+\frac{1}{96}(-1920g_1^2g_2-4320g_1g_2^2+34\,560g_2^3\\&\quad+8g_1^2g_3+21\,528g_2^2g_3-2337g_3^3).\end{aligned} \tag{23}$$

The anomalous dimensions are

$$\begin{aligned}\gamma_\theta&=\frac{1}{60}(2g_1^2-3g_2^2),\\\gamma_\sigma&=\frac{1}{240}(-8g_1^2+72g_2^2-3g_3^2).\end{aligned} \tag{24}$$

We find an IR stable fixed point where

$$(g_1,g_2,g_3)=i\sqrt{\frac{6\epsilon}{1651}}(3,2,8). \tag{25}$$

The two anomalous dimensions are equal at this fixed point, $\gamma_\theta=\gamma_\sigma=-(3/8255)\epsilon$, suggesting that the Sp(4) symmetry is enhanced to OSp(1|4). This is indeed the case, since the interaction in (22) combines at the fixed point into $\sim i(\sigma^2+2\theta^1\bar\theta^1+2\theta^2\bar\theta^2)^{5/2}$.

The IR scaling dimension is

$$\Delta_\theta=\Delta_\sigma=\frac{d-2}{2}+\gamma_\theta=\frac{2}{3}-\frac{8261}{16\,510}\epsilon+O(\epsilon^2). \tag{26}$$

Substituting $\epsilon=1/3$ gives the answer $\Delta_{\text{one loop}}=(8253/16\,510)\approx 0.499\,88$. This is extremely close to the free dimension $1/2$ due to the smallness of the coefficient of ϵ in the anomalous dimension. We can try to improve on this estimate using the two-sided Padé including the $2+\epsilon$ expansion for the $N=-3$ sigma model:

$$\Delta(2+\epsilon)=\frac{2}{5}\epsilon+\frac{2}{25}\epsilon^2+\mathcal{O}(\epsilon^3). \tag{27}$$

A Padé approximant for the scaling dimension as a function of d,

$$\Delta_P(d)=\frac{16(d-2)(32\,900-8219d)}{1875\,420+d(-756\,484+74\,007d)}, \tag{28}$$

is consistent with the expansions near $10/3$ and 2 dimensions. It gives $\Delta_P(3)\approx 0.485$ which is somewhat lower than $\Delta_{\text{one loop}}$. It would be interesting to calculate the $O(\epsilon^2)$ correction to Δ using the two-loop diagrams in the quintic theory (22). We expect this to reduce the uncertainty in estimating the scaling dimension in $d=3$. It would be also useful to study the critical exponents using the functional renormalization group approach [34] to theory (22).

Let us also discuss the theory with Sp(6) symmetry and seventh-order interactions:

$$S=\int d^dx\Big(\sum_{i=1}^{2}\partial_\mu\theta^i\partial^\mu\bar\theta^i+\frac{1}{2}(\partial_\mu\sigma)^2+\frac{g_1}{90}\sigma(\theta^i\bar\theta^i)^3 + \frac{g_2}{36}\sigma^3(\theta^i\bar\theta^i)^2+\frac{g_3}{120}\sigma^5\theta^i\bar\theta^i+\frac{g_4}{5040}\sigma^7\Big). \tag{29}$$

Using Gracey's results [22] for the $O(N)$ symmetric theory with seventh-order interactions, and continuing them to $N=-6$, we find an IR fixed point in $(14/5)-\epsilon$ dimensions. In the normalization of couplings such that

$$\gamma_\theta=\frac{1}{15\,120}(-24g_1^2+200g_2^2-45g_3^2), \tag{30}$$

$$\gamma_\sigma=\frac{1}{30\,240}(48g_1^2-1800g_2^2+1350g_3^2-15g_4^2), \tag{31}$$

it is located at

$$(g_1,g_2,g_3,g_4)=i\cdot 0.012\,4124\sqrt{\epsilon}(15,6,8,48). \tag{32}$$

At this fixed point $\gamma_\theta=\gamma_\sigma\approx 0.000\,011\epsilon$, and we observe the enhancement of the Sp(6) symmetry to OSp(1|6).

Our calculations provide evidence for the consistency of our proposal for $M>1$, but a lot remains to be done. In particular, it would be interesting to formulate a Monte Carlo approach to the OSp(1|4) lattice model, where all the integrations in the partition function are over the Grassmann variables θ_x^1, $\bar\theta_x^1$, θ_x^2, $\bar\theta_x^2$. The results could be then compared with our prediction that the scaling exponents have exactly mean field values in four dimensions, but exhibit small deviations from them in three dimensions. Also, perhaps a conformal bootstrap approach to the critical exponents can be attempted along the lines of the method [35], which is applicable to nonunitary theories.

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