Fast numerical solutions to direct and inverse scattering for bi-anisotropic periodic Maxwell's equations

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Abstract. This paper is concerned with the numerical solution to the direct and inverse electromagnetic scattering problem for bi-anisotropic periodic structures. The direct problem can be reformulated as an integro-differential equation. We study the existence and uniqueness of solution to the latter equation and analyze a spectral Galerkin method to solve it. This spectral method is based on a periodization technique which allows us to avoid the evaluation of the quasiperiodic Green's tensor and to use the fast Fourier transform in the numerical implementation of the method. For the inverse problem, we study the orthogonality sampling method to reconstruct the periodic structures from scattering data generated by only two incident fields. The sampling method is fast, simple to implement, regularization free, and very robust against noise in the data. Numerical examples for both direct and inverse problems are presented to examine the efficiency of the numerical solvers.

1. Introduction

We study in this paper numerical methods for solving both direct and inverse electromagnetic scattering problems for bi-anisotropic periodic media. This work is motivated by applications of electromagnetic scattering from complex periodic media (e.g. bi-anisotropic media, chiral media) in optics and metamaterials; and nondestructive evaluations [**13, 15, 9, 27**].

Results on the well-posedness of the direct problem for both bounded and periodic scattering media can be found in [**5, 2, 29, 7, 8, 30, 26**]. To our knowledge results on numerical methods for direct and inverse scattering problems for electromagnetic complex media are limited. The authors in [**3**] studied numerical analysis of a finite element method for solving the direct problem for periodic chiral media. Spectral Galerkin methods for the integro-differential equation formulation of the direct scattering problem were studied in [**24**] for chiral bounded objects and in [**23**] for bi-anisotropic bounded objects. The advantages of the spectral Galerkin method studied in [**23**] are that this spectral method, which is based on a periodization technique, allows us to avoid the evaluation of the quasiperiodic Green's tensor and to use the fast Fourier transform in the numerical implementation of

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the method. In this paper we study this spectral method to solve the direct problem for bi-anisotropic periodic media by extending the results in [**21, 23**]. More precisely, we follow the periodization process of [**21**] for the integro-differential equation. The Garding estimates for the integro-differential equation can be done similarly as [**23**]. The uniqueness of solution, the mapping properties of the periodized integro-differential operators, and the numerical examples for the Galerkin method are the new ingredients of the study for the direct scattering problem in this paper.

For the inverse problem, we refer to [**19, 20, 17, 6**] for some uniqueness results in the case of bounded scattering objects. To numerically solving the inverse problem, the factorization method was studied in [**12**] for the case of chiral bounded objects. This method was also studied in [**22, 25**] for periodic chiral and bi-anisotropic periodic media. However, the factorization method requires the scattering data associated with multiple incident fields. In this paper we implement the orthogonality sampling method (OSM) to solve the inverse problem with only a few incident fields. The OSM we implement is also very stable against noise in the data. The OSM was introduced by Potthast in [**28**] and has attracted increasing interests thanks to its computational efficiency. We refer to [**10, 1, 11, 16**] for some other results on the OSM for inverse scattering problems.

The paper is organized as follows. The direct problem and its basic setup are presented in Section 2. We discuss in Section 3 the integro-differential formulation for the direct problem and its well-posedness. A spectral Galerkin method, its convergence analysis are presented in Section 4. Numerical examples of the direct solver are presented in Section 5. The inverse problem and the OSM are studied in Section 6.

2. The problem setup

We are concerned with the scattering problem for a bi-anisotropic medium which is 2π-periodic in x_1, x_2 and bounded in x_3 (here x_1, x_2, x_3 are the three coordinates of a vector $\mathbf{x} = (x_1, x_2, x_3)^\top$ in $\mathbb{R}^3$). Let $D_0 \subset \mathbb{R}^3$ be the interior of the medium. The problem is governed by the time-harmonic Maxwell's equations with frequency $\omega > 0$

$$\operatorname{curl}\mathbf{E} + i\omega\mathbf{B} = 0, \quad \operatorname{curl}\mathbf{H} - i\omega\mathbf{D} = 0, \quad \text{in } \mathbb{R}^3, \tag{2.1}$$

where the electric field $\mathbf{E}$, the magnetic field $\mathbf{H}$, the electric flux density $\mathbf{D}$ and the magnetic flux density $\mathbf{B}$ are three dimensional vector-valued functions, along with the constitutive relations

$$\mathbf{B} = \mu\mathbf{H} + \xi\sqrt{\varepsilon_0\mu_0}\ \mathbf{E}, \quad \mathbf{D} = \varepsilon\mathbf{E} + \overline{\xi}\sqrt{\varepsilon_0\mu_0}\ \mathbf{H} \tag{2.2}$$

where the permittivity ε and the permeability μ are 3×3 matrix-valued bounded functions. We assume that these functions are 2π-periodic in x_1 and x_2 and that they are respectively equal $\mu_0 I_3$ and $\varepsilon_0 I_3$ in $\mathbb{R}^3 \setminus \overline{D_0}$ (I_3 is the identity matrix). The function ξ is related to the magnetoelectric effect (see [**18**]) which is also a 3×3 matrix-valued bounded function. This function is also assumed to be 2π-periodic in x_1 and x_2 and to be zero in $\mathbb{R}^3 \setminus \overline{D_0}$.

We introduce the relative quantities

$$\varepsilon_r := \frac{\varepsilon}{\varepsilon_0}, \quad \mu_r := \frac{\mu}{\mu_0}$$

and the scaled quantities (with the same notations as the original ones)

$$\mathbf{E} := \sqrt{\varepsilon_0}\ \mathbf{E}, \quad \mathbf{H} := \sqrt{\mu_0}\ \mathbf{H}.$$

Using these new quantities and plugging (2.2) into (2.1) we obtain

$$\text{curl}\,\mathbf{E} - ik(\mu_r \mathbf{H} + \xi \mathbf{E}) = 0, \quad \text{curl}\,\mathbf{H} + ik(\varepsilon_r \mathbf{E} + \overline{\xi}\mathbf{H}) = 0 \tag{2.3}$$

where $k := \omega\sqrt{\varepsilon_0\mu_0}$ is the wave number.

Assume that μ_r is invertible almost everywhere in $\mathbb{R}^3$, by the first equation in (2.3) we can write

$$\mathbf{H} = -\frac{i}{k}\mu_r^{-1}\text{curl}\,\mathbf{E} - \mu_r^{-1}\xi\mathbf{E}. \tag{2.4}$$

Plugging this into the second one and rearranging the resulting equation we obtain

$$\text{curl}\,(\mu_r^{-1}\text{curl}\,\mathbf{E}) + ik\left[\overline{\xi}\mu_r^{-1}\text{curl}\,\mathbf{E} - \text{curl}\,(\mu_r^{-1}\xi\mathbf{E})\right] - k^2(\varepsilon_r - \overline{\xi}\mu_r^{-1}\xi)\mathbf{E} = 0. \tag{2.5}$$

Let $\mathbf{d} = (d_1, d_2, d_3) \in \mathbb{R}^3$ be the wave vector satisfying $d_3 \neq 0$ and $|\mathbf{d}| = k$, we consider the plane electromagnetic wave given by

$$\mathbf{E}^{in} = \frac{1}{\omega\varepsilon_0}\ (\mathbf{s}\times\mathbf{d})e^{i\mathbf{d}\cdot\mathbf{x}}, \quad \mathbf{H}^{in} = \mathbf{s}e^{i\mathbf{d}\cdot\mathbf{x}} \tag{2.6}$$

where $\mathbf{s}$ is the polarization vector satisfying $\mathbf{s}\cdot\mathbf{d} = 0$. The interaction between this plane wave and the medium gives rise to the scattered wave $(\mathbf{E}^{sc}, \mathbf{H}^{sc})$, and the total wave $(\mathbf{E}, \mathbf{H})$ given by $\mathbf{E} = \mathbf{E}^{in} + \mathbf{E}^{sc}$, $\mathbf{H} = \mathbf{H}^{in} + \mathbf{H}^{sc}$ satisfies (2.4)–(2.5). From now on we will be working with $\mathbf{E}$ only since $\mathbf{E}$ and $\mathbf{H}$ are intimately related by (2.4).

For simplicity we denote $\mathbf{u} := \mathbf{E}^{sc}$. Note that by construction, $\mathbf{E}^{in}$ satisfies

$$\text{curl}\,\text{curl}\,\mathbf{E}^{in} - k^2\mathbf{E}^{in} = 0,$$

thus (2.5) can be rewritten as

$$\begin{aligned}\text{curl}^2\mathbf{u} - k^2\mathbf{u} = k^2 &\left[(P - \overline{\xi}\mu_r^{-1}\xi)(\mathbf{E}^{in} + \mathbf{u}) + \frac{i}{k}\overline{\xi}\mu_r^{-1}(\text{curl}\,\mathbf{E}^{in} + \text{curl}\,\mathbf{u})\right] \\ &+ \text{curl}\,\left[Q(\text{curl}\,\mathbf{E}^{in} + \text{curl}\,\mathbf{u}) - ik\mu_r^{-1}\xi(\mathbf{E}^{in} + \mathbf{u})\right]\end{aligned} \tag{2.7}$$

where $P := \varepsilon_r - I$ and $Q := I - \mu_r^{-1}$ are the contrasts. It is clear that outside of $\overline{D_0}$, ε_r and μ_r equal the identity matrix, so $P = Q = 0$. Thus we have

$$\text{supp}(P) \cup \text{supp}(Q) \subset \overline{D_0}.$$

We now define that for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, 0) \in \mathbb{R}^3$, a function $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ is called $\boldsymbol{\alpha}$-*quasiperiodic* if for any $n = (n_1, n_2, 0) \in \mathbb{Z}^3$

$$\mathbf{v}(x_1 + n_1 2\pi, x_2 + n_2 2\pi, x_3) = e^{2\pi i\alpha\cdot n}\mathbf{v}(x_1, x_2, x_3), \quad \mathbf{x} \in \mathbb{R}^3.$$

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, 0) = (d_1, d_2, 0)$ and $\boldsymbol{\alpha}_m = (\alpha_1 + m_1, \alpha_2 + m_2, 0)$ for $m \in \mathbb{Z}^2$. Then it can be seen that the plane wave $\mathbf{E}^{in}$ is $\boldsymbol{\alpha}$-quasiperiodic. It is well-known that we will seek for the unique solution $\mathbf{u}$ of the direct problem as an $\boldsymbol{\alpha}$-quasiperiodic function.

Next we prescribe a radiation condition for $\mathbf{u}$. Recall that the medium D_0 is bounded in x_3, thus let $h > 0$ be such that

$$h > \sup\{|x_3| : \mathbf{x} \in D_0\} \tag{2.8}$$

and for $m \in \mathbb{Z}^2$ define

$$\beta_m = \begin{cases} \sqrt{k^2 - |\boldsymbol{\alpha}_m|^2}, & |\boldsymbol{\alpha}_m| \le k \\ i\sqrt{|\boldsymbol{\alpha}_m|^2 - k^2}, & |\boldsymbol{\alpha}_m| > k \end{cases},$$

then $\mathbf{u}$ is called radiating if it can be expressed as the following Rayleigh expansion

$$\mathbf{u}(\mathbf{x}) = \sum_{m \in \mathbb{Z}^2} \widehat{\mathbf{u}}_m^{\pm} e^{i(\boldsymbol{\alpha}_m \cdot \mathbf{x} \pm \beta_m (x_3 \mp h))} \quad \text{for } x_3 \gtrless \pm h \tag{2.9}$$

where

$$\widehat{\mathbf{u}}_m^{\pm} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-i\boldsymbol{\alpha}_m \cdot \mathbf{x}} \mathbf{u}(x_1, x_2, \pm h) \mathrm{d}x_1 \mathrm{d}x_2, \quad m \in \mathbb{Z}^2.$$

Note that all but finitely many terms in (2.9) are exponentially decaying, which helps us easily deduce absolute convergence of the series for every $\mathbf{x}$. Moreover, we can choose $\boldsymbol{\alpha}$ and k such that β_m are nonzero for all $m \in \mathbb{Z}^2$, and this will be the case for the remaining part of this paper.

We now introduce notation and assumptions related to the spatial domains and function spaces that we will be using throughout this paper. First, due to periodicity of all the introduced quantities and the condition (2.9), we can restrict our problem from $\mathbb{R}^3$ to one period

$$\Omega = (-\pi, \pi)^2 \times \mathbb{R}.$$

Also, for any $r > 0$, we define a truncation of Ω as

$$\Omega_r = (-\pi, \pi)^2 \times (-r, r).$$

Next we introduce some Sobolev spaces for vector-valued functions

$$\begin{aligned} L^2(\mathcal{O})^3 &= \{\mathbf{w} = (w_1, w_2, w_3)^T : w_1, w_2, w_3 \in L^2(\mathcal{O})\}, \\ H(\mathrm{curl}\,, \mathcal{O}) &= \{\mathbf{w} \in L^2(\mathcal{O}, \mathbb{C}^3) : \mathrm{curl}\,\mathbf{w} \in L^2(\mathcal{O}, \mathbb{C}^3)\}, \\ H_{\mathrm{loc}}(\mathrm{curl}\,, \mathcal{O}) &= \{\mathbf{w} : \mathbf{w}|_V \in H(\mathrm{curl}\,, V) \text{ for all } V \subset \mathcal{O} \text{ open, bounded}\}, \\ H_{\boldsymbol{\alpha}}(\mathrm{curl}\,, \mathcal{O}) &= \{\mathbf{w} \in H(\mathrm{curl}\,, \mathcal{O}) : \mathbf{w} = \mathbf{W}|_{\mathcal{O}} \text{ for some } \boldsymbol{\alpha}\text{-quasiperiodic} \\ &\qquad \text{function } \mathbf{W} \in H_{\mathrm{loc}}(\mathrm{curl}\,, \mathbb{R}^3)\}, \\ H_{\boldsymbol{\alpha},\mathrm{loc}}(\mathrm{curl}\,, \mathcal{O}) &= \{\mathbf{w} \in H_{\mathrm{loc}}(\mathrm{curl}\,, \mathcal{O}) : \mathbf{w} = \mathbf{W}|_{\mathcal{O}} \text{ for some } \boldsymbol{\alpha}\text{-quasiperiodic} \\ &\qquad \text{function } \mathbf{W} \in H_{\mathrm{loc}}(\mathrm{curl}\,, \mathbb{R}^3)\}, \end{aligned}$$

along with $L^\infty(\mathbb{R}^3)^{3\times 3}$ which is the space of 3×3 matrix-valued functions whose components are functions in $L^\infty(\mathbb{R}^3)$. We define the following norm on this space

$$\|\gamma\|_\infty = \max_{i,j=\overline{1,3}} \|\gamma_{ij}\|_\infty$$

for $\gamma \in L^\infty(\mathbb{R}^3)^{3\times 3}$.

Let $D = D_0 \cap \Omega$. It is clear that D is open and bounded. Finally, the following assumption holds true throughout the theoretical analysis in this paper.

Assumption 2.1. Assume that D has Lipschitz boundary; ε_r, μ_r, μ_r^{-1}, ξ are in $L^\infty(\mathbb{R}^3)^{3\times 3}$, symmetric almost everywhere in $\mathbb{R}^3$ and ξ is real-valued. Moreover,

(1) There exist $c_1, c_2 > 0$ such that for all $\mathbf{z} \in \mathbb{C}^3$

$$\mathrm{Re}\,(\mu_r^{-1})\mathbf{z} \cdot \overline{\mathbf{z}} \ge c_1 |\mathbf{z}|^2, \quad \mathrm{Re}\,(\varepsilon_r - \xi\mu_r^{-1}\xi)\mathbf{z} \cdot \overline{\mathbf{z}} \ge c_2 |\mathbf{z}|^2$$

a.e in $\mathbb{R}^3$ and that

$$\| |\mu_r^{-1}\xi|_F \|_\infty < c_1 c_2.$$

(2) There exist $c_3, c_4 > 0$ such that for all $\mathbf{z} \in \mathbb{C}^3$

$$-\mathrm{Im}\,(\mu_r^{-1})\mathbf{z}\cdot\bar{\mathbf{z}} \geq c_3|\mathbf{z}|^2, \qquad \mathrm{Im}\,(\varepsilon_r - \xi\mu_r^{-1}\xi)\mathbf{z}\cdot\bar{\mathbf{z}} \geq c_4|\mathbf{z}|^2$$

a.e in $\mathbb{R}^3$ and that

$$\frac{1}{2}\left(\|\mu_r^{-1}\xi\|_\infty^2 + 1\right) \leq \min\{c_3, c_4\}.$$

Here the norm $\||\cdot|_F\|_\infty$ is defined as, for a 3×3 matrix-valued function $A = (a_{ij})$,

$$\||A|_F\|_\infty = \left\|\left(\sum_{i,j=1}^{3} |a_{ij}|^2\right)^{1/2}\right\|_\infty .$$

3. Integro-differential formulation

In this section, we will reformulate the direct problem into an integro-differential equation. Define the operators

$$\mathcal{S}\mathbf{u} = (P - \xi\mu_r^{-1}\xi)\mathbf{u} + \frac{i}{k}\xi\mu_r^{-1}\mathrm{curl}\,\mathbf{u}, \quad \mathcal{T}\mathbf{u} = Q\mathrm{curl}\,\mathbf{u} - ik\mu_r^{-1}\xi\mathbf{u}$$

and let $\mathbf{f} = \mathcal{S}\mathbf{E}^{in}$, $\mathbf{g} = \mathcal{T}\mathbf{E}^{in}$. Then it is clear that $\mathcal{S}\mathbf{u}$, $\mathcal{T}\mathbf{u}$, $\mathbf{f}$ and $\mathbf{g}$ are compactly supported in D and (2.7) can be written as

$$\mathrm{curl}^{\,2}\mathbf{u} - k^2\mathbf{u} = k^2(\mathcal{S}\mathbf{u} + \mathbf{f}) + \mathrm{curl}\,(\mathcal{T}\mathbf{u} + \mathbf{g}). \tag{3.1}$$

Note that (3.1) needs to be solved in the variational sense, therefore we will be seeking solutions in $H_{\boldsymbol{\alpha},\mathrm{loc}}(\mathrm{curl}\,,\Omega)$ and $\mathbf{u}$ is called a solution of (3.1) in the variational sense if it satisfies

$$\int_\Omega (\mathrm{curl}\,\mathbf{u}\cdot\mathrm{curl}\,\overline{\mathbf{v}} - k^2\mathbf{u}\cdot\overline{\mathbf{v}})\,d\mathbf{x} = k^2\int_D (\mathcal{S}\mathbf{u}+\mathbf{f})\cdot\overline{\mathbf{v}}\,d\mathbf{x} + \int_D (\mathcal{T}\mathbf{u}+\mathbf{g})\cdot\mathrm{curl}\,\overline{\mathbf{v}}\,d\mathbf{x} \tag{3.2}$$

for all $\mathbf{v} \in H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega)$ with compact support.

Let G be the $\boldsymbol{\alpha}$-quasiperiodic Green's function of the Helmholtz equation, then G has the Fourier expansion (see [**4**])

$$G(\mathbf{x}) = \frac{i}{8\pi^2}\sum_{m\in\mathbb{Z}^2}\frac{1}{\beta_m}e^{i(\boldsymbol{\alpha}_m\cdot\mathbf{x} + \beta_m|x_3|)}$$

for $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{x} \neq (2\pi n_1, 2\pi n_2, 0)$, $n_1, n_2 \in \mathbb{Z}$.

The variational problem (3.2) along with the radiation condition (2.9) is equivalent to the following integro-differential equation

$$\mathbf{u} - \mathcal{A}\mathcal{S}\mathbf{u} - \mathcal{B}\mathcal{T}\mathbf{u} = \mathcal{A}\mathbf{f} + \mathcal{B}\mathbf{g} \quad \text{in } \Omega \tag{3.3}$$

where

$$\mathcal{A}\mathbf{h}(\mathbf{x}) := (k^2 + \nabla\mathrm{div})\int_D G(\mathbf{x}-\mathbf{y})\mathbf{h}(\mathbf{y})d\mathbf{y},$$

$$\mathcal{B}\mathbf{h}(\mathbf{x}) := \mathrm{curl}\,\int_D G(\mathbf{x}-\mathbf{y})\mathbf{h}(\mathbf{y})d\mathbf{y}$$

in the sense that, if a radiating $\mathbf{u} \in H_{\boldsymbol{\alpha},\mathrm{loc}}(\mathrm{curl}\,,\Omega)$ solves (3.2) then $\mathbf{u}|_D$ solves (3.3). Conversely, if $\mathbf{u} \in H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,D)$ solves (3.3) in then $\mathbf{u}$ can be extended by the right-hand side of (3.3) to a radiating solution of (3.2) in $H_{\boldsymbol{\alpha},\mathrm{loc}}(\mathrm{curl}\,,\Omega)$. The proof for this equivalence is similar to that in [**14**].

The operator $I - \mathcal{AS} - \mathcal{BT}$ on the left-hand side of (3.3) satisfies a Garding-type estimate, and thus the Fredholm property. Note that $\overline{D} \subset \Omega_h$, so we only need to know $\mathbf{u}$ on Ω_h and then extend it to the whole Ω using integration. On the solution space $H_{\boldsymbol{\alpha}}(\mathrm{curl}\,, \Omega_h)$, we define the inner product

$$(\cdot,\cdot)_{H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega_h)} = k^2(\cdot,\cdot)_{L^2(\Omega_h)^3} + (\mathrm{curl}\,\cdot, \mathrm{curl}\,\cdot)_{L^2(\Omega_h)^3}$$

where $(\cdot,\cdot)_{L^2(\Omega_h)^3}$ is the usual inner product in $L^2(\Omega_h)^3$.

Under Assumption 2.1, the following Fredholm property can be proved in a similar manner as in [**23**].

Theorem 3.1. *There exist a constant $C > 0$ and a compact operator K on $H_{\boldsymbol{\alpha}}(\mathrm{curl}\,, \Omega_h)$ such that*

$$\mathrm{Re}\,(\mathbf{u} - \mathcal{AS}\mathbf{u} - \mathcal{BT}\mathbf{u}, \mathbf{u})_{H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega_h)} \geq C\|\mathbf{u}\|^2_{H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega_h)} + \mathrm{Re}\,(K\mathbf{u}, \mathbf{u})_{H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega_h)}.$$

Since the Fredholm property holds, existence of solution to (3.3) is equivalent to uniqueness, therefore we show that it has a unique solution.

Theorem 3.2. *The integro-differential equation (3.3) has a unique solution in $H_{\boldsymbol{\alpha}}(\mathrm{curl}\,, \Omega_h)$.*

Proof. Suppose (3.3) has two solutions $\mathbf{u}_1, \mathbf{u}_2$ in $H_{\boldsymbol{\alpha}}(\mathrm{curl}\,, \Omega_h)$. Let $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, we have

$$\mathbf{u} - \mathcal{AS}\mathbf{u} - \mathcal{BT}\mathbf{u} = 0 \tag{3.4}$$

in Ω_h. In particular, $\mathbf{u}|_D$ solves the (3.4) in D. Therefore $\mathbf{u}|_D$ can be extended by the right-hand side of (3.4) to a radiating solution $\tilde{\mathbf{u}} \in H_{\boldsymbol{\alpha},\mathrm{loc}}(\mathrm{curl}\,, \Omega)$ of the variational problem

$$\int_\Omega (\mathrm{curl}\,\tilde{\mathbf{u}} \cdot \mathrm{curl}\,\overline{\mathbf{v}} - k^2\tilde{\mathbf{u}} \cdot \overline{\mathbf{v}})\, d\mathbf{x} = \int_D (k^2(P - \xi\mu_r^{-1}\xi)\tilde{\mathbf{u}} + ik\xi\mu_r^{-1}\mathrm{curl}\,\tilde{\mathbf{u}}) \cdot \overline{\mathbf{v}}\, d\mathbf{x} + \int_D (Q\mathrm{curl}\,\tilde{\mathbf{u}} - ik\mu_r^{-1}\xi\tilde{\mathbf{u}}) \cdot \mathrm{curl}\,\overline{\mathbf{v}}\, d\mathbf{x} \tag{3.5}$$

for all $\mathbf{v} \in H_{\boldsymbol{\alpha}}(\mathrm{curl}\,, \Omega)$ with compact support. Let $\phi \in C_0^\infty(\mathbb{R})$ be a smooth cut-off function such that $\phi(t) = 1$ if $|t| \leq h$ and $\phi(t) = 0$ if $|t| \geq 2h$. Using the test function $\mathbf{v}(\mathbf{x}) = \phi(x_3)\tilde{\mathbf{u}}(\mathbf{x})$ in (3.5) yields

$$\int_{\Omega_{2h}} (\mathrm{curl}\,\tilde{\mathbf{u}} \cdot \mathrm{curl}\,\overline{\phi\tilde{\mathbf{u}}} - k^2\tilde{\mathbf{u}} \cdot \overline{\phi\tilde{\mathbf{u}}})\, d\mathbf{x} = \int_D (k^2(P - \xi\mu_r^{-1}\xi)\tilde{\mathbf{u}} + ik\xi\mu_r^{-1}\mathrm{curl}\,\tilde{\mathbf{u}}) \cdot \overline{\tilde{\mathbf{u}}}\, d\mathbf{x} + \int_D (Q\mathrm{curl}\,\tilde{\mathbf{u}} - ik\mu_r^{-1}\xi\tilde{\mathbf{u}}) \cdot \mathrm{curl}\,\overline{\tilde{\mathbf{u}}}\, d\mathbf{x}. \tag{3.6}$$

Since $\tilde{\mathbf{u}}$ is divergence-free and solves $\mathrm{curl}\,^2\tilde{\mathbf{u}} - k^2\tilde{\mathbf{u}} = 0$ on $\Omega_{2h} \setminus \Omega_h$, we have

$$\begin{aligned}
&\int_{\Omega_{2h}} (\mathrm{curl}\,\tilde{\mathbf{u}} \cdot \mathrm{curl}\,\overline{\phi\tilde{\mathbf{u}}} - k^2\tilde{\mathbf{u}} \cdot \overline{\phi\tilde{\mathbf{u}}})\, d\mathbf{x} \\
&= \int_{\Omega_h} (|\mathrm{curl}\,\tilde{\mathbf{u}}|^2 - k^2|\tilde{\mathbf{u}}|^2)\, d\mathbf{x} + \left(\int_{\{x_3=h\}} - \int_{\{x_3=-h\}}\right) (e_3 \times \mathrm{curl}\,\tilde{\mathbf{u}}) \cdot \overline{\tilde{\mathbf{u}}}\, ds(\mathbf{x}) \\
&= \int_{\Omega_h} (|\mathrm{curl}\,\tilde{\mathbf{u}}|^2 - k^2|\tilde{\mathbf{u}}|^2)\, d\mathbf{x} + \left(\int_{\{x_3=h\}} - \int_{\{x_3=-h\}}\right) \left(\tilde{u}_3 \frac{\partial \overline{\tilde{u}_3}}{\partial x_3} - \overline{\tilde{u}_1}\frac{\partial \tilde{u}_1}{\partial x_3} - \overline{\tilde{u}_2}\frac{\partial \tilde{u}_2}{\partial x_3}\right) ds(\mathbf{x}).
\end{aligned}$$

Using radiation condition (2.9) on the boundary term then taking the imaginary part of both sides gives

$$\mathrm{Im}\int_{\Omega_{2h}} (\mathrm{curl}\,\tilde{\mathbf{u}}\cdot\mathrm{curl}\,\overline{\phi\tilde{\mathbf{u}}} - k^2\tilde{\mathbf{u}}\cdot\overline{\phi\tilde{\mathbf{u}}})\,d\mathbf{x} = -4\pi^2\sum_{m:\beta_m>0}\beta_m(|\hat{u}_m^+|^2+|\hat{u}_m^-|^2)\le 0.$$

On the other hand, by Assumption 2.1,

$$\mathrm{Im}\left(\int_D (k^2(P-\xi\mu_r^{-1}\xi)\tilde{\mathbf{u}} + ik\xi\mu_r^{-1}\mathrm{curl}\,\tilde{\mathbf{u}})\cdot\overline{\tilde{\mathbf{u}}}\,d\mathbf{x} + \int_D (Q\mathrm{curl}\,\tilde{\mathbf{u}} - ik\mu_r^{-1}\xi\tilde{\mathbf{u}})\cdot\mathrm{curl}\,\overline{\tilde{\mathbf{u}}}\,d\mathbf{x}\right)$$
$$\ge k^2c_2\|\tilde{\mathbf{u}}\|^2 + c_1\|\mathrm{curl}\,\tilde{\mathbf{u}}\|^2 + k\mathrm{Re}\,(\xi\mu_r^{-1}\mathrm{curl}\,\tilde{\mathbf{u}},\tilde{\mathbf{u}}) - k\mathrm{Re}\,(\mu_r^{-1}\xi\tilde{\mathbf{u}},\mathrm{curl}\,\tilde{\mathbf{u}}),$$

here $\|\cdot\|$ and $(\cdot,\cdot)$ denote the $L^2(D)$ norm and $L^2(D)$ inner product, respectively. Using Cauchy-Schwarz inequality and the fact that μ_r^{-1} and ξ are symmetric we have

$$k\mathrm{Re}\,(\xi\mu_r^{-1}\mathrm{curl}\,\tilde{\mathbf{u}},\tilde{\mathbf{u}}) \ge -\|\xi\mu_r^{-1}\mathrm{curl}\,\tilde{\mathbf{u}}\|\|k\tilde{\mathbf{u}}\| \ge -\frac{1}{2}\left(\||\mu_r^{-1}\xi|_F\|^2_{L^\infty(D)}\|\mathrm{curl}\,\tilde{\mathbf{u}}\|^2 + k^2\|\tilde{\mathbf{u}}\|^2\right)$$

and

$$-k\mathrm{Re}\,(\mu_r^{-1}\xi\tilde{\mathbf{u}},\mathrm{curl}\,\tilde{\mathbf{u}}) \ge -\|k\mu_r^{-1}\xi\tilde{\mathbf{u}}\|\|\mathrm{curl}\,\tilde{\mathbf{u}}\| \ge -\frac{1}{2}\left(k^2\||\mu_r^{-1}\xi|_F\|^2_{L^\infty(D)}\|\tilde{\mathbf{u}}\|^2 + \|\mathrm{curl}\,\tilde{\mathbf{u}}\|^2\right).$$

Therefore

$$\mathrm{Im}\left(\int_D (k^2(P-\xi\mu_r^{-1}\xi)\tilde{\mathbf{u}} + ik\xi\mu_r^{-1}\mathrm{curl}\,\tilde{\mathbf{u}})\cdot\overline{\tilde{\mathbf{u}}}\,d\mathbf{x} + \int_D (Q\mathrm{curl}\,\tilde{\mathbf{u}} - ik\mu_r^{-1}\xi\tilde{\mathbf{u}})\cdot\mathrm{curl}\,\overline{\tilde{\mathbf{u}}}\,d\mathbf{x}\right)$$
$$\ge C_1\|\mathrm{curl}\,\tilde{\mathbf{u}}\|^2 + k^2C_2\|\tilde{\mathbf{u}}\|^2,$$

where

$$C_1 = c_1 - \frac{1}{2}(\||\mu_r^{-1}\xi|_F\|^2_{L^\infty(D)} + 1) > 0,$$
$$C_2 = c_2 - \frac{1}{2}(\||\mu_r^{-1}\xi|_F\|^2_{L^\infty(D)} + 1) > 0.$$

Combining the estimates for the two sides of (3.6) we have

$$C_1\|\mathrm{curl}\,\tilde{\mathbf{u}}\|^2 + k^2C_2\|\tilde{\mathbf{u}}\|^2 \le 0,$$

and thus $\tilde{\mathbf{u}} = 0$ in D or $\mathbf{u}|_D = 0$. Due to (3.4), $\mathbf{u} = 0$ in $H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega_h)$, which means $\mathbf{u}_1 = \mathbf{u}_2$ in $H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega_h)$. □

4. Periodization of the integro-differential equation

We will be focusing on solving

$$\mathbf{u} - \mathcal{A}\mathcal{S}\mathbf{u} - \mathcal{B}\mathcal{T}\mathbf{u} = \mathcal{A}\mathbf{f} + \mathcal{B}\mathbf{g}, \quad \mathbf{u}\in H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega_h). \tag{4.1}$$

The idea is to transform (4.1) into a periodic equation so that we can utilize all the tools of periodic integral equations, especially the fast Fourier transform (FFT), while ensure that the solution to this periodic equation can be used to find $\mathbf{u}$ that solves (4.1).

For $R > 2h$ let

$$\mathcal{K}_R(\mathbf{x}) = G(\mathbf{x}), \quad \mathbf{x}\in\Omega_R\setminus\{0\}$$

and then extend $\mathcal{K}_R$ $2R$-periodically in x_3 and $\boldsymbol{\alpha}$-quasiperiodically in x_1 and x_2 to the entire $\mathbb{R}^3$. Since G is already $\boldsymbol{\alpha}$-quasiperiodic and smooth everywhere but at its singularities, extending along the x_1 and x_2 axes will not affect the smoothness of $\mathcal{K}_R$. However doing so along the x_3 axis does as G is not periodic in x_3. Thus next we smoothen $\mathcal{K}_R$ by defining

$$\mathcal{K}(\mathbf{x}) = \mathcal{K}_R(\mathbf{x})\chi(x_3), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \{(n_1 2\pi, n_2 2\pi, n_3 2R) : n_1, n_2, n_3 \in \mathbb{Z}\}$$

where $\chi : \mathbb{R} \to [0,1]$ is a $2R$-periodic smooth function satisfying

$$\chi(t) = \begin{cases} 1, & -2h \le t \le 2h \\ 0, & t = \pm R \end{cases} \qquad \text{and} \quad \chi'(\pm R) = \chi''(\pm R) = \chi'''(\pm R) = 0.$$

The explicit formula for the Fourier coefficients of $\mathcal{K}_R$ can be found in [**21**]. This helps us avoid the evaluation of $\mathcal{K}_R$ in the spectral domain. We also extend P, Q, ξ from Ω_R to $\mathbb{R}^3$ 2π-periodically in x_1, x_2 and $2R$-periodically in x_3, and extend $\mathbf{f}$, $\mathbf{g}$ from Ω_R to $\mathbb{R}^3$ $\boldsymbol{\alpha}$-quasiperiodically in x_1, x_2 and $2R$-periodically in x_3. We obtain the following periodic integro-differential equation

$$\mathbf{u} - \mathcal{A}_p \mathcal{S}\mathbf{u} - \mathcal{B}_p \mathcal{T}\mathbf{u} = \mathcal{A}_p \mathbf{f} + \mathcal{B}_p \mathbf{g}, \quad \mathbf{u} \in H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,, \Omega_R). \tag{4.2}$$

Next we will analyze the periodic integral operators in the equation. To this end, for $\lambda \ge 0$, we consider periodic Sobolev spaces $H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)$ and $H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,, \Omega_R)$ as spaces consisting of functions in $L^2(\Omega_R)^3$ that satisfy

$$\|\mathbf{w}\|^2_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)} = \sum_{\mathbf{j}\in\mathbb{Z}^3} (1 + |\mathbf{j}|^2)^\lambda |\widehat{\mathbf{w}}(\mathbf{j})|^2 < \infty,$$

$$\|\mathbf{w}\|^2_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} = \sum_{\mathbf{j}\in\mathbb{Z}^3} (1 + |\mathbf{j}|^2)^\lambda \left(k^2 |\widehat{\mathbf{w}}(\mathbf{j})|^2 + \left| \tilde{\mathbf{j}} \times \widehat{\mathbf{w}}(\mathbf{j}) \right|^2 \right) < \infty,$$

respectively, where $\tilde{\mathbf{j}} = \left(\alpha_1 + j_1, \alpha_2 + j_2, \frac{\pi j_3}{R}\right)$. Note that $H^0_{\boldsymbol{\alpha},p}(\Omega_R) = L^2(\Omega_R)^3$ and $H^0_{\boldsymbol{\alpha},p}(\mathrm{curl}\,, \Omega_R) = H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,, \Omega_R)$. Here the orthonormal basis for $L^2(\Omega_R)^3$ is defined as

$$\phi_{\mathbf{j}}(\mathbf{x}) = \frac{1}{\sqrt{8R\pi^2}} \exp\left(i\alpha_{\tilde{j}} \cdot \mathbf{x} + i\frac{\pi j_3}{R} x_3 \right), \quad \mathbf{j} = (j_1, j_2, j_3) \in \mathbb{Z}^3 \tag{4.3}$$

where $\tilde{j} = (j_1, j_2)$, and for $\mathbf{w} \in L^2(\Omega_R)^3$, the Fourier coefficients $\widehat{\mathbf{w}}(\mathbf{j})$ are given by

$$\widehat{\mathbf{w}}(\mathbf{j}) = \int_{\Omega_r} \mathbf{w}(\mathbf{x}) \overline{\phi_{\mathbf{j}}(\mathbf{x})} d\mathbf{x}. \tag{4.4}$$

In fact

$$\|\mathbf{w}\|^2_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} = k^2 \|\mathbf{w}\|^2_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)} + \|\mathrm{curl}\,\mathbf{w}\|^2_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)}.$$

Note that the extension of a function $\mathbf{w}$ in $H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,, \Omega_R)$ to the entire $\mathbb{R}^3$ using the Fourier expansion

$$\mathbf{w}(\mathbf{x}) = \sum_{\mathbf{j}\in\mathbb{Z}^3} \widehat{\mathbf{w}}(\mathbf{j}) \phi_{\mathbf{j}}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3$$

is $\boldsymbol{\alpha}$-quasiperiodic in x_1, x_2 and $2R$-periodic in x_3.

Let $H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)^{3\times3}$ be the space of 3×3 matrix-valued functions whose columns are functions in $H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)$. Now we prescribe an assumption on regularity of the parameters as well as the solution $\mathbf{u}$ of (4.2).

Assumption 4.1. Assume that $\varepsilon_r, \mu_r, \mu_r^{-1}, \xi \in H^{\eta+\frac{1}{2}+\epsilon}_{\boldsymbol{\alpha},p}(\Omega_R)^{3\times3}$ for some $\epsilon > 0$ and $\eta > \frac{1}{2}$. Moreover, $\mathbf{u} \in H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,, \Omega_R)$ for some λ such that $0 \le \lambda < \eta - \frac{1}{2}$.

Next, for $\mathbf{h} \in H^{\lambda}_{\boldsymbol{\alpha},p}(\Omega_R)$, we define

$$\mathcal{A}_p\mathbf{h}(\mathbf{x}) := (k^2 + \nabla \mathrm{div}) \int_D \mathcal{K}(\mathbf{x}-\mathbf{y})\mathbf{h}(\mathbf{y})d\mathbf{y},$$
$$\mathcal{B}_p\mathbf{h}(\mathbf{x}) := \mathrm{curl} \int_D \mathcal{K}(\mathbf{x}-\mathbf{y})\mathbf{h}(\mathbf{y})d\mathbf{y}$$

as periodized versions of $\mathcal{A}$ and $\mathcal{B}$ respectively.

Lemma 4.2. *$\mathcal{A}_p$ and $\mathcal{B}_p$ are bounded linear operators from $H^{\lambda}_{\boldsymbol{\alpha},p}(\Omega_R)$ to $H^{\lambda}_{\boldsymbol{\alpha},p}(\mathrm{curl}\,, \Omega_R)$.*

Proof. The linearity is obvious. From the Fourier coefficients of $\mathcal{K}$ in [**21**] there exists $C_1, C_2 > 0$ such that $|\widehat{\mathcal{K}}(\mathbf{j})| \le C_1|\mathbf{j}|^{-2}$ and $|\tilde{\mathbf{j}}| \le C_2|\mathbf{j}|$ for all $\mathbf{j} \in \mathbb{Z}^3\setminus\{0\}$. Let $\mathbf{v} \in H^{\lambda}_{\boldsymbol{\alpha},p}(\Omega_R)$,

$$\|\mathcal{A}_p\mathbf{v}\|^2_{H^{\lambda}_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} = \sum_{\mathbf{j}\in\mathbb{Z}^3} (1+|\mathbf{j}|^2)^{\lambda} \left(k^2 \left|\widehat{\mathcal{A}_p\mathbf{v}}(\mathbf{j})\right|^2 + \left|\tilde{\mathbf{j}} \times \widehat{\mathcal{A}_p\mathbf{v}}(\mathbf{j})\right|^2 \right)$$
$$= \sum_{\mathbf{j}\in\mathbb{Z}^3} (1+|\mathbf{j}|^2)^{\lambda} \left(k^2 \left| k^2\widehat{\mathcal{K}}(\mathbf{j})\widehat{\mathbf{v}}(\mathbf{j}) + \tilde{\mathbf{j}}\left(\tilde{\mathbf{j}}\cdot\widehat{\mathcal{K}}(\mathbf{j})\widehat{\mathbf{v}}(\mathbf{j})\right)\right|^2 + \left|\tilde{\mathbf{j}} \times k^2\widehat{\mathcal{K}}(\mathbf{j})\widehat{\mathbf{v}}(\mathbf{j})\right|^2 \right)$$
$$\le k^2(2k^4 + 2|\boldsymbol{\alpha}|^4 + k^2|\boldsymbol{\alpha}|^2)|\widehat{\mathcal{K}}(0)|^2|\widehat{\mathbf{v}}(0)|^2 + C_1^2k^2\left(2k^4 + 2C_2^4 + C_2^2k^2\right) \sum_{\mathbf{j}\in\mathbb{Z}^3\setminus\{0\}} (1+|\mathbf{j}|^2)^{\lambda}|\widehat{\mathbf{v}}(\mathbf{j})|^2.$$

Hence let $C^2 = k^2 \max\left\{(2k^4 + 2|\boldsymbol{\alpha}|^4 + k^2|\boldsymbol{\alpha}|^2)|\widehat{\mathcal{K}}(0)|^2, C_1^2\left(2k^4 + 2C_2^4 + C_2^2k^2\right)\right\}$ we have

$$\|\mathcal{A}_p\mathbf{v}\|_{H^{\lambda}_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} \le C\|\mathbf{v}\|_{H^{\lambda}_{\boldsymbol{\alpha},p}(\Omega_R)}.$$

On the other hand

$$\|\mathcal{B}_p\mathbf{v}\|^2_{H^{\lambda}_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} = \sum_{\mathbf{j}\in\mathbb{Z}^3} (1+|\mathbf{j}|^2)^{\lambda} \left(k^2 \left|\widehat{\mathcal{B}_p\mathbf{v}}(\mathbf{j})\right|^2 + \left|\tilde{\mathbf{j}} \times \widehat{\mathcal{B}_p\mathbf{v}}(\mathbf{j})\right|^2 \right)$$
$$= \sum_{\mathbf{j}\in\mathbb{Z}^3} (1+|\mathbf{j}|^2)^{\lambda} \left[k^2 \left|\tilde{\mathbf{j}} \times \widehat{\mathcal{K}}(\mathbf{j})\widehat{\mathbf{v}}(\mathbf{j})\right|^2 + \left|\tilde{\mathbf{j}} \times \left(\tilde{\mathbf{j}} \times \widehat{\mathcal{K}}(\mathbf{j})\widehat{\mathbf{v}}(\mathbf{j})\right|^2\right)\right]$$
$$\le (k^2|\boldsymbol{\alpha}|^2 + |\boldsymbol{\alpha}|^4)|\widehat{\mathcal{K}}(0)|^2|\widehat{\mathbf{v}}(0)|^2 + C_1^2\left(C_2^2k^2 + C_2^4\right) \sum_{\mathbf{j}\in\mathbb{Z}^3\setminus\{0\}} (1+|\mathbf{j}|^2)^{\lambda}|\widehat{\mathbf{v}}(\mathbf{j})|^2.$$

Let $C'^2 = \max\left\{(k^2|\boldsymbol{\alpha}|^2 + |\boldsymbol{\alpha}|^4)|\widehat{\mathcal{K}}(0)|^2, C_1^2\left(C_2^2k^2 + C_2^4\right)\right\}$ we have

$$\|\mathcal{B}_p\mathbf{v}\|_{H^{\lambda}_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} \le C'\|\mathbf{v}\|_{H^{\lambda}_{\boldsymbol{\alpha},p}(\Omega_R)}.$$

□

The operators $\mathcal{A}_p\mathcal{S}$ and $\mathcal{B}_p\mathcal{T}$ have some interesting mapping properties. First the following lemma will be handy.

Lemma 4.3. *If $A \in H^{\gamma}_{\boldsymbol{\alpha},p}(\Omega_R)^{3\times3}$ and $\mathbf{v} \in H^{\nu}_{\boldsymbol{\alpha},p}(\Omega_R)$ where $\gamma > \frac{1}{2}$ and $0 \le \nu < \gamma - \frac{1}{2}$ then there exists $C > 0$ such that*

$$\|A\mathbf{v}\|_{H^{\nu}_{\boldsymbol{\alpha},p}(\Omega_R)} \le C\|\mathbf{v}\|_{H^{\nu}_{\boldsymbol{\alpha},p}(\Omega_R)}.$$

PROOF. Suppose $A = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}$, $v = \begin{bmatrix}v_1 & v_2 & v_3\end{bmatrix}^T$ then

$$\|A\mathbf{v}\|_\nu^2 = \left\|\sum_{j=1}^{3} \mathbf{a}_j v_j\right\|_\nu^2 \leq 3\sum_{j=1}^{3}\|\mathbf{a}_j v_j\|_\nu^2. \tag{4.5}$$

For $j = 1, 2, 3$ we have

$$\begin{aligned}
\|\mathbf{a}_j v_j\|^2_{H^\nu_{\boldsymbol{\alpha},p}(\Omega_R)} &= \sum_{\mathbf{j}\in\mathbb{Z}^3}(1+|\mathbf{j}|^2)^\nu |\widehat{\mathbf{a}_j v_j}(\mathbf{j})|^2 = \sum_{\mathbf{j}\in\mathbb{Z}^3}(1+|\mathbf{j}|^2)^\nu \left|\sum_{\mathbf{l}\in\mathbb{Z}^3}\widehat{\mathbf{a}}_j(\mathbf{j}-\mathbf{l})\widehat{v}_j(\mathbf{l})\right|^2 \\
&\leq \sum_{\mathbf{j}\in\mathbb{Z}^3}\left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{j}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}}_j(\mathbf{j}-\mathbf{l})||\widehat{v}_j(\mathbf{l})|\right)^2 \\
&\leq 2^{2\nu}\sum_{\mathbf{j}\in\mathbb{Z}^3}\left(\sum_{\mathbf{l}\in\mathbb{Z}^3}\left[(1+|\mathbf{j}-\mathbf{l}|^2)^{\frac{\nu}{2}} + (1+|\mathbf{l}|^2)^{\frac{\nu}{2}}\right]|\widehat{\mathbf{a}}_j(\mathbf{j}-\mathbf{l})||\widehat{v}_j(\mathbf{l})|\right)^2 \\
&\leq 2^{2\nu+1}(S_1+S_2)
\end{aligned}$$

where

$$\begin{aligned}
S_1 &= \sum_{\mathbf{j}\in\mathbb{Z}^3}\left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{j}-\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}}_j(\mathbf{j}-\mathbf{l})||\widehat{v}_j(\mathbf{l})|\right)^2, \\
S_2 &= \sum_{\mathbf{j}\in\mathbb{Z}^3}\left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}}_j(\mathbf{j}-\mathbf{l})||\widehat{v}_j(\mathbf{l})|\right)^2.
\end{aligned}$$

Applying the Cauchy - Schwarz inequality consecutively gives

$$\begin{aligned}
S_1 &\leq \sum_{\mathbf{j}\in\mathbb{Z}^3}\left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{j}-\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}}_j(\mathbf{j}-\mathbf{l})||\widehat{v}_j(\mathbf{l})|^2\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{j}-\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}}_j(\mathbf{j}-\mathbf{l})|\right) \\
&= \left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}}_j(\mathbf{l})|\right)\sum_{\mathbf{l}\in\mathbb{Z}^3}|\widehat{v}_j(\mathbf{l})|^2\sum_{\mathbf{j}\in\mathbb{Z}^3}(1+|\mathbf{j}-\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}}_j(\mathbf{j}-\mathbf{l})| \\
&\leq \left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}}_j(\mathbf{l})|\right)^2\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^\nu|\widehat{v}_j(\mathbf{l})|^2
\end{aligned}$$

and

$$\begin{aligned}
S_2 &\leq \sum_{\mathbf{j}\in\mathbb{Z}^3}\left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^\nu|\widehat{\mathbf{a}}_j(\mathbf{j}-\mathbf{l})||\widehat{v}_j(\mathbf{l})|^2\sum_{\mathbf{l}\in\mathbb{Z}^3}|\widehat{\mathbf{a}}_j(\mathbf{j}-\mathbf{l})|\right) \\
&= \left(\sum_{\mathbf{l}\in\mathbb{Z}^3}|\widehat{\mathbf{a}}_j(\mathbf{l})|\right)\sum_{\mathbf{l}\in\mathbb{Z}^3}|\widehat{v}_j(\mathbf{l})|^2\sum_{\mathbf{j}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^\nu|\widehat{\mathbf{a}}_j(\mathbf{j}-\mathbf{l})||\widehat{v}_j(\mathbf{l})|^2 \\
&\leq \left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}}_j(\mathbf{l})|\right)^2\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^\nu|\widehat{v}_j(\mathbf{l})|^2.
\end{aligned}$$

Thus

$$\|\mathbf{a}_j v_j\|^2_{H^\nu_{\boldsymbol{\alpha},p}(\Omega_R)} \le 2^{2\nu+2}\left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}_j}(\mathbf{l})|\right)^2\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\nu}|\widehat{v_j}(\mathbf{l})|^2.$$

Also by the Cauchy - Schwarz inequality

$$\left(\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\frac{\nu}{2}}|\widehat{\mathbf{a}_j}(\mathbf{l})|\right)^2 \le \sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\nu-\gamma}\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\gamma}|\widehat{\mathbf{a}_j}(\mathbf{l})|^2 = C_1^2\|\mathbf{a}_j\|^2_{H^\gamma_{\boldsymbol{\alpha},p}(\Omega_R)},$$

where $C_1^2=\sum_{\mathbf{l}\in\mathbb{Z}^3}(1+|\mathbf{l}|^2)^{\nu-\gamma}$ (this series converges since $\nu-\gamma<-\frac{1}{2}$). Hence

$$\|\mathbf{a}_j v_j\|^2_{H^\nu_{\boldsymbol{\alpha},p}(\Omega_R)} \le 2^{2\nu+2}C_1^2\|\mathbf{a}_j\|^2_{H^\gamma_{\boldsymbol{\alpha},p}(\Omega_R)}\|\mathbf{v}\|^2_{H^\nu_{\boldsymbol{\alpha},p}(\Omega_R)}. \tag{4.6}$$

By (4.5) and (4.6)

$$\|A\mathbf{v}\|^2_{H^\nu_{\boldsymbol{\alpha},p}(\Omega_R)} = \sum_{j=1}^3\|\mathbf{a}_j v_j\|^2_{H^\nu_{\boldsymbol{\alpha},p}(\Omega_R)} \le 3.2^{2\nu+2}C_1^2\max_j\left\{\|\mathbf{a}_j\|^2_{H^\gamma_{\boldsymbol{\alpha},p}(\Omega_R)}\right\}\|\mathbf{v}\|^2_{H^\nu_{\boldsymbol{\alpha},p}(\Omega_R)}.$$

Let $C=\sqrt{3}.2^{\nu+1}C_1\max_j\left\{\|\mathbf{a}_j\|_{H^\gamma_{\boldsymbol{\alpha},p}(\Omega_R)}\right\}$, we have

$$\|A\mathbf{v}\|_{H^\nu_{\boldsymbol{\alpha},p}(\Omega_R)} \le C\|\mathbf{v}\|_{H^\nu_{\boldsymbol{\alpha},p}(\Omega_R)}.$$

□

LEMMA 4.4. *The matrix-valued functions P, Q, $\xi\mu_r^{-1}$, $\xi\mu_r^{-1}\xi$, $\mu_r^{-1}\xi$ belong to $H^\eta_{\boldsymbol{\alpha},p}(\Omega_R)^{3\times3}$. Moreover, the operators $\mathcal{S}$, $\mathcal{T}$ are bounded linear operators from $H^\lambda_{\boldsymbol{\alpha},p}(\operatorname{curl},\Omega_R)$ to $H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)$.*

PROOF. For the first part, we only prove for $\xi\mu_r^{-1}\xi$, the other cases are either similar or straightforward. Suppose $\xi=\begin{bmatrix}\boldsymbol{\xi}_1 & \boldsymbol{\xi}_2 & \boldsymbol{\xi}_3\end{bmatrix}$, then $\mu_r^{-1}\xi=\begin{bmatrix}\mu_r^{-1}\boldsymbol{\xi}_1 & \mu_r^{-1}\boldsymbol{\xi}_2 & \mu_r^{-1}\boldsymbol{\xi}_3\end{bmatrix}$. For $j=1,2,3$, since $\mu_r^{-1}\in H^{\eta+\frac{1}{2}+\epsilon}_{\boldsymbol{\alpha},p}(\Omega_R)^{3\times3}$ and $\boldsymbol{\xi}_j\in H^\eta_{\boldsymbol{\alpha},p}(\Omega_R)$, by Lemma 4.3 there exists $c>0$ such that

$$\|\mu_r^{-1}\boldsymbol{\xi}_j\|_{H^\eta_{\boldsymbol{\alpha},p}(\Omega_R)} \le c\|\boldsymbol{\xi}_j\|_{H^\eta_{\boldsymbol{\alpha},p}(\Omega_R)} < \infty.$$

Similarly we can show that $\|\bar{\xi}\mu_r^{-1}\boldsymbol{\xi}_j\|_{H^\eta_{\boldsymbol{\alpha},p}(\Omega_R)} \le c'\|\mu_r^{-1}\boldsymbol{\xi}_j\|_{H^\eta_{\boldsymbol{\alpha},p}(\Omega_R)} < \infty$ for some $c'>0$. Thus $\xi\mu_r^{-1}\xi\in H^\eta_{\boldsymbol{\alpha},p}(\Omega_R)^{3\times3}$.

Linearity of $\mathcal{S}$ and $\mathcal{T}$ is clear, we only prove their boundedness. For $\mathbf{u}\in H^\lambda_{\boldsymbol{\alpha},p}(\operatorname{curl},\Omega_R)$, $\operatorname{curl}\mathbf{u}\in H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)$. Therefore by lemma 4.3

$$\begin{aligned}\|\mathcal{S}\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)} &\le \|(P-\xi\mu_r^{-1}\xi)\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)} + \left\|\frac{i}{k}\xi\mu_r^{-1}\operatorname{curl}\mathbf{u}\right\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)}\\ &\le C_1k\|\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)} + C_2\|\operatorname{curl}\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)}\\ &\le C\|\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\operatorname{curl},\Omega_R)}\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{T}\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)} &\le \|Q\operatorname{curl}\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)} + \|ik\mu_r^{-1}\xi\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)}\\ &\le C_1'\|\operatorname{curl}\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)} + C_2'k\|\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\Omega_R)}\\ &\le C'\|\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\operatorname{curl},\Omega_R)}\end{aligned}$$

where $C_1, C_1', C_2, C_2', C, C'$ are positive numbers. □

The following theorem gives us the the relation between the periodic equation (4.2) and the original one as well as its Fredholm property.

THEOREM 4.5. *If* $\mathbf{u} \in H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)$ *is a solution to (4.2) then* $\mathbf{u}|_{\Omega_h} \in H_{\boldsymbol{\alpha}}(\mathrm{curl}\,,\Omega_h)$ *solves (4.1). Moreover, there exist a constant* $C > 0$ *and a compact operator* K *on* $H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)$ *such that*

$$\mathrm{Re}\,(\mathbf{u}-\mathcal{A}_p\mathcal{S}\mathbf{u}-\mathcal{B}_p\mathcal{T}\mathbf{u},\mathbf{u})_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} \geq C\|\mathbf{u}\|^2_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}+\mathrm{Re}\,(K\mathbf{u},\mathbf{u})_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}.$$

The proof can be done using the same reasoning as in Theorem 5.2 and Theorem 5.3 in [**21**].

5. A spectral Galerkin method

For $N \in \mathbb{N}$, define

$$\mathbb{Z}^3_N = \left\{\mathbf{j} = \begin{bmatrix} j_1 & j_2 & j_3 \end{bmatrix}^T \in \mathbb{Z}^3 : -\frac{N}{2} < j_1, j_2, j_3 \leq \frac{N}{2}\right\}.$$

Now we consider

$$\mathcal{T}_N = \mathrm{span}\left\{\phi_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^3_N\right\}$$

which is a finite dimensional subspace of $H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)$ and let $P_N : H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R) \to \mathcal{T}_N$ be the orthogonal projection onto $\mathcal{T}_N$, i.e

$$P_N\mathbf{u} = \sum_{\mathbf{j}\in\mathbb{Z}^3_N} \widehat{\mathbf{u}}(\mathbf{j})\phi_{\mathbf{j}}(\mathbf{x}).$$

We approximate (4.2) by the following Galerkin equation

(5.1)
$$(\mathbf{u}_N-\mathcal{A}_p\mathcal{S}\mathbf{u}_N-\mathcal{B}_p\mathcal{T}\mathbf{u}_N,\mathbf{w}_N)_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} = (\mathcal{A}_p\mathbf{f}+\mathcal{B}_p\mathbf{g},\mathbf{w}_N)_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}, \quad \mathbf{u}_N \in \mathcal{T}_N$$

for all $\mathbf{w}_N \in \mathcal{T}_N$, which is equivalent to

$$\mathbf{u}_N - P_N\mathcal{A}_p\mathcal{S}\mathbf{u}_N - P_N\mathcal{B}_p\mathcal{T}\mathbf{u}_N = P_N\mathcal{A}_p\mathbf{f} + P_N\mathcal{B}_p\mathbf{g}, \quad \mathbf{u}_N \in \mathcal{T}_N. \tag{5.2}$$

We will see that convergence is achieved without smoothness of the parameters, but the smoother they are the higher order of convergence we will get.

LEMMA 5.1. *There exists* $C_\lambda > 0$ *such that for all* $\mathbf{w} \in H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)$

$$\|P_N\mathbf{w} - \mathbf{w}\|_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} \leq C_\lambda N^{-\lambda}\|\mathbf{w}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}.$$

PROOF. For $\mathbf{w} \in H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)$

$$\begin{aligned}
\|P_N\mathbf{w} - \mathbf{w}\|^2_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} &= \sum_{\mathbf{j}\notin\mathbb{Z}^3_N} \left(k^2|\widehat{\mathbf{w}}(\mathbf{j})|^2 + \left|\tilde{\mathbf{j}} \times \widehat{\mathbf{w}}(\mathbf{j})\right|^2\right) \\
&= \sum_{\mathbf{j}\notin\mathbb{Z}^3_N} \frac{1}{(1+|\mathbf{j}|^2)^\lambda}(1+|\mathbf{j}|^2)^\lambda \left(k^2|\widehat{\mathbf{w}}(\mathbf{j})|^2 + \left|\tilde{\mathbf{j}} \times \widehat{\mathbf{w}}(\mathbf{j})\right|^2\right) \\
&\leq \frac{1}{\left(1+\frac{3N^2}{4}\right)^\lambda} \sum_{\mathbf{j}\notin\mathbb{Z}^3_N} (1+|\mathbf{j}|^2)^\lambda \left(k^2|\widehat{\mathbf{w}}(\mathbf{j})|^2 + \left|\tilde{\mathbf{j}} \times \widehat{\mathbf{w}}(\mathbf{j})\right|^2\right) \\
&\leq \left(\frac{4}{7}\right)^\lambda N^{-2\lambda}\|\mathbf{w}\|^2_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}.
\end{aligned}$$

□

Recall that the equation (4.2) satisfies a Garding estimate, we have the following convergence theorem for the Galerkin method.

THEOREM 5.2. *There exists $N_0 > 0$ such that for all $N \geq N_0$, the equation (5.2) has a unique solution $\mathbf{u}_N$ and the following error estimate holds*

$$\|\mathbf{u}_N - \mathbf{u}\|_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} \leq C_\lambda N^{-\lambda} \|\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}$$

where $C_\lambda > 0$ is a constant depending on λ and $\mathbf{u}$ is the solution to (4.2).

PROOF. Since (4.2) satisfies the Garding estimate in Theorem 4.5, by Theorem 4.2.9 in [**31**], there exists N_0 and a positive constant C independent of N such that for all $N \geq N_0$,

$$\|\mathbf{u}_N - \mathbf{u}\|_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} \leq C \inf_{\mathbf{w}_N \in \mathcal{T}_N} \|\mathbf{w}_N - \mathbf{u}\|_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)}.$$

Moreover, according to Lemma 5.1,

$$\inf_{\mathbf{w}_N \in \mathcal{T}_N} \|\mathbf{w}_N - \mathbf{u}\|_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} \leq \|P_N \mathbf{u} - \mathbf{u}\|_{H_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)} \leq C_\lambda N^{-\lambda} \|\mathbf{u}\|_{H^\lambda_{\boldsymbol{\alpha},p}(\mathrm{curl}\,,\Omega_R)},$$

therefore the desired estimate holds. □

By this theorem, we can conclude that the spectral Galerkin method converges with arbitrarily high order provided $\mathbf{u}$ is sufficiently smooth. The discretization of equation (5.2) as well as the process of calculating the Fourier coefficients of the involved quantities can be done similarly as in [**21**], therefore are omitted here.

6. Numerical examples

We now present numerical results for the Galerkin method. The computational domain is the rectangular box $(-\pi, \pi)^2 \times (-1, 1)$ $(h = 1)$ which is partitioned uniformly in each dimension into N^3 grid points for some $N \in \mathbb{N}$. The wave number $k = 3\pi/2$. We choose the following $\boldsymbol{\alpha}$-quasiperiodic plane wave is used as the incident electric field

$$\mathbf{E}^{in}(\mathbf{x}) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{ikx_3}.$$

We ran the simulation in MATLAB and observed the scattered electric field $\mathbf{u}$ at $\mathbf{x} = (0, 0, 1.5)$ when N varies from 20 to 100 with step size 10. For each value of N, we calculated the relative error between $\mathbf{u}$ of the current step and that of the previous one, that means

$$\mathrm{error}_j = \frac{|\mathbf{u}_{N_j}(\mathbf{x}) - \mathbf{u}_{N_{j-1}}(\mathbf{x})|}{|\mathbf{u}_{N_{j-1}}(\mathbf{x})|}, \quad j = 1, 2, \ldots, 8.$$

Here $N_0 = 20$ and $N_j = N_{j-1} + 10$. We consider three test cases in the numerical simulations.

6.1. Small smooth spheres.

The first case is a medium consisting of a small sphere in each period with the geometry given by

$$D = \left\{ \mathbf{x} \in \Omega_h : x_1^2 + x_2^2 + x_3^2 < 0.2^2 \right\}$$

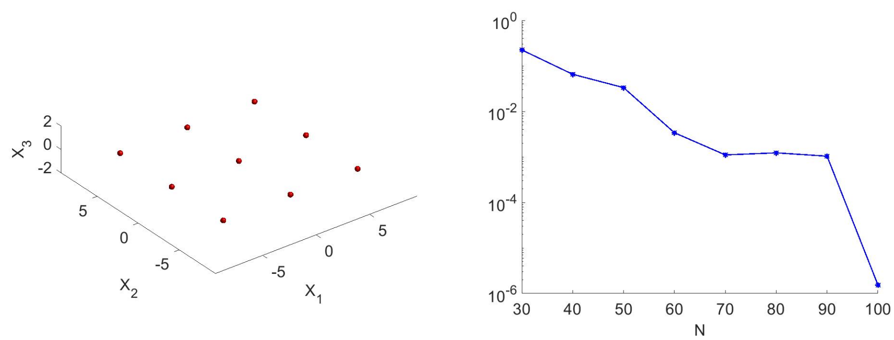
and, for $\mathbf{x} \in \Omega_h$, the material parameters are

$$\begin{aligned}
\varepsilon_r(\mathbf{x}) &= I + \gamma_0(\mathbf{x})\mathrm{diag}\left\{0.2, 0.4, 0.1\right\},\\
\mu_r^{-1}(\mathbf{x}) &= I + \gamma_0(\mathbf{x})\mathrm{diag}\left\{0.3, 0.1, 0.2\right\},\\
\xi(\mathbf{x}) &= \gamma_0(\mathbf{x})\mathrm{diag}\left\{0.01, 0.02, 0.05)\right\},
\end{aligned}$$

where $\gamma_0 : \mathbb{R}^3 \to [0,1]$ is a smooth function that equals 1 at the center of the sphere and 0 outside it. More specifically, for $\mathbf{x} \in D$,

$$\gamma_0(\mathbf{x}) = \exp\left(1 - \frac{0.2^2}{0.2^2 - x_1^2 - x_2^2 - x_3^2}\right).$$

The three-dimensional image for the geometry as well as the relative errors for increasing values of N are shown in Figure 1. The error decreases as N increases and the difference between $N = 90$ and $N = 100$ is very slim.



(A) Periodic structure (9 periods) (B) Relative errors of $\mathbf{u}$ at $(0, 0, 1.5)$

FIGURE 1. Solving the direct scattering problem for periodically aligned spheres.

6.2. Smooth rectangular boxes. Next we consider a medium that consists of one rectangular box in each period. The geometry is given by

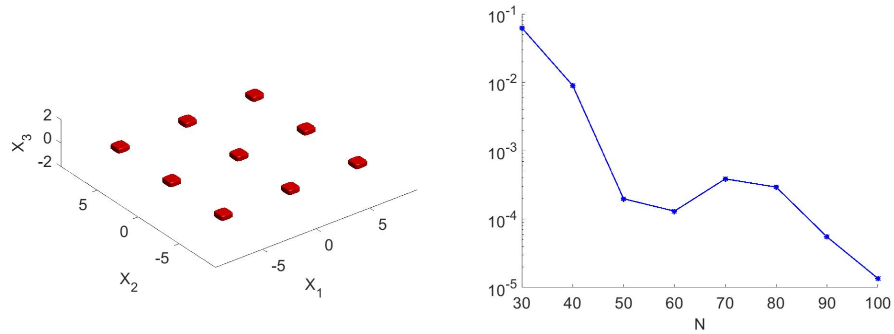
$$D = \{\mathbf{x} \in \Omega_h : |x_1| < 0.5, |x_2| < 0.5, |x_3| < 0.2\}$$

and, for $\mathbf{x} \in \Omega_h$, the material parameters are

$$\begin{aligned}
\varepsilon_r(\mathbf{x}) &= I + \gamma_1(x_1)\gamma_2(x_2)\gamma_3(x_3)\mathrm{diag}\left\{0.2, 0.4, 0.1\right\},\\
\mu_r^{-1}(\mathbf{x}) &= I + \gamma_1(x_1)\gamma_2(x_2)\gamma_3(x_3)\mathrm{diag}\left\{0.3, 0.1, 0.2\right\},\\
\xi(\mathbf{x}) &= \gamma_1(x_1)\gamma_2(x_2)\gamma_3(x_3)\mathrm{diag}\left\{0.01, 0.02, 0.05)\right\},
\end{aligned}$$

where $\gamma_1, \gamma_2, \gamma_3 : \mathbb{R} \to [0,1]$ are smooth functions such that their product equals 1 at the centers of the box and 0 outside it. For $\mathbf{x} \in D$, they are given by

$$\begin{aligned}
\gamma_1(x_1) &= \exp\left(1 - \frac{0.5^2}{0.5^2 - x_1^2}\right)\\
\gamma_2(x_2) &= \exp\left(1 - \frac{0.5^2}{0.5^2 - x_2^2}\right)\\
\gamma_3(x_3) &= \exp\left(1 - \frac{0.2^2}{0.2^2 - x_3^2}\right).
\end{aligned}$$



(A) Periodic structure (9 periods) (B) Relative errors of $\mathbf{u}$ at $(0,0,1.5)$

FIGURE 2. Solving the direct scattering problem for periodically aligned boxes.

Figure 2 shows the true geometry and relative errors for the rectangular box. Overall, the error decreases from $N = 30$ to $N = 100$. There is an abnormal rise in error from $N = 60$ to $N = 70$, but the error is already very small at this point so this might be due to roundoff errors.

6.3. Smooth flat strip. The last scattering medium is a strip that spans the entire period horizontally. The geometry is

$$D = \{\mathbf{x} \in \Omega_h : |x_3| < 0.4\}$$

and, for $\mathbf{x} \in \Omega_h$, the material parameters are

$$\begin{aligned} \varepsilon_r(\mathbf{x}) &= I + \gamma_4(\mathbf{x})\mathrm{diag}\,\{0.2, 0.4, 0.1\}\,, \\ \mu_r^{-1}(\mathbf{x}) &= I + \gamma_4(\mathbf{x})\mathrm{diag}\,\{0.3, 0.1, 0.2\}\,, \\ \xi(\mathbf{x}) &= \gamma_4(\mathbf{x})\mathrm{diag}\,\{0.01, 0.02, 0.05)\}\,, \end{aligned}$$

where

$$\gamma_4(\mathbf{x}) = \begin{cases} \exp\left(1 - \frac{0.4^2}{0.4^2 - x_3^2}\right) & \text{if } \mathbf{x} \in D, \\ 0 & \text{if } \mathbf{x} \in \Omega_h \setminus D. \end{cases}$$

In this case, the error gradually decreases as N increases, see Figure 3.

In summary, the spectral method converges numerically for all examples. The periodization of the integro-differential equation along with the use of the fast Fourier transform helps significantly improve computational time and accuracy since we can avoid direct computation of the $\boldsymbol{\alpha}$-quasiperiodic Green's function.

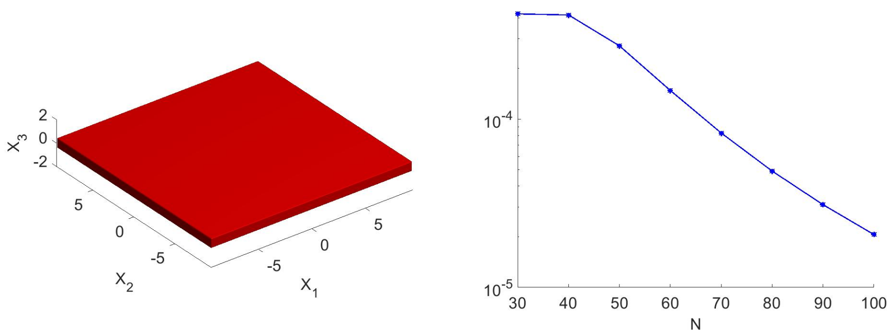
7. The inverse problem

We consider the inverse problem of reconstructing the geometry of periodic scattering objects using data generated by solving the direct problem. For $s > h$, let

$$\Gamma_{\pm s} = (-\pi, \pi)^2 \times \{\pm s\}$$

be the set where data is measured. Moreover, we consider two incident electric fields $\mathbf{E}^{in}_{\pm}(\mathbf{x})$ which are $\boldsymbol{\alpha}$-quasiperiodic plane waves of the form

$$\mathbf{E}^{in}_{\pm}(\mathbf{x}) = \mathbf{q}e^{i(\alpha_1 x_1 + \alpha_2 x_2 \pm \beta x_3)}$$



(A) Periodic structure (9 periods) (B) Relative errors of $\mathbf{u}$ at $(0, 0, 1.5)$

FIGURE 3. Solving the direct scattering problem for a flat strip

where $\beta = \sqrt{k^2 - \alpha_1^2 - \alpha_2^2}$ and $\mathbf{q} \in \mathbb{R}^3$ is the polarization vector satisfying $\mathbf{q} \cdot \begin{bmatrix}\alpha_1 & \alpha_2 & \beta\end{bmatrix}^T = 0$. Note that this choice of the polarization vector $\mathbf{q}$ is appropriate according to 2.6, in fact, it is proportional to $\mathbf{s} \times \mathbf{d}$ when $\mathbf{s} = \mathbf{q} \times \mathbf{d}$. These two incident fields can be understood as one propagating upward and one propagating downward.

Inverse problem. Given the scattered electric field $\mathbf{u}_\pm(\mathbf{x})$ for $\mathbf{x} \in \Gamma_s = \Gamma_{+s} \cup \Gamma_{-s}$ corresponding to $\mathbf{E}^{in}_\pm(\mathbf{x})$, respectively. Determine the geometry D of the scattering object.

7.1. The imaging functional. For $\mathbf{p} \in \mathbb{R}^3$, $\mathbf{z} \in \Omega_h$, and $p > 0$, define

$$\mathcal{I}(\mathbf{z}) = \left| \int_{\Gamma_s} \mathbf{u}_+(\mathbf{x}) \cdot \mathbf{p}\overline{G}(\mathbf{x}, \mathbf{z})\, ds(\mathbf{x}) \right|^p + \left| \int_{\Gamma_s} \mathbf{u}_-(\mathbf{x}) \cdot \mathbf{p}\overline{G}(\mathbf{x}, \mathbf{z})\, ds(\mathbf{x}) \right|^p. \tag{7.1}$$

The definition of $\mathcal{I}(\mathbf{z})$ as in (7.1) is inspired by that of the orthogonality sampling method solving the inverse scattering problem for bi-anisotropic bounded objects in [**23**]. Based on our numerical experiments, $\mathcal{I}(\mathbf{z})$ also behaves as an imaging functional for the inverse problem in this paper, meaning it attains large values when $\mathbf{z} \in D$ and is close to 0 when $\mathbf{z} \notin D$. Therefore, we use it as a tool to solve the inverse problem numerically. The analysis of the imaging functional will be addressed in the future.

7.2. Numerical results. We use the same set of parameters as in the numerical study for the direct problem, which means $h = 1$, $k = 3\pi/2$, $\alpha_1 = \alpha_2 = 0$, $\beta = k$. The two incident waves are

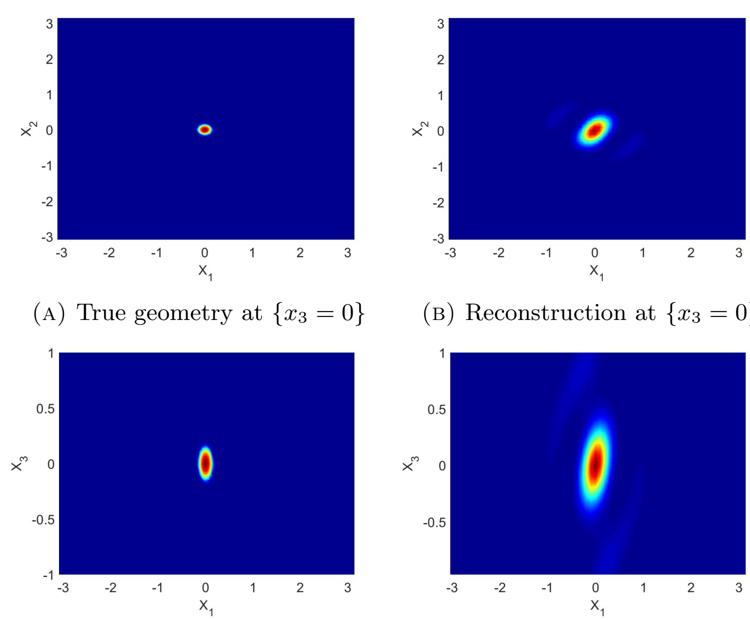
$$\mathbf{E}^{in}_\pm(\mathbf{x}) = \begin{bmatrix}1\\1\\0\end{bmatrix} e^{\pm ikx_3}.$$

The scattered electric fields are measured on $\Gamma_{\pm 1.5} = (-\pi, \pi)^2 \times \{\pm 1.5\}$, each contains 32×32 points of measurement. The sampling domain $(-\pi, \pi)^2 \times (-1, 1)$ is uniformly partitioned into a $64 \times 64 \times 64$ grid. The direct problem is solved on the same domain but with grid density $32 \times 32 \times 32$. We also added 50% of artificial noise to the data obtained from the direct solver.

We used a discretized version of the imaging functional $\mathcal{I}(\mathbf{z})$ for the reconstruction

$$\mathcal{I}(\mathbf{z}) = \left| \sum_{j_1=1}^{32} \sum_{j_2=1}^{32} \mathbf{u}_+(x_1^{j_1}, x_2^{j_2}, +1.5) \cdot \mathbf{p}\overline{G}\left((x_1^{j_1}, x_2^{j_2}, +1.5), \mathbf{z}\right) + \right.$$
$$\left. + \sum_{j_1=1}^{32} \sum_{j_2=1}^{32} \mathbf{u}_+(x_1^{j_1}, x_2^{j_2}, -1.5) \cdot \mathbf{p}\overline{G}\left((x_1^{j_1}, x_2^{j_2}, -1.5), \mathbf{z}\right) \right|^p +$$
$$+ \left| \sum_{j_1=1}^{32} \sum_{j_2=1}^{32} \mathbf{u}_-(x_1^{j_1}, x_2^{j_2}, +1.5) \cdot \mathbf{p}\overline{G}\left((x_1^{j_1}, x_2^{j_2}, +1.5), \mathbf{z}\right) + \right.$$
$$\left. + \sum_{j_1=1}^{32} \sum_{j_2=1}^{32} \mathbf{u}_-(x_1^{j_1}, x_2^{j_2}, -1.5) \cdot \mathbf{p}\overline{G}\left((x_1^{j_1}, x_2^{j_2}, -1.5), \mathbf{z}\right) \right|^p,$$

where $\mathbf{p} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ and the exponent $p = 4$. We then normalized the values of $\mathcal{I}(\mathbf{z})$ by dividing it by the largest value over all $\mathbf{z} \in \Omega_h$, thus we ignored all the constant factors in its discretization including the grid step size. We consider two periodic scattering media. The first is a medium consisting of a small smooth sphere in each period identical to that in the direct problem. Cross-sections of the true and reconstructed geometries are shown in Figure 4.



(A) True geometry at $\{x_3 = 0\}$ (B) Reconstruction at $\{x_3 = 0\}$

(C) True geometry at $\{x_2 = 0\}$ (D) Reconstruction at $\{x_2 = 0\}$

FIGURE 4. Reconstruction of a small sphere (in one period) using two incident fields.

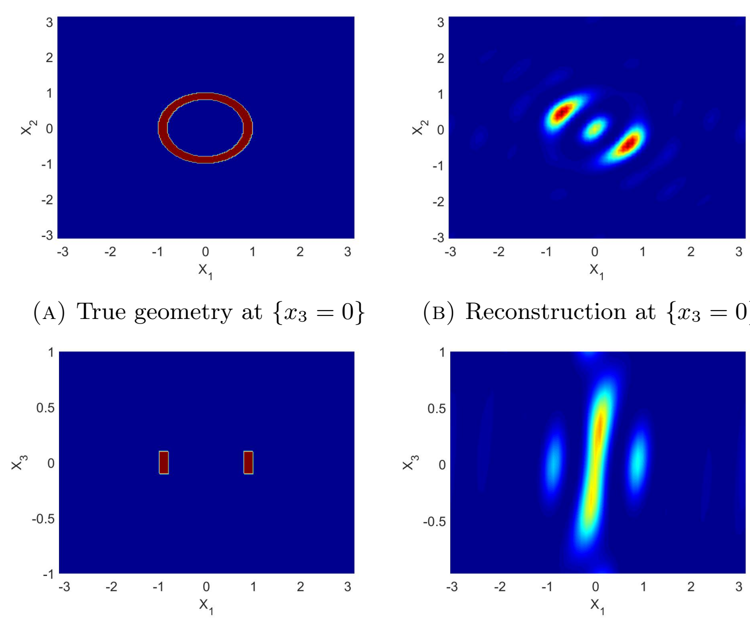
For this small smooth object, the imaging functional was able to give a good reconstruction even though we only used two incident waves. The second medium consists of a non-smooth ring-like object in each period. Each ring is in fact a relatively thin hollow cylinder with small height. The geometry is given by

$$D = \{x \in \Omega_h : 0.8^2 < x_1^2 + x_2^2 < 1, |x_3| < 0.1\}.$$

For $\mathbf{x} \in D$, the material parameters are

$$\begin{aligned} \varepsilon_r(\mathbf{x}) &= I + \operatorname{diag}\{0.2, 0.4, 0.1\}, \\ \mu_r^{-1}(\mathbf{x}) &= I + \operatorname{diag}\{0.3, 0.1, 0.2\}, \\ \xi(\mathbf{x}) &= \operatorname{diag}\{0.01, 0.02, 0.05)\}, \end{aligned}$$

For $\mathbf{x} \in \Omega_h \setminus D$, $\varepsilon_r = \mu_r^{-1} = I$ and $\xi = 0$. The true and reconstructed cross-sections are shown in Figure 5.



(A) True geometry at $\{x_3 = 0\}$ (B) Reconstruction at $\{x_3 = 0\}$

(C) True geometry at $\{x_2 = 0\}$ (D) Reconstruction at at $\{x_2 = 0\}$

FIGURE 5. Reconstruction of a ring (in one period) with two incident fields.

This ring is an interesting object but challenging to reconstruct when there are only a small number of incident waves used. Regardless, the imaging functional gives us some decent information about the shape of the ring, especially in the $\{x_3 = 0\}$ cross-section.

The reconstruction method using $\mathcal{I}(\mathbf{z})$ is viable for reconstructing relatively small and thin objects when data is provided for two incident waves. The most promising aspects of the method are its simple implementation and stability. In its discretized form, calculating the imaging functional consists of only dot products and summations. As for stability, it can deal with very high levels of noise compared to some other methods, as we can see from the above examples.

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