Inner workings of fractional quantum Hall parent Hamiltonians:
A matrix product state point of view

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We study frustration-free Hamiltonians of fractional quantum Hall (FQH) states from the point of view of the matrix product state (MPS) representation of their ground and excited states. There is a wealth of solvable models relating to FQH physics, which, however, is mostly derived and analyzed from the vantage point of first-quantized "analytic clustering properties." In contrast, one obtains long-ranged frustration-free lattice models when these Hamiltonians are studied in an orbital basis, which is the natural basis for the MPS representation of FQH states. The connection between MPS-like states and frustration-free parent Hamiltonians is the central guiding principle in the construction of solvable lattice models, but thus far, only for short-range Hamiltonians and MPSs of finite bond dimension. The situation in the FQH context is fundamentally different. Here we expose the direct link between the infinite-bond-dimension MPS structure of Laughlin–conformal field theory (CFT) states and their parent Hamiltonians. While focusing on the Laughlin state, generalizations to other CFT-MPSs will become transparent.

I. INTRODUCTION

The study of the fractional quantum Hall (FQH) effect is known for its bootstrap approach of producing phase diagrams via beautiful many-body wave functions [1–7] that are pulled apparently out of thin air and linked to effective field theory using a sophisticated array of methods. This seemingly by-passes the traditional Hamiltonian starting point of condensed matter physics. In truth, many of the most preeminent FQH trial wave functions admit [3,5,7–14] parent Hamiltonians with very useful properties. Such Hamiltonians do not only cement the status of certain states as incompressible representatives of viable phases in the FQH regime. On top of that, they also identify a tower of zero-energy (zero-mode) states that describes both quasihole and edge degrees of freedom, and whose angular momentum spectrum is in one-to-one correspondence with the spectrum of a conformal field theory (CFT). This represents the almost ideal scenario of unambiguously extracting low-energy physics from microscopic principles. Arguably, there is no other regime of correlated electron physics beyond one spatial dimension with similar analytic control.

At the same time, one may argue that the picture is not quite complete. Unlike in one dimension, where this is possible for many special Hamiltonians, there still lacks an analytic proof of the existence of an energy gap, a most defining feature of all of quantum Hall physics, but one that is not shared by all model Hamiltonians constructed in this context [6,15]. Indeed, as traditionally constructed, the FQH model Hamiltonians bear little resemblance with solvable models in one dimension. Typically, they are formulated as ultrashort-ranged *k*-body interactions in the presence of some form of Landau level (LL) projection. This description makes efficient use of analytic clustering properties of prototypical trial wave functions. On the other hand, it leaves LL projection quite implicit, masking the physical degrees of freedom [16]. This is not ideal for making statements about spectral properties beyond the zero-mode space. A radical departure from this is the study of FQH Hamiltonians in the LL occupation number basis, i.e., in second quantization [13,17–21]. In this approach, FQH parent Hamiltonians are indeed formally cast as one-dimensional lattice models, albeit, crucially, with interactions that are not strictly finite-ranged. At the pure wave function level, the last decade has seen similar radically new directions, driven by the insight that FQH model wave functions are matrix product states (MPSs) [22–28]. The MPS description likewise tends to lead to a description in the occupation number basis. Naively, these developments should elevate our understanding of FQH parent Hamiltonians to a level similar to that of traditional frustration-free lattice models in one dimension. However, the infinite-ranged nature of these lattice Hamiltonians, and the undoubtedly related fact that no finite bond dimension is associated to the corresponding FQH-MPS, represent considerable obstacles. Indeed, here we argue that the following may be the decisive difference between traditional frustration-free 1D lattice models whose spectral feature are well understood, and the FQH model Hamiltonians in question: The pertinent 1D framework [29] heavily capitalizes on the fact that the existence of frustration-free lattice models is understood *as a consequence* of the MPS structure of the respective states they stabilize. In other words, traditional frustration-free 1D lattice models are constructed to capitalize on the MPS

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structure of certain wave functions, whereas quantum Hall model Hamiltonians are constructed to capitalize on analytic clustering properties of first-quantized wave functions. So far, the only exception to the latter statement may be the composite fermion state Hamiltonians constructed in Ref. [13], although this construction also did not make contact with an MPS. To the best of our knowledge, the existence of a frustration-free parent Hamiltonian in the FQH regime has thus far not been demonstrated using an orbital-basis MPS description.

In this paper, we intend to develop such an angle of attack. Starting from a CFT, and the resulting MPS as constructed in the literature, we will demonstrate the frustration-free character of the second-quantized, "lattice" parent Hamiltonian directly from the orbital-basis MPS formulation of its zero modes. While focusing on Laughlin states as the key example, the generic nature of most our reasoning will be evident.

The remainder of this paper is organized as follows. In Sec. II we review the MPS formalism for FQH states in the context of the Laughlin state. In Sec. III A we review the second-quantized representation of fractional quantum Hall parent Hamiltonians. In Sec. III B we present the induction step that is the heart of our formalism. It can be seen to be quite readily generalizable. Section III C discusses those aspects that are, on the other hand, truly sensitive to details of the Hamiltonian and the CFT in questions. Formally, this is the induction beginning of our method. In Sec. III D, we discuss generic methods to establish the completeness of a given class of CFT-MPSs within the zero-mode space of the corresponding parent Hamiltonian. In Sec. III E we take a detour and discuss variational (non-eigen) quasiparticle states that emerge naturally from the action of second-quantized operator algebras on the MPS. We conclude in Sec. IV.

II. MPS REPRESENTATION OF LAUGHLIN STATES

Being the most fundamental fractional quantum Hall state, the Laughlin state also has the simplest matrix product representation. Before we can elaborate on its relation to the existence of a local parent Hamiltonian, we first review the derivation of this MPS representation from CFT [22,23,25,27]. The filling factor $\nu = 1/q$ Laughlin state [1] for a system of N particles has the wave function

$$\psi_L(z_1 \cdots z_N) = \prod_{i<j} (z_i - z_j)^q, \tag{1}$$

where we suppress the exponential factor $\exp[-\frac{1}{4}\sum_i |z_i|^2]$ and focus on the polynomial part. As is well established in the literature [2,30–33], and again well covered recently with MPSs in mind [23–28], Eq. (1) can be written as the CFT correlator of a chiral free massless bosonic theory, or "Coulomb gas,"

$$\psi_L(z_1 \cdots z_N) = \langle V_{\sqrt{q}}(z_N) \cdots V_{\sqrt{q}}(z_1) \rangle, \tag{2}$$

where $V_{\sqrt{q}}(z) =: e^{i\sqrt{q}\phi(z)}:$ is the holomorphic (chiral) part of the vertex operator. More specifically, it is a primary field in a chiral free massless bosonic CFT in $1+1d$ with $U(1)$ charge $\sqrt{q}$ (see below). Here we will only summarize defining properties of this theory and refer the interested reader to

Refs. [32,33]. The chiral bosonic field can be given the mode expansion defined by

$$\phi(z) = \phi_0 - ia_0 \log(z) + i \sum_{n \neq 0} \frac{1}{n} a_n z^{-n}, \tag{3}$$

where the a_n's obey the algebra

$$[\phi_0, a_0] = i, \quad [a_n, a_m] = n\delta_{n+m,0}. \tag{4}$$

These operators represent neutral excitations of the CFT. They act on states as follows:

$$a_n |N\rangle = 0, \quad n > 0, \quad a_0 |N\rangle = \sqrt{q} N |N\rangle, \tag{5}$$

where $|N\rangle$ is a primary state of the bosonic theory. These states together with the descendant states,

$$\{a_{-n_1} a_{-n_2} \cdots |N\rangle\}, \quad n_i \in \mathbb{N}, \tag{6}$$

form a basis of the Fock space of this theory. The $U(1)$ charge $\sqrt{q}N$, measured by a_0 and shifted by $\exp(i\sqrt{q}\phi_0)$, will also play the role of being proportional to the particle number in the quantum Hall state to be constructed. The a_n's operators are the raising operators that generate the neutral excitations of the chiral free massless bosonic CFT.

The following steps now result in an MPS form for the Laughlin state Eq. (2). The vertex operator admits the mode expansion,

$$V_{\sqrt{q}}(z) = \sum_\lambda V_{-\lambda-h} z^\lambda, \tag{7}$$

where $h = q/2$ is the conformal dimension of V and

$$V_{-\lambda} = \oint \frac{dz}{2\pi i} z^{-\lambda-1} V_{\sqrt{q}}(z). \tag{8}$$

The $V_{\sqrt{q}}(z)$ operator and its modes commute (anticommute) for q even (odd), i.e., when the Jastrow factor in Eq. (1) is describing a symmetric (antisymmetric) function.

This mode expansion allows one to write the correlators (2) of V's in terms of powers of z's:

$$\psi_N(z_1 \cdots z_N) = \sum_{\{\lambda_i\}} \langle \alpha_{\text{out}} | V_{-\lambda_N-h} \cdots V_{-\lambda_1-h} | \alpha_{\text{in}} \rangle \prod_{i=1}^N z_i^{\lambda_i}. \tag{9}$$

Equation (7) formally includes all powers (positive and negative) of z, but will only contain non-negative powers for the appropriate choice of "out" and "in" states $\langle \alpha_{\text{out}} |$ and $|\alpha_{\text{in}}\rangle$, respectively, which we will now discuss. This will lead to the introduction of the background charge operator. As we will see, the $\langle \alpha_{\text{out}} |$ state sets the "maximum" value of the angular momentum, $\lambda_{\max} = q(N-1)$, of the Laughlin state, i.e., the largest angular momentum that will appear in the resulting orbital-basis MPS. Correspondingly, we choose the $|\alpha_{\text{in}}\rangle$ state to have "minimum" angular momentum $\lambda_{\min} = 0$.

More concretely, the in-state $|\alpha_{\text{in}}\rangle$ is defined by the action on the vacuum $|0\rangle$ of the field $V_{\alpha_{\text{in}}\sqrt{q}}$ at the origin,

$$|\alpha_{\text{in}}\rangle = \lim_{z \to 0} V_{\alpha_{\text{in}}\sqrt{q}}(z) |0\rangle = e^{i\alpha_{\text{in}}\sqrt{q}\phi_0} |0\rangle, \tag{10}$$

where the modes have the property $V_{-\lambda-h} |\alpha_{\text{in}}\rangle = 0$ for $\lambda < q\alpha_{\text{in}}$, to ensure the absence of singularities [32,33]. Thus,

$\alpha_{\text{in}} = 0$ indeed leads to $\lambda_{\min} = 0$. On the other hand, the out state is defined via the following limit,

$$\langle \alpha_{\text{out}} | = \lim_{z \to \infty} z^{q\alpha_{\text{out}}} \langle 0 | V_{-\alpha_{\text{out}}\sqrt{q}}(z) = \langle 0 | e^{-i\sqrt{q}\alpha_{\text{out}}\phi_0}, \quad (11)$$

where $\langle \alpha_{\text{out}} | V_{-\lambda-h} = 0$ for $\lambda > q(\alpha_{\text{out}} - 1)$, i.e., $\lambda_{\max} = q(\alpha_{\text{out}} - 1)$. Given the above, and the number of vertex operators in (2), we must now enforce the neutrality condition for the bosonic CFT, which asserts that $-\alpha_{\text{out}}\sqrt{q} + N\sqrt{q} + \alpha_{\text{in}}\sqrt{q} = 0$. This condition must be satisfied; otherwise the correlator vanishes. Since $\alpha_{\text{in}} = 0$, we get $\alpha_{\text{out}} = N$. This is indeed consistent with $\lambda_{\max} = q(N - 1)$ being the highest occupied single particle angular momentum in the Laughlin state. Through Eq. (11), this introduces a background charge operator,

$$O_{bg} = e^{-i\sqrt{q}N\phi_0}, \quad (12)$$

to be added in the CFT correlator, which enforces the neutrality condition. This operator is closely related to the one generating a uniform charge distribution, $\tilde{O}_{bg} = \exp[-i\rho\sqrt{q}\int d^2z\,\phi(z)]$, where ρ is the electric charge density. The only difference between these two operators lies in the log term in Eq. (3). This term generates the Gaussian factor of the Laughlin wave function, modulo an everywhere singular gauge transformation [2,23,26,34]. As is customary, we will usually drop this Gaussian factor, which is merely a spectator in all of the following. We will thus achieve charge neutrality through the operator O_{bg} in Eqs. (11) and (12).

The Laughlin wave function (1) is symmetric (antisymmetric) for q even (odd). This is also manifest in Eq. (9) by the (anti)commutation of the modes $V_{-\lambda-h}$. It is somewhat customary in the literature, though not strictly necessary for our purposes, to restrict sums over $\{\lambda_i\}$ by passing from unordered sequences $\{\lambda_i\}$ to ordered sequences (λ_i) via

$$(\lambda_i) : \begin{cases} 0 \leqslant \lambda_1 \leqslant \cdots \lambda_N \leqslant \lambda_{\max}, & \text{for bosons,} \\ 0 \leqslant \lambda_1 < \cdots \lambda_N \leqslant \lambda_{\max}, & \text{for fermions;} \end{cases} \quad (13)$$

i.e., we now wish to restrict the sum in Eq. (9) to only the ordered sets (λ_i). Indeed, using the (anti)symmetric commutators of the modes of the vertex operator, we may equivalently write

$$\psi_N(z_1 \cdots z_N) = \sum_{(\lambda_i)} \langle \alpha_{\text{out}} | V_{-\lambda_N-h} \cdots V_{-\lambda_1-h} | \alpha_{\text{in}} \rangle$$

$$\times \frac{N!}{\prod_\lambda l_\lambda!} \frac{1}{N!} \sum_{\sigma \in S_N} (\operatorname{sgn}\sigma)^q \prod_{i=1}^N z_i^{\lambda_{\sigma_i}}, \quad (14)$$

where l_λ is the number of occurrences of λ among the λ_i. The ordering of the modes in Eq. (14) must be consistent with (13) because of their (anti)commutation relations. Note that $l_\lambda = 0, 1$ for fermions only.

The second line in the last equation can be identified as an occupation number eigenstate in the angular momentum basis of the Fock space. As a consequence, the last expression is manifestly the second-quantized description of the Laughlin state in this basis,

$$|\psi_N\rangle = \sum_{(\lambda_i)} \langle \alpha_{\text{out}} | V_{-\lambda_N-h} \cdots V_{-\lambda_1-h} | \alpha_{\text{in}} \rangle \, |(\lambda_i)\rangle, \quad (15)$$

where

$$\langle z_1, \ldots, z_N | (\lambda_i) \rangle = \frac{1}{N! \prod_\lambda l_\lambda!} \sum_{\sigma \in S_N} (\operatorname{sgn}\sigma)^q \prod_{i=1}^N z_i^{\lambda_{\sigma_i}}, \quad (16)$$

and, in (15) and (16), we dropped an overall normalization factor $N!$ present in (14), for convenience. We point out that as a basis of the Fock space, the last equation is not only lacking the Gaussian factor, which is a trivial (λ_i)-independent multiplicative factor, but is in any case not normalized (even with the Gaussian factor in place). This is to say, we may think of this basis as created by pseudofermion operators c_λ, c_λ^*, or their bosonic counterparts, where c_λ^* creates a particle in the un-normalized state $z^\lambda \exp(-|z|^2/4)$, and the associated destruction operators c_λ may be defined such that $[c_\lambda, c_{\lambda'}^*]_\pm = \delta_{\lambda,\lambda'}$ holds, as in Ref. [13] (with upper [lower] sign for fermions [bosons]). We use this basis for convenience; however, it is trivial to convert the coefficients in the above expansion to ones referring to a normalized basis, where the λ dependence of proper single-particle normalization factors is given by $(2^\lambda \lambda!)^{-1/2}$. In the following, we will for simplicity refer to the operators c_λ, $c_{\lambda'}^*$ as pseudoparticle operators, referring to both the fermionic and bosonic case on equal footing. They differ from ordinary ladder operators chiefly through the fact that the Hermitian adjoint $(c_\lambda)^\dagger$ is only *proportional* to c_λ^*. The particle number operator can, however, be expressed as $\hat{N} = \sum_\lambda c_\lambda^* c_\lambda$. We refer the reader to Ref. [13] for details.

With these remarks, Eq. (15) defines an MPS representation for the N-particle Laughlin state in the basis $|(\lambda_i)\rangle$ [23–28]. It is common in the literature [25] to further represent the MPS coefficients, $\langle \alpha_{\text{out}} | V_{-\lambda_N-h} \cdots V_{-\lambda_1-h} | \alpha_{\text{in}} \rangle$, in terms of the "site-dependent" matrices $A^{l_\lambda}[\lambda] \propto (V_{-\lambda-h})^{l_\lambda}$: $\langle \alpha_{\text{out}} | A^{l_{qN}}[qN] \cdots A^{l_0}[0] | \alpha_{\text{in}} \rangle$, passing to a "site-dependent MPS" with basis states labeled by orbital occupancies l_λ, thus trading the "external labels" λ_i that list the occupied angular momenta for new labels l_λ, which list the occupancies of all possible angular momenta. While one may clearly switch back and forth between these representations, it will be advantageous for us to work with the original MPS carrying λ_i-labels, Eq. (15).

In the present work, we want to make direct contact between the MPS representation of the Laughlin state (15) and the existence of a frustration-free parent Hamiltonian. Such parent Hamiltonians in the fractional quantum Hall regime traditionally have the benefit of not only counting the incompressible ground state in question among their zero modes, but also quasihole-type excitations, which increase the angular momentum of the state. This increase is infinitesimal if the quasihole approaches the edge of the system, in which case the zero mode describes a gapless edge excitation. Though strictly speaking nondispersing, the physical character of these gapless edge modes becomes apparent when a confining potential is added, which may be taken to be proportional to the total angular momentum. In this way, one recovers gapless, linearly dispersing edge modes, whose mode counting one expects to be in one-to-one correspondence with the counting of modes in the CFT from which the incompressible ground state is constructed in the manner of Eq. (15). Indeed, one can construct variational MPS states for a set of edge excitations in a natural manner (see below). It will be an added benefit of this

work that a closed formalism will emerge to establish these MPS-edge modes as a complete set of zero modes of some parent Hamiltonian. In this way, it will become most manifest that zero-mode counting, which has a long tradition [35–38] in the field, is in one-to-one correspondence with mode counting in the associated CFT. This "zero-mode paradigm" is indeed what one expects of a "good" parent Hamiltonian.

We proceed by reviewing how variational edge modes naturally emerge in the MPS formalism.

In the conformal edge theory, a complete set of edge excitations at fixed particle number is generated by products of the mode operators a_{-n}, as in Eq. (6). It is thus natural, from a variational point of view, to replace $|\alpha_{out}\rangle = |N\rangle$ with Eq. (6) in the MPS (9) in order to describe edge excitations. This leads to the following form:

$$|\psi_N^{a_n\cdots}\rangle = \sum_{(\lambda_i)} \langle N|a_n \cdots V_{-\lambda_N-h} \cdots V_{-\lambda_1-h}|0\rangle \, |(\lambda_i)\rangle, \quad (17)$$

where $a_n \ldots$ is short for any products of M mode operators $a_{n_1}a_{n_2} \ldots a_{n_M}$. While the above motivates the expression (17), it does not establish this expression from a Hamiltonian principle, which we will achieve below. As a stepping stone, it is worthwhile to motivate Eq. (17) in a slightly different manner.

In first quantization, it is well known that a complete set of edge excitations of the Laughlin state can be generated by multiplication of the incompressible Laughlin state with general products of power-sum polynomials [18,35,39–41]. The equivalent of this operation in second quantization, which is more useful for present purposes, is given by the action of the operators $p_n = \sum_{i=1}^{N} c_{\lambda_i+n}^* c_{\lambda_i}$, which facilitate multiplication with the power-sum polynomial $\sum_{k=1}^{N} z_k^n$ [18,41]. The effect of these operators on the Laughlin state in the MPS form (9) can be readily worked out:

$$\sqrt{q}p_n |\psi_N\rangle = \sqrt{q} \sum_{(\lambda_i)} \langle N|V_{-\lambda_1-h} \cdots V_{-\lambda_N-h}|0\rangle$$

$$\times \sum_{i=1}^{N} |\lambda_1, \ldots, \lambda_i + n, \ldots, \lambda_N\rangle$$

$$= \sqrt{q} \sum_{(\lambda_i)} (\langle N|V_{-\lambda_1+n-h} \cdots V_{-\lambda_N-h}|0\rangle + \cdots) |\{\lambda_i\}\rangle$$

$$= \sum_{(\lambda_i)} \langle N|a_n V_{-\lambda_1-h} \cdots V_{-\lambda_N-h}|0\rangle \, |\{\lambda_i\}\rangle = \left|\psi_N^{a_n}\right\rangle,$$

$$(18)$$

where we used the commutation relation

$$[a_n, V_{-\lambda-h}] = \sqrt{q}\, V_{-\lambda+n-h}, \quad (19)$$

which follows from the more elementary CFT identity [32]

$$[a_n, V_{\sqrt{q}}(z)] = \sqrt{q}z^n V_{\sqrt{q}}(z), \quad (20)$$

together with Eq. (8) for the modes of the vertex operator. p_n adds n to the Laughlin state total angular momentum and increases λ_{max} from $q(\alpha_{out} - 1)$ to $q(\alpha_{out} - 1) + n$.

Equation (18) can be iterated arbitrarily many times to give the general edge excitation (17). This now establishes these states (in 17) as a complete set of zero modes in the MPS form, making a parallel with the power-sum polynomials

in first quantization. In the following section, we will make the completeness of the set of zero modes in the MPS more transparent. This now establishes these states as a complete set of zero modes *if* we take as given that the p_n generate such when acting on $|\psi_N\rangle$. While this can be shown in various ways, we will do so here without leaving the MPS framework.

Moreover, the above establishes a one-to-one correspondence between operator algebras acting on the "real" Hilbert space describing microscopic degrees of freedom and modes representing the "virtual" degrees of freedom of the MPS. It is the fact that this kind of translation is feasible that will make the developments of the following section possible, where we explore the action of a Hamiltonian defined in terms of microscopic electron operators on the MPS states (17).

III. THE ZERO-MODE PROPERTY

A. Hamiltonian and setup

We now derive the fact that Laughlin states admit parent Hamiltonians with the claimed features from the MPS structure of the states (17). To this end, we begin by reviewing pseudopotentials in second quantization. From first quantization, it is well known that the $\nu = 1/q$-Laughlin state is annihilated by 2-particle projection operators onto relative angular momentum m known as Haldane pseudopotentials [8], with $m < q$ and m even (odd) for q even (odd). The second-quantized version of such a Hamiltonian is of the general form [17,42,43]

$$H_{\frac{1}{q}} = \sum_{\substack{0 \leqslant m < q \\ (-1)^m = (-1)^q}} \sum_R Q_R^{m\dagger} Q_R^m, \quad (21)$$

where Q_R^m is an operator that annihilates a pair of particles with total angular momentum $2R$. The Hamiltonian is frustration free, in that the ground state is a zero mode of each of the (noncommuting) positive-semidefinite terms $Q_R^{m\dagger} Q_R^m$. The Q_R^m can be chosen such that $Q_R^{m\dagger} Q_R^m$ is precisely the mth Haldane pseudopotential. However, as we will be interested only in the zero-mode space, and the zero-mode condition for a ket $|\psi\rangle$ can be stated as

$$Q_R^m |\psi\rangle = 0 \quad (22)$$

for all pertinent m and R, one may form linearly independent new linear combinations of the Q_R^m. The resulting Hamiltonian (21) then has the same zero-mode space as the sum of the original Haldane pseudopotentials. With this in mind, it was established in [43] that we may choose

$$Q_R^m = \sum_x x^m c_{R-x} c_{R+x}. \quad (23)$$

The above operators appear long-ranged in the distance x only in pseudofermion/boson language. When the normalization factors are restored, turning pseudoparticles into ordinary electron destruction operators via $c_k \rightarrow \mathcal{N}_k c_k$, the action of Eq. (23) is seen to be exponentially decaying in x. Indeed, for $m = 0$ (bosons) and $m = 1$ (fermions), $\sum_R Q_R^{m\dagger} Q_R^m$ with Q_R^m chosen as in Eq. (23) *is* (proportional to) the Haldane pseudopotential with index m. While the relation is more complicated for larger m, all statements about zero modes

that follow do equally apply to Eq. (23) as well as the original Haldane pseudopotential's parent Hamiltonian. Moreover, while we have the disk geometry in mind, all of the following applies also to the other genus-zero geometries, i.e., the cylinder and sphere, where the Laughlin state is the same in terms of suitably defined pseudofermions (which is to say that its first-quantized wave function is given by the same polynomial).

We are now in a position to demonstrate that the general MPS (17) is a zero mode of the Hamiltonian (21). We do so by induction in particle number N. We present the induction step first. That is, we will first demonstrate

$$Q_R^m |\psi_N^{a_n \cdots}\rangle = 0, \tag{24}$$

for all half-integer $R \geqslant 0$, all m included in the sum (21), and all strings $a_n \ldots$, as long as the same statement is known for $N - 1$ in place of N. We note that this is a stronger induction assumption than that in Ref. [17], which was used there to relate the zero-mode property to a second-quantized recursion relation related to Read's order parameter formalism [44] for the Laughlin state. Unlike in Ref. [17], we make an induction assumption for all zero modes, not just the incompressible Laughlin state. In this way, our reasoning becomes completely independent of particular properties of the p_n discussed earlier. This is the main reason why the present analysis can be generalized to other CFT-related quantum Hall states and their parent Hamiltonians.

The two-body operator (23) obeys the following identity, independently of the form factors (here, x^m) [17]:

$$Q_R^m \frac{1}{N} \sum_{\lambda \geqslant 0} c_\lambda^* c_\lambda = \frac{2}{N} Q_R^m + \frac{1}{N} \sum_{\lambda \geqslant 0} c_\lambda^* Q_R^m c_\lambda. \tag{25}$$

The key insight in utilizing this equation is the realization that c_λ acting on any N-particle state of the form (17) yields a state to which the induction hypothesis applies, i.e., a linear combination of $N - 1$ particle states of the form (17). Then, when acting with Eq. (25) on any state $|\psi_N^{a_n \cdots}\rangle$, the last term on the right vanishes by the induction assumption, and the remaining terms yield $(1 - 2/N) Q_R^m |\psi_N^{a_n \cdots}\rangle = 0$; therefore, Eq. (24) follows if $N > 2$.

B. The induction step

To complete the induction step, we must therefore evaluate

$$\sqrt{N} c_\lambda |\psi_N^{a_n \cdots}\rangle$$
$$= \sum_{(\lambda_i)_{i=1\ldots N-1}} \langle N | a_n \ldots V_{-\lambda-h} V_{-\lambda_{N-1}-h} \cdots V_{-\lambda_1-h} |0\rangle$$
$$\times |(\lambda_i)_{i=1\ldots N-1}\rangle. \tag{26}$$

In the above, it was used that $|(\lambda_i)_{i=1\ldots N}\rangle$ as defined in Eq. (16) equals $(\sqrt{N!} \prod_\lambda l_\lambda!)^{-1} c_{\lambda_1}^* \ldots c_{\lambda_N}^* |0\rangle$, and that the modes of $V_{\sqrt{q}}(z)$ commute (anticommute) for even (odd) q so we can pull the mode corresponding to the annihilated angular momentum λ to the left. For fermions, the sign generated by the anticommutation of the modes of the vertex cancel exactly the sign due to the anticommutation of the pseudofermion operators. Let us, for simplicity, first consider the case where $a_n \ldots = a_n$ is a

single mode operator. The relation (19) is used to apply $V_{-\lambda-h}$ directly on $\langle N |$, $\langle N | a_n V_{-\lambda-h} V_{-\lambda_N-h} \cdots V_{-\lambda_1-h} |0\rangle \rightarrow \langle N | (V_{-\lambda-h} a_n + \sqrt{q} V_{-\lambda+n-h}) V_{-\lambda_N-h} \cdots V_{-\lambda_1-h} |0\rangle$, generating an extra term involving $V_{-\lambda-h+n}$ and no a_n mode. The modes of the vertex operator modify the final state in the following way:

$$\langle N | V_{-\lambda-h} = \langle 0 | e^{-i\sqrt{q} N \phi_0} V_{-\lambda-h} = \langle N - 1 | \sum_{l=0}^{q(N-1)-\lambda} b_{q(N-1)-\lambda}^l, \tag{27}$$

where

$$b_k^l = \frac{(-\sqrt{q})^l}{l!} \sum_{i_1 + \cdots + i_l = k} \frac{a_{i_1}}{i_1} \frac{a_{i_2}}{i_2} \cdots \frac{a_{i_l}}{i_l}, \; i_j > 0 \Rightarrow k \geqslant l;$$

$$b_k^0 = \begin{cases} 1, & k = 0, \\ 0, & k > 0. \end{cases} \tag{28}$$

The proof of this result will be left to Appendix A. The operators b_k^l are a collection of l neutral excitations corresponding to the addition of angular momentum k to the Laughlin state; i.e., they are in the algebra generated by the modes a_n, $n > 0$. Inserting Eq. (27) into (26) we find

$$\sqrt{N} c_\lambda |\psi_N^{a_n}\rangle = \sum_{(\lambda_i)_{i=1\ldots N-1}} \langle N - 1 | \left(\sum_{l=0}^{q(N-1)-\lambda} b_{q(N-1)-\lambda}^l a_n \right.$$
$$\left. + \sqrt{q} \sum_{l=0}^{q(N-1)-\lambda+n} b_{q(N-1)-\lambda+n}^l \right)$$
$$\times V_{-\lambda_N-h} \cdots V_{-\lambda_1-h} |0\rangle |(\lambda_i)_{i=1\ldots N-1}\rangle, \tag{29}$$

which, by the induction assumption, is a linear combination of $N - 1$ particle zero modes. Therefore, the action of Q_R^m on $c_\lambda |\psi_N^{a_n}\rangle$ is zero.

We can, of course, similarly treat $c_\lambda |\psi_N\rangle$. Moreover, using the same method (and induction in j), we generally have that

$$\langle N | a_{n_1} \ldots a_{n_j} V_{-\lambda-h} = \langle N - 1 | A, \tag{30}$$

where A is an element of the algebra generated by the a_n, $n > 0$. The induction hypothesis then implies just as before that $c_\lambda |\psi_N^{a_{n_1} \ldots a_{n_j}}\rangle$ is a zero mode. This completes the induction step. One may recognize the generality of these arguments, using only general properties of the operators Q_R^m, and, despite our limitation to the free chiral bosonic case, arguably the CFT in question. Indeed, the case of k-body generalization of the Q_R^m is also quite analogous, the only difference being that the induction needs to start at $N = k$. Therefore, the only aspect that truly depends on details of the Q_R^m-like operators and on the CFT is the induction beginning. Below we continue discussing this for the Laughlin states.

C. The zero-mode property for two particles

Now let us prove the zero-mode property (24) for $N = 2$. In this case we can prove that a state with arbitrary number of edge excitations can always be written as a linear combination of

$$|\psi_2^{n,l}\rangle = \frac{1}{2} \sum_{\{\lambda_1, \lambda_2\}} \langle 2 | V_{-\lambda_2+n-h} V_{-\lambda_1+l-h} |0\rangle |\lambda_1, \lambda_2\rangle, \tag{31}$$

for arbitrary n and l, where we have returned to the nonordered set $\{\lambda_1, \lambda_2\}$ with the addition of the factor $1/2$. For two excitations a_l, $l > 0$, and a_n, $n > 0$, we have, according to (19),

$$\frac{1}{q} \langle a_l a_n V_{-\lambda_2 - h} V_{-\lambda_1 - h} \rangle$$
$$= \langle V_{-\lambda_2 - h} V_{-\lambda_1 + n + l - h} \rangle + \langle V_{-\lambda_2 + l - h} V_{-\lambda_1 + n - h} \rangle$$
$$+ \langle V_{-\lambda_2 + n - h} V_{-\lambda_1 + l - h} \rangle + \langle V_{-\lambda_2 + n + l - h} V_{-\lambda_1 - h} \rangle. \quad (32)$$

Adding more than two excitations to the state will only result in terms of the same general type, and, moreover, the case of zero and one excitations yield, of course, special instances thereof. Hence, it is enough to prove $Q_R^m |\psi_2^{n,l}\rangle = 0$, with Q_R^m given by (23), for non-negative integers integer n and l:

$$\langle 0 | Q_R^m | \psi_2^{n,l} \rangle = \sum_{x=-R}^{R} x^m \frac{1}{2} \sum_{\{\lambda_1, \lambda_2\}} \langle 2, 0 | V_{-\lambda_2 + n - h} V_{-\lambda_1 + l - h} | 0 \rangle$$
$$\times [\delta_{R+x, \lambda_2} \delta_{R-x, \lambda_1} + (-1)^q \delta_{R+x, \lambda_1} \delta_{R-x, \lambda_2}]$$
$$= \delta_{q, 2R-n-l} \sum_{x=-R}^{R} x^m (-1)^{R+x-n} \binom{q}{R+x-n}$$
$$= \delta_{q, 2R-n-l} \sum_{x'=-n}^{q+l} \left(x' - \frac{q-n+l}{2} \right)^m (-1)^{x'} \binom{q}{x'}$$
$$= 0. \quad (33)$$

Here, the first line is found by acting with the operator Q_R^m on the two-particle state (31). The second line is found using the result for the correlator of two modes of the vertex operator given in Appendix B.

The final line is found using the more elementary result $\sum_{0 \leqslant j \leqslant q} (x - j)^q (-1)^j \binom{q}{j} = q! \; \forall x$ [45], and differentiating it with respect to x, finding that the sum vanishes for all $m < q$, as expected.

This concludes our demonstration of how the existence of frustration-free parent Hamiltonians for the Laughlin states can be seen as a consequence of their MPS structure.

D. Considerations of completeness

Having established the existence of a parent Hamiltonian with zero modes Eq. (17), we would also like to remark on how to establish the completeness of these zero modes without leaving the present framework. It has been known for some time [46,47] that the expansion of Laughlin states, and similarly other quantum Hall states, in Salter determinants $|(\lambda_i)\rangle$ or their bosonic equivalents is characterized by certain root states $|(\lambda_i^r)\rangle$. The defining property of $|(\lambda_i^r)\rangle$ is that it has nonzero amplitude in the expansion of the underlying zero mode, and that all other $|(\lambda_i)\rangle$ with this property can be obtained from $|(\lambda_i^r)\rangle$ via sequences of inward-squeezing processes $c_i^* c_j^* c_{j+r} c_{i-r}$, where $i < j$, $r > 0$. The root state agrees with the thin-torus limit [42,46,48–53]. Furthermore, $|(\lambda_i^r)\rangle$ is subject to rules known as generalized Pauli principles (GPPs) that are characteristic of the quantum Hall state and its quasihole excitations. In the case of the Laughlin state, the GPP dictates that there is no more than 1 particle in q adjacent sites. The root state of the incompressible Laughlin state is

simply the densest Slater determinant consistent with this rule, having occupancy $100100100100\ldots$ for $q = 3$.

It is worth noting that these concepts have nontrivial generalizations to multicomponent and mixed-LL states [19,21], leading to entanglement at root level. The root state can then be thought of as the "entangled DNA" of the quantum Hall state, encoding much and more of its topological properties. In essence, the GPP becomes an "entangled Pauli principle" (EPP).

For both single Landau level and mixed-Landau level states, a general framework exists to derive the GPP/EPP as necessary conditions on root states of zero modes of the respective parent Hamiltonian of the general form (21) [13,17,19,21,43], assuming one exists. (Again, generalization to k-particle operators is possible.) One consequence of this framework is that there cannot be more (linearly independent) zero modes than root states consistent with the GPP/EPP. Thus, whenever one has found one zero mode for every permissible root pattern, one is assured of the completeness of such a set of zero modes. This can be applied in the present situation: For MPS quantum Hall states, Estienne *et al.* [25] have derived the systematics of extracting root states. This shows that the states (17) realize all possible root patterns consistent with the GPP, and are thus the complete set of zero modes for the Hamiltonian (21).

E. Quasielectrons

In principle, quasielectron states are expected to emerge as finite-energy eigenstates of the respective parent Hamiltonian. Unfortunately, for all Hamiltonians known to us describing fractional quantum Hall states, such finite-energy eigenstates have not been obtained exactly. Many variational constructs exist [1,54–57]. Here we elaborate on a construction that seems both natural to us and has a simple MPS representation. Consider again Eq. (18). As we have seen, the operators p_n generate all possible zero modes by acting on the incompressible Laughlin state. Moreover, via the second-quantized framework developed here, the action of such operators easily translates into MPSs. One natural approach is to consider the action of the adjoints, or, related to that (by form factors arising from the pseudoparticle nature of the c_λ, c_λ^*), the operators p_{-n}, $n > 0$. A similar approach was taken recently in Ref. [57], where a quasielectron operator has been constructed that exactly fractionalizes, in that q applications exactly correspond to the local addition of one electron in the lowest Landau level, mirroring the celebrated property [58] of Laughlin's quasi*hole* operator. Here we take a simplified approach, and consider the action of the p_{-n} operators on the incompressible Laughlin state. This lowers angular momentum, and so leads to states with nonzero energy in principle, at least for finite particle number. In complete analogy with Eq. (18), one finds

$$\sqrt{q} p_{-n} |\psi_N\rangle = \sqrt{q} \sum_{(\lambda_i)} \langle N | V_{-\lambda_1 - h} \cdots V_{-\lambda_N - h} | 0 \rangle$$
$$\sum_{i=1}^{N} |\lambda_1, \ldots, \lambda_i - n, \ldots, \lambda_N\rangle$$

$$= \sum_{(\lambda_i)} \langle N| V_{-\lambda_1-h} \cdots V_{-\lambda_N-h} \, a_{-n}|0\rangle \, |\{\lambda_i\}\rangle$$

$$=: \left| \psi_N^{a_{-n}} \right\rangle, \tag{34}$$

with obvious generalization to the action of products of p_{-n} operators. Having well-defined angular momentum, one can interpret the above as a basis for extended quasielectron states, with the benefit of having *both* a nice MPS structure *and* a known generating principle in terms of second-quantized electron operators. We expect that this basis may be key to understanding relations between different quasielectron constructions found in the literature, such as Refs. [28,56], which emphasizes MPS formalism, and Ref. [57], which emphasizes second-quantized operator algebras. We stress again that for quasiholes and in the context of the Haldane pseudopotential parent Hamiltonian, Eq. (18) is an exact quasihole basis, whereas for quasiparticles, Eq. (34) is variational in character. Detailed energetic considerations necessitate putting back the Gaussian factor dropped below Eq. (12) [59].

IV. CONCLUSION

In this work we have explicitly linked this existence of a frustration-free parent Hamiltonian of the Laughlin states to the MPS structure of these states. In doing so, we discussed the action of certain second-quantized operator algebras on CFT-derived MPSs, which, in general, also lends itself to a discussion of quasielectrons of *either* sign. While we leave details to future work [60], the framework is expected to generalize to many existing parent Hamiltonians in the FQH regime. We argued that this elevates the theory of these Hamiltonians to a level that is more on par with the theory of frustration-free lattice models in one dimension. Specifically, FQH parent Hamiltonians have recently been constructed that are not obviously determined by clustering properties of first-quantized wave functions [13], counter to the tradition of the field. It is, however, not obvious how to generalize the construction of Ref. [13]. The present construction is expected to make such generalization possible. We are hopeful that this will spur developments leading to a wealth of new solvable models.

APPENDIX A

In this Appendix we will show Eq. (27). The normal ordering expansion of the holomorphic part of the vertex operator is

$$V_{\sqrt{q}}(z) = e^{i\sqrt{q}\phi_0} e^{\sqrt{q}a_0 \log(z)} \prod_{n=1}^{\infty} e^{\frac{\sqrt{q}}{n}\frac{a_{-n}}{z^{-n}}} e^{-\frac{\sqrt{q}}{n}\frac{a_n}{z^n}}. \tag{A1}$$

The action of the vertex operator in the dual state is then

$$\langle 0| \, e^{-i\sqrt{q}N\phi_0} V_{-\lambda-h} = \langle 0| \, e^{-i\sqrt{q}N\phi_0} \oint \frac{dz}{2\pi i} \frac{V_{\sqrt{q}}(z)}{z^{\lambda+1}}$$

$$= \langle N-1| \oint \frac{dz}{2\pi i} z^{q(N-1)-\lambda-1} \sum_{l=0}^{\infty} \frac{1}{l!} \left(\sum_{n=1}^{\infty} \frac{-\sqrt{q}}{n} \frac{a_n}{z^n} \right)^l, \tag{A2}$$

where we have used the Taylor expansion

$$e^{-\sqrt{q}\sum_{n=1}^{\infty} \frac{1}{n}\frac{a_n}{z^n}} = \sum_{l=0}^{\infty} \frac{1}{l!} \left(\sum_{n=1}^{\infty} \frac{-\sqrt{q}}{n} \frac{a_n}{z^n} \right)^l. \tag{A3}$$

Expanding again in powers of z each term with power l in the above equation one finds

$$\frac{1}{l!} \left(\sum_{n=1}^{\infty} \frac{-\sqrt{q}}{n} a_n z^{-n} \right)^l = \frac{1}{l!} \frac{(-\sqrt{q}a_1)^l}{z^l} + \frac{1}{l!} \frac{(-\sqrt{q})^l}{z^{(l+1)}} \left(\sum_{i_1+\cdots+i_l=l+1} \frac{a_{i_1}}{i_1} \frac{a_{i_2}}{i_2} \cdots \frac{a_{i_l}}{i_l} \right) + \cdots = \sum_{k=l}^{\infty} b_k^l z^{-k}, \tag{A4}$$

with b_k^l defined in (28), yielding

$$\langle N| V_{-\lambda-h} = \langle N-1| \sum_{\substack{l=0 \\ k=l}}^{\infty} \sum_{k=l}^{\infty} b_k^l \oint \frac{dz}{2\pi i} z^{q(N-1)-\lambda-1-k}$$

$$= \langle N-1| \sum_{l=0}^{\infty} \sum_{k=l}^{\infty} b_k^l \delta_{q(N-1)-\lambda-k,0}. \tag{A5}$$

The last sum can be evaluated using the definition of b_k^l and the Kronecker delta. As a consequence Eq. (27) is found.

APPENDIX B

In this Appendix we will calculate the correlator of two modes of the vertex operator,

$$
\begin{aligned}
\langle 2,0|V_{-a-h}V_{-b-h}|0\rangle &= \frac{1}{(2\pi i)^2}\oint \frac{dz_1}{z_1^{a+1}}\oint \frac{dz_2}{z_2^{b+1}}\langle 2|V(z_1)V(z_2)|0\rangle \\
&= \frac{1}{(2\pi i)^2}\oint \frac{dz_1}{z_1^{a+1}}\oint \frac{dz_2}{z_2^{b+1}}(z_1-z_2)^q \\
&= \sum_{k=0}^{q}\binom{q}{k}\frac{(-1)^k}{(2\pi i)^2}\oint \frac{dz_1}{2\pi i}\frac{z_1^{q-k}}{z_1^{a+1}}\oint \frac{dz_2}{2\pi i}\frac{z_2^k}{z_2^{b+1}} \\
&= \sum_{k=0}^{q}(-1)^k\binom{q}{k}\delta_{q-k,a}\delta_{k,b} = \frac{\delta_{q,a+b}}{(-1)^b}\binom{q}{b},
\end{aligned}
\tag{B1}
$$

where we have used the binomial expansion in the second line to find the third one.

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