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Validated Numerical Approximation of Stable Manifolds for Parabolic Partial Differential Equations

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Abstract

This paper develops validated computational methods for studying infinite dimensional stable manifolds at equilibrium solutions of parabolic PDEs, synthesizing disparate errors resulting from numerical approximation. To construct our approximation, we decompose the stable manifold into three components: a finite dimensional slow component, a fast-but-finite dimensional component, and a strongly contracting infinite dimensional "tail". We employ the parameterization method in a finite dimensional projection to approximate the slow-stable manifold, as well as the attached finite dimensional invariant vector bundles. This approximation provides a change of coordinates which largely removes the nonlinear terms in the slow stable directions. In this adapted coordinate system we apply the Lyapunov-Perron method, resulting in mathematically rigorous bounds on the approximation errors. As a result, we obtain significantly sharper bounds than would be obtained using only the linear approximation given by the eigendirections. As a concrete example we illustrate the technique for a 1D Swift-Hohenberg equation.

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1 Introduction

In this paper we develop a novel method for representing the *infinite dimensional* stable manifold of an equilibrium solution of a parabolic PDE. The method makes extensive use of numerical calculations, results in an approximation valid in an explicitly prescribed neighborhood of the equilibrium, and comes equipped with mathematically rigorous bounds on all truncation and discretization errors. The work is motivated by our intention to use this method as an ingredient in further mathematically rigorous computer assisted proofs (see also Sect. 1.2). The method is able to provide validated bounds on the linear approximation of the stable manifold by the stable eigenspace, but gives dramatically improved results when combined with a nonlinear change of coordinates which "flattens out" a finite dimensional slow stable manifold. The main tools used here are the Lyapunov-Perron method, a parameterization method for slow-stable manifolds and their invariant normal bundles (see [60]), and an iterative strategy for bootstrapping Gronwall's inequality in subspaces associated with various linear growth rates.

We remark first on the need for the present work, noting that while the abstract theory for invariant manifolds of compact semi-flows is well developed, there are obstacles preventing its direct application in computer assisted proofs. One complication stems from the fact that in a given example we generally do not have explicit formulas for either the equilibrium or the eigendecomposition of the linearized operator: instead we have approximations. To perform computer assisted proofs, these approximation errors must be incorporated into the set-up from the start.

A second difficulty concerns localizing the estimates, which is necessary because the nonlinearities are not globally Lipschitz. Moreover, in infinite dimensions we do not generally have access to smooth cut-off functions. Finally, even in situations where it is possible to apply the general theory, this typically leads to bounds that are valid in an inconveniently small neighborhood of the equilibrium.

To overcome these difficulties, we project the Lyapunov-Perron operator into various judiciously chosen subspaces, corresponding to collections of approximate eigendirections. The assumption that the PDE is parabolic gives that the spectrum is comprised entirely of isolated eigenvalues (of finite multiplicity) which "accumulate to minus infinity". More precisely, for any $M \in \mathbb{R}$ there are only finitely many eigenvalues with real part greater than M. We choose an approximation of the (finite dimensional) unstable subspace, and split the approximate stable space into finite dimensional "slow" and infinite dimensional "fast" parts. As a subtle refinement, we further decompose the finite dimensional stable eigenspace into slow-finite dimensional stable and fast-finite dimensional stable subspaces.

We remark that the Lyapunov-Perron operator acts on candidate functions α, which map (an approximation of) the linear stable eigenspace to the (approximate) unstable eigenspace. The main technical difficulty is to choose the domain of the candidate functions so as to maximize the portion of the manifold represented, while minimizing the final error bounds. To manage this problem we take domains which are products of balls, having aspect ratios

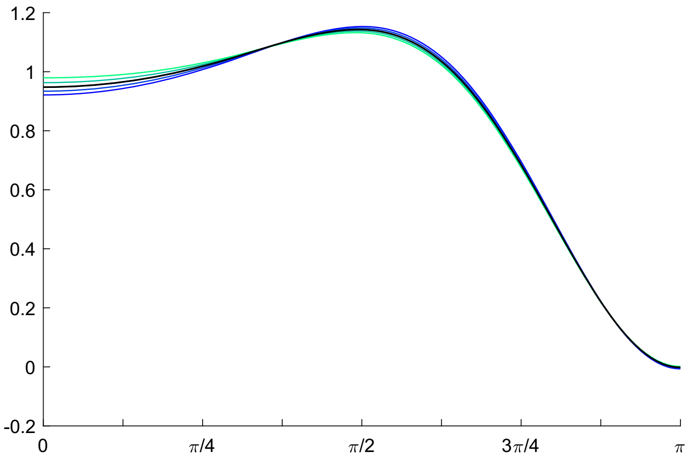

Fig. 1 A verified numerical approximation of an unstable equilibrium (black curve) for the Swift-Hohenberg PDE (1) with $\beta_1 = 0.05$ and $\beta_2 = -0.35$ and several (numerical approximations of) "points"—that is functions—along its verified slow stable manifold. Near this slow stable manifold we find a description of the full, co-dimension 1, stable manifold, with validated computer assisted error bounds (Color figure online)

determined by the growth rates in the various subspaces. We perform an explicit change of coordinates, which may be linear or nonlinear, and which provides more flexibility in choosing a good domain for the stable manifold approximation.

To show that the Lyaponuv-Perron operator is a contraction we need explicit bounds on the projections of the nonlinearities onto the specified subspaces. To obtain effective bounds, i.e. bounds that guarantee contraction for functions defined on a reasonably large neighborhood of the equilibrium, a naive Gronwall estimate does not suffice. Instead we take a more refined approach, in which we bootstrap a system of Gronwall inequalities (roughly, decomposed along eigendirections) exploiting the different decay rates in different directions. The applications to computer assisted proofs of transverse connecting orbits we have in mind (see again Sect. 1.2), introduce the additional technical complication that we would like a $C^{1,1}$ description of the stable manifold.

1.1 Example Results for Swift-Hohenberg

The utility of the method is best illustrated through application to an explicit example. To this end we provide a complete numerical implementation of our method for the Swift-Hohenberg PDE

$$u_t = -\beta_1 u_{xxxx} + \beta_2 u_{xx} + u - u^3, \tag{1}$$

posed on a one-dimensional spatial domain $x \in [0, \pi]$ with Neumann boundary conditions

$$u_x(0) = u_x(\pi) = 0 \qquad \text{and} \qquad u_{xxx}(0) = u_{xxx}(\pi) = 0.$$

The parameters of the problem are $\beta_1 > 0$ and $\beta_2 \in \mathbb{R}$. For comparison, we illustrate the use of our method for both a linear, and a nonlinear change of variables near the equilibrium. As a result, we obtain stable manifold theorems of varying accuracy, and in neighborhoods of the equilibrium having various sizes and shapes.

For example, in Theorem 6.4 we focus on a non-trivial equilibrium solution of Swift-Hohenberg with Morse index 1. The equilibrium solution is illustrated in Fig. 1. To obtain the results described in Theorem 6.4, we represent the local stable manifold as the graph of a function over the stable eigenspace. We take a 31 dimensional Galerkin projection, so that the stable eigenspace is decomposed into a 30 dimensional finite part, and an infinite dimensional remainder. The domain of the graph is taken to be the product of a box of radius 2.2×10^{-2} in 30 dimensional subspace, and a box of radius 10^{-5} in the tail. The chart for the local stable manifold has C^0 norm bound by 3.36×10^{-3}. That is, the true stable manifold has distance no more than 3.36×10^{-3} away from the stable eigenspace, over the box just described.

Contrast this with the results described in Theorem 7.1. In this case we use the nonlinear change of coordinates discussed in Sect. 2.4, and represent the local stable manifold as the graph of a function over a one dimensional slow-stable manifold and its 29 dimensional invariant stable vector bundles. This time the domain of the graph is the product of three boxes: a box of radius 3.18×10^{-2} in the slow stable direction, a box of radius 10^{-6} in the remaining 29 dimensions of the finite dimensional eigenspace, and a box of radius 10^{-10} in the tail. The chart for the local stable manifold has C^0 norm bound by 7.34×10^{-12}. That is, the true stable manifold is 7.34×10^{-12} close to the slow stable manifold and its stable vector bundles over the box just described.

Comparing the results of Theorem 6.4 with the results of Theorem 7.1 illustrate the power of the techniques developed in the present work. The two representaitons of the infinite dimensional stable manifold are valid in neighborhoods having size on the order of 10^{-2} away from the equilibrium (in some directions). Exploiting the nonlinear change of variables improves the validated error bounds by nine order of magnitude in the unstable directions (bounds on the graph) and by five orders of magnitude in the stable tail directions. These are by far the most accurate mathematically rigorous computer assisted error bounds for an infinite dimensional manifold appearing in the literature up until now. More details and comparisons are found in Sects. 6.3 and 7.6.

1.2 Motivation: Saddle-to-Saddle Connects for Parabolic PDEs

When viewed as ODEs on Banach spaces, nonlinear parabolic PDEs fit well within the qualitative theory of dynamical systems. Theorems regarding the stability of equilibria, periodic orbits, and their attached invariant manifolds follow in analogy with the finite dimensional case. Connecting orbits between invariant sets serve as a kind of a road map to the global dynamics, illuminating transitions between distinct regions of the phase space and signaling global bifurcations. Such orbits are main ingredients in forcing theorems like those of Smale and Shilnikov: theorems which guarantee the existence of rich dynamics. Connecting orbits are essential for defining geometric chain groups and boundary operators in the homology theories of Witten and Floer. In short, proving the existence of connection orbits provides critical information about the global dynamics generated by the PDE.

Yet, precisely because of their global and nonlinear nature, connecting orbits are difficult to work with analytically. These difficulties are compounded in infinite dimensional settings. In specific applications researchers typically perform numerical calculations to gain insights into the properties of important invariant objects. Recent progress in computer-assisted methods of proof for infinite dimensional systems brings the mathematically rigorous quantitative study of connecting orbits for PDEs within the realm of possibility.

We refer for example to the work of [18,53] for some examples of computer assisted proofs for connecting orbits in PDEs. In particular the authors study connections frome saddle to attracting equilibrium solutions. The works just mentioned study the finite dimensional unstable manifold attached to an equilibrium, and develop mathematically rigorous tools for extending this manifold into a trapping neighborhood of a sink. Similarly, in a nonconservative nonlinear Schrödinger equation, the work [36] computes connecting orbits from saddle equilibria to a center equilibrium. In each of the studies just mentioned the authors obtain explicit and mathematically rigorous bounds on the basin of attraction of the limiting equilibrium—which is an open set.

Controlling the asymptotic behavior of a connecting orbit requires an explicit description of the local stable and unstable manifolds of the equilibrium solutions (or other limiting invariant sets). The major obstacle to extending the methods of [18,36,53] to the general case of a saddle-to-saddle connection is obtaining an explicit description of the local stable manifold. It is worth mentioning that rigorous numerical integration of a PDE is a nontrivial task, and invariably suffers from the so called wrapping effects resulting from the accumulation of numerical error. Consequently, in computer assisted arguments involving connecting orbits it is desirable to minimize integration time by absorbing as much of the connecting orbit into the local stable and unstable manifolds as possible. This motivates out interest in the nonlinear coordinate changes utilized in the present work.

We refer the interested reader also to the related work of [19], where saddle-to-saddle connections are established using topological methods based on Conley Index theory and its connection matrix. Being topological in nature these methods require much less in the way of C^1 information, resulting in a softer description of the dynamics. The challenge in applying these methods is the rigorous calculation of index information for macroscopic regions in the infinite dimensional phase space.

The computational framework developed here is rather general, and will be useful for describing invariant manifolds in a variety of other settings. We have in mind examples such as (un)stable and center-(un)stable manifolds in delay differential equations and partial differential equations on domains in $\mathbb{R}^n$, as well as stable and unstable manifolds in strongly indefinite problems, where both the dimension and the co-dimension of the manifold are infinite dimensional (e.g. [14]). In [56] a similar methodology is used to construct a local representation for a co-dimension 0 center-stable manifold of the homogeneous equilibrium in a complex-valued nonlinear heat equation.

Remark 1.1 (Inertial Manifolds) It is a well known fact that many infinite dimensional dynamical systems, for example those generated by parabolic PDEs including the one studied below, admit inertial manifolds: finite dimensional flow invariant manifolds containing all the invariant dynamics, including the connecting orbits discussed above [25,38,54]. An alternative strategy to the one above would be to construct computer assisted error bounds for the inertial manifold, and to study the dynamics of the resulting lower dimensional system.

Moreover, such bounds could be constructed using arguments similar to those developed in the present work. For example the usual existence proofs for inertial manifolds involve tools like fixed point arguments and Gronwall inequalities. It is even possible that a non-linear change of coordinates, similar to the one that developed in Sect. 7, could be constructed based on existing powerful computational methods for approximating inertial manifolds [17,24,39,57].

It is however important to remark that, even after an inertial manifold approximation has been constructed and equipped with mathematically rigorous computer assisted error bounds, one would have to prove that the finite dimensional subsystem had the desired dynamics (equilibrium solutions/periodic orbits/connecting orbits/etcetera). In some cases this could probably be done by hand, yet in general one has to expect it to require further computer assisted arguments. It is not obvious then that an inertial manifold approach would lead to simpler calculations/proofs than the ones envisioned above. One hopes that in time both research programs are implemented and their various strengths compared.

1.3 Related Work

The present work grows out of the thriving literature on methods of computer assisted proof in dynamical systems theory going back to the first proofs of the Feigenbaum conjectures [21,22,42,43], the first proofs of chaotic motions in the Lorenz equations [27,46–48] and for Chua's circuit [26], and the computer assisted resolution of Smale's 14th problem [58,59]. In particular, we build on the substantial literature on computer assisted proofs for studying the dynamics of parabolic PDEs. A thorough review of this literature beyond the scope of the present work, and we refer the reader to the work of [1,2,4,28,49,51,52,62,65,72]. See also the book of [50], and the review articles [29,42,63]

A number of techniques for computer assisted proofs involving finite dimensional invariant manifolds have emerged from this literature. One family of methods for proving existence of unstable manifolds involves checking a number of geometric covering and cone conditions near the equilibrium in the same spirit as Fenichel theory [11,12,71]. Since time reversal is well defined for ODEs, equivalent bounds for stable manifolds follow as a trivial corollary. Applications of these methods to the study of stable manifolds for PDEs requires substantial modification and have—to the best of our knowledge—not yet appeared in the literature. We refer the interested reader to the recent work of [70] where, following [26,27,46–48], the authors bypass consideration of stable/unstable manifolds and provide a direct computer assisted proof of the existence of a geometric horseshoe in the Kuramoto-Sivashinsky equation, by studying covering relations in a Poincaré section.

Another technique for obtaining validated bounds on invariant manifolds which has been applied successfully in a number of finite dimensional settings is the parametrization method [8–10], see also to the book [32] for detailed discussions of the method and its applications. Briefly, the idea is to study a conjugacy equation between the dynamics on the manifold and the linear dynamics in an eigenspace. The conjugacy equation is reduced to a set of linear homological equations via recursive power matching, and one obtains a high order Taylor expansions for the manifold, as well as remainder estimates on the truncation errors in the tail of the series. This method recovers both the embedding of the manifold and the dynamics on it, and is very effective for representing invariant manifolds far beyond a small neighborhood of the equilibrium, periodic orbit, or invariant torus, where the linear approximation is valid.

There is a substantial literature devoted to validated numerics based on the parameterization method for invariant manifolds of ODEs. We refer the interested reader to the works of [3,6,13,37,45,64] for more a complete discussion. Such methods have also been extended for studying finite dimensional invariant manifolds of infinite dimensional systems. The case of compact infinite dimensional maps is treated in [44], the case of PDEs is studied in [53], and DDEs are considered in [30,33].

However, there is an obstruction to applying the parameterization method to infinite dimensional manifolds in PDEs, which is that the existence of a conjugacy depends certain non-resonance conditions between the eigenvalues. There are techniques to deal with the case of a finite number of resonant eigenvalues [8,61]. Nonetheless, to describe an infinite dimensional manifold one will have an infinite number of resonance conditions to check, which seems to be a major obstruction. Indeed, there is no good reason to think that a parabolic PDE can in practice satisfy infinitely many non-resonance conditions.

Instead, we consider the two widespread approaches for studying infinite dimensional invariant manifolds in Banach spaces: these are the graph transform method (e.g. see [5]), and the Lyapunov-Perron method (e.g. see [16]). We refer to [23, Section 1.4] for a comparison of these methods, but the important point to mention here is that the graph transform method is most natural for discrete time dynamical systems.

Indeed, in [20], a graph transform-type argument was used to obtain validated computer assisted error bounds for the infinite dimensional stable manifold of a compact infinite dimensional map generated by convolution against a smooth kernel. The result just cited was a significant motivation for the present work. The graph transform method applies to continuous time systems by considering the implicitly defined time-1 map generated by the semi-flow. But this requires direct access to the time-1 maps, which are defined only implicitly by the PDE. Because of this, we have opted to work with the Lyapunov-Perron method. The present work extends the work of [20] to parabolic PDEs, exploiting geometric techniques in the projection space which allow us to obtain validated results on much larger domains.

1.4 Organization of the Present Work

The outline of the paper is as follows. In Sect. 2 we discuss the notation to be used in this paper, and the level of generality to be considered. Abstractly, we assume that our approximate (un)stable eigenspaces are decomposed into further subspaces, with (potentially) different time scales. This corresponds to our plan to develop distinct methods of approximation along the slow-stable, fast-but finite-stable, and infinite-stable eigenvalues. We intend to compute $C^{1,1}$ bounds on our manifold, and here we define a number of constants relating to our nonlinearity $\mathcal{N}$.

In Sect. 3 we discuss how we explicitly bootstrap Gronwall's inequality to get component-wise bounds on the exponential tracking problem. This iterative bootstrapping of Gronwall's inequality is described in Algorithm 3.11. The approach is quite versatile, and we apply the same procedure several times in different scenarios. A general description for where this approach can be taken is described in Algorithm A.5.

In Sect. 4 we discuss the Lyapunov-Perron Operator Ψ, which is given in Definition 2.11. We formulate conditions for when Ψ maps a ball of $C^{0,1}$ functions into itself in Theorem 4.2, and for when Ψ maps a ball of $C^{1,1}$ functions into itself in Theorem 4.11.

In Sect. 5 we obtain the necessary estimates to show that the Lyapunov-Perron Operator is a contraction mapping. In Definition 5.2 we define a norm in which we wish to prove we have a contraction mapping. We then give conditions for when we have a contraction in Theorem 5.9, and the results of Sects. 3–5 are summarized in Theorem 5.11.

In Sect. 6 we apply our results to the Swift-Hohenberg equation, obtaining the appropriate estimates for a linear change of variables at a nonlinear equilibrium. Finally in Sect. 7 we discuss how to get the estimates to work using a nonlinear change of coordinates at a nontrivial equilibrium. Computer assisted proofs of a stable manifold theorem using a linear approximation and a nonlinear approximation are given in Theorem 6.4 and Theorem 7.1 respectively, and the source code is available online [68].

2 Background and Notation

A useful first step in studying stable/unstable manifolds is to perform a change of coordinates taking the equilibrium to zero and aligning the (possible generalized) eigendirections with the coordinate axes. For ordinary differential equations (ODEs) such a transformation always exists. Nevertheless, in a particular problem it may be impractical to compute this transformation exactly due to the lack of explicit formulas and the finite numerical precision. For PDEs, the situation is even worse, as the desired change of coordinates is infinite dimensional. In the present work we settle for coordinate transformations which move the origin approximately to zero, and approximately align the coordinate axes with eigendirections. This is achieved by computing good numerical approximations of the equilibrium and the eigendata for a finite dimensional Galerkin projection, and approximating the eigendata in the infinite dimensional complement via the linearization of the homogeneous equilibrium. To obtain mathematically rigorous results it is necessary to quantify these errors, and formalizing this discussion requires a good deal of notation.

2.1 Parabolic PDEs and Semigroup Operators

Let X be a Banach space with norm $|\cdot| = |\cdot|_X$, and consider the differential equation

$$\dot{\mathrm{x}} = \tilde{\Lambda}\mathrm{x} + \tilde{\mathcal{N}}(\mathrm{x}), \tag{2}$$

where $\tilde{\Lambda} : \mathrm{Dom}(\tilde{\Lambda}) \subseteq X \to X$ is a densely defined linear operator with bounded inverse, and $\tilde{\mathcal{N}} \in C^2_{\mathrm{loc}}(X, X)$. We will need explicit bounds on $D\tilde{\mathcal{N}}(0)$ and a local (uniform) bound on the second derivative(s). See Proposition 2.6 below. Assume that $\tilde{h} \in X$ is a hyperbolic equilibrium solution of Eq. (2), where we think of $\tilde{h}$ as being small. Making the change of variables $\mathrm{x} \to \mathrm{x} + \tilde{h}$ leads to the differentail equation

$$\dot{\mathrm{x}} = \Lambda\mathrm{x} + L\mathrm{x} + \hat{\mathcal{N}}(\mathrm{x}). \tag{3}$$

where

$$\Lambda := \tilde{\Lambda}, \qquad L := D\tilde{\mathcal{N}}(\tilde{h}), \qquad \hat{\mathcal{N}}(\mathrm{x}) := \tilde{\mathcal{N}}(\tilde{h} + \mathrm{x}) - \tilde{\mathcal{N}}(\tilde{h}) - D\tilde{\mathcal{N}}(\tilde{h})\mathrm{x}. \tag{4}$$

Equation (3) has that the origin is an equilibrium solution and that $\hat{\mathcal{N}}(0) = 0$ and $D\hat{\mathcal{N}}(0) = 0$.

Definition 2.1 (**Stable and unstable decomposition**) Let $X = X_s \times X_u$ denote the decomposition of X into stable and unstable eigenspaces of the operator Λ. Fix integers $m_s, m_u \in \mathbb{N}$, and define two index sets $I := \{1, 2, \ldots, m_s\}$ and $I' := \{1', 2', \ldots, m'_u\}$. For $i \in I$ and $i' \in I'$, assume that $X_i \subseteq X_s$ and $X_{i'} \subseteq X_u$ are closed subspaces of X with:

$$X_s := \prod_{1 \leq i \leq m_s} X_i, \qquad X_u := \prod_{1' \leq i' \leq m'_u} X_{i'}.$$

Remark 2.2 (**primed and un-primed indices**) Throughout the paper we use a primed notation, such as i' or j', to index over the unstable eigenspace X_u and un-primed indices for the stable. It is sometimes convenient to have an index ranging over all stable and unstable indices, so we define $\mathbf{I} := I \cup I'$ and write $\mathbf{i} \in \mathbf{I}$ to signify that $\mathbf{i}$ may be a primed or un-primed index.

For the projections onto the subspaces X_i, $X_{i'}$, X_s and X_u we use the notation π_i, $\pi_{i'}$, π_s and π_u, respectively. Since these subspaces are closed, the projection maps are bounded linear operators. That is, there exist constants $p_s, p_u, p_{\mathbf{i}} < \infty$ so that

$$\|\pi_s\| \le p_s \qquad \|\pi_u\| \le p_u \qquad \|\pi_{\mathbf{i}}\| \le p_{\mathbf{i}}. \tag{5}$$

We use the notation, $\mathrm{x}_{\mathbf{i}} = \pi_{\mathbf{i}}\mathrm{x}$, $\mathrm{x}_s = \pi_s \mathrm{x}$, etc, hence $\mathrm{x} = \mathrm{x}_s + \mathrm{x}_u$, $\mathrm{x}_s = \sum_{i\in I} \mathrm{x}_i$ and $\mathrm{x}_u = \sum_{i'\in I'} \mathrm{x}_{i'}$, as well as $\mathrm{x} = \sum_{\mathbf{i}\in\mathbf{I}} \mathrm{x}_{\mathbf{i}}$.

Assume that Λ is invariant along the subspaces $X_i, X_{i'}$. That is to say, assume that there exist $\Lambda_i : X_i \to X_i$ and $\Lambda_{i'} : X_{i'} \to X_{i'}$ such that

$$\Lambda \mathrm{x} = \sum_{i\in I} \Lambda_i \mathrm{x}_i + \sum_{i'\in I'} \Lambda_{i'} \mathrm{x}_{i'}.$$

Furthermore, assume there are constants $\lambda_i < 0$ such that for $1 \le i \le m_s$

$$|e^{\Lambda_i t}\mathrm{x}_i| \le e^{\lambda_i t}|\mathrm{x}_i|, \qquad t \ge 0, \mathrm{x}_i \in X_i, \tag{6}$$

and $\lambda_{i'} > 0$ such that for $1' \le i' \le m'_u$

$$|e^{\Lambda_{i'} t}\mathrm{x}_{i'}| \le e^{\lambda_{i'} t}|\mathrm{x}_{i'}|, \qquad t \le 0, \mathrm{x}_{i'} \in X_{i'}. \tag{7}$$

In particular, this implies that the norm on X aligns well with flow of Λ on the subspaces X_i in the sense that the vector field Λ_i points inwards on the boundary of the unit ball in X_i.

The linear operator L is decomposed in the following manner: for all $\mathbf{i}, \mathbf{j} \in \mathbf{I}$, define the bounded linear operators $L_{\mathbf{i}}^{\mathbf{j}} : X_{\mathbf{j}} \to X_{\mathbf{i}}$ by

$$[L\mathrm{x}]_{\mathbf{i}} = \sum_{\mathbf{j}\in\mathbf{I}} L_{\mathbf{i}}^{\mathbf{j}} \mathrm{x}_{\mathbf{j}}.$$

Restricting Λ and L to X_s and X_u gives operators

$$\begin{aligned} &\Lambda_s \mathrm{x}_s : X_s \to X_s \qquad && L_s^s \mathrm{x}_s : X_s \to X_s \qquad && L_s^u \mathrm{x}_u : X_u \to X_s \\ &\Lambda_u \mathrm{x}_u : X_u \to X_u \qquad && L_u^s \mathrm{x}_s : X_s \to X_u \qquad && L_u^u \mathrm{x}_u : X_u \to X_u \end{aligned}$$

defined by

$$\begin{aligned} \Lambda_s \mathrm{x}_s &:= \sum_{i\in I} \Lambda_i \mathrm{x}_i \qquad & L_s^s \mathrm{x}_s &:= \sum_{i,j\in I} L_i^j \mathrm{x}_j \qquad & L_s^u \mathrm{x}_u &:= \sum_{i\in I, j'\in I'} L_i^{j'} \mathrm{x}_{j'} \\ \Lambda_u \mathrm{x}_u &:= \sum_{i'\in I'} \Lambda_{i'} \mathrm{x}_{i'} \qquad & L_u^s \mathrm{x}_s &:= \sum_{i'\in I, j\in I} L_{i'}^j \mathrm{x}_j \qquad & L_u^u \mathrm{x}_u &:= \sum_{i'\in I', j'\in I'} L_{i'}^{j'} \mathrm{x}_{j'}. \end{aligned}$$

Assume that $-(\Lambda_u + L_u^u)$ and $(\Lambda_s + L_s^s)$ are negative operators, in the sense that there exist constants C_s, C_u and $\lambda_s < 0$ and $\lambda_u > 0$ so that

$$|e^{(\Lambda_s + L_s^s)t}\mathrm{x}_s| \le C_s e^{\lambda_s t}|\mathrm{x}_s|, \qquad t \ge 0, \mathrm{x}_s \in X_s, \tag{8}$$

$$|e^{(\Lambda_u + L_u^u)t}\mathrm{x}_u| \le C_u e^{\lambda_u t}|\mathrm{x}_u|, \qquad t \le 0, \mathrm{x}_u \in X_u. \tag{9}$$

Calculation of these constants is discussed in Sect. B, and an explicit example is given in Sect. 6.

Remark 2.3 For both the prime and non-prime spatial indices we employ Einstein summation notation, writing

$$L_i^j \mathrm{x}_j \equiv \sum_{j\in I} L_i^j \mathrm{x}_j, \qquad \text{and} \qquad L_i^{j'} \mathrm{x}_{j'} \equiv \sum_{j'\in I'} L_i^{j'} \mathrm{x}_{j'}.$$

For other indices, for example sums over $\mathbf{I} = I \cup I'$, we write the summation explicitly.

We now project the nonlinear terms into the subspaces just defined, and write $\hat{\mathcal{N}}_{\mathbf{i}} := \pi_{\mathbf{i}} \circ \hat{\mathcal{N}}(\mathrm{x})$ for $\mathbf{i} \in \mathbf{I}$. Then $\hat{\mathcal{N}}_s(\mathrm{x}) := \pi_s \circ \hat{\mathcal{N}}(\mathrm{x})$ and $\hat{\mathcal{N}}_u(\mathrm{x}) := \pi_u \circ \hat{\mathcal{N}}(\mathrm{x})$. For $\mathbf{i} \in \mathbf{I}$ let

$$\mathcal{N}_{\mathbf{i}}(\mathrm{x}_s, \mathrm{x}_u) := L_{\mathbf{i}}^{j} \mathrm{x}_j + L_{\mathbf{i}}^{j'} \mathrm{x}_{j'} + \hat{\mathcal{N}}_{\mathbf{i}}(\mathrm{x}_s, \mathrm{x}_u). \tag{10}$$

We write

$$\mathcal{N}_s := \sum_{i \in I} \mathcal{N}_i, \qquad \mathcal{N}_u := \sum_{i' \in I'} \mathcal{N}_{i'}, \qquad \mathcal{N} := \mathcal{N}_s + \mathcal{N}_u.$$

Equation (3) becomes

$$\dot{\mathrm{x}}_i = \Lambda_i \mathrm{x}_i + \mathcal{N}_i(\mathrm{x}_s, \mathrm{x}_u), \tag{11}$$

$$\dot{\mathrm{x}}_{i'} = \Lambda_{i'} \mathrm{x}_{i'} + \mathcal{N}_{i'}(\mathrm{x}_s, \mathrm{x}_u). \tag{12}$$

We study functions defined on certain a certain products of balls containing the origin in the various subspaces.

Definition 2.4 Fix positive vectors $r_s \in \mathbb{R}^{m_s}$ and $r_u \in \mathbb{R}^{m_u}$, and define the closed balls $B_s(r_s) \subseteq X_s$ and $B_u(r_u) \subseteq X_u$ given by

$$B_s(r_s) := \{\mathrm{x}_s \in X_s : |\mathrm{x}_i| \leq r_i \text{ for } i \in I\}$$
$$B_u(r_u) := \left\{\mathrm{x}_u \in X_u : |\mathrm{x}_{i'}| \leq r_{i'} \text{ for } i' \in I'\right\}.$$

When the vectors r_s, r_u are understood, we abbreviate to $B_s \equiv B_s(r_s)$ and $B_u \equiv B_u(r_u)$. Below we define bounds on our nonlinearity $\mathcal{N}$ over balls of fixed radius.

Definition 2.5 Suppose $r_s \in \mathbb{R}^{m_s}$ and $r_u \in \mathbb{R}^{m_u}$.
For $\mathrm{x}_s \in B_s(r_s)$, $\mathrm{x}_u \in B_u(r_u)$ and $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbf{I}$ define

$$\mathcal{N}_{\mathbf{j}}^{\mathbf{i}}(\mathrm{x}_s, \mathrm{x}_u) := \frac{\partial}{\partial \mathrm{x}_{\mathbf{i}}} \mathcal{N}_{\mathbf{j}}(\mathrm{x}_s, \mathrm{x}_u), \qquad \|\mathcal{N}_{\mathbf{j}}^{\mathbf{i}}\|_{(r_s, r_u)} := \sup_{\mathrm{x}_s \in B_s(r_s)} \sup_{\mathrm{x}_u \in B_u(r_u)} \|\mathcal{N}_{\mathbf{j}}^{\mathbf{i}}(\mathrm{x}_s, \mathrm{x}_u)\|$$

$$\mathcal{N}_{\mathbf{j}}^{\mathbf{ik}}(\mathrm{x}_s, \mathrm{x}_u) := \frac{\partial^2}{\partial \mathrm{x}_{\mathbf{i}} \partial \mathrm{x}_{\mathbf{k}}} \mathcal{N}_{\mathbf{j}}(\mathrm{x}_s, \mathrm{x}_u), \quad \|\mathcal{N}_{\mathbf{j}}^{\mathbf{ik}}\|_{(r_s, r_u)} := \sup_{\mathrm{x}_s \in B_s(r_s)} \sup_{\mathrm{x}_u \in B_u(r_u)} \|\mathcal{N}_{\mathbf{j}}^{\mathbf{ik}}(\mathrm{x}_s, \mathrm{x}_u)\|.$$

Proposition 2.6 *Fix $r_s \in \mathbb{R}^{m_s}$, and $r_u \in \mathbb{R}^{m_u}$, and suppose that $|\tilde{h}_{\mathbf{i}}| < \epsilon_{\mathbf{i}}$. Assume that the constants $\tilde{D}_{\mathbf{j}}^{\mathbf{i}}$ and $\tilde{C}_{\mathbf{j}}^{\mathbf{ik}}$ satisfy*

$$\tilde{D}_{\mathbf{j}}^{\mathbf{i}} \geq \|\tilde{\mathcal{N}}_{\mathbf{j}}^{\mathbf{i}}(0, 0)\|, \qquad \tilde{C}_{\mathbf{j}}^{\mathbf{ik}} \geq \|\tilde{\mathcal{N}}_{\mathbf{j}}^{\mathbf{ik}}\|_{(r_s + \epsilon_s, r_u + \epsilon_u)}.$$

For $\mathbf{i}, \mathbf{j}, \mathbf{k} \in I \cup I'$ define constants $\hat{C}_{\mathbf{j}}^{\mathbf{i}}$, $D_{\mathbf{j}}^{\mathbf{i}}$, $C_{\mathbf{j}}^{\mathbf{i}}$, and $C_{\mathbf{j}}^{\mathbf{ik}}$ as below:

$$D_{\mathbf{j}}^{\mathbf{i}} := \tilde{D}_{\mathbf{j}}^{\mathbf{i}} + \tilde{C}_{\mathbf{j}}^{\mathbf{i}l} \epsilon_l + \tilde{C}_{\mathbf{j}}^{\mathbf{i}l'} \epsilon_{l'}, \qquad C_{\mathbf{j}}^{\mathbf{ik}} := \tilde{C}_{\mathbf{j}}^{\mathbf{ik}}$$
$$\hat{C}_{\mathbf{j}}^{\mathbf{i}} := \tilde{C}_{\mathbf{j}}^{\mathbf{i}l} r_l + \tilde{C}_{\mathbf{j}}^{\mathbf{i}l'} r_{l'} \qquad C_{\mathbf{j}}^{\mathbf{i}} := \hat{C}_{\mathbf{j}}^{\mathbf{i}} + D_{\mathbf{j}}^{\mathbf{i}}.$$

Then for L and $\hat{\mathcal{N}}$ defined in (4) and $\mathcal{N}$ defined in (10) we have the bounds

$$D_{\mathbf{j}}^{\mathbf{i}} \geq \|L_{\mathbf{j}}^{\mathbf{i}}\| \qquad C_{\mathbf{j}}^{\mathbf{ik}} \geq \|\mathcal{N}_{\mathbf{j}}^{\mathbf{ik}}\|_{(r_s, r_u)} \tag{13a}$$

$$\hat{C}_{\mathbf{j}}^{\mathbf{i}} \geq \|\hat{\mathcal{N}}_{\mathbf{j}}^{\mathbf{i}}\|_{(r_s, r_u)} \qquad C_{\mathbf{j}}^{\mathbf{i}} \geq \|\mathcal{N}_{\mathbf{j}}^{\mathbf{i}}\|_{(r_s, r_u)}. \tag{13b}$$

The proof follows directly from the definitions.

2.2 Regularity of the Candidate Functions

Our goal is to find a chart $\alpha : B_s \to X_u$ such that the graph $\{(\xi, \alpha(\xi)) : \xi \in B_s\}$ is a local stable manifold attached to the origin of the differential Eq. (3). The desired chart is formulated as a fixed point of the Lyapunov-Perron operator in Sect. 2.3. In preparation for that formulation we now specify the appropriate spaces of candidate functions.

Remark 2.7 In Sect. 2.1 there is notational symmetry between the stable and unstable eigenspaces. For the *stable* manifold the main parameter is the stable radius r_s, which determines the domain of the chart α. On the other hand, the unstable radius r_u in the codomain of α follows from a Lipschitz assumption on the chart. To highlight this distinction, in the contexts of the Lyapunov-Perron operators and the associated charts we denote the radius in the stable subspace by the parameter ρ.

Let $\rho \in \mathbb{R}^{m_s}$ and $\alpha \in C^0(B_s(\rho), X_u)$. Define the Lipschitz constants of α relative to the subspaces X_i and $X_{i'}$ by

$$\text{Lip}(\alpha)^i_{i'} := \sup_{\xi \in B_s} \sup_{\substack{0 \neq \zeta_i \in X_i \\ \xi + \zeta_i \in B_s}} \frac{|\alpha_{i'}(\xi + \zeta_i) - \alpha_{i'}(\xi)|}{|\zeta_i|}.$$

Observe that if α is Fréchet differentiable, then $\sup_{\xi \in B_s(\rho)} \|\alpha^i_{i'}(\xi)\| = \text{Lip}(\alpha)^i_{i'}$. Here we employ the notation of Definition 2.5, so that superscripts attached directly to α denote partial derivatives. Let $C^{0,1}(B_s(\rho), X_u)$ denote the set of all Lipschitz continuous functions on $B_s(\rho)$, taking values in X_u. Similarly, let $C^{1,1}(B_s(\rho), X_u) \subset C^{0,1}(B_s(\rho), X_u)$ denote the set of all continuously differentiable functions whose derivative is Lipschitz continuous.

Definition 2.8 Fix positive tensors $\rho \in \mathbb{R}^{m_s}$, $P \in \mathbb{R}^{m_s} \otimes \mathbb{R}^{m_u}$ and $\bar{P} \in (\mathbb{R}^{m_s})^{\otimes 2} \otimes \mathbb{R}^{m_u}$, and define the function spaces

$$\mathcal{B}^{0,1}_{\rho,P} := \{\alpha \in C^{0,1}(B_s(\rho), X_u) : \alpha(0) = 0,\ \text{Lip}(\alpha)^i_{i'} \le P^i_{i'}\},$$
$$\mathcal{B}^{1,1}_{\rho,P,\bar{P}} := \{\alpha \in C^{1,1}(B_s(\rho), X_u) : \alpha(0) = 0,\ \text{Lip}(\alpha)^i_{i'} \le P^i_{i'},\ \text{Lip}(\partial_i \alpha)^j_{i'} \le \bar{P}^{ij}_{i'}\}.$$

Note that for all $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$ and $\xi, \zeta \in B_s$ we have: $|\alpha_{i'}(\xi) - \alpha_{i'}(\zeta)| \le P^i_{i'}|\xi_i - \zeta_i|$. For a positive vector ρ and positive tensor P, the range of the $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$ lies in a ball $B_u(r_u)$ with r_u given by $r_{i'} = P^i_{i'}\rho_i$.

Definition 2.9 Let the vector ρ and tensor P be as in Definition 2.8. Define r_u by $r_{i'} := P^i_{i'}\rho_i$. For constants C^i_j, $\hat{C}^i_j$ and D^i_j such that the bounds (13) hold with $r_s = \rho$, define positive tensors

$$H^i_j := C^i_j + C^{i'}_j P^i_{i'}, \qquad H^i_{j'} := C^i_{j'} + C^{i'}_{j'} P^i_{i'}, \qquad \hat{H}^i_j := \hat{C}^i_j + (\hat{C}^{i'}_j + D^{i'}_j) P^i_{i'},$$

and the positive scalar:

$$\hat{\mathcal{H}} := \sup_{\alpha \in \mathcal{B}^{0,1}_{\rho,P}} \sup_{\mathrm{x}_s \in B_s(\rho)} \|\tfrac{\partial}{\partial \mathrm{x}_s} L^u_s \alpha(\mathrm{x}_s) + \tfrac{\partial}{\partial \mathrm{x}_s} \hat{\mathcal{N}}_s(\mathrm{x}_s, \alpha(\mathrm{x}_s))\|.$$

The tensor H provides the following bound: fix ρ, P and $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$, $\xi, \zeta \in B_s(\rho)$. Then for each $\mathbf{j} \in \mathbf{I}$ we have

$$|\mathcal{N}_{\mathbf{j}}(\xi, \alpha(\xi)) - \mathcal{N}_{\mathbf{j}}(\zeta, \alpha(\zeta))| \le H^i_{\mathbf{j}} |\xi_i - \zeta_i|. \tag{14}$$

Proposition 2.10 *Fix ρ and P as in Definition 2.9. If the norm on X has $|\mathrm{x}| = \sum_{\mathbf{i}\in\mathbf{I}} |\mathrm{x}_\mathbf{i}|$, then $\hat{\mathcal{H}} \le \max_{i\in I} \sum_{j\in I} \hat{H}^i_j$.*

Proof. Fix $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$ and $\mathrm{x}_s \in B_s(\rho)$. Then

$$\left\| \tfrac{\partial}{\partial \mathrm{x}_i} L^u_s \alpha(\mathrm{x}_s) \right\| = \left\| \sum_{j\in I} \tfrac{\partial}{\partial \mathrm{x}_i} L^{n'}_j \alpha^i_{n'}(\mathrm{x}_s) \right\| \le \sum_{j\in I} D^{n'}_j P^i_{n'},$$

$$\left\| \tfrac{\partial}{\partial \mathrm{x}_i} \hat{\mathcal{N}}_s\,(\mathrm{x}_s, \alpha(\mathrm{x}_s)) \right\| \le \left\| \sum_{j\in I} \hat{\mathcal{N}}^i_j(\mathrm{x}_s, \alpha(\mathrm{x}_s)) + \hat{\mathcal{N}}^{n'}_j(\mathrm{x}_s, \alpha(\mathrm{x}_s))\alpha^i_{n'}(\mathrm{x}_s) \right\| \le \sum_{j\in I} \hat{C}^i_j + \hat{C}^{n'}_j P^i_{n'}.$$

It now follows from the hypothesis on the norm of X that $\|\pi_\mathbf{i}\| = 1$ for all $\mathbf{i} \in \mathbf{I}$. Then

$$\begin{aligned}
&\left\| \tfrac{\partial}{\partial \mathrm{x}_s} L^u_s \alpha(\mathrm{x}_s) + \tfrac{\partial}{\partial \mathrm{x}_s} \hat{\mathcal{N}}_s\,(\mathrm{x}_s, \alpha(\mathrm{x}_s)) \right\| \\
&= \sup_{u\in X_s, |u|=1} \left| \sum_{i\in I} \left(\tfrac{\partial}{\partial \mathrm{x}_i} L^u_s \alpha(\mathrm{x}_s) + \tfrac{\partial}{\partial \mathrm{x}_i} \hat{\mathcal{N}}_s\,(\mathrm{x}_s, \alpha(\mathrm{x}_s)) \right) u_i \right| \\
&\le \sup_{u\in X_s, |u|=1} \sum_{i,j\in I} \left(D^{n'}_j P^i_{n'} + \hat{C}^i_j + \hat{C}^{n'}_j P^i_{n'} \right) |u_i|.
\end{aligned}$$

In the righthand side of the previous inequality we recognize $\hat{H}^i_j$. Hence

$$\sum_{i,j\in I} \hat{H}^i_j |u_i| = \sum_{i\in I} \Big(\sum_{j\in I} \hat{H}^i_j \Big) |u_i| \le \sum_{i\in I} \Big(\max_{n\in I} \sum_{j\in I} \hat{H}^n_j \Big) |u_i| = \Big(\max_{i\in I} \sum_{j\in I} \hat{H}^i_j \Big) |u|. \tag{15}$$

Taking the sup over $u \in X_s$, $|u| = 1$ gives

$$\left\| \tfrac{\partial}{\partial \mathrm{x}_s} L^u_s \alpha(\mathrm{x}_s) + \tfrac{\partial}{\partial \mathrm{x}_s} \hat{\mathcal{N}}_s\,(\mathrm{x}_s, \alpha(\mathrm{x}_s)) \right\| \le \max_{i\in I} \sum_{j\in I} \hat{H}^i_j. \quad \square$$

2.3 Overview of the Lyapunov-Perron Approach

Having established the necessary notation, we are prepared to formalize the discussion. Namely, we transform the problem of studying the local stable manifold into the problem of finding a fixed point of the Lyapunov-Perron operator. Excellent general references on the Lyapunov-Perron approach include books [15,34,54].

This operator is an endomorphism on charts $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$. Given such an α, define $x(t, \xi, \alpha)$ to be the solution of the projected differential equation

$$\dot{\mathrm{x}}_s = \Lambda_s \mathrm{x}_s + \mathcal{N}_s(\mathrm{x}_s, \alpha(\mathrm{x}_s)), \tag{16}$$

with initial condition $\xi \in B_s(\rho)$ at time $t = 0$. In Sect. 3 we show that if Λ_s sufficiently dominates the nonlinearity $\mathcal{N}_s$, then solutions of the projected system (16) do not blow up for any $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$. In fact, solutions of the projected system approach 0 as $t \to \infty$.

Assuming for the moment this is true, consider the pair $(x(t, \xi, \alpha), \alpha(x(t, \xi, \alpha)))$. If Eq. (12) is satisfied for all $i' \in I'$, then by construction Eq. (11) is satisfied for all $i \in I$. Hence the pair $(x(t, \xi, \alpha), \alpha(x(t, \xi, \alpha)))$ is a solution to the full system (3), and moreover the map $\xi \mapsto (\xi, \alpha(\xi))$ is a chart for a local invariant manifold of the origin.

To find α solving Eq. (12) for all $i' \in I'$, we exploit the variation of constants formula and defining the Lyapunov-Perron operator.

Definition 2.11 Fix a positive vector $\rho \in \mathbb{R}^{m_s}$ and a positive tensor P. The Lyapunov Perron operator $\Psi : \mathcal{B}^{0,1}_{\rho,P} \to \mathrm{Lip}(B_s(\rho), X_u)$ is given by

$$\Psi[\alpha](\xi) := -\int_0^\infty e^{-\Lambda_u t}\mathcal{N}_u(x(t,\xi,\alpha),\alpha(x(t,\xi,\alpha)))dt, \qquad \text{for all } \alpha \in \mathcal{B}^{0,1}_{\rho,P}. \tag{17}$$

Remark 2.12 (**Dynamics on the graph of** α) A fixed point of Ψ is a coordinate chart for a local invariant manifold of the origin. Showing this is the stable manifold requires an additional argument. This is part of the power of the approach, as by modifying the assumptions one can study other attached invariant manifolds like center and center-stable manifolds. For an example involving computer assisted proofs see [56].

Let $\mathbb{E}_s, \mathbb{E}_u \subseteq X$ denote the stable and unstable eigenspaces of the operator $\Lambda + L$. If either $\dim(X_s) = \dim(\mathbb{E}_s) < \infty$ or $\dim(X_u) = \dim(\mathbb{E}_u) < \infty$, then $\alpha = \Psi[\alpha]$ is a chart for a local stable manifold of the origin. In practice this is established by correctly counting with multiplicity the finite number of stable/unstable eigenvalues of $\Lambda + L$. We consider this case in Sects. 6 and 7.

If, on the other hand, both $\dim(\mathbb{E}_s) = \infty$ and $\dim(\mathbb{E}_u) = \infty$, then the desired result is obtained by showing that the family of operators $\Lambda + sL$ does not have any eigenvalues crossing the imaginary axis for $s \in [0, 1]$. This is the approach taken in [67] and it could be extended to studying strongly indefinite problems as typically appear in elliptic problems, see e.g. [14].

In Sect. 4 we show that, for an appropriate choice of constants, Ψ is simultaneously an endomorphism on the balls $\mathcal{B}^{0,1}_{\rho,P}$ and $\mathcal{B}^{1,1}_{\rho,P,\bar{P}}$. In Sect. 5 we show that Ψ is a contraction in a C^0-like norm (see Definition 5.2) and use the Banach Fixed Point Theorem to establish the existence of a unique fixed point.

2.4 Good Coordinates: Parameterization of Slow Stable Manifolds and Attached Invariant Frame Bundles

In this section we describe a method for high order computation of slow stable manifolds, as well as some attached invariant frame bundles describing the stable and unstable directions normal to the slow stable manifold. Our approach is based on the parameterization method of [8–10], and especially on the notion of slow spectral submanifolds discussed in the references just cited. See also the works of [7,31,40,55,60], and the book [32].

The theorem below is extracted from the results of [8,10]. The version we state assumes that the eigenvalues are real and have geometric multiplicity one. These assumptions are not necessary, but simplify the presentation. In the applications considered in Sect. 7, these assumptions have to be checked. In slight abuse of notation, to align with the existing literature we use P to denote the parametrizaton of a slow stable manifold; this should not be confounded with the positive tensor denoted by the same symbol in previous subsection.

Theorem 2.13 (Slow-stable manifold parameterization) *Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be a real analytic vector field, and $p_0 \in \mathbb{R}^d$ be a hyperbolic equilibrium point whose differential $DF(p_0)$ is diagonalizable. Let $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ denote the eigenvalues of $DF(p_0)$ and suppose that $\lambda_1, \ldots, \lambda_{m_{slow}}$ with $m_{slow} < d$ are the slow stable eigenvalues. Let $\xi_1, \ldots, \xi_{m_{slow}} \in \mathbb{R}^d$*

denote the associated slow stable eigenvectors. Write

$$\Lambda_{slow} = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_{m_{slow}} \end{pmatrix}, \quad \text{and} \quad \Lambda = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_d \end{pmatrix},$$

to denote respectively the $m_{slow} \times m_{slow}$ and $d \times d$ matrices of the slow stable eigenvalues and all the eigenvalues of $DF(p_0)$. Suppose that $P\colon [-1, 1]^{m_{slow}} \to \mathbb{R}^d$ is a smooth solution of the invariance equation

$$F(P(\theta)) = DP(\theta)\Lambda_{slow}\theta, \qquad \theta \in [-1, 1]^{m_{slow}}, \tag{18}$$

subject to the first order constraints $P(0) = p_0$ and $\partial_j P(0) = \xi_j$, $1 \leq j \leq m_{slow}$. Then P parameterizes the m_{slow} dimensional smooth slow manifold attached to p_0.

It follows from the results of [8] that Eq. (18) has analytic solution as long as for all $(m_1, \dots, m_{\text{slow}}) \in \mathbb{N}^{m_{\text{slow}}}$ with $m_1 + \ldots + m_{\text{slow}} \geq 2$, the non-resonance conditions $m_1\lambda_1 + \ldots + m_{\text{slow}}\lambda_{m_{\text{slow}}} \neq \lambda_j$ for $1 \leq j \leq d$, are satisfied. Observe that this reduces to a finite number of conditions. Moreover, the solution is unique up to the choice of the scalings of the eigenvectors $\xi_1, \dots, \xi_{m_{\text{slow}}}$.

To control the fast dynamics we exploit the “slow manifold Floquet theory” developed in [60]. The idea is to study certain linearized invariance equations describing the stable/unstable bundles attached to the slow stable manifold. These invariant bundles describe the linear approximation of the full stable manifold near the slow stable manifold, and in addition they provide control over the normal and tangent directions. Combining the stable, unstable, and tangent bundles provides a frame bundle for the phase space in a tubular region surrounding the slow manifold – the “good coordinates” exploited in Sect. 7. The idea is illustrated in Fig. 2.

Computation of the invariant frame bundles is facilitated by the following theorem, the main result of [60]. Note that we apply this theorem only in a finite dimensional Galerkin projection of our PDE.

Theorem 2.14 (Slow-stable manifold Floquet normal form) *Let $F\colon \mathbb{R}^d \to \mathbb{R}^d$, $p_0 \in \mathbb{R}^d$, $DF(p_0)$, $\lambda_1, \dots, \lambda_d$, $\xi_1, \dots, \xi_d$, $m_{slow} < d$, Λ_{slow}, Λ, and $P\colon [-1, 1]^{m_{slow}} \to \mathbb{R}^d$ be as in Theorem 2.13. Assume that for $1 \leq j \leq d$ the functions $q_j\colon [-1, 1]^{m_{slow}} \to \mathbb{R}^d$ are smooth solutions of the equations*

$$DF(P(\theta))q_j(\theta) = \lambda_j q_j(\theta) + Dq_j(\theta)\Lambda_{slow}\theta, \tag{19}$$

for $\theta \in [-1, 1]^{m_{slow}}$, subject to the constraints $q_j(0) = \xi_j$. Let $GL(\mathbb{R}^d)$ denote the collection of all non-singular $d \times d$ matrices with real entries. Define $Q\colon [-1, 1]^{m_{slow}} \to GL(\mathbb{R}^d)$ by

$$Q(\theta) = [q_1(\theta)| \dots |q_d(\theta)] .$$

Then

1. *For all $\theta \in [-1, 1]^{m_{slow}}$ the collection of vectors $q_1(\theta)$, ..., $q_d(\theta)$ span $\mathbb{R}^d$. That is, Q takes values in $GL(\mathbb{R}^d)$ and hence parameterizes a frame bundle.*
2. *For all $t \geq 0$ and for all $\theta \in [-1, 1]^{m_{slow}}$, the derivative of the flow along the slow stable manifold factors as*
$$M(t) = Q(e^{\Lambda_{slow}t}\theta)e^{\Lambda t}Q^{-1}(\theta), \tag{20}$$
where $M(t)$ is the solution of the equation of first variation for F along $P(\theta)$:
$$M'(t) = DF(P(\theta))M(t), \qquad \text{for all } t \geq 0,$$

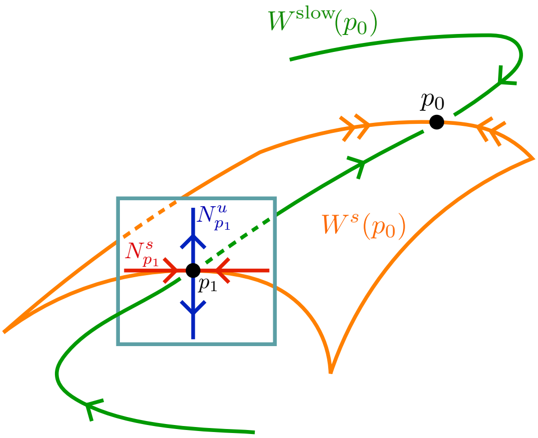

Fig. 2 *Slow stable manifold and attached frame bundles:* the figure illustrates an equilibrium solution p_0 and its slow stable manifold in green. The orange surface illustrates the full stable manifold, of which the slow manifold is a submanifold. At each point on the slow manifold there are invariant stable/unstable normal bundles. The stable normal bundle describes the stable manifold of p_0 near W^{slow}. Taking the stable, unstable, and tangent bundles gives a frame for the entire space. Theorem 2.14 provides an explicit method for computing these structures (Color figure online)

with $M(0)$ the identity matrix.

Considering (20) one column at a time gives that the frame bundles $q(\theta)_j$, $1 \leq j \leq d$ satisfy the invariance equation

$$M(t)q_j(\theta) = e^{\lambda_j t} q_j\left(e^{\Lambda_{\text{slow}} t}\theta\right), \qquad \text{for } \theta \in [-1, 1]^{m_{\text{slow}}}.$$

This says that the flow along $P(\theta)$ leaves the direction of q_j invariant (maps the bundle into itself) but expands vectors at an exponential rate of λ_j. It follows that if $q_{m_{\text{slow}}+1}(\theta), \ldots, q_{m_s}(\theta)$ are the parameterized vector bundles associated with the stable eigenvalues which have not been designated as slow (the so called *fast stable* directions), then for each $\theta \in [-1, 1]^{m_{\text{slow}}}$ these invariant bundles are the fastest contracting directions near $P(\theta)$, and hence they describe $W^s(p_0)$ near $P(\theta)$.

We now define a nonlinear change of coordinates which, to first order, diagonalizes the vector field F near $P(\theta)$. Let $d = m_{\text{slow}} + m_{\text{fast}} + m_{\text{unst}}$. Define the coordinate change $K\colon [-1, 1]^{m_{\text{slow}}} \times [-\epsilon_f, \epsilon_f]^{m_{\text{fast}}} \times [-\epsilon_u, \epsilon_u]^{m_{\text{unst}}} \to \mathbb{R}^d$ by

$$K(\theta, \phi_f, \phi_u) := P(\theta) + Q_f(\theta)\phi_f + Q_u(\theta)\phi_u,$$

i.e. K is a diffeomorphism with $K(0, 0, 0) = p_0$ and $DK(0, 0, 0) = Q(0)$, the matrix of eigenvectors. Here θ is the coordinate in the slow stable manifold, Q_f and ϕ_f denote the fast stable directions, and Q_u and ϕ_u denote the unstable directions. Recall that the defining relations for P, Q_f and Q_u are

$$F(P(\theta)) = DP(\theta)\Lambda_{\text{slow}}\theta, \tag{21}$$

$$DF(P(\theta))Q_f(\theta) = DQ_f(\theta)\Lambda_{\text{slow}}\theta + Q_f(\theta)\Lambda_{\text{fast}}, \tag{22}$$

$$DF(P(\theta))Q_u(\theta) = DQ_u(\theta)\Lambda_{\text{slow}}\theta + Q_u(\theta)\Lambda_{\text{unst}}. \tag{23}$$

We use K to pull back the vector field $F\colon \mathbb{R}^d \to \mathbb{R}^d$, resulting in

$$\begin{pmatrix} \theta' \\ \phi_f' \\ \phi_u' \end{pmatrix} = DK^{-1}(\theta, \phi_f, \phi_u)\, F(K(\theta, \phi_f, \phi_u)) = \begin{pmatrix} \Lambda_{\text{slow}}\theta + N_\theta(\theta, \phi_f, \phi_u) \\ \Lambda_{\text{fast}}\phi_f + N_{\phi_f}(\theta, \phi_f, \phi_u) \\ \Lambda_{\text{unst}}\phi_u + N_{\phi_u}(\theta, \phi_f, \phi_u) \end{pmatrix},$$

where each of the $N_k(\theta, \phi_f, \phi_u)$ is *quadratic in ϕ_f and ϕ_u*, for $k = \theta, \phi_f, \phi_u$.

To see this, and to obtain explicitly the form of N_k, expanding about $P(\theta)$ results in

$$\begin{aligned} F(K(\theta, \phi_f, \phi_u)) &= F(P(\theta) + Q_f(\theta)\phi_f + Q_u(\theta)\phi_u) \\ &= F(P(\theta)) + DF(P(\theta))\left[Q_f(\theta)\phi_f + Q_u(\theta)\phi_u\right] + R(\theta, \phi_f, \phi_u), \end{aligned} \tag{24}$$

where the remainder term R is quadratic in ϕ_f and ϕ_u. For the first two terms in (24) we use the defining relations for P, Q_f and Q_u as well as the definition of K to rewrite

$$\begin{aligned} F(P(\theta)) + DF(P(\theta))\left[Q_f(\theta)\phi_f + Q_u(\theta)\phi_u\right] &= DP(\theta)\Lambda_{\text{slow}}\theta \\ &\quad + DQ_f(\theta)(\Lambda_{\text{slow}}\theta, \phi_f) + Q_f(\theta)\Lambda_{\text{fast}}\phi_f \\ &\quad + DQ_u(\theta)(\Lambda_{\text{slow}}\theta, \phi_u) + Q_u(\theta)\Lambda_{\text{unst}}\phi_u \\ &= DK(\theta, \phi_f, \phi_u)\begin{pmatrix} \Lambda_{\text{slow}}\theta \\ \Lambda_{\text{fast}}\phi_f \\ \Lambda_{\text{unst}}\phi_u \end{pmatrix}. \end{aligned}$$

Then

$$DK^{-1}(\theta, \phi_f, \phi_u)\, F(K(\theta, \phi_f, \phi_u)) = \begin{pmatrix} \Lambda_{\text{slow}}\theta \\ \Lambda_{\text{fast}}\phi_f \\ \Lambda_{\text{unst}}\phi_u \end{pmatrix} + DK^{-1}(\theta, \phi_f, \phi_u)\, R(\theta, \phi_f, \phi_u),$$

hence

$$N(\theta, \phi_f, \phi_u) = DK(\theta, \phi_f, \phi_u)^{-1} R(\theta, \phi_f, \phi_u),$$

As R is quadratic in ϕ_f and ϕ_u, so is N. Once again we refer to Fig. 2 for the geometric interpretation of the coordinate change.

Note that the invariance Eq. (18) and the invariant bundle Eq. (19) do not have to be solved exactly. Given any approximate solutions, defects are defined by considering the invariance equations defining the objects. The numerical approximations exploit formal power series methods which have been discussed in many places. In particular, we use the numerical schemes discussed in [60] freely throughout Sect. 7.

3 Exponential Tracking

Remark 3.1 Throughout this section, $\rho \in \mathbb{R}^{m_s}$ denotes a positive vector (the radius of the domain of the local stable manifold chart candidates) and $P \in \mathbb{R}^{m_s} \otimes \mathbb{R}^{m_u}$ denotes a positive tensor (bounding the subspace-Lipschitz constants of our charts).

To begin the analysis we first derive estimates on $x(t, \xi, \alpha)$, the solution of the projected system (16).

Proposition 3.2 *Let $\xi, \zeta \in B_s(\rho)$. If $x(t, \xi, \alpha)$ and $x(t, \zeta, \alpha)$ stay inside B_s for all $t \in [0, T]$, then*

$$|x(t, \xi, \alpha) - x(t, \zeta, \alpha)| \le C_s|\xi - \zeta|e^{(\lambda_s + C_s\hat{\mathcal{H}})t} \qquad \textit{for all } t \in [0, T].$$

Proof. Recall from (16) that

$$\dot{\mathrm{x}}_s = \Lambda_s \mathrm{x}_s + L_s^s \mathrm{x}_s + L_s^u \alpha(\mathrm{x}_s) + \hat{\mathcal{N}}_s(\mathrm{x}_s, \alpha(\mathrm{x}_s)).$$

Define $x(t) = x(t, \xi, \alpha)$ and $z(t) = x(t, \zeta, \alpha)$. By variation of constants, we have that

$$x(t) = e^{(\Lambda_s + L_s^s)t}\xi + \int_0^t e^{(\Lambda_s + L_s^s)(t-\tau)} \left(L_s^u \alpha(x(\tau)) + \hat{\mathcal{N}}_s\left(x(\tau), \alpha(x(\tau))\right) \right) d\tau.$$

From (8), we have that $|e^{(\Lambda_s + L_s^s)t}\xi_s| \leq C_s |e^{\lambda_s t}\xi_s|$. Let $U(t) = |x(t) - z(t)|$, so that

$$e^{-\lambda_s t} U(t) \leq C_s |\xi - \zeta| + \int_0^t C_s e^{-\lambda_s \tau} \left| L_s^u (\alpha(x(\tau)) - \alpha(z(\tau))) \right| d\tau$$
$$+ \int_0^t C_s e^{-\lambda_s \tau} \left| \hat{\mathcal{N}}_s\left(x(\tau), \alpha(x(\tau))\right) - \hat{\mathcal{N}}_s\left(z(\tau), \alpha(z(\tau))\right) \right| d\tau. \qquad (25)$$

Recall from Definition 2.9 the definition of $\hat{\mathcal{H}}$. Applying the mean value theorem gives

$$\left| L_s^u (\alpha(x(\tau)) - \alpha(z(\tau))) \right| + \left| \hat{\mathcal{N}}_s\left(x(\tau), \alpha(x(\tau))\right) - \hat{\mathcal{N}}_s\left(z(\tau), \alpha(z(\tau))\right) \right| \leq \hat{\mathcal{H}} |x(\tau) - z(\tau)|.$$

Plugging this bound into (25) gives

$$e^{-\lambda_s t} U(t) \leq C_s |\xi - \zeta| + \int_0^t C_s \hat{\mathcal{H}} e^{-\lambda_s \tau} U(\tau) d\tau.$$

By Gronwall's inequality, it follows that $e^{-\lambda_s t} U(t) \leq C_s |\xi - \zeta| \exp\{C_s \hat{\mathcal{H}} t\}$, which we rewrite as

$$U(t) \leq C_s |\xi - \zeta| e^{(\lambda_s + C_s \hat{\mathcal{H}})t}.$$

□

From the proof of Proposition 3.2, it is clear that $\lambda_s + C_s \hat{\mathcal{H}} < 0$ implies the solution limits to zero. Taking $\zeta = 0$, this shows that points in $B_s(\frac{1}{C_s}\rho)$ stay in $B_s(\rho)$ for all time. A sharper version of Proposition 3.2 follows by taking into account the rates in the different subspaces of X_s. Consider for example the decomposition $X_s = X_{\text{slow}} \times X_{\text{fast}}$ and the initial condition $\xi = (\xi_{\text{slow}}, \xi_{\text{fast}}) \in X_{\text{slow}} \times X_{\text{fast}}$. Solving the linear system, and exploiting the bound from (6), gives that $|e^{\Lambda_{\text{slow}} t}\xi_{\text{slow}}| \leq e^{\lambda_{\text{slow}} t}|\xi_{\text{slow}}|$, and that $|e^{\Lambda_{\text{fast}} t}\xi_{\text{fast}}| \leq e^{\lambda_{\text{fast}} t}|\xi_{\text{fast}}|$. If $0 > \lambda_{\text{slow}} \gg \lambda_{\text{fast}}$, we expect that solutions of Eq. (16) have a component $x_{\text{fast}}(t, \xi, \alpha)$ that initially decreases very quickly.

This intuition motivates the definition of the characteristic "control" rates, arising from each subspace in the stable eigenspace, by which solutions to (16) grow/shrink. The effect of coupling the various subspaces together is controlled by the constant $\gamma_0 = \lambda_s + C_s \hat{\mathcal{H}}$, the exponent derived in Proposition 3.2.

Definition 3.3 For integers $0 \leq k \leq m_s$, define constants γ_k (control rates) as

$$\gamma_k := \begin{cases} \lambda_s + C_s \hat{\mathcal{H}} & \text{if } k = 0 \\ \lambda_k + H_k^k & \text{otherwise.} \end{cases}$$

Assume the ordering $\gamma_k > \gamma_{k+1}$.

In practice the ordering of γ_k is always satisfied by suitably (re)arranging the subspaces X. The strictness of the ordering indicates that on the balls chosen, the nonlinearities do not spoil the subspace splitting. Using these exponential rates, we estimate the components of $|x(t, \xi, \alpha)|$ using tensors $G_{j,k}^n$ defined as follows.

Condition 3.4 *A tensor* $G \in (\mathbb{R}^{m_s})^{\otimes 2} \otimes \mathbb{R}^{m_s+1}$ *satisfies Condition 3.4 on the interval* $[0, T]$ *if:*

$$|x_j(t, \xi, \alpha) - x_j(t, \zeta, \alpha)| \le \sum_{\substack{n \in I \\ 0 \le k \le m_s}} e^{\gamma_k t} G^n_{j,k} |\xi_n - \zeta_n|, \tag{26}$$

for all $t \in [0, T]$, *all* $\xi, \zeta \in B_s(\rho)$ *and all* $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$.

Remark 3.5 Since $|x_j| \le p_j |x|$, with p_j defined in (5), by Proposition 3.2 the tensor

$$\widehat{G}^n_{j,k} := \begin{cases} p_j C_s & \text{for } k = 0, \\ 0 & \text{for } k \ne 0, \end{cases}$$

satisfies Condition 3.4.

Note that while this tensor $\widehat{G}$ is non-negative, a generic tensor G satisfying Condition 3.4 can, and in practice will, have negative components.

Additionally, we remark that while this estimate is typically initially worse than the bound given by Proposition 3.2, an explicit bootstrapping argument allows us to obtain tighter component-wise bounds on solutions of Eq. (16). The bootstrapping argument applies variation of constants to Eq. (16) in each subspace, focusing on improving the bound one component at a time. To begin, we first prove the following proposition.

Proposition 3.6 *Let* $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$ *and* $\xi, \zeta \in B_s(\rho)$. *Define* $u_i(t) := |x_i(t, \xi, \alpha) - x_i(t, \zeta, \alpha)|$ *for* $i \in I$. *If* $x(t, \xi, \alpha), x(t, \zeta, \alpha) \in B_s(\rho)$ *for* $t \in [0, T]$, *then for each* $j \in I$ *and all* $t \in [0, T]$ *we have*

$$e^{-\lambda_j t} u_j(t) \le |\xi_j - \zeta_j| + \int_0^t e^{-\lambda_j \tau} \sum_{i \in I} H^i_j u_i(\tau) d\tau. \tag{27}$$

Proof. By variation of constants

$$x_j(t, \xi, \alpha) = e^{\Lambda_j t} \xi_j + \int_0^t e^{\Lambda_j (t-\tau)} \mathcal{N}_j\left(x(\tau, \xi, \alpha), \alpha(x(\tau, \xi, \alpha))\right) d\tau.$$

Then

$$|\mathcal{N}_j(x(t, \xi, \alpha), \alpha(x(t, \xi, \alpha))) - \mathcal{N}_j(x(t, \zeta, \alpha), \alpha(x(t, \zeta, \alpha)))| \le H^i_j u_i(t) \qquad \text{for all } t \ge 0.$$

Together with the estimate $|e^{\Lambda_j t} \xi_j| \le e^{\lambda_j t} |\xi_j|$ for $t \ge 0$ this gives

$$e^{-\lambda_j t} u_j(t) \le |\xi_j - \zeta_j| + \int_0^t e^{-\lambda_j \tau} \sum_{i \in I} H^i_j u_i(\tau) d\tau. \qquad \square$$

Given a tensor G satisfying Condition 3.4, we obtain sharper component-wise estimates by the following theorem.

Theorem 3.7 *Let* $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$ *and let* $\xi, \zeta \in B_s(\rho)$. *Suppose* G *satisfies Condition 3.4, and fix* $j \in I$. *If* $G^n_{i,j} = 0$ *for all* $n \in I$ *and* $i \in I - \{j\}$, *then*

$$|x_j(t, \xi, \alpha) - x_j(t, \zeta, \alpha)| \le |\xi_j - \zeta_j| e^{\gamma_j t} + \sum_{\substack{n, i \in I, i \ne j \\ 0 \le m \le m_s, m \ne j}} \frac{e^{\gamma_m t} - e^{\gamma_j t}}{\gamma_m - \gamma_j} H^i_j G^n_{i,m} |\xi_n - \zeta_n|. \tag{28}$$

That is, for $j \in I$ and $\mathcal{T}_j : (\mathbb{R}^{m_s})^{\otimes 2} \otimes \mathbb{R}^{m_s+1} \to \mathbb{R}^{m_s} \otimes \mathbb{R}^{m_s+1}$ defined by

$$\left[\mathcal{T}_j(G)\right]_k^n := \begin{cases} \sum_{n,i\in I, i\neq j} (\gamma_k - \gamma_j)^{-1} H_j^i G_{i,k}^n & \text{if } k \neq j, \\ \delta_k^n - \sum_{\substack{n,i\in I, i\neq j \\ 0\le m\le m_s, m\neq j}} (\gamma_m - \gamma_j)^{-1} H_j^i G_{i,m}^n & \text{if } k = j, \end{cases} \tag{29}$$

replacing $G_{j,k}^n$ by $\left[\mathcal{T}_j(G)\right]_k^n$ results in a new tensor G satisfying Condition 3.4.

Two lemmas aid in the proof.

Lemma 3.8 (see [41, p.4]) *Let $u, V, h \in C^0\big([0,\infty),[0,\infty)\big)$ and suppose that*

$$u(t) \le V(t) + \int_0^t h(s)u(s)ds.$$

If V is differentiable, then

$$u(t) \le V(0)\exp\left\{\int_0^t h(s)ds\right\} + \int_0^t V'(s)\exp\left\{\int_s^t h(\tau)d\tau\right\}ds.$$

Lemma 3.9 *Fix constants $c_0, c_1, c_2 \in \mathbb{R}$ with $c_1, c_2 \ge 0$ and define $\mu_0 = c_0 + c_2$. For constants μ_k, a_k with $\mu_k \neq \mu_0$ for $k = 1, \ldots, K$, we set*

$$v(s) = \sum_{k=1}^K e^{\mu_k s} a_k.$$

Suppose that $v(t) \ge 0$ for $t \ge 0$, and assume

$$e^{-c_0 t}u_0(t) \le \left(c_1 + \int_0^t e^{-c_0 s}v(s)ds\right) + \int_0^t c_2 e^{-c_0 s}u_0(s)ds.$$

Then

$$u_0(t) \le c_1 e^{\mu_0 t} + \sum_{k=1}^K \frac{a_k}{\mu_k - \mu_0}\left(e^{\mu_k t} - e^{\mu_0 t}\right). \tag{30}$$

Furthermore, the sum in the righthand side is non-negative for all $t \ge 0$.

Proof Lemma 3.8 gives

$$\begin{aligned} e^{-c_0 t}u_0(t) &\le c_1 e^{c_2 t} + \int_0^t e^{-c_0 s}v(s)e^{c_2(t-s)}ds. \\ &= c_1 e^{c_2 t} + e^{c_2 t}\int_0^t \sum_{k=1}^n a_k e^{(\mu_k - c_0 - c_2)s}ds \\ &= c_1 e^{c_2 t} + e^{c_2 t}\sum_{k=1}^n \frac{a_k}{\mu_k - \mu_0}\left(e^{(\mu_k-\mu_0)t} - 1\right). \end{aligned} \tag{31}$$

Multiplying each side by $e^{c_0 t}$ gives the desired inequality (30).

Since $v(t)$ is nonnegative, so is the integrand. Hence the sum in the righthand side of (31) is non-negative for all $t \ge 0$. ◻

Proof of Theorem 3.7 Fix $j \in J$ and rewrite (27) as

$$e^{-\lambda_j t} u_j(t) \le |\xi_j - \zeta_j| + \sum_{i \in I, i \neq j} \int_0^t e^{-\lambda_j s} H^i_j u_i(s) ds + \int_0^t e^{-\lambda_j s} H^j_j u_j(s) ds. \tag{32}$$

Since G satisfies Condition 3.4 we have

$$\begin{aligned} \sum_{i \in I, i \neq j} H^i_j u_i(t) &\le \sum_{i \in I, i \neq j} H^i_j \sum_{\substack{n \in I \\ 0 \le m \le m_s}} e^{\gamma_m t} G^n_{i,m} |\xi_n - \zeta_n| \\ &= \sum_{0 \le m \le m_s} e^{\gamma_m t} \sum_{n, i \in I, i \neq j} H^i_j G^n_{i,m} |\xi_n - \zeta_n| \\ &= \sum_{0 \le m \le m_s, m \neq j} e^{\gamma_m t} \sum_{n, i \in I, i \neq j} H^i_j G^n_{i,m} |\xi_n - \zeta_n|, \end{aligned} \tag{33}$$

where the final equality follows from the assumption that $G^n_{i,j} = 0$ whenever $i \neq j$. Defining

$$v(s) = \sum_{0 \le m \le m_s, m \neq j} e^{\gamma_m s} a_m, \qquad \text{with} \qquad a_m := \sum_{n, i \in I, i \neq j} H^i_j G^n_{i,m} |\xi_n - \zeta_n|,$$

and combining (32) with (33) leads to

$$\begin{aligned} e^{-\lambda_j t} u_j(t) &\le |\xi_j - \zeta_j| + \int_0^t e^{-\lambda_j s} \sum_{0 \le m \le m_s, m \neq j} e^{\gamma_m s} a_m ds + \int_0^t e^{-\lambda_j s} H^j_j u_j(s) ds. \\ &= |\xi_j - \zeta_j| + \int_0^t e^{-\lambda_j s} v(s) ds + \int_0^t H^j_j e^{-\lambda_j s} u_j(s) ds. \end{aligned}$$

Now apply Lemma 3.9 with $u_0 = u_j$, $c_0 = \lambda_j$, $c_1 = |\xi_j - \zeta_j|$, $c_2 = H^j_j$. Re-indexing $\{\mu_k\}_{1 \le k \le K} = \{\gamma_m\}_{0 \le m \le m_s, m \neq j}$, we see that $\gamma_m \neq \lambda_j + H^j_j = \gamma_j$ for $m \neq j$ follows from the strict ordering assumption of Definition 3.3. Then the assumption in Lemma 3.9 is satisfied. Applying Lemma 3.9 is justified, and leads to the result (28). □

Theorem 3.7 lets us pick a $j \in I$, and replace a bound of the form (26) with the same bound, where $G^n_{j,k}$ is replaced by $[\mathcal{T}_j(G)]^n_k$, possibly producing a sharper bound. Note that in Theorem 3.7, we impose that for a fixed $j \in I$ we have $G^n_{i,j} = 0$ for all $n \in I$ and $i \in I - j$. Without this assumption, we would end up with terms of the form $te^{\gamma_j t}$ in (28). We choose to avoid this, as we prefer to work with a finite set of exponentially decaying functions as the basis of our estimates.

However, we also need to deal with the case $G^n_{i,j} \neq 0$ for some $i \neq j$ and some $n \in I$. This problem is solved by modifying such an "ill-conditioned" G before replacing it with $\mathcal{T}_j(G)$. Namely, if $G^n_{i,j} \neq 0$ then, depending on the sign of $G^n_{i,j}$ we estimate $(G^n_{i,j})e^{\gamma_j t}$ from above by either $G^n_{i,j} e^{\gamma_{j-1} t}$ or $G^n_{i,j} e^{\gamma_{j+1} t}$ for $t \ge 0$. Here we use the ordering $\gamma_0 > \cdots > \gamma_{m_s}$ asserted in Definition 3.3. To be precise, for any fixed $j \in I$, define the modified tensor

$$[\mathcal{Q}_j(G)]^n_{i,k} := \begin{cases} 0 & \text{if } k = j \\ G^n_{i,k} + G^n_{i,j} & \text{if } k = j-1, \text{ and } G^n_{i,j} > 0 \\ G^n_{i,k} + G^n_{i,j} & \text{if } k = j+1, \text{ and } G^n_{i,j} < 0 \\ G^n_{i,k} & \text{otherwise.} \end{cases} \tag{34}$$

Note that if $j = m_s$ and $G^n_{i,j} < 0$, then we are effectively employing the estimate $G^n_{i,j} e^{\gamma_{m_s} t} < 0$.

The following lemma summarizes the preceding discussion.

Lemma 3.10 *Fix $j \in I$. If G satisfies Condition 3.4, then $\mathcal{Q}_j(G)$ satisfies Condition 3.4.*

Thus, starting from an initial bound of the form (26) with tensor $\widehat{G}$ given in Remark 3.5, we iteratively improve the bound using the following algorithm.

Algorithm 3.11 *Let $N_{bootstrap} \in \mathbb{N}$ be a computational parameter.*

$G \leftarrow \widehat{G}$
for $1 \le i \le N_{bootstrap}$ ***do***
  for $1 \le j \le m_s$ ***do***
    $G^n_{j,k} \leftarrow \left[\mathcal{T}_j \circ \mathcal{Q}_j(G)\right]^n_k$
  end for
end for
return G

In practice Algorithm 3.11 quickly converges to a fixed tensor G. For example $N_{\text{bootstrap}} \le 5$ is sufficient for the applications to folllow.

Theorem 3.12 *Let $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$, and suppose that the coefficients $G^n_{j,k}$ are output by Algorithm 3.11. Fix initial conditions $\xi, \zeta \in B_s(\rho)$. If $x(\tau, \xi, \alpha)$ and $x(\tau, \zeta, \alpha)$ stay inside $B_s(\rho)$ for all $t \in [0, T]$, then*

$$|x_j(t, \xi, \alpha) - x_j(t, \zeta, \alpha)| \le \sum_{\substack{n \in I \\ 0 \le k \le m_s}} e^{\gamma_k t} \cdot G^n_{j,k} \, |\xi_n - \zeta_n| \qquad \text{for all } t \in [0, T]. \tag{35}$$

Furthermore, if α is differentiable then $\left\| \frac{\partial}{\partial \xi_n} x_j(t, \xi, \alpha) \right\| \le \sum_{0 \le k \le m_s} e^{\gamma_k t} G^n_{j,k}$ for all $t \in [0, T]$.

The proof of Theorem 3.12 is by induction on $N_{\text{bootstrap}}$, with Proposition 3.2 taking care of the base case ($N_{\text{bootstrap}} = 0$), and Theorem 3.7 taking care of the inductive step. We omit the details.

Now, in Proposition 3.2 the assumption that $\gamma_0 < 0$ gives only that points $\xi \in B_s(C_s^{-1}\rho)$ have solutions to (16) staying in $B_s(\rho)$ for all $t \ge 0$. The following proposition gives conditions which extend the result to all points $\xi \in B_s(\rho)$.

Proposition 3.13 *Suppose that $\gamma_0 < 0$ and that $G^n_{j,k}$ is the output of Algorithm 3.11. If*

$$\rho_j \ge \sum_{\substack{n \in I \\ 0 \le k \le m_s}} e^{\gamma_k t} G^n_{j,k} \rho_n, \tag{36}$$

for all $t \ge 0$, then for all $\xi \in B_s(\rho)$ and $t \ge 0$ we have $x(t, \xi, \alpha) \in B_s(\rho)$ for all $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$.

Proof Fix $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$, $0 < \epsilon < 1$, and $\xi \in B_s(\epsilon\rho)$. Define $T = \sup\{t \ge 0 : x(t, \xi, \alpha) \in B_s(\rho)\}$. Assume that $T < +\infty$. We show by contradiction that $T = +\infty$.

Since $x(0, \xi, \alpha) \in B_s(\epsilon\rho)$ and $x(t, \xi, \alpha)$ is continuous in t, it follows that $T > 0$. By Proposition 3.12 we have for all $t \in [0, T)$ that

$$|x_j(t, \xi, \alpha)| \le \sum_{0 \le k \le m_s} e^{\gamma_k t} G^n_{j,k} \, |\xi_n| \le \epsilon \sum_{0 \le k \le m_s} e^{\gamma_k t} G^n_{j,k} \rho_n \le \epsilon \, \rho_j.$$

Hence $x(t, \xi, \alpha) \in B_s(\epsilon\rho)$ for all $t \in [0, T)$, and so by continuity $x(T, \xi, \alpha) \in B_s(\epsilon\rho)$. Since $x(T, \xi, \alpha)$ is in the interior of $B_s(\rho)$, the solution of (16) starting at $x(T, \xi, \alpha)$ stays inside the ball $B_s(\rho)$ for some positive amount of time. But this contradicts the definition of T as the supremum of $\{t \geq 0 : x(t, \xi, \alpha) \in B_s(\rho)\}$. Hence, if $0 < \epsilon < 1$ and $\xi \in B_s(\epsilon\rho)$, then $x(t, \xi, \alpha) \in B_s(\rho)$ for all $t \geq 0$.

By continuity of solutions, this result extends to initial conditions on the boundary of $B_s(\rho)$. □

Remark 3.14 In practice we verify the hypothesis of Proposition 3.13 in three steps:

1. For some $T_2 > 0$, we check that $\rho_j > \sum_{n \in I, 0 \leq k \leq m_s} e^{\gamma_k T_2} |G^n_{j,k}| \rho_n$, and hence (36) is satisfied for all $t \geq T_2$.
2. For some $0 < T_1 < T_2$, we use interval arithmetic to verify the inequality (36) for $T_1 \leq t \leq T_2$.
3. To prove inequality (36) for $t \in [0, T_1]$, we both prove that the inequality holds at $t = 0$ (explained below), and show using interval arithmetic that the derivative of the right-hand side of (36) is negative:

$$\sum_{\substack{n \in I \\ 0 \leq k \leq m_s}} \gamma_k e^{\gamma_k t} G^n_{j,k} \rho_n < 0 \qquad \text{for } t \in [0, T_1].$$

   To prove that inequality (36) holds at $t = 0$, we fix $j \in I$. If G is the final output of Algorithm 3.11, then there is a tensor $\widetilde{G} \in (\mathbb{R}^{m_s})^{\otimes 2} \otimes \mathbb{R}^{m_s+1}$ for which $G^n_{j,k} \leftarrow \left[\mathcal{T}_j \circ \mathcal{Q}_j(\widetilde{G})\right]^n_k$. It is assigned at step j of the inner for-loop of the algorithm, and at step $N_{\text{bootstrap}}$ of the outer for-loop. Letting $\bar{G} := \mathcal{Q}_j(\widetilde{G})$, it follows from the definition of $\mathcal{T}_j$ in (29) that

$$\sum_{\substack{n \in I \\ 0 \leq k \leq m_s}} e^{\gamma_k t} G^n_{j,k} |\xi_n| = |\xi_j| e^{\gamma_j t} + \sum_{\substack{n, i \in I, i \neq j \\ 0 \leq k \leq m_s, k \neq j}} \frac{e^{\gamma_k t} - e^{\gamma_j t}}{\gamma_k - \gamma_j} H^i_j \bar{G}^n_{i,k} |\xi_n|.$$

   Evaluating at $t = 0$, we have

$$|x_j(0, \xi, \alpha)| = |\xi_j| = \sum_{0 \leq k \leq m_s} G^n_{j,k} |\xi_n|.$$

   Taking $|\xi_n| = \rho_n$ for all $n \in I$, it follows that $\rho_j = \sum_{0 \leq k \leq m_s} G^n_{j,k} \rho_n$. Hence (36) is satisfied at $t = 0$ for all $j \in I$.

Remark 3.15 When inequality (36) fails to be true, we cannot be sure that all solutions of Equation (16) stay inside the ball $B_s(\rho)$ for all time. There are two common reasons for why this happens: first, the nonlinearity may be too large and solutions leave the ball never to return; second, solutions to Eq. (16) may temporarily leave the ball, reenter, and then converge to zero.

If inequality (36) fails to be true because of the first reason, then ρ should be made smaller.

If inequality (36) fails to be true because of the second reason, it is often because $B_s(\rho)$ is too wide in one direction and too thin in another. If we suspect this to be true, then to better align the box with the flow, we iteratively select a new value of ρ using the map $\rho_j \mapsto \sup_{0 \leq t \leq T} \sum_k e^{\gamma_k t} G^n_{j,k} \rho_n$. In practice, this heuristic is effective for finding a value of ρ for which (36) is satisfied.

Algorithm 3.11 can be applied in more general situations. The two conditions necessary to construct such an algorithm are Condition 3.4 and Proposition 3.6. These are all generalized in Appendix A leading to an algorithm used in Sect. 4.2 to obtain bounds on $\frac{\partial}{\partial \xi_i} x(t, \xi, \alpha)$, and in Sect. 5 to construct bounds on $|x(t, \xi, \alpha) - x(t, \xi, \beta)|$ for charts $\alpha, \beta \in \mathcal{B}^{0,1}_{\rho,P}$.

4 Lyapunov-Perron Operator

In this section we show that the Lyapunov-Perron operator Ψ is an endomorphism on balls $\mathcal{B}^{0,1}_{\rho,P}$ and $\mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ for appropriately chosen constants.

Remark 4.1 Throughout this section, we fix a positive vector $\rho \in \mathbb{R}^{m_s}$ and a positive tensor $P \in \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s}$, and fix $G \in (\mathbb{R}^{m_s})^{\otimes 2} \otimes \mathbb{R}^{m_s+1}$ as the output of Algorithm 3.11 taken with $N_{bootstrap} \geq 1$. Furthermore, we assume that the hypotheses of Proposition 3.13 are satisfied, and in particular that inequality (36) holds for all $t \geq 0$. Hence G satisfies Condition 3.4 on the interval $[0, \infty)$.

Throughout this section we adopt Einstein summation convention for indices of I and I'.

4.1 Endomorphism on $\mathcal{B}^{0,1}_{\rho,P}$

The next theorem provides a straightforward bound on $\text{Lip}(\Psi[\alpha])$ for $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$.

Theorem 4.2 *Define $\tilde{P} \in \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s}$ component-wise by:*

$$\tilde{P}^n_{i'} := \sum_{0 \leq k \leq m_s} (\lambda_{i'} - \gamma_k)^{-1} H^i_{i'} G^n_{i,k}.$$

If $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$, then $\text{Lip}(\Psi[\alpha])^n_{i'} \leq \tilde{P}^n_{i'}$. If $\tilde{P}^j_{j'} \leq P^j_{j'}$ then $\Psi : \mathcal{B}^{0,1}_{\rho,P} \to \mathcal{B}^{0,1}_{\rho,P}$ is well defined.

Proof Fix $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$ and $\xi, \zeta \in B_s(\rho)$. Define $x(t) := x(t, \xi, \alpha)$ and $z(t) := x(t, \zeta, \alpha)$. Our goal is to prove that $|\Psi[\alpha]_{i'}(\xi) - \Psi[\alpha]_{i'}(\zeta)| \leq \tilde{P}^n_{i'} |\xi_n - \zeta_n|$. From the definition of Ψ we have

$$\Psi[\alpha](\xi) - \Psi[\alpha](\zeta) = -\int_0^\infty e^{-\Lambda_u t} \left[\mathcal{N}_u(x(t), \alpha(x(t))) - \mathcal{N}_u(z(t), \alpha(z(t)))\right] dt.$$

Using the bound (14), and the fact that G satisfies Condition 3.4 on $[0, \infty)$, we obtain

$$\begin{aligned} |\Psi[\alpha]_{i'}(\xi) - \Psi[\alpha]_{i'}(\zeta)| &\leq \int_0^\infty e^{-\lambda_{i'} t} H^i_{i'} |x_i(t) - z_i(t)| dt \\ &\leq \int_0^\infty e^{-\lambda_{i'} t} \sum_{0 \leq k \leq m_s} e^{\gamma_k t} H^i_{i'} G^n_{i,k} |\xi_n - \zeta_n| \, dt \\ &= \sum_{0 \leq k \leq m_s} (\lambda_{i'} - \gamma_k)^{-1} H^i_{i'} G^n_{i,k} |\xi_n - \zeta_n| . \end{aligned}$$

For $\tilde{P}^n_{i'}$ as defined above, it follows that

$$|\Psi[\alpha]_{i'}(\xi) - \Psi[\alpha]_{i'}(\zeta)| \leq \tilde{P}^n_{i'} |\xi_n - \zeta_n| .$$

Hence $\mathrm{Lip}(\Psi[\alpha])^n_{i'} \le \tilde{P}^n_{i'}$. Since $\mathcal{N}(0) = 0$, direct evaluation shows that $\Psi[\alpha](0) = 0$. Hence $\Psi[\alpha] \in \mathcal{B}^{0,1}_{\rho,P}$. □

Remark 4.3 Ideally, we would like to choose a tensor P as small as possible while still satisfying the inequality $\tilde{P}^j_{i'} \le P^j_{i'}$. In practice, we find a nearly optimal P by iteratively mapping $P^j_{i'} \mapsto \tilde{P}^j_{i'}$. This has the effect that if $\tilde{P}^j_{i'} \le P^j_{i'}$, then the new value of P will be smaller. Since the bounds for H and G improve with smaller P, the inequality $\tilde{P}^j_{i'} \le P^j_{i'}$ will likely be satisfied for the new P. On the other hand, if P is too small and $\tilde{P}^j_{i'} \le P^j_{i'}$ is not satisfied, then the new value of P will be larger, and the inequality will hopefully be satisfied at the next iterate of the algorithm.

Note that the definitions of H and G depend on P, and so these constants need to be recomputed every time. Nevertheless, this iterative process provides an effective, algorithmic method for selecting appropriate $P^j_{i'}$.

Using second derivative bounds on $\mathcal{N}_u$ sharpens Theorem 4.2 as below.

Proposition 4.4 *Define $\tilde{P} \in \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s}$ component-wise by:*

$$\tilde{P}^n_{i'} := \left(D^i_{i'} + D^{j'}_{i'} P^i_{j'}\right) \sum_{0 \le k \le m_s} (\lambda_{i'} - \gamma_k)^{-1} G^n_{i,k} + \left(\hat{C}^{ij}_{i'} + \hat{C}^{j'j}_{i'} P^i_{j'}\right) \sum_{0 \le k_1, k_2 \le m_s} (\lambda_{i'} - \gamma_{k_1} - \gamma_{k_2})^{-1} G^m_{j,k_1} G^n_{i,k_2} \rho_m.$$

If $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$, then $\mathrm{Lip}(\Psi[\alpha])^n_{i'} \le \tilde{P}^n_{i'}$. If $\tilde{P}^j_{j'} \le P^j_{j'}$ then $\Psi : \mathcal{B}^{0,1}_{\rho,P} \to \mathcal{B}^{0,1}_{\rho,P}$ is well defined.

Proof By the mean value theorem we have (recall that $\mathcal{N}^i_{i'} = \frac{\partial}{\partial x_i} \mathcal{N}_{i'}$)

$$|\mathcal{N}_{i'}(x, \alpha(x)) - \mathcal{N}_{i'}(z, \alpha(z))| \le \left[\sup_{\substack{y \in B_s(\rho),\ j \in I \\ |y_j| \le \max\{|x_j|, |z_j|\}}} \|\mathcal{N}^i_{i'}(y, \alpha(y))\| \right] |x_i - z_i|.$$

We estimate $\max\{|x_j(t)|, |z_j(t)|\}$ using the tensor G (which satisfies Condition 3.4), and since $\max\{|\xi_m|, |\zeta_m|\} \le \rho_m$, we have

$$\begin{aligned} \sup_{\substack{y \in B_s(\rho),\ j \in I \\ |y_j| \le \max\{|x_j(t)|, |z_j(t)|\}}} \|\mathcal{N}^i_{i'}(y, \alpha(y))\| &\le D^i_{i'} + D^{j'}_{i'} P^i_{j'} + (\hat{C}^{ij}_{i'} + \hat{C}^{j'j}_{i'} P^i_{j'}) \max\{|x_j(t)|, |z_j(t)|\} \\ &\le D^i_{i'} + D^{j'}_{i'} P^i_{j'} + (\hat{C}^{ij}_{i'} + \hat{C}^{j'j}_{i'} P^i_{j'}) \sum_{0 \le k \le m_s} e^{\gamma_k t} G^m_{j,k} \rho_m. \end{aligned}$$

Using Condition 3.4 gives

$$|\mathcal{N}_{i'}(x, \alpha(x)) - \mathcal{N}_{i'}(z, \alpha(z))| \le \left(D^i_{i'} + D^{j'}_{i'} P^i_{j'}\right) \sum_{0 \le k \le m_s} e^{\gamma_k t} G^n_{i,k} |\xi_n - \zeta_n| + \left(\hat{C}^{ij}_{i'} + \hat{C}^{j'j}_{i'} P^i_{j'}\right) \sum_{0 \le k_1, k_2 \le m_s} e^{(\gamma_{k_1} + \gamma_{k_2}) t} G^m_{j,k_1} G^n_{i,k_2} \rho_m |\xi_n - \zeta_n|.$$

We obtain the desired result by integration:

$$\begin{aligned}|\Psi[\alpha]_{i'}(\xi)-\Psi[\alpha]_{i'}(\zeta)| &\le \int_0^\infty e^{-\lambda_{i'}t}|\mathcal{N}_{i'}(x,\alpha(x))-\mathcal{N}_{i'}(z,\alpha(z))|\,dt\\ &\le \left(D^i_{i'}+D^{j'}_{i'}P^i_{j'}\right)\sum_{0\le k\le m_s}(\lambda_{i'}-\gamma_k)^{-1}G^n_{i,k}\,|\xi_n-\zeta_n|\\ &\quad+\left(\hat{C}^{ij}_{i'}+\hat{C}^{j'j}_{i'}P^i_{j'}\right)\sum_{0\le k_1,k_2\le m_s}(\lambda_{i'}-\gamma_{k_1}-\gamma_{k_2})^{-1}\\ &\quad G^m_{j,k_1}G^n_{i,k_2}\rho_m\,|\xi_n-\zeta_n|\,.\end{aligned}$$

□

4.2 Endomorphism on $\mathcal{B}^{1,1}_{\rho,P,\bar{P}}$

We now bound the Lipschitz constant of the derivative of the local stable manifold. To do this, we show that Ψ maps $\mathcal{B}^{1,1}_{\rho,P,\bar{P}}$, a ball of functions with Lipschitz derivative, into itself. Hence, if there are any fixed points $\Psi[\alpha]=\alpha\in\mathcal{B}^{1,1}_{\rho,P,\bar{P}}$, then by Definition 2.8 they satisfy $\mathrm{Lip}(\partial_i\alpha)^j_{i'}\le\bar{P}^{ij}_{i'}$. To show that $\Psi:\mathcal{B}^{1,1}_{\rho,P,\bar{P}}\to\mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ we first derive bounds on the difference $\frac{\partial}{\partial\xi_i}x_j(t,\eta,\alpha)-\frac{\partial}{\partial\xi_i}x_j(t,\zeta,\alpha)$ for $i,j\in I$. In particular, we are interested in finding a tensor K as follows.

Condition 4.5 *Define* $\{\mu_k\}_{k=1}^{N_\mu}=\{\gamma_k\}_{k=0}^{m_s}\cup\{\gamma_{k_1}+\gamma_{k_2}\}_{k_1,k_2=0}^{m_s}$. *A tensor* $K\in(\mathbb{R}^{m_s})^{\otimes 3}\otimes\mathbb{R}^{N_\mu}$ *is said to satisfy Condition 4.5 if*

$$\left\|\frac{\partial}{\partial\xi_i}x_j(t,\eta,\alpha)-\frac{\partial}{\partial\xi_i}x_j(t,\zeta,\alpha)\right\|\le\sum_{k=1}^{N_\mu}e^{\mu_k t}K^{il}_{j,k}|\eta_l-\zeta_l|,$$

for all $\alpha\in\mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ *and* $\eta,\zeta\in B_s(\rho)$ *and* $i,j\in I$.

The bound is obtained using an approach analogous to the one discussed in Sect. 3. Since we use this approach in Sects. 3, 4, and 5, we present in Appendix A a generalization which encompasses all cases. In Proposition 4.6 we define a tensor S analogous to H given in Definition 2.9. In Proposition 4.7 we derive an *a priori* bound, constructing an initial tensor K satisfying Condition 4.5 (cf. Proposition 3.2). In Proposition 4.9 we derive a system of integral inequalities (cf. Proposition 3.6 and Condition A.2). Then, as described in Theorem 4.10, we apply Algorithm A.5 (cf. Algorithm 3.11) to bootstrap Gronwall's inequality, and obtain successively sharper tensors K satisfying Condition 4.5. Finally, in Proposition 4.11, we give conditions guaranteeing that $\Psi:\mathcal{B}^{1,1}_{\rho,P,\bar{P}}\to\mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ is a well defined map.

Proposition 4.6 *Let* $\alpha\in\mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ *and* $\eta,\zeta\in B_s(\rho)$. *Define* $x=x(t,\eta,\alpha)$, $z=x(t,\zeta,\alpha)$, $x^i_j=\frac{\partial}{\partial\xi_i}x_j(t,\eta,\alpha)$, *and likewise for* z^i_j. *Fix* $\mathbf{j}\in\mathbf{I}$, *and define*

$$S^{nm}_{\mathbf{j}}:=(C^{nm}_{\mathbf{j}}+C^{nm'}_{\mathbf{j}}P^m_{m'})+C^{n'}_{\mathbf{j}}P^{nm}_{n'}+(C^{n'm}_{\mathbf{j}}+C^{n'm'}_{\mathbf{j}}P^m_{m'})P^n_{n'}.$$

Then

$$\left\|\frac{\partial}{\partial\xi_i}\left(\mathcal{N}_{\mathbf{j}}(x,\alpha(x))-\mathcal{N}_{\mathbf{j}}(z,\alpha(z))\right)\right\|\le S^{nm}_{\mathbf{j}}|x_m-z_m|\,\|z^i_n\|+H^n_{\mathbf{j}}\|x^i_n-z^i_n\|.$$

Proof We have

$$\frac{\partial}{\partial \xi_i}\mathcal{N}_{\mathbf{j}}(x,\alpha(x)) = \left(\mathcal{N}_{\mathbf{j}}^{n}(x,\alpha(x)) + \mathcal{N}_{\mathbf{j}}^{n'}(x,\alpha(x))\alpha_{n'}^{n}(x)\right)\cdot x_n^i, \tag{37}$$

and split the estimate into four parts:

$$\begin{aligned}\frac{\partial}{\partial \xi_i}\left(\mathcal{N}_{\mathbf{j}}(x,\alpha(x)) - \mathcal{N}_{\mathbf{j}}(z,\alpha(z))\right) = &\left(\mathcal{N}_{\mathbf{j}}^{n}(x,\alpha(x)) - \mathcal{N}_{\mathbf{j}}^{n}(z,\alpha(z))\right)\cdot z_n^i \\ &+ \mathcal{N}_{\mathbf{j}}^{n'}(x,\alpha(x))\left(\alpha_{n'}^{n}(x) - \alpha_{n'}^{n}(z)\right) z_n^i \\ &+ \left(\mathcal{N}_{\mathbf{j}}^{n'}(x,\alpha(x)) - \mathcal{N}_{\mathbf{j}}^{n'}(z,\alpha(z))\right)\alpha_{n'}^{n}(z) z_n^i \\ &+ \left(\mathcal{N}_{\mathbf{j}}^{n}(x,\alpha(x)) + \mathcal{N}_{\mathbf{j}}^{n'}(x,\alpha(x))\alpha_{n'}^{n}(x)\right)\cdot (x_n^i - z_n^i).\end{aligned}$$

Each term is bound separately, as

$$\begin{aligned}\left(\mathcal{N}_{\mathbf{j}}^{n}(x,\alpha(x)) - \mathcal{N}_{\mathbf{j}}^{n}(z,\alpha(z))\right)\cdot z_n^i &\leq (C_{\mathbf{j}}^{nm} + C_{\mathbf{j}}^{nm'}P_{m'}^{m})|x_m - z_m|\,\|z_n^i\|, \\ \mathcal{N}_{\mathbf{j}}^{n'}(x,\alpha(x))\left(\alpha_{n'}^{n}(x) - \alpha_{n'}^{n}(z)\right) z_n^i &\leq C_{\mathbf{j}}^{n'} P_{n'}^{nm}|x_m - z_m|\,\|z_n^i\|, \\ \left(\mathcal{N}_{\mathbf{j}}^{n'}(x,\alpha(x)) - \mathcal{N}_{\mathbf{j}}^{n'}(z,\alpha(z))\right)\alpha_{n'}^{n}(z) z_n^i &\leq (C_{\mathbf{j}}^{n'm} + C_{\mathbf{j}}^{n'm'}P_{m'}^{m})P_{n'}^{n}|x_m - z_m|\,\|z_n^i\|, \\ \left(\mathcal{N}_{\mathbf{j}}^{n}(x,\alpha(x)) + \mathcal{N}_{\mathbf{j}}^{n'}(x,\alpha(x))\alpha_{n'}^{n}(x)\right)(x_n^i - z_n^i) &\leq (C_{\mathbf{j}}^{n} + C_{\mathbf{j}}^{n'}P_{n'}^{n})\,\|x_n^i - z_n^i\|.\end{aligned}$$

The result follows by collecting all terms. □

Proposition 4.7 *Define a tensor* $\widetilde{K} \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes (\mathbb{R}^{m_s+1})^{\otimes 2}$ *as*

$$\widetilde{K}_{j,k_1k_2}^{il} = \left(\gamma_{k_1} + \gamma_{k_2} - \gamma_0\right)^{-1} C_s p_j S_j^{nm} G_{m,k_1}^{l} G_{n,k_2}^{i}.$$

Then we have

$$\left\|\tfrac{\partial}{\partial \xi_i}x(t,\eta,\alpha) - \tfrac{\partial}{\partial \xi_i}x(t,\zeta,\alpha)\right\| \leq \sum_{\substack{0\leq k_1,k_2\leq m_s\\ j\in I}} \left(e^{(\gamma_{k_1}+\gamma_{k_2})t} - e^{\gamma_0 t}\right)\widetilde{K}_{j,k_1k_2}^{il}|\eta_l - \zeta_l|,$$

for all $\alpha \in \mathcal{B}_{\rho,P,\bar{P}}^{1,1}$ *and* $\eta,\zeta \in B_s(\rho)$ *and* $i \in I$.

The indices in tensor notation $\widetilde{K}_{j,k_1k_2}^{il}$ are interpreted as follows. The superscripts correspond to derivatives, the subscript to the left of the comma corresponds to subspace projections, and the subscript to the right of the comma correspond to exponentials.

Proof. Define $x = x(t,\eta,\alpha)$ and $z = x(t,\zeta,\alpha)$. Let $x^i = \frac{\partial}{\partial \xi_i}x(t,\eta,\alpha)$ and likewise for z^i. By variation of constants, we have that

$$\begin{aligned}x^i(t) - z^i(t) = &\int_0^t e^{(\Lambda_s+L_s^s)(t-\tau)}\frac{\partial}{\partial \xi_i}L_s^u\big(\alpha(x(\tau)) - \alpha(z(\tau))\big)d\tau. \\ &+ \int_0^t e^{(\Lambda_s+L_s^s)(t-\tau)}\frac{\partial}{\partial \xi_i}\left(\hat{\mathcal{N}}_s(x(\tau),\alpha(x(\tau))) - \hat{\mathcal{N}}_s(z(\tau),\alpha(z(\tau)))\right)d\tau.\end{aligned} \tag{38}$$

Expanding the partial derivatives appearing in (38), and dropping the τ dependence in the notation in the right hand side, gives

$$\frac{\partial}{\partial \xi_i} L_s^u \alpha(x(\tau)) = \sum_{j \in I} L_j^{n'} \alpha_{n'}^n(x) x_n^i$$

$$\frac{\partial}{\partial \xi_i} \hat{\mathcal{N}}_s\big(x(\tau), \alpha(x(\tau))\big) = \sum_{j \in I} \Big(\hat{\mathcal{N}}_j^n(x, \alpha(x)) + \hat{\mathcal{N}}_j^{n'}(x, \alpha(x)) \alpha_{n'}^n(x) \Big) \cdot x_n^i.$$

In Proposition 4.6 we demonstrated how the tensor S offers a $C^{1,1}$ bound on $\mathcal{N}_{\mathbf{j}} = L_{\mathbf{j}}^s + L_{\mathbf{j}}^u + \hat{\mathcal{N}}_{\mathbf{j}}$, for $\mathbf{j} \in \mathbf{I}$. By using (8) we obtain, in analogy with the proof of Proposition 4.6,

$$e^{-\lambda_s t} \|x^i - z^i\| \leq \int_0^t C_s e^{-\lambda_s \tau} \sum_{j \in I} p_j S_j^{nm} |x_m - z_m| \, \|z_n^i\| d\tau + \int_0^t e^{-\lambda_s \tau} C_s \hat{\mathcal{H}} \|x^i - z^i\| d\tau.$$

It then follows from Proposition 3.12 that

$$e^{-\lambda_s t} \|x^i - z^i\| \leq \int_0^t C_s e^{-\lambda_s \tau} \sum_{\substack{0 \leq k_1, k_2 \leq m_s \\ j \in I}} e^{(\gamma_{k_1} + \gamma_{k_2})\tau} p_j S_j^{nm} G_{m,k_1}^l G_{n,k_2}^i |\eta_l - \zeta_l| d\tau$$
$$+ \int_0^t e^{-\lambda_s \tau} C_s \hat{\mathcal{H}} \|x^i - z^i\| d\tau.$$

By Lemma 3.9 we infer that

$$\|x^i - z^i\| \leq \sum_{\substack{0 \leq k_1, k_2 \leq m_s \\ j \in I}} \frac{e^{(\gamma_{k_1} + \gamma_{k_2})t} - e^{\gamma_0 t}}{\gamma_{k_1} + \gamma_{k_2} - \gamma_0} C_s p_j S_j^{nm} G_{m,k_1}^l G_{n,k_2}^i |\eta_l - \zeta_l|. \qquad \square$$

Remark 4.8 Define $\{\mu_k\}_{k=1}^{N_\mu} = \{\gamma_{k_1}\}_{k_1=0}^{m_s} \cup \{\gamma_{k_1} + \gamma_{k_2}\}_{k_1,k_2=0}^{m_s}$, with $N_\mu = (m_s+1)(m_s+4)/2$. Let $\widetilde{K}$ be defined as in Proposition 4.7, and define a tensor $\widehat{K} \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{N_\mu}$ by

$$\widehat{K}_{j,k}^{il} := \begin{cases} p_j \sum_{m \in I} \widetilde{K}_{m,k_1 k_2}^{il} + \widetilde{K}_{m,k_2 k_1}^{il} & \text{if } \mu_k = \gamma_{k_1} + \gamma_{k_2} \text{ for } 0 \leq k_1, k_2 \leq m_s, \\ -p_j \sum_{m \in I} \sum_{0 \leq k_1, k_2 \leq m_s} \widetilde{K}_{m,k_1 k_2}^{il} + \widetilde{K}_{m,k_2 k_1}^{il} & \text{if } \mu_k = \gamma_0, \\ 0 & \text{if } \mu_k = \gamma_{k_1}, \text{ for } 1 \leq k_1 \leq m_s. \end{cases}$$

It follows from Proposition 4.7 that $\widehat{K}$ satisfies Condition 4.5.

We now establish componentwise Lipschitz bounds on the derivatives.

Proposition 4.9 *Let $\alpha \in \mathcal{B}_{\rho,P,\bar{P}}^{1,1}$ and define $x(t) = x(t, \eta, \alpha)$ and $z(t) = z(t, \zeta, \alpha)$ for some $\eta, \zeta \in B_s(\rho)$. Let $x_j^i(t) = \frac{\partial}{\partial \xi_i} x_j(t, \eta, \alpha)$ and likewise for $z_j^i(t)$. Then*

$$e^{-\lambda_j t} \|x_j^i - z_j^i\| \leq \int_0^t e^{-\lambda_j \tau} \sum_{0 \leq k_1, k_2 \leq m_s} e^{(\gamma_{k_1} + \gamma_{k_2})\tau} S_j^{nm} G_{m,k_1}^l G_{n,k_2}^i |\eta_l - \zeta_l| d\tau$$
$$+ \int_0^t e^{-\lambda_j \tau} H_j^n \|x_n^i - z_n^i\| d\tau.$$

Proof By variation of constants, we have that

$$x_j^i(t) = e^{\Lambda_j t} \delta_j^i + \int_0^t e^{\Lambda_j (t-\tau)} \left(\frac{\partial}{\partial \xi_i} \mathcal{N}_j(x(\tau), \alpha(x(\tau))) \right) d\tau,$$

where δ^i_j is the Kronecker delta. Taking the difference $x^i_j - z^i_j$ we obtain

$$x^i(t) - z^i(t) = \int_0^t e^{\Lambda_s(t-\tau)} \frac{\partial}{\partial \xi_i} \Big(\mathcal{N}_j(x(\tau), \alpha(x(\tau))) - \mathcal{N}_j(z(\tau), \alpha(z(\tau))) \Big) d\tau.$$

From Proposition 4.6 we have

$$e^{-\lambda_j t} \|x^i_j - z^i_j\| \le \int_0^t e^{-\lambda_j \tau} S^{nm}_j |x_m - z_m| \, \|z^i_n\| d\tau + \int_0^t e^{-\lambda_j \tau} H^n_j \|x^i_n - z^i_n\| d\tau.$$

Plugging in the bounds on $|x_m - z_m|$ and $\|z^i_n\|$ from Proposition 3.12, we obtain the desired result. □

Theorem 4.10 *Let $\{\mu_k\}_{k=1}^{N_\mu}$ and let the tensor $\widehat{K} \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{N_\mu}$ be as defined in Remark 4.8. When K is the output of Algorithm A.5 taken with input $\widehat{K}$ and some $N_{bootstrap} \ge 1$, then K satisfies Condition 4.5.*

The proof of Theorem 4.10 follows from the argument outlined in Appendix A, where Conditions A.1 and A.2 correspond to Proposition 4.9 and Condition 4.5 respectively.

Theorem 4.11 *Let $\bar{P} \in \mathbb{R}^{m_u} \otimes (\mathbb{R}^{m_s})^{\otimes 2}$ and assume $K \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{N_\mu}$ satisfies Condition 4.5. Define the tensor $\tilde{P} \in \mathbb{R}^{m_u} \otimes (\mathbb{R}^{m_s})^{\otimes 2}$ as*

$$\tilde{P}^{il}_{j'} := \sum_{0 \le k_1, k_2 \le m_s} (\lambda_{j'} - \gamma_{k_1} - \gamma_{k_2})^{-1} S^{nm}_{j'} G^l_{m,k_1} G^i_{n,k_2} + \sum_{1 \le k \le N_\mu} (\lambda_{j'} - \mu_k)^{-1} H^n_{j'} K^{il}_{n,k}. \tag{39}$$

Then for all $\alpha \in \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ we have $\mathrm{Lip}(\partial_i \Psi[\alpha])^l_{j'} \le \tilde{P}^{il}_{j'}$. If $\tilde{P}^{il}_{j'} \le \bar{P}^{il}_{j'}$ then $\Psi : \mathcal{B}^{1,1}_{\rho,P,\bar{P}} \to \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ is well defined.

Proof Let $\eta, \zeta \in B_s(\rho)$ and define $x(t) = x(t, \eta, \alpha)$ and $z(t) = x(t, \zeta, \alpha)$. Define $x^i_j(t) = \frac{\partial}{\partial \xi_i} x_j(t, \eta, \alpha)$ and likewise for $z^i_j(t)$. From Definition 2.11 we have

$$\Psi[\alpha](\eta) - \Psi[\alpha](\zeta) = -\int_0^\infty e^{-\Lambda_u t} \big(\mathcal{N}_u(x(t), \alpha(x(t))) - \mathcal{N}_u(z(t), \alpha(z(t))) \big) dt.$$

Using Proposition 4.6 gives

$$\left\| \Psi[\alpha]^i_{j'}(\eta) - \Psi[\alpha]^i_{j'}(\zeta) \right\| \le \int_0^\infty e^{-\lambda_{j'} t} \left(S^{nm}_{j'} |x_m - z_m| \, \|z^i_n\| + H^n_{j'} \|x^i_n - z^i_n\| \right) dt.$$

Plugging in the bounds on $|x_m - z_m|$ and $\|z^i_n\|$ from Proposition 3.12, as well as the bounds on $|x^i_n - z^i_n|$ from Proposition 4.9, gives

$$\begin{aligned} \left\| \Psi[\alpha]^i_{j'}(\eta) - \Psi[\alpha]^i_{j'}(\zeta) \right\| &\le \int_0^\infty e^{-\lambda_{j'} t} \sum_{0 \le k_1, k_2 \le m_s} e^{(\gamma_{k_1} + \gamma_{k_2}) t} S^{nm}_{j'} G^l_{m,k_1} G^i_{n,k_2} |\xi_l - \zeta_l| dt \\ &\quad + \int_0^\infty e^{-\lambda_{j'} t} \sum_{1 \le k \le N_\mu} e^{\mu_k t} H^n_{j'} K^{il}_{n,k} |\eta_l - \zeta_l| dt \\ &= \tilde{P}^{il}_{j'} |\eta_l - \zeta_l|. \end{aligned}$$

Hence, we have obtained the desired bound $\mathrm{Lip}(\partial_i \Psi[\alpha])^l_{j'} \le \tilde{P}^{il}_{j'}$. □

5 Contraction Mapping

Remark 5.1 Throughout this section, suppose all the assumptions on the positive vector $\rho \in \mathbb{R}^{m_s}$, the positive tensor $P \in \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s}$, and the tensor $G \in (\mathbb{R}^{m_s})^{\otimes 2} \otimes \mathbb{R}^{m_s+1}$ made in Remark 4.1 are in force. Additionally, fix a tensor $K \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{m_u} \otimes \mathbb{R}^{N_\mu}$ satisfying Condition 4.5, and a positive tensor $\bar{P} \in \mathbb{R}^{m_u} \otimes (\mathbb{R}^{m_s})^{\otimes 2}$. Assume the hypotheses of Theorems 4.4 and 4.11 are satisfied, so that both $\Psi : \mathcal{B}^{0,1}_{\rho,P} \to \mathcal{B}^{0,1}_{\rho,P}$ and $\Psi : \mathcal{B}^{1,1}_{\rho,P,\bar{P}} \to \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ are well defined maps.

5.1 Bounding the Difference Between Two Projected Systems

We show that the Lyapunov-Perron operator is a contraction mapping in an appropriate norm. Note that the norm is weaker than the one used to define $\mathcal{B}^{0,1}_{\rho,P}$ in Definition 2.8.

Definition 5.2 For $\alpha \in \mathcal{E} := \{\alpha \in Lip(B_s(\rho), X_u) : \alpha(0) = 0\}$ define the semi-norms

$$\|\alpha\|^i_{i'\mathcal{E}} := \sup_{\xi \in B_s(r); \xi_i \neq 0} \frac{|\alpha_{i'}(\xi) - \alpha_{i'}(\xi - \xi_i)|}{|\xi_i|},$$

where $i \in I$ and $i' \in I'$. The semi-norms define a norm by

$$\|\alpha\|_{\mathcal{E}} := \sum_{i \in I, i' \in I'} \|\alpha\|^i_{i'\mathcal{E}}.$$

Note that $\|\alpha\|^i_{i'\mathcal{E}} \leq Lip(\alpha)^i_{i'}$ and $|\alpha(\xi)| \leq \sum_{i' \in I'} \|\alpha\|^i_{i'\mathcal{E}} |\xi_i| \leq \|\alpha\|_{\mathcal{E}} |\xi| \big(\max_{i \in I} p_i\big)$. With this norm both $\mathcal{B}^{0,1}_{\rho,P}$ and $\mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ are complete metric spaces (cf. [15, Chapter 4]).

Before showing that Ψ is a contraction, we need to derive estimates on $x(t, \xi, \alpha) - x(t, \xi, \beta)$, the difference between two solutions of the projected system of Eq. (16) for two different maps $\alpha, \beta \in \mathcal{B}^{0,1}_{\rho,P}$. Classically, this results in an estimate of the form $|x(t, \xi, \alpha) - x(t, \xi, \beta)| \leq k e^{\gamma t} |\xi| \|\alpha - \beta\|_{\mathcal{E}}$, for some constants k and γ. This estimate can be notably tightened, as at time zero $|x(0, \xi, \alpha) - x(0, \xi, \beta)| = |\xi - \xi| = 0$. A bound on $|x(t, \xi, \alpha) - x(t, \xi, \beta)|$ is obtained below, using a tensor F as now described.

Condition 5.3 *Fix some $\gamma_{-1} > \gamma_0$ and define $\{\mu_k\}_{k=1}^{m_s+2} = \{\gamma_k\}_{k=-1}^{m_s}$. A tensor $F \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s+2}$ is said to satisfy Condition 5.3 if*

$$|x_m(t, \xi, \alpha) - x_m(t, \xi, \beta)| \leq \sum_{-1 \leq k \leq m_s} e^{\gamma_k t} F^{ni'}_{mi,k} \|\alpha - \beta\|^i_{i'\mathcal{E}} |\xi_n|,$$

for all $\alpha, \beta \in \mathcal{B}^{0,1}_{\rho,P}$ and $\xi \in B_s(\rho)$ and $m \in I$.

We obtain the tensor F by applying the bootstrapping method as in Sects. 3 and 4, which is presented in a general setting in Appendix A. However, in this section we encounter a resonance problem involving γ_0, and augment $\{\gamma_k\}_{k=0}^{m_s}$, defining

$$\gamma_{-1} := \gamma_0/2.$$

In this manner we obtain an indexed set $\{\mu_k\}_{k=1}^{N_\mu} = \{\gamma_k\}_{k=-1}^{m_s}$. The exact choice of γ_{-1} is somewhat arbitrary; it should satisfy $\lambda_{1'} > \gamma_{-1} > \gamma_0$, and $(\gamma_{-1} - \gamma_0)^{-1}$ should not be too large. We augment the tensor G fixed in Remark 4.1 by defining $G^n_{i,-1} = 0$ for all $i, n \in I$.

To overcome the resonance problem we use the map $\mathcal{Q}_0$ (following the notation convention from Appendix A) defined as

$$\mathcal{Q}_0(G)^n_{i,k} = \begin{cases} G^n_{i,0} & \text{if } k = -1 \\ 0 & \text{if } k = 0 \\ G^n_{i,k} & \text{if } 1 \le k \le m_s \end{cases} \qquad \text{for } i, n \in I. \tag{40}$$

In Proposition 5.4 and Remark 5.5 below, we identify an initial tensor $\widehat{F}$ satisfiying Condition 5.3.

Proposition 5.4 *Fix $\alpha, \beta \in \mathcal{B}^{0,1}_{\rho,P}$ and some $\gamma_{-1} > \gamma_0$. Define $\mathcal{Q}_0$ as in (40), and the tensor $\widetilde{F} \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s+2}$ as*

$$\widetilde{F}^{ni'}_{ji,k} := \begin{cases} C_s(\gamma_k - \gamma_0)^{-1} p_j C^{i'}_j \mathcal{Q}_0(G)^n_{i,k} & \text{if } k \ne 0, \\ 0 & \text{if } k = 0. \end{cases}$$

Then

$$|x(t,\xi,\alpha) - x(t,\xi,\beta)| \le \sum_{-1 \le k \le m_s, j \in I} \left(e^{\gamma_k t} - e^{\gamma_0 t}\right) \widetilde{F}^{ni'}_{ji,k} \|\alpha - \beta\|^i_{i'\mathcal{E}} |\xi_n|,$$

for all $\alpha, \beta \in \mathcal{B}^{0,1}_{\rho,P}$, and $\xi \in B_s(\rho)$.

Proof. Fix an initial condition $\xi \in B_s(\rho)$ and define $x(t) := x(t,\xi,\alpha)$ and $y(t) := x(t,\xi,\beta)$. Variation of constants gives

$$x(t) - y(t) = \int_0^t e^{(\Lambda_s + L^s_s)(t-\tau)} \Big(L^u_s \alpha(x(\tau)) + \hat{\mathcal{N}}_s(x(\tau), \alpha(x(\tau))) \\ - L^u_s \beta(y(\tau)) - \hat{\mathcal{N}}_s(y(\tau), \beta(y(\tau)))\Big)\, d\tau.$$

By the usual splitting $\alpha(x) - \beta(y) = [\alpha(x) - \alpha(y)] + [\alpha(y) - \beta(y)]$ and the definition of $\hat{\mathcal{H}}$ we obtain

$$\begin{aligned} &\left| L^u_s \alpha(x) + \hat{\mathcal{N}}_s(x, \alpha(x)) - L^u_s \beta(y) - \hat{\mathcal{N}}_s(y, \beta(y)) \right| \\ &\quad \le \hat{\mathcal{H}} |x - y| \\ &\qquad + \left| L^u_s \alpha(y) + \hat{\mathcal{N}}_s(y, \alpha(y)) - L^u_s \beta(y) - \hat{\mathcal{N}}_s(y, \beta(y)) \right|. \end{aligned}$$

Set $E^i_{i'} := \|\alpha - \beta\|^i_{i'\mathcal{E}}$. Since $|\alpha_{i'}(y) - \beta_{i'}(y)| \le E^i_{i'} |y_i|$ we have

$$\left| L^u_s \alpha(y) + \hat{\mathcal{N}}_s(y, \alpha(y)) - L^u_s \beta(y) - \hat{\mathcal{N}}_s(y, \beta(y)) \right| \le \sum_{j \in I} p_j (\hat{C}^{i'}_j + D^{i'}_j) E^i_{i'} |y_i|.$$

Combining these estimates gives

$$e^{-\lambda_s t} |x(t) - y(t)| \le \int_0^t C_s e^{-\lambda_s \tau} \sum_{j \in I} p_j C^{i'}_j E^i_{i'} |y_i(\tau)| d\tau + \int_0^t C_s e^{-\lambda_s \tau} \hat{\mathcal{H}} |x(\tau) - y(\tau)| d\tau.$$

We would like to use the bound $|y_i(\tau)| \le \sum_{0 \le k \le m_s} e^{\gamma_k \tau} G^n_{i,k} |\xi_n|$ from Theorem 3.12, and apply Lemma 3.9. However, this integral inequality has a resonance when γ_0. The problem is overcome by replacing G with $\mathcal{Q}_0(G)$, so that

$$e^{-\lambda_s t}|x(t)-y(t)| \le \int_0^t C_s e^{-\lambda_s \tau} \sum_{-1\le k\le m_s;\, j\in I} p_j C_j^{i'} E_{i'}^i e^{\gamma_k \tau} \mathcal{Q}_0(G)_{i,k}^n |\xi_n| d\tau$$
$$+ \int_0^t C_s e^{-\lambda_s \tau} \hat{\mathcal{H}} |x(\tau)-y(\tau)| d\tau.$$

By Lemma 3.9, we infer that

$$|x(t)-y(t)| \le C_s \sum_{-1\le k\le m_s;\, j\in I} \frac{e^{\gamma_k t}-e^{\gamma_0 t}}{\gamma_k-\gamma_0} p_j C_j^{i'} \mathcal{Q}_0(G)_{i,k}^n E_{i'}^i |\xi_n|. \qquad \square$$

Remark 5.5 For some fixed $\gamma_{-1} > \gamma_0$, define the tensor $\widetilde{F} \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s+2}$ as in Proposition 5.4. Define the tensor $\widehat{F} \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s+2}$ by

$$\widehat{F}_{mi,k}^{ni'} := \begin{cases} p_m \sum_{j\in I} \widetilde{F}_{ji,k}^{ni'} & \text{if } k \neq 0, \\ -p_m \sum_{j\in I} \sum_{-1\le k_1\le m_s} \widetilde{F}_{ji,k_1}^{ni'} & \text{if } k = 0. \end{cases}$$

It follows that $\widehat{F}$ satisfies Condition 5.3.

We refine the initial norm estimate from Proposition 5.4 using the following auxiliary proposition.

Proposition 5.6 *Fix $\alpha, \beta \in \mathcal{B}_{\rho,P}^{0,1}$ and an initial condition $\xi \in B_s$. Define*

$$u_i(t) := |x_i(t,\xi,\alpha) - x_i(t,\xi,\beta)|$$
$$E_{i'}^i := \|\alpha-\beta\|_{i'\mathcal{E}}^i$$
$$V_j(t) := \int_0^t e^{-\lambda_j \tau} \sum_{0\le k\le m_s} e^{\gamma_k \tau} E_{i'}^i C_j^{i'} G_{i,k}^n |\xi_n| d\tau.$$

Then

$$e^{-\lambda_j t} u_j(t) \le V_j(t) + \int_0^t e^{-\lambda_j \tau} H_j^i u_i(\tau) d\tau\,. \tag{41}$$

Proof Let $x(t) := x(t,\xi,\alpha)$ and $y(t) := x(t,\xi,\beta)$. By variation of constants we have

$$x_j(t) - y_j(t) = \int_0^t e^{\Lambda_j(t-\tau)} \big(\mathcal{N}_j(x(\tau),\alpha(x(\tau))) - \mathcal{N}_j(y(\tau),\beta(y(\tau)))\big) d\tau,$$

and the triangle inequality gives

$$|\alpha_{i'}(x) - \beta_{i'}(y)| \le |\alpha_{i'}(y) - \beta_{i'}(y)| + |\alpha_{i'}(x) - \alpha_{i'}(y)|$$
$$\le \|\alpha-\beta\|_{i'\mathcal{E}}^i |y_i| + P_{i'}^i |x_i - y_i|,$$

hence

$$\big|\mathcal{N}_j(x,\alpha(x)) - \mathcal{N}_j(y,\beta(y))\big| \le C_j^{i'} E_{i'}^i |y_i| + H_j^i |x_i - y_i|. \tag{42}$$

Applying the bound from Theorem 3.12 gives

$$e^{-\lambda_j t}|x_j - y_j| \le \int_0^t e^{-\lambda_j \tau} \left(C_j^{i'} E_{i'}^i |y_i| + H_j^i |x_i - y_i| \right) d\tau$$
$$= \int_0^t e^{-\lambda_j \tau} C_j^{i'} E_{i'}^i |y_i| d\tau + \int_0^t e^{-\lambda_j \tau} H_j^i |u_i| d\tau$$

$$\leq \int_0^t e^{-\lambda_j \tau} \sum_{0 \leq k \leq m_s} C_j^{i'} E_{i'}^i e^{\gamma_k \tau} G_{i,k}^n |\xi_n| ds + \int_0^t e^{-\lambda_j \tau} H_j^i u_i(\tau) d\tau.$$

Recalling the definition of $V_j(t)$, the above inequality is of the form stated in (41). □

Theorem 5.7 *Define $N_\lambda = m_s$ and $\{\mu_k\}_{k=1}^{N_\mu} = \{\gamma_k\}_{k=-1}^{m_s}$. Let $\widehat{F} \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s+2}$ denote the tensor defined in Remark 5.5. When F is the output of Algorithm A.5 taken with input $\widehat{F}$ and some $N_{bootstrap} \geq 1$, then F satisfies Condition 5.3.*

Proof By Proposition 5.4 the initial tensor F satisfies Condition 5.3. We note that Proposition 5.6 is a special case of Condition A.1 and Condition 5.3 is a special case of Condition A.2. Hence Proposition A.6 applies, yielding the result. □

5.2 Contraction Mapping

The tensor J below, which takes $m_s \times m_u$ matrices to $m_s \times m_u$ matrices, provides a bound on $\|\Psi[\alpha] - \Psi[\beta]\|_{i'\mathcal{E}}^{i}$.

Definition 5.8 Define the tensor $J \in (\mathbb{R}^{m_s} \otimes \mathbb{R}^{m_u})^{\otimes 2}$ by

$$J_{j'i}^{i'n} := \sum_{-1 \leq k \leq m_s} (\lambda_{j'} - \gamma_k)^{-1} \left(C_{j'}^{i'} G_{i,k}^n + H_{j'}^m F_{mi,k}^{ni'} \right). \tag{43}$$

Theorem 5.9 *If the tensor $F \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s+2}$ satisfies Condition 5.3, then $\|\Psi[\alpha] - \Psi[\beta]\|_{j'\mathcal{E}}^{n} \leq J_{j'i}^{i'n} \|\alpha - \beta\|_{i'\mathcal{E}}^{i}$ for all $\alpha, \beta \in \mathcal{B}_{\rho,P}^{0,1}$.*

Proof Fix charts $\alpha, \beta \in \mathcal{B}_{\rho,P}^{0,1}$ and choose $\xi \in B_s(\rho)$. Define $x := x(t, \xi, \alpha)$, and $y := x(t, \xi, \beta)$. By the definition of the Lyapunov-Perron operator, we have

$$\Psi[\alpha](\xi) - \Psi[\beta](\xi) = -\int_0^\infty e^{-\Lambda_u t} \left[\mathcal{N}_u(x, \alpha(x)) - \mathcal{N}_u(y, \beta(y))\right] dt.$$

Using (42) with the estimates provided in Conditions 3.4 and 5.3, we obtain

$$\begin{aligned} |\Psi[\alpha]_{j'}(\xi) - \Psi[\beta]_{j'}(\xi)| &\leq \int_0^\infty e^{-\lambda_{j'} t} \left(C_{j'}^{i'} E_{i'}^i |y_i| + H_{j'}^i |x_i - y_i| \right) dt \\ &\leq \int_0^\infty e^{-\lambda_{j'} t} \sum_{-1 \leq k \leq m_s} e^{\gamma_k t} E_{i'}^i \left(C_{j'}^{i'} G_{i,k}^n + H_{j'}^m F_{mi,k}^{ni'} \right) |\xi_n| dt. \end{aligned}$$

Integrating gives

$$|\Psi[\alpha]_{j'}(\xi) - \Psi[\beta]_{j'}(\xi)| \leq E_{i'}^i J_{j'i}^{i'n} |\xi_n|,$$

where the coefficients $J_{j'i}^{i'n}$ are defined as in (43). It follows that $\|\Psi[\alpha] - \Psi[\beta]\|_{j'\mathcal{E}}^{n} \leq E_{i'}^i J_{j'i}^{i'n}$. □

Remark 5.10 The tensor J is a linear operator which maps $m_s \times m_u$ matrices to $m_s \times m_u$ matrices. If we represent an $m_s \times m_u$ matrix E as an $m_s \cdot m_u$ dimensional vector $\tilde{E}$ with components $\tilde{E}_{(i'-1)m_s+i} = E_{i'}^i$, then the action of J can be represented as a $m_s m_u \times m_s m_u$ matrix $\tilde{J}$ with components $\tilde{J}_{(j'-1)m_s+n}^{(i'-1)m_s+i} \equiv J_{j'i}^{i'n}$.

We are principally interested in whether the Lyapunov-Perron operator Ψ has a unique fixed point. By Theorem 5.9, this will be true if an iterative application of J to any $m_s \times m_u$ matrix E limits to zero, that is

$$\lim_{k\to\infty} \underbrace{J \circ \cdots \circ J}_{k} \cdot E = 0.$$

This limits to zero if and only if the spectral radius of J, denoted by $\rho(J)$, is less than 1. Since J is finite dimensional, $\rho(J)$ is equal to the absolute value of the eigenvalue with largest magnitude. This is bounded as $\rho(J) \leq \|J^k\|^{1/k}$ for any positive integer $k \geq 1$, and any matrix norm $\|\cdot\|$.

The theorem below collects the major results thus far.

Theorem 5.11 *Take the assumptions made in Remarks 4.1 and 5.1. Suppose the tensor* $F \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s+2}$ *satisfies Condition 5.3 and define* $J \in (\mathbb{R}^{m_s} \otimes \mathbb{R}^{m_u})^{\otimes 2}$ *as in Definition 5.8. If the spectral radius of* J *is less than* 1*, then there exists a unique fixed point* $\alpha \in \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ *for which* $\Psi[\alpha] = \alpha$*. Furthermore, the graph*

$$M_{loc} := \{(\mathrm{x}_s, \alpha(\mathrm{x}_s)) \in X_s \times X_u : \mathrm{x}_s \in B_s(\rho)\}$$

is an invariant manifold under the flow (3), and points in M_{loc} *converge asymptotically to* 0*.*

In addition, suppose that $\tilde{h}$ *is an equilibrium solution to (2) satisfying* $|\tilde{h}_{\mathbf{i}}| < \epsilon_{\mathbf{i}}$ *for* $\mathbf{i} \in \mathbf{I}$*, and that* $\epsilon_i < \rho_i$ *for* $i \in I$*. Define* $\tilde{\alpha}(\mathrm{x}_s) := \alpha(\mathrm{x}_s - \tilde{h}_s) + \tilde{h}_u$*. The graph*

$$\widetilde{M}_{loc} := \{(\mathrm{x}_s, \tilde{\alpha}(\mathrm{x}_s)) \in X_s \times X_u : \mathrm{x}_s \in B_s(\rho - \epsilon_s)\}$$

is an invariant manifold under the flow (2), and points in $\widetilde{M}_{loc}$ *converge asymptotically to* $\tilde{h}$*. Moreover, we have the estimates*

$$|\tilde{\alpha}_{i'}(\mathrm{x}_s)| \leq P^i_{i'}(|\mathrm{x}_i| + \epsilon_i) + \epsilon_{i'} \qquad \|\tilde{\alpha}^i_{i'}(\mathrm{x}_s)\| \leq P^i_{i'} \qquad \mathrm{Lip}(\partial_i \tilde{\alpha})^j_{i'} \leq \bar{P}^{ij}_{i'},$$

for all $\mathrm{x}_s \in B_s(\rho - \epsilon_s)$ *and* $i, j \in I$ *and* $i' \in I'$*.*

Proof We infer from the assumptions made in Remarks 4.1 and 5.1, all of which can be verified *a posteriori*, that the map $\Psi : \mathcal{B}^{1,1}_{\rho,P,\bar{P}} \to \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ is a well defined endomorphism. Since the spectral radius of J is less than 1, there exists a unique fixed point $\alpha \in \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ for which $\Psi[\alpha] = \alpha$, see Remark 5.10. As discussed in Sect. 2.3, the fixed point of the Lyapunov-Perron operator provides us with a chart for a local invariant manifold for the differential equation defined in (3). By construction $\alpha(0) = 0$, hence the origin is contained in the manifold. It follows from the proof of Proposition 3.13 that points in M_{loc} converge asymptotically to the origin.

As (3) is conjugate to (2) via the change of variables $\mathrm{x} \to \mathrm{x} + \tilde{h}$, it follows that $\tilde{\alpha}(\mathrm{x}_s)$ is a graph for a local invariant manifold (having a slightly smaller domain) for the differential equation defined in (2). Furthermore this manifold contains the equilibrium $\tilde{h}$, a point to which trajectories in $\widetilde{M}_{\mathrm{loc}}$ are asymptotically attracted. The error estimates follow by virtue of $\alpha \in \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$. □

As discussed at the end of in Sect. 2.3, the fixed point of the Lyapunov-Perron operator provides us with a chart for the local stable manifold provided we have captured all stable eigenvalues.

6 Application I: Linear Change of Variables

6.1 The Swift-Hohenberg Equation

Consider the Swift-Hohenberg Eq. (1) of Sect. 1.1. Since the boundary conditions are Neumann, we will expand the spatial variable using Fourier cosine series. Proceeding formally (we do not yet specify the norms) define the space of one-sided sequence of real numbers, denoted $Y = \mathbb{R}^{\mathbb{N}}$. Given a one parameter curve $a \in C(\mathbb{R}, Y)$, define a path of Fourier cosine series by

$$u(t, x) = a_0(t) + 2\sum_{k=1}^{\infty} a_k(t)\cos(kx).$$

Taking the expansion above as an *ansatz*, and plugging it into Eq. (1) leads to the system of infinitely many coupled scalar ordinary differential equations

$$\dot{a}_k = (-\beta_1 k^4 - \beta_2 k^2 + 1)a_k - (a * a * a)_k. \tag{44}$$

Here, the discrete convolution $*$ for $a, b \in Y$ is defined by

$$(a * b)_k = \sum_{\substack{k_1+k_2=k \\ k_1,k_2\in\mathbb{Z}}} a_{|k_1|} b_{|k_2|}.$$

We endow Y with the "analytic" norm corresponding to cosine series with geometrically decaying coefficients. So, for $a \in Y$ let

$$|a|_{\ell^1_\nu} := \sum_{k=0}^{\infty} |a_k|\omega_k(\nu),$$

where

$$\omega_k(\nu) = \omega_k := \begin{cases} 1 & k = 0 \\ 2\nu^k & k \geq 1. \end{cases}$$

With $\nu > 1$ define

$$\ell^1_\nu = \left\{ a \in Y \;:\; |a|_{\ell^1_\nu} < \infty \right\},$$

and note that ℓ^1_ν is a commutative Banach algebra, in the sense that

$$\|a * b\|^1_\nu \leq \|a\|^1_\nu \|b\|^1_\nu, \qquad \text{for all } a, b \in \ell^1_\nu.$$

We rewrite (44) as a (densely defined) vector field $F \colon \ell^1_\nu \to \ell^1_\nu$ given by

$$F(a) := \mathfrak{L}a - a * a * a, \tag{45}$$

where $\mathfrak{L}$ is the diagonal linear operator

$$\mathfrak{L}(a)_k := (-\beta_1 k^4 - \beta_2 k^2 + 1)a_k, \qquad \text{for all } k \geq 0. \tag{46}$$

Fix some $N \in \mathbb{N}$ and define a Galerkin projection $\pi_N : \ell^1_\nu \to \mathbb{R}^{N+1} \subseteq \ell^1_\nu$ by

$$\pi_N(a) := (a_0, a_1 \ldots a_{N-1}, a_N, 0, 0, 0, \ldots). \tag{47}$$

We define the Galerkin projection of F by $F_N := \pi_N \circ F \circ \pi_N$.

Remark 6.1 (**Normal form**) To enter into the notational framework established in Sect. 2 we define a change of variables conjugating the differential Eq. (44) to one of the type given in Eq. (2). Note that (45) has the desired form at the homogeneous equilibrium solution $0 \in \ell^1_\nu$, but that a change of variables is required when a is non-trivial. After performing the change of variables, we will bound the constants needed to satisfy the hypotheses of Theorem 5.11.

Remark 6.2 (**First order data**) We exploit the extensive literature on computer assisted proofs for equilibrium solutions to partial differential equations, and provide computer assisted proofs for the existence, local uniqueness, and bounds on the accuracy of the numerical approximation. Such techniques rely on solving the finite dimensional problem $F_N(\bar{a}) = 0$, and use an implicit function type argument to show that there is a point $\tilde{a} \in \ell^1_\nu$ close to $\bar{a}$ for which $F(\tilde{a}) = 0$. We use the techniques described in [35,66]. Similar ideas are used to solve the linearized equations at $\tilde{a}$, providing enclosures of the necessary eigendata. The Morse index of the stationary point $\tilde{a}$, denoted n_u, is established rigorously using a straightforward implementation based on the ideas and techniques from [67,69].

In a more theoretical setting we would use the sectorial nature of $\mathcal{L}$ to decompose ℓ^1_ν as a Cartesian product of eigenspaces of $DF(\tilde{a})$. In the more constructive setting of the present work we do not have direct access to this data. Instead, we numerically compute approximate eigenspaces associated with the Galerkin projection. Suppose then that $A^\dagger_N \in \mathrm{Mat}(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ is a matrix of real numbers having that $A^\dagger_N \approx DF_N(\bar{a})$.

Assume for the moment (this assumption will have to be checked in practice) that $A^\dagger_N$ has n_u unstable eigenvalues (i.e. it captures the correct Morse index, see Remark 6.2). Let $\{\mu_{k'}\}_{k'=1'}^{n'_u}$ denote positive numbers approximating the unstable eigenvalues of $A^\dagger_N$, and $\{\mu_k\}_{k=1}^{n_f}$ with $n_f = N + 1 - n_u$ denote negative numbers approximating the stable eigenvalues. Without loss of generality, suppose that these numbers are ordered as

$$\mu_{n'_u} \geq \cdots \geq \mu_{1'} > 0 > \mu_1 \geq \cdots \geq \mu_{n_f}.$$

Remark 6.3 (**Gradient structure**) The Swift-Hohenberg PDE is a gradient system, hence $A^\dagger_N$ has real eigenvalues with $N+1$ linearly independent eigenvectors. Indeed, this is most easily established by working with the slightly adapted $\widetilde{F}$ rather than F directly, where

$$\widetilde{F}(a)_k = \begin{cases} F(a)_0/2 & \text{for } k = 0, \\ F(a)_k & \text{for } k \geq 1, \end{cases}$$

so that $D\widetilde{F}_N(\bar{a})$ is symmetric with respect to the standard inner product on $\mathbb{R}^{N+1}$. However, this is a minor technical point.

Consider now the Swift-Hohenberg equation at parameter values such that $m_u = 1$, and choose a decomposition of the stable eigenspace having $m_s = 2$. We decompose X into subspaces

$$X_{1'} := \mathbb{R}^{n'_u} \qquad X_1 := \mathbb{R}^{n_f} \qquad X_2 := \{a \in \ell^1_\nu : a_k = 0 \text{ for } k \leq N\},$$

and have that $X_u := X_{1'}$ and $X_s := X_1 \times X_2$ and $X = X_u \times X_s$. We sometimes employ the notational shorthand $X_f := X_1$ and $X_\infty := X_2$.

Note that the map π_N defined in (47), is the projection $\pi_N : X \to X_N \subseteq X$ where $X_N := X_{1'} \times X_1 \cong \mathbb{R}^{N+1}$. Define $\pi_\infty : X \to X_\infty$ by $\pi_\infty x := x - \pi_N x$. A Schauder basis $\{\hat{e}_n\}_{n\in\mathbb{N}}$ for X is given by

$$X_{1'} := \mathrm{span}\{\hat{e}_0, \ldots, \hat{e}_{n_u-1}\} \quad X_1 := \mathrm{span}\{\hat{e}_{n_u}, \ldots, \hat{e}_N\} \quad X_2 := \overline{\mathrm{span}\{\hat{e}_{N+1}, \hat{e}_{N+2}, \ldots\}},$$

so that every $\phi \in X$ has a unique representation $\phi = \sum_{n=0}^{\infty} \phi_n \hat{e}_n$.

We are now ready to construct a linear change of variables from X to ℓ_ν^1. Fix $Q_u \in \mathrm{Mat}(\mathbb{R}^{n_u}, \mathbb{R}^{N+1})$ and $Q_f \in \mathrm{Mat}(\mathbb{R}^{n_f}, \mathbb{R}^{N+1})$ as matrices whose columns are numerical approximations of unstable/stable eigenvectors of $A_N^\dagger$. For $\phi = (\phi_u, \phi_f, \phi_\infty) \in X_u \times X_f \times X_\infty$, define the linear map $Q : X \to \ell_\nu^1$ by

$$Q(\phi) = Q_u\phi_u + Q_f\phi_f + \phi_\infty. \tag{48}$$

We endow X with a Banach space structure as follows. Let $\phi^N = \pi_N\phi$ and let Q^N be the $(N+1)\times(N+1)$ invertible matrix given by $Q^N = [Q_u, Q_f]$. Define the transformation $Q\colon X \to \ell_\nu^1$ by

$$[Q\phi]_n = \begin{cases} [Q^N\phi^N]_n & 0 \le n \le N, \\ \phi_n & n > N+1, \end{cases}$$

for $\phi \in X$. Denote the columns of Q by q_n, $n \in \mathbb{N}$. Note that $q_n = e_n$ when $n \ge N+1$ and that $q_n = Q_n^N$, the n-the column of Q^N, for $0 \le n \le N$. Define the norm on X by

$$\begin{aligned} |\phi|_X &:= \sum_{n=0}^{N} \left|\phi_n Q\hat{e}_n\right|_{\ell_\nu^1} \\ &= \sum_{n=0}^{N} |\phi_n||q_n|_{\ell_\nu^1} + \sum_{n=N+1}^{\infty} |\phi_n|\omega_n \\ &= \sum_{n=0}^{N} |\phi_n||q_n|_{\ell_\nu^1} + |\phi_\infty|_{\ell_\nu^1}. \end{aligned} \tag{49}$$

Note that $|\phi|_X = \sum_{\mathbf{i}\in\mathbf{I}} |\phi_{\mathbf{i}}|$ for $\phi \in X$, so that with this norm, X satisfies the hypotheses of Proposition 2.10.

We also require explicit formulas for the induced norms on several collections of operators in $\mathcal{L}(X, X)$, $\mathcal{L}(X, \ell_\nu^1)$ and $\mathcal{L}(\ell_\nu^1, X)$. Suppose that M^N is a $(N+1)\times(N+1)$ matrix and define the linear operator $M\colon X \to X$ by

$$[M\phi]_n = \begin{cases} [M^N\phi^N]_n & 0 \le n \le N, \\ 0 & n \ge N+1. \end{cases}$$

A standard calculation shows that

$$\|M\|_{\mathcal{L}(X,X)} = \sup_{|\phi|_X=1} |M\phi|_X \le \max_{0\le k\le N} \frac{|M_k^N|_X}{|q_k|_{\ell_\nu^1}}, \tag{50}$$

where M_k^N denotes the k-th column of M^N. Similarly, for Ω^N an $(N+1)\times(N+1)$ matrix define the linear operator $\Omega\colon X \to \ell_\nu^1$ by

$$[\Omega\phi]_n = \begin{cases} [\Omega^N\phi^N]_n & 0 \le n \le N, \\ \phi_n & n \ge N+1. \end{cases}$$

Again, a standard calculation shows that

$$\|\Omega\|_{\mathcal{L}(X,\ell_\nu^1)} = \sup_{|\phi|_X=1} |\Omega\phi|_{\ell_\nu^1} \le \max\left(\max_{0\le k\le N} \frac{|\Omega_k^N|_{\ell_\nu^1}}{|q_k|_{\ell_\nu^1}}, 1\right), \tag{51}$$

where Ω_k^N denotes the k-th column of Ω^N. From this it follows that $\|Q\|_{\mathcal{L}(X,\ell_\nu^1)} = 1$.

To compute the norm of $Q^{-1} : \ell_\nu^1 \to X$, let B^N denote the matrix inverse of Q^N. The action of Q^{-1} is expressed as

$$\left[Q^{-1}a\right]_n = \begin{cases} \left[B^N a^N\right]_n & 0 \le n \le N, \\ a_n & n \ge N+1. \end{cases}$$

Then

$$\|Q^{-1}\|_{\mathcal{L}(\ell_\nu^1, X)} = \sup_{|a|_{\ell_\nu^1}=1} |Q^{-1}a|_X \le \max\left(\max_{0\le k\le N} \frac{|B_k^N|_X}{\omega_k}, 1 \right). \tag{52}$$

Now, for any $\mathbf{i} \in \mathbf{I}$, we define projection maps $\pi_{\mathbf{i}} : X \to X_{\mathbf{i}}$. Again, π_∞ coincides with its usual definition. By our choice of norm on X, we have $\|\pi_{\mathbf{i}}\|_{\mathcal{L}(X,X_{\mathbf{i}})} = 1$. Recalling the definitions of $p_u, p_s, p_{\mathbf{i}}$ in Eq. (5), we have that $p_u = p_s = p_{\mathbf{i}} = 1$. Lastly, we define Λ by

$$\Lambda_{1'} := \operatorname{diag}\{\mu_{n_u}, \ldots, \mu_{1'}\}, \qquad \Lambda_1 := \operatorname{diag}\{\mu_1, \ldots, \mu_{n_f}\}, \qquad \Lambda_2 := \mathfrak{L} \circ \pi_\infty.$$

We show that the norm on X, as defined above, is well aligned with the semigroup $e^{\Lambda t}$. Fix a point $\phi = (\phi_u, \phi_f, \phi_\infty) \in X$ and write $\phi_u = (\phi_0, \ldots, \phi_{n_u-1})$ and $\phi_f = (\phi_{n_u}, \ldots, \phi_N)$ and $\phi_\infty = (\phi_{N+1}, \phi_{N+2}, \ldots)$. Then for $t \in \mathbb{R}$ we have

$$\begin{aligned} e^{\Lambda_{1'}t}\phi_u &= \sum_{1\le k\le n_u} e^{\mu_{k'}t}\phi_{k-1}\hat{e}_{k-1}, \\ e^{\Lambda_1 t}\phi_f &= \sum_{1\le k\le n_f} e^{\mu_k t}\phi_{k+n_u-1}\hat{e}_{k+n_u-1}, \\ e^{\Lambda_2 t}\phi_\infty &= \sum_{k=N+1}^{\infty} e^{(-\beta_1 k^4-\beta_2 k^2+1)t}\phi_k\hat{e}_k. \end{aligned}$$

Define $\lambda_{1'}$, λ_1, and λ_2 as

$$\lambda_{1'} := \operatorname{Re}\mu_{1'}, \qquad \lambda_1 := \operatorname{Re}\mu_1, \qquad \lambda_2 := -\beta_1(N+1)^4 - \beta_2(N+1)^2 + 1. \tag{53}$$

It follows that $\lambda_{1'} \le \operatorname{Re}\mu_{k'}$ for $1' \le k' \le n_u'$, and $\lambda_1 \ge \operatorname{Re}\mu_k$ for $1 \le k \le n_f$, and $\lambda_2 \ge (-\beta_1 k^4 - \beta_2 k^2 + 1)$ for $k \ge N+1$. Choose N sufficiently large so that $-\beta_1 k^4 - \beta_2 k^2 + 1$ is negative and decreasing for $k \ge N+1$. Then

$$\begin{aligned} |e^{\Lambda_{1'}t}\phi_u|_X &\le \sum_{0\le k\le n_u-1} e^{\lambda_{1'}t}|Q\phi_k|_{\ell_\nu^1}, && \text{for } t \le 0, \\ |e^{\Lambda_1 t}\phi_f|_X &\le \sum_{n_u\le k\le N} e^{\lambda_1 t}|Q\phi_k|_{\ell_\nu^1}, && \text{for } t \ge 0, \\ |e^{\Lambda_2 t}\phi_\infty|_X &\le \sum_{k=N+1}^{\infty} e^{\lambda_2 t}|Q\phi_k|_{\ell_\nu^1} && \text{for } t \ge 0. \end{aligned}$$

From Eq. (49), we have that (6) and (7) are satisfied.

6.2 Bounds for the Linear Change of Coordinates

The estimates necessary for completing the argument are obtained following the instructions outlined below, which summarizes the discussion of the previous sections.

1. For $U \subset X$, define a change of variables $K : U \to \ell^1_\nu$ such that $K(0) = \bar{a}$. For the equilibrium $\tilde{h} = K^{-1}(\tilde{a})$, obtain bounds $|\pi_{\mathbf{i}}\tilde{h}| \leq \epsilon_{\mathbf{i}}$ for $\mathbf{i} \in \mathbf{I}$.
2. Pull back the vector field from ℓ^1_ν to U, creating the conjugate differential equation

$$\dot{\mathrm{x}} = DK(\mathrm{x})^{-1}F(K(\mathrm{x})).$$

   Define $\tilde{\mathcal{N}} \in C^2_{loc}(U, X)$ as $\tilde{\mathcal{N}}(\mathrm{x}) := DK(\mathrm{x})^{-1}F(K(\mathrm{x})) - \Lambda\mathrm{x}$.
3. Obtain constants $\tilde{C}^{\mathbf{ik}}_{\mathbf{j}}(r_s, r_u)$ which bound $\|\tilde{\mathcal{N}}^{\mathbf{ik}}_{\mathbf{j}}\|_{(r_s+\epsilon_s, r_u+\epsilon_u)}$ for $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbf{I}$.
4. Obtain constants $\tilde{D}^{\mathbf{i}}_{\mathbf{j}}$ which bound $\|\tilde{\mathcal{N}}^{\mathbf{i}}_{\mathbf{j}}(0)\|$ for $\mathbf{i}, \mathbf{j} \in \mathbf{I}$.
5. Obtain constants C_s, λ_s which satisfy Eq. (8) to bound $e^{(\Lambda_s + L^s_s)t}$.

In the remainder of this section we explain how to follow the outline above, arriving at a linear change of coordinates K. The results of the a calculation are presented in Sect. 6.3.

6.2.1 Estimate 1: Defining a Change of Variables

Define the affine change of coordinates $K : X \to \ell^1_\nu$ by

$$K(\phi) := \bar{a} + Q\phi. \tag{54}$$

Let $|\bar{a} - \tilde{a}|_{\ell^1_\nu} \leq \epsilon$ be a bound on the distance between the approximate solution and true equilibrium solutions, and define $\epsilon_{\mathbf{i}} := \epsilon \|\pi_{\mathbf{i}} Q^{-1}\|_{\mathcal{L}(\ell^1_\nu, X_{\mathbf{i}})}$ for $\mathbf{i} \in \mathbf{I}$ as needed in Proposition 2.6.

6.2.2 Estimate 2: Defining the Conjugate Differential Equation

Applying the change of coordinates defined in (54) to the Swift-Hohenberg equation leads to

$$\dot{\phi} = \Lambda\phi + \tilde{\mathcal{N}}(\phi) \qquad \text{with} \qquad \tilde{\mathcal{N}}(\phi) := DK(\phi)^{-1}F(K(\phi)) - \Lambda\phi. \tag{55}$$

We note that the form of $\tilde{\mathcal{N}}$ as given is not easy to work with, and expand $\tilde{\mathcal{N}}$ into an affine part and a purely nonlinear part. Define functions $E, R : X \to \ell^1_\nu$ as

$$E(\phi) := F(\bar{a}) + DF(\bar{a})Q\phi - Q\Lambda\phi, \qquad R(\phi) := -3\bar{a} * (Q\phi)^{*2} - (Q\phi)^{*3}.$$

Then $E + R = F \circ K - DK \cdot \Lambda$, where $DK(\phi) = Q$ for all $\phi \in X$. It follows that $\tilde{\mathcal{N}}(\phi) = Q^{-1}(E(\phi) + R(\phi))$.

6.2.3 Estimate 3: Bounding $\tilde{\mathcal{N}}^{\mathbf{ij}}_{\mathbf{k}}$

All second derivatives of E are zero. Hence $\partial_{\mathbf{i}}\partial_{\mathbf{j}}\pi_{\mathbf{k}}\tilde{\mathcal{N}} = \tilde{\mathcal{N}}^{\mathbf{ij}}_{\mathbf{k}} = (Q^{-1}R)^{\mathbf{ij}}_{\mathbf{k}}$ for $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbf{I}$. For $\phi \in X$, define

$$\mathbf{Q} := Q\phi = Q_f\phi_f + Q_u\phi_u + \phi_\infty, \tag{56}$$

and note that each term in R itself contains a term of the form $\mathbf{Q} * \mathbf{Q}$. Set

$$\mathbf{Q}^2 := \mathbf{Q} * \mathbf{Q} \qquad \text{and} \qquad \mathbf{Q}^3 := \mathbf{Q} * \mathbf{Q} * \mathbf{Q}.$$

Then $R(\phi) = -3\bar{a} * \mathbf{Q}^2 - \mathbf{Q}^3$.

The derivatives of $\mathbf{Q}$ are

$$\partial_f\mathbf{Q} \cdot h_f = Q_f h_f, \qquad \partial_u\mathbf{Q} \cdot h_u = Q_u h_u, \qquad \partial_\infty\mathbf{Q} \cdot h_\infty = h_\infty,$$

where $h_f \in X_f$, $h_u \in X_u$ and $h_\infty \in X_\infty$. Since $\|Q\|_{\mathcal{L}(X,\ell^1_\nu)} = 1$, we have $\|\partial_{\mathbf{i}}\mathbf{Q}\|_{\mathcal{L}(X,\ell^1_\nu)} = 1$ for $\mathbf{i} \in \mathbf{I}$. As $\partial_{\mathbf{i}}\mathbf{Q}$ is a linear operator, the second derivatives $\partial_{\mathbf{ij}}\mathbf{Q}$ vanish for all $\mathbf{i}, \mathbf{j} \in \mathbf{I}$.

The derivatives of $\mathbf{Q}^2$ and $\mathbf{Q}^3$ are given by

$$\partial_{\mathbf{ij}}\mathbf{Q}^2 = 2\partial_{\mathbf{i}}\mathbf{Q} * \partial_{\mathbf{j}}\mathbf{Q} \qquad \text{and} \qquad \partial_{\mathbf{ij}}\mathbf{Q}^3 = 6\mathbf{Q} * \partial_{\mathbf{i}}\mathbf{Q} * \partial_{\mathbf{j}}\mathbf{Q},$$

so that

$$\partial_{\mathbf{ij}} R = -6(\bar{a} + \mathbf{Q}) * \partial_i \mathbf{Q} * \partial_j \mathbf{Q}.$$

Recall that $\|\partial_{\mathbf{i}}\mathbf{Q}\|_{\mathcal{L}(X,\ell^1_\nu)} = 1$ for all $\mathbf{i} \in \mathbf{I}$. Fixing $\phi = (\phi_u, \phi_s) \in B_u(r_u) \times B_s(r_s)$ with $r_s = (r_f, r_\infty)$ gives $|Q\phi| \le r_u + r_f + r_\infty$. Define

$$C^{\mathbf{ij}}_{\mathbf{k}} := 6\|\pi_{\mathbf{k}} Q^{-1}\|_{\mathcal{L}(\ell^1_\nu, X)} \left(|\bar{a}| + r_u + r_f + r_\infty + \epsilon_u + \epsilon_f + \epsilon_\infty\right). \tag{57}$$

Then $\|\tilde{\mathcal{N}}^{\mathbf{ij}}_{\mathbf{k}}\|_{(r_s+\epsilon_s, r_u+\epsilon_u)} \le C^{\mathbf{ij}}_{\mathbf{k}}$ for $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbf{I}$.

6.2.4 Estimate 4: Bounding $\tilde{\mathcal{N}}^{\mathbf{i}}_{\mathbf{j}}(0)$

Since $\partial_{\mathbf{i}} R(0) = 0$ and $\partial_\phi DK(\phi)^{-1} E(\phi) = Q^{-1} DF(\bar{a}) Q - \Lambda$, we have

$$\tilde{\mathcal{N}}^{\mathbf{i}}_{\mathbf{j}}(0) = \pi_{\mathbf{j}} \left(Q^{-1} DF(\bar{a}) Q - \Lambda\right) \pi_{\mathbf{i}}.$$

Approximate $DF(\bar{a})$ by the operator $A^\dagger : \ell^1_\nu \to \ell^1_\nu$ defined by

$$(A^\dagger v)_k := \begin{cases} (A^\dagger_N v)_k & k \le N \\ (\mathfrak{L} v)_k & k > N, \end{cases}$$

for $v \in \ell^1_\nu$. We bound $\tilde{\mathcal{N}}^{\mathbf{i}}_{\mathbf{j}}(0)$ by adding and subtracting $Q^{-1} A^\dagger Q$ to obtain

$$\left\|\tilde{\mathcal{N}}^{\mathbf{i}}_{\mathbf{j}}(0)\right\|_{\mathcal{L}(X,X)} \le \left\|\pi_{\mathbf{j}} Q^{-1} \left(DF(\bar{a}) - A^\dagger\right) Q \pi_{\mathbf{i}}\right\|_{\mathcal{L}(X,X)} + \left\|\pi_{\mathbf{j}} \left(Q^{-1} A^\dagger Q - \Lambda\right) \pi_{\mathbf{i}}\right\|_{\mathcal{L}(X,X)}. \tag{58}$$

To bound the right summand in (58), note that $\pi_{\mathbf{j}} \left(Q^{-1} A^\dagger Q - \Lambda\right) \pi_{\mathbf{i}}$ vanishes when either $\mathbf{i} = \infty$ or $\mathbf{j} = \infty$, hence the right-summand in (58) is computed directly using (50). The left summand in (58) is bounded by considering four cases, depending on whether $\mathbf{i}$ or $\mathbf{j}$ equals ∞. Each of these terms involves

$$\left(DF(\bar{a})h - A^\dagger h\right)_k = \begin{cases} -3(\bar{a} * \bar{a} * \pi_\infty h)_k + ((DF_N(\bar{a}) - A^\dagger_N)\pi_N h)_k & 0 \le k \le N \\ -3(\bar{a} * \bar{a} * h)_k & k \ge N+1. \end{cases} \tag{59}$$

For the case $\mathbf{i} = \infty$ and $\mathbf{j} = \infty$, since ℓ^1_ν is a Banach algebra and π_∞ projects onto the modes $k \ge N+1$, we use (59) and obtain

$$\left|\pi_\infty \left(DF(\bar{a}) - A^\dagger\right) h\right| \le 3|\bar{a} * \bar{a}|_{\ell^1_\nu} |h|_{\ell^1_\nu}.$$

Hence $\left\|\pi_\infty \left(DF(\bar{a}) - A^\dagger\right)\right\|_{\mathcal{L}(\ell^1_\nu, \ell^1_\nu)} \le 3|\bar{a} * \bar{a}|_{\ell^1_\nu}$. Define

$$\tilde{D}^{\mathbf{i}}_\infty := 3|\bar{a} * \bar{a}|_{\ell^1_\nu}, \tag{60}$$

so that $\|\tilde{\mathcal{N}}^{\mathbf{i}}_\infty(0)\|_{\mathcal{L}(X,X)} \le \tilde{D}^{\mathbf{i}}_\infty$ for all $\mathbf{i} \in \mathbf{I}$.

For the case $\mathbf{i} \neq \infty$ and $\mathbf{j} \neq \infty$, we note that the operator $\pi_{\mathbf{j}} \left(Q^{-1} DF(\bar{a}) Q - \Lambda \right) \pi_{\mathbf{i}}$ is represented by an $(N+1) \times (N+1)$ matrix and explicitly bound the norm. Define

$$\tilde{D}^{\mathbf{i}}_{\mathbf{j}} := \left\| \pi_{\mathbf{j}} \left(Q^{-1} DF(\bar{a}) Q - \Lambda \right) \pi_{\mathbf{i}} \right\|_{\mathcal{L}(X,X)}. \tag{61}$$

It follows that $\|\tilde{\mathcal{N}}^{\mathbf{i}}_{\mathbf{j}}(0)\| \leq \tilde{D}^{\mathbf{i}}_{\mathbf{j}}$ for all $\mathbf{i}, \mathbf{j} \in \mathbf{I} - \{\infty\}$.

For the case $\mathbf{i} = \infty$ and $\mathbf{j} \neq \infty$, it follows from (59) that

$$\pi_{\mathbf{j}} [DF(\bar{a}) - A^{\dagger}]_k = 0 \qquad \text{for } k > 3N,$$

where we recall that the subscript k denotes the k-th column. Since $Q\pi_\infty = \pi_\infty$, using the appropriate analogue of (50) for a matrix of a larger size, we set

$$\tilde{D}^{\infty}_{\mathbf{j}} := \max_{N+1 \leq k \leq 3N} \frac{|\pi_{\mathbf{j}} Q^{-1} [DF(\bar{a}) - A^{\dagger}]_k|_X}{\omega_k}. \tag{62}$$

It follows that $\|\tilde{\mathcal{N}}^{\infty}_{\mathbf{j}}(0)\| \leq \tilde{D}^{\infty}_{\mathbf{j}}$ for all $\mathbf{j} \in \mathbf{I} - \{\infty\}$.

For the case $\mathbf{i} \neq \infty$ and $\mathbf{j} = \infty$, we note that since $\pi_\infty Q^{-1} = \pi_\infty$ and $\pi_\infty A^{\dagger} \pi_N = 0$, we have

$$\pi_{\mathbf{j}} Q^{-1} \left(DF(\bar{a}) - A^{\dagger} \right) Q \pi_{\mathbf{i}} = \pi_\infty DF(\bar{a}) Q \pi_{\mathbf{i}}.$$

Recalling the formula in (50), we set

$$\tilde{D}^{\mathbf{i}}_{\infty} := \max_{0 \leq k \leq N} \frac{|[\pi_\infty DF(\bar{a}) Q \pi_{\mathbf{i}}]_k|_X}{|q_k|_{\ell^1_\nu}}. \tag{63}$$

It follows that $\|\tilde{\mathcal{N}}^{\infty}_{\mathbf{j}}(0)\| \leq \tilde{D}^{\infty}_{\mathbf{j}}$ for all $\mathbf{j} \in \mathbf{I} - \{\infty\}$. With $\tilde{D}^{\mathbf{i}}_{\mathbf{j}}$ as in Eqs. (60), (61), (62) and (63), we have bounds on $\|\tilde{\mathcal{N}}^{\mathbf{i}}_{\mathbf{j}}(0)\|_{\mathcal{L}(X,X)}$ for all $\mathbf{i}, \mathbf{j} \in \mathbf{I}$.

6.2.5 Estimate 5: Semigroup Bounds

To find C_s and λ_s as needed in (8), we use Proposition B.1 and Remark B.3. Define $D^{\mathbf{i}}_{\mathbf{j}} := \tilde{D}^{\mathbf{i}}_{\mathbf{j}} + \tilde{C}^{\mathbf{i}l}_{\mathbf{j}} \epsilon_l + \tilde{C}^{\mathbf{i}l'}_{\mathbf{j}} \epsilon_{l'}$ for $\mathbf{i}, \mathbf{j} \in \mathbf{I}$ as in Proposition 2.6, and let

$$\begin{aligned}
&\mu_1 := \lambda_1 && \delta_a := D^f_f && \delta_b := D^\infty_f \\
&\mu_\infty := \lambda_2 = \lambda_\infty && \delta_c := D^f_\infty && \delta_d := D^\infty_\infty && \varepsilon := \sum_{\tilde{\mu}_k \in \sigma(\Lambda_1)} \frac{|\mu_\infty|^{-1}}{1 - |\mu_\infty|^{-1}(\delta_d + |\tilde{\mu}_k|)}.
\end{aligned}$$

Note that $\|\Lambda_\infty^{-1}\| = |\mu_\infty|^{-1}$. Assume that the spectral gap conditions

$$1 > |\mu_\infty|^{-1} \Big(\delta_d + \sup_{\tilde{\mu}_k \in \sigma(\Lambda_1)} |\tilde{\mu}_k| \Big), \qquad \mu_1 > \mu_\infty + \delta_d + \varepsilon \delta_b \delta_c (1 + \varepsilon^2 \delta_b \delta_c), \tag{64}$$

are satisfied. (These must be checked in explicit examples). It then follows from Proposition B.1 and Remark B.3 that

$$\|e^{(\Lambda_s + L^s_s)t}\| \leq C_s e^{\lambda_s t},$$

where

$$\begin{aligned}
C_s &:= (1 + \varepsilon \delta_b)^2 (1 + \varepsilon \delta_c)^2 \\
\lambda_s &:= \mu_1 + \delta_a C_s + \Delta \\
\Delta &:= \varepsilon \delta_b \delta_c \max \left\{ 1 + \varepsilon \delta_c (1 + \varepsilon \delta_b), \varepsilon \delta_b (2 + \varepsilon^2 \delta_b \delta_c) \right\}.
\end{aligned}$$

6.3 Numerical Results

Following the steps given in Sect. 6.2 allows us to prove a variety of stable manifold theorems. In Theorem 6.4 below we present one such result, for the equilibrium displayed in Fig. 1. Here we choose ρ_f, the radius of the domain $B_s(\rho) \subseteq X_f \times X_\infty$ projected into the finite dimensional subspace X_f , as large as possible. A number of additional results are presented in Sect. 7.6.

Theorem 6.4 *Consider the Swift-Hohenberg Eq. (1) with parameters $\beta_1 = 0.05$, and $\beta_2 = -0.35$. Let $\nu = 1.001$ and suppose that $\bar{a} \in \ell^1_\nu$ is an approximate equilibrium solution, $\epsilon = 1.61 \times 10^{-14}$ close in the ℓ^1_ν norm to a true equilibrium solution. Fixing the Galerkin projection dimension at $N = 30$, and following the instructions described in Sect. 6.2.1, we bound $\epsilon_s \leq 10^{-14} \cdot (4.97, 1.61)$. Let $\rho = \left(2.2 \times 10^{-2}, 10^{-5}\right)$, and define $B_s(\rho - \epsilon_s)$ as in Definition 2.4, and I, I', and $\mathbf{I} = I \cup I'$ as in Remark 2.2. Let*

$$P = \left(0.153,\ 1.38 \times 10^{-5}\right) \qquad \text{and} \qquad \bar{P} = \begin{pmatrix} 16.9 \times 10^{-0} & 1.37 \times 10^{-3} \\ 1.37 \times 10^{-3} & 2.14 \times 10^{-4} \end{pmatrix},$$

be tensors as in Definition 2.8.

Then, there exists a unique $\tilde{\alpha} \in C^{1,1}(B_s(\rho - \epsilon_s), X_u)$, such that the local stable manifold of $\tilde{a} \in \ell^1_\nu$ is given by

$$\mathrm{x}_s \mapsto K\left(\mathrm{x}_s, \tilde{\alpha}(\mathrm{x}_s)\right),$$

for K as given in (54). Moreover, $\tilde{\alpha}$ has

$$|\tilde{\alpha}_{i'}(\xi)| \leq 3.36 \times 10^{-3} \qquad \|\tilde{\alpha}^i_{i'}(\xi)\| \leq P^i_{i'} \qquad \mathrm{Lip}(\partial_i \tilde{\alpha})^j_{i'} \leq \bar{P}^{ij}_{i'},$$

for all $\xi \in B_s(\rho - \epsilon_s)$, $i, j \in I$, $i' \in I'$ and $\mathbf{i} \in \mathbf{I}$.

Proof In script `main.m` we calculate all of the constants and verify all of the hypotheses in Theorem 5.11. In particular we have a contraction constant $\|J\| < 0.356$. The entire computation took about 4 seconds and was run on MATLAB 2019a with INTLAB on a i7-8750H processor. □

Remark 6.5 (**Performance: timing and conditioning**) One valuable indicator of performance for the computations just discussed is to compare the runtime of the non-rigorous portions of the computation. This gives an impression of the cost of passing from "good numerics" to a computer assisted proof. For example, of the roughly 4 second runtime for the proof of Theorem 6.4, roughly one second is spent on the numerical approximation of the equilibrium solution (Newton's method) and the numerical approximation of the eigenvalues and eigenvectors (computational linear algebra). Then the validation stage takes roughly three times longer than the non-rigorous linear approximation of the equilibrium and stable manifold. While this is only the result of a single computation, it gives a rough sense of the cost (in time) of the validation stage.

Another valuable performance indicator is the conditioning of the algorithm. Since Theorem 6.4 involves the linear approximation of the stable manifold by the eigenspace, we expect that the approximation is quadratically good. Then an excellent condition number for the algorithm is the constant of proportionality. For example, in the calculation above the approximation is valid on a subset of a ball of size 2.2×10^{-2} about the equilibrium, and the bound on the linear approximation is roughly 3.36×10^{-3}. This suggests that the condition number for this calculation is about 15. Note that this is roughly the size of the largest entry of $\bar{P}$. Recalling the definitions in Sect. 6.2, we see that the entries of $\bar{P}$—and hence the conditioning of the algorithm—are determined by the size of the derivative of F in a neighborhood of the equilibrium, and the size of the spectral gap. This is a heuristic observation which, while difficult to fully justify thanks to iterative procedure of defining $\bar{P}$, is still useful.

7 Application II: Nonlinear Change of Variables

In this section we improve the approximation of the stable manifolds in certain directions, by making the nonlinear change of coordinates discussed in Sect. 2.4. Again, we consider the example of the Swift-Hohenberg Eq. (1). We employ the notation established in Sect. 6.1, with some minor adjustments. In particular, we use $m_u = 1$ and $m_s = 3$. Recalling the notation of Sect. 2.4, set $n_u = m_{\text{unst}}$, $n_\theta := m_{\text{slow}}$, $n_f = m_{\text{fast}} + m_{\text{slow}}$, and $N = n_u + n_f - 1$, and define

$$X_{1'} := \mathbb{R}^{n_u} \qquad X_1 := \mathbb{R}^{n_\theta} \qquad X_2 := \mathbb{R}^{n_f - n_\theta} \qquad X_3 := \{a \in \ell^1_\nu : a_k = 0 \text{ for } k \leq N\}.$$

We write $X_u := X_{1'}$ and $X_s := X_1 \times X_2 \times X_3$ and $X = X_u \times X_s$, and use the notational shorthand $X_\theta := X_1$ (slow stable), $X_f := X_2$ (fast but finite stable) and $X_\infty := X_3$ (stable tail). The map π_N, as defined in (47), is a projection operator $\pi_N : X \to X_N \subseteq X$, where we define $X_N := X_{1'} \times X_1 \times X_2 \cong \mathbb{R}^{N+1}$. Define $\pi_\infty : X \to X_\infty$ by $\pi_\infty x := x - \pi_N x$, and Λ as

$$\Lambda_{1'} := \operatorname{diag}\{\mu_{n'_u}, \ldots, \mu_{1'}\}, \quad \Lambda_1 := \operatorname{diag}\{\mu_1, \ldots, \mu_{n_\theta}\}, \quad \Lambda_2 := \operatorname{diag}\{\mu_{n_\theta+1}, \ldots, \mu_{n_f}\}, \quad \Lambda_3 := \mathfrak{L} \circ \pi_\infty,$$

with μ defined in Sect. 6.1, and $\mathfrak{L}$ defined in (46). Define $\lambda_{\mathbf{i}}$ for $\mathbf{i} \in \mathbf{I}$ by

$$\begin{aligned} &\lambda_{1'} := \mu_{1'} \quad \lambda_1 := \mu_1, \quad \lambda_2 := \mu_{n_\theta+1}, \\ &\lambda_3 := -\beta_1 (N+1)^4 - \beta_2 (N+1)^2 + 1. \end{aligned} \tag{65}$$

Repeating the argument given at the end of Sect. 6.1 in this context gives that the inequalities of Eqs. (6) and (7) are satisfied. We now follow the scheme for stable manifold validation outlined in Sect. 6.2.

7.1 Estimate 1: Defining a Change of Variables

Using the parameterization method, and the good coordinates discussed in Sect. 2.4, we approximate a slow stable manifold and finite dimensional invariant normal bundles

$$\begin{aligned} P &: [-1,1]^{n_\theta} \to X_N, \\ Q_f(\theta) &: [-1,1]^{n_\theta} \to \operatorname{Mat}(\mathbb{R}^{n_f - n_\theta}, X_N) \\ Q_u(\theta) &: [-1,1]^{n_\theta} \to \operatorname{Mat}(\mathbb{R}^{n_u}, X_N). \end{aligned}$$

These are chosen to approximately solve (21)–(22). The error terms

$$E_\theta : [-1,1]^{n_\theta} \to \ell^1_\nu \qquad\qquad E_f : [-1,1]^{n_\theta} \to \mathcal{L}(X_f, \ell^1_\nu) \tag{66a}$$

$$E_u : [-1,1]^{n_\theta} \to \mathcal{L}(X_u, \ell^1_\nu) \qquad\qquad E_\infty : [-1,1]^{n_\theta} \to \mathcal{L}(X_\infty, \ell^1_\nu), \tag{66b}$$

are defined by

$$E_\theta(\theta) := F(P(\theta)) - DP(\theta)\Lambda_\theta \theta \tag{67a}$$

$$E_f(\theta) := DF(P(\theta))Q_f(\theta) - DQ_f(\theta)\Lambda_\theta\theta - Q_f(\theta)\Lambda_f \tag{67b}$$
$$E_u(\theta) := DF(P(\theta))Q_u(\theta) - DQ_u(\theta)\Lambda_\theta\theta - Q_u(\theta)\Lambda_u \tag{67c}$$
$$E_\infty(\theta) := DF(P(\theta))\pi_\infty - \Lambda_\infty. \tag{67d}$$

Define $U := B(r_s + \epsilon_s, r_u + \epsilon_u) \subseteq X_u \times [-1, 1]^{n_\theta} \times X_f \times X_\infty$, a normal frame bundle $Q : [-1, 1]^{n_\theta} \to \mathcal{L}(X/X_1, \ell^1_\nu)$, and a local diffeomorphism $K : U \subseteq X \to \ell^1_\nu$ by

$$Q(\theta)\phi := Q_f(\theta)\phi_f + Q_u(\theta)\phi_u + \phi_\infty \tag{68}$$
$$K(\theta, \phi) := P(\theta) + Q(\theta)\phi. \tag{69}$$

We define the norm $|\cdot|_X$ as in (49) relative to the linear map $Q_0 : X \to \ell^1_\nu$ defined by

$$Q_0 \cdot (h_\theta, h_\phi) := DK(0, 0) \cdot (h_\theta, h_\phi) = \partial_\theta P(0)h_\theta + Q(0)h_\phi, \tag{70}$$

where $h_\theta \in X_\theta$ and $h_\phi \in X_u \times X_f \times X_\infty$.

While we do not have an explicit expression for the inverse function K^{-1}, we can bound the norm of $\tilde{h} = K^{-1}(\tilde{a})$ as follows. Note that $K^{-1}(a) = Q_0^{-1}(a - \bar{a}) + \mathcal{O}(|a - \bar{a}|^2)$. If $|\bar{a} - \tilde{a}|_{\ell^1_\nu} \leq \epsilon$ bounds the distance between the approximate and true solutions, we apply standard techniques from rigorous numerics (cf Remark 6.2) to bound $|\pi_{\mathbf{i}}\tilde{h}| \leq \epsilon_{\mathbf{i}}$ for $\mathbf{i} \in \mathbf{I}$ as needed in Proposition 2.6, in terms of ϵ, $\|\pi_{\mathbf{i}} Q_0^{-1}\|$, and the polynomial coefficients of $K(\theta, \phi)$.

7.2 Estimate 2: Defining the Conjugate Differential Equation

Applying the coordinate change of Eq. (69) to the Swift-Hohenberg equation leads to

$$\dot{\mathrm{x}} = \Lambda \mathrm{x} + \tilde{\mathcal{N}}(\mathrm{x}), \qquad \tilde{\mathcal{N}}(\mathrm{x}) := DK(\mathrm{x})^{-1}F(K(\mathrm{x})) - \Lambda \mathrm{x}, \tag{71}$$

for $\mathrm{x} \in U$. We now perform a Taylor expansion of $F(K(\mathrm{x}))$ in $\mathrm{x} \in U$. To simplify the notation, for $\mathrm{x} = (\theta, \phi)$ where $\theta \in [-1, 1]^{n_\theta}$ and $\phi \in X_u \times X_f \times X_\infty$, define

$$\mathbf{P} := P(\theta) \qquad \mathbf{Q} := Q(\theta)\phi. \tag{72}$$

Starting from (45), expand $F(K(\theta, \phi))$ as

$$\begin{aligned} F(K(\theta, \phi)) &= \mathfrak{L}[\mathbf{P} + \mathbf{Q}] - (\mathbf{P} + \mathbf{Q})^3 \\ &= \left(\mathfrak{L}\mathbf{P} - \mathbf{P}^3\right) + \left(\mathfrak{L}\mathbf{Q} - 3\mathbf{P}^2 * \mathbf{Q}\right) - 3\mathbf{P} * \mathbf{Q}^2 - \mathbf{Q}^3, \end{aligned}$$

where the powers denote products of convolutions. Note that for $a, h \in \ell^1_\nu$, the derivative of F is given by

$$DF(a) \cdot h = \mathfrak{L}h - 3(a * a * h),$$

so that

$$F(\mathbf{P}) = \mathfrak{L}\mathbf{P} - \mathbf{P}^3, \qquad DF(\mathbf{P}) \cdot \mathbf{Q} = \mathfrak{L}\mathbf{Q} - 3(\mathbf{P}^2 * \mathbf{Q}).$$

Defining a remainder term $R : U \subseteq X \to \ell^1_\nu$ by

$$\begin{aligned} \mathbf{R} = R(\theta, \phi) &:= -3P(\theta) * (Q(\theta)\phi) * (Q(\theta)\phi) - (Q(\theta)\phi) * (Q(\theta)\phi) * (Q(\theta)\phi) \\ &= -3\mathbf{P} * \mathbf{Q}^2 - \mathbf{Q}^3, \end{aligned} \tag{73}$$

simplifies $F(K(\theta, \phi))$ as

$$F(K(\theta, \phi)) = F(\mathbf{P}) + DF(\mathbf{P}) \cdot \mathbf{Q} + \mathbf{R}. \tag{74}$$

The (approximate) conjugacy relations in (67) (approximately) linearize the non-remainder components in (74). More precisely, we have that

$$\begin{aligned}
&F(P(\theta)) + DF(P(\theta))\left[Q_f(\theta)\phi_f + Q_u(\theta)\phi_u + \phi_\infty\right] \\
&\quad = E_\theta(\theta) + DP(\theta)\Lambda_\theta\theta \\
&\qquad + E_f(\theta)\phi_f + DQ_f(\theta)(\Lambda_\theta\theta, \phi_f) + Q_f(\theta)\Lambda_f\phi_f \\
&\qquad + E_u(\theta)\phi_u + DQ_u(\theta)(\Lambda_\theta\theta, \phi_u) + Q_u(\theta)\Lambda_u\phi_u \\
&\qquad + E_\infty(\theta)\phi_\infty + \Lambda_\infty\phi_\infty \\
&\quad = E(\theta, \phi) + DK(\theta, \phi_f, \phi_u, \phi_\infty)\begin{pmatrix} \Lambda_\theta\theta \\ \Lambda_f\phi_f \\ \Lambda_u\phi_u \\ \Lambda_\infty\phi_\infty \end{pmatrix},
\end{aligned}$$

where $E : U \to \ell^1_\nu$ is defined by

$$E(\theta, \phi) := E_\theta(\theta) + E_f(\theta)\phi_f + E_u(\theta)\phi_u + E_\infty(\theta)\phi_\infty. \tag{75}$$

It follows that for $\mathrm{x} \in U$, we have

$$\begin{aligned}
DK(\mathrm{x})^{-1}F(K(\mathrm{x})) &= DK(\mathrm{x})^{-1}\big(E(\mathrm{x}) + DK(\mathrm{x})\Lambda\mathrm{x} + R(\mathrm{x})\big) \\
&= \Lambda\mathrm{x} + DK(\mathrm{x})^{-1}\left(E(\mathrm{x}) + R(\mathrm{x})\right).
\end{aligned}$$

Thus, the differential equation is decomposed into a diagonalized part and nonlinear error terms. It follows that

$$\tilde{\mathcal{N}}(\theta, \phi) = DK(\theta, \phi)^{-1}\left(E(\theta, \phi) + R(\theta, \phi)\right). \tag{76}$$

7.3 Estimate 3: Bounding $\tilde{\mathcal{N}}^{ij}_k$

Throughout this section, consider points in the ball $(\theta, \phi) \in U = B(r_s + \epsilon_s, r_u + \epsilon_u)$, and assume that $|\phi_u| \le r_u + \epsilon_u$, $|\phi_f| \le r_f + \epsilon_f$, and $|\phi_\infty| \le r_\infty + \epsilon_\infty$. Additionally, choosing $\delta_\theta \in (0, 1]$ such that if $|\theta|_X \le r_\theta + \epsilon_\theta$, we have that $(\theta)_k \le \delta_\theta$ for all components $1 \le k \le n_\theta$, whereby $U = B(r_s + \epsilon_s, r_u + \epsilon_u) \subseteq X_u \times [-\delta_\theta, \delta_\theta]^{n_\theta} \times X_f \times X_\infty$.

7.3.1 Bounding the Derivatives of *DK* and its Inverse

Fix $h = (h_\theta, h_f, h_u, h_\infty) \in X_\theta \times X_f \times X_u \times X_\infty$. We have that

$$DK(\theta, \phi) \cdot h = \big(\partial_\theta P(\theta) + \partial_\theta Q_f(\theta)\phi_f + \partial_\theta Q_u(\theta)\phi_u\big)h_\theta + Q_f(\theta)h_f + Q_u(\theta)h_u + h_\infty. \tag{77}$$

Define the maps

$$\begin{aligned}
A_0(\theta) \cdot h &:= \partial_\theta P(\theta)h_\theta + Q_f(\theta)h_f + Q_u(\theta)h_u + h_\infty, \\
A_1(\theta, \phi) \cdot h &:= \partial_\theta Q_f(\theta)\phi_f h_\theta + \partial_\theta Q_u(\theta)\phi_u h_\theta.
\end{aligned}$$

Then $DK = A_0 + A_1$.

The norm of A_1 is controlled by taking $|\phi|$ small. Assume $A_0(\theta)$ is invertible for all $\theta \in [-\delta_\theta, \delta_\theta]^{n_\theta}$ with inverse $B(\theta) := A_0(\theta)^{-1}$. Indeed, the action of the operator $A_0(\theta) : X_N \times X_\infty \to \ell^1_\nu \cong X_N \times X_\infty$ leaves both subspaces X_N and X_∞ invariant. The action of the operator $A_0(\theta)$ in the finite dimensional component is represented by a polynomial in θ

with $(N+1) \times (N+1)$ matrix coefficients. Its action in the infinite dimensional component is precisely the identity map. Hence the operator $B(\theta) = A_0(\theta)^{-1}$ is an infinite power series in θ, with Taylor coefficients defined recursively by power matching. We compute finitely many of these coefficients by solving the recursion relations.

The inverse $DK^{-1} : \ell^1_\nu \to X$ now has

$$DK(\theta, \phi)^{-1} = B(\theta)\big(I + A_1(\theta, \phi)B(\theta)\big)^{-1}.$$

Bounds on the derivatives of $DK(\theta, \phi)^{-1}$ are obtained by the product rule. We first compute finitely many terms in the power series expansion of $B(\theta)$, and bound the Taylor remainder and its derivatives using a Neumann series argument similar to the one given below to bound $\big(I + A_1(\theta, \phi)B(\theta)\big)^{-1}$. Indeed, for ϕ sufficiently small the Neumann series provides the bound

$$\begin{aligned}\|\big(I + A_1(\theta, \phi)B(\theta)\big)^{-1}\| &\le \frac{1}{1 - \|A_1(\theta, \phi)B(\theta)\|_{\mathcal{L}(\ell^1_\nu, \ell^1_\nu)}} \\ &\le \Big[1 - (|\phi_f| + |\phi_u|)\|\partial_\theta Q(\theta)\|_{\mathcal{L}(X_\theta \otimes X, \ell^1_\nu)} \|B(\theta)\|_{\mathcal{L}(\ell^1_\nu, X)}\Big]^{-1}.\end{aligned}$$

Derivatives of $\big(I + A_1(\theta, \phi)B(\theta)\big)^{-1}$ are bound using the fact that for any smooth path of invertible matrices, it holds that

$$\frac{\partial Y^{-1}}{\partial t} = -Y^{-1}\frac{\partial Y}{\partial t}Y^{-1}.$$

Applying the product rule gives

$$\frac{\partial^2 Y^{-1}}{\partial t \partial s} = Y^{-1}\left(\frac{\partial Y}{\partial s}Y^{-1}\frac{\partial Y}{\partial t} - \frac{\partial^2 Y}{\partial t \partial s} + \frac{\partial Y}{\partial t}Y^{-1}\frac{\partial Y}{\partial s}\right)Y^{-1}.$$

Hence, to bound the derivatives of $\big(I + A_1(\theta, \phi)B(\theta)\big)^{-1}$, it suffices to bound the inverse and the derivatives of $I + A_1(\theta, \phi)B(\theta)$.

For fixed $(\theta, \phi) \in U$ and $\mathbf{i} \in \mathbf{I}$, we see that the nontrivial first derivatives $\partial_{\mathbf{i}} A_1(\theta, \phi) : X \otimes X_{\mathbf{i}} \to \ell^1_\nu$ are given by

$$\partial_\theta A_1(\theta, \phi) = \partial_{\theta\theta} Q_f(\theta)\phi_f + \partial_{\theta\theta} Q_u(\theta)\phi_u, \quad \partial_\star A_1(\theta, \phi) = \partial_\theta Q_\star(\theta) \quad \text{for } \star \in \{f, u\}.$$

For fixed $(\theta, \phi) \in U$, and $\mathbf{i}, \mathbf{j} \in \mathbf{I}$, compute the nontrivial second derivatives $\partial_{\mathbf{i}}\partial_{\mathbf{j}} A_1(\theta, \phi) : X \otimes X_{\mathbf{i}} \otimes X_{\mathbf{j}} \to \ell^1_\nu$, by

$$\partial_{\theta\theta} A_1(\theta, \phi) = \partial_{\theta\theta\theta} Q_f(\theta)\phi_f + \partial_{\theta\theta\theta} Q_u(\theta)\phi_u, \quad \partial_{\theta\star} A_1(\theta, \phi) = \partial_{\theta\theta} Q_\star(\theta) \quad \text{for } \star \in \{f, u\}.$$

Note that $\partial_\infty DK^{-1} = 0$. Furthermore, $\pi_\infty DK^{-1} = \pi_\infty$, so that $\pi_\infty \partial_{\mathbf{i}}(DK^{-1}) = 0$ for all $\mathbf{i} \in \mathbf{I}$. Then bounds on DK^{-1} and its derivatives follow from bounds on

$$\|\pi_\circ B(\theta)\|_{\mathcal{L}(\ell^1_\nu, X)} \qquad \left\|\pi_\circ \tfrac{\partial^k}{\partial\theta^k} B(\theta)\right\|_{\mathcal{L}\left(X_\theta^{\otimes k} \otimes \ell^1_\nu, X\right)} \qquad \left\|\pi_\circ \tfrac{\partial^k}{\partial\theta^k} Q(\theta)\right\|_{\mathcal{L}\left(X \otimes X_\theta^{\otimes k}, \ell^1_\nu\right)}, \tag{78}$$

where $\pi_\circ \in \{\pi_N, \pi_\infty\}$ and $k = 1, 2, 3$. Since we have either explicit expressions (we may take a supremum over $\theta \in [-\delta_\theta, \delta_\theta]^{n_\theta}$ using interval arithmetic) or explicit bounds for each of these, we obtain the necessary explicit bounds on DK^{-1} and its derivatives. Note that bounds on $\pi_{\mathbf{k}} DK(\theta, \phi)^{-1} = \pi_{\mathbf{k}} B(\theta)\big(I + A_1(\theta, \phi)B(\theta)\big)^{-1}$ are improved by bounding $\|\pi_{\mathbf{k}} B(\theta)\|_{\mathcal{L}(\ell^1_\nu, X)}$ for $\mathbf{k} \in \mathbf{I}$, and likewise for the derivatives.

7.3.2 Bounding *E*

To bound $E : U \to \ell^1_\nu$ defined in (75), see also (66) and (67), we note first that these bounds are calculated in the $|\cdot|_{\ell^1_\nu}$ norm, whereas bound on E_f, E_u, E_∞ are calculated in the $\|\cdot\|_{\mathcal{L}(X,\ell^1_\nu)}$ norm. We have that

$$\partial_\theta E(\theta,\phi)\cdot h = \Big(\partial_\theta E_\theta(\theta) + \partial_\theta E_f(\theta)\phi_f + \partial_\theta E_u(\theta)\phi_u + \partial_\theta E_\infty(\theta)\phi_\infty\Big)\cdot h_\theta.$$

The other first derivatives of E are

$$\partial_\star E(\theta,\phi)\cdot h = E_\star(\theta)\cdot h_f, \qquad \text{for } \star \in \{f,u,\infty\}.$$

The nontrivial second derivatives of E are

$$\begin{aligned}\partial_{\theta\theta} E(\theta,\phi)\cdot(h^1,h^2) &= \big(\partial_{\theta\theta}E_\theta + \partial_{\theta\theta}E_f\phi_f + \partial_{\theta\theta}E_u\phi_u + \partial_{\theta\theta}E_\infty\phi_\infty\big)\cdot(h^1_\theta,h^2_\theta),\\ \partial_{\theta\star}E(\theta,\phi)\cdot(h^1,h^2) &= \partial_\theta E_\star(\theta)\cdot(h^1_\theta,h^2_\star), \qquad \text{for } \star\in\{f,u,\infty\}.\end{aligned}$$

Recall that we have an explicit finite dimensional polynomial representation for the functions E_θ, E_f and E_u. For E_∞ and its derivatives we have

$$\begin{aligned}E_\infty(\theta)\cdot\phi_\infty &= -3P(\theta)*P(\theta)*\phi_\infty\\ \partial_\theta E_\infty(\theta)\cdot(\phi_\infty,h_\theta) &= -6\,(\partial_\theta P(\theta)h_\theta)*P(\theta)*\phi_\infty\\ \partial_{\theta\theta}E_\infty(\theta)\cdot(\phi_\infty,h^1_\theta,h^2_\theta) &= -6(\partial_{\theta\theta}P(\theta)\cdot(h^1_\theta,h^2_\theta))*P(\theta)*\phi_\infty - 6(\partial_\theta P(\theta)h^1_\theta)*(\partial_\theta P(\theta)h^2_\theta)*\phi_\infty.\end{aligned}$$

Using the bounds on $|\phi|$, the explicit expressions for the polynomials P, Q, and the expressions above, we obtain bounds on E over all of $U\subseteq X$. In summary, we have bounds on E and its derivatives, and bound

$$\left\|\pi_\circ\frac{\partial^k}{\partial\theta^k}E_\theta(\theta)\right\|_{\mathcal{L}\left(X_\theta^{\otimes k},\ell^1_\nu\right)} \qquad\qquad \left\|\pi_\circ\frac{\partial^k}{\partial\theta^k}E_\star(\theta)\right\|_{\mathcal{L}\left(X\otimes X_\theta^{\otimes k},\ell^1_\nu\right)}, \tag{79}$$

where $\pi_\circ\in\{\pi_N,\pi_\infty\}$, $\star\in\{u,f,\infty\}$, and the supremum is taken over $\theta\in[-\delta_\theta,\delta_\theta]^{n_\theta}$. Here for $k=0,1,2$, $X^{\otimes k}$ is the k-fold tensor product of X, and $X^{\otimes 0}$ is the trivial vector space.

7.3.3 Bounding *R*

Recalling (72) and (73), we have

$$\mathbf{P}:=P(\theta),\qquad \mathbf{Q}:=Q_f(\theta)\phi_f+Q_u(\theta)\phi_u+\phi_\infty,\qquad \mathbf{R}:=-3\mathbf{P}*\mathbf{Q}^2-\mathbf{Q}^3.$$

To calculate bounds on $R(\theta,\phi)=\mathbf{R}$ and its derivatives, we start by calculating the derivatives of $\mathbf{Q}$. These are

$$\partial_\theta\mathbf{Q}\cdot h = \big(\partial_\theta Q_f\phi_f+\partial_\theta Q_u\phi_u\big)\cdot h_\theta,\qquad \partial_\star\mathbf{Q}\cdot h = Q_\star\cdot h_\star \ \text{ for } \star\in\{f,u\},\qquad \partial_\infty\mathbf{Q}\cdot h = h_\infty.$$

The nonvanishing second derivatives of $\mathbf{Q}$ are given by

$$\partial_{\theta\theta}\mathbf{Q}\cdot(h^1,h^2)=\big(\partial_{\theta\theta}Q_f\phi_f+\partial_{\theta\theta}Q_u\phi_u\big)\cdot(h^1_\theta,h^2_\theta),\qquad \partial_{\star\theta}\mathbf{Q}\cdot(h^1,h^2)=\partial_\theta Q_\star\cdot(h^1_\theta,h^2_\star)\ \text{ for } \star\in\{f,u\}.$$

The only nonvanishing derivatives of $\mathbf{P}$ are with respect to θ. Then, bounds on $\mathbf{Q}^2$, $\mathbf{Q}^3$,$\mathbf{P}*\mathbf{Q}^2$, and their partial derivatives are obtained using the product rule.

Using that $\mathbf{R} = -3\mathbf{P} * \mathbf{Q}^2 - \mathbf{Q} * \mathbf{Q}^2$, we have expressions for all of the first and second derivatives of R. Hence, to bound R and its derivatives, it suffices to bound

$$\left\| \pi_\circ \frac{\partial^k}{\partial \theta^k} P(\theta) \right\|_{\mathcal{L}\left(X_\theta^{\otimes k}, \ell_\nu^1\right)} \qquad \left\| \pi_\circ \frac{\partial^k}{\partial \theta^k} Q_\star(\theta) \right\|_{\mathcal{L}\left(X \otimes X_\theta^{\otimes k}, \ell_\nu^1\right)}, \tag{80}$$

where we take $\pi_\circ \in \{\pi_N, \pi_\infty\}$, $\star \in \{u, f\}$, $k = 0, 1, 2$, and the supremum over $\theta \in [-\delta_\theta, \delta_\theta]^{n_\theta}$. The rest of the bounds follow by applying the product rule (as detailed above), the Banach algebra property of ℓ_ν^1, and the bounds on $|\phi|$ which result from restricting to the ball $B(r_s + \epsilon_s, r_u + \epsilon_u)$.

7.3.4 Bounding $\tilde{\mathcal{N}}$

The derivatives of $\tilde{\mathcal{N}} = DK^{-1}(E + R)$ are calculated using the product rule. Exploiting the formulas derived in Sect. 7.3 facilitates implementation of the constants $\tilde{C}_{\mathbf{j}}^{\mathbf{ik}}$ bounding $\|\tilde{\mathcal{N}}_{\mathbf{j}}^{\mathbf{ik}}\|_{(r_s+\epsilon_s, r_u+\epsilon_u)}$, for $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbf{I}$ needed to apply Proposition 2.6.

7.4 Estimate 4: Bounding $\tilde{\mathcal{N}}_{\mathbf{j}}^{\mathbf{i}}(0)$

We now compute a tensor $\tilde{D}$ bounding $\|\tilde{\mathcal{N}}(0)\|$, as needed in Proposition 2.6. We infer from the computations in Sect. 7.3 that $\mathbf{Q}^2(\theta, 0) = 0$, $D\mathbf{Q}^2(\theta, 0) = 0$, $D(\mathbf{Q} * \mathbf{Q}^2) = 0$, and $D(\mathbf{P} * \mathbf{Q}^2) = 0$ when $\phi = 0$. Hence $DR(\theta, 0) = 0$. Since $R(\theta, 0) = 0$ as well, we infer that

$$\partial_{\mathbf{i}} \tilde{\mathcal{N}}(0) = DK(0)^{-1} \partial_{\mathbf{i}} E(0, 0) + (\partial_{\mathbf{i}} DK(0)^{-1}) E(0, 0) \qquad \text{for } \mathbf{i} \in \mathbf{I}. \tag{81}$$

The first summand in (81) is similar to the term studied in Sect. 6.2.4. To see this, starting from (75), compute the first derivatives of E at $(\theta, \phi) = (0, 0)$ to obtain

$$\partial_\theta E(0, 0) \cdot h = \partial_\theta E_\theta(0) \cdot h_\theta, \qquad \partial_\star E(0, 0) \cdot h = E_\star(0) \cdot h_\star \quad \text{for } \star \in \{f, u, \infty\}.$$

We deduce from the definition of E in (67) and the substitution $P(0) = \bar{a}$, that

$$\begin{aligned} \partial_\theta E_\theta(0) \pi_\theta &= (DF(\bar{a}) \partial_\theta P(0) - \partial_\theta P(0) \Lambda_\theta)\, \pi_\theta, \\ E_\star(0) \pi_\star &= (DF(\bar{a}) Q_\star(0) - Q_\star \Lambda_\star)\, \pi_\star \qquad \text{for } \star \in \{f, u, \infty\}. \end{aligned}$$

Using Q_0 as defined in (70), we obtain the simplification

$$\partial_{\mathbf{i}} E(0, 0) h = (DF(\bar{a}) Q_0 - Q_0 \Lambda)\, \pi_{\mathbf{i}} \qquad \text{for } \mathbf{i} \in \mathbf{I}.$$

Finally, the first summand in (81) simplifies to

$$DK(0, 0)^{-1} \partial_{\mathbf{i}} E(0, 0) = \left(Q_0^{-1} DF(\bar{a}) Q_0 - \Lambda \right) \pi_{\mathbf{i}} \qquad \text{for } \mathbf{i} \in \mathbf{I}.$$

We then bound $\|\pi_{\mathbf{j}} \left(Q_0^{-1} DF(\bar{a}) Q_0 - \Lambda \right) \pi_{\mathbf{i}}\|_{\mathcal{L}(X,X)}$ as in Sect. 6.2.4, with the trivial addition that the projection map π_θ must also be considered.

To bound the second summand in (81), note that $E(0, 0) = E_\theta(0)$, for which we have an explicit expression. From a calculation in the same vein as in Sect. 7.3.1, we obtain

$$\left(\partial_{\mathbf{i}} DK(0)^{-1} \right) E(0, 0) = -Q_0^{-1} \left(\partial_{\mathbf{i}} DK(0) \right) Q_0^{-1} E_\theta(0).$$

Then

$$\partial_\theta DK(0) = \partial_\theta A_0(0), \qquad \partial_\star DK(0) = \partial_\theta Q_\star(0) \quad \text{for } \star \in \{f, u, \infty\}.$$

The norm $|E_\theta(0)|_{\ell^1_\nu}$ is quite small in practice, and it suffices to obtain a rough bound on the norm of $\partial_{\mathbf{i}} DK(0)^{-1}$. Thus, for $\mathbf{i}, \mathbf{j} \in \mathbf{I}$ we bound the components of (81) as

$$\tilde{D}^{\mathbf{i}}_{\mathbf{j}} := \|\pi_{\mathbf{j}} \left(Q_0^{-1} DF(\bar{a}) Q_0 - \Lambda \right) \pi_{\mathbf{i}} \|_{\mathcal{L}(X,X)} + \left\| \pi_{\mathbf{j}} Q_0^{-1} \right\|_{\mathcal{L}(\ell^1_\nu, X)} \|\partial_{\mathbf{i}} DK(0)\|_{\mathcal{L}(X_{\mathbf{i}} \otimes X, \ell^1_\nu)} \left| \pi_N Q_0^{-1} E_\theta(0) \right|_X .$$

There are some additional cancellations, as $\pi_{\mathbf{j}} Q_0^{-1} (\partial_{\mathbf{i}} DK(0)) = 0$ when $\mathbf{i} = \infty$ or $\mathbf{j} = \infty$.

7.5 Estimate 5: Semigroup Bounds

The constants C_s and λ_s are obtained by applying Theorem B.1 as in Sect. 6.2.5. The only difference is that X_s is decomposed into 3 subspaces in Sect. 7 (as opposed to 2 subspaces in the linear case). We argue as follows. Define $D^{\mathbf{i}}_{\mathbf{j}} := \tilde{D}^{\mathbf{i}}_{\mathbf{j}} + \tilde{C}^{\mathbf{i}l}_{\mathbf{j}} \epsilon_l + \tilde{C}^{\mathbf{i}l'}_{\mathbf{j}} \epsilon_{l'}$ as in Proposition 2.6, and

$$\mu_1 := \lambda_1 \qquad \delta_a := \max_{1 \le i \le m_s - 1} \sum_{1 \le j \le m_s - 1} D^i_j \qquad \delta_b := \sum_{1 \le j \le m_s - 1} D^{m_s}_j ,$$

$$\mu_\infty := \lambda_3 = \lambda_\infty \qquad \delta_c := \max_{1 \le i \le m_s - 1} D^i_{m_s} \qquad \delta_d := D^{m_s}_{m_s} .$$

The rest of the computation for C_s and λ_s are exactly as described in Sect. 6.2.5.

7.6 Conclusion and Numerical Results

We recall that the parameter $\rho = (\rho_\theta, \rho_f, \rho_\infty)$ determines the size of the domain

$$B_s(\rho) = \left\{ (\mathrm{x}_\theta, \mathrm{x}_f, \mathrm{x}_\infty) \in X_s : |\mathrm{x}_\theta| \le \rho_\theta, |\mathrm{x}_f| \le \rho_f, |\mathrm{x}_\infty| \le \rho_\infty \right\},$$

for the candidate charts $\alpha \in \mathcal{B}_{\rho, P, \bar{P}}$, where X_s is decomposed in terms of the eigenspaces X_θ, X_f, and X_∞ of Λ_s corresponding to the slow stable eigenvalues, the fast-but-finite stable eigenvalues, and the remaining infinite stable eigenvalues respectively. This parameter ρ has a significant impact on nearly every aspect of our analysis.

For a given application it may be advantageous to choose certain components of $\rho = (\rho_\theta, \rho_f, \rho_\infty)$ large and others small. For example, we generically expect connecting orbits to have a larger projection into the slow-stable subspace X_θ and a smaller projection into the other stable subspaces. In Theorem 7.1, we present one such result, taking ρ_θ as large as possible. The parameters are the same as the ones used to produce Fig. 1. This nonlinear approximation of the stable manifold produces significantly better error estimates than a linear approximation: the C^0 error bounds in Theorem 7.1 are of size 7.43×10^{-12}, whereas the approximate manifold in Theorem 6.4 has C^0 error bounds of 3.36×10^{-3}.

Theorem 7.1 *Consider the Swift-Hohenberg Eq.* (1) *with parameters* $\beta_1 = 0.05$*, and* $\beta_2 = -0.35$*. Let* $\nu = 1.001$ *and suppose that* $\bar{a} \in \ell^1_\nu$ *is an approximate equilibrium solution,* $\epsilon = 1.61 \times 10^{-14}$ *close in the* ℓ^1_ν *norm to a true equilibrium solution. Using the techniques discussed in Sect. 2.4, we compute a slow stable manifold and finite dimensional (un)stable bundles, represented by Taylor polynomials of degree 20. Fixing the Galerkin projection dimension at* $N = 30$*, and following the instructions described in Sect. 6.2.1, we bound* $\epsilon_s \le 10^{-14} \cdot (1.85, 4.51, 1.61)$*. Let*

$$\rho = \left(3.18 \times 10^{-2} \ 10^{-6} \ 10^{-10}\right),$$

and

$$P = \begin{pmatrix} 9.43 \times 10^{-11} \\ 4.41 \times 10^{-6} \\ 3.31 \times 10^{-6} \end{pmatrix} \qquad \bar{P} = \begin{pmatrix} 1.30 \times 10^{-9} & 5.60 \times 10^{-5} & 1.04 \times 10^{-4} \\ 5.60 \times 10^{-5} & 2.72 \times 10^{-0} & 8.20 \times 10^{-4} \\ 1.04 \times 10^{-4} & 8.20 \times 10^{-4} & 1.41 \times 10^{-4} \end{pmatrix},$$

be tensors as in Definition 2.8. Define $B_s(\rho - \epsilon_s)$ as in Definition 2.4, I, I', and $\mathbf{I} = I \cup I'$ as in Remark 2.2.

Then, there exists a unique $\tilde{\alpha} \in C^{1,1}(B_s(\rho - \epsilon_s), X_u)$ so that the local stable manifold of $\tilde{a} \in \ell^1_\nu$ is given by

$$\mathrm{x}_s \mapsto K\left(\mathrm{x}_s, \tilde{\alpha}(\mathrm{x}_s)\right),$$

for K as in (69). Moreover, $\tilde{\alpha}$ has

$$|\tilde{\alpha}_{i'}(\xi)| \leq 7.43 \times 10^{-12} \qquad \|\tilde{\alpha}^i_{i'}(\xi)\| \leq P^i_{i'} \qquad \mathrm{Lip}(\partial_i \tilde{\alpha})^j_{i'} \leq \bar{P}^{ij}_{i'},$$

for all $\xi \in B_s(\rho - \epsilon_s)$ and $i, j \in I$, $i' \in I'$ and $\mathbf{i} \in \mathbf{I}$.

Proof In script `main_NL.m` we calculate all of the constants and verify all of the hypotheses in Theorem 5.11. In particular we have a contraction constant $\|J\| < 5.86 \times 10^{-6}$. It takes approximately 11 s to construct the slow-stable manifold and normal bundles, 23 s to compute the bounds detailed in Sect. 7, and 12 s to compute all the bounds in Sects. 3–5 needed to validate the stable manifold. These we run on MATLAB 2019a with INTLAB on a i7-8750H processor. □

Remark 7.2 (**Performance: timing and conditioning**) As in Remark 6.5 after Theorem 6.4, we consider briefly the timing and conditioning of the calculations required for the proof of Theorem 7.1. Note that, while the proof of Theorem 7.1 takes roughly ten times longer than the proof of Theorem 6.4, the cost in time of the validation, when compared to the cost in time of the non-rigorous calculations is very similar. That is, the numerical computation of the parameterized bundles takes about 11 s, which is roughly one quarter of the full computation time. On the other hand, since Theorem 7.1 involves complex higher order approximation schemes, it is less clear how to define a useful condition number for the argument. However, we give a more nuanced discussion of the final error relative to the algorithm inputs below.

The nonlinear approximation in Theorem 7.1 is optimized to produce a larger validated part of the manifold in the direction of the slow stable eigenvector, as this is where we would generically expect to find connecting orbits. Note that in Theorem 7.1 the gap between eigenvalues of $\Lambda_{1'}$, Λ_1 and Λ_2 is not very large:

$$\lambda_{1'} = 1.01, \qquad \lambda_1 = -1.41, \qquad \lambda_2 = -1.99, \qquad \lambda_3 = -4.58 \times 10^4.$$

We took the slow-stable eigenspace to be one dimensional. If a particular application required a stable manifold which was wider along the second slowest stable eigendirection, we could increase ρ_f at a cost of also increasing P, $\bar{P}$, etc. These error estimates could be improved somewhat by splitting X_f into two subspaces. Moreover, we could significantly increase the radius of our approximation along the second slowest stable eigendirection by using a higher dimensional slow stable manifold.

From the classical theory [15], we expect our derivative bound $P \geq \|D\alpha\|$ to be at least as large as the ratio between the derivative of the nonlinearity and the spectral gap, roughly

$$|P| \gtrsim \frac{\|D\mathcal{N}\|}{\lambda_u - \lambda_s} \gtrsim \frac{\|L\| + \|D^2\mathcal{N}\|\rho}{\lambda_u - \lambda_s}.$$

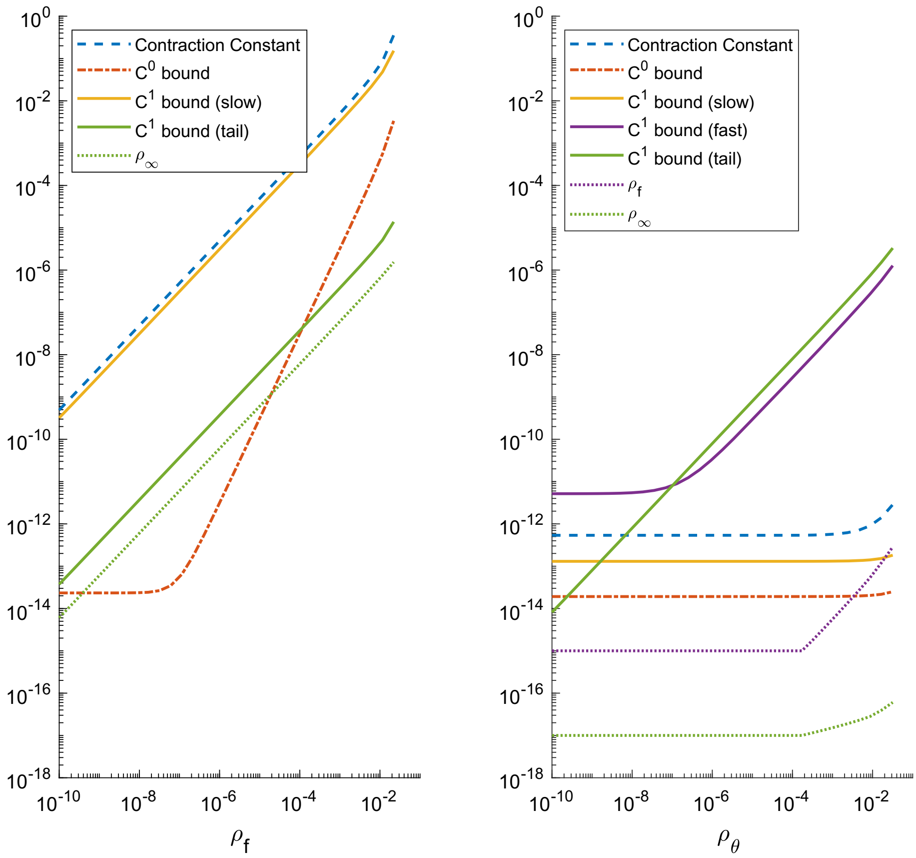

Fig. 3 (Left) Using the estimates from Sect. 6, the bounds produced by a computer assisted proof for a range of radii $\rho_f \in [10^{-10}, 0.022]$, with ρ_∞ chosen to be as small as possible. (Right) Using the estimates from Sect. 7, the bounds produced for a range of radii $\rho_\theta \in [10^{-10}, 0.0318]$, with ρ_f and ρ_∞ chosen to be as small as possible. Note that the nonlinear approximation yields smaller C^0 error bounds (red dash-dotted lines) (Color figure online)

We expect that this bound should increase linearly with ρ, and be bounded below by $\|L\|$, the error from not perfectly splitting $X_u \times X_s$ into eigenspaces. This scaling is observed in Fig. 3, where we display the error bounds in Theorems 6.4 and 7.1 as functions of ρ. The nonlinear approximation maintains small error bounds, despite taking ρ_θ large. This is because the change of variables prepares the nonlinearity so that $\|\partial_\theta D\mathcal{N}\|$ is small. Note that one should be mindful in comparing the two graphs in Fig. 3, as in Theorem 6.4 we split $X_s = X_f \times X_\infty$ with $\dim(X_f) = N$, and in Theorem 7.1 we split $X_s = X_\theta \times X_f \times X_\infty$ with $\dim(X_\theta) = 1$ and $\dim(X_f) = N - 1$.

When using the linear approximation we see that for a large range of ρ_f, the contraction constant, the tensor P, and the minimal choice of ρ_∞, all scale linearly with ρ_f. The C^0 error of the manifold, given by $|\tilde{\alpha}_{i'}| \leq P^i_{i'}(\rho_i + \epsilon_i) + \epsilon_{i'}$ in Theorem 5.11, is dominated by the error in validating the equilibrium until $\rho_f \approx 10^{-7}$, where it begins to scale quadratically with ρ_f. The $C^{1,1}$ error bounds on the norm of the components of $\bar{P}$ do not improve much for $\rho < 10^{-3}$, and increase quite rapidly for $\rho_f > 10^{-2}$.

For the nonlinear approximation, the error in validating the equilibrium dominates the C^0 bound until $\rho_\theta \approx 10^{-2}$, the point after which P_u^θ increases marginally. The contraction constant scales similarly, begining to increase around $\rho_\theta \approx 10^{-3}$. The C^1 bounds in the X_f and X_∞ subspaces are bounded below by the accuracy of the decomposition into eigenspaces of $DF(\bar{a})$, and increase linearly with ρ_θ. For the whole range of admissible ρ_θ, both ρ_f and ρ_∞ can be taken exceedingly small, without contributing significantly to the overall error.

We do not expect to validate a global stable manifold with the Lyapunov-Perron approach; if ρ is too large, the various hypotheses of Theorem 5.11 may no longer be satisfied. For example, we may be unable to prove the image of Ψ is contained within $\mathcal{B}^{0,1}_{\rho,P}$ or $\mathcal{B}^{1,1}_{\rho,P,\bar{P}}$, as detailed in Theorems 4.2 or 4.4. Other causes for failure would be if $\|J\| > 1$ whereby Ψ is not a contraction mapping, or if we are unable to prove solutions $x(t,\xi,\alpha)$ are contained within $B_s(\rho)$ for all $t \geq 0$ as required by Proposition 3.13. When using a linear approximation, many of these hypotheses all simultaneously fail for larger values of ρ. In contrast, for the nonlinear approximation in Sect. 7, the dominant limiting factor is the condition $\gamma_0 = \lambda_s + C_s\hat{\mathcal{H}} < 0$ as required in Proposition 3.13. Overall, the framework developed in Sects. 2 – 5 allow us to leverage our estimates on our approximate stable manifold made in Sects. 6–7.

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A General Strategy for Bootstrapping Gronwall's Inequality

We generalize the bootstrapping argument used in Sect. 3 so that it can be applied in Sects. 4 and 5. To unify the class of functions we wish to bound, and the set of assumptions we make on these functions, we define Condition A.1 below. In a slight abuse of notation, here we define $\mathcal{B}$ to be a tensor, distinct from its previous usage as a ball of functions in Definition 2.8.

Condition A.1 *Fix $\lambda_1, \ldots, \lambda_{N_\lambda} \in \mathbb{R}$, fix $H \in \mathbb{R}^{N_\lambda} \otimes \mathbb{R}^{N_\lambda}$ and define $\gamma_k := \lambda_k + H_k^k$ for $1 \leq k \leq N_\lambda$. For $N_\mu \in \mathbb{N}$, fix some $\mu_k \in \mathbb{R}$ for $1 \leq k \leq N_\mu$. Assume that $\{\gamma_j\}_{j=1}^{N_\lambda} \subseteq \{\mu_k\}_{k=1}^{N_\mu}$, and suppose that both $\gamma_k > \gamma_{k+1}$ and $\mu_k > \mu_{k+1}$. Assume further that $\mu_1 > \gamma_1$.*

For $M \in \mathbb{N}$, and $N_i \in \mathbb{N}$ for $1 \leq i \leq M$ and basis elements $e_{n_i} \in \mathbb{R}^{N_i}$ where $1 \leq n_i \leq N_i$, we fix tensors

$$\mathcal{A} \in \Big(\bigotimes_{i=1}^{M} \mathbb{R}^{N_i}\Big) \otimes \mathbb{R}^{N_\lambda} \otimes \mathbb{R}^{N_\mu}, \qquad \mathcal{B} \in \Big(\bigotimes_{i=1}^{M} \mathbb{R}^{N_i}\Big) \otimes \mathbb{R}^{N_\lambda}$$

component-wise by

$$\mathcal{A}_{j,k} := A_{j,k}^{n_1\ldots n_M} \cdot e_{n_1} \otimes \cdots \otimes e_{n_M}, \qquad \mathcal{B}_j := B_j^{n_1\ldots n_M} \cdot e_{n_1} \otimes \cdots \otimes e_{n_M}.$$

For this arrangement of constants, we say that a pair (u, ω) satisfies Condition A.1 on a time interval $[0, T]$ if the functions $u = (u_j)_{j=1}^{N_\lambda}$ and the positive tensor $\omega \in \bigotimes_{i=1}^{M} \mathbb{R}^{N_i}$ satisfy the inequalities

$$e^{-\lambda_j t}u_j(t) \leq \mathcal{B}_j\omega + \int_0^t e^{-\lambda_j \tau} \sum_{0\leq k\leq N_\mu} e^{\mu_k \tau}\mathcal{A}_{j,k}\omega\, d\tau + \int_0^t e^{-\lambda_j \tau} H_j^i u_i(\tau)\, d\tau \quad \textit{for all } t \in [0, T]. \tag{82}$$

In all cases where we consider constants satisfying Condition A.1, we take $N_\lambda = m_s$, and $\lambda_1, \ldots, \lambda_{N_\lambda}$ as in (6), and H_j^i as in Definition 2.9. Hence, the definition of γ_k here coincides with that given in Definition 3.3. For the other variables, we take them in the various sections according to the following table.

We note that for $\mathcal{A}_{j,k}$ in Sect. 4 we use a double index (k_1, k_2) to index over the elements of $\{\mu_k\}$. For a system given as in Condition A.1 we are interested in finding a tensor $\mathcal{G}$ satisfying Condition A.2 below.

Condition A.2 *Given μ as in Assumption A.1 and a pair (u, ω) of functions $u = (u_j)_{j=1}^{N_\lambda}$ on $[0, T]$ and a positive tensor $\omega \in \bigotimes_{i=1}^M \mathbb{R}^{N_i}$, we say that the tensor $\mathcal{G} \in \left(\bigotimes_{i=1}^M \mathbb{R}^{N_i}\right) \otimes \mathbb{R}^{N_\lambda} \otimes \mathbb{R}^{N_\mu}$ with components*

$$\mathcal{G}_{j,k} := G_{j,k}^{n_1\ldots n_M} e_{n_1} \otimes \cdots \otimes e_{n_M},$$

satisfies Condition A.2 if $u_j(t) \leq \sum_{k=1}^{N_\mu} e^{\mu_k t}\mathcal{G}_{j,k}\omega$ for all $t \in [0, T]$.

From these two conditions, we can bootstrap our bounds on a tensor $\mathcal{G}$.

Proposition A.3 *Assume the pair (u, ω) satisfies Condition A.1 on $[0, T]$ and assume $\mathcal{G}$ satisfies Condition A.2. Fix $1 \leq j \leq N_\lambda$. If $\mathcal{A}_{j,k} = 0$ and $\mathcal{G}_{i,k} = 0$ whenever $\mu_k = \gamma_j$, then we have:*

$$u_j(t) \leq e^{\gamma_j t}\mathcal{B}_j\omega + \sum_{\substack{1\leq k\leq N_\mu \\ \mu_k\neq\gamma_j}} \frac{e^{\mu_k t} - e^{\gamma_j t}}{\mu_k - \gamma_j}\Big(\mathcal{A}_{j,k} + \sum_{\substack{1\leq i\leq N_\lambda \\ i\neq j}} H_j^i \mathcal{G}_{i,k}\Big)\omega \quad \textit{for all } t \in [0, T]. \tag{83}$$

In other words, define a map $\mathcal{T}_{j,k} : \left(\bigotimes_{i=1}^M \mathbb{R}^{N_i}\right) \otimes \mathbb{R}^{N_\lambda} \otimes \mathbb{R}^{N_\mu} \to \bigotimes_{i=1}^M \mathbb{R}^{N_i}$ by:

$$\mathcal{T}_{j,k}(\mathcal{A}, \mathcal{B}, \mathcal{G}) := \begin{cases} (\mu_k - \gamma_j)^{-1}\Big(\mathcal{A}_{j,k} + \sum\limits_{\substack{1\leq i\leq N_\lambda \\ i\neq j}} H_j^i \mathcal{G}_{i,k}\Big) & \textit{if } \mu_k \neq \gamma_j \\ \mathcal{B}_j - \sum\limits_{\substack{0\leq m\leq N_\mu \\ \mu_m\neq\gamma_j}} (\mu_m - \gamma_j)^{-1}\Big(\mathcal{A}_{j,m} + \sum\limits_{\substack{1\leq i\leq N_\lambda \\ i\neq j}} H_j^i \mathcal{G}_{i,m}\Big) & \textit{if } \mu_k = \gamma_j. \end{cases} \tag{84}$$

Then $\mathcal{G}$ also satisfies Condition A.2 if we replace $\mathcal{G}_{j,k}$ by $\mathcal{T}_{j,k}(\mathcal{A}, \mathcal{B}, \mathcal{G})$ for all k.

Proof of Proposition A.3 Splitting $H_j^i u_i = \sum_{i\neq j} H_j^i u_i + H_j^j u_j$, we write (82) as

$$e^{-\lambda_j t}u_j(t) \leq \mathcal{B}_j\omega + \int_0^t e^{-\lambda_j \tau} v(\tau, \omega) d\tau + \int_0^t e^{-\lambda_j \tau} H_j^j u_j(\tau) d\tau.$$

where

$$v(\tau, \omega) = \sum_{\substack{1\leq k\leq N_\mu \\ \mu_k\neq\gamma_j}} e^{\mu_k \tau}\mathcal{A}_{j,k}\omega + \sum_{\substack{1\leq i\leq N_\lambda \\ i\neq j}} H_j^i u_i(\tau).$$

	Section 3	Section 4	Section 5
u_j	$\lvert x_j(t,\xi,\alpha)-x_j(t,\zeta,\alpha)\rvert$	$\lVert \partial_i x_j(t,\eta,\alpha)-\partial_i x_j(t,\zeta,\alpha)\rVert$	$\lvert x_j(t,\xi,\alpha)-x_j(t,\xi,\beta)\rvert$
ω	$\lvert \xi_n-\zeta_n\rvert$	$\lvert \eta_l-\zeta_l\rvert$	$\lvert \xi_{n_1}\rvert \otimes \lVert \alpha-\beta\rVert^{n_3}_{n_2',\mathcal{E}}$
$\mathcal{A}_{j,k}$	0	$S_j^{nm} G^l_{m,k_1} G^i_{n,k_2}$	$C_j^{n_2'} G^{n_1}_{n_3,k}$
$\mathcal{B}_j$	δ_j^n	0	0
$\{\mu_k\}$	$\{\gamma_k\}_{k=0}^{m_s}$	$\{\gamma_k\}_{k=0}^{m_s} \cup \{\gamma_{k_1}+\gamma_{k_2}\}_{k_1,k_2=0}^{m_s}$	$\{\gamma_k\}_{k=-1}^{m_s}$

By plugging in the bound assumed in Condition A.2, we obtain

$$v(\tau,\omega) \le \sum_{\substack{1\le k\le N_\mu\\ \mu_k\ne\gamma_j}} e^{\mu_k\tau}\Big(\mathcal{A}_{j,k}\omega + \sum_{\substack{1\le i\le N_\lambda\\ i\ne j}} H^i_j \mathcal{G}_{i,k}\omega\Big).$$

By applying Lemma 3.9 we obtain (83).

In order to obtain tensors satisfying the requirement that $\mathcal{A}_{j,k}, \mathcal{G}_{i,k} = 0$ whenever $\mu_k = \gamma_j$, we define an operator $\mathcal{Q}_j$ as below.

Proposition A.4 *Fix $1 \le j \le N_\lambda$ and define a map $\mathcal{Q}_j : \big(\bigotimes_{i=1}^M \mathbb{R}_+^{N_i}\big) \otimes \mathbb{R}^{N_\lambda} \otimes \mathbb{R}^{N_\mu} \to \big(\bigotimes_{i=1}^M \mathbb{R}_+^{N_i}\big) \otimes \mathbb{R}^{N_\lambda} \otimes \mathbb{R}^{N_\mu}$ by*

$$\mathcal{Q}_j(\mathcal{G})^{n_1\ldots n_M}_{i,k} := \begin{cases} 0 & \text{if } \mu_k = \gamma_j \\ G^{n_1\ldots n_M}_{i,k} + G^{n_1\ldots n_M}_{i,(k+1)} & \text{if } \mu_{k+1} = \gamma_j, \text{ and } G^{n_1\ldots n_M}_{i,(k+1)} > 0 \\ G^{n_1\ldots n_M}_{i,k} + G^{n_1\ldots n_M}_{i,(k-1)} & \text{if } \mu_{k-1} = \gamma_j, \text{ and } G^{n_1\ldots n_M}_{i,(k-1)} < 0 \\ G^{n_1\ldots n_M}_{i,k} & \text{otherwise.} \end{cases}$$

Then $\mathcal{Q}_j(\mathcal{G})_{i,k} = 0$ whenever $\mu_k = \gamma_j$. Furthermore, if $\mathcal{G}$ satisfies Condition A.2 then $\mathcal{Q}_j(\mathcal{G})$ satisfies Condition A.2.

We are able to generalize Algorithm 3.11 as follows.

Algorithm A.5 *Take as input all the constants in Condition A.1, an input tensor $\widehat{\mathcal{G}}$ satisfying Condition A.2, and a computational parameter $N_{bootstrap}$. The algorithm outputs a tensor $\mathcal{G}$.*

$\mathcal{G} \leftarrow \widehat{\mathcal{G}}$
for $1 \le i \le N_{bootstrap}$ ***do***
  for $1 \le j \le m_s$ ***do***
    $\mathcal{G}_{j,k} \leftarrow \mathcal{T}_{j,k}(\mathcal{Q}_j(\mathcal{A}), \mathcal{B}, \mathcal{Q}_j(\mathcal{G}))$
  end for
end for
return $\mathcal{G}$

Proposition A.6 *If the input tensor $\widehat{\mathcal{G}}$ to Algorithm A.5 satisfies Condition A.2, then the output tensor $\mathcal{G}$ satisfies Condition A.2.*

The proof of Proposition A.4 follows from the assumption that $\mu_k > \mu_{k+1}$. The proof of Proposition A.6 follows from an induction argument which uses Proposition A.3 for the inductive step. Both proofs are left to the reader.

B Semigroup Estimates for Fast-Slow Systems

In Eq. (8) we require constants C_s, λ_s satisfying

$$|e^{(\Lambda_s+L_s^S)t}\mathrm{x}_s| \le C_s e^{\lambda_s t}|\mathrm{x}_s|, \qquad t \ge 0, \mathrm{x}_s \in X_s. \tag{85}$$

Our assumption that $\lambda_s < 0$, and moreover that $\gamma_0 = \lambda_s + C_s\hat{\mathcal{H}} < 0$, is essential. In Proposition 3.13 this is used to prove that solutions $x(t,\xi,\alpha)$ stay inside the ball $B_s(\rho)$ for all $t \geq 0$. While our method of bootstrapping Gronwall's inequality greatly mitigates the effect of these constants C_s, λ_s on our final estimates, for the Lyapunov-Perron operator to be well defined it is essential that we prove $\gamma_0 < 0$.

There are two types of estimates which we will apply to obtain pairs (C_s, λ_s) satisfying (85). First, for linear operators $A, B \in \mathcal{L}(X, X)$ with $|e^{At}\mathrm{x}| \leq ke^{\lambda t}|\mathrm{x}|$ for all $\mathrm{x} \in X$ and $t \geq 0$, and $\|B\| < \infty$, we have (the proof is analogous to the one of Proposition 3.2)

$$|e^{(A+B)t}\mathrm{x}| \leq ke^{(\lambda+k\|B\|)t}|\mathrm{x}|, \qquad \text{for all } t \geq 0, \mathrm{x} \in X. \tag{86}$$

This estimate by itself is not enough, as the largest eigenvalue of Λ_s is often small in comparison with $\|L_s^s\|$. For example, in Sect. 6 we showed that $|e^{\Lambda_i t}\mathrm{x}_i| \leq e^{\lambda_i t}|\mathrm{x}_i|$ and $\|L_j^i\| \leq D_j^i$ with values

$$\lambda_1 = -1.41, \qquad \lambda_2 = -4.58 \times 10^4, \qquad D_s^s = \begin{pmatrix} 4 \times 10^{-10} & 1.6 \\ 1.6 & 5.7 \end{pmatrix}.$$

Since $\lambda_1 + \|L_s^s\| > 0$, just an estimate of the type in (86) with A the diagonal part of D_s^s and B the off-diagonal part will not suffice. We further note that our estimates for D_s^s do not improve with a larger Galerkin projection dimension. Hence we want to change basis to diagonalize $\Lambda_s + L_s^s$, at least approximately, and then take advantage of the identity $e^{PJP^{-1}t} = Pe^{Jt}P^{-1}$ in our estimates. To motivate our construction, we first consider a 2×2 matrix

$$M = \begin{pmatrix} \lambda_1 & \delta_b \\ \delta_c & \lambda_\infty \end{pmatrix}.$$

If λ_∞ is much larger in absolute value than the other matrix entries, then the eigenvalues of M are approximately given by λ_1 and λ_∞. In particular, if $|\delta_b\delta_c| < |\lambda_1\lambda_\infty|$ and $\lambda_1, \lambda_\infty < 0$, then all of the eigenvalues of M have negative real part. Below in Theorem B.1 we prove an analogous theorem where we replace λ_1 by a finite dimensional matrix, and λ_∞ by an infinite dimensional linear operator. This is the second type of estimate that we use to find pairs (C_s, λ_s) satisfying (85).

Theorem B.1 *Consider Banach spaces $\mathbb{C}^N$ and X_∞ with arbitrary norms, and their product $\mathbb{C}^N \times X_\infty$ with norm $|(x_N, x_\infty)| = (|x_N|^p + |x_\infty|^p)^{1/p}$ for any $1 \leq p \leq \infty$.*

Consider the linear operators $M, \Lambda, L : \mathbb{C}^N \times X_\infty \to \mathbb{C}^N \times X_\infty$ given by

$$M = \Lambda + L, \qquad \Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_\infty \end{pmatrix}, \qquad L = \begin{pmatrix} L_1^1 & L_1^\infty \\ L_\infty^1 & L_\infty^\infty \end{pmatrix}. \tag{87}$$

We require Λ to be densely defined and L to be bounded. Suppose that Λ_1 is diagonal *and that Λ_∞ has a bounded inverse.*

Fix constants $\mu_1, \mu_\infty, C_1, C_\infty \in \mathbb{R}$ such that for all $t \geq 0$ we have

$$\|e^{\Lambda_1 t}\| \leq C_1 e^{\mu_1 t}, \qquad \|e^{\Lambda_\infty t}\| \leq C_\infty e^{\mu_\infty t}.$$

Fix constants $\delta_1, \delta_b, \delta_c, \delta_d, \varepsilon > 0$ such that

$$\|L_1^1\| \leq \delta_a, \qquad \|L_1^\infty\| \leq \delta_b, \qquad \|L_\infty^1\| \leq \delta_c, \qquad \|L_\infty^\infty\| \leq \delta_d,$$

and set

$$\varepsilon := \sum_{\lambda \in \sigma(\Lambda_1)} \frac{\|\Lambda_\infty^{-1}\|}{1 - \|\Lambda_\infty^{-1}\|(\delta_d + |\lambda|)}.$$

Assume that the inequalities

$$\|\Lambda_\infty^{-1}\|\left(\delta_d+\sup_{\lambda_k\in\sigma(\Lambda_1)}|\lambda_k|\right)<1,\qquad \mu_\infty+C_\infty\left(\delta_d+\varepsilon\delta_b\delta_c(1+\varepsilon^2\delta_b\delta_c)\right)<\mu_1, \tag{88}$$

are satisfied. Then we have

$$\|e^{Mt}\|\le C_s e^{\lambda_s t},$$

where

$$\begin{aligned}C_s &:= (1+\varepsilon\delta_b)^2(1+\varepsilon\delta_c)^2\max\{C_1,C_\infty\}\\ \lambda_s &:= \mu_1+C_s\delta_a+\Delta\max\{C_1,C_\infty\}\\ \Delta &:= \varepsilon\delta_b\delta_c\left(1+\varepsilon(2\delta_b+\delta_c)+\varepsilon^2\delta_b\delta_c(1+\varepsilon\delta_b)\right).\end{aligned}$$

First we prove a lemma for general Banach spaces which allows us to approximately diagonalize our matrix. When $|\cdot|$ denotes the norm on a Banach space, then by $|\cdot|_*$ we denote the norm on its dual.

Lemma B.2 *For a Banach space X_∞ consider the linear operator $M_1:\mathbb{C}^N\times X_\infty\to\mathbb{C}^N\times X_\infty$ defined as*

$$M_1=\begin{pmatrix}A & B\\ C & D\end{pmatrix}.$$

Suppose that $\sigma(A)\cap\sigma(D)=\emptyset$ and that A has distinct eigenvalues $\lambda_1,\dots,\lambda_N$ with eigenvectors $v_1,\dots,v_N$, and dual eigenvectors $u_1,\dots,u_N$ (the corresponding eigenvectors of A^*). Normalize the vectors so that $u_i^*v_j=\delta_{ij}$, the Kronecker delta.

We define $W_b:X_\infty\to\mathbb{C}^N$ and $W_c:\mathbb{C}^N\to X_\infty$ as a sum of products between vectors in their codomains, and dual vectors acting on their domains:

$$W_b:=\sum_{k=1}^N v_k\left[(D^*-\lambda_k^*I_\infty)^{-1}B^*u_k^*\right],\qquad W_c:=\sum_{k=1}^N-\left[(D-\lambda_kI_\infty)^{-1}Cv_k\right]u_k^*,$$

where $D^*:X_\infty^*\to X_\infty^*$ and $B^*:(\mathbb{C}^N)^*\to X_\infty^*$ are the dual transformations. Define invertible operators $P_b,P_c:\mathbb{C}^N\times X_\infty\to\mathbb{C}^N\times X_\infty$ by

$$P_b=\begin{pmatrix}I_N & W_b\\ 0 & I_\infty\end{pmatrix}\qquad\qquad P_c=\begin{pmatrix}I_N & 0\\ W_c & I_\infty\end{pmatrix}.$$

Then

$$(P_cP_b)^{-1}M_1(P_cP_b)=\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}+E,$$

where

$$E=\begin{pmatrix}(I_N+W_bW_c)BW_c & BW_cW_b+W_bW_cB(I+W_cW_b)\\ -W_cBW_c & -W_cB(I_\infty+W_cW_b)\end{pmatrix}.$$

Proof. First we show that

$$P_b^{-1}\begin{pmatrix}A & B\\ 0 & D\end{pmatrix}P_b=\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix},\qquad\qquad P_c^{-1}\begin{pmatrix}A & 0\\ C & D\end{pmatrix}P_c=\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}. \tag{89}$$

We begin with the second equality in (89), and calculate

$$P_c^{-1}\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} P_c = \begin{pmatrix} A & 0 \\ -W_c A + C + DW_c & D \end{pmatrix}.$$

We compute the action of $-W_c A + C + DW_c$ on an eigenvector v_k of A as follows:

$$(-W_c A + C + DW_c)v_k = Cv_k + (D - \lambda_k I_\infty)W_c v_k.$$

To see that the right hand side is equal to zero, we calculate, using $u_i^* v_j = \delta_{ij}$,

$$W_c v_k = -(D - \lambda_k I_\infty)^{-1} C v_k.$$

Since the eigenvectors $v_1 \dots v_N$ span $\mathbb{C}^N$, then $-W_c A + C + DW_c = 0$, yielding the desired equality.

The argument is analogous for the first identity in (89). Again we begin by calculating

$$P_b^{-1}\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} P_b = \begin{pmatrix} A & AW_b + B - W_b D \\ 0 & D \end{pmatrix}.$$

Hence, we would like to show the map $(AW_b + B - W_b D) : X_\infty \to \mathbb{C}^N$ is the zero map, which we do by arguing that $u_k^*(AW_b + B - W_b D) = 0$ for all k. The latter follows from a calculation similar to the one performed above.

Finally, we calculate $(P_c P_b)^{-1} M_1 P_c P_b$ as follows:

$$\begin{aligned}
(P_c P_b)^{-1} M_1 (P_c P_b) &= P_b^{-1}\left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + P_c^{-1}\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} P_c\right) P_b \\
&= P_b^{-1}\left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} + \begin{pmatrix} BW_c & 0 \\ -W_c B W_c & -W_c B \end{pmatrix}\right) P_b \\
&= \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} (I_N + W_b W_c) B W_c & BW_c W_b + W_b W_c B(I + W_c W_b) \\ -W_c B W_c & -W_c B(I_\infty + W_c W_b) \end{pmatrix}.
\end{aligned}$$

□

Proof of Theorem B.1. Let $M = M_1 + M_2$, where

$$M_1 := \begin{pmatrix} A & B \\ C & D \end{pmatrix} := \begin{pmatrix} \Lambda_1 & L_1^\infty \\ L_\infty^1 & \Lambda_\infty + L_\infty^\infty \end{pmatrix}, \qquad M_2 := \begin{pmatrix} L_1^1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We will apply Lemma B.2 to the matrix M_1. Since we have assumed that Λ_1 is diagonal we may take $u_k = v_k = e_k$, the standard basis vectors in $\mathbb{C}^N$. We begin by proving $\|W_b\| \le \varepsilon\delta_b$ and $\|W_c\| \le \varepsilon\delta_c$. We first calculate

$$(D - \lambda_k I_\infty)^{-1} = (\Lambda_\infty + L_\infty^\infty - \lambda_k I_\infty)^{-1} = (I_\infty + \Lambda_\infty^{-1}(L_\infty^\infty - \lambda_k I_\infty))^{-1}\Lambda_\infty^{-1}.$$

By our hypothesis, we are allowed to apply the Neumann series and we obtain

$$\|(D - \lambda_k I_\infty)^{-1}\| \le \frac{\|\Lambda_\infty^{-1}\|}{1 - \|\Lambda_\infty^{-1}\|(\delta_d + |\lambda_k|)}. \tag{90}$$

We note that the same estimate holds for the dual operator $(D^* - \lambda_k^* I_\infty)^{-1}$.

We now show that $\|W_b\| \le \varepsilon\delta_b$. Namely, by using that $\|u_k^*\|_{(\mathbb{C}^N)^*} = \|v_k\|_{\mathbb{C}^N} = 1$ we find that

$$\|W_b\| = \sup_{x \in X_\infty, \|x\|=1} \left\| \sum_{\lambda_k \in \sigma(\Lambda_1)} v_k \left[(D^* - \lambda_k^* I_\infty)^{-1} B^* u_k^T\right] x \right\|_{\mathbb{C}^N}$$

$$
\begin{aligned}
&\leq \sup_{x\in X_\infty, \|x\|=1} \sum_{\lambda_k\in\sigma(\Lambda_1)} \left| \left[(D^* - \lambda_k^* I_\infty)^{-1} B^* u_k^T \right] x \right| \\
&\leq \sum_{\lambda_k\in\sigma(\Lambda_1)} \left\| (D^* - \lambda_k^* I_\infty)^{-1} B^* \right\|_{\mathcal{L}((\mathbb{C}^N)^*, X_\infty^*)} \\
&\leq \|B^*\| \sum_{\lambda_k\in\sigma(\Lambda_1)} \frac{\|\Lambda_\infty^{-1}\|}{1 - \|\Lambda_\infty^{-1}\|(\delta_d + |\lambda_k|)}.
\end{aligned}
$$

Hence, by plugging in $\|B^*\| = \|L_1^\infty\|$ we obtain $\|W_b\| \leq \varepsilon\delta_b$. The proof of the estimate $\|W_c\| \leq \varepsilon\delta_c$ is analogous. Next, we note that

$$
\|P_b\|, \|P_b^{-1}\| \leq 1 + \varepsilon\delta_b \qquad\qquad \|P_c\|, \|P_c^{-1}\| \leq 1 + \varepsilon\delta_c.
$$

By Lemma B.2 we have

$$
(P_c P_b)^{-1}(M_1 + M_2)(P_c P_b) = M_3 + M_4 + (P_b P_b)^{-1} M_2 (P_c P_b), \tag{91}
$$

where

$$
\begin{aligned}
M_3 &:= \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_\infty + L_\infty^\infty - W_c L_1^\infty (I_\infty + W_c W_b) \end{pmatrix}, \\
M_4 &:= \begin{pmatrix} (I_N + W_b W_c) L_1^\infty W_c & L_1^\infty W_c W_b + W_b W_c L_1^\infty (Id + W_c W_b) \\ -W_c L_1^\infty W_c & 0 \end{pmatrix}.
\end{aligned}
$$

For $(x_N, x_\infty) \in \mathbb{C}^N \times X_\infty$ we see that

$$
e^{M_3 t}(x_N, x_\infty) = \left(e^{\Lambda_1 t} x_N, e^{(\Lambda_\infty + L_\infty^\infty - W_c L_1^\infty (I_\infty + W_c W_b))t} x_\infty \right).
$$

We also have $\|L_\infty^\infty - W_c L_1^\infty (I_\infty + W_c W_b)\| \leq \delta_d + \varepsilon\delta_b\delta_c(1 + \varepsilon_b\varepsilon_c)$. By applying the estimate (86) we obtain, for all $t \geq 0$,

$$
\begin{aligned}
\|e^{\Lambda_1 t} x_N\| &\leq C_1 e^{\mu_1 t} \|x_N\|, \\
\|e^{(\Lambda_\infty + L_\infty^\infty - W_c L_1^\infty (I_\infty + W_c W_b))t} x_\infty\| &\leq C_\infty e^{(\mu_\infty + C_\infty[\delta_d + \varepsilon\delta_b\delta_c(1+\varepsilon_b\varepsilon_c)])t} \|x_\infty\|.
\end{aligned}
$$

From our assumption in (88) that $\mu_1 > \mu_\infty + C_\infty[\delta_d + \varepsilon\delta_b\delta_c(1 + \varepsilon^2\delta_b\delta_c)]$, we obtain, for any p-norm, $1 \leq p \leq \infty$, on the product $\mathbb{C}^N \times X_\infty$,

$$
\|e^{M_3 t}(x_N, x_\infty)\| \leq \max\{C_1, C_\infty\} e^{\mu_1 t} \|(x_N, x_\infty)\|.
$$

We may estimate the norm of the components of M_4 as

$$
\begin{aligned}
\|(I_N + W_b W_c) L_1^\infty W_c\| &\leq \varepsilon\delta_b\delta_c(1 + \varepsilon^2\delta_b\delta_c), \\
\| - W_c L_1^\infty W_c\| &\leq \varepsilon^2\delta_b\delta_c^2, \\
\|L_1^\infty W_c W_b + W_b W_c L_1^\infty (Id + W_c W_b)\| &\leq \varepsilon^2\delta_b^2\delta_c(2 + \varepsilon^2\delta_b\delta_c).
\end{aligned}
$$

We then obtain the bound

$$
\|M_4\| \leq \Delta := \varepsilon\delta_b\delta_c \left(1 + \varepsilon(2\delta_b + \delta_c) + \varepsilon^2\delta_b\delta_c(1 + \varepsilon\delta_b) \right)
$$

by summing the component bounds.

We now perform the final estimate. By using (91) we obtain

$$
e^{Mt} = (P_c P_b) \exp\left\{ \left[M_3 + M_4 + (P_c P_b)^{-1} M_2 (P_c P_b) \right] t \right\} (P_c P_b)^{-1}.
$$

By then applying (86) to the sum of M_3 and the bounded operator $M_4 + (P_c P_b)^{-1} M_2 (P_c P_b)$ we obtain, with $C_{1,\infty} := \max\{C_1, C_\infty\}$,

$$\|e^{Mt}\| \le \|P_c P_b\| \cdot \|(P_c P_b)^{-1}\| C_{1,\infty} \exp\left\{\mu_1 + C_{1,\infty} \left\| M_4 + (P_c P_b)^{-1} M_2 (P_c P_b) \right\| t\right\}.$$

Defining $C_s = \max\{C_1, C_\infty\}(1 + \varepsilon\delta_b)^2(1 + \varepsilon\delta_c)^2$ and plugging in our bounds, we finally infer

$$\|e^{Mt}\| \le C_s e^{(\mu_1 + C_s\delta_a + \Delta \max\{C_1, C_\infty\})t}.$$

□

Remark B.3 If we use the $p = 1$ norm for the product space $\mathbb{C}^N \times X_\infty$ then our bound for Δ can be sharpened to

$$\|M_4\| \le \varepsilon\delta_b\delta_c \max\left\{1 + \varepsilon\delta_c(1 + \varepsilon\delta_b), \varepsilon\delta_b(2 + \varepsilon^2\delta_b\delta_c)\right\}.$$

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