Parameterized stable/unstable manifolds for periodic solutions of implicitly defined dynamical systems

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Abstract

We develop a multiple shooting parameterization method for studying stable/unstable manifolds attached to periodic orbits of systems whose dynamics is determined by an implicit rule. We represent the local invariant manifold using high order polynomials and show that the method leads to efficient numerical calculations. We implement the method for several example systems in dimension two and three. The resulting manifolds provide useful information about the orbit structure of the implicit system even in the case that the implicit relation is neither invertible nor single-valued.

Key words. Implicitly defined dynamical systems, computational methods, invariant manifolds, periodic orbits, parameterization method

1 Introduction

A smooth diffeomorphism $F\colon \mathbb{R}^d \to \mathbb{R}^d$ generates a dynamical system by the rule

$$x_{n+1} = F(x_n), \qquad n \in \mathbb{N},$$

with initial condition $x_0 \in \mathbb{R}^d$. The infinite sequence $\{x_n\}_{n=0}^\infty$ is called the (forward) orbit of x_0 generated by F, and the fundamental objective of dynamical systems theory is to understand the qualitative features of the set of all orbits generated by F. This analysis often begins by considering simple invariant sets like fixed points, periodic orbits, and their attached invariant manifolds. If this program goes well one may move on to bifurcations of these objects, or to the study of more exotic invariant sets like connecting orbits, horseshoes, strange attractors, and invariant tori.

A interesting generalization, whose motivation and literature are discussed briefly in Section 1.2, is to consider a smooth map $T\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, and to study the properties of the mapping F defined by the implicit rule

$$y = F(x) \qquad \text{if} \qquad T(y, x) = 0.$$

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This immediately raises delicate questions about the domain of F, or whether F is even single valued. Of course the implicit function theorem provides valuable information. That is, if $(\bar{y}, \bar{x}) \in \mathbb{R}^d \times \mathbb{R}^d$ has $T(\bar{y}, \bar{x}) = 0$, and $D_1 T(\bar{y}, \bar{x})$ is an isomorphism, then there exists an open set $U \subset \mathbb{R}^d$, $\bar{x} \in U$, and a smooth diffeomorphism $F \colon U \to \mathbb{R}^d$ with $F(\bar{x}) = \bar{y}$ and having that

$$T(F(x), x) = 0, \qquad \text{for all } x \in U.$$

We say that T implicitly defines the diffeomorphism F near $\bar{x}$.

We remark that for fixed $\bar{x} \in \mathbb{R}^d$, the solution $\bar{y}$ of $T(y, x) = 0$ may not be unique. Hence F depends on the choice of $\bar{y}$. Nevertheless, for a given choice of $\bar{y}$ the associated "branch" of F is a perfectly well defined diffeomorphism. This and other fundamental notions for implicitly defined maps are discussed in Appendix B.

To iterate the procedure suppose $\tilde{x} \in F(U)$ and $\tilde{y} \in \mathbb{R}^d$ have $T(\tilde{y}, \tilde{x}) = 0$, with $D_1 T(\tilde{y}, \tilde{x})$ an isomorphism. Then there is an open set an open set $U \subset \mathbb{R}^d$, $\tilde{x} \in U$, and a diffeomorphism $\tilde{F} \colon \tilde{U} \to \mathbb{R}^d$ with $\tilde{F}(x)$ a branch of solutions of $T(y, x) = 0$ having $\tilde{F}(\tilde{x}) = \tilde{y}$. Now, for any $x \in U \cap F^{-1}(\tilde{U})$, the composition of x under the two implicitly defined maps F and $\tilde{F}$ is well defined.

Continuing in this way, suppose that $y_1 \neq \ldots \neq y_N \in \mathbb{R}^d$ have

$$\begin{aligned} T(y_2, y_1) &= 0 \\ T(y_3, y_2) &= 0 \\ &\vdots \\ T(y_N, y_{N+1}) &= 0 \\ T(y_1, y_N) &= 0, \end{aligned} \tag{1}$$

with the linear maps $D_1 T(y_2, y_1)$, ..., $D_1 T(y_1, y_N)$ all isomorphisms. Then there exist neighborhoods $U_j \subset \mathbb{R}^d$, and diffeomorphisms $F_j \colon U_j \to \mathbb{R}^d$ so that each F_j is a branch of solutions of $T(y, x) = 0$ having $F_j(y_j) = y_{j+1}$ for $j = 1, N$ (with the understanding that $y_{N+1} = y_1$). Indeed, by taking the U_j disjoint, there is no reason to think of the F_j different maps. Rather, we think of $y_1, \ldots, y_N$ as a period-N orbit of a single mapping $F \colon U_1 \cup \ldots \cup U_N \to \mathbb{R}^d$, and have that F is a diffeomorphism in a neighborhood of each of the points $y_1, \ldots, y_N$.

It makes sense in this context to consider the linear stability of the periodic orbit of F, and to study in turn the attached invariant manifolds. The main goal of the present work is to develop efficient numerical procedures for computing high order polynomial approximations of the local stable/unstable manifold attached to a periodic orbit of an implicitly defined dynamical system. We also implement and profile the results in some example applications.

We stress that our approach does not require information about the implicitly defined diffeomorphism F, much less any compositions of F. Rather, we develop a multiple shooting scheme for the invariant manifolds which depends only on the mapping T and the choice of periodic solution $x_1, \ldots, x_N$ for the system given by Equation (1). A power matching argument leads to linear systems of equations for the jets of the manifold, reducing the calculation of the stable/unstable manifold parameterizations to a problem in linear algebra. As we will see, computing the local invariant manifolds to high order leads to polynomial approximations valid in fairly large neighborhoods of the periodic orbit. This in turn provides insights into the existence of more global dynamical objects like heteroclinic/homoclinic connecting orbits for the implicitly defined system.

The remainder of the paper is organized as follows. In the next section, Section 1.1, we introduce the main examples considered in the remainder of the present work. In Section 1.2 we briefly discuss the literature concerning generalized notions of a dynamical system. In Section 2 we review some basic notions from the parameterization method for invariant manifolds, and in Section 3 we review how these notions work for fixed points of implicitly defined dynamical systems. We then introduce the multiple shooting parameterization method for stable/unstable manifolds attached to periodic orbits of implicitly defined maps. In Section 4 we develop power series solutions for the parameterized manifolds, and in Section 5 we implement numerical methods based on the power series approach. Some conclusions are given in Section 6. The Appendices A, B, and C fill in some background details and develop some more technical extensions of the methods developed in the main text.

All the MATLAB codes discussed in the present work are on github at

`https://github.com/aneupanetims2016/Implicit-map-archana`

1.1 A class of examples: perturbations of explicitly defined maps

We now describe a class of examples sufficient for the needs of the present work. Other potential applications are mentioned in Section 6

Let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be a C^k diffeomorphism (or real analytic with real analytic inverse if $k = \omega$). Then for any $x_0 \in \mathbb{R}^d$, f defines a dynamical system by the rule

$$f(x_n) = x_{n+1},$$

for $n = 0, 1, 2, \ldots$ Define the function $T\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$T(y, x) = y - f(x),$$

and note that for a given $\bar{x} \in \mathbb{R}^d$, $\bar{y}$ solves the equation $T(y, \bar{x}) = 0$ if and only if

$$\bar{y} = f(\bar{x}).$$

In this case, the problem $T(y, x) = 0$ implicitly defines the dynamical system generated by the diffeomorphism $f(x)$. Now let U, V be open subsets of $\mathbb{R}^d$ and $H\colon U \times V \to \mathbb{R}^d$ be a C^k function. Consider the one parameter family of problems $T_\epsilon\colon U \times V \to \mathbb{R}^d$ by

$$T_\epsilon(y, x) = y - f(x) + \epsilon H(y, x), \tag{2}$$

and note that for any $(\bar{y}, \bar{x}) \in V \times U$ we have that

$$D_1 T_\epsilon(\bar{y}, \bar{x}) = \mathrm{Id} + \epsilon D_1 H(\bar{y}, \bar{x}).$$

Note that $D_1 T_0(\bar{y}, \bar{x}) = \mathrm{Id}$, so that –by the implicit function theorem – there is a $\delta > 0$ and a smooth curve $y\colon (-\delta, \delta) \to V \subset \mathbb{R}^d$ so that $y(0) = \bar{y}$ and

$$T_\epsilon(y(\epsilon), \bar{x}) = 0,$$

for all $\epsilon \in (-\delta, \delta)$.

Moreover, for a possibly smaller $\delta > 0$ we have that

$$D_1 T_\epsilon(y(\epsilon), \bar{x}) = \mathrm{Id} + \epsilon D_1 H(y(\epsilon), \bar{x}),$$

is invertible for each $\epsilon \in (-\delta, \delta)$, by the Neumann theorem [44, 40]. Then there exists an $r > 0$ and a family of functions $F_\epsilon \colon B_r(\bar{x}) \times (-\delta, \delta) \to \mathbb{R}^d$ so that

$$F_0(x) = f(x),$$

and

$$T_\epsilon(F_\epsilon(x), x) = 0,$$

for all $x \in B_r(\bar{x})$ and $\epsilon \in (-\delta, \delta)$. The family F_ϵ depends smoothly on ϵ and is C^k for each fixed ϵ. Moreover, for small $\epsilon \neq 0$ and $\bar{x} \in U$ we take $\bar{y} = f(\bar{x})$ as an approximate zero for $T_\epsilon(y, \bar{x})$ and apply Newton's method to find $y(\epsilon)$ so that $T_\epsilon(y(\epsilon), \bar{x}) = 0$. That is, for small ϵ we can compute images of the implicitly defined mapping $F_\epsilon(x)$ using Newton's method. For larger ϵ we perform numerical continuation from the $\epsilon = 0$ case.

Note that we make no attempt to guarantee that for a given ϵ the $\bar{y}$ we find is globally unique. Indeed, it may not be. What is required for our purposes is the local uniqueness given by the implicit function theorem. This is expected to apply whenever the Newton method is successful (as Newton will struggle or fail when a solution is degenerate) and is enough to give a unique local branch of F having $F(\bar{x}) = \bar{y}$. Finally, suppose that $\bar{x} \in U$ is a hyperbolic fixed point of f, and recall that $F_0(x) = f(x)$. Since F_ϵ depends smoothly on ϵ, it follows that for small $\epsilon \neq 0$ the map $F_\epsilon(x)$ has a hyperbolic fixed point near $\bar{x}$ by the usual perturbation argument for maps.

The discussion just given shows that problems of the form given in Equation (2) provide a natural class of examples - perturbations of diffeomorphisms - for which our method applies. Two specific examples are when f is either the classic Hénon map or its three dimensional generalization to the Lomelí map. These are discussed briefly now.

1.1.1 Example 1: the Hénon map

Let $\theta_{\mathbf{1}} = (x_1, y_1), \theta_{\mathbf{2}} = (x_2, y_2)$ denote points in the plane. The Hénon map is a two parameter family of quadratic mappings defined by

$$f(\theta_{\mathbf{1}}) = f(x_1, y_1) = \begin{pmatrix} 1 + y_1 - \alpha x_1^2 \\ \beta x_1 \end{pmatrix} \tag{3}$$

The mapping is a classic example of a system with complicated dynamics, and was originally introduced in [35]. See also the books of [22, 61]. We define an implicit Hénon system T_ϵ : $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T_\epsilon(\theta_{\mathbf{2}}, \theta_{\mathbf{1}}) = T_\epsilon(x_2, y_2, x_1, y_1) = \begin{pmatrix} x_2 - (1 - \alpha x_1^2 + y_1 + \epsilon x_2^5) \\ y_2 - \beta x_1 + \epsilon y_2^5 \end{pmatrix}. \tag{4}$$

Here we have choosen, somewhat arbitrarily, the perturbation term

$$H(y, x) = \begin{pmatrix} x^5 \\ y^5 \end{pmatrix},$$

to be of high enough polynomial order that it is difficult (if not impossible) to work out useful formulas for the implicitly defined system.

The equation for a fixed point is $T_\epsilon(\mathbf{x}, \mathbf{x}) = 0$, or

$$\left(\begin{array}{c} x_1 - (1 - \alpha x_1^2 + y_1 + \epsilon x_1^5) \\ y_1 - \beta x_1 + \epsilon y_1^5 \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \end{array} \right).$$

Similarly, the multiple shooting equations for a period two orbit are

$$\begin{aligned} T_\epsilon(\theta_2, \theta_1) &= 0 \\ T_\epsilon(\theta_1, \theta_2) &= 0, \end{aligned} \tag{5}$$

or

$$\begin{aligned} x_2 - (1 - \alpha x_1^2 + y_1 + \epsilon x_2^5) &= 0 \\ y_2 - \beta x_1 + \epsilon y_2^5 &= 0 \\ x_1 - (1 - \alpha x_2^2 + y_2 + \epsilon x_1^5) &= 0 \\ y_1 - \beta x_2 + \epsilon y_1^5 &= 0 \end{aligned} \tag{6}$$

The implicit equations for fixed points or periodic orbits are solved using Newton's method. Eigenvalues and eigenvectors we compute using the approach outlined in Section B.1.

For classical parameters $a, b \in \mathbb{R}$ the Hénon map has a pair of hyperbolic fixed points, each with one stable and one unstable eigenvalue. Then for small epsilon the same is true for the perturbation. The numerical value of the unperturbed fixed points serve as initial guesses for the perturbed fixed points in the Newton method. Similar comments hold for periodic orbits.

1.1.2 Example 2: the Lomelí map

We consider also the five parameter family of maps $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(x, y, z) = \left(\begin{array}{c} z + Q(x, y) \\ x \\ y \end{array} \right) \tag{7}$$

where $Q(x, y) = \rho + \gamma x + a x^2 + bxy + cy^2$ and one usually takes $a + b + c = 1$. The system is known as *the Lomelí map*, and it is a normal form quadratic volume preserving maps with quadratic inverse. In that sense it can be thought of as a three dimensional generalization of the area preserving Hénon map. The map was first introduced in [49], and was subsequently studied by a number of authors including [49, 23, 55, 57, 11, 28].

Let $\theta_1 = (x_1, y_1, z_1), \theta_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$. We consider the dynamics implicitly defined by the map $T_\epsilon : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\begin{aligned} T_\epsilon(\theta_2, \theta_1) &= T_\epsilon(x_2, y_2, z_2, x_1, y_1, z_1) \\ &= \left(\begin{array}{c} x_2 - \rho - \tau x_1 - z_1 - a x_1^2 - b x_1 y_1 - c y_2^2 + \epsilon \left(\alpha y_2^5 + \beta z_2^5 \right) \\ y_2 - x_1 + \epsilon \gamma z_2^5 \\ z_2 - y_1 \end{array} \right). \end{aligned} \tag{8}$$

Note that T_ϵ is analytic in all variables. We remark that the perturbation is chosen so that the system still preserves volume.

Fixed of points of the implicit Lomelí system (8) are obtained as solutions of

$$\begin{pmatrix} x - \rho - \tau x - z - ax^2 - bxy - cy^2 + \epsilon(\alpha y^5 + \beta z^5) \\ y - x + \epsilon\gamma z^5 \\ z - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

Similarly, a period four orbit for the Lomelí system solves the equations

$$\begin{aligned} T_\epsilon(\theta_2, \theta_1) &= 0 \\ T_\epsilon(\theta_3, \theta_2) &= 0 \\ T_\epsilon(\theta_4, \theta_3) &= 0 \\ T_\epsilon(\theta_1, \theta_4) &= 0. \end{aligned} \tag{10}$$

More explicitly, this is

$$\begin{aligned} x_2 - \rho - \tau x_1 - z_1 - ax_1^2 - bx_1y_1 - cy_1^2 + \epsilon(\alpha y_2^5 + \beta z_2^5) &= 0 \\ y_2 - x_1 + \epsilon\gamma z_2^5 &= 0 \\ z_2 - y_1 &= 0 \\ x_3 - \rho - \tau x_2 - z_2 - ax_2^2 - bx_2y_2 - cy_2^2 + \epsilon(\alpha y_3^5 + \beta z_3^5) &= 0 \\ y_3 - x_2 + \epsilon\gamma z_3^5 &= 0 \\ z_3 - y_2 &= 0 \\ x_4 - \rho - \tau x_3 - z_3 - ax_3^2 - bx_3y_3 - cy_3^2 + \epsilon(\alpha y_4^5 + \beta z_4^5) &= 0 \\ y_4 - x_3 + \epsilon\gamma z_4^5 &= 0 \\ z_4 - y_3 &= 0 \\ x_1 - \rho - \tau x_4 - z_4 - ax_4^2 - bx_4y_4 - cy_4^2 + \epsilon(\alpha y_1^5 + \beta z_1^5) &= 0 \\ y_1 - x_4 + \epsilon\gamma z_1^5 &= 0 \\ z_1 - y_4 &= 0 \end{aligned} \tag{11}$$

The equations for fixed and periodic orbits are again amenable to Newton's method, and the multipliers $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$, and associated eigenvectors $\xi_{j1}, \xi_{j2}, \xi_{j3} \in \mathbb{C}^3$, $1 \leq i \leq 4$ are are computed as discussed in Section A.3,

1.2 Generalized notions of dynamical systems

Generalizations of nonlinear dynamics to the setting of relations instead of functions, where neither uniqueness of forward or backward iterates is required, appeared in the early 1990's in the work of Akin [2] and McGehee [53]. The Ph.D. dissertation of Sander generalized stable/unstable manifold theory to the setting of relations [25], and work by Lerman [45] and Wather [65] studied transverse homoclinic/heteroclinic phenomena in the setting of non-invertible dynamical systems, with a view toward applications to semi-flows in infinite dimensions. Further work by Sander [63, 62, 64] studied homoclinic bifurcations for noninvertible maps and relations.

The ideas of the authors mentioned above have been applied to generalized dynamical systems coming from applications to population dynamics [3], iterated difference methods/numerical algorithms [50, 25], delay differential equations [65], adaptive control [1], discrete variational problems [27, 67], and economic theory [41, 42, 43, 54]. Indeed, this list is far from comprehensive and the interested reader will find a wealth of additional references in the works just cited. We mention also the recent book on dynamical systems defined by implicit rules [51], where many further examples and references are found.

A complementary approach to the study of generalized dynamics, based on functional analytic rather than topological tools, is given by the parameterization method. The idea of the parameterization method is to consider the equation defining a (semi-)conjugacy between a subset of the given system, and some simpler model problem. Example include stable/unstable manifolds attached to fixed or periodic orbits, or a quasiperiodic family of orbits - that is an invariant torus. The equations describing special solutions often have nicer properties than the Cauchy problem describing a generic orbit. While this observation is important for a classical dynamical system defined by an invertible map, it can be even more useful when studying dynamical systems which are not invertible, are ill posed, or are not even single valued.

The parameterization method was originally developed for studying non-resonant invariant manifolds attached to fixed points of infinite dimensional maps between Banach spaces in a series of papers by Cabré, Fontich, and de la Llave [6, 7, 8], though the approach has roots going back to the Nineteenth Century (see appendix B of [8]). The method has since been extended to the study of parabolic fixed points [4], invariant tori and their stable/unstable fibers [31, 30, 32, 10, 38], for stable/unstable manifolds attached to periodic solutions of ordinary differential equations [37, 13, 59], and to develop KAM arguments without action angle variables [18, 9]. See also the recent book of Haro, Canadell, Figueras, Luque and Mondelo [29] for much more complete overview.

The parameterization method can also be extended to generalized dynamical systems like those mentioned in the first paragraph of this section. We refer for example to the work of [14, 15] on stable and center manifolds for ill-posed problem, the work of [20, 68] on invariant tori for ill-posed PDEs and state dependent delay differential equations [34, 33], the work of [17, 16] on periodic orbits and their isochrons in state dependent perturbations of ODEs, and the related work of [12, 26] on computer assisted existence proofs for periodic orbits in the Boussinesq equation and in some state dependent delay differential equations.

Remark 1.1. The work of [19], which develops numerical methods for computing stable/unstable manifolds attached to fixed points of implicitly defined discrete time dynamical systems, is the jumping off point for the present study – which extends their method to periodic orbits. Another paper closely related to the present work is [28], where the authors develop a multiple shooting parameterization method for computing stable/unstable manifolds attached to periodic orbits of diffeomorphisms. The main contribution of the present work is to extend the methods of [28] to the more general setting of implicitly defined systems, and to illustrate their implementation in examples. This also extends the applicability of the parameterization method for implicit systems beyond the foundations laid in [19].

2 A brief overview of the parameterization methods for maps

We review some basic results about the parameterization method for maps.

2.1 Parameterization of stable/unstable manifolds attached to fixed points

In this section we recall some basic results from the work of [6, 7, 8]. In fact, we paraphrase these results, simplifying them to the finite dimensional setting of the present work. The reader interested in infinite dimensional dynamics can consult the references just cited for theorems formulated in full generality. Moreover, we recall that spec(x) refers to the eigenvalues of $DF(x)$, and refer to Section A.2 for a review of notions and notation related to stability of fixed points.

Lemma 2.1 (Parameterization method for fixed points in $\mathbb{R}^d$). *Suppose that $U \subset \mathbb{R}^d$ is an open set, that $F \colon U \to \mathbb{R}^d$ is a $C^k(U)$ mapping with $k = 1, 2, 3, \ldots, \infty, \omega$, that $x_* \in U$ is a fixed point of F, and that $DF(x_*)$ is invertible. Take $d_s = dim(\mathbb{E}^s)$ to be the dimension of the stable (generalized) eigenspace/the number of stable eigenvalues (counted with multiplicity).*

Let $\alpha, \beta > 0$ have that

$$|\lambda| \leq \alpha < 1,$$

for all $\lambda \in spec_s(x_*)$ and

$$1 < \beta \leq |\lambda|,$$

for all $\lambda \in spec_u(x_*)$. Let $L \in \mathbb{N}$ be the smallest natural number with

$$\alpha^L < \frac{1}{\beta},$$

and assume that

$$L + 1 < k.$$

Then there exists an open set $D_s \subset \mathbb{R}^{d_s}$ with $0 \in D_s$, a polynomial $K \colon D_s \to \mathbb{R}^{d_s}$ of degree not more than L, and a C^k mapping $P \colon D_s \to \mathbb{R}^d$ so that

1. $$P(0) = x_*,$$

2. *The columns of $DP(0)$ span $\mathbb{E}^s$, and*

3. $$F(P(\theta)) = P(K(\theta)), \tag{12}$$

   for all $\theta \in D_s$.

Moreover, P is unique up to the choice of the scalings of the columns of $DP(0)$.

Several additional comments are in order. First, we remark that the columns of $DP(0)$ can be taken as stable (generalized) eigenvectors of $DF(x_*)$, so that $DP(0)$ is unique up to the choice of the scalings of these vectors. The theorem says that once these scalings are fixed, the parameterization P is uniquely determined.

Note also that if $k = \infty$ or $k = \omega$ then $L + 1 < k$ is automatically satisfied. Consider the case when $k = \omega$, that is F (real) analytic at x_*, and suppose that the scalings of $DP(0)$ are fixed. Then P, and hence its power series expansion at 0, is uniquely determined. In this case K and P are worked out by power matching arguments, and these arguments lead in turn to practical numerical schemes. In fact, the scalings of the eigenvectors can be chosen so that the power series coefficients of P decay at a desired exponential rate. Numerical schemes for determining the optimal scalings of eigenvectors are developed in [5].

The following lemma allows us to determine the polynomial mapping K a-priori, in the case that some (generic) non-resonance conditions hold between the stable eigenvalues.

Lemma 2.2 (Non-resonant eigenvalues implies K linear). *Let $\lambda_1, \ldots, \lambda_{d_s} \in \mathbb{C}$ denote the stable eigenvalues of $DF(x_*)$, and assume that each has multiplicity exactly one. Moreover, assume that for all $(n_1, \ldots, n_{d_s}) \in \mathbb{N}^{d_s}$ with*

$$2 \leq n_1 + \ldots + n_{d_s} \leq L,$$

we have that

$$\lambda_1^{n_1} \ldots \lambda_{d_s}^{n_{d_s}} \notin spec_s(x_*). \tag{13}$$

Then we can choose K to be the linear mapping

$$K(\theta) = \Lambda\theta,$$

where $\theta = (\theta_1, \ldots, \theta_{d_s}) \in \mathbb{R}^{d_s}$ and Λ is the $d_s \times d_s$ matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_{d_s-1} & 0 \\ 0 & 0 & \ldots & 0 & \lambda_{d_s} \end{pmatrix}.$$

That is, Λ is the matrix with the stable eigenvalues on the diagonal entries and zeros in all other entries.

We say that the stable eigenvalues are non-resonant when the condition given by Equation (13) is satisfied. We say there is a resonance at $(n_1, \ldots, n_{d_s}) \in \mathbb{N}^{d_s}$ if

$$\lambda_1^{n_1} \ldots \lambda_{d_s}^{n_{d_s}} \in \mathrm{spec}_s(x_*).$$

In this case, the polynomial K is required to have a monomial term of the form $c\,\theta_1^{n_1} \ldots \theta_{d_s}^{n_{d_s}}$ with non-zero $c \in \mathbb{R}^{d_s}$. That is, even in the resonant case the form of the polynomial K can be determined by examining the resonances between the stable eigenvalues. Numerical procedures for determining P and K in the resonant case are discussed in [66].

It is worth remarking that when the stable eigenvalues are non-resonant, Equation (12) reduces to

$$F(P(\theta)) = P(\Lambda\theta), \qquad \theta \in D_s \subset \mathbb{R}^{d_s}, \tag{14}$$

so that P is now the only unknown in the equation. Indeed, the equation is viewed as requiring a conjugacy between the dynamics on the image of P and the diagonal linear map given by the stable eigenvalues.

We also note that the domain D_s can be chosen so that $\Lambda_s D_s \subset D_s$. In this case, since Equation (14) holds, it is easy to see that P parameterizes a local stable manifold. To see this, let $\theta \in D_s$. Since P is continuous (in fact C^k) we have that

$$\begin{aligned} \lim_{n\to\infty} F^n(P(\theta)) &= \lim_{n\to\infty} F(P(\Lambda^n\theta)) \\ &= F\left(P\left(\lim_{n\to\infty} \Lambda^n\theta\right)\right) \\ &= F(P(0)) \\ &= F(x_*) \\ &= x_*, \end{aligned}$$

so that image$(P) \subset W^s(x_*)$. Noting that image(P) is a d_s dimensional manifold tangent to $\mathbb{E}^s$ at x_* gives equality rather than inclusion.

Remark 2.3 (Generality)**.** Lemma 2.1 follows trivially from Theorem 1.1 of [6, 7, 8]. In the much more general work just cited U is taken to be an open subset of a Banach space, and the infinite dimensional complications result in more delicate spectral assumptions. The finite dimensional setting of the present work, and the fact that we parameterize the full stable manifold simplify somewhat the statement of Lemma.

Remark 2.4 (Unstable manifold parameterization)**.** Note that in Lemma 2.1, the assumption that $DF(x_*)$ is invertible implies that F is a local diffeomorphism. Then, in a small enough neighborhood of x_* there is a well defined C^k inverse mapping F^{-1}. Let Σ denote the diagonal matrix of unstable eigenvalues of $DF(x_*)$, so that Σ^{-1} is the matrix of stable eigenvalue of $DF^{-1}(x_*)$. Assume that these stable eigenvalues (entries of Σ^{-1}) are non-resonant. Then there exists an open set D_u and a C^k mapping $Q\colon D_u \to \mathbb{R}^d$ so that

$$F^{-1}(Q(\sigma)) = Q(\Sigma^{-1}\sigma), \qquad \sigma \in D_u.$$

Applying F to both sides of the equation and composing with Σ leads to the equation

$$Q(\Sigma\sigma) = F(Q(\sigma)), \qquad \sigma \in D_u.$$

In other words, the unstable parameterization Q satisfies exactly the same invariance equation as the stable parameterization P. Only the conjugating matrix changes, in the sense that the matrix of stable eigenvalues Λ is replaced by the matrix of unstable eigenvalues Σ.

2.2 Stable/unstable manifolds attached to periodic orbits

The material in this section provides a brief review of the techniques developed in [28] for parameterization of stable/unstable manifolds attached to periodic orbits of an explicitly given diffeomorphism. The main idea is to exploit multiple shooting schemes which avoid function compositions.

Let $x_1, , ..., x_N \in \mathbb{R}^d$ be the points along a hyperbolic period N orbit. Let $\lambda_1, \ldots, \lambda_{d_s}$ denote the stable multipliers of the periodic orbit, and let

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_{d_s-1} & 0 \\ 0 & 0 & \ldots & 0 & \lambda_{d_s} \end{pmatrix},$$

denote the $d_s \times d_s$ diagonal matrix of stable multipliers (similarly Σ denote the $d_u \times d_u$ diagonal matrix of unstable multipliers). For $1 \leq j \leq d_s$, let $\xi_{j,1}, \ldots, \xi_{j,d_s} \subset \mathbb{C}^d$ denote the eigenvectors of $DF(x_j)$ associated with the eigenvalue λ_j.

Assume that the stable multipliers are non-resonant, in the sense of Lemma 2.2. Then, by Lemma 2.2, there is an open set $D_s \subset \mathbb{R}^{d_s}$ and are unique $P_1, \dots P_N \colon D_s \to \mathbb{R}^d$ so that

$$\begin{aligned} P_1(0) &= x_1 \\ &\vdots \\ P_N(0) &= x_N \end{aligned}$$

and

$$\begin{aligned} DP_1(0) &= [\xi_{1,1}, \dots, \xi_{1,d_s}] \\ &\vdots \\ DP_N(0) &= [\xi_{N,1}, \dots, \xi_{N,d_s}], \end{aligned}$$

having that

$$\begin{aligned} F^N(P_1(\theta)) &= P_1(\Lambda\theta) \\ &\vdots \\ F^N(P_N(\theta)) &= P_N(\Lambda\theta) \end{aligned} \tag{15}$$

Note that we are treating the periodic point as a fixed point of the composition map, so that Lemmas 2.2 and 2.1 apply directly.

On the other hand, the presence of composition mapping F^N is precisely what makes these equations difficult, as F^N is in general a much more complicated map than F. The main result of [28] (see Section 3) is that the parameterizations admit a composition free formulation.

Lemma 2.5 (Composition free invariance equations)**.** *Under the hypotheses above (non-degenerate periodic orbit and non-resonant multipliers), the functions $P_1, \dots, P_N \colon D_s \to \mathbb{R}^d$ satisfy the system of composition free equations*

$$\begin{aligned} F(P_1(\theta)) &= P_2\left(\tilde{\Lambda}\theta\right) \\ F(P_2(\theta)) &= P_3\left(\tilde{\Lambda}\theta\right) \\ &\vdots \\ F(P_{N-1}(\theta)) &= P_N\left(\tilde{\Lambda}\theta\right) \\ F(P_N(\theta)) &= P_1\left(\tilde{\Lambda}\theta\right) \end{aligned}$$

where

$$\tilde{\Lambda} = \begin{pmatrix} \sqrt[N]{\lambda_1} & 0 & \dots & 0 & 0 \\ 0 & \sqrt[N]{\lambda_2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \sqrt[N]{\lambda_{d_s-1}} & 0 \\ 0 & 0 & \dots & 0 & \sqrt[N]{\lambda_{d_s}} \end{pmatrix},$$

is the diagonal matrix of N-th roots of the multipliers. (Here it is sufficient to choose any branch of the N-th root).

On easily checks that if $P_1, \ldots, P_N$ satisfy the invariance equations in Lemma 2.5, then they solve Equations (15). From the perspective of numerical calculations it is much easier to solve simultaneously the system of equations given in Lemma 2.5 than it is to apply the parameterization method directly to the composition mapping F^N. This is illustrated by examples in [28]. Note also that the N-th roots of the multipliers are the eigenvalues of the derivative of the multiple shooting map, see Equation (31).

3 Parameterization methods for implicitly defined maps

Recall that implicitly defined dynamical system were discussed in the introduction and are reviewed in more detail in Section B. We now discuss the parametrization method for fixed points of implicit maps as introduced in [19], and then extend these ideas via a multiple shooting scheme to periodic orbits of implicit systems. For the sake of clarity let us recall that $T\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a smooth mapping, and that we are interested in the implicitly defined dynamical system F is given by the rule

$$F(x) = y \qquad \text{if and only if} \quad T(y, x) = 0.$$

Then x_* is a fixed point if F if and only if $T(x_*, x_*) = 0$. See Equation (1) in the introduction for the implicit equations satisfied by a periodic orbit.

3.1 Stable/unstable manifolds attached to implicit fixed points

Before introducing new results for periodic orbits of implicitly defined maps, we first review the main result of[19] for fixed points.

Theorem 3.1. *Suppose that $U, V \subset \mathbb{R}^d$ are open sets and that $T\colon U \times V \to \mathbb{R}^d$ is a C^k mapping with fixed point $x_* \in U \cap V$, that is*

$$T(x_*, x_*) = 0.$$

Assume that

- *$D_1T(x_*, x_*)$ is invertible.*
- *Let $\lambda_1, \ldots, \lambda_{d_s} \in \mathbb{C}$ denote the stable eigenvalues and $\xi_1, \ldots, \xi_{d_s} \in \mathbb{C}^d$ associated eigenvectors of $-D_1T(x_*, x_*)^{-1} D_2T(x_*, x_*)$. Assume that the stable eigenvalues are distinct (otherwise choose the appropriate ξ_j as generalized eigenvectors).*
- *Let*

  $$\alpha = \max_{1 \le j \le d_s} |\lambda_j|,$$

  $$\beta = \max_{\lambda \in \mathrm{spec}_u(x_*)} \left|\lambda^{-1}\right|,$$

  and $2 \le L$ be the smallest integer so that

  $$\alpha^L \beta < 1.$$

  Assume that $L + 1 \le k$.

- *Assume that for all* $(n_1, \ldots, n_{d_s}) \in \mathbb{N}^{d_s}$ *with* $2 \le n_1 + \ldots + n_{d_s} \le L$ *we have that*

$$\lambda_1^{n_1} \ldots \lambda_{d_s}^{n_{d_s}} \ne \lambda_j$$

for $1 \le j \le \lambda_{d_s}$.

Then there exists an open set $D_s \subset \mathbb{R}^{d_s}$ *with* $0 \in D_s$*, and a* C^k *mapping* $P\colon D_s \to \mathbb{R}^d$ *so that*

$$P(0) = x_*,$$

$$DP(0) = [\xi_1, \ldots, \xi_{d_s}],$$

and

$$T(P(\Lambda\theta), P(\theta))) = 0, \qquad \theta \in D_s \tag{16}$$

where Λ *is the* $d_s \times d_s$ *matrix with the stable eigenvalues on the diagonal entries and zero entries elsewhere.* P *parameterizes a local stable manifold attached to the fixed point* x_* *of the implicitly defined mapping* F*.* P *is unique up to the choices of the scalings of the eigenvectors.*

The proof is a simple matter of translating the assumptions about T, its derivative, and its eigenvalues/eigenvectors into equivalent statements about F, and then applying Lemma 2.1 to the implicitly defined mapping F. Recalling for example that $F(x) = y$ if and only if $T(y, x) = 0$, then by letting $y = P(\Lambda\theta)$ and $x = P(\theta)$, Equation (16), is equivalent to

$$F(P(\theta)) = P(\Lambda\theta), \qquad \theta \in D_s,$$

and this is precisely Equation (14).

3.2 Stable/unstable manifolds attached to implicit periodic orbits

We now introduce a multiple shooting version of the parameterization method for periodic orbits of implicitly defined systems. We remark that the multipliers and eigenvectors for such an orbit are computed as discussed in Section B.1.

Theorem 3.2. *Suppose that* $U, V \subset \mathbb{R}^d$ *are open sets and that* $T\colon U \times V \to \mathbb{R}^d$ *is a* C^k *mapping, and that* $x_1, \ldots, x_N \in U \cap V$ *have*

$$\begin{aligned} T(x_2, x_1) &= 0 \\ &\vdots \\ T(x_{N-1}, x_N) &= 0 \\ T(x_1, x_N) &= 0 \end{aligned}$$

Assume that:

- *the matrices* $D_1T(x_2, x_1), \ldots, D_1T(x_N, x_{N-1}), D_1T(x_1, x_N)$ *are invertible.*
- *Let* $\lambda_1, \ldots, \lambda_{d_s} \in \mathbb{C}$ *denote the stable multipliers and for* $1 \le j \le N$ *let* $\xi_{j,1}, \ldots, \xi_{j,d_s} \in \mathbb{C}^d$ *denote associated eigenvectors. Assume that the stable multipliers are distinct (otherwise choose the appropriate generalized eigenvectors).*

- *Let*

$$\alpha = \max_{1\le j\le d_s} |\lambda_j|,$$

$$\beta = \max_{\lambda\in \mathrm{spec}_u(x_*)} \left|\lambda^{-1}\right|,$$

and $2 \le L$ *be the smallest integer so that*

$$\alpha^L \beta < 1.$$

Assume that $L + 1 \le k$.

- *Assume that for all* $(n_1, \ldots, n_{d_s}) \in \mathbb{N}^{d_s}$ *with* $2 \le n_1 + \ldots + n_{d_s} \le L$ *we have that*

$$\lambda_1^{n_1} \ldots \lambda_{d_s}^{n_{d_s}} \ne \lambda_j$$

for $1 \le j \le \lambda_{d_s}$.

Then there exists an open set $D_s \subset \mathbb{R}^{d_s}$ *with* $0 \in D_s$, *and* C^k *mappings* $P_1, \ldots, P_N \colon D_s \to \mathbb{R}^d$ *so that*

$$P_1(0) = x_1, \ldots, P_N(0) = x_N$$

$$DP_1(0) = [\xi_{1,1}, \ldots, \xi_{1,d_s}], \ldots, DP_N(0) = [\xi_{N,1}, \ldots, \xi_{N,d_s}],$$

and

$$\begin{aligned} T(P_2(\tilde\Lambda\theta), P_1(\theta))) &= 0 \\ T(P_3(\tilde\Lambda\theta), P_2(\theta))) &= 0 \\ &\vdots \\ T(P_N(\tilde\Lambda\theta), P_{N-1}(\theta))) &= 0 \\ T(P_1(\tilde\Lambda\theta), P_N(\theta))) &= 0 \end{aligned} \tag{17}$$

for all $\theta \in D_s$. *Here* $\tilde\Lambda$ *is the* $d_s \times d_s$ *matrix with* N*-th roots of the stable eigenvalues on the diagonal entries and zero entries elsewhere.* P_j *parameterizes a local stable manifold attached to the periodic point* x_j *of the implicitly defined mapping* F. *The* P_j *are unique up to the choices of the scalings of the eigenvectors.*

The theorem follows by applying Lemma 2.5 to the implicit map F defined by $T(y, x) = 0$. We remark that the knowledge the P_j exist tells us that it is reasonable to develop numerical methods to find them. Moreover, the fact that they solve a functional equation leads to efficient numerical methods and a-posteriori error bounds. Indeed, if T is analytic then the P_j are analytic as well, and it makes sense to look for power series solutions of the functional equations. This topic is pursued in the next section.

4 Formal series solution of equation (17)

In this section we illustrate the formal series calculations which allow us to compute stable/unstable manifolds using the parameterization method. In particular, we derive the linear recurrence equations for the power series coefficients of the functions solving Equation (17). We illustrate the method for several examples of one dimensional stable/unstable manifolds attached to implicitly defined fixed and periodic points. These calculations involve only power series of one variable. Similar calculations for two dimensional manifolds, involving power series of two variables, are given in the Appendices.

4.1 Operations on formal power series

We recall some basic facts about manipulating power series. Consider two infinite sequences of complex numbers $\{a_n\}_{n=0}^\infty, \{b_n\}_{n=0}^\infty \subset \mathbb{C}$ and the corresponding power series

$$f(z) = \sum_{n=0}^\infty a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^\infty b_n z^n.$$

Suppose that $\lambda \in \mathbb{C}$. Then

$$f(\lambda z) = \sum_{n=0}^\infty \lambda^n a_n z^n.$$

Also, for any $\alpha, \beta \in \mathbb{C}$ the linear combination $\alpha f + \beta g$ has power series

$$\alpha f(z) + \beta g(z) = \sum_{n=0}^\infty (\alpha a_n + \beta b_n) z^n.$$

Moreover, the product of two power series is given by the Cauchy product

$$f(z)g(z) = \sum_{n=0}^\infty (a * b)_n z^n,$$

where

$$(a * b)_n = \sum_{k_1+k_2=n} a_{k_1} b_{k_2} = \sum_{k=0}^{n} a_{n-k} b_k.$$

Higher order products are defined analogously. For example suppose that $f_1, \ldots, f_N$ are power series given by

$$f_i(z) = \sum_{n=0}^\infty a_n^i z^n, \qquad 1 \le i \le N.$$

Then

$$f_1(z) \ldots f_N(z) = \sum_{n=0}^\infty \left(a^1 * \ldots * a^N\right)_n z^n,$$

where the N-th Cauchy product is given by

$$\begin{aligned}(a^1 * \ldots * a^N)_n &= \sum_{k_1+\ldots+k_N=n} a_{k_1}^1 \ldots a_{k_N}^N \\ &= \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{N-2}=0}^{k_{N-3}} \sum_{k_{N-1}=0}^{k_{N-2}} a_{n-k_1}^1 a_{k_1-k_2}^2 \ldots a_{k_{N-2}-k_{N-1}}^{N-1} a_{k_{N-1}}^N.\end{aligned}$$

Note that the first form of the sum is easier to read, but that the second form is easily implemented in computer programs as a loop.

Another important operation is the extraction of the coefficients of n-th order from the n-th term of a Cauchy product. For example, we have that

$$(a * b)_n = b_0 a_n + a_0 b_n + \sum_{k=1}^{n-1} a_{n-k} b_k.$$

We write

$$(\widehat{a * b})_n = \sum_{k=1}^{n-1} a_{n-k} b_k,$$

to denote the terms in the Cauchy product depending only on lower order terms. Note that this is

$$(\widehat{a * b})_n = (a * b)_n - a_0 b_n - b_0 a_n = \sum_{\substack{k_1+k_2=n \\ k_1,k_2 \neq n}} a_{k_1} b_{k_2}$$

Similarly, define

$$(a^1 \widehat{* \ldots *} a^N)_n = (a^1 * \ldots * a^N)_n - a_0^1 \ldots a_0^{N-1} a_n^N - \ldots - a_0^2 \ldots a_0^N a_n^1,$$

which is equivalent to

$$(a^1 \widehat{* \ldots *} a^N)_n = \sum_{\substack{k_1+\ldots+k_N=n \\ k_1,\ldots,k_N \neq n}} a_{k_1}^1 \ldots a_{k_N}^N.$$

4.2 An overview of the power matching strategy

In pursuit of a formal series solution of Equation (17), suppose that $x_1, \ldots, x_N \in \mathbb{R}^d$ is a period N-orbit for the implicitly defined dynamics. That is, we assume that $T \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a smooth function and that $x_1, \ldots, x_N$ solve Equation (1). In the discussion to follow, let λ denote a stable/unstable multiplier and $\xi_1, \ldots, \xi_N \in \mathbb{R}^d$ be an associated collection of stable/unstable eigenvectors, computed as described in Section A.3.

Since we want to solve a functional equation (Equation (17)) with prescribed first order data, we look for a power series solution of form

$$P_j(\theta) = \sum_{n=0}^{\infty} p_n^j \theta^n. \qquad 1 \le j \le N.$$

Here, for each $n \in \mathbb{N}$ and $1 \le j \le N$, the power series coefficient $p_n^j \in \mathbb{R}^d$. Note that if the P_j are the solutions of Equation (17), then for $1 \le j \le N$ we have that

$$p_0^j = x_j, \qquad \text{and} \qquad p_1^j = \xi_j.$$

Supposing that T is analytic in both variables (otherwise we proceed formally) write

$$Q_j(\theta) = T(P_{j+1}(\tilde{\Lambda}\theta), P_j(\theta)) = \sum_{n=0}^{\infty} q_n^j \theta^n = 0, \tag{18}$$

where it is understood that $j_{N+1} = j_1$, and where the q_n^j depend on the coefficients of the P_j in a possibly complicated way. Nevertheless, since $Q_j(\theta) = 0$, we have that

$$q_n^j = 0, \tag{19}$$

for all $n \geq 0$. Since our unknowns are the coefficients p_n^j, and since the q_n^j depend on them, we use Equation (19) to derive recurrence relations for the coefficients of the P_j. The following example is meant to provide some insight into this procedure. Detailed calculations for non-trivial examples are given in the following sections, and the appendices.

Example 4.1. As a simple example, consider the nonlinear mapping $T\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$T(y, x) = x + y + xy + \frac{1}{2}x^2.$$

Let $F\colon U \subset \mathbb{R} \to \mathbb{R}$ denote the mapping defined by the implicitly by the requirement that $F(x) = y$ if and only if

$$T(y, x) = 0.$$

Suppose now that $x_j \in \mathbb{R}$, $1 \leq j \leq N$ are periodic for F. That is, assume that

$$\begin{aligned} T(x_2, x_1) &= 0 \\ &\vdots \\ T(x_1, x_N) &= 0, \end{aligned}$$

with

$$\partial_1 T(x_{j+1}, x_j) \neq 0,$$

for $1 \leq j \leq N$, again with the understanding that $x_{N+1} = x_1$. Suppose in addition that the periodic orbit has multiplier $-1 < \lambda < 1$. In this case the stable manifold is the union of a one dimensional neighborhoods of the points x_j, and we seek

$$\begin{aligned} P_1(\theta) &= \sum_{n=0}^{\infty} p_n^1 \theta^n \\ &\vdots \\ P_N(\theta) &= \sum_{n=0}^{\infty} p_n^N \theta^n, \end{aligned}$$

satisfying Equation (17).

Our aim is to work out the coefficients of the $Q_j(\theta)$ defined in Equation (18). To this end, consider the component equation

$$Q_j(\theta) = \sum_{n=0}^{\infty} q_n^j \theta^n = T\left(P_{j+1}(\lambda\theta), P_j(\theta)\right) = 0,$$

which becomes

$$
\begin{aligned}
Q_j(\theta) &= P_j(\theta) + P_{j+1}(\lambda\theta) + P_j(\theta)P_{j+1}(\lambda\theta) + \frac{1}{2}P_j(\theta)^2 \\
&= \sum_{n=0}^{\infty} p_n^j \theta^n + \sum_{n=0}^{\infty} p_n^{j+1}\lambda^n\theta^n + \left(\sum_{n=0}^{\infty} p_n^j\theta^n\right)\left(\sum_{n=0}^{\infty} p_n^{j+1}\lambda^n\theta^n\right) + \frac{1}{2}\left(\sum_{n=0}^{\infty} p_n^j\theta^n\right)^2 \\
&= \sum_{n=0}^{\infty}\left(p_n^n + \lambda^n p_n^{j+1} + \sum_{k=0}^{n}\lambda^k p_{n-k}^j p_k^{j+1} + \sum_{k=0}^{n}\frac{1}{2}p_{n-k}^j p_k^j\right)\theta^n.
\end{aligned}
$$

Matching like powers results in

$$
q_n^j = p_n^j + \lambda^n p_n^{j+1} + \sum_{k=0}^{n}\lambda^k p_{n-k}^j p_k^{j+1} + \sum_{k=0}^{n}\frac{1}{2}p_{n-k}^j p_k^j,
$$

for $n \geq 2$ (the first order coefficients are already constrained). Recalling that $q_n^j = 0$ and isolating the p_n^j and p_n^{j+1} terms on the left hand side of the equality leads to

$$
\begin{aligned}
p_n^j + \lambda^n p_n^{j+1} + \lambda^n p_0^j p_n^{j+1} + p_0^{j+1}p_n^j + p_0^j p_n^j &= -\sum_{k=1}^{n-1}\lambda^k p_{n-k}^j p_k^{j+1} - \sum_{k=1}^{n-1}\frac{1}{2}p_{n-k}^j p_k^j \\
&= -(\widehat{p^j * p^{j+1}})_n - \frac{1}{2}(\widehat{p^j * p^j})_n
\end{aligned}
$$

or

$$
\begin{pmatrix} 1 + p_0^{j+1} + p_0^j & \lambda^n(1 + p_0^j) \end{pmatrix}\begin{bmatrix} p_n^j \\ p_n^{j+1} \end{bmatrix} = s_n^j
$$

where s_n depends only on lower order coefficients.

Since $p_0^j = x_j$ for $1 \leq j \leq N$, one easily checks that the entries of the row vector on the left hand side of the equation depend on derivatives of T evaluated along the periodic orbit. More precisely, we have that

$$
\begin{pmatrix} \frac{\partial}{\partial x}T(x_{j+1}, x_j) & \lambda^n \frac{\partial}{\partial y}T(x_{j+1}, x_j) \end{pmatrix}\begin{bmatrix} p_n^j \\ p_n^{j+1} \end{bmatrix} = s_n^j
$$

By combining the results for each of the components, we see that the n-th order coefficients solve a linear equation of the form

$$
A_n \begin{bmatrix} p_n^1 \\ \vdots \\ p_n^N \end{bmatrix} = \begin{bmatrix} s_n^1 \\ \vdots \\ s_n^N \end{bmatrix},
$$

for $n \geq 2$. Since the first order terms are known, we can solve for $n = 2$. Once these have been obtained, we solve for $n = 3$. And so on.

The problem of determining the power series coefficients can be solved quite generally for multivariable power series by exploiting the Faa di Bruno formula (multivariable generalization of the Leibniz rule). See for example the arguments for maps in [6], or the arguments for unstable manifolds of delay differential equations in [36]. This approach however leads to formulas which may be cumbersome in practice, and we find it illuminating to consider the procedure in the context of specific examples. We illustrate the formal series computation of the power series coefficients for parameterizations of some one and two dimensional stable/unstable manifolds attached to fixed and periodic orbits in polynomial examples in two and three dimensions. It is fairly straightforward to generalize these computations to any polynomial system. Computations for non-polynomial systems are handled using automatic differentiation for power series. Non-polynomial nonlinearities are discussed in detail in [29]. See also [39, 19, 28].

4.3 A worked example: fixed points of an implicit Hénon system

We now derive a formal series solution of the invariance equation given in Equation (16) for the stable/unstable manifold attached to a fixed point of the implicit Hénon system given in Equation (4). Since the Hénon mapping is on $\mathbb{R}^2$ and the fixed points will have one dimensional stable/unstable eigenspace, this provides a simple example where the attached invariant manifolds have dimension less than that of the phase space.

Let $\mathbf{x}_* \in \mathbb{R}^2$ have $T_\epsilon(\mathbf{x}_*, \mathbf{x}_*) = 0$, and suppose that $\lambda \in \mathbb{C}$ is the stable eigenvalue and that $\xi \in \mathbb{C}^2$ is an associated eigenvector. Indeed, note that $\lambda \in \mathbb{R}$ (as the only other eigenvalue is unstable), so that we can choose $\xi \in \mathbb{R}^2$. The eigendata is computed numerically following the discussion in Section 1.1.1.

Motivated by Theorem 3.1 we seek $P\colon (-\tau, \tau) \to \mathbb{R}^2$ so that

$$P(0) = \mathbf{x}_*, \qquad P'(0) = \xi,$$

and

$$T_\epsilon(P(\lambda\theta), P(\theta)) = 0,$$

for $\theta \in (-\tau, \tau)$. Observe that since λ is the only stable eigenvalue, the resonance conditions of Theorem 3.1 are automatically satisfied.

Since T_ϵ is analytic in both variables we look for analytic P of the form

$$P(\theta) = \begin{pmatrix} \sum_{n=0}^\infty a_n \theta^n \\ \sum_{n=0}^\infty b_n \theta^n \end{pmatrix},$$

and note that

$$T_\epsilon(P(\lambda\theta), P(\theta)) = T_\epsilon\left(\sum_{n=0}^\infty \lambda^n a_n \theta^n, \sum_{n=0}^\infty \lambda^n b_n \theta^n, \sum_{n=0}^\infty a_n \theta^n, \sum_{n=0}^\infty b_n \theta^n\right) = 0$$

has component equations

$$\begin{aligned} \sum_{n=0}^\infty \lambda^n a_n \theta^n - 1 + \alpha\left[\sum_{n=0}^\infty a_n \theta^n\right]^2 - \sum_{n=0}^\infty b_n \theta^n - \epsilon\left[\sum_{n=0}^\infty \lambda^n a_n \theta^n\right]^5 &= 0 \\ \sum_{n=0}^\infty \lambda^n b_n \theta^n - \beta \sum_{n=0}^\infty a_n \theta^n + \epsilon\left[\sum_{n=0}^\infty \lambda^n b_n \theta^n\right]^5 &= 0. \end{aligned} \tag{20}$$

Define the infinite sequence $\{\delta_n\}_{n=0}^{\infty}$ by

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & n \geq 1 \end{cases},$$

to represent the power series coefficients of the constant function taking the value 1. We rewrite Equation (20) in terms of Cauchy products as

$$\sum_{n=0}^{\infty} \left[\lambda^n a_n - \delta_n + \alpha(a * a)_n - b_n - \epsilon\lambda^n (a * a * a * a * a)_n\right] \theta^n = 0$$
$$\sum_{n=0}^{\infty} \left[\lambda^n b_n - \beta a_n + \epsilon\lambda^n (b * b * b * b * b)_n\right] \theta^n = 0.$$

Recalling the Cauchy "hat products" defined in Section 4.1, we observe that

$$(a * a)_n = 2a_0 a_n + (\widehat{a * a})_n,$$

and that

$$(a * a * a * a * a)_n = 5a_0^4 a_n + (a * a \widehat{* a *} a * a)_n,$$

and similarly for the coefficients involving the 5-th power of b. Matching like powers of θ in both sides of (20), and recalling that the first order coefficients $n = 0$ and $n = 1$ are already known, we obtain for $n \geq 2$

$$\begin{aligned} a_n\lambda^n - b_n + 2\alpha a_0 a_n + \alpha(\widehat{a * a})_n - 5\epsilon a_0^4 \lambda^n a_n - \epsilon\lambda^n (a * a \widehat{* a *} a * a)_n = 0 \\ \lambda^n b_n - \beta a_n + 5\epsilon b_0^4 \lambda^n b_n + \epsilon\lambda^n (b * b \widehat{* b *} b * b)_n = 0 \end{aligned} \tag{21}$$

and note that the "hat" products depend only on terms of order lower that n.

Isolating terms of order n on the left and lower order terms on the right leads to the Homological equations

$$\begin{pmatrix} \lambda^n + 2\alpha a_0 - 5\epsilon a_0^4 \lambda^n & -1 \\ -\beta & 5\epsilon b_0^4 \lambda^n + \lambda^n \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} S_n^1 \\ S_n^2 \end{pmatrix} \tag{22}$$

for $n \geq 2$, where,

$$\begin{aligned} S_n^1 &= -\alpha(\widehat{a * a})_n + \epsilon\lambda^n (a * a \widehat{* a *} a * a)_n \\ S_n^2 &= -\epsilon\lambda^n (b * b \widehat{* b *} b * b)_n. \end{aligned} \tag{23}$$

This is a linear equation for (a_n, b_n), where the right hand side depends only on terms of lower order. We can solve the homological equations to any desired order, provided that the matrices are invertible.

Remark 4.2 (Non-resonances and uniqueness)**.** Again, if the fixed point is a saddle, then λ^n is never resonant, and Equation (22) has a unique solution for all $n \geq 2$. It follows that the formal power series solution is unique up to the choice of the scaling of the eigenvector. This comment in fact holds generally. See [6].

4.4 A second worked example: period two orbit of implicit Hénon

Suppose now that $\mathbf{x}_1 = (x_1, y_1)$ and $\mathbf{x}_2 = (x_2, y_2)$ is a period two point for the implicit Hénon system, which is computed numerically – along with its first order data – as discussed in Section 1.1.1. Motivated by Theorem 3.2, we seek parameterizations $P, Q\colon (-\tau, \tau) \to \mathbb{R}^2$ so that

$$\begin{aligned} T_\epsilon(Q(\lambda\theta), P(\theta)) &= 0 \\ T_\epsilon(P(\lambda\theta), Q(\theta)) &= 0. \end{aligned} \tag{24}$$

Letting

$$P(\theta) = \sum_{n=0}^{\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \theta^n, \qquad Q(\theta) = \sum_{n=0}^{\infty} \begin{pmatrix} c_n \\ d_n \end{pmatrix} \theta^n,$$

Equation (24) becomes

$$\begin{aligned} \sum_{n=0}^{\infty} a_n \lambda^n \theta^n - \left(1 - \alpha \left[\sum_{n=0}^{\infty} c_n \theta^n\right]^2 + \sum_{n=0}^{\infty} d_n \theta^n + \epsilon \left[\sum_{n=0}^{\infty} a_n \lambda^n \theta^n\right]^5\right) &= 0 \\ \sum_{n=0}^{\infty} b_n \lambda^n \theta^n - \beta \sum_{n=0}^{\infty} c_n \theta^n + \epsilon \left[\sum_{n=0}^{\infty} b_n \lambda^n \theta^n\right]^5 &= 0 \\ \sum_{n=0}^{\infty} c_n \lambda^n \theta^n - \left(1 - \alpha \left[\sum_{n=0}^{\infty} a_n \theta^n\right]^2 + \sum_{n=0}^{\infty} b_n \theta^n + \epsilon \left[\sum_{n=0}^{\infty} c_n \lambda^n \theta^n\right]^5\right) &= 0 \\ \sum_{n=0}^{\infty} d_n \lambda^n \theta^n - \beta \sum_{n=0}^{\infty} a_n \theta^n + \epsilon \left[\sum_{n=0}^{\infty} d_n \lambda^n \theta^n\right]^5 &= 0 \end{aligned} . \tag{25}$$

Expanding the powers as Cauchy products and extracting the terms of order n, we have

$$\sum_{n=0}^{\infty} \begin{pmatrix} \lambda^n a_n - \delta_n + \alpha(c * c)_n - d_n - \epsilon\lambda^n (a * a * a * a * a)_n \\ \lambda^n b_n - \beta c_n + \epsilon\lambda^n (b * b * b * b * b)_n \\ \lambda^n c_n - \delta_n + \alpha(a * a)_n - b_n - \epsilon\lambda^n (c * c * c * c * c)_n \\ \lambda^n d_n - \beta a_n + \epsilon(d * d * d * d * d)_n \end{pmatrix} \theta^n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{26}$$

Extracting from the Cauchy products terms of order n and matching like powers of θ leads to the equations

$$\begin{aligned} \lambda^n a_n + 2\alpha c_0 c_n + \alpha(\widehat{c * c})_n - d_n - \epsilon\lambda^n 5 a_0^4 a_n - \epsilon\lambda^n (a * a \widehat{* a *} a * a)_n &= 0 \\ \lambda^n b_n - \beta c_n + \epsilon\lambda^n 5 b_0^4 b_n + \epsilon\lambda^n (b * b \widehat{* b *} b * b)_n &= 0 \\ \lambda^n c_n + 2\alpha a_0 a_n + \alpha(\widehat{a * a})_n - b_n - \epsilon\lambda^n 5 c_0^4 c_n - \epsilon\lambda^n (c * c \widehat{* c *} c * c)_n &= 0 \\ \lambda^n d_n - \beta a_n + \epsilon\lambda^n 5 d_0^4 d_n + \epsilon\lambda^n (d * d \widehat{* d *} d * d)_n &= 0 \end{aligned}$$

for $n \geq 2$. Observing that these equations are linear in (a_n, b_n, c_n, d_n) we isolate the terms of order n on the left and have the homological equations

$$\begin{pmatrix} \lambda^n - 5\epsilon a_0^4 \lambda^n & 0 & 2\alpha c_0 & -1 \\ 0 & \lambda^n + 5\epsilon b_0^4 \lambda^n & -\beta & 0 \\ 2\alpha a_0 & -1 & \lambda^n - 5\epsilon c_0^4 \lambda^n & 0 \\ -\beta & 0 & 0 & \lambda^n + 5\epsilon d_0^4 \lambda^n \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix} \tag{27}$$

Where

$$\begin{aligned} S_1 &= -\alpha(\widehat{c * c})_n + \epsilon\lambda^n(a * a \widehat{* a *} a * a)_n \\ S_2 &= -\epsilon\lambda^n(b * b \widehat{* b *} b * b)_n \\ S_3 &= -\alpha(\widehat{a * a})_n + \epsilon\lambda^n(c * c \widehat{* c *} c * c)_n \\ S_4 &= -\epsilon\lambda^n(d * d \widehat{* d *} d * d)_n. \end{aligned} \tag{28}$$

Once the period two point and its eigenvectors are known, so that we have the first and second order coefficients, we solve the homological equations for $2 \leq n \leq N$ to find the coefficients of the parameterization to order N. Indeed, the scheme just described generalizes to manifolds attached to periodic orbits of any period in an obvious way.

5 Numerical Results

We illustrate the utility of the explicit homological equations derived in the previous section with some example calculations.

5.1 Numerical example: stable/unstable manifolds attached to fixed points of the implicit Hénon system

As a first example we consider stable/unstable manifolds attached to fixed points of the implicit Hénon system defined in Equation (4). We compute a fixed point, and its stable/unstable eigenvalues and eigenvectors as discussed in Section B.1. The results are summarized in Figure 1. This first order data allows us to compute the Taylor coefficients of parameterizations of the manifolds order by order, by recursively solving the homological equations. Some results are reported for the unstable manifold in Figure 1.

The results in the Figure illustrate the fact that, while small changes in ϵ result in small changes in the first order data, the global dynamics are greatly affected. Note also that the scaling of the eigenvector has to be decreased as ϵ increases. This reflects the fact that the domain of analyticity of the parameterization shrinks as ϵ increases. See also the remark below. We note that while the parameterized manifold is not terribly large (roughly order one) many terms are needed to conjugate the nonlinear to the linear dynamics.

The program which generates the results discussed here is

```
henonPaperEx_fixedPoint.m
```

First order data: implicit Hénon			
parameter	fixed point	eigenvalues	eigenvectors
$\epsilon = 0.01$	$p_0 \approx \begin{pmatrix} 0.6317 \\ 0.1895 \end{pmatrix}$	$\lambda_u \approx -1.939$ $\lambda_s \approx 0.1559$	$\xi_u \approx \begin{pmatrix} -0.9882 \\ 0.1529 \end{pmatrix}$ $\xi_s \approx \begin{pmatrix} -0.4612 \\ -0.8873 \end{pmatrix}$
$\epsilon = 0.03$	$p_0 \approx \begin{pmatrix} 0.6326 \\ 0.1898 \end{pmatrix}$	$\lambda_u \approx -1.971$ $\lambda_s \approx 0.1559$	$\xi_u \approx \begin{pmatrix} -0.9886 \\ 0.1505 \end{pmatrix}$ $\xi_s \approx \begin{pmatrix} -0.4613 \\ -0.8873 \end{pmatrix}$
$\epsilon = 0.0315$	$p_0 \approx \begin{pmatrix} 0.6326 \\ 0.1898 \end{pmatrix}$	$\lambda_u \approx -1.973$ $\lambda_s \approx 0.1559$	$\xi_u \approx \begin{pmatrix} -0.9886 \\ 0.1503 \end{pmatrix}$ $\xi_s \approx \begin{pmatrix} -0.4613 \\ -0.8872 \end{pmatrix}$
$\epsilon = 0.04$	$p_0 \approx \begin{pmatrix} 0.6330 \\ 0.1900 \end{pmatrix}$	$\lambda_u \approx -1.987$ $\lambda_s \approx 0.1560$	$\xi_u \approx \begin{pmatrix} -0.9888 \\ 0.1492 \end{pmatrix}$ $\xi_s \approx \begin{pmatrix} -0.4613 \\ -0.8872 \end{pmatrix}$

Table 1: **Fixed point/stability data:** the table reports the location and stability of one of the fixed points of the implicit Hénon system as the parameter ϵ varies. Data is given to four decimal places. More accurate values (approximately machine precision) are obtained by running the programs.

Remark 5.1 (Loss of the hypotheses of the implicit function theorem)**.** Following the discussion in Section 1.1, we see that the implicit Hénon equations define a local diffeomorphism whenever

$$D_1 T_\epsilon(x_2, y_2) = \mathrm{Id} + \epsilon \begin{pmatrix} -5\epsilon x^4 & 0 \\ 0 & 5\epsilon y^4 \end{pmatrix},$$

is invertible. For $\epsilon > 0$ the matrix is singular on the vertical line through

$$x_*(\epsilon) = \left(\frac{1}{5\epsilon} \right)^{1/4}.$$

Note that when $\epsilon = 0.01$ we have that

$$x_*(0.01) \approx 2.115,$$

and the singular line is far from the attractor. However as ϵ increases the singular line moves closer to the attractor, disrupting the assymptotic dynamics dramatically. In particular note that

$$x_*(0.0315) \approx 1.587,$$

and

$$x_*(0.04) \approx 1.495,$$

so that the singular line eventually moves into the attractor, creating the jumps, or breaks see in the bottom left and right frames of Figure 1.

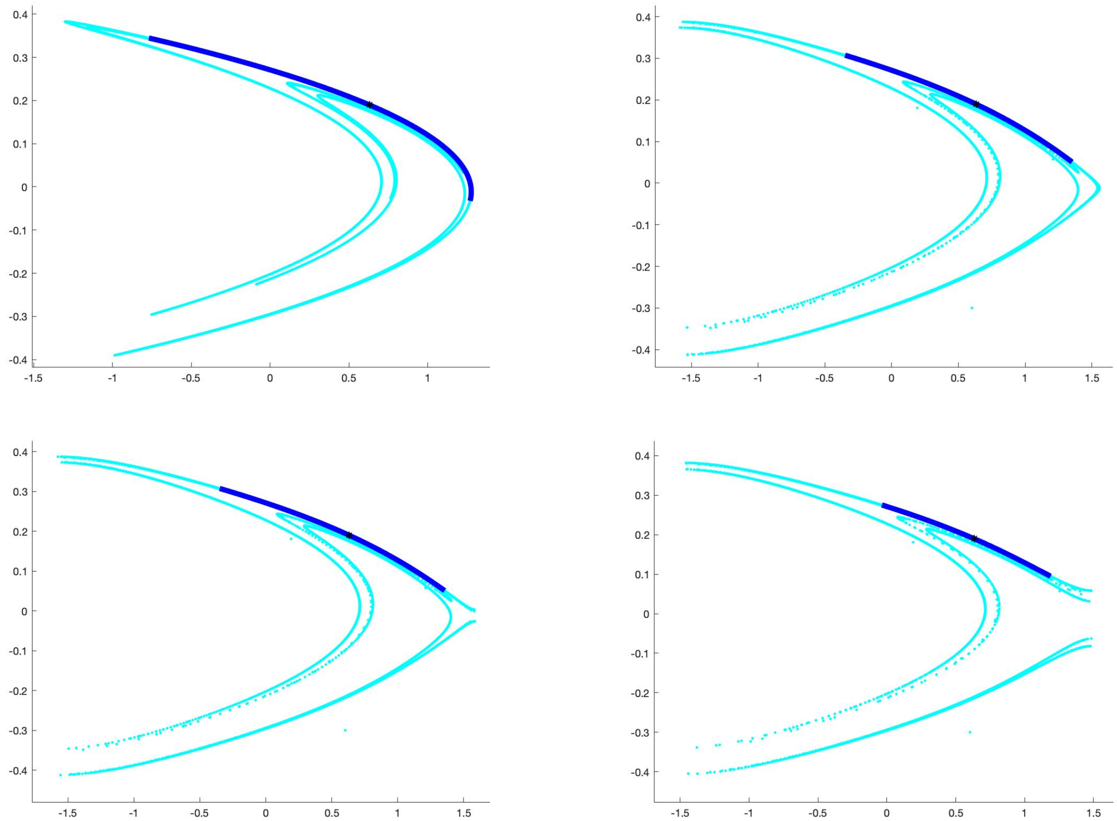

Figure 1: **Implicit Hénon –stable/unstable manifolds attached to fixed points:** four calculations of the local unstable manifold of the fixed point with data as in Table 1. The local unstable manifold is colored dark blue, and eight of its forward iterates are lighter. In each case we computed $N = 75$ Taylor coefficients, with the eigenvector scalings as reported below. Top left: $\epsilon = 0.01$. The eigenvector is scaled by $\alpha = 1.0$. Top right: $\epsilon = 0.031$. The eigenvector is scaled by $\alpha = 0.85$. Bottom left: $\epsilon = 0.0315$. The eigenvector is scaled by $\alpha = 0.8$. Bottom right: $\epsilon = 0.04$. The eigenvector is scaled by $\alpha = 0.6$. These scalings insure that the highest order coefficient computed has magnitude on the order of machine epsilon.

5.2 Numerical example: stable/unstable manifolds attached to periodic orbits of the implicit Hénon system

We now illustrate the computation of the stable/unstable manifolds attached a period two point for the implicit Hénon systems. For the period two problem we consider only the two larger values of ϵ. When $\epsilon = 0.0315$ there is a period two orbit located at

$$p_1 \approx \begin{pmatrix} -0.4945 \\ 0.2940 \end{pmatrix} \qquad p_2 \approx \begin{pmatrix} 0.9802 \\ -0.1483 \end{pmatrix}$$

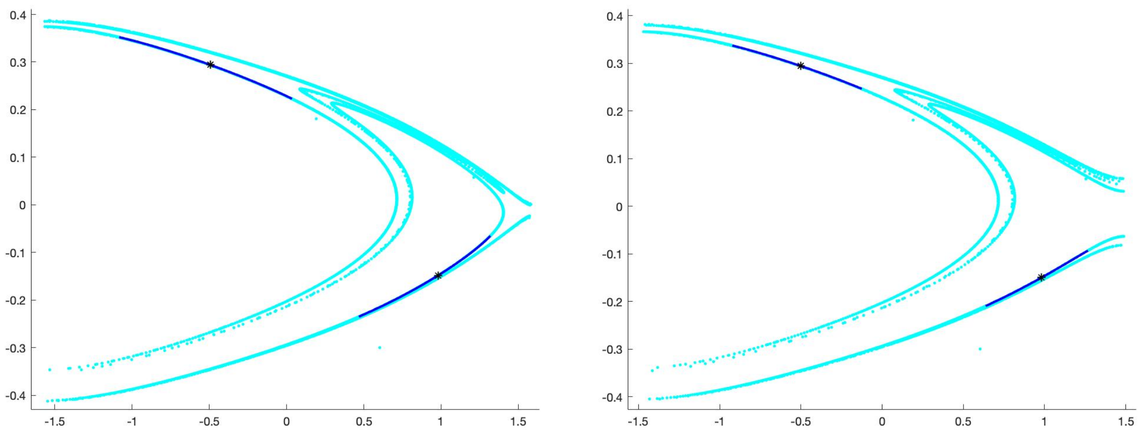

Figure 2: **Implicit Hénon –stable/unstable manifolds attached to period 2 orbits:** two calculations of the local unstable manifolds colored with light blue attached to a period two orbit of the implicit Hénon system. In each case we computed $N = 50$ Taylor coefficients, with eigenvector scalings as reported below. Left: $\epsilon = 0.0315$. The eigenvector is scaled by $\alpha = 0.75$. Right: $\epsilon = 0.04$. The eigenvector is scaled by $\alpha = 0.5$. These scalings ensure that the highest order coefficient computed has magnitude on the order of machine epsilon.

with multipliers

$$\lambda_u \approx -3.807, \qquad \text{and} \qquad \lambda_s \approx -0.0279.$$

We choose the square roots

$$\tilde{\lambda}_u \approx 1.951i, \qquad \text{and} \qquad \tilde{\lambda}_s \approx 0.1670i.$$

and eigenvectors

$$\xi_1^u \approx \begin{pmatrix} 0.7868 \\ -0.0919 \end{pmatrix} \qquad \xi_2^u \approx \begin{pmatrix} -0.5982 \\ -0.1210 \end{pmatrix} \qquad \xi_1^s \approx \begin{pmatrix} 0.3958 \\ -0.5076 \end{pmatrix} \quad \text{and} \quad \xi_2^s \approx \begin{pmatrix} 0.2829 \\ 0.7110 \end{pmatrix}.$$

Similarly, when $\epsilon = 0.04$ the data is

$$p_1 \approx \begin{pmatrix} -0.4995 \\ 0.2943 \end{pmatrix} \qquad p_2 \approx \begin{pmatrix} 0.9814 \\ -0.1499 \end{pmatrix}$$

with multipliers

$$\lambda_u \approx -4.080, \qquad \text{and} \qquad \lambda_s \approx -0.0274.$$

We choose the square roots

$$\tilde{\lambda}_u \approx 2.020i, \qquad \text{and} \qquad \tilde{\lambda}_s \approx 0.165i.$$

and eigenvectors

$$\xi_1^u \approx \begin{pmatrix} 0.7800 \\ -0.0902 \end{pmatrix} \qquad \xi_2^u \approx \begin{pmatrix} -0.6083 \\ -0.1158 \end{pmatrix} \qquad \xi_1^s \approx \begin{pmatrix} 0.3923 \\ -0.5107 \end{pmatrix} \quad \text{and} \quad \xi_2^s \approx \begin{pmatrix} 0.2821 \\ 0.711 \end{pmatrix}.$$

The results are reported with only four significant figures. More accurate data is obtained by running the computer programs.

In both cases these are taken as initial data for computation of the stable/unstable parameterizations, whose Taylor coefficients for orders $2 \leq n \leq N$ are found by recursive solution or the homological equations defined explicitloy in Equations (27) and (28). The resulting local manifolds and a number of forward iterations are illustrated in Figure 2. See Remark 5.1 for the explication of the "jump/break" in the attractor.

The programs which generate the results discussed here are

```
more_iteration.m
```

and

```
henonForPaper_per2.m
```

Remark 5.2 (Heteroclinic/homoclinic connections: infinite forward and backward time orbits)**.** Figures 3 and 4 illustrate local parameterizations of the the stable and unstable manifolds attached to the fixed points and the period two orbit of the implicit Hénon system with $\epsilon = 0.04$, without and with that application of two iterates of the implicit dynamics. At this parameter value the singular value has moved into the basin of attraction and strongly disrupts the system. Nevertheless, the intersection of unstable and stable manifolds illustrated in the figure suggest the existence of heteroclinic and homoclinic orbits: that is, dynamics which exist for all forward and backward time. The figures illustrates that, even though simulating the system for long times is very difficult (the intersection of the singular set with the attractor disrupts iteration schemes based on Newton's method) we nevertheless obtain a great deal of useful information about the global dynamics by studying the parameterized manifolds.

5.3 Numerical example: stable/unstable manifolds attached to fixed points of the implicit Lomelí system

In this section we compute and extend the two dimensional local stable/unstable manifolds attached to fixed points of the implicit Lomelí system defined by Equation (8) with parameter values $\rho = 0.3\overline{4}$, $\tau = 1.\overline{3}$, $a = 0.5$, $b = 0.5$, $c = 1$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, and $\epsilon = 0.01$. We also compute the two dimensional local stable/unstable manifolds associated with a period four orbit. The results illustrated in Figures 5 and 6 are obtained by solving order by order the homological equations given in Equations (37) and (39) respectively.

The local manifolds in Figures 5 have been iterated (forward for the unstable manifolds and backwards for the stable) and seem to intersect transversally. This suggests that the heteroclinic arcs of the $\epsilon = 0$ system studied in [57] persist into the implicit system at least for small ϵ. Numerical values of the fixed points, period orbits, and their first order data can be found by running the computer programs.

The program generating the results discussed here is

```
TwoD_Manifold_period4.m
```

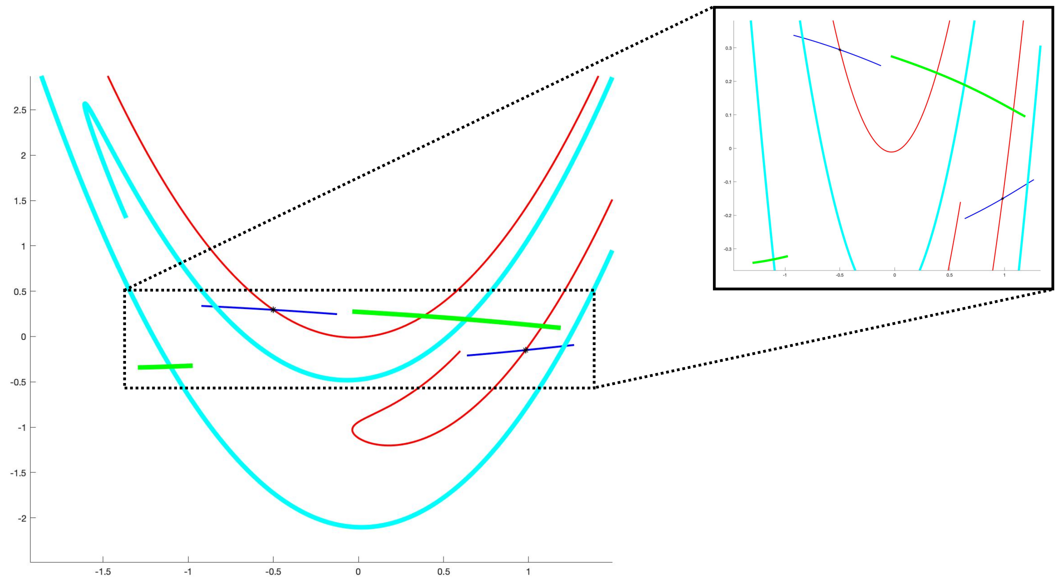

Figure 3: **Implicit Hénon – connecting orbits:** Stable and unstable manifolds when $\epsilon = 0.04$. The green curves represent the unstable manifolds of the two fixed points. The blue curves represent unstable manifolds attached to the period two orbit. Similarly, the cyan curves represent the stable manifolds of the two fixed points, and the red curves the stable manifolds of the period two orbit. All curves are plots of polynomial approximations of the local manifolds computed using the parameterization method; no iteration has been applied to "grow" the manifolds. Note that the blue and cyan curves, as well as the green and the red curves already intersect. These intersections provide numerical evidence for the existence of transverse connecting orbits from the period two orbit to the fixed point and from the fixed point to the period two. These connections also appear to be isolated away from the singular set, so that their existence would imply the existence of a geometric horseshoe (heteroclinic cycle).

6 Conclusions

In this work we have developed a multiple shooting method for studying invariant manifolds attached to periodic orbits of implicitly defined dynamical systems, effectively extending the parameterization method to this setting. After some preliminary formal series calculations are performed "by hand", our approach reduces the computation of the parameterizations, the basic linear algebra and facilitates polynomial approximation to any desired order. By judiciously adjusting the scalings of the eigenvectors, the method can be used to compute fairly large portions of the attached local stable/unstable manifolds of the fixed/periodic orbits. In some examples these large local manifolds parameterizations already indicate the existence of heteroclinic and homoclinic connecting orbits-for the implicitly defined dynamics. In other examples, some globalization methods can be applied after the initial parameterization.

An interesting direction for further research would be to use the methods developed here to

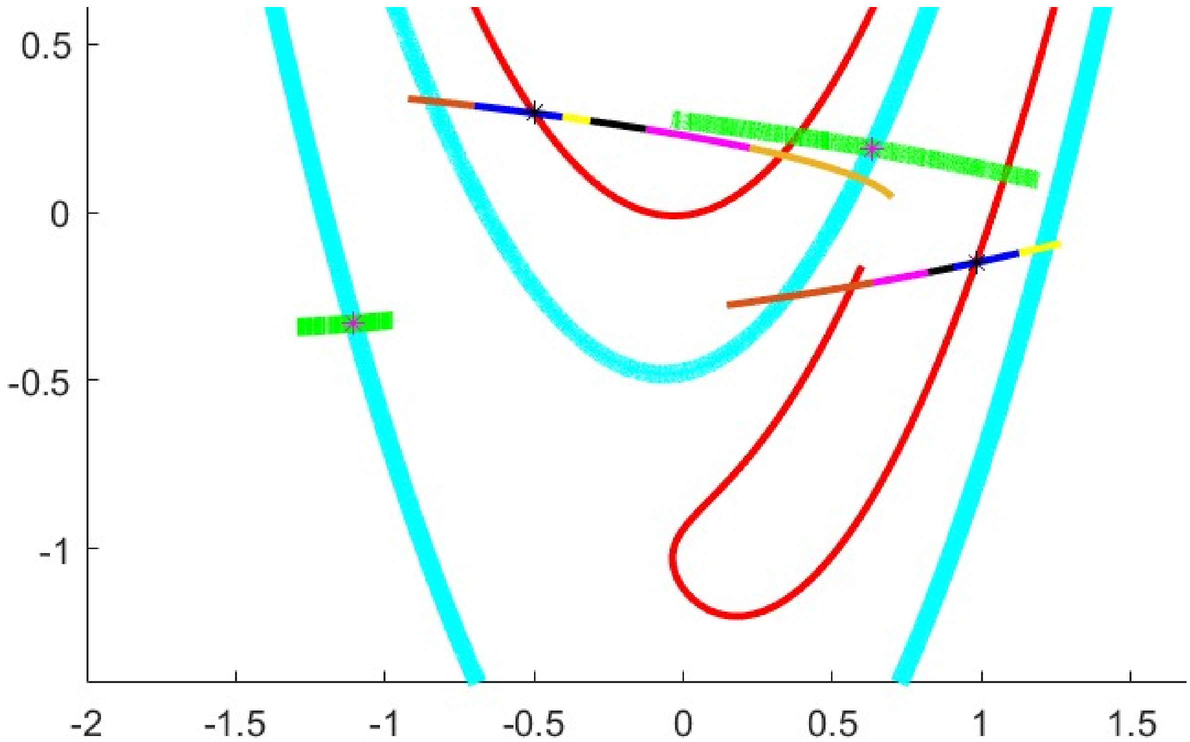

Figure 4: **Implicit Hénon – more connecting orbits:** In this figure the local unstable manifolds of the period two points have been iterated twice, again for the $\epsilon = 0.04$ system. Iterates of different manifold segments are shown in matching colors, so that yellow is the image of yellow, orange of orange, brown of brown, and black of black. After "growing" the local manifolds under iteration we now see that the unstable manifolds of the period two points intersect the stable manifolds of the period two, providing evidence for another geometric horseshoe (homoclinic tangle).

study problems in crystalline lattices, like the Frenkel Kontorova model [24, 19]. While constant solutions of such models can be studied by finding fixed points, non-trivial equilibrium solutions appear as periodic solutions of some implicitly defined maps. Connecting orbits between periodic solutions describe traveling waves in the lattice. Moreover, the methods developed in the present work are amenable to mathematically rigorous computer assisted validation methods similar to those discussed in [58, 56, 5, 47]. Combining the methods of the present work with the techniques of the references just cited would lead computer assisted methods of proof for theorems about Frenkel Kontorova and other such problems.

Another interesting direction of research is to extend the methods of the present work to infinite dimensional implicitly defined dynamical systems, like delay differential equations. For example, with $\tau > 0$ and $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ a smooth function, a delay differential equation of the form

$$y'(t) = f(y(t), y(t - \tau)),$$

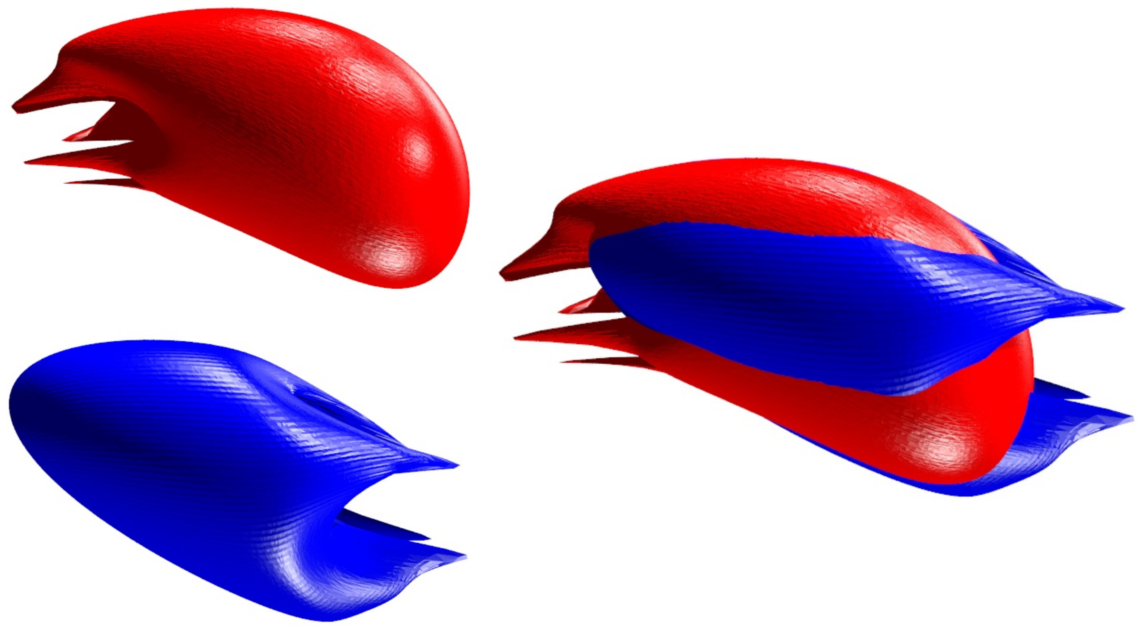

Figure 5: **Implicit Lomelí systems– stable/unstable manifolds attached to fixed points:** the local invariant manifold parameterizations and a number of forward/backward iterations. The image on the right illustrates both manifolds superimposed together, and suggests that the manifolds intersect transversally.

can be rewritten as a *step map*

$$T(y(t), x(t)) = y(t) - x(0) - \int_{-\tau}^{t} f(y(s), x(s))\, ds, \tag{29}$$

where $x(s)$ is the history function defined on $[-\tau, 0]$. That is, given x, if y has $T(y, x) = 0$ then $y(t - \tau)$ is a solution of the delay differential equation on the interval $[0, \tau]$ with history $x(t)$ given on $[-\tau, 0]$.

The interested reader can consult the papers [21, 48, 46] where the authors study the dynamics generated by some delay differential equations by considering discretization of the implicitly defined dynamical system defined by the zeros of Equation (29). In particular, computer assisted proofs of periodic orbits for delay equations are given in the last reference just cited, using a multiple shooting setup much like the one considered in the present work. The authors of [36] are currently adapting the methods of the present work to the infinite dimensional setting of delay equations to study homoclinic chaos in systems like Mackey-Galss [52].

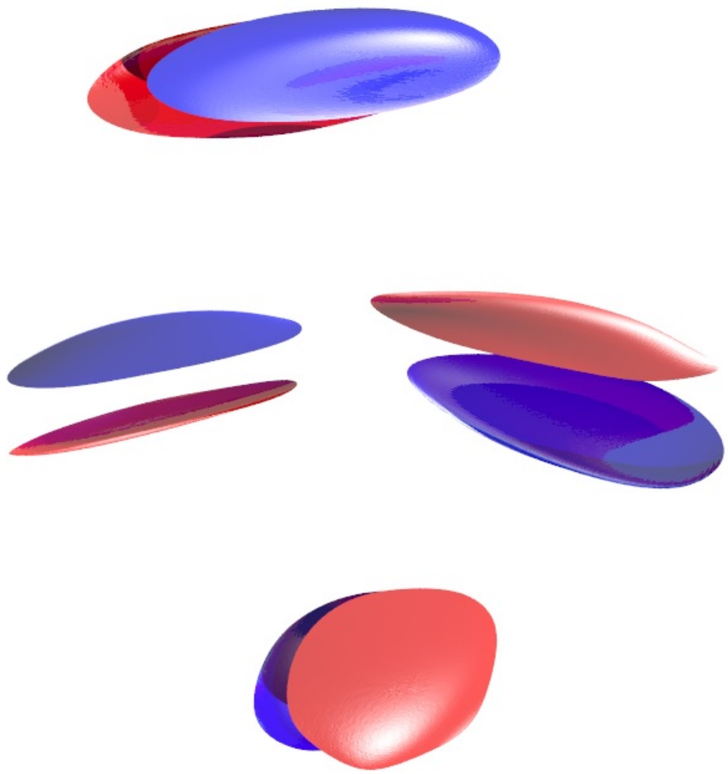

Figure 6: **Implicit Lomelí systems– stable/unstable manifolds attached to a period 4 orbit:** the local invariant manifold parameterizations.

7 Acknowledgments

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A Definitions and Background

In this section we review some basic definitions from the qualitative theory of nonlinear dynamical systems. We also review the main results from [6, 7, 8] about the parameterization method for fixed points of local diffeomorphisms, and results from [28] extending these results to periodic orbits. The reader familiar with this material may want to skim or skip this section upon first reading, referring back to it only as needed.

A.1 Discrete time semi-dynamical systems: Maps

The material in this section is standard, and an excellent reference is [61]. Suppose that $U \subset \mathbb{R}^d$ is an open set and $F\colon U \to U$ is a $C^k(U)$ mapping, with $k = 0, 1, 2, \ldots, \infty, \omega$. For $x_0 \in U$, define the sequence $x_1 = F(x_0)$, $x_2 = F(x_1)$, and in general $x_{n+1} = F(x_n)$ for $n \geq 0$. We refer to the set $\{x_n\}_{n=0}^{\infty}$ as the *forward orbit of* x_0 *under* F, and write orbit(x_0, F) to denote this set. Let $F^0(x) = x$, $F^1(x) = F(x)$, $F^2(x) = F(F(x))$ and in general $F^n(x)$ denote the composition of F with itself n times applied to x. When F is understood we simply write orbit(x_0) and talk about the orbit of x_0. Then

$$\text{orbit}(x_0) = \bigcup_{n=0}^{\infty} F^n(x_0).$$

A sequence $\{x_n\}_{n=-\infty}^{0} \subset U$ with $F(x_{-1}) = x_0$ and $F(x_n) = x_{n+1}$ for all $n < 0$ is *a backward orbit of* x_0 under F. The pair (U, F) is referred to as a *semi-dynamical system*, as, while forward orbits are uniquely defined, backwards orbits need not exist and when they do exist they need not be unique.

A.2 Local stable/unstable manifolds for fixed points/periodic orbits

Let $F \in C^k(U)$ with $k \geq 1$ and suppose that $x_* \in U$ is a fixed point, so that

$$F(x_*) = x_*.$$

We write spec$(x_*) = \{\lambda_1, \ldots, \lambda_d\} \subset \mathbb{C}$ to denote the set of eigenvalues of $DF(x_*)$. Let $\xi_1, \ldots, \xi_d \in \mathbb{C}^d$ be an associated choice of (possibly generalized) eigenvectors. Let $D_1 \subset \mathbb{C}$ denote the open unit disk in the complex plane, and S_1 denote the unit circle. Define

$$\begin{aligned}
\text{spec}_s(x_*) &= \text{spec}(x_*) \cap D_1 \\
\text{spec}_c(x_*) &= \text{spec}(x_*) \cap S_1 \\
\text{spec}_u(x_*) &= \text{spec}(x_*) \backslash \left(\text{spec}_s(x_*) \cup \text{spec}_c(x_*)\right),
\end{aligned}$$

and note that spec$_s(x_*)$ is the set of eigenvalues with complex absolute value less than one, spec$_c(x_*)$ is the set of eigenvalues with complex absolute value equal to on, and spec$_u(x_*)$ is the set of eigenvalues with complex absolute value greater than one. There are referred to as the stable, center, and unstable eigenvalues respectively, and we note that any of two of these sets could be empty. If spec$_c(x_*) = \emptyset$ then we say that x_* is a hyperbolic fixed point.

Define the vector spaces $\mathbb{E}^s$, $\mathbb{E}^c$, and $\mathbb{E}^u$ to be the span of the stable, the center, and the unstable eigenvectors respectively. These are referred to as the stable, center, and unstable eigenspaces of $DF(x_*)$, and they are invariant linear subspaces for the dynamics induced by $DF(x_*)$. It is a classical fact that they are tangent to corresponding locally invariant nonlinear manifolds of F in a neighborhood of x_*. Let $d_s = \dim(\mathbb{E}^s)$, $d_c = \dim(\mathbb{E}^c)$, and $d_u = \dim(\mathbb{E}^u)$ denote the dimension of the stable/center/unstable eigenspaces, or equivalently the number (counted with multiplicity) of stable/center/unstable eigenvalues.

Define the sets

$$\begin{aligned}
W^s(x_*) &= \left\{ x \in U \, : \, \lim_{n\to\infty} F^n(x) = x_* \right\} \\
W^u(x_*) &= \left\{ x \in U \, : \text{there exists a backward orbit } \{x_n\} \text{ of } x \text{ with } \lim_{n\to-\infty} x_n = x_* \right\}.
\end{aligned}$$

These are referred to as the stable and unstable sets for x_* respectively. In a similar fashion, for any open set $V \subset U$ with $x_* \in V$, define

$$\begin{aligned}
W^s_{\text{loc}}(x_*, V) &= \{x \in V : F^n(x) \in V \text{ for all } n \geq 0, \text{ and } F^n(x) \to x_* \text{ as } n \to \infty\} \\
W^u_{\text{loc}}(x_*, V) &= \left\{ x \in V : \text{there is a backward orbit for } x \text{ in } V \text{ with } \lim_{n\to-\infty} x_n \to x_* \right\},
\end{aligned}$$

and note that for any $V \subset U$ we have that $W^s_{\text{loc}}(x_*, V) \subset W^s(x_*)$, and $W^u_{\text{loc}}(x_*, V) \subset W^u(x_*)$.

The following stable manifold theorem says that if x_* is hyperbolic then there exist local stable/unstable sets with especially nice properties.

Theorem A.1 (Local stable manifold theorem)**.** *Suppose that x_* is a hyperbolic fixed point for F. Then there exists an open set $V \subset U$ with $x_* \in V$ so that $W^s_{loc}(x_*, V)$ and $W^u_{loc}(x_*, V)$ are respectively d_s and d_u dimensional embedded disks – as smooth as F– and tangent at $x*$ to $\mathbb{E}^s$ and $\mathbb{E}^u$ respectively.*

The theorem gives that the stable/unstable sets are locally smooth manifolds. If F is a diffeomorphism then the full stable/unstable sets are obtained by iterating F and F^{-1}, hence the stable/unstable sets are smooth manifolds (which can nevertheless be embedded in U in very complicated ways). However, if F is not invertible the global stable/unstable sets might misbehave in a number of ways.

- **Connectedness:** While the unstable set must be connected (image of a disk is connected under iteration of a continuous map) the stable set can in general be disconnected. The unstable set can have self intersections.
- **Dimension:** both the stable/unstable sets can increase in dimension outside a neighborhood of x_*.
- **Smoothness:** the stable/unstable sets need not be smooth manifolds away from x_*. At points where $DF(x)$ has an isolated non-singularity the set can develop corners or cusps.

Examples of each of these phenomena are discussed in [64], and many explicit examples are given. See also [25].

A.3 Multiple shooting for periodic orbits

With $U \subset \mathbb{R}^d$ an open set, and $F\colon U \to \mathbb{R}^d$ a smooth mapp, suppose that $x_1, \ldots, x_N \in U$ have

$$\begin{aligned} F(x_1) &= x_2 \\ &\vdots \\ F(x_{N-1}) &= x_N \\ F(x_N) &= x_1 \end{aligned}$$

Then $\{x_1, \ldots, x_N\}$ is a periodic orbit for F. If the x_j, $1 \leq j \leq N$ are distinct, then N is the least period. We refer to x_j, $1 \leq j \leq N$ as a period N point. If $DF(x_j)$ is invertible for each $1 \leq j \leq N$ we say that the periodic orbit is non-degenerate.

Note that $\bar{x} \in U$ is a period N point for F if and only if $\bar{x}$ is a fixed point of the composition F^N. If the orbit of $\bar{x}$ is non-degenerate and least period N then $DF^N(\bar{x})$ is invertible. We note that if $\{x_1, \ldots, x_N\}$ is a non-degenerate periodic orbit then the matrices $DF^N(x_j)$, $1 \leq j \leq N$ have the same eigenvalues. These are also referred to as the multipliers of the periodic orbit.

If $DF^N(\bar{x})$ has no eigenvalues on the unit circle we say that the periodic orbit is hyperbolic and Theorem A.1 applies to the composition mapping F^N. In particular, there are local stable and unstable manifolds attached to the points of the periodic orbit.

Let $U^N = U \times \ldots \times U \subset \mathbb{R}^{Nd}$ denote the product of N copies of U. Define $G\colon U^N \to \mathbb{R}^{Nd}$ by

$$G(x_1, x_2, \ldots, x_{N-1}, x_N) = \begin{pmatrix} F(x_N) \\ F(x_1) \\ F(x_2) \\ \vdots \\ F(x_{N-1}) \end{pmatrix}. \tag{30}$$

and observe that if $(x_1, \ldots, x_N) \in \mathbb{R}^{Nd}$ is a fixed point of G then $\{x_1, \ldots, x_N\}$ is a period N orbit for F. We refer to G as a multiple shooting map for a period N orbit of F. In practice numerically computing fixed points of G using Newton's method is more stable than computing fixed points of F^N[28] also with Newton. This is because the condition number of DF^N grows exponentially with N. While DG is a larger matrix, it has a much better condition number, and modern linear algebra routines easily solve the Newton step. Taking advantage of the sparse structure of DG (which we have not done) leads to even great improvements.

Note also that if

$$DG(x_1, \ldots, x_N) = \begin{pmatrix} 0 & 0 & \ldots & 0 & DF(x_N) \\ DF(x_1) & 0 & \ldots & 0 & 0 \\ 0 & DF(x_2) & \ldots & 0 & 0 \\ 0 & 0 & \ldots & DF(x_{N-1}) & 0 \end{pmatrix} \tag{31}$$

is invertible then the periodic orbit is non-degenerate. In fact, $\lambda \in \mathbb{C}$ is an eigenvalue of $DG(x_1, \ldots, x_N)$ if and only if λ^N is an eigenvalue of $DF^N(x_j)$. Moreover, one can check that if $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{C}^{dN}$ is an eigenvector associated with the eigenvalue λ of the matrix $DG(x_1, \ldots, x_N)$, then for $1 \leq j \leq N$ we have that (λ^N, ξ_j) is an eigenvalue/eigenvector pair for the matrix $DF^N(x_j)$. That is, the multipliers of the periodic orbit and the eigenspaces of $DF^N(x_j)$ are easily recovered from the eigenvalues/eigenvectors of $DG(x_1, \ldots, x_N)$. The interested reader will find a more thorough discussion of the relationship between multiple shooting maps and periodic orbits in [28].

B Implicitly defined dynamical systems

Let $U, V \subset \mathbb{R}^d$ be open sets and suppose that $T\colon V \times U \to \mathbb{R}^d$ is a smooth function. We are interested in the existence of open sets $D \subset U$, $R \subset V$ and a mapping $F\colon D \to R \subset \mathbb{R}^d$ defined by the rule

$$F(x) = y, \tag{32}$$

if and only if for a fixed given input $x \in D$, y solves the equation

$$T(y, x) = 0, \tag{33}$$

with $y \in R$. We say that the mapping F is *implicitly defined* by the rule given in Equation (33). Note that F need not be one-to-one or even single valued globally. However, if for a fixed $\bar{x} \in U$ we have that there are $\bar{y}, \tilde{y} \in U$ so that $T(\bar{y}, \bar{x}) = T(\tilde{y}, \bar{x}) = 0$, then we are only interested in the case when there exist neighborhoods $\bar{R}, \tilde{R} \subset V$ with $\bar{y} \in \bar{R}$, $\tilde{y} \in \tilde{R}$ and $\bar{R} \cap \tilde{R} = \emptyset$.

Existence and regularity of implicitly defined maps is subtle, yet– as already mentioned in the introduction – the implicit function theorem provides sufficient conditions for the existence of a single valued branch. More precisely, let $D_1T(y, x)$ and $D_2T(y, x)$ denote the partial derivatives of T with respect to the first and second variables respectively, and suppose that $T(x_1, x_0) = 0$ (note that $D_1T(y, x), D_2T(y, x)$ are $d \times d$ matrices). If $D_1T(x_1, x_0)$ is invertible then, by the implicit function theorem [60], there exists an $r > 0$ and a function $F\colon B_r(x_0) \subset U \to \mathbb{R}^d$ so that $F(x_0) = x_1$ and

$$T(F(x), x) = 0, \tag{34}$$

for all $x \in B_r(x_0)$. Moreover, the mapping F is as smooth as T. By differentiating (34) we have that $DF(x_0)$ solves the equation

$$D_1T(x_1, x_0)DF(x_0) = -D_2T(x_1, x_2), \tag{35}$$

with $D_1T(x_1, x_0)$ invertible. Of course the map F depends on the choice of x_1. However, once we choose a solution x_1, the deffeomorphism F is well defined and unique locally. The discussion above motivates the following definition.

Definition 1. We say that $\bar{x} \in U$ is a *regular point for T* if there exists $\bar{y} \in V$ such that

$$T(\bar{y}, \bar{x}) = 0,$$

and $D_1T(\bar{y}, \bar{x})$ is invertible. Note that, by the implicit function theorem as above, $\bar{x}$ is in the interior of $D = \mathrm{dom}(F)$. Moreover, F is a local diffeomorphism of a neighborhood of $\bar{x}$ into a neighborhood of $\bar{y}$.

Remark B.1 (Numerical evaluation of F)**.** Evaluation of $F(x)$ requires solving the nonlinear equation $T(y, x) = 0$ with x given. In practice we use Newton's method as follows. Let $\bar{x}$ be fixed and y_0 be an approximate solution in the sense that

$$\|T(y_0, \bar{x})\| \approx 0.$$

For $n \geq 0$, define

$$y_{n+1} = y_n + \Delta_n,$$

where Δ_n solves the linear equation

$$D_1T(y_n, \bar{x})\Delta_n = -T(y_n, \bar{x}).$$

Convergence and error analysis for the algorithm is a classic topic (see any book on numerical analysis). For the moment it is enough to remark that the algorithm is expected to perform well close enough to a regular point, as invertibility of the derivative is an open property. In practice, if after N steps of the algorithm we have a numerical approximation y_N with $\|T(y_N, \bar{x})\| < \tau_{\text{tol}}$ (i.e. defect smaller than some prescribed tolerance) then we consider the algorithm to have converged. We take y_N as our numerical solution and have $F(\bar{x}) \approx y_N$.

B.1 Fixed and periodic points

Assume that $x_* \in U \cap V \subset \mathbb{R}^d$ is a regular point for T having

$$T(x_*, x_*) = 0.$$

Then there exists an open neighborhood $D \subset U$ of x_* and a diffromorphism $F\colon D \to \mathbb{R}^d$ with

$$F(x_*) = x_*.$$

In this case, there may be points near x_* with well defined forward orbits under F. To further study this question we consider the stability of x_*.

Exploiting the formula for the derivative in Equation (35), we have that

$$DF(x_*) = -D_1T(x_*, x_*)^{-1} D_2T(x_*, x_*),$$

where $D_1T(x_*, x_*)$ is invertible thanks to the assumption that x_* is a regular point for F. If the stable and/or center eigenspaces of $DF(x_*)$ are non-empty, then the attached local stable and/or center manifolds are natural places to look for orbits which remain in a neighborhood of x_* under forward iteration of F. (Likewise the unstable manifold is a natural place to look for points with backward orbits). We focus for a moment on the stable manifold.

Let $\lambda_1, \ldots, \lambda_{d_s} \in \mathbb{C}$ be the stable eigenvalues of $DF(x_*)$ and $\xi_1, \ldots, \xi_{d_s} \in \mathbb{C}^d$ denote associated stable eigenvectors. Note (λ, ξ) is an eigenpair for $DF(x_*)$ if and only if they solve the generalized eigenvalue problem

$$D_2T(x_*, x_*)\xi = \lambda D_1T(x_*, x_*)\xi.$$

From a numerical point of view, this equation has the advantage of not requiring the inversion of any matrix. Once the the fixed point and eigendata are known we can apply the algorithms based on the parameterization method discussed in the main body of the paper to compute the stable (or unstale) manifolds.

In a similar fashion, suppose that $x_1, \ldots, x_N \in U \cap V$ have

$$\begin{aligned} T(x_2, x_1) &= 0 \\ T(x_3, x_2) &= 0 \\ &\vdots \\ T(x_N, x_{N-1}) &= 0 \\ T(x_1, x_N) &= 0, \end{aligned}$$

with the $x_1, \ldots, x_N$ distinct. If each of $x_1, \ldots, x_N$ is a regular point for T, then the collection is a periodic orbit (of least period N) for an implicitly defined map F, whose domain can be taken as a union of neighborhoods of the periodic orbit. Again, we are interested in the existence of well defined orbits near $x_1, \ldots, x_N$, so we consider the stability of the periodic orbit.

To find the multipliers and eigenvectors, proceed as follows. Recall from Section A.2 that the multipliers are found by computing the eigenvalues and eigenvectors of the derivative of the multiple shooting map. The formula for the derivative is in Equation (31), and exploiting again the formula for the derivative of F given in Equation (35), and the fact that the periodic orbit is non-degenerate, the non-zero entries of $DG(x_1, \ldots, x_N)$ are

$$\begin{aligned} DF(x_1) &= -D_1T(x_2, x_1)^{-1} D_2T(x_2, x_1) \\ DF(x_2) &= -D_1T(x_3, x_2)^{-1} D_2T(x_3, x_2) \\ &\vdots \\ DF(x_{N-1}) &= -D_1T(x_N, x_{N-1})^{-1} D_2T(x_N, x_{N-1}) \\ DF(x_N) &= -D_1T(x_1, x_N)^{-1} D_2T(x_1, x_N). \end{aligned}$$

It is an exercise to write the associated generalized eigenvalue problem and avoid the matrix inversion. Once the stable (or unstable) eigendata is determined, the stable (or unstable) manifold can be computed using the parameterization method developed in the body of the present work.

C Parameterization of two dimensional invariant manifolds

In this appendix we provide the details for higher dimensional stable/unstable manifolds, focusing on the case of two dimensions. These calculations involve power series of two variables.

C.1 Formal power series of two variables

Let f and g be two variable power series if the form

$$f(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} z_1^m z_2^n, \qquad \text{and} \qquad g(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{mn} z_1^m z_2^n.$$

We have that

$$\alpha f(z_1, z_2) + \beta g(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha a_{mn} + \beta b_{mn}) z_1^m z_2^n,$$

$$f(\lambda_1 z_1, \lambda_2 z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_1^m \lambda_2^n a_{mn} z_1^m z_2^n,$$

and

$$f(z_1, z_2) g(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a * b)_{mn} z_1^m z_2^n,$$

where the coefficients of the two variable Cauchy product are given by

$$\begin{aligned}(a * b)_{mn} &= \sum_{\substack{j_1 + j_2 = m \\ k_1 + k_2 = n}} a_{j_1 k_1} b_{j_2 k_2} \\ &= \sum_{j=0}^{m} \sum_{k=0}^{n} a_{m-j, n-k} b_{jk}.\end{aligned}$$

If $f_1, \ldots, f_N$ are power series given by

$$f_i(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}^i z_1^m z_2^n \qquad 1 \le i \le N,$$

then

$$f_1(z_1, z_2) \ldots f_N(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a^1 * \ldots * a^N)_{mn} z_1^m z_2^n,$$

where

$$\begin{aligned}(a^1 * \ldots * a^N)_{mn} &= \sum_{\substack{j_1 + \ldots + j_N = m \\ k_1 + \ldots + k_N = n}} a_{j_1 k_1}^1 \ldots a_{j_N k_N}^N \\ &= \sum_{j_1=0}^{m} \sum_{j_2=0}^{j_1} \cdots \sum_{j_{N-1}=0}^{j_{N-2}} \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{N-1}=0}^{k_{N-2}} a_{m-j_1, n-k_1}^1 \ldots a_{j_{N-1} k_{N-1}}^N\end{aligned}$$

For coefficient extraction define

$$(\widehat{a * b})_{mn} = (a * b)_{mn} - b_{00} a_{mn} - a_{00} b_{mn},$$

and similarly

$$(a^1 \widehat{* \ldots *} a^N)_{mn} = (a^1 * \ldots * a^N)_{mn} - a_{00}^1 \ldots a_{00}^{N-1} a_{mn}^N - \ldots - a_{00}^2 \ldots a_{00}^N a_{mn}^1,$$

to be the Cauchy product of order m, n with the m, n-th order coefficients removed.

C.2 Parameterized stable/unstable manifolds attached to fixed points of the implicit Lomelí system

Consider the implicit Lomelí system defined in Equation (8). At the parameter values studied in the present work the Lomelí map has a pair of hyperbolic fixed points. One of the fixed points has 2d unstable and 1d stable manifold, while for the other it is vice versa. For small $\epsilon \neq 0$ these features persist into the implicit system, and we will compute the formal series expansion for the parameterization of a two dimensional stable manifold of the implicit system. We focus on the case of complex conjugate eigenvalues, but the real distinct case is similar.

So, let $\mathbf{x}_* = (x_*, y_*, z_*) \in \mathbb{R}^3$ denote the fixed point, $\lambda_1, \lambda_2 \in \mathbb{C}$ the stable eigenvalues, and $\xi_1, \xi_2 \in \mathbb{C}^3$ be associated stable eigenvectors. Note that $\lambda_2 = \overline{\lambda_1}$ and we choose eigenvectors with the same symmetry. This data is computed numerically as outlined in Section 1.1.2.

Let

$$B_r(0) = \left\{ (\theta_1, \theta_2) \in \mathbb{R}^2 \,|\, \sqrt{\theta_1^2 + \theta_2^2} < 1 \right\}.$$

Motivated again by Theorem 3.1, we seek a smooth function $\mathbf{P} : \mathcal{B}_1(0) \to \mathbb{R}^3$ solving the invariance equation

$$T_\epsilon(\mathbf{P}(\lambda_1\theta_1, \lambda_2\theta_2), \mathbf{P}(\theta_1, \theta_2)) = 0, \qquad (\theta_1, \theta_2) \in B_r(0), \tag{36}$$

of the form

$$\mathbf{P}(\theta_1, \theta_2) = \begin{pmatrix} P(\theta_1, \theta_2) \\ Q(\theta_1, \theta_2) \\ R(\theta_1, \theta_2) \end{pmatrix} = \begin{pmatrix} \sum_{m=0}^\infty \sum_{n=0}^\infty u_{mn}\theta_1^m\theta_2^n \\ \sum_{m=0}^\infty \sum_{n=0}^\infty v_{mn}\theta_1^m\theta_2^n \\ \sum_{m=0}^\infty \sum_{n=0}^\infty w_{mn}\theta_1^m\theta_2^n \end{pmatrix}.$$

The first order constraints require that

$$\begin{pmatrix} u_{00} \\ v_{00} \\ w_{00} \end{pmatrix} = \begin{pmatrix} x_* \\ y_* \\ z_* \end{pmatrix}, \qquad \text{and that} \qquad \begin{pmatrix} u_{10} \\ v_{10} \\ w_{10} \end{pmatrix} = \xi_1 \quad \text{and} \quad \begin{pmatrix} u_{01} \\ v_{01} \\ w_{01} \end{pmatrix} = \xi_2.$$

To work out the higher order terms we plug the power series into the invariance equation and have

$$\begin{pmatrix} P(\lambda_1\theta_1, \lambda_2\theta_2) - \rho - \tau P(\theta_1, \theta_2) - R(\theta_1, \theta_2) - N(\theta_1, \theta_2) + \epsilon H_1(\lambda_1\theta_1, \lambda_2\theta_2) \\ Q(\lambda_1\theta_1, \lambda_2\theta_2) - P(\theta_1, \theta_2) + \epsilon\gamma H_2(\lambda_1\theta_1, \lambda_2\theta_2) \\ R(\lambda_1\theta_1, \lambda_2\theta_2) - Q(\theta_1, \theta_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

where

$$N(\theta_1, \theta_2) = aP(\theta_1, \theta_2)^2 + bP(\theta_1, \theta_2)Q(\theta_1, \theta_2) + cQ(\theta_1, \theta_2)^2,$$
$$H_1(\lambda_1\theta_1, \lambda_2\theta_2) = \alpha Q(\lambda_1\theta_1, \lambda_2\theta_2)^5 + \beta R(\lambda_1\theta_1, \lambda_2\theta_2)^5,$$

and

$$H_2(\lambda_1\theta_1, \lambda_2\theta_2) = \gamma R(\lambda_1\theta_1, \lambda_2\theta_2)^5.$$

Define $\{\delta_{mn}\}_{m+n=0}^\infty$ by

$$\delta_{mn} = \begin{cases} 1 & m = n = 0 \\ 0 & \text{otherwise} \end{cases},$$

the power series coefficients of the constant function taking value one. Then

$$\sum_{m=0}^\infty \sum_{n=0}^\infty \begin{pmatrix} \lambda_1^m\lambda_2^n u_{mn} - \rho\delta_{mn} - \tau u_{mn} - w_{mn} - N_{mn} + \epsilon H_{mn}^1 \\ \lambda_1^m\lambda_2^n v_{mn} - u_{mn} + \epsilon H_{mn}^2 \\ \lambda_1^m\lambda_2^n w_{mn} - v_{mn} \end{pmatrix} \theta_1^m\theta_2^n = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$\begin{aligned} N_{mn} &= a(u*u)_{mn} + b(u*v)_{mn} + c(v*v)_{mn} \\ &= 2au_{00}u_{mn} + a(\widehat{u*u})_{mn} + bu_{00}v_{mn} + bv_{00}u_{mn} + b(\widehat{u*v})_{mn} + 2cv_{00}v_{mn} + c(\widehat{v*v})_{mn} \\ H_{mn}^1 &= \alpha\lambda_1^m\lambda_2^n(v*v*v*v*v)_{mn} + \beta\lambda_1^m\lambda_2^n(w*w*w*w*w)_{mn} \\ &= 5\alpha\lambda_1^m\lambda_2^n v_{00}^4 v_{mn} + \alpha\lambda_1^m\lambda_2^n(v*v\widehat{*v*}v*v)_{mn} \\ &\quad + 5\beta\lambda_1^m\lambda_2^n w_{00}w_{mn} + \beta\lambda_1^m\lambda_2^n(w*w\widehat{*w*}w*w)_{mn} \\ &\text{and} \\ H_{mn}^2 &= 5\gamma\lambda_1^m\lambda_2^n w_{00}w_{mn} + \gamma\lambda_1^m\lambda_2^n(w*w\widehat{*w*}w*w)_{mn}. \end{aligned}$$

Recall the definition of the Cauchy "hat products" given in Appendix C.1. We define

$$\hat{N}_{mn} = a(\widehat{u * u})_{mn} + b(\widehat{u * v})_{mn} + c(\widehat{v * v})_{mn}$$
$$\hat{H}^1_{mn} = \alpha\lambda_1^m\lambda_2^n(v * v \,\widehat{* v *}\, v * v)_{mn} + \beta\lambda_1^m\lambda_2^n(w * w \,\widehat{* w *}\, w * w)_{mn}$$
and
$$\hat{H}^2_{mn} = \gamma\lambda_1^m\lambda_2^n(w * w \,\widehat{* w *}\, w * w)_{mn},$$

so that

$$N_{mn} = 2au_{00}u_{mn} + bu_{00}v_{mn} + bv_{00}u_{mn} + 2cv_{00}v_{mn} + \hat{N}_{mn},$$
$$H^1_{mn} = 5\alpha\lambda_1^m\lambda_2^n v_{00}^4 v_{mn} + 5\beta\lambda_1^m\lambda_2^n w_{00}w_{mn} + \hat{H}^1_{mn},$$

and

$$H^2_{mn} = 5\gamma\lambda_1^m\lambda_2^n w_{00}w_{mn} + \hat{H}^2_{mn},$$

are all terms of order mn plus lower order terms. Matching like powers of θ leads to

$$\begin{aligned}\lambda_1^m\lambda_2^n u_{mn} - \tau u_{mn} - w_{mn} - 2au_{00}u_{mn} - bu_{00}v_{mn} - bv_{00}u_{mn} - 2cv_{00}v_{mn} - \hat{N}_{mn}& \\ +5\epsilon\alpha\lambda_1^m\lambda_2^n v_{00}^4 v_{mn} + 5\epsilon\beta\lambda_1^m\lambda_2^n w_{00}w_{mn} + \epsilon\hat{H}^1_{mn} &= 0 \\ \lambda_1^m\lambda_2^n v_{mn} - u_{mn} + 5\epsilon\gamma\lambda_1^m\lambda_2^n w_{00}w_{mn} + \epsilon\hat{H}^2_{mn} &= 0 \\ \lambda_1^m\lambda_2^n w_{mn} - v_{mn} &= 0\end{aligned}$$

This leads to linear homological equations for (u_{mn}, v_{mn}, w_{mn}) when $m + n \geq 2$ of the form

$$A_{mn}\begin{pmatrix} u_{mn} \\ v_{mn} \\ w_{mn}\end{pmatrix} = \begin{pmatrix} S^1_{mn} \\ S^2_{mn} \\ 0\end{pmatrix}, \tag{37}$$

where

$$A_{mn} = \begin{pmatrix} \lambda_1^m\lambda_2^n - \tau - 2au_{00} - bv_{00} & -bu_{00} - 2cv_{00} + 5\epsilon\alpha v_{00}^4\lambda_1^m\lambda_2^n & -1 + 5\epsilon\beta w_{00}^4\lambda_1^m\lambda_2^n \\ -1 & \lambda_1^m\lambda_2^n & 5\epsilon\gamma w_{00}^4\lambda_1^m\lambda_2^n \\ 0 & -1 & \lambda_1^m\lambda_2^n \end{pmatrix},$$

and the components of the right hand side are given by

$$S^1_{mn} = \hat{N}_{mn} - \epsilon\hat{H}^1_{mn}$$
$$S^2_{mn} = -\epsilon\hat{H}^2_{mn}.$$

Note that the homological equations can be solved order by order for $2 \leq m + n \leq N$ to any desired N, as long as the A_{mn} are invertible. It is easy to check that in the case of a pair of complex conjugate stable (or unstable) eigenvalues, when all other eigenvalues are not stable (or not unstable), then a resonance is impossible, and A_{mn} is invertible for all $m + n \geq 2$.

C.3 Parameterized stable/unstable manifolds attached to period four points of the implicit Lomelí system

Once again consider the implicit Lomelí system defined in Equation (8). We consider the case of a period four orbit with stable saddle-focus stability. That is, we assume that the periodic orbit has that $\lambda_{i1} = \overline{\lambda_{i2}}$ with $|\lambda_{i1}| < 1$ for $i = 1, 2, 3, 4$. Let

$$\tilde{\lambda}_1 = (\lambda_{11})^{1/4}, \qquad \text{and} \qquad \tilde{\lambda}_2 = (\lambda_{12})^{1/4}.$$

Motivated by Theorem 3.2, we seek smooth functions $P_i : \mathcal{B}_1^2(0) \to \mathbb{R}^3$ for $1 \leq i \leq 4$ having

$$P_i(0) = \mathbf{x}_i^*,$$
$$\frac{\partial}{\partial_j}P_i(0) = \xi_{ij},$$

for $1 \le j \le 4$ and

$$\begin{aligned} T_\epsilon(P_1(\tilde\lambda_1 z_1, \tilde\lambda_2 z_2), P_4(z_1, z_2)) &= 0 \\ T_\epsilon(P_2(\tilde\lambda_1 z_1, \tilde\lambda_2 z_2), P_1(z_1, z_2)) &= 0 \\ T_\epsilon(P_3(\tilde\lambda_1 z_1, \tilde\lambda_2 z_2), P_2(z_1, z_2)) &= 0 \\ T_\epsilon(P_4(\tilde\lambda_1 z_1, \tilde\lambda_2 z_2), P_3(z_1, z_2)) &= 0. \end{aligned} \tag{38}$$

We write

$$P_i(z_1, z_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} p^i_{n_1,n_2} z_1^{n_1} z_2^{n_2} = \begin{pmatrix} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} u^i_{n_1,n_2} z_1^{n_1} z_2^{n_2} \\ \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} v^i_{n_1,n_2} z_1^{n_1} z_2^{n_2} \\ \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} w^i_{n_1,n_2} z_1^{n_1} z_2^{n_2} \end{pmatrix},$$

and have that

$$P_i(\lambda_1 z_1, \lambda_2 z_2) = \begin{pmatrix} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u^i_{m,n} \tilde\lambda_1^m \tilde\lambda_2^n z_1^m z_2^n \\ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} v^i_{m,n} \tilde\lambda_1^m \tilde\lambda_2^n z_1^m z_2^n \\ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} w^i_{m,n} \tilde\lambda_1^m \tilde\lambda_2^n z_1^m z_2^n \end{pmatrix},$$

where

$$p^i_{0,0} = \begin{pmatrix} u^{(i)}_{00} \\ v^{(i)}_{00} \\ w^{(i)}_{00} \end{pmatrix} = \mathbf{x}^*_i, \qquad p^i_{1,0} = \begin{pmatrix} u^{(i)}_{10} \\ v^{(i)}_{10} \\ w^{(i)}_{10} \end{pmatrix} = \xi_{i1}, \qquad p^i_{0,1} = \begin{pmatrix} u^{(i)}_{01} \\ v^{(i)}_{01} \\ w^{(i)}_{01} \end{pmatrix} = \xi_{i2},$$

for $i = 1, 2, 3, 4$.

Let $v_{mn} = (u^1_{mn}, v^1_{mn}, w^1_{mn}, \dots, u^4_{mn}, v^4_{mn}, w^4_{mn})$ and plug the power series for $P_i(z_1, z_2)$ and $P_i(\lambda_1 z_1, \lambda_2 z_2)$ into the equation (38). Expanding Cauchy products, matching like powers of z_1, z_2, extracting the coefficients of order m, n from the Cauchy products, and isolating them from the lower order terms – just as in the other formal series calculations above – leads to the homological equation for $(m+n)$th term as follows

$$\mathcal{A}_{mn}\mathbf{v}_{mn} = \mathbf{S}_{mn}, \tag{39}$$

where the explicit formulas for $\mathcal{A}_{mn}$ and S_{mn} are recorded in Section C.4, as they are needed in the numerical implementation. Once again, solving these equations order by order gives the coefficients $\mathbf{v}_{mn}$ of the parameterizations to any desired accuracy. Moreover, the parameterizations of manifolds attached to longer period orbits are similar.

C.4 Explicit formulas for the period four homological equations of the implicit Lomelí system

We have that $\mathcal{A}_{mn}$ is the 12 by 12 matrix

$$\mathcal{A}_{mn} = \begin{pmatrix}
A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} & A_{1,7} & A_{1,8} & A_{1,9} & A_{1,10} & A_{1,11} & A_{1,12} \\
A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} & A_{2,7} & A_{2,8} & A_{2,9} & A_{2,10} & A_{2,11} & A_{2,12} \\
A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & A_{3,5} & A_{3,6} & A_{3,7} & A_{3,8} & A_{3,9} & A_{3,10} & A_{3,11} & A_{3,12} \\
B_{1,1} & B_{1,2} & B_{1,3} & B_{1,4} & B_{1,5} & B_{1,6} & B_{1,7} & B_{1,8} & B_{1,9} & B_{1,10} & B_{1,11} & B_{1,12} \\
B_{2,1} & B_{2,2} & B_{2,3} & B_{2,4} & B_{2,5} & B_{2,6} & B_{2,7} & B_{2,8} & B_{2,9} & B_{2,10} & B_{2,11} & B_{2,12} \\
B_{3,1} & B_{3,2} & B_{3,3} & B_{3,4} & B_{3,5} & B_{3,6} & B_{3,7} & B_{3,8} & B_{3,9} & B_{3,10} & B_{3,11} & B_{3,12} \\
C_{1,1} & C_{1,2} & C_{1,3} & C_{1,4} & C_{1,5} & C_{1,6} & C_{1,7} & C_{1,8} & C_{1,9} & C_{1,10} & C_{1,11} & C_{1,12} \\
C_{2,1} & C_{2,2} & C_{2,3} & C_{2,4} & C_{2,5} & C_{2,6} & C_{2,7} & C_{2,8} & C_{2,9} & C_{2,10} & C_{2,11} & C_{2,12} \\
C_{3,1} & C_{3,2} & C_{3,3} & C_{3,4} & C_{3,5} & C_{3,6} & C_{3,7} & C_{3,8} & C_{3,9} & C_{3,10} & C_{3,11} & C_{3,12} \\
D_{1,1} & D_{1,2} & D_{1,3} & D_{1,4} & D_{1,5} & D_{1,6} & D_{1,7} & D_{1,8} & D_{1,9} & D_{1,10} & D_{1,11} & D_{1,12} \\
D_{2,1} & D_{2,2} & D_{2,3} & D_{2,4} & D_{2,5} & D_{2,6} & D_{2,7} & D_{2,8} & D_{2,9} & D_{2,10} & D_{2,11} & D_{2,12} \\
D_{3,1} & D_{3,2} & D_{3,3} & D_{3,4} & D_{3,5} & D_{3,6} & D_{3,7} & D_{3,8} & D_{3,9} & D_{3,10} & D_{3,11} & D_{3,12}
\end{pmatrix}$$

with entries

$$A_{1,1} = \lambda_1^{n_1}\lambda_2^{n_2} \quad A_{1,2} = 5\epsilon\alpha v_{00}^{(1)}\lambda_1^{n_1}\lambda_2^{n_2} \quad A_{1,3} = 5\epsilon\beta w_{00}^{(1)}\lambda_1^{n_1}\lambda_2^{n_2} \quad A_{1,4} = 0 \quad A_{1,5} = 0 \quad A_{1,6} = 0$$
$$A_{1,7} = 0 \quad A_{1,8} = 0 \quad A_{1,9} = 0 \quad A_{1,10)} = -\gamma - 2\alpha u_{00}^{(4)} - bv_{00}^{(4)} \quad A_{1,11} = -bu_{00}^{(4)} - 2cv_{00}^{(4)} \quad A_{1,12} = -1$$
$$A_{2,1} = 0 \quad A_{2,2} = \lambda_1^{n_1}\lambda_2^{n_2} \quad A_{2,3} = 5\epsilon\gamma w_{00}^{(1)}\lambda_1^{n_1}\lambda_2^{n_2} \quad A_{2,4} = 0 \quad A_{2,5} = 0 \quad A_{2,6} = 0$$
$$A_{2,7} = 0 \quad A_{2,8} = 0 \quad A_{2,9} = 0 \quad A_{2,10} = -1 \quad A_{2,11} = 0 \quad A_{2,12} = 0$$
$$A_{3,1} = 0 \quad A_{3,2} = 0 \quad A_{3,3} = \lambda_1^{n_1}\lambda_2^{n_2} \quad A_{3,4} = 0 \quad A_{3,5} = 0 \quad A_{3,6} = 0$$
$$A_{3,7} = 0 \quad A_{3,8} = 0 \quad A_{3,9} = 0 \quad A_{3,10} = 0 \quad A_{3,11} = -1 \quad A_{3,12} = 0$$
$$B_{1,1} = -\gamma - 2\alpha u_{00}^{(1)} - bv_{00}^{(1)} \quad B_{1,2} = -bu_{00}^{(1)} - 2cv_{00}^{(1)} \quad B_{1,3} = -1 \quad B_{1,4} = \lambda_1^{n_1}\lambda_2^{n_2} \quad B_{1,5} = 5\epsilon\alpha v_{00}^{(2)}\lambda_1^{n_1}\lambda_2^{n_2}$$
$$B_{1,6} = 5\epsilon\beta w_{00}^{(2)}\lambda_1^{n_1}\lambda_2^{n_2} \quad B_{1,7} = 0 \quad B_{1,8} = 0 \quad B_{1,9} = 0 \quad B_{1,10} = 0 \quad B_{1,11} = 0 \quad B_{1,12} = 0$$
$$B_{2,1} = -1 \quad B_{2,2} = 0 \quad B_{2,3} = 0 \quad B_{2,4} = 0 \quad B_{2,5} = \lambda_1^{n_1}\lambda_2^{n_2} \quad B_{2,6} = 5\epsilon\gamma w_{00}^{(2)}\lambda_1^{n_1}\lambda_2^{n_2}$$
$$B_{2,7} = 0 \quad B_{2,8} = 0 \quad B_{2,9} = 0 \quad B_{2,10} = 0 \quad B_{2,11} = 0 \quad B_{2,12} = 0$$
$$B_{3,1} = 0 \quad B_{3,2} = -1 \quad B_{3,3} = 0 \quad B_{3,4} = 0 \quad B_{3,5} = 0 \quad B_{3,6} = \lambda_1^{n_1}\lambda_2^{n_2}$$
$$B_{3,7} = 0 \quad B_{3,8} = 0 \quad B_{3,9} = 0 \quad B_{3,10} = 0 \quad B_{3,11} = 0 \quad B_{3,12} = 0$$
$$C_{1,1} = 0 \quad C_{1,2} = 0 \quad C_{1,3} = 0 \quad C_{1,4} = -\gamma - 2\alpha u_{00}^{(2)} - bv_{00}^{(2)} \quad C_{1,5} = -bu_{00}^{(4)} - 2cv_{00}^{(4)} \quad C_{1,6} = 0$$
$$C_{1,7} = \lambda_1^{n_1}\lambda_2^{n_2} \quad C_{1,8} = 5\epsilon\alpha v_{00}^{(3)}\lambda_1^{n_1}\lambda_2^{n_2} \quad C_{1,9} = 5\epsilon\beta w_{00}^{(3)}\lambda_1^{n_1}\lambda_2^{n_2} \quad C_{1,10} = 0 \quad C_{1,11} = 0 \quad C_{1,12} = 0$$
$$C_{2,1} = 0 \quad C_{2,2} = 0 \quad C_{2,3} = 0 \quad C_{2,4} = -1 \quad C_{2,5} = 0 \quad C_{2,6} = 0 \quad C_{2,7} = 0$$
$$C_{2,8} = \lambda_1^{n_1}\lambda_2^{n_2} \quad C_{2,9} = 5\epsilon\gamma w_{00}^{(3)}\lambda_1^{n_1}\lambda_2^{n_2} \quad C_{2,10} = 0 \quad C_{2,11} = 0 \quad C_{2,12} = 0$$
$$C_{3,1} = 0 \quad C_{3,2} = 0 \quad C_{3,3} = 0 \quad C_{3,4} = 0 \quad C_{3,5} = -1 \quad C_{3,6} = 0$$
$$C_{3,7} = 0 \quad C_{3,8} = 0 \quad C_{3,9} = \lambda_1^{n_1}\lambda_2^{n_2} \quad C_{3,10} = 0 \quad C_{3,11} = 0 \quad C_{3,12}$$
$$D_{1,1} = 0 \quad D_{1,2} = 0 \quad D_{1,3} = 0 \quad D_{1,4} = 0 \quad D_{1,5} = 0 \quad D_{1,6} = 0$$
$$D_{1,7} = -\gamma - 2\alpha u_{00}^{(3)} - bv_{00}^{(3)} \quad D_{1,8} = -bu_{00}^{(3)} - 2cv_{00}^{(3)} \quad D_{1,9} = -1 \quad D_{1,10} = \lambda_1^{n_1}\lambda_2^{n_2}$$
$$D_{1,11} = 5\epsilon\alpha v_{00}^{(4)}\lambda_1^{n_1}\lambda_2^{n_2} \quad D_{1,12} = 5\epsilon\beta w_{00}^{(4)}\lambda_1^{n_1}\lambda_2^{n_2}$$
$$D_{2,1} = 0 \quad D_{2,2} = 0 \quad D_{2,3} = 0 \quad D_{2,4} = 0 \quad D_{2,5} = 0 \quad D_{2,6} = 0$$
$$D_{2,7} = -1 \quad D_{2,8} = 0 \quad D_{2,9} = 0 \quad D_{2,10} = 0 \quad D_{2,11} = \lambda_1^{n_1}\lambda_2^{n_2} \quad D_{2,12} = 5\epsilon\gamma w_{00}^{(4)}\lambda_1^{n_1}\lambda_2^{n_2}$$
$$D_{3,1} = 0 \quad D_{3,2} = 0 \quad D_{3,3} = 0 \quad D_{3,4} = 0 \quad D_{3,5} = 0 \quad D_{3,6} = 0$$
$$D_{3,7} = 0 \quad D_{3,8} = -1 \quad D_{3,9} = 0 \quad D_{3,10} = 0 \quad D_{3,11} = 0 \quad D_{3,12} = \lambda_1^{n_1}\lambda_2^{n_2}.$$

Similarly, the right hand side

$$\mathbf{S}_{mn} = (S^1_{mn}, S^2_{mn}, 0, S^3_{mn}, S^4_{mn}, 0, S^5_{mn}, S^6_{mn}, 0, S^7_{mn}, S^8_{mn}, 0)^T$$

has components

$$\begin{aligned}
S^1_{mn} &= a(\widehat{u_4 * u_4})_{mn} + b(\widehat{u_4 * v_4})_{mn} + c(\widehat{v_4 * v_4})_{mn} \\
&- \alpha\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(v_1 * v_1 \widehat{* v_1 *} v_1 * v_1)_{mn} - \beta\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(w_1 * w_1 \widehat{* w_1 *} w_1 * w_1)_{mn} \\
S^2_{mn} &= \gamma\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(w_1 * w_1 \widehat{* w_1 *} w_1 * w_1)_{mn} \\
S^3_{mn} &= a(\widehat{u_1 * u_1})_{mn} + b(\widehat{u_1 * v_1})_{mn} + c(\widehat{v_1 * v_1})_{mn} \\
&- \alpha\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(v_2 * v_2 \widehat{* v_2 *} v_2 * v_2)_{mn} - \beta\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(w_2 * w_2 \widehat{* w_2 *} w_2 * w_2)_{mn} \\
S^4_{mn} &= \gamma\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(w_2 * w_2 \widehat{* w_2 *} w_2 * w_2)_{mn} \\
S^5_{mn} &= a(\widehat{u_2 * u_2})_{mn} + b(\widehat{u_2 * v_2})_{mn} + c(\widehat{v_2 * v_2})_{mn} \\
&- \alpha\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(v_3 * v_3 \widehat{* v_3 *} v_3 * v_3)_{mn} - \beta\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(w_3 * w_3 \widehat{* w_3 *} w_3 * w_3)_{mn} \\
S^6_{mn} &= \gamma\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(w_3 * w_3 \widehat{* w_3 *} w_3 * w_3)_{mn} \\
S^7_{mn} &= a(\widehat{u_3 * u_3})_{mn} + b(\widehat{u_3 * v_3})_{mn} + c(\widehat{v_3 * v_3})_{mn} \\
&- \alpha\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(v_4 * v_4 \widehat{* v_4 *} v_4 * v_4)_{mn} - \beta\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(w_4 * w_4 \widehat{* w_4 *} w_4 * w_4)_{mn} \\
S^8_{mn} &= \gamma\epsilon\tilde{\lambda}_1^m\tilde{\lambda}_2^n(w_4 * w_4 \widehat{* w_4 *} w_4 * w_4)_{mn}.
\end{aligned}$$

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