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Fourier Dimension Estimates for Sets of Exact Approximation Order: the Well-Approximable Case

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We obtain a Fourier dimension estimate for sets of exact approximation order introduced by Bugeaud for certain approximation functions ψ. This Fourier dimension estimate implies that these sets of exact approximation order contain normal numbers.

1 Introduction and Background

1.1 Hausdorff and Fourier Dimension

Let $E \subset \mathbb{R}$ be a compact set. Frostman's lemma [6] implies that the Hausdorff dimension $\dim_H(E)$ of E is the supremum over all values of $s < 1$ such that E supports a Borel probability measure μ satisfying the condition

$$\int |\widehat{\mu}(\xi)|^2 |\xi|^{s-1} d\xi < \infty.$$

This condition essentially says, up to an ϵ-loss in the exponent, that $|\widehat{\mu}(\xi)|$ decays, in an L^2-average sense, at least as quickly as $|\xi|^{-s/2}$.

A related notion of dimension is given by the Fourier dimension. For a compact set E, the **Fourier dimension** of E, denoted $\dim_F(E)$, is the supremum over all $s < 1$ such that E supports a measure μ satisfying the *pointwise* Fourier decay condition

$$|\widehat{\mu}(\xi)| \leq |\xi|^{-s/2}.$$

Clearly, we have that $\dim_F(E) \leq \dim_H(E)$ for every compact subset $E \subset \mathbb{R}$. If a compact set E satisfies $\dim_F(E) = \dim_H(E)$, then we say that E is a compact **Salem set**.

In fact, it is nontrivial to construct examples of Salem sets. The earliest constructions of Salem sets are random constructions, such as random Cantor sets of Salem [19]. Körner [14] shows that the Fourier dimension of a set $E \subset \mathbb{R}$ can take any value from 0 to $\dim_H(E)$.

1.2 Metric Diophantine Approximation

A classical result of Jarnìk and Besicovitch [2] [10] concerns the Hausdorff dimension of the set of τ-approximable numbers. The τ-approximable numbers are the set

$$E(\tau) := \{x : |x - p/q| \leq q^{-\tau} \text{ for infinitely many pairs of integers } (p, q).\}$$

For $\tau \leq 2$, it is easy to see using the Dirichlet principle that $E(\tau) = \mathbb{R}$. Jarnìk and Besicovitch show that for $\tau > 2$, $\dim_H(E(\tau)) = \frac{2}{\tau}$.

Kaufman [12] shows that, in fact, the set $E(\tau)$ has Fourier dimension equal to $\frac{2}{\tau}$, implying that $E(\tau)$ is a Salem set. Notably, this is the first explicit non-random construction of a Salem set of Hausdorff dimension other than 0 or 1 in $\mathbb{R}$.

In fact, Jarnìk's result can be extended to a more general type of set. Suppose that $\psi(q) : \mathbb{N} \to \mathbb{R}^+$ is a decreasing function satisfying the condition that

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$$\lambda^+(\psi) = -\limsup_{q\to\infty} \frac{\log \psi(q)}{\log q} \geq 2.$$

If we define the set $W(\psi,\theta)$ to be the set of inhomogeneous well-approximable numbers defined by

$$\left| x - \frac{p-\theta}{q} \right| \leq \psi(q) \quad \text{for infinitely many pairs } (p,q) \text{ of relatively prime integers,}$$

then a result of Levesley [15] implies that this set has Hausdorff dimension equal to $\frac{2}{\lambda^+}$. In fact, Levesley computes the Hausdorff dimension even in the case of simultaneous or dual approximation.

In fact, for limsup sets such as $W(\psi,\theta)$, statements about the Hausdorff dimension can be deduced from the mass transference principle of Beresnevich and Velani [1]. For the specific case of $W(\psi,\theta)$, the Hausdorff dimension can be computed by applying the mass transference principle with a result of Khintchine [13] in the homogeneous case and a result of Szüsz [20] in the inhomogeneous case.

The sets $W(\psi,\theta)$ are shown to be Salem sets in the homogeneous case $\theta = 0$ by Bluhm [3], and in the inhomogeneous case $\theta \neq 0$ by Hambrook [7] and independently by Zafeiropoulos [21].

Fourier dimension calculations are of interest in metric Diophantine approximation because of a celebrated result of Davenport, Erdős, and Leveque [5]. This result concerns the presence of normal numbers in subsets $E \subset \mathbb{R}$.

Specifically, Davenport, Erdős and Leveque show that if μ is a positive Borel probability measure, and a is a positive integer, then the sequence $\{a^j x\}_{n=1}^{\infty}$ is uniformly distributed modulo 1 almost everywhere with respect to the measure μ if

$$\sum_{N=1}^{\infty} N^{-3} \sum_{j=1}^{N} \sum_{k=1}^{N} \widehat{\mu}(m(a^j - a^k)) < \infty \tag{1}$$

for every nonzero integer $m \in \mathbb{Z}$. If we crudely assume $|\widehat{\mu}(\xi)| \leq |\xi|^{-s/2}$ for every $\xi \in \mathbb{R}$, the sum in k in (1) is essentially a geometric sum, so we have that the sum in j is bounded above by an m-dependent constant times N, and the sum certainly converges for all nonzero integers m, and all integers $a \geq 2$. Therefore, if μ is a Borel probability measure such that $|\widehat{\mu}(\xi)| \leq |\xi|^{-s/2}$ for some $s > 0$, then μ-almost every point is a normal number. Note that a far weaker assumption on μ suffices to locate normal numbers; see e.g. [16].

In particular, any set E of positive Fourier dimension must contain normal numbers. It is therefore of interest to find Fourier dimension estimates for subsets of $\mathbb{R}$ arising in Diophantine approximation. Of course, the well-approximable numbers, being a Salem set of positive dimension, contain normal numbers. In fact, Kaufman [11] also shows a Fourier dimension result for sets of *badly*-approximable numbers.

The badly approximable numbers consist of those $x \in \mathbb{R}$ such that the partial quotients in the continued fraction expansion of x are bounded. Given a finite set $S \subset \mathbb{N}$ with at least two elements, we use the term S-badly-approximable numbers to refer to those real numbers x such that the partial quotients of the continued fraction expansion of x all lie in the finite set S. Kaufman [11] shows that, if S is a finite set such that the Hausdorff dimension of the S-badly-approximable numbers is greater than $2/3$, then the S-badly approximable numbers have positive Fourier dimension.

The method used by Kaufman to estimate the Fourier dimension of the badly approximable numbers is very different from the method used to estimate the Fourier dimension of the well-approximable numbers. For the well-approximable numbers, Kaufman's argument relies on the cancellation of the exponential sum

$$\sum_{p=0}^{q} e(ps/q) \tag{2}$$

for any integers s, q such that q does not divide s; since s has a small number of divisors, the sum of (2) over all $M/2 \leq q < M$ will also be small.

In contrast, Kaufman's Fourier dimension estimate for the badly-approximable numbers [11] follows a rather different argument. This argument relies on constructing a certain random measure on bounded integer sequences whose pushforward under the continued fraction map satisfies the relevant Fourier decay condition, which is established via a van der Corput-type lemma.

For the set of S-approximable numbers, Queffelec and Ramaré [17] improve the $2/3$ requirement on the Hausdorff dimension, which ensured positive Fourier dimension, to $1/2$; in particular, this condition holds if $S = \{1,2\}$. Hochman and Shmerkin [8] show that, without any Hausdorff dimension assumption, the set of S-badly-approximable numbers contains normal numbers for any finite set $S \subset \mathbb{N}$ with at least two elements. In a recent work, Sahlsten and Stevens [18] improved on all of these results by showing, without any Hausdorff dimension assumption, that the S-badly-approximable numbers have positive Fourier dimension for any finite set $S \subset \mathbb{N}$ with at least two elements.

Of note is that, while Kaufman's argument for the well-approximable numbers [12] works just as well in the inhomogeneous setting, there does not seem to be an easy way to modify Kaufman's argument for the badly approximable numbers, [11], to this case. Doing so would require a satisfactory analogue of the continued fraction expansion for the inhomogeneous version of the badly approximable numbers.

1.3 Approximation to Exact Order

Bugeaud [4] introduces sets of exact approximation order. We will now define an inhomogeneous analogue.

Definition 1.1 (Sets of exact approximation order). Given a function $\psi : \mathbb{N} \to (0, \infty)$ and a real number $\theta \in [0, 1)$, define the set $\mathrm{Exact}(\psi, \theta)$ to be the set of real numbers x satisfying the pair of conditions:

$$\left| x - \frac{p-\theta}{q} \right| \leq \psi(q) \qquad \text{for infinitely many pairs } (p, q) \text{ of relatively prime integers}$$
$$\left| x - \frac{p-\theta}{q} \right| \leq \psi(q) - c\psi(q) \quad \text{for only finitely many pairs } (p, q) \text{ of relatively prime integers and any } c > 0.$$

□

Observe that the set $\mathrm{Exact}(\psi, 0)$ is not a limsup set, so the mass transference principle of Beresnevich and Velani cannot be applied to compute the Hausdorff dimension. Nonetheless, Bugeaud [4] computes the Hausdorff dimension of the set $\mathrm{Exact}(\psi, 0)$ for certain functions ψ. Specifically, Bugeaud considers functions ψ such that the function $x^2\psi(x)$ is nonincreasing. Bugeaud shows that the Hausdorff dimension of $\mathrm{Exact}(\psi, 0)$ is $\frac{2}{\lambda^+}$, where

$$\lambda^+(\psi) = -\limsup_{q\to\infty} \frac{\log \psi(q)}{\log q} \geq 2.$$

The upper Hausdorff dimension bound follows trivially from the Jarnìk-Besicovitch theorem. For the lower bound, Bugeaud considers a subset of $\mathrm{Exact}(\psi, 0)$ consisting of numbers whose continued fractions have partial quotients that typically grow very slowly, except for some exceptional partial quotients that are larger.

Observe that the set $\mathrm{Exact}(\psi, \theta)$ is invariant under translations by integers. Therefore, we can view $\mathrm{Exact}(\psi, \theta)$ as a subset of the torus $[0, 1)$ in a natural way. We will use the notation $\mathrm{Exact}^{[0,1)}(\psi, \theta)$ to refer to this subset of the torus.

Let $\theta \in [0, 1)$. We define the **Diophantine approximation exponent** of θ to be the supremum over values of γ such that the equation

$$0 < \left| \theta - \frac{p}{q} \right| \leq q^{-\gamma}$$

has infinitely many solutions for integers p and q. Note that if θ is rational, then $\gamma = 1$. If $\gamma(\theta)$ is the Diophantine approximation exponent of an irrational number θ, and $\eta > 0$, then, for any integers p and q with $q > 0$,

$$|q\theta - p| \geq q^{-\gamma(\theta)-\eta+1},$$

provided q is sufficiently large depending on η and θ. Since θ is irrational, we can then observe that there exists a constant $D_{\eta,\theta} > 0$ such that for $q > 0$,

$$\|q\theta\| \geq D_{\eta,\theta} q^{-\gamma(\theta)+1-\eta}, \tag{3}$$

where $\|q\theta\|$ is the distance from $q\theta$ to the nearest integer.

Theorem 1.2. Let $\theta \in [0, 1)$ be either 0 or an irrational number with finite Diophantine exponent γ. Let $\psi : \mathbb{N} \to (0, \infty)$ be a positive, decreasing function such that the limit

$$\lambda(\psi) := -\lim_{q\to\infty} \frac{\log \psi(q)}{\log q} \tag{4}$$

exists and is finite. Suppose $\tau = \lambda(\psi)$ is such that

$$\tau > \frac{2 + \gamma^2 + \sqrt{(2 + \gamma^2)^2 - 4}}{2}. \tag{5}$$

Then $\dim_F \mathrm{Exact}(\psi,\theta)$ is positive, and therefore, $\mathrm{Exact}(\psi,\theta)$ contains normal numbers. Moreover, we have the inequality

$$\dim_F \mathrm{Exact}(\psi,\theta) \geq \alpha := \frac{2(\beta-\tau)}{\tau(\beta-1)}, \tag{6}$$

where

$$\beta := \gamma^{-2}(\tau-1)^2. \tag{7}$$

□

For technical reasons, we do not consider non-zero rational θ in Theorem 1.2. An indication of these technicalities is made at Remark 2.2, which follows the relevant proof.

One can observe that the condition (5) implies that $\beta > \tau$; this implies that the right side of (6) is positive. Indeed, to see this, one considers solutions to the quadratic equation $\beta(\tau) - \tau = 0$ in τ, which is equivalent to

$$\tau^2 - (2+\gamma^2)\tau + 1 = 0.$$

The largest solution of this equation is

$$\tau_+ = \frac{(2+\gamma^2) + \sqrt{(2+\gamma^2)^2 - 4}}{2}$$

so that, for $\tau > \tau_+$, we must have $\beta(\tau) > \tau$. Let us make some quick observations about Theorem 1.2.

Remark 1.3. If $\theta = 0$, we are able to take $\gamma = 1$. In this case, the inequality (5) reduces to

$$\tau > \frac{3+\sqrt{5}}{2}. \tag{8}$$

□

Remark 1.4. For a fixed γ, observe that in the regime $\tau \to \infty$, we have that $\beta = \gamma^{-2}\tau^2 + O(\tau)$. Therefore,

$$\lim_{\tau\to\infty} \frac{\frac{2(\beta-\tau)}{\tau(\beta-1)}}{2/\tau} = 1.$$

This means that, if τ is large, Theorem 1.2 "nearly" shows the set $\mathrm{Exact}(\psi,\theta)$ is a Salem set. □

In fact, our proof yields a slightly more general Fourier dimension estimate than the one in Theorem 1.2. In order to state this estimate, we will introduce sets of tight approximation order.

Definition 1.5 (Sets of tight approximation order)**.** Let $\psi_1, \psi_2 : \mathbb{N} \to (0,\infty)$ be a pair of functions such that $\psi_2(q) \leq \psi_1(q)$ for all q. The set $\mathrm{Tight}(\psi_1,\psi_2,\theta)$ consists of those real numbers x satisfying the conditions

$$\left| x - \frac{p-\theta}{q} \right| \leq \psi_1(q) \qquad \text{for infinitely many pairs } (p,q) \text{ of relatively prime integers,}$$

$$\left| x - \frac{p-\theta}{q} \right| \leq \psi_1(q) - c\psi_2(q) \quad \text{for only finitely many pairs } (p,q) \text{ of relatively prime integers and any } c > 0.$$

□

Remark 1.6. The set $\mathrm{Exact}(\psi,\theta)$ is the same as the set $\mathrm{Tight}(\psi,\psi,\theta)$. □

As is the case for $\mathrm{Exact}(\psi,\theta)$, the set $\mathrm{Tight}(\psi_1,\psi_2,\theta)$ is invariant under translations by integers. So $\mathrm{Tight}(\psi_1,\psi_2,\theta)$ can naturally be associated to a subset of the torus, which we denote by $\mathrm{Tight}^{[0,1)}(\psi_1,\psi_2,\theta)$.

Given appropriate conditions on ψ_1 and ψ_2, we are able to estimate the Fourier dimension of the set $\mathrm{Tight}(\psi_1,\psi_2,\theta)$.

Theorem 1.7. Let $\theta \in [0,1)$ be either 0 or an irrational number with finite Diophantine exponent γ. Let $\psi_1(q)$ and $\psi_2(q)$ be decreasing functions with $\psi_2(q) \leq \psi_1(q)$ for all q, such that $\psi_1(q) - \psi_2(q)$ is decreasing, and such that the limits $\lambda(\psi_1)$ and $\lambda(\psi_2)$ exist and are finite. Let $\tau_1 = \lambda(\psi_1)$, and $\tau_2 = \lambda(\psi_2)$.

Let

$$\beta := \gamma^{-2}(\tau_1 - 1)^2. \tag{9}$$

Suppose $\tau_1 = \lambda(\psi_1)$ and $\tau_2 = \lambda(\psi_2)$ are such that $\beta > \tau_2$. Then $\dim_F \mathrm{Tight}(\psi_1, \psi_2, \theta)$ is positive; moreover, we have the inequality

$$\dim_F \mathrm{Tight}(\psi_1, \psi_2, \theta) \geq \alpha := \frac{2(\beta - \tau_2)}{\tau_2(\beta - 1)}. \tag{10}$$

□

Remark 1.8. Because $\psi_2(q) \leq \psi_1(q)$ for all q, it follows that $\tau_2 \geq \tau_1$. In particular, the condition $\beta > \tau_2$ implies that $\beta > \tau_1$. Combining this inequality with (9) gives the condition

$$\tau_2 \geq \tau_1 > \frac{2 + \gamma^2 + \sqrt{(2 + \gamma^2)^2 - 4}}{2}. \tag{11}$$

In particular, since $\gamma \geq 1$, we will always have

$$\tau_2 \geq \tau_1 > \frac{3 + \sqrt{5}}{2}. \tag{12}$$

□

Remark 1.9. It is likely that Bugeaud's proof [4] can be adapted to show that $\mathrm{Tight}(\psi_1, \psi_2, 0)$ has Hausdorff dimension $\frac{2}{\tau_2}$. For more general θ, a simple covering argument shows that $\mathrm{Tight}(\psi_1, \psi_2, \theta)$ has Hausdorff dimension at most $\frac{2}{\tau_2}$. □

Remark 1.10. If the quantity β could have been chosen to be an arbitrarily large real number rather than $\gamma^{-2}(\tau_1 - 1)^2$, then we would have $\dim_F \mathrm{Tight}(\psi_1, \psi_2, \theta) = \frac{2}{\tau_2}$, showing together with Remark 1.9 that $\mathrm{Tight}(\psi_1, \psi_2, \theta)$ is a Salem set. However, the choice of β is dictated by Lemma 2.1. This places limits on the growth rate of the sequence $\{M_j\}_{j=1}^\infty$ introduced in Section 6, leading to some interference in the Fourier transform between scales that cannot be eliminated. This problem does not occur in the case of the well-approximable numbers, where the corresponding sequence can be chosen to grow arbitrarily quickly, allowing for the interference between scales to be minimized. □

2 An elementary Diophantine approximation lemma

The key to adapting Kaufman's argument to the set $\mathrm{Tight}(\psi_1, \psi_2, \theta)$ is an elementary lemma in Diophantine approximation. This lemma states that if a real number x is approximable by rationals at two "fairly close" scales, then x cannot be approximable at any intermediate scale.

Let

$$\beta_\epsilon = \gamma^{-2}(\tau_1 - 1)^2 - \epsilon. \tag{13}$$

Lemma 2.1. Let $\tau_2 \geq \tau_1 > 2$, $0 < c < 1$, $\epsilon > 0$ be real numbers, and let $x \in \mathbb{R}$. Suppose $\psi_1 : \mathbb{N} \to \mathbb{R}^+$ and $\psi_2 : \mathbb{N} \to \mathbb{R}^+$ are decreasing functions satisfying (4), with $\lambda(\psi_1) = \tau_1$ and $\lambda(\psi_2) = \tau_2$, chosen such that $\psi_1 - \psi_2$ is also decreasing. Let $\theta \in [0, 1)$ be either 0 or an irrational number with Diophantine approximation exponent γ. Let us suppose that for some $x \in \mathbb{R}$ such that there exist pairs (p_1, q_1) and $(p_2, q_2) \in \mathbb{Z} \times \mathbb{N}$ for which

$$\psi_1(q_1) - c\psi_2(q_1) \leq \left| x - \frac{p_1 - \theta}{q_1} \right| \leq \psi_1(q_1)$$
$$\psi_1(q_2) - c\psi_2(q_2) \leq \left| x - \frac{p_2 - \theta}{q_2} \right| \leq \psi_1(q_2),$$

where $Q(\epsilon) < q_1 < q_2 < q_1^{\beta_\epsilon}$ and q_2 is prime. Then x does not satisfy any inequality of the form

$$\left| x - \frac{p - \theta}{q} \right| < \psi_1(q) - c\psi_2(q),$$

for any integer pair $(p, q) \in \mathbb{Z} \times \mathbb{N}$ with $q_1 < q < q_2$.

□

Proof. Suppose q, q_1, q_2, p, p_1, and p_2 are positive integers such that q_2 is prime and $Q(\epsilon) < q_1 < q < q_2$, and suppose that x satisfies the inequalities

$$\psi_1(q_1) - c\psi_2(q_1) \leq \left| x - \frac{p_1 - \theta}{q_1} \right| \leq \psi_1(q_1) \tag{14}$$

$$\psi_1(q_2) - c\psi_2(q_2) \leq \left| x - \frac{p_2 - \theta}{q_2} \right| \leq \psi_1(q_2) \tag{15}$$

$$\left| x - \frac{p - \theta}{q} \right| < \psi_1(q) - c\psi_2(q). \tag{16}$$

We will prove the lemma by showing that, provided we choose our $Q(\epsilon) = Q(\epsilon, \theta, \psi_1)$ sufficiently large, we must have that $q_2 \geq q_1^{\beta_\epsilon}$.

We will split into two cases depending on whether $\theta = 0$.

Case 1 Here we consider $\theta = 0$. In this case, (14) reduces to

$$\psi_1(q_1) - c\psi_2(q_1) \leq \left| x - \frac{p_1}{q_1} \right| \leq \psi_1(q_1). \tag{17}$$

Observe that if there exist p, q, with $q > q_1$, such that (16) holds, then we must have $\frac{p}{q} \neq \frac{p_1}{q_1}$; this follows from (17) and the fact that $\psi_1 - c\psi_2$ is decreasing.

Thus, we have the inequality

$$\left| \frac{p}{q} - \frac{p_1}{q_1} \right| = \frac{|pq_1 - p_1 q|}{qq_1} \geq \frac{1}{qq_1},$$

since the numerator is a nonzero integer. On the other hand, equations (17) and (16) imply by the triangle inequality that

$$\left| \frac{p}{q} - \frac{p_1}{q_1} \right| \leq \psi_1(q_1) + \psi_1(q) - c\psi_2(q) \leq 2\psi_1(q_1),$$

where the last inequality follows from the fact that ψ_1 is decreasing.

Combining these inequalities gives that

$$\frac{1}{qq_1} \leq 2\psi_1(q_1). \tag{18}$$

We consider some small $\eta > 0$, later to be fixed. At this stage, we select some Q_η such that $\frac{\log(2\psi_1(q))}{\log q} < -\tau_1 + \eta$ for all $q \geq Q_\eta$. Then, it follows from (18), that if $q_1 \geq Q_\eta$,

$$q \geq q_1^{\tau_1 - 1 - \eta}.$$

Because $q_2 > q$ and q_2 is prime, it follows that $\frac{p}{q} \neq \frac{p_2}{q_2}$, so a similar argument reveals that

$$q_2 \geq q^{\tau_1 - 1 - \eta}$$

and thus, we combine to get that

$$q_2 \geq q_1^{(\tau_1 - 1 - \eta)^2}.$$

So if we choose η sufficiently small that $(\tau_1 - 1 - \eta)^2 > (\tau_1 - 1)^2 - \epsilon$, we must have

$$q_2 \geq q_1^{\beta_\epsilon},$$

as desired.

Case 2 Now, we will assume $\theta \in [0,1)$ is an irrational number with Diophantine approximation exponent γ. As before, we will consider some small $\eta > 0$, later to be fixed. In this case, we simply observe that if (14) and (16) both hold, then we must have, by the triangle inequality and the fact that ψ_1 is decreasing, that

$$\begin{aligned} \left|\frac{p_1-\theta}{q_1} - \frac{p-\theta}{q}\right| &\leq 2\psi_1(q_1), \\ \text{or, equivalently,} \quad \left|\frac{qp_1-pq_1-(q-q_1)\theta}{qq_1}\right| &\leq 2\psi_1(q_1). \end{aligned} \tag{19}$$

Now, $qp_1 - pq_1$ is an integer, as is $(q-q_1)$. Therefore, if θ has Diophantine exponent γ, then, by (3), we have $||(q-q_1)\theta|| \geq D_{\eta,\theta}(q-q_1)^{-\gamma+1-\eta/2}$. Here, $||\cdot||$ denotes the distance to the nearest integer. If we then take $q > Q_{\eta,1}$, where $Q_{\eta,1}$ is chosen to be sufficiently large, we have that $||(q-q_1)\theta|| \geq q^{-\gamma+1-\eta}$. Thus, for such q, we have

$$\left|\frac{qp_1-pq_1-(q-q_1)\theta}{qq_1}\right| \geq \frac{q^{-\gamma+1-\eta}}{qq_1}. \tag{20}$$

By combining inequalities (19) and (20), we get

$$q^{-\gamma-\eta}q_1^{-1} \leq 2\psi_1(q_1).$$

Now, observe that if $q_1 \geq Q_\eta$, for some suitably large Q_η chosen as in Case 1, then we have from the fact that $\tau_1 = \lambda(\psi_1)$ that $2\psi_1(q_1) \leq q_1^{-\tau_1+\eta}$. Thus, for $q_1 > \max(Q_{\eta,1}, Q_\eta)$, we have

$$q^{-\gamma-\eta}q_1^{-1} \leq q_1^{-\tau_1+\eta}$$

and solving for q yields

$$q \geq q_1^{(\gamma+\eta)^{-1}(\tau_1-1-\eta)}.$$

By a similar argument, we can observe

$$q_2 \geq q^{(\gamma+\eta)^{-1}(\tau_1-1-\eta)}.$$

Combining these inequalities gives

$$q_2 \geq q_1^{(\gamma+\eta)^{-2}(\tau_1-1-\eta)^2}.$$

Thus we can see, provided we take $\eta > 0$ sufficiently small depending on ϵ, we have the inequality

$$q_2 \geq q_1^{\beta_\epsilon},$$

as desired. ■

Remark 2.2. Besides $\theta = 0$, we do not present any argument for rational θ. More specifically, we are unable to obtain a lower bound of the form (20), since $qp_1 - pq_1 - (q-q_1)\theta$ might vanish, so our argument doesn't carry over to this case. □

For the purposes of the rest of the argument, it will be important to have $\beta_\epsilon > \tau_2$ for sufficiently small ϵ. In particular, $\beta_\epsilon > \tau_1$, which explains the restriction (5).

3 A periodization trick

In order to establish Theorem 1.7, we must construct, for any $\epsilon > 0$, a finite Borel measure μ_ϵ supported on $\text{Tight}(\psi_1,\psi_2,\theta) \cap [0,1]$ such that $\widehat{\mu}_\epsilon(\xi) \leq C(1+|\xi|)^{-\alpha/2+\epsilon}$ for all $\xi \in \mathbb{R}$. However, it is convenient to evaluate $\widehat{\mu}_\epsilon(\xi)$ at only integer values of ξ and for this purpose, we consider the set $\text{Tight}^{[0,1)}(\psi_1,\psi_2,\theta)$, a subset of the torus. To this end, we will construct a measure $\mu_\epsilon^{[0,1)}$ on the torus with support contained in $\text{Tight}^{[0,1)}(\psi_1,\psi_2,\theta)$. Observe that, as $\mu_\epsilon^{[0,1)}$ is a measure on the torus, it has a corresponding Fourier-Stieltjes series, the coefficients of which will be denoted $\widehat{\mu_\epsilon^{[0,1)}}(\xi)$. The measure $\mu_\epsilon^{[0,1)}$ can be associated to a 1-periodic measure μ_ϵ^P supported on the real numbers.

Lemma 3.1. Suppose that $\mu_\epsilon^{[0,1)}$ is a measure on the torus with support contained in $\text{Tight}^{[0,1)}(\psi_1,\psi_2,\theta)$, with the property that $|\widehat{\mu_\epsilon^{[0,1)}}(\xi)| \leq C_1(1+|\xi|)^{-\alpha/2+\epsilon}$ for all $\xi \in \mathbb{Z}\backslash\{0\}$. Let $\phi \in C_c^\infty$ be any smooth function supported in $[0,1]$. Then there exists a C_2 not depending on ξ such that $\widehat{\phi\mu_\epsilon^P}(\xi) \leq C_2(1+|\xi|)^{-\alpha/2+\epsilon}$ for all $\xi \in \mathbb{R}$. □

Proof. Observe that the Fourier transform of μ_ϵ^P, viewed as a tempered distribution on $\mathbb{R}$, is given by

$$\sum_{s=-\infty}^{\infty} \widehat{\mu}^{[0,1)}(s)\delta_s,$$

where δ_s is the Dirac mass centered at s.

Therefore, we can make sense of $\widehat{\phi\mu_\epsilon^P}$ as the convolution of $\widehat{\mu_\epsilon^P}$ and $\widehat{\phi}$. This convolution is equal to

$$\int \widehat{\phi}(\xi - s)d\widehat{\mu_\epsilon^P}(s) = \sum_{s=-\infty}^{\infty} \widehat{\phi}(\xi - s)\widehat{\mu_\epsilon^{[0,1)}}(s).$$

We now apply our assumption on $\widehat{\mu_\epsilon^{[0,1)}}(s)$, as well as the Schwartz bound on $\widehat{\phi}$, to conclude

$$|\widehat{\phi\mu_\epsilon^P}(\xi)| \lesssim \sum_{s=-\infty}^{\infty} (1 + |\xi - s|)^{-100}(1 + |s|)^{-\alpha/2+\epsilon}. \tag{21}$$

We will now write the sum in (21) as $S_1 + S_2$, where

$$\begin{aligned} S_1 &= \sum_{|s-\xi|\le|\xi|/2} (1 + |\xi - s|)^{-100}(1 + |s|)^{-\alpha/2+\epsilon}. \\ S_2 &= \sum_{|s-\xi|>|\xi|/2} (1 + |\xi - s|)^{-100}(1 + |s|)^{-\alpha/2+\epsilon}. \end{aligned}$$

We will first estimate S_1. Observe that if $|s - \xi| \le |\xi|/2$, we must have $1 + |s| \ge |\xi|/2$. Therefore, we have

$$\begin{aligned} S_1 &\lesssim (1 + |\xi|/2)^{-\alpha/2+\epsilon} \sum_{|u|\le|\xi|/2} (1 + |u|)^{-100} \\ &\lesssim (1 + |\xi|)^{-\alpha/2+\epsilon}. \end{aligned}$$

This gives the desired estimate for S_1. It remains to estimate S_2. In order to estimate S_2, one can observe that the inequality $|s - \xi| \ge |\xi|/2$ implies that $|s - \xi| \ge |s|/4$. Indeed, for $|s| < 2|\xi|$, $|s - \xi| \ge |\xi|/2 > |s|/4$ and, for $|s| \ge 2|\xi|$, $|s - \xi| \ge |s|/2$. Applying this estimate gives

$$\begin{aligned} S_2 &\le \sum_{|s-\xi|\ge|\xi|/2} (1 + |\xi|/2)^{-99}(1 + |s|/4)^{-1}(1 + |s|)^{-\alpha/2+\epsilon} \\ &\lesssim (1 + |\xi|)^{-99} \sum_{|s-\xi|\ge|\xi|/2} (1 + |s|)^{-1-\alpha/2+\epsilon} \\ &\le (1 + |\xi|)^{-99} \sum_{s=-\infty}^{\infty} (1 + |s|)^{-1-\alpha/2+\epsilon} \\ &\lesssim (1 + |\xi|)^{-99}. \end{aligned}$$

Adding S_1 and S_2 gives the result. ■

We will need one more result that goes in the other direction—a result that allows us to lift compactly supported, bounded, measurable functions f on $\mathbb{R}$ to bounded, measurable functions $f^{[0,1)}$ the torus. We emphasize that the following lemma allows us to control the Fourier coefficients of $f^{[0,1)}$ by knowing $\widehat{f}(s)$ for *integer* values s.

Let $f \in L^\infty(\mathbb{R})$ be compactly supported. Define f^P by

$$f^P(x) = \sum_{j\in\mathbb{Z}} f(x + j).$$

The assumptions on f guarantee that $f^P(x)$ converges a.e. to a 1-periodic function in $L^\infty(\mathbb{R})$. This function can naturally be associated to a function $f^{[0,1)}$ on the torus.

Lemma 3.2. Let $f \in L^\infty(\mathbb{R})$ be a compactly supported function, and define $f^{[0,1)}$ as above. Then $\widehat{f^{[0,1)}}(s) = \widehat{f}(s)$ for all integers s. □

Proof. We have

$$\begin{aligned}
\widehat{f^{[0,1)}}(s) &= \int_0^1 e^{-2\pi i s x} f^{[0,1)}(x)\,dx \\
&= \int_0^1 e^{-2\pi i s x} f^{P}(x)\,dx \\
&= \int_0^1 e^{-2\pi i s x} \sum_{j\in\mathbb{Z}} f(x+j)\,dx \\
&= \sum_{j\in\mathbb{Z}} \int_j^{j+1} e^{-2\pi i s (x-j)} f(x)\,dx \\
&= \sum_{j\in\mathbb{Z}} \int_j^{j+1} e^{-2\pi i s x} f(x)\,dx \\
&= \int_{\mathbb{R}} e^{-2\pi i s x} f(x)\,dx \\
&= \widehat{f}(s).
\end{aligned}$$

■

4 A single-scale estimate

Lemma 3.1 reduces the proof of Theorem 1.7 to finding a measure μ supported on $\mathrm{Tight}(\tau_1,\tau_2,\theta)\cap[0,1)$ such that $|\widehat{\mu}(s)| \le |s|^{-\alpha/2+\epsilon}$ for integers s. This measure will be constructed as a weak-limit of products of functions, each of which is a sum of smoothed indicator functions of balls of an appropriate scale.

For now, we consider functions supported on $\mathbb{R}$. We will later lift these functions to the torus. We define a function g_M at scale M which we use to construct our measure supported in the exact order set. The function we consider is supported in the set

$$\bigcup_{\substack{M\le q<2M \\ \text{q prime}}} \bigcup_{0\le p<q} \left\{ x : \psi_1(q) - c_M\psi_2(q) \le \left| x - \frac{p-\theta}{q} \right| \le \psi_1(q) \right\} \subset \mathbb{R}, \tag{22}$$

where we take

$$c_M = M^{-\epsilon/100}. \tag{23}$$

Observe that $0 < c_M < 1$, with c_M close to 0 if M is chosen sufficiently large. For the remainder of this section, we typically suppress the dependence on M and write c for c_M, though this dependence will be recalled at the appropriate points.

For a given prime $q \in [M, 2M)$, the interval $I_{q,p} = \{x : \psi_1(q) - c\psi_2(q) \le x - \frac{p-\theta}{q} \le \psi_1(q)\}$ can be expressed as

$$I_{q,p} = x_{q,p} + c\psi_2(q)[-1/2, 1/2],$$

where $x_{q,p} = \frac{p-\theta}{q} + \psi_1(q) - (c/2)\psi_2(q)$.

Let ϕ be a smooth, nonnegative function with $\operatorname{supp}\phi \subset [-1/2,1/2]$ for which $|\widehat{\phi}(\xi)| \le C\exp\left(-|\xi|^{3/4}\right)$ for large $|\xi|$. Such a function ϕ is provided by Ingham [9], who in fact constructs a real-valued function whose square satisfies the desired properties.

We define

$$g_M(x) = \sum_{\substack{M\le q<2M \\ \text{q prime}}} \sum_{0\le p<q} \phi_{p,q}(x),$$

where

$$\phi_{p,q}(x) := (c\psi_2(q))^{-1}\phi\left((c\psi_2(q))^{-1}(x - x_{q,p})\right).$$

Observe that the Fourier transform of $\phi_{p,q}$ satisfies

$$\widehat{\phi_{p,q}}(s) = e(sx_{q,p})\widehat{\phi}(c\psi_2(q)s), \tag{24}$$

where $e(x) = e^{2\pi i x}$. We then set $f_M(x) = g_M(x)/\widehat{g_M}(0)$ so that $\widehat{f_M}(0) = 1$.

Lemma 4.1. Let g_M, f_M be defined as above. Let $\tau_2 = \lambda(\psi_2)$. Then

$$\widehat{f}_M(0) = 1, \tag{25}$$

$$\widehat{f}_M(s) = 0 \qquad \text{if } 1 \le |s| < M, \tag{26}$$

$$|\widehat{f}_M(s)| \le C_\epsilon M^{-1+\epsilon} \qquad \text{if } M \le |s| \le M^{\tau_2(1+\epsilon/2)}, \tag{27}$$

$$|\widehat{f}_M(s)| \le \exp\left(-\left|\frac{s}{M^{\tau_2}}\right|^{1/2}\right) \qquad \text{if } |s| \ge M^{\tau_2(1+\epsilon/2)}. \tag{28}$$

□

Proof. Equation (25) follows directly from our normalization of f_M.

For the proof of (26), (27), and (28), we will begin by computing $\widehat{g}_M(s)$ explicitly. From (24), we have

$$\widehat{g}_M(s) = \sum_{\substack{M \le q < 2M \\ q \text{ prime}}} \sum_{0 \le p < q} e(s x_{q,p}) \widehat{\phi}(c\psi_2(q)s). \tag{29}$$

By plugging in the value for $x_{q,p}$, we obtain

$$\begin{aligned}\widehat{g}_M(s) &= \sum_{\substack{M \le q < 2M \\ q \text{ prime}}} \sum_{0 \le p < q} e\left(s\left(\frac{p-\theta}{q} + \psi_1(q) - (c/2)\psi_2(q)\right)\right)\widehat{\phi}(c\psi_2(q)s) \\ &= \sum_{\substack{M \le q < 2M \\ q \text{ prime}}} e\left(s\left(\frac{-\theta}{q} + \psi_1(q) - (c/2)\psi_2(q)\right)\right)\widehat{\phi}(c\psi_2(q)s) \sum_{0 \le p < q} e\left(\frac{ps}{q}\right).\end{aligned}$$

The inner sum $\sum_{0 \le p < q} e\left(\frac{ps}{q}\right)$ is a geometric series that evaluates to 0 unless $q|s$, in which case it evaluates to q. Therefore, we have

$$\widehat{g}_M(s) = \sum_{\substack{M \le q < 2M \\ q \text{ prime} \\ q|s}} e\left(s\left(\frac{-\theta}{q} + \psi_1(q) - (c/2)\psi_2(q)\right)\right)\widehat{\phi}(c\psi_2(q)s) \cdot q.$$

For $0 < |s| < M$ this sum is empty, establishing that $\widehat{g}_M(s) = 0$ and giving (26). If $s = 0$, the sum in q consists of all $M \le q < 2M$, giving

$$\widehat{g}_M(0) = \widehat{\phi}(0) \sum_{\substack{M \le q < 2M \\ q \text{ prime}}} q \sim \frac{M^2}{\log M}, \tag{30}$$

by the prime number theorem.

For other values of s, we use the triangle inequality to give the estimate

$$|\widehat{g}_M(s)| \lesssim M \sum_{\substack{M \le q < 2M \\ q \text{ prime} \\ q|s}} |\widehat{\phi}(c\psi_2(q)s)|. \tag{31}$$

Combining (30) and (31) gives the estimate

$$|\widehat{f}_M(s)| \lesssim M^{-1} \log M \sum_{\substack{M \le q < 2M \\ q \text{ prime} \\ q|s}} |\widehat{\phi}(c\psi_2(q)s)|. \tag{32}$$

We now consider the regime where $M \le |s| < M^{\tau_2(1+\epsilon/2)}$. Observe that the number of terms q in the sum on the right hand side of (32) is bounded above by $\frac{\log|s|}{\log M}$. Using the bound $|\widehat{\phi}(c\psi_2(q)s)| \lesssim 1$, we find

$$|\widehat{f}_M(s)| \lesssim M^{-1} \log|s| \lesssim M^{-1} \log M \le C_\epsilon M^{-1+\epsilon},$$

establishing (27).

For $|s| > M^{\tau_2(1+\epsilon/2)}$, we take advantage of our choice of ϕ. First, observe that, because ψ_2 is decreasing, we can bound $c\psi_2(q)|s|$ from below by $c\psi_2(2M)|s|$. By (23), this is $M^{-\epsilon/100}\psi_2(2M)|s|$. Provided that M is sufficiently large depending on ϵ, we can use (4) to conclude that

$$|c\psi_2(q)s| \geq 2^{-\tau_2-\epsilon/100}M^{-\tau_2-\epsilon/50}|s|.$$

Therefore, applying our assumption on ϕ, we have that

$$\left|\widehat{\phi}(c\psi_2(q)s)\right| \leq C\exp\left(-2^{-3/4(\tau_2+\epsilon/100)}M^{-3/4(\tau_2+\epsilon/50)}|s|^{3/4}\right). \tag{33}$$

Now, we estimate the series (32) using the bound (33). There are no more than M terms in this sum, so, for an appropriate constant C,

$$|\widehat{f}_M(s)| \leq C\log M\exp\left(-2^{-3/4(\tau_2+\epsilon/100)}M^{-3/4(\tau_2+\epsilon/50)}|s|^{3/4}\right).$$

Since we are in the regime $|s| > M^{\tau_2(1+\epsilon/2)}$,

$$|\widehat{f}_M(s)| \leq C\log M\exp\left(-\left(|s|^{1/2}M^{-\tau_2/2}\right)\left(2^{-3/4(\tau_2+\epsilon/100)}M^{\epsilon\tau_2/8-3\epsilon/200}\right)\right).$$

Because the exponent $\epsilon\tau_2/8 - 3\epsilon/200$ is positive, it follows that for M sufficiently large, we have the bound

$$|\widehat{f}_M(s)| \leq \exp\left(-|s|^{1/2}M^{-\tau_2/2}\right),$$

establishing the desired bound (28). ■

5 A Convolution Stability Lemma

In this section, we establish a convolution stability lemma. This lemma will later be used in Section 6, in combination with Lemma 4.1, applied at different scales as part of an induction argument, to complete the construction of the measure with the Fourier decay required to complete the proof of of Theorem 1.7.

To this end, we will consider a sequence $\{M_j\}_{j=1}^{\infty}$ of positive numbers whose growth rate is dictated by Lemma 2.1.

The convolution stability lemma will provide an estimate for $F * G$, where functions F and G satisfy certain bounds following Lemma 4.1. In practice, the function G will be $\widehat{f}_{M_1} * \widehat{f}_{M_2} * \cdots * \widehat{f}_{M_j}$ for some appropriate j, and F will be taken to be $\widehat{f}_{M_{j+1}}$.

In this section, we will assume that $\tau_1 = \lambda(\psi_1)$ and $\tau_2 = \lambda(\psi_2)$ satisfy the condition $\beta > \tau_2$. Recall we defined $\beta_\epsilon = \gamma^{-2}(\tau_1 - 1)^2 - \epsilon = \beta - \epsilon$ in the equation (13). We consider only those ϵ small enough so that $\beta_\epsilon > \tau_2$.

For the next lemma, given a small $\epsilon > 0$, it will be convenient to define an auxiliary exponent $\delta(\epsilon)$ given by the equation

$$\delta(\epsilon) := \frac{\beta_\epsilon(1-\epsilon) - \tau_2(1+\epsilon)^2}{\tau_2(\beta_\epsilon - 1)(1+\epsilon)}. \tag{34}$$

Observe that, since $\beta_\epsilon \to \beta$ as $\epsilon \to 0$ and $\beta > \tau_2$, we have

$$\lim_{\epsilon\to 0}\delta(\epsilon) = \frac{\beta - \tau_2}{\tau_2(\beta - 1)} > 0.$$

Thus there exists an $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, then $\delta(\epsilon) - \epsilon > 0$. Note that for such ϵ, we have that $\delta(\epsilon) > 0$ and thus

$$\beta_\epsilon > \frac{(1+\epsilon)^2}{1-\epsilon}\tau_2 > (1+\epsilon)\tau_2. \tag{35}$$

Lemma 5.1 (Convolution Stability Lemma)**.** Given suitably large $M_1 \in \mathbb{N}$, we define the sequence $\{M_j\}_{j=1}^{\infty}$ by

$$M_j = M_1^{\beta_\epsilon^{j-1}},$$

where $\beta_\epsilon > \tau_2 \geq 2$. For $\epsilon > 0$, we define the quantity $\delta = \delta(\epsilon)$ as in (34). Suppose ϵ is small enough to ensure that $\delta - \epsilon > 0$. Let $F, G : \mathbb{Z} \to \mathbb{C}$ be functions satisfying the following estimates:

$$F(0) = 1, \tag{36}$$
$$F(s) = 0 \qquad \text{if } 1 \leq |s| < M_{j+1}, \tag{37}$$
$$|F(s)| \leq C_\epsilon M_{j+1}^{-1+\epsilon} \qquad \text{if } M_{j+1} \leq |s| \leq M_{j+1}^{\tau_2(1+\epsilon/2)}, \tag{38}$$
$$|F(s)| \leq \exp\left(-\left|\frac{s}{M_{j+1}^{\tau_2}}\right|^{1/2}\right) \qquad \text{if } |s| \geq M_{j+1}^{\tau_2(1+\epsilon/2)}. \tag{39}$$

and

$$G(0) \leq 2, \tag{40}$$
$$|G(s)| \leq 2|s|^{-\delta+\epsilon} \qquad \text{if } |s| \leq M_j^{\tau_2(1+\epsilon)}, \tag{41}$$
$$|G(s)| \leq \exp\left(-\frac{1}{2}\left|\frac{s}{M_j^{\tau_2}}\right|^{1/2}\right) \qquad \text{if } |s| \geq M_j^{\tau_2(1+\epsilon)}. \tag{42}$$

Then, provided that M_1 is sufficiently large depending on ϵ, we have the following three conclusions:

(a)

$$|F * G(s) - G(s)| \leq M_{j+1}^{-\delta} \quad \text{if } |s| \leq M_j^{\tau_2(1+\epsilon)}$$

(b)

$$|F * G(s)| \leq |s|^{-\delta+\epsilon} \quad \text{if } M_j^{\tau_2(1+\epsilon)} < |s| \leq M_{j+1}^{\tau_2(1+\epsilon)}$$

(c)

$$|F * G(s)| \leq \exp\left(-\frac{1}{2}\left|\frac{s}{M_{j+1}^{\tau_2}}\right|^{1/2}\right) \quad \text{if } |s| > M_{j+1}^{\tau_2(1+\epsilon)}.$$

□

For reference, in Figure 1 we give a sketch of F and G corresponding to some index j. The usage of $\approx$ in this figure indicates an ϵ-loss in the exponent on M_j or M_{j+1}.

Note that in region A, we have that $|G(s)| \lesssim |s|^{-\delta}$. In region B, $G(s)$ decays rapidly. In regions A and B (not including 0), F vanishes. In region C, F decays rapidly.

Remark 5.2. Lemma 5.1 is called a **convolution stability lemma** because of the bound (a), which shows that for small values of s, the convolution $F * G$ will be very close to G.

The bound (b) will dictate the Fourier decay of the infinite product measure supported on our set. Note that the bound $|s|^{-\delta+\epsilon}$ is significantly worse than the bound $M_{j+1}^{-1+\epsilon}$ available for F in this region—this is the reason for the loss in Fourier dimension compared to the set of well-approximable numbers from Kaufman's argument.

The bound (c) will allow for the convolution stability lemma to be applied inductively. Although this bound gets slightly worse at each stage of the induction, it will always be good enough to match the conditions required for G at the next stage of the induction.

□

Proof. We will first prove (a). To this end, we assume $|s| \leq M_j^{\tau_2(1+\epsilon)}$. We write

$$F * G(s) = \sum_{t\in\mathbb{Z}} F(t)G(s-t).$$

The main contribution to this sum will come from the $t = 0$ term, which is precisely $G(s)$. Additionally, there is no contribution for $1 \leq |t| < M_{j+1}$ because $F(t) = 0$ there. Thus, we see that

$$|F * G(s) - G(s)| \leq \left|\sum_{M_{j+1}\leq|t|\leq M_{j+1}^{\tau_2(1+\epsilon/2)}} F(t)G(s-t)\right| + \left|\sum_{|t|>M_{j+1}^{\tau_2(1+\epsilon/2)}} F(t)G(s-t)\right|. \tag{43}$$

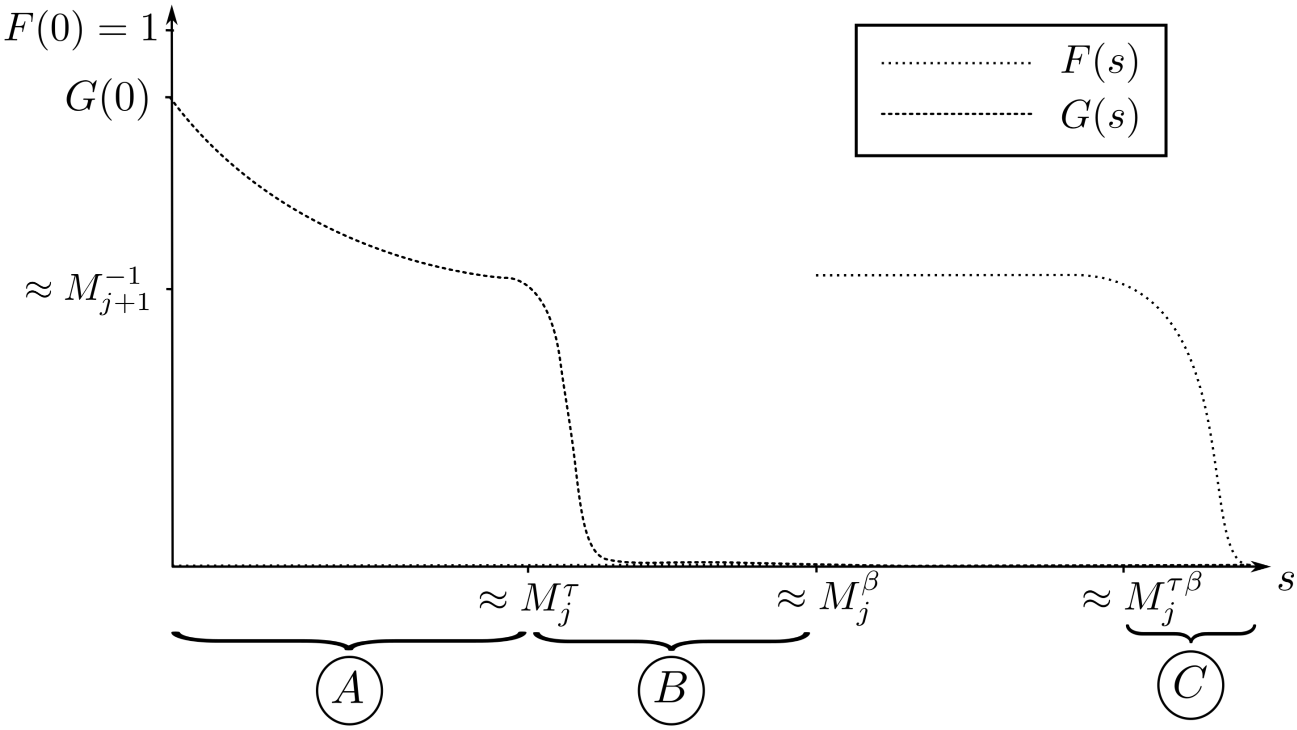

Fig. 1. Sketch of F and G

We now estimate the size of the first term on the right hand side of (43) and consider the corresponding $M_{j+1} \leq |t| \leq M_{j+1}^{\tau_2(1+\epsilon/2)}$. We have $|F(t)| \leq C_\epsilon M_{j+1}^{-1+\epsilon}$. Furthermore, because $|s| \leq M_j^{\tau_2(1+\epsilon)}$, it follows from (35) that we have $|s-t| \geq \frac{M_{j+1}}{2}$ provided M_1 is chosen large enough. Thus, we see

$$|G(s-t)| \leq \exp\left(-\frac{1}{2}\left(\frac{M_{j+1}}{2M_j^{\tau_2}}\right)^{1/2}\right)$$
$$= \exp\left(-\frac{1}{2}\frac{M_j^{\frac{\beta_\epsilon-\tau_2}{2}}}{\sqrt{2}}\right).$$

Recall that $\beta_\epsilon - \tau_2 > 0$. Combining the bounds on F and G and counting the number of terms in the sum, the first sum of (43) is bounded by

$$C_\epsilon \exp\left(-\frac{1}{2}\frac{M_j^{\frac{\beta_\epsilon-\tau_2}{2}}}{\sqrt{2}}\right) M_{j+1}^{-1+\epsilon+\tau_2(1+\epsilon/2)} = C_\epsilon \exp\left(-\frac{1}{2}\frac{M_{j+1}^{\frac{\beta_\epsilon-\tau_2}{2\beta_\epsilon}}}{\sqrt{2}}\right) M_{j+1}^{-1+\epsilon+\tau_2(1+\epsilon/2)} \leq \frac{1}{2}M_{j+1}^{-\delta}, \tag{44}$$

provided M_1 is chosen large enough.

The final step in proving (a) is to bound the second term on the right hand side of (43). For $|t| > M_{j+1}^{\tau_2(1+\epsilon/2)}$, we still have $|s-t| \geq M_j^{\tau_2(1+\epsilon)}$, so we can apply the tail estimate for F, $|F(t)| \leq \exp\left(-\left|\left(t/M_{j+1}^{\tau_2}\right)\right|^{1/2}\right)$. We simply bound $|G(s-t)| \leq 2$ to obtain the estimate

$$\left|\sum_{|t|>M_{j+1}^{\tau_2(1+\epsilon/2)}} F(t)G(s-t)\right| \leq 2\sum_{|t|\geq M_{j+1}^{\tau_2(1+\epsilon/2)}} \exp\left(-\left|\frac{t}{M_{j+1}^{\tau_2}}\right|^{1/2}\right) \leq \frac{1}{2}M_{j+1}^{-\delta}, \tag{45}$$

by comparing to the corresponding integral, if M_1 is sufficiently large. Summing the bounds (44) and (45) completes the proof of (a).

We now prove (b). We here consider those s with $M_j^{\tau_2(1+\epsilon)} < |s| \leq M_{j+1}^{\tau_2(1+\epsilon)}$. We bound $F * G$ by writing

$$\left|\sum_{t\in\mathbb{Z}} F(s-t)G(t)\right|$$

$$\leq \left| \sum_{|t| \leq M_j^{\tau_2(1+\epsilon)}} F(s-t)G(t) \right| + \left| \sum_{|t| \geq M_j^{\tau_2(1+\epsilon)}} F(s-t)G(t) \right| \tag{46}$$

The main contribution to this bound will be the first term. For such values of $|t| \leq M_j^{\tau_2(1+\epsilon)}$, we have the estimate $|G(t)| \leq 2|t|^{-\delta+\epsilon}$ for $t \neq 0$ and $|G(t)| \leq 2$ at $t = 0$.

We continue our analysis of the first term of (46), where $|t| \leq M_j^{\tau_2(1+\epsilon)}$. Because $s \neq t$, one of the bounds (37), (38), (39) will apply; in any case, we have that $|F(s-t)| \leq C_\epsilon M_{j+1}^{-1+\epsilon}$. As a result, for appropriate constants C and K,

$$\left| \sum_{|t| \leq M_j^{\tau_2(1+\epsilon)}} F(s-t)G(t) \right| \leq M_{j+1}^{-1+\epsilon} \left(2C_\epsilon + 4C_\epsilon \sum_{t=1}^{M_j^{\tau_2(1+\epsilon)}} |t|^{-\delta+\epsilon} \right).$$

$$\leq K M_{j+1}^{-1+\epsilon+(-\delta+\epsilon+1)\frac{\tau_2}{\beta_\epsilon}(1+\epsilon)} \leq C|s|^{\frac{1}{\tau_2(1+\epsilon)}(-1+\epsilon+(-\delta+\epsilon+1)\frac{\tau_2}{\beta_\epsilon}(1+\epsilon))} = C|s|^{-\delta} \leq \frac{1}{2}|s|^{-\delta+\epsilon},$$

since our choice of δ at (34) guarantees that the penultimate exponent on $|s|$ above is $-\delta$ and we can absorb the constant C in to the $|s|^\epsilon$ term, which is possible if M_1 is sufficiently large.

We now consider the second term in the bound (46). The relevant t are those with $|t| \geq M_j^{\tau_2(1+\epsilon)}$, including the case in which $t = s$. For such t, we have a bound of 1 on $|F(s-t)|$ and a bound of $\exp\left(-\frac{1}{2}\left|\frac{t}{M_j^{\tau_2}}\right|^{1/2}\right)$ for $|G(t)|$. By comparison with a corresponding integral and ensuring that M_1 is sufficiently large, we have the estimate

$$\sum_{|t| \geq M_j^{\tau_2(1+\epsilon)}} F(s-t)G(t) \leq C \exp\left(-\frac{1}{4} M_j^{\tau_2 \epsilon/2}\right) \leq \frac{1}{2}|s|^{-\delta+\epsilon}$$

The established bounds on each of the terms in (46) combine to show $|F * G(s)| \leq |s|^{-\delta+\epsilon}$, completing the proof of (b).

It remains to prove (c). Let s be such that $|s| > M_{j+1}^{\tau_2(1+\epsilon)}$. Writing the convolution, we use the preliminary bound for $F * G(s)$ given by

$$\left| \sum_{t \in \mathbb{Z}} F(s-t)G(t) \right|$$

$$\leq \left| \sum_{\substack{|t| \leq 2|s| \\ |s-t| \geq \frac{|s|}{2}}} F(s-t)G(t) \right| + \left| \sum_{\substack{|t| \leq 2|s| \\ |s-t| < \frac{|s|}{2}}} F(s-t)G(t) \right| + \left| \sum_{|t| > 2|s|} F(s-t)G(t) \right|. \tag{47}$$

The thrust of the proof is that, because $|s|$ is so large, we are always in a situation for which the tail bounds on either F or G will apply.

We consider the first term of (47), where $|t| \leq 2|s|$ and $|s-t| \geq \frac{|s|}{2}$. For such t, including $t = 0$, we have the bound $|G(t)| \leq 2$. On the other hand, for such t, we certainly have $|s-t| \geq \frac{1}{2} M_{j+1}^{\tau_2(1+\epsilon)} \geq M_{j+1}^{\tau_2(1+\epsilon/2)}$, provided M_1 is taken sufficiently large. Thus, we have

$$|F(s-t)| \leq \exp\left(-\left|\frac{s}{2M_{j+1}^{\tau_2}}\right|^{1/2}\right).$$

Summing over $|t| \leq 2|s|$, we see

$$\left| \sum_{\substack{|t| \leq 2|s| \\ |s-t| \geq \frac{|s|}{2}}} F(s-t)G(t) \right| \leq 4|s| \exp\left(-\left|\frac{s}{2M_{j+1}^{\tau_2}}\right|^{1/2}\right) \leq \frac{1}{3} \exp\left(-\frac{1}{2}\left|\frac{s}{M_{j+1}^{\tau_2}}\right|^{1/2}\right),$$

provided M_1 is large enough depending on ϵ.

Next, we will bound the second term of (47). The relevant t are such that $|s-t| < \frac{|s|}{2}$ (including the $t = s$ term). Note that for such t, we certainly have $|t| \geq \frac{|s|}{2} \geq M_j^{\tau_2(1+\epsilon)}$, if M_1 is large enough. We also have

$|F(s-t)| \leq 1$, and $|G(t)| \leq \exp\left(-\frac{1}{2}\left|\frac{s}{2M_j^{\tau_2}}\right|^{1/2}\right)$. Observe that the total number of values of t summed is at most $|s|$. If M_1 is sufficiently large, keeping in mind that $|s| \geq M_{j+1}^{\tau_2(1+\epsilon)}$, we observe that $|s|$ is much less than $\frac{1}{3}\exp(\frac{1}{2}|s|^{1/2}((2M_j)^{-\tau_2/2} - M_{j+1}^{-\tau_2/2}))$. Thus,

$$\left|\sum_{\substack{|t|\leq 2|s| \\ |s-t|<\frac{|s|}{2}}} F(s-t)G(t)\right| \leq |s|\exp\left(-\frac{1}{2}\left|\frac{s}{2M_j^{\tau_2}}\right|^{1/2}\right) \leq \frac{1}{3}\exp\left(-\frac{1}{2}\left|\frac{s}{M_{j+1}^{\tau_2}}\right|^{1/2}\right).$$

It remains to bound the third term in (47). Here, $|t| \geq 2|s|$, and we can see that $|s-t| \geq \frac{|t|}{2}$. We will use an estimate of 1 for $|G(t)|$ and a bound of $\exp\left(-\left|\frac{t}{2M_{j+1}^{\tau_2}}\right|^{1/2}\right)$ for $|F(s-t)|$, as we can apply the tail bound on F. Hence,

$$\left|\sum_{|t|>2|s|} F(s-t)G(t)\right| \leq \sum_{|t|\geq 2|s|} \exp\left(-\left|\frac{t}{2M_{j+1}^{\tau_2}}\right|^{1/2}\right) \leq \frac{1}{3}\exp\left(-\frac{1}{2}\left|\frac{s}{M_{j+1}^{\tau_2}}\right|^{1/2}\right),$$

by comparison with the corresponding integral.

We arrive at the desired bound, (c), by summing the three terms. ■

6 Construction of the Measure

In this section, we complete the proof of Theorem 1.7. Recall from (34) that

$$\delta = \frac{\beta_\epsilon(1-\epsilon) - \tau_2(1+\epsilon)^2}{\tau_2(\beta_\epsilon - 1)(1+\epsilon)}.$$

The stated lower bound on the Fourier dimension, $\alpha = \frac{2(\beta-\tau_2)}{\tau_2(\beta-1)}$, was given in (10). To prove Theorem 1.7, it suffices to construct a measure μ_ϵ on the torus $[0,1)$, supported in $\mathrm{Tight}^{[0,1)}(\psi_1,\psi_2,\theta)$ and with decay $|\widehat{\mu}_\epsilon(s)| \lesssim |s|^{-\delta+\epsilon}$. Indeed, once we have constructed such a measure μ_ϵ, we can apply Lemma 3.1, which gives the desired measure on $\mathbb{R}$ supported in $\mathrm{Tight}(\psi_1,\psi_2,\theta)$ and with Fourier decay bounded in magnitude by $|s|^{-\delta+\epsilon}$, which gives the result.

Let $\epsilon > 0$, and let M_1 be a number so large that Lemma 5.1 applies (with $M_{j+1} = M_j^{\beta_\epsilon}$ for all $j > 1$), and sufficiently large that $\sum_{j=1}^\infty M_j^{-\delta} = \sum_{j=1}^\infty M_1^{-\beta_\epsilon^{j-1}\delta} < \frac{1}{100}$. For each j, define f_{M_j} as in Lemma 4.1, and $f_{M_j}^{[0,1)}$ as in Lemma 3.2. We define the function $\mu_\epsilon^{(k)} = \prod_{j=1}^k f_{M_j}^{[0,1)}$. In the proof, we conflate the function $\mu_\epsilon^{(k)}$ with the absolutely continuous measure whose Radon-Nikodym derivative is $\mu_\epsilon^{(k)}$. We claim that the measures $\mu_\epsilon^{(k)}$ have a subsequence with a weak limit μ_ϵ with the desired properties.

The proof of this will require us to estimate $\widehat{\mu}_\epsilon^{(k)}(s)$ for integer values s. We will obtain the following estimate by applying Lemma 5.1 inductively.

Lemma 6.1. Let $M_0 = 0$ for convenience. We have the following estimates on $\widehat{\mu}_\epsilon^{(k)}(s)$ for any integers $k \geq 1$ and $s \in \mathbb{Z}$:

$$1 - \sum_{j=1}^k M_j^{-\delta} < \widehat{\mu}_\epsilon^{(k)}(0) < 1 + \sum_{j=1}^k M_j^{-\delta}, \tag{48}$$

$$|\widehat{\mu}_\epsilon^{(k)}(s)| < |s|^{-\delta+\epsilon} + \sum_{j=J}^k M_j^{-\delta} < 2|s|^{-\delta+\epsilon} \quad \text{if } M_{J-1}^{\tau_2(1+\epsilon)} < |s| \leq M_J^{\tau_2(1+\epsilon)} \text{ for } 1 \leq J \leq k, \tag{49}$$

$$|\widehat{\mu}_\epsilon^{(k)}(s)| \leq \exp\left(-\frac{1}{2}\left|\frac{s}{M_k^{\tau_2}}\right|^{1/2}\right) \quad \text{if } |s| > M_k^{\tau_2(1+\epsilon)}. \tag{50}$$

□

Proof. We prove this lemma by induction. The base case of this lemma is implied by Lemma 4.1 applied to $f_{M_1}^{[0,1)}$. So we need only show the inductive step.

Suppose, for some $k > 1$, we have that $\mu_\epsilon^{(k-1)}$ satisfies the estimates in Lemma 6.1. We must show that $\mu_\epsilon^{(k)}$ also satisfies these estimates. Our tool for this is Lemma 5.1. Observe that, by definition, we have that

$$\mu_\epsilon^{(k)} = \mu_\epsilon^{(k-1)} f_{M_k}^{[0,1)}.$$

Therefore, by the convolution rule for the Fourier transform, we have

$$\widehat{\mu}_\epsilon^{(k)} = \widehat{\mu}_\epsilon^{(k-1)} * \widehat{f_{M_k}^{[0,1)}}.$$

The estimates (48), (49), and (50) imply that $\widehat{\mu}_\epsilon^{(k-1)}$ is able to serve as the function G in Lemma 5.1. Note that (48) for $\mu_\epsilon^{(k)}$ follows immediately by combining (a) of Lemma 5.1 and (48) for $\mu_\epsilon^{(k-1)}$. Similarly, for $1 \leq J \leq k-1$, we have that (49) holds for $\mu_\epsilon^{(k)}$ by combining (a) of Lemma 5.1 and (49) for $\mu_\epsilon^{(k-1)}$. The $J = k$ case of the estimate (49) for $\mu_\epsilon^{(k)}$ is an immediate consequence of (b) of Lemma 5.1. Finally, the estimate (50) is given by (c) of Lemma 5.1. ■

We are now in a position to define our measure μ_ϵ. It is clear from the Banach-Alaoglu theorem that some subsequence of the $\mu_\epsilon^{(k)}$ converges weakly to some measure μ_ϵ. The estimate (48) shows that the weak-limit of this subsequence is a nonzero finite measure, and it is clear from the fact that $\mu_\epsilon^{(k)} \geq 0$ for all k that $\mu_\epsilon \geq 0$. The estimate (49) implies that $|\widehat{\mu}_\epsilon(s)| \leq 2|s|^{-\delta+\epsilon}$ for all s. Therefore, in order to establish the Fourier dimension bound, the only statement it remains to prove about μ_ϵ is that its support is contained in $\mathrm{Tight}^{[0,1)}(\psi_1, \psi_2, \theta)$.

Lemma 6.2. The measure μ_ϵ is supported on $\mathrm{Tight}^{[0,1)}(\psi_1, \psi_2, \theta)$. □

Proof. The measures $\mu_\epsilon^{(k)}$ have nested, decreasing support, so we must have

$$\operatorname{supp} \mu_\epsilon \subset \bigcap_k \operatorname{supp} \mu_\epsilon^{(k)} = \bigcap_k \operatorname{supp} f_{M_k}^{[0,1)}.$$

So it is sufficient to prove that

$$\bigcap_k \operatorname{supp} f_{M_k}^{[0,1)} \subset \mathrm{Tight}^{[0,1)}(\psi_1, \psi_2, \theta).$$

In particular, according with Definition 1.5, it suffices for us to show that, for $x \in \bigcap_k \operatorname{supp} f_{M_k}^{[0,1)}$,

$$\left| x - \frac{p-\theta}{q} \right| \leq \psi_1(q) \quad \text{for infinitely many pairs } (p,q) \text{ of relatively prime integers,}$$
$$\left| x - \frac{p-\theta}{q} \right| \leq \psi_1(q) - c\psi_2(q) \quad \text{for only finitely many pairs } (p,q) \text{ of relatively prime integers and any } c > 0.$$

We recall that, by construction, $\operatorname{supp} f_{M_k}$ is contained in the set given at (22):

$$\bigcup_{\substack{M_k \leq q < 2M_k \\ q \text{ prime}}} \bigcup_{0 \leq p < q} \left\{ x : \psi_1(q) - c_{M_k}\psi_2(q) \leq \left| x - \frac{p-\theta}{q} \right| \leq \psi_1(q) \right\} \subset \mathbb{R},$$

with $c_{M_k} = M_k^{-\epsilon/100}$. Let $c < 1$. There exists $K = K(\epsilon, c)$ such that $M_K^{-\epsilon/100} < c$.

Suppose $x \in \bigcap_k \operatorname{supp} f_{M_k}^{[0,1)}$. Let x^* in $\mathbb{R}$ be the element of the interval $[0,1)$ that is congruent modulo 1 to x. Then for any k, there exists an integer z_k such that $x^* + z_k \in \operatorname{supp} f_{M_k}$. This means that, for any $k > K$, there exists a pair (p_k, q_k) with $0 \leq p_k < q_k$ and $M_k \leq q_k < 2M_k$ and q_k prime such that $\psi_1(q_k) - c\psi_2(q_k) < \left| x^* - \frac{p_k - z_k q_k - \theta}{q_k} \right| < \psi_1(q_k)$. If we set $p_k' = p_k - z_k q_k$, this gives, for every $k > K$, a pair (p_k', q_k), with $M_k \leq q_k < 2M_k$, such that $\psi_1(q_k) - c\psi_2(q_k) < \left| x^* - \frac{p_k' - \theta}{q_k} \right| < \psi_1(q_k)$ and, in particular, there are infinitely many integer pairs (p,q) for which $\left| x^* - \frac{p-\theta}{q} \right| < \psi_1(q)$. It remains to show that there are only finitely many integer pairs (p,q) for which $\left| x^* - \frac{p-\theta}{q} \right| \leq \psi_1(q) - c\psi_2(q)$. By our choice of q_k, since $q_{k+1} \lesssim q_k^{\beta_\epsilon}$, it follows that there exists $K'(\epsilon)$ such that $q_{k+1} \leq q_k^{\beta_{\epsilon/2}}$ for every $k > K'(\epsilon)$. Therefore, Lemma 2.1 applied with $\beta_{\epsilon/2}$ in place of β_ϵ shows that, provided $k > K''$ for an appropriate value $K''(\epsilon, c)$, there can be no integer pair (p, q) such that $q_k < q < q_{k+1}$ and $\left|x^* - \frac{p-\theta}{q}\right| \leq \psi_1(q) - c\psi_2(q)$. Because this works for all $k > K''' := \max(K, K', K'')$, this shows that $\left|x^* - \frac{p-\theta}{q}\right| > \psi_1(q) - c\psi_2(q)$ for all pairs (p, q) with $q > q_{K'''}$, so there are only finitely many pairs (p, q) with $\left|x^* - \frac{p-\theta}{q}\right| \leq \psi_1(q) - c\psi_2(q)$, establishing the result. ■

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