

Shortened Universal Cycles for Permutations

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Abstract

Kitaev, Potapov, and Vajnovszki [On shortening u-cycles and u-words for permutations, Discrete Appl. Math, 2019] described how to shorten universal words for permutations, to length $n! + n - i(n-1)$ for any $i \in [(n-2)!]$, by introducing incomparable elements. They conjectured that it is also possible to use incomparable elements to shorten universal cycles for permutations to length $n! - i(n-1)$ for any $i \in [(n-2)!]$. In this note we prove their conjecture. The proof is constructive, and, on the way, we also show a new method for constructing universal cycles for permutations.

1 Introduction

A *universal cycle* for a family $\mathcal{F}$ of combinatorial objects is a cyclic sequence whose consecutive substrings of a given length n represent each object of $\mathcal{F}$ exactly once. The canonical type of universal cycle is a De Bruijn sequence, which is a universal cycle for the words of length n over an alphabet $\mathcal{A}$. De Bruijn sequences have been widely studied [7]. Chung, Diaconis, and Graham [5] introduced the notion of universal cycles for other combinatorial objects such as permutations, sets, and set partitions. A *universal word* for $\mathcal{F}$ is the non-cyclic analogue of a universal cycle. A universal cycle has length $|\mathcal{F}|$, while a universal word has length $|\mathcal{F}| + (n-1)$.

In this paper, we are focusing on permutations. We use $[n]$ to denote $\{1, 2, \dots, n\}$ and S_n to denote all permutations of $[n]$. A *universal word* for S_n is a word w over $\mathbb{N}$ such that each permutation in S_n is order-isomorphic to exactly one consecutive substring of length n . Notice that entries in w are not restricted to $[n]$. For example, 14524314 is a universal word for S_3 . The permutations in S_3 represented by w from left to right are 123, 231, 312, 132, 321, and 213. See Figure 1 for a way to plot the word that depicts the order of its letters. We use similar figures throughout the paper to indicate relative order of letters. A *universal cycle* for S_n is a cyclic universal word. For example, 145243 is a universal cycle for S_3 . Note that a universal word has length $n! + (n-1)$, and a universal cycle has length $n!$. Hurlbert [15] showed that universal cycles for S_n exist for all n . Chung, Diaconis, and Graham [5] conjectured that it is sufficient to use $n+1$ distinct numbers in a universal cycle for S_n , which would be best possible for $n \geq 3$. This conjecture was proved constructively by Johnson [20]. In addition, constructions of universal cycles for permutations, and variations thereof, have been studied in [1, 8, 9, 12–14, 16, 18, 21, 24]. Universal cycles for permutations have applications in various areas such as molecular biology [6], computer vision [19], robotics [22], and psychology [23].

Universal cycles, for permutations and more generally for $\mathcal{F}$, are useful because they represent the elements of $\mathcal{F}$ compactly. In recent years there has been interest in shortening universal cycles to compress

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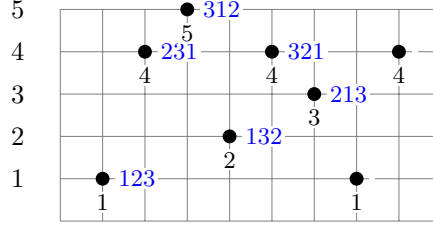


Figure 1: A universal word 14524314 for S_3 depicted in a grid from left to right. A blue label to the right of a node indicates a permutation of S_3 starting at the node.

information even further. (These efforts are not to be confused with the study of shorthand universal cycles for permutations, which are shorthand in the sense of using only n distinct symbols but have the same length, $n!$, as universal cycles for S_n ; see [8, 12, 21]. Here shortening means reducing the length of the cycle.) De Bruijn sequences have been shortened, to *universal partial cycles* and *universal partial words*, using a wildcard symbol $\diamond$ that covers any letter of the alphabet, so that a window of length n may cover more than one word of length n ; see [4, 10, 11]. Graph universal cycles, introduced in [2], and graph universal cycles for permutations, introduced in [3], have been shortened in [17].

Kitaev, Potapov, and Vajnovszki [18] shortened universal words for permutations in two different ways: with wildcard symbols and with incomparable elements. Similarly to universal partial words and universal partial cycles, they considered using the wildcard symbol $\diamond$, which yielded many nonexistence results, but they created shortened universal words for permutations using a wildcard symbol $\diamond_{\{a,b\}}$ that covers either of the two elements a and b .

Most relevantly to our work, in [18], Kitaev, Potapov, and Vajnovszki used incomparable elements at distance $n - 1$ to shorten universal words for permutations of $[n]$ to lengths $n! + n - 1 - i(n - 1)$ for each $i \in [(n - 2)!]$. A word w of length n with incomparable elements *covers* all permutations of length n that are linear extensions of the order given by w . In the case of incomparable elements at distance $n - 1$, w covers two permutations. For example, the word 2132 covers the two permutations 2143 and 3142. We also say that a longer word v containing w as a consecutive substring *covers* the permutations that w covers. For example, 4321324 contains 2132 as a consecutive substring, so 4321324 covers 2143 and 3142; the other 4-permutations that 4321324 covers are 4321, 3214, 4213, and 1324.

Theorem 1 (Kitaev, Potapov, and Vajnovszki [18]). *Using incomparable elements at distance $n - 1$, one can obtain shortened universal words for S_n of lengths $n! + n - 1 - i(n - 1)$ for each $0 \leq i \leq (n - 2)!$.*

They conjecture [18, Conjecture 8] that their result may be strengthened by obtaining shortened universal cycles instead of shortened universal words. Here we prove their conjecture.

Theorem 2. *For $n \geq 3$ and each $0 \leq i \leq (n - 2)!$, using incomparable elements at distance $n - 1$, one can obtain a shortened universal cycle for S_n of length $n! - i(n - 1)$.*

Our proof is constructive and does not attempt to control the number of distinct symbols used. Examples of shortened universal cycles for S_4 arising from the construction include

$$(1, 2, 3, 12, 4, 10, 9, 8, 11, 7, 8, 6, 9, 10, 6, 5, 9, 7, 10, 8, 3) \quad \text{and} \quad (1, 2, 3, 12, 6, 8, 7, 6, 8, 5, 9, 8, 10, 11, 8, 10, 6, 3).$$

See the Appendix for more detailed illustrations. The running time is $O((n!)^2)$, which comes from finding the Euler tour in a graph with $n!$ edges and $(n - 1)!$ vertices in quadratic time, and finding a cyclic word representing the tour, which involves relabeling at most $n!$ symbols at most $n!$ times. The memory required to store the output is $O(n!(\log(n!)))$.

Getting from Theorem 1 to Theorem 2 requires several background definitions which we provide forthwith. For a word w over comparable letters, we denote by $\text{red}(w)$ the word obtained from w by replacing each copy

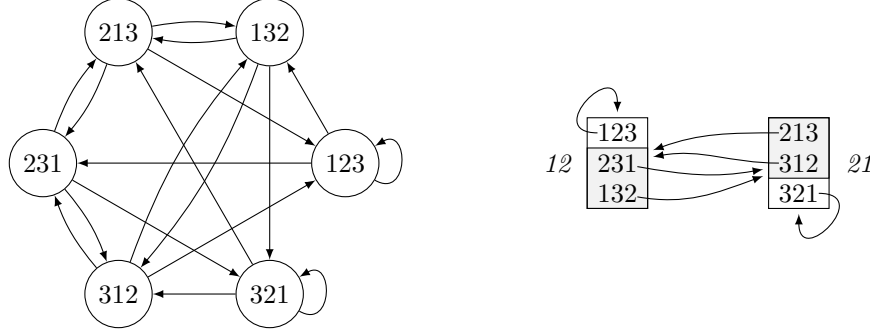


Figure 2: The graph of overlapping 3-permutations on the left and the cluster graph for 3-permutations on the right.

of the i th smallest element of w by i . For example, $\text{red}(37361) = 24231$. We say that w *reduces to* $\text{red}(w)$, and words w and v satisfying $\text{red}(w) = \text{red}(v)$ are *order-isomorphic*.

For any n , the *graph of overlapping n -permutations* is a directed graph on $n!$ vertices, each vertex corresponding to one permutation in S_n . There is an edge from $x = x_1x_2 \cdots x_n$ to $y = y_1y_2 \cdots y_n$ iff $\text{red}(x_2x_3 \cdots x_n) = \text{red}(y_1y_2 \cdots y_{n-1})$, see Figure 2. Notice that a universal cycle for S_n gives a Hamiltonian cycle in the graph of overlapping n -permutations.

The cluster graph, where a universal cycle corresponds to an Eulerian tour rather than a Hamiltonian cycle, has also been used to study universal cycles for S_n ; see [5]. The *cluster graph* for n -permutations, denoted here by G , identifies all permutations x and y where $\text{red}(x_1 \cdots x_{n-1}) = \text{red}(y_1 \cdots y_{n-1})$. This means the edges are grouped by their origin. Its vertices are the clusters of n -permutations whose first $n-1$ entries are order-isomorphic, and its edges are the n -permutations: it has an edge from σ to τ for each n -permutation x satisfying $\sigma = \text{red}(x_1 \cdots x_{n-1})$ and $\tau = \text{red}(x_2 \cdots x_n)$. This means that each cluster contains n permutations, and the cluster graph is a directed multigraph. For example, on the right in Figure 2 is the cluster graph for 3-permutations, where each edge corresponds to the 3-permutation at its head. Compressing the parallel edges of the cluster graph for n -permutations to single edges yields the graph of overlapping $(n-1)$ -permutations. For example, the cluster graph for 4-permutations is shown in Figure 3, and after compressing parallel edges it is the same as the graph of overlapping 3-permutations shown on the left of Figure 2. The cluster graph G is balanced and strongly connected, as observed in [5].

Any Eulerian tour in the cluster graph gives a Hamiltonian cycle in the graph of overlapping n -permutations. It is conjectured that these Hamiltonian cycles can be extended to universal cycles for n -permutations; see [5, 15].

For words, translating a Hamiltonian cycle in the De Bruijn graph to a De Bruijn sequence is straightforward since the universal cycle uses just n letters. This is not the case for n -permutations. A universal cycle or a universal word for n permutations may use many more distinct entries than n . Building a universal cycle for n -permutations by following a Hamiltonian cycle in the graph of overlapping n -permutations could possibly lead to a situation where the beginning and the end are not compatible. To illustrate this potential misalignment, consider the following simple example. Recall S_3 has a universal word 14524314 that can be turned into universal cycle by removing the last two letters. On the other hand 14625415 is still a universal word, but removing the last two letters does not turn it into a universal cycle since it would contain a sub-word 414. It is, however, easy to construct a universal word by following a Hamiltonian path.

The proof of Theorem 1 utilizes the cluster graph of n -permutations and performs a compression on the cluster graph. We use the same shortening ideas as Kitaev, Potapov, and Vajnovszki [18]. In order to obtain a universal cycle instead of a universal word, we identify a particular part of the cluster graph that allows for the beginning and the end of the word coming from traversing an Eulerian tour to be connected. We use many of the lemmas as Kitaev, Potapov, and Vajnovszki [18], we include also proofs for completeness.

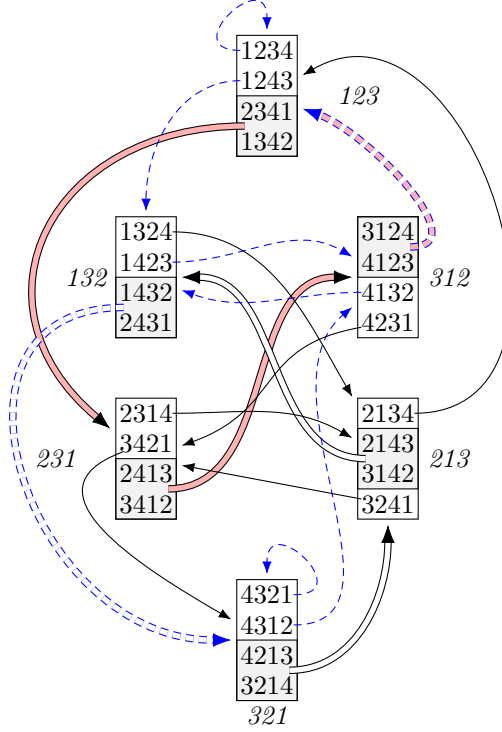


Figure 3: Cluster graph for 4-permutations. Grey boxes indicate pairs of twins. A double edge represents parallel edges corresponding to the permutations in the gray box. Double edges with filled insides and white insides form two cycles that could be used for shortening the universal cycle. Blue dashed edges denote permutations in the family $\mathcal{P}^*$, defined in the upcoming equation (3). They are used in the gluing part depicted in Figure 7.

2 Proof of Theorem 2

We start by introducing and restating some technical definitions. Then we describe several properties of the cluster graph from Kitaev, Potapov, and Vajnovszki [18]. Finally, we describe the procedure to obtain a shortened universal cycle.

Two permutations $\pi_1 \cdots \pi_n$ and $\sigma_1 \cdots \sigma_n$ are called *twins* if they belong to the same cluster and $|\pi_1 - \pi_n| = |\sigma_1 - \sigma_n| = 1$. For example, 3142 and 2143 are twins, and 134562 and 234561 are twins. Pairs of twins are depicted in Figure 3 in gray boxes, and they correspond to a pair of parallel edges.

The cluster graph G is balanced and strongly connected, as observed in [5]. Recall G has an edge e from σ to τ for each n -permutation x satisfying $\sigma = \text{red}(x_1 \cdots x_{n-1})$ and $\tau = \text{red}(x_2 \cdots x_n)$. We write $L(e)$ for x . Later, when we compress pairs of twins and have only one edge e from σ to τ , we write $L(e)$ for $x = x_1 \cdots x_n \in \mathbb{N}^n$, where $x_1 = x_n$, $\sigma = \text{red}(x_1 \cdots x_{n-1})$, and $\tau = \text{red}(x_2 \cdots x_n)$.

The proof of Theorem 2 is constructive. Here is a sketch of the proof, omitting compression of twins.

Proof sketch (omitting compression of twins).

1. In the cluster graph G , we identify a cycle given by permutations $\mathcal{P}$ defined in the upcoming Eq (1).
2. We remove them from G and find a word w whose n -windows are the permutations of $S_n \setminus \mathcal{P}$, exactly once each.

- (a) Lemma 5 guarantees that $G - \mathcal{P}$ is strongly connected, which we use to show $G - \mathcal{P}$ is Eulerian.
- (b) Given an Eulerian trail of $G - \mathcal{P}$, Lemma 6 shows how to build w .

3. Finally, we extend w to a cyclic word that includes $\mathcal{P}$ in Lemma 7.

A series of Lemmas in [18] describe the following additional properties of the cluster graph.

Lemma 3 (Kitaev, Potapov, and Vajnovszki [18]). *The following are true for each cluster graph G .*

- (i) *Each cluster has exactly one pair of twins.*
- (ii) *For each cluster X , there exists a unique cluster Y such that there are two edges from X to Y . Also, there are no clusters X and Y such that there are three or more edges from X to Y .*
- (iii) *For each cluster Y , there exists a unique cluster X such that there are two edges from X to Y .*
- (iv) *Any of the disjoint cycles formed by the double edges goes through exactly $n - 1$ distinct clusters.*

Let $\mathcal{P}$ be the following set of $2n$ permutations:

$$\mathcal{P} = \left\{ \begin{array}{ll} (n, n-1, \dots, 2, 1), & (1, 2, \dots, n-2, n, n-1), \\ (n, n-1, \dots, 3, 1, 2), & (1, 2, \dots, n-3, n, n-2, n-1), \\ (n, n-1, \dots, 4, 1, 2, 3), & (1, 2, \dots, n-4, n, n-3, n-1, n-2), \\ \vdots & \vdots \\ (n, n-1, \dots, k, 1, \dots, k-1), & (1, 2, \dots, k, n, k+1, n-1, n-2, \dots, k+2), \\ \vdots & \vdots \\ (n, n-1, 1, 2, \dots, n-2), & (1, n, 2, n-1, n-2, \dots, 3), \\ (n, 1, 2, \dots, n-1), & (n, 1, n-1, n-2, \dots, 2), \\ (1, 2, \dots, n), & (1, n, n-1, n-2, \dots, 2) \end{array} \right\}. \quad (1)$$

The permutations in $\mathcal{P}$ form a tour in the cluster graph when read first from top to bottom of the first column, and then from top to bottom of the second column. For example, when $n = 6$ we have

$$\mathcal{P} = \left\{ \begin{array}{ll} (6, 5, 4, 3, 2, 1), & (1, 2, 3, 4, 6, 5), \\ (6, 5, 4, 3, 1, 2), & (1, 2, 3, 6, 4, 5), \\ (6, 5, 4, 1, 2, 3), & (1, 2, 6, 3, 5, 4), \\ (6, 5, 1, 2, 3, 4), & (1, 6, 2, 5, 4, 3), \\ (6, 1, 2, 3, 4, 5), & (6, 1, 5, 4, 3, 2), \\ (1, 2, 3, 4, 5, 6), & (1, 6, 5, 4, 3, 2) \end{array} \right\}. \quad (2)$$

We will use $\mathcal{P}$ to glue together the ends of a universal word for $S_n \setminus \mathcal{P}$, creating a universal cycle for S_n . To accommodate shortening, we also need to consider twins of permutations in $\mathcal{P}$. The only permutations in $\mathcal{P}$ that have twins are $(n, 1, 2, \dots, n-1)$ and $(1, n, n-1, n-2, \dots, 2)$. Let

$$\mathcal{P}^* = \mathcal{P} \cup \{(n-1, 1, 2, 3, \dots, n-2, n), (2, n, n-1, \dots, 4, 3, 1)\}. \quad (3)$$

Hence $\mathcal{P}^*$ is obtained from $\mathcal{P}$ by adding twin permutations to those already in $\mathcal{P}$. Blue dashed edges in Figure 3 correspond to $\mathcal{P}^*$.

Remark 4. By inspection of cases, for $n \geq 4$, the permutations in $\mathcal{P}^*$ do not contain a 3-window that is order-isomorphic to 213 or 231.

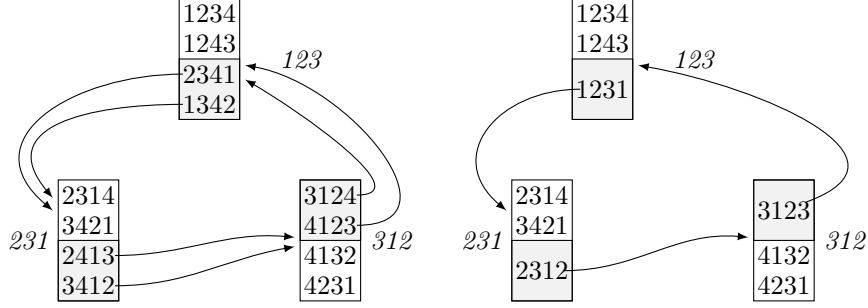


Figure 4: The effect of compressing one cycle in the cluster graph for $n = 4$. See Figure 3 for the entire cluster graph.

If TW is a set of twins in a cluster graph G , the *compressed cluster graph* for TW is obtained from G by replacing each pair of parallel edges in TW , labeled by L as $(x_1x_2 \cdots x_{n-1}x_n)$ and $(x_nx_2 \cdots x_{n-1}x_1)$, by one edge e with $L(e) = \text{red}(x_1x_2 \cdots x_{n-1}x_1)$. See Figure 4 for an illustration where TW forms a cycle.

Lemma 5. *For $n \geq 4$, the graph obtained from a compressed cluster graph by removing all edges corresponding to $\mathcal{P}^*$ is strongly connected.*

Proof. Notice that a compressed cluster graph is obtained from the (uncompressed) cluster graph G by replacing parallel edges with single edges. Hence it is enough to show that $G - \mathcal{P}^*$ is strongly connected. We prove there exists a walk between any two vertices in $G - \mathcal{P}^*$.

Let $a = a_1a_2 \cdots a_{n-1}$ and $b = b_1b_2 \cdots b_{n-1}$ be permutations in S_{n-1} corresponding to two vertices in $G - \mathcal{P}^*$. We construct a word Q that corresponds to a walk in $G - \mathcal{P}^*$ starting at a and ending at b . We define $Q = axb'$, where x is an integer and b' is created from b as follows:

$$x = \begin{cases} n & \text{if } a_{n-2} > a_{n-1}, \\ 0 & \text{if } a_{n-2} < a_{n-1}, \end{cases} \quad b'_i = \begin{cases} b_i + n & \text{if } b_i > (b_1 + b_2)/2, \\ b_i - n & \text{otherwise.} \end{cases}$$

See Figure 5 for sketch of the construction of a general Q and for a particular example see Figure 6. Notice that $\text{red}(b') = b$ and $\{\text{red}(a_{n-2}a_{n-1}x), \text{red}(xb_1b_2)\} \subseteq \{231, 213\}$. Hence any n -permutation in Q contains x and contains a consecutive triple that reduces to 213 or 231. Neither of these are contained in any permutation in $\mathcal{P}^*$ by Claim 4. Therefore, Q corresponds to a path in $G - \mathcal{P}^*$ from a to b , and hence $G - \mathcal{P}^*$ is strongly connected. $\square$

For a word w , an n -window of w is a substring of n consecutive letters of w . A *trail* in a directed multigraph is a walk without repeated edges.

Lemma 6. *Let $T = (e_1, e_2, \dots, e_\ell)$ be a trail in a compressed cluster graph for n -permutations. There exists a word $w = w_1 \cdots w_{\ell+n-1}$, where $\text{red}(w_i \cdots w_{i+n-1}) = L(e_i)$ for all $1 \leq i \leq \ell$.*

Proof. If $\ell = 1$, we let $w = L(e_1)$. If $\ell \geq 2$, let $w' = w'_1 \cdots w'_{\ell+n-2}$ be a word for $(e_1, e_2, \dots, e_{\ell-1})$ by induction. Let $a_1a_2 \cdots a_n$ be $L(e_\ell)$. If $a_n = a_1$, i.e. e_ℓ is a compressed edge, then w is obtained from w' by appending w'_ℓ , which is the letter corresponding to a_1 . Otherwise, $a_1 \neq a_n$, and we may need to modify w' and determine a letter to append as follows.

If $a_n = n$, let $x = \max\{w'_i : 1 \leq i \leq \ell + n - 2\} + 1$. If $a_n < n$, let i be the index such that $a_i = a_n + 1$ and $x = w'_{\ell-1+i}$ be the letter in w' corresponding to a_i . Then we define $w = w_1 \cdots w_{\ell+n-1}$ as

$$w_i = \begin{cases} w'_i & \text{if } w'_i < x \text{ and } i \leq \ell + n - 2, \\ w'_i + 1 & \text{if } w'_i \geq x \text{ and } i \leq \ell + n - 2, \\ x & \text{if } i = \ell + n - 1. \end{cases}$$

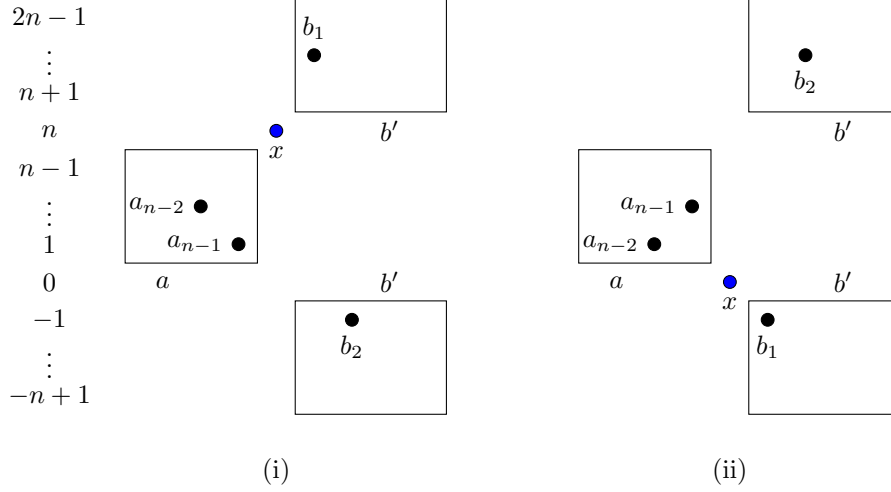


Figure 5: Transitioning between two clusters in Lemma 5. (i) depicts $a_{n-1} < a_{n-2}$ and $b_1 > b_2$ and (ii) depicts $a_{n-1} > a_{n-2}$ and $b_1 < b_2$. The remaining two cases are similar.

Since all n -windows of $w_1 \cdots w_{\ell+n-2}$ are order isomorphic to n -windows of $w'_1 \cdots w'_{\ell+n-2}$, w still represents the same permutations as w' in the part before the last letter. The choice of x makes the last n -window in w be order isomorphic to $L(e_\ell)$. $\square$

The next lemma describes how to use $\mathcal{P}$ to turn a particular word w into a cyclic word z . It adds $n+1$ letters at the beginning of w . The result is a cyclic word covering the permutations in w and also all permutations in $\mathcal{P}$. A result of the operation, where w is indicated by a box, can be seen in the Appendix for $n=4$.

Lemma 7. *Let $n \geq 4$. Let $\mathcal{P}'$ be such that $\mathcal{P} \subseteq \mathcal{P}' \subseteq \mathcal{P}^*$. Let $w = w_1 w_2 \cdots w_k$ be a word with $k \geq n-1$ that covers each n -permutation in S_n at most once, and let $\mathcal{W}$ be the set of n -permutations covered by w . If $\text{red}(w_1 w_2 \cdots w_{n-1}) = \text{red}(w_{k-n+2} \cdots w_{k-1} w_k) = (n-1, \dots, 2, 1)$ and $\mathcal{W} \cap \mathcal{P}' = \emptyset$, then there exists a cyclic word z of length $k+n+1$ such that each permutation in $\mathcal{W} \cup \mathcal{P}'$ is covered by exactly one n -window of z . The permutations in $\mathcal{P}' \setminus \mathcal{P}$ are covered in a compressed way.*

Proof. The cyclic word $z = z_1 z_2 \cdots z_{n+1+k}$ can be defined as follows; see Figure 7 for guidance:

$$\begin{aligned}
z_i &= \min\{w_1, w_2, \dots, w_k\} - n - 1 + i \text{ for } 1 \leq i \leq n-1, \\
z_n &= \max\{w_1, w_2, \dots, w_k\} + 1, \\
z_{n+1} &= \begin{cases} z_{n-1} + 1 & \text{if } (2, n, n-1, \dots, 4, 3, 1) \notin \mathcal{P}', \\ w_{n-1} & \text{otherwise,} \end{cases} \\
z_{n+1+i} &= w_i \text{ for } 1 \leq i \leq k-1, \\
z_{n+1+k} &= \begin{cases} w_k & \text{if } (n-1, 1, 2, 3, \dots, n-2, n) \notin \mathcal{P}', \\ z_{n-1} & \text{otherwise.} \end{cases}
\end{aligned}$$

The n -windows of the cyclic word z that start at indices $k+3$ through $n+1$ (the first $n-1$ of which wrap around to the beginning of z) cover each permutation in $\mathcal{P}'$ exactly once, in the order given by reading the left column in (1) first and the right column second, and with the permutations in $\mathcal{P}' \setminus \mathcal{P}$ covered in a compressed way. Notice that the possible shortenings are obtained by n -windows starting at z_{n+1} and z_{n+1+k} .

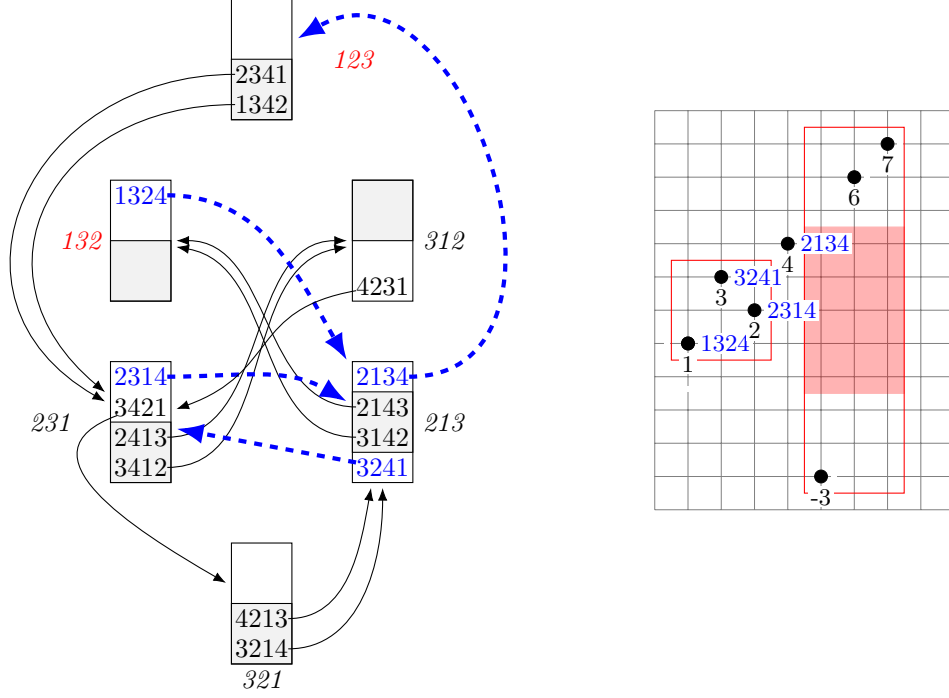


Figure 6: Cluster graph for 4-permutations with $\mathcal{P}^*$ deleted. This figure demonstrates using the procedure in Lemma 5 to obtain a walk from $a = 132$ to $b = 123$, depicted using dashed edges. Here $n = 4$, $x = 4$, $b' = (-3, 6, 7)$, and $Q = (1, 3, 2, 4, -3, 6, 7)$. Reducing the 4-windows of Q yields the walk $(1324, 3241, 2314, 2134)$. On the right is a picture of Q with red rectangles bounding regions for a and b' . The filled red rectangle indicates a region not containing any letter from b' regardless of the letters in b .

Since $z_{n+2} \cdots z_{n+k} = w_1 \cdots w_{k-1}$, all permutations in $\mathcal{W}$ except the last one containing w_k are covered by z . In the case that $z_{n+1+k} = w_k$, the last one is there as well. We need to check that $\text{red}(w_{k-n+1} \cdots w_k) = \text{red}(z_{k+2} \cdots z_{n+1+k})$ when $z_{n+1+k} = z_{n-1}$. By hypothesis, $w_{k-n+2} > \cdots > w_{k-1} > w_k$ so, if $w_k \geq w_{k-n+1}$, then $\text{red}(w_{k-n+1} \cdots w_k) = (1, n, n-1, \dots, 2) \in \mathcal{P}$ or $\text{red}(w_{k-n+1} \cdots w_k) = (1, n-1, n-2, \dots, 1)$, in both cases contradicting $\mathcal{P} \cap \mathcal{W} = \emptyset$. Therefore, $w_k < w_{k-n+1}$, so $w_k < \min\{w_{k-n+1}, \dots, w_{k-1}\}$, and decreasing w_k to $z_{n+1+k} = z_{n-1} = \min\{w_1, w_2, \dots, w_k\} - 2$ does not change the relative order of $w_{k-n+1} \cdots w_k$. $\square$

Now we are ready to prove Theorem 2, restated below.

Theorem 2. For $n \geq 3$ and each $0 \leq i \leq (n-2)!$, using incomparable elements at distance $n-1$, one can obtain a shortened universal cycle for S_n of length $n! - i(n-1)$.

Proof of Theorem 2. If $n = 3$, then possible universal cycles are for example 1232 and 145243 for $i = 1$ and $i = 0$, respectively. Let $n \geq 4$ and $0 \leq i \leq (n-2)!$. Let G be the cluster graph for n -permutations. By Lemma 3, G contains $(n-1)!/(n-1) = (n-2)!$ disjoint cycles formed by double edges. Let C_i be a union of i of these cycles. Let $\mathcal{P}'$ be the set satisfying $\mathcal{P} \subseteq \mathcal{P}' \subseteq \mathcal{P}^*$, where each permutation in $\mathcal{P}^* \setminus \mathcal{P}$ is included in $\mathcal{P}'$ if and only if it is a twin in C_i . Let G_i be the compressed cluster graph for the set of twins in C_i . Notice that G_i has $n! - i(n-1)$ edges and is still Eulerian since cycles of double edges were replaced by cycles of single edges. Let T be a tour in G_i corresponding to all of the permutations in $\mathcal{P}'$. Let G'_i be obtained from G_i by removing T . Then G'_i is balanced because G_i and T are balanced, and G'_i is strongly connected as Lemma 5 implies G'_i has a strongly connected spanning subgraph, so G'_i is Eulerian. Hence, using Lemma 6

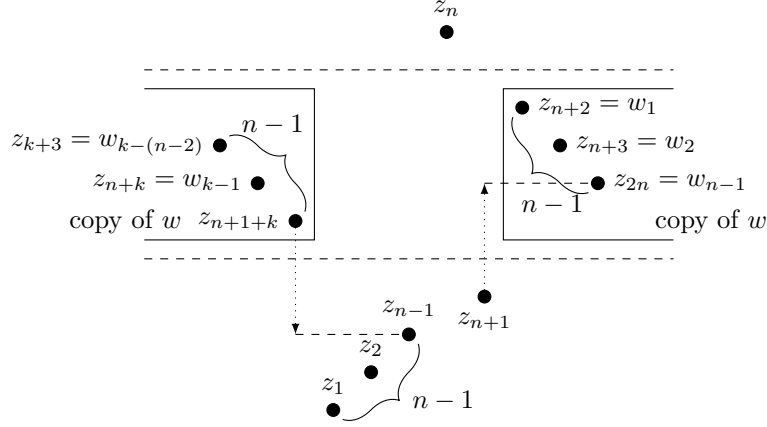


Figure 7: The word z from Lemma 7 for $n = 4$ depicted. Dashed lines indicate relative order of entries. Arrows indicate the “otherwise” cases for z_{n+1+k} and z_{n+1} .

with an Eulerian trail of G'_i that starts and ends at $(n-1, n-2, \dots, 1)$, there exists a word $w = w_1 w_2 \dots w_k$ whose n -windows are $L(e)$ for the edges e of G'_i in the order of the Eulerian trail, and which in particular has $\text{red}(w_1 w_2 \dots w_{n-1}) = \text{red}(w_{k-n+2} \dots w_k) = (n-1, n-2, \dots, 1)$. By Lemma 7, there exists a cyclic word z , which covers each n -permutation exactly once, and covers the twins in C_i in a compressed way. This cyclic word z is a shortened universal cycle for S_n of length $n! - i(n-1)$. $\square$

3 Conclusion

In this note we proved a conjecture of Kitaev, Potapov, and Vajnovszki [18] on shortening universal words by adding repeated elements. Our construction does not control how many different entries are in the resulting universal cycle. It would be interesting to investigate the smallest number of symbols needed to create a shortened universal cycle of a given length. As suggested in [9], it would also be interesting to determine a greedy algorithm for constructing shortened universal cycles for S_n .

The case $i = 0$ in Theorem 2 gives a new way to construct universal cycles for S_n . It is an open question whether every Eulerian tour of the cluster graph for n -permutations corresponds to a universal cycle for S_n ; see [5, 15]. Our proof shows that the Eulerian tours containing the specific tour for $\mathcal{P}$ all correspond to universal cycles for S_n .

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References

- [1] M. Albert and J. West. Universal cycles for permutation classes. *Discrete Mathematics & Theoretical Computer Science*, DMTCS Proceedings vol. AK, 21st International Conference on Formal Power Series

- and Algebraic Combinatorics (FPSAC 2009), Jan. 2009. URL: <https://dmtdcs.episciences.org/2727>.
- [2] G. Brockman, B. Kay, and E. E. Snively. On universal cycles of labeled graphs. *Electronic Journal of Combinatorics*, 17(1):#R4, 2010. doi:10.37236/276.
 - [3] A. Cantwell, J. Geraci, A. Godbole, and C. Padilla. Graph universal cycles of combinatorial objects. *Adv. in Appl. Math.*, 127:Paper No. 102166, 14, 2021. doi:10.1016/j.aam.2021.102166.
 - [4] H. Z. Q. Chen, S. Kitaev, T. Mütze, and B. Y. Sun. On universal partial words. *Discrete Mathematics & Theoretical Computer Science*, Vol. 19 no. 1, May 2017. URL: <https://dmtdcs.episciences.org/3690>, doi:10.23638/DMTCS-19-1-16.
 - [5] F. Chung, P. Diaconis, and R. Graham. Universal cycles for combinatorial structures. *Discrete Mathematics*, 110(1):43 – 59, 1992. doi:10.1016/0012-365X(92)90699-G.
 - [6] P. E. C. Compeau, P. A. Pevzner, and G. Tesler. How to apply de Bruijn graphs to genome assembly. *Nature Biotechnology*, 29(11):987–991, Nov. 2011. doi:10.1038/nbt.2023.
 - [7] D. Gabric, A. Gündoğan, T. Mütze, E. Sala, J. Sawada, A. Williams, and D. Wong. De bruijn sequence and universal cycle constructions, 2022. URL: <http://debruijnsequence.org>.
 - [8] D. Gabric, J. Sawada, A. Williams, and D. Wong. A successor rule framework for constructing k -ary de Bruijn sequences and universal cycles. *IEEE Transactions on Information Theory*, 66(1):679–687, 2020. doi:10.1109/TIT.2019.2928292.
 - [9] A. L. L. Gao, S. Kitaev, W. Steiner, and P. B. Zhang. On a greedy algorithm to construct universal cycles for permutations. *Internat. J. Found. Comput. Sci.*, 30(1):61–72, 2019. doi:10.1142/S0129054119400033.
 - [10] B. Goeckner, D. Fillmore, R. Kirsch, K. Martin, and D. McGinnis. The existence and structure of universal partial cycles. In preparation.
 - [11] B. Goeckner, C. Groothuis, C. Hettle, B. Kell, P. Kirkpatrick, R. Kirsch, and R. Solava. Universal partial words over non-binary alphabets. *Theoretical Computer Science*, 713:56–65, 2018. doi:10.1016/j.tcs.2017.12.022.
 - [12] A. E. Holroyd, F. Ruskey, and A. Williams. Shorthand universal cycles for permutations. *Algorithmica*, 64(2):215–245, 2012. doi:10.1007/s00453-011-9544-z.
 - [13] V. Horan and G. Hurlbert. s -Overlap Cycles for Permutations. *Bull. Inst. Combin. Appl.*, 69:60–67, 2013. arXiv:1301.1270.
 - [14] V. Horan and G. Hurlbert. Universal cycles for weak orders. *SIAM J. Discrete Math.*, 27(3):1360–1371, 2013. doi:10.1137/120886807.
 - [15] G. H. Hurlbert. *Universal cycles: On beyond de Bruijn*. ProQuest LLC, Ann Arbor, MI, 1990. Thesis (Ph.D.)—Rutgers The State University of New Jersey - New Brunswick. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:9034910.
 - [16] B. Jackson. Universal cycles of k -subsets and k -permutations. *Discrete Mathematics*, 117(1):141–150, 1993. doi:10.1016/0012-365X(93)90330-V.
 - [17] R. Kirsch, C. Sibley, and E. Sprangel. Graph universal cycles: Compression and connections to universal cycles. In preparation.

- [18] S. Kitaev, V. N. Potapov, and V. Vajnovszki. On shortening u-cycles and u-words for permutations. *Discrete Appl. Math.*, 260:203–213, 2019. doi:[10.1016/j.dam.2019.01.025](https://doi.org/10.1016/j.dam.2019.01.025).
- [19] J. Pagès, J. Salvi, C. Collewet, and J. Forest. Optimised de bruijn patterns for one-shot shape acquisition. *Image and Vision Computing*, 23(8):707–720, Aug. 2005. doi:[10.1016/j.imavis.2005.05.007](https://doi.org/10.1016/j.imavis.2005.05.007).
- [20] J. Robert Johnson. Universal cycles for permutations. *Discrete Mathematics*, 309(17):5264–5270, 2009. Generalisations of de Bruijn Cycles and Gray Codes/Graph Asymmetries/Hamiltonicity Problem for Vertex-Transitive (Cayley) Graphs. doi:[10.1016/j.disc.2007.11.004](https://doi.org/10.1016/j.disc.2007.11.004).
- [21] F. Ruskey and A. Williams. An explicit universal cycle for the $(n - 1)$ -permutations of an n -set. *ACM Trans. Algorithms*, 6(3):Art. 45, 12, 2010. doi:[10.1145/1798596.1798598](https://doi.org/10.1145/1798596.1798598).
- [22] E. Scheinerman. Determining planar location via complement-free de Bruijn sequences using discrete optical sensors. *IEEE Transactions on Robotics and Automation*, 17(6):883–889, 2001. doi:[10.1109/70.976017](https://doi.org/10.1109/70.976017).
- [23] H.-S. Sohn, D. L. Bricker, J. R. Simon, and Y. Hsieh. Optimal sequences of trials for balancing practice and repetition effects. *Behavior Research Methods, Instruments, & Computers*, 29(4):574–581, Dec. 1997. doi:[10.3758/bf03210610](https://doi.org/10.3758/bf03210610).
- [24] D. Wong. A new universal cycle for permutations. *Graphs Combin.*, 33(6):1393–1399, 2017. doi:[10.1007/s00373-017-1778-3](https://doi.org/10.1007/s00373-017-1778-3).

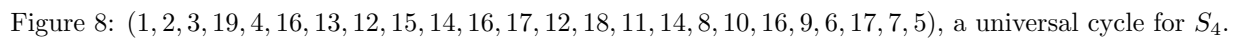
Appendix

3.1 Constructions

Here we describe Figures 8, 9, and 10 depicting constructions coming from Theorem 2 for $n = 4$ of three different lengths. Recall that the construction of w in the proof of Theorem 2 uses an Eulerian tour but does not explicitly say which one. Hence there are many possible outputs of the described algorithm. Here we present one for each length of the resulting universal cycle.

In addition to writing a sequence of numbers forming a universal cycle, we depict the word in a grid, where each node corresponds to a letter. The letter is depicted below as a number. The number with four digits to the right of each node is a permutation of 1234 whose representation starts at the given letter. Since the word is cyclic, the first three nodes are repeated at the end as gray nodes to make it easier to see the cycle. Blue permutations are in $\mathcal{P}^*$. Red sequences are compressed permutations, where the first and last entries are the same. The black rectangle denotes the word w as in Figure 7.

We made no attempt to minimize the number of letters used in the depicted universal cycles.



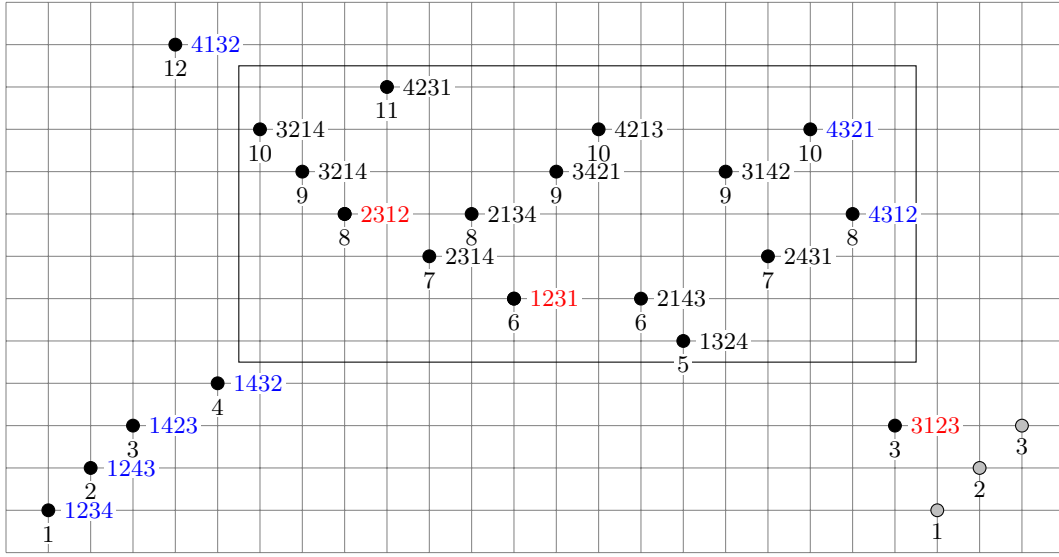


Figure 9: (1, 2, 3, 12, 4, 10, 9, 8, 11, 7, 8, 6, 9, 10, 6, 5, 9, 7, 10, 8, 3), a universal cycle for S_4 shortened by 3.

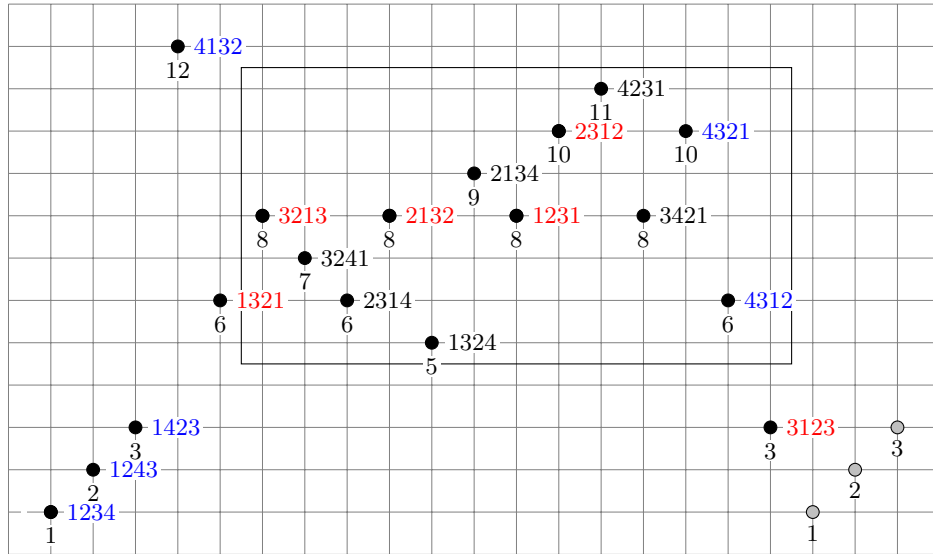


Figure 10: (1, 2, 3, 12, 6, 8, 7, 6, 8, 5, 9, 8, 10, 11, 8, 10, 6, 3), a universal cycle for S_4 shortened by 6.