Planar Turán Numbers of Cycles: A Counterexample

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Abstract

The planar Turán number $\mathrm{ex}_{\mathcal{P}}(C_\ell, n)$ is the largest number of edges in an n-vertex planar graph with no ℓ-cycle. For $n \in \{3, 4, 5, 6\}$, upper bounds on $\mathrm{ex}_{\mathcal{P}}(C_\ell, n)$ are known that hold with equality infinitely often. Ghosh, Györi, Martin, Paulo, and Xiao [arxiv:2004.14094] conjectured an upper bound on $\mathrm{ex}_{\mathcal{P}}(C_\ell, n)$ for every $\ell \geq 7$ and n sufficiently large. We disprove this conjecture for every $\ell \geq 11$. We also propose two revised versions of the conjecture.

1 Introduction

The Turán number $\mathrm{ex}(n, H)$ for a graph H is the maximum number of edges in an n-vertex graph with no copy of H as a subgraph. Turán famously showed that $\mathrm{ex}(n, K_\ell) \leq (1-\frac{1}{\ell-1})\frac{n^2}{2}$; for example, see [1, Chapter 32]. The Erdős–Stone Theorem [8, Exercise 10.38] generalizes this result, by asymptotically determining $\mathrm{ex}(n, H)$ for every non-bipartite graph H: $\mathrm{ex}(n, H) = (1 - \frac{1}{\chi(H)-1})\frac{n^2}{2} + o(n^2)$; here $\chi(H)$ is the chromatic number of H. Dowden [3] considered the problem when restricting to n-vertex graphs that are planar. The *planar Turán number* $\mathrm{ex}_{\mathcal{P}}(n, H)$ for a graph H is the maximum number of edges in an n-vertex planar graph with no copy of H as a subgraph (not necessarily induced). This parameter has been investigated for various graphs H in [6] and [4]; but here we focus mainly on cycles. It is well-known that if G is an n-vertex planar graph with no triangle, then G has at most $2n - 4$ edges; further, this bound is achieved by every planar graph with each face of length 4. Thus, $\mathrm{ex}_{\mathcal{P}}(n, C_3) = 2n - 4$ for all $n \geq 4$. Dowden [3] proved that $\mathrm{ex}_{\mathcal{P}}(n, C_4) \leq \frac{15(n-2)}{7}$ for all $n \geq 4$ and $\mathrm{ex}_{\mathcal{P}}(n, C_5) \leq \frac{12n-33}{5}$ for all $n \geq 11$. He also gave constructions showing that both of these bounds are tight infinitely often. $\mathrm{ex}_{\mathcal{P}}(n, H)$

For each $k \in \{4, 5\}$, form Θ_k from C_k by adding a chord of the cycle. Lan, Shi, and Song [7] showed that $\mathrm{ex}_{\mathcal{P}}(n, \Theta_4) \leq \frac{12(n-2)}{5}$ for all $n \geq 4$, that $\mathrm{ex}_{\mathcal{P}}(n, \Theta_5) \leq \frac{5(n-2)}{2}$ for all $n \geq 5$, and that $\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq \frac{18(n-2)}{7}$ for all $n \geq 7$. The bounds for Θ_4 and Θ_5 are tight infinitely often. However, the bound for C_6 was strengthened by Ghosh, Györi, Martin, Paulos, and Xiao [5], who showed that $\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq \frac{5n-14}{2}$ for all $n \geq 18$. They also showed that this bound is sharp infinitely often. In the same paper, Ghosh et al. conjectured a bound on $\mathrm{ex}_{\mathcal{P}}(n, C_\ell)$ for each $\ell \geq 7$ and each sufficiently large n. In this note, we disprove their conjecture.

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Conjecture 1 ([5]; now disproved)**.** *For each $\ell \geq 7$, for n sufficiently large, if G is an n-vertex planar graph with no copy of C_ℓ, then $e(G) \leq \frac{3(\ell-1)}{\ell}n - \frac{6(\ell+1)}{\ell}$. That is, $ex_{\mathcal{P}}(n, C_\ell) \leq \frac{3(\ell-1)}{\ell}n - \frac{6(\ell+1)}{\ell}$.*

In fact, we disprove the conjecture in a strong way.

Theorem 1. *For each $\ell \geq 11$ and each n sufficiently large (as a function of ℓ), we have $ex_{\mathcal{P}}(n, C_\ell) > \frac{3(\ell-1)}{\ell}n - \frac{6(\ell+1)}{\ell}$. Furthermore, if there exists a function $s : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $ex_{\mathcal{P}}(n, C_\ell) \leq \frac{3(s(\ell)-1)}{s(\ell)}n$ for all ℓ and all n sufficiently large (as a function of ℓ), then $s(\ell) = \Omega(\ell^{\lg_2 3})$.*

We prove the first statement of Theorem 1 in Section 2, and sketch a proof of the second statement in Section 3. Our constructions modify that outlined by Ghosh et al. [5]. The main building blocks, which we call *gadgets*, are triangulations, in which every cycle has length less than ℓ. Clearly, a set of vertex-disjoint gadgets will have no C_ℓ. To increase the average degree, we can identify vertices on the outer faces of these gadgets as long as we avoid creating cycles. We can also allow ourselves to create cycles among the gadgets as long as each created cycle has length more than ℓ. So we must find the way to do this most efficiently.

Our notation is standard, but for completeness we record a few things here. We let $e(G)$ and $n(G)$ denote the numbers of edges and vertices in a graph G. We write C_ℓ for a cycle of length ℓ.

2 Disproving the Conjecture: a First Construction

To disprove Conjecture 1, we start with a planar graph in which each face has length $\ell + 1$ (and each cycle has length at least $\ell + 1$), and then we "substitute" a gadget for each vertex. As a first step, we construct the densest planar graphs with a given girth g, for each fixed $g \geq 6$. We will also need that our dense graphs have maximum degree 3, as we require in the following definition.

Definition 1. If G is a planar graph of girth g with each vertex of degree 2 or 3, and $e(G) = \frac{g}{g-2}(n-2)$, then G is a *dense graph of girth g*. dense graph

An easy counting argument shows that if G is an n-vertex dense graph of girth g, where $n = (5g-10)\frac{k}{2} - g + 4$ (for some positive even integer k), then G has $10k - 8$ vertices of degree 3 and all other vertices of degree 2.

Lemma 2. *Fix an integer $g \geq 3$. If G is an n-vertex planar graph with girth g, then $e(G) \leq \frac{g}{g-2}(n-2)$. For each $g \geq 6$, this bound holds with equality infinitely often; specifically, it holds with equality if k is a positive even integer and $n = (5g-10)\frac{k}{2} - g + 4$. In fact, for each such k and n, there exists a graph G that attains this bound and that has every vertex of degree 2 or 3.*

Proof of Lemma 2. Let G be an n-vertex planar graph with girth g. Denote by n, e, and f the numbers of vertices, edges, and faces in G. Every face boundary contains a cycle,[1] so every face boundary has length at least g. Thus, $2e \geq gf$. Substituting into Euler's formula and simplifying gives the desired bound: $e \leq \frac{g}{g-2}(n-2)$.

Now we construct graphs for which the bound holds with equality. Before giving our full construction, we sketch a simpler construction which has the desired properties except that it has

[1]To see this, form G' from G by deleting all cut-edges. Since each component of G' is 2-connected, each face boundary *is* either a cycle or a disjoint union of cycles (if G' is disconnected). Note that each face boundary in G contains all edges of a face boundary in G'.

maximum degree 6 (rather than each degree being 2 or 3, as we require). Begin with a 4-connected n-vertex planar triangulation with maximum degree 6. We will find a set M of edges such that every triangular face contains exactly one edge in M. To see that such a set exists, we consider the planar dual G^*. Since G is a triangulation and 2-connected, G^* is 3-regular. By Tutte's Theorem, G^* contains a perfect matching M^* (in fact, this was proved earlier by Petersen). The set M of edges in G corresponding to the edges of M^* in G^* has the desired property: each triangle of G contains exactly one edge of M.

To get the desired graph G' with each face of length g, we replace each edge of G not in M with a path of length $\lfloor (g+1)/3 \rfloor$ and replace each edge of G in M with a path of length $g - 2\lfloor (g+1)/3 \rfloor$. Now each face of G' has length $2\lfloor (g+1)/3 \rfloor + (g - 2\lfloor (g+1)/3 \rfloor) = g$. Thus, for G' the inequality $2e(G') \geq gf(G')$ in the initial paragraph holds with equality. So $e(G') = \frac{g}{g-2}(n(G') - 2)$. Since each non-facial cycle of G has length at least 4, each non-facial cycle of G' has length at least g.

Now we show how to also guarantee that each vertex of G' has degree 2 or 3. The construction is similar, except that it starts from a particular planar graph G with every face of length 6 and every vertex of degree 2 or 3. Again, we find a subset M of edges such that each face of G contains exactly one edge of M. To form G' from G, we replace each edge not in M with a path of length $\lfloor (g+1)/6 \rfloor$ and we replace each edge in M with a path of length $g - 5\lfloor (g+1)/6 \rfloor$. Thus, each face of G' has length exactly $5\lfloor (g+1)/6 \rfloor + (g - 5\lfloor (g+1)/6 \rfloor) = g$.

It will turn out that each non-facial cycle of G has either (i) length at least 10 or (ii) length at least 8 and at least one edge in M. The corresponding non-facial cycle in G' thus has length at least g. In Case (ii) this follows from the calculation in the previous paragraph. In Case (i), when $g \geq 10$ this holds because $10\lfloor (g+1)/6 \rfloor \geq 10(g-4)/6 \geq g$. So consider Case (i) when $g \leq 9$. Since each path in G' replacing an edge in G has length at least 1, each non-facial cycle in G' has length at least 10, which is at least g since $g \leq 9$. Thus, what remains is to construct our graph G, specify the set of edges M, and check that each non-facial cycle in G either has length at least 10 or has length 8 and includes an edge in M.

We construct an infinite family of planar graphs G_k on $10k-2$ vertices, with $5k-2$ faces (each of length 6), and with all vertices of degree 2 or 3; here k is an arbitrary positive even integer. Figure 1 shows G_k. (By Euler's formula, each G_k has 6 vertices of degree 2 and $10k-8$ vertices of degree 3.) Each of k "diagonal columns" contains 10 vertices, except for the first and last, which each contain one vertex fewer. We write $v_{i,j}$ to denote the jth vertex down from the top in column i, except that we start column 1 with $v_{1,2}$. So $V = \{v_{i,j} \mid 1 \leq i \leq k,\ 1 \leq j \leq 10, (i,j) \notin \{(1,1),(k,10)\}\}$. The edge set consists of the boundary cycles of $4(k-1)$ 6-faces in the hexagonal grid, $k-1$ "curved edges" $v_{i,1}v_{i-1,10}$, when $2 \leq i \leq k$, as well two "end edges" $v_{1,2}v_{1,7}$ and $v_{k,4}v_{k,9}$. The matching M contains $v_{i,4}v_{i+1,3}$ and $v_{i,8}v_{i+1,7}$ when $1 \leq i \leq k-1$, edge $v_{i,1}v_{i-1,10}$ for each odd $i \geq 3$ if $k \geq 4$, and the end edges $v_{1,2}v_{1,7}$ and $v_{k,4}v_{k,9}$. It is easy to check that the only vertices with degree 2 are $v_{1,3}, v_{1,5}, v_{1,9}, v_{k,2}, v_{k,6}, v_{k,8}$; the remaining $10k-8$ vertices all have degree 3.

We now show that every non-facial cycle has either (i) length at least 10 or (ii) length at least 8 and at least one edge in M. We denote by $C_2, C_3, \ldots, C_{5k-5}$ the facial cycles that do not use any end-edge. Informally, C_2 is the "top left" of these (containing $v_{1,2}$), and subscripts increase as we move down the first diagonal and then wrap around toroidally with the facial cycle containing $v_{1,10}$ and two curved edges (see Figure 1), and continue on to the facial cycle containing $v_{k,9}$. Formally, each of these is C_k, where X denotes its vertex set and $k := \max\{j/2 : v_{i,j} \in X\} + 5 * \min\{i-1 : v_{i,j} \in X\} + (|\{i : v_{i,j} \in X\}| - 2)$. The facial cycles containing the left end-edge are C_0 and C_1, and those containing the right end-edge are C_{5k-4} and C_{5k-3}.

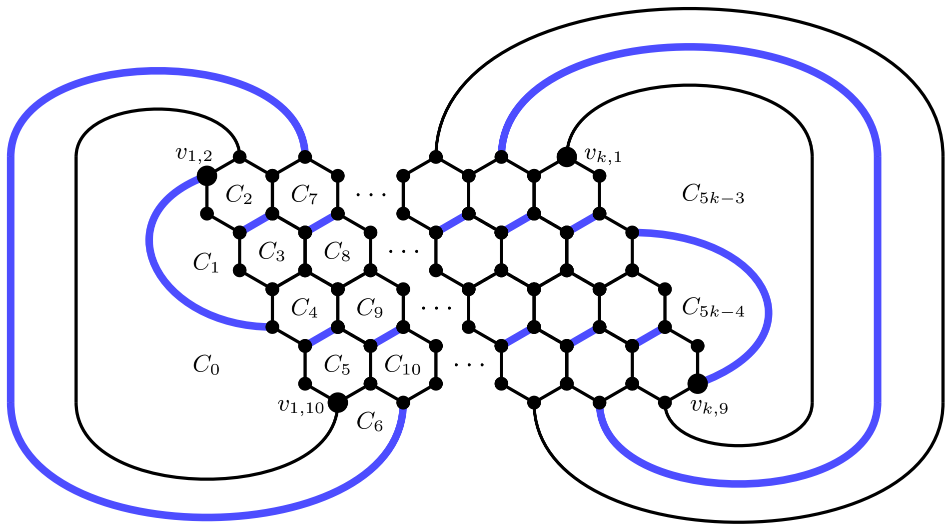

Figure 1: The planar graph G_k has $10k-2$ vertices, $15k-6$ edges, and every face of length 6. Every vertex of G_k has degree 2 or 3 and every non-facial cycle either (i) has length at least 10 or (ii) has length 8 and includes a blue edge. The set of blue edges intersects every face exactly once.

Note that the edge-set of any non-facial cycle C is the symmetric difference of the edge-sets of the facial cycles "inside" (or "outside") of C. Consider first a non-facial cycle C that does not contain any end-edge. Pick the side of C that does not contain the right end-edge; take the symmetric difference of the edge-sets of the facial cycles on this side incrementally, in order of increasing subscripts. The symmetric difference of the first two facial cycles has size at least 10 and this size never decreases. Now consider the non-facial cycles that contain exactly one end-edge; by (rotational) symmetry, assume it is the left end-edge. For these cycles, take the symmetric difference incrementally as above for the side not containing the right end-edge; the symmetric difference of the first two facial cycles has size at least 8 and again this size never decreases.

Finally, consider a non-facial cycle C that contains both end-edges. Now take the symmetric difference incrementally as above for the side of C that includes C_1; the size of the symmetric difference is now initially at least 8, and never decreases until the final facial cycle (C_{5k-4} or C_{5k-3}) is added and the symmetric difference is complete. The final facial cycle C' may reduce the size of the symmetric difference by at most 4, but the final symmetric difference still has size at least 12 (due to the position of C' relative to C_1, and the fact that $k \geq 2$).

To finish the proof, we should verify that $|V(G')| = (5g-10)\frac{k}{2} - g + 4$, as claimed. By construction, each vertex of G' has degree 2 or 3. Each vertex with degree 3 in G' also has degree 3 in G, and we have exactly $10k-8$ of these. Let n, e, and f denote the numbers of vertices, edges, and faces in G'. Now summing degrees gives

$$3(10k-8) + 2(n - (10k-8)) = 2e = gf = \frac{g}{g-2}(2n-4),$$

where the last two equalities hold as at the start of the proof. Thus, $n = (5g-10)\frac{k}{2} - g + 4$. $\square$

Definition 2. Let G be a 2-connected plane graph, with every vertex of degree 2 or 3. Let B be a plane graph with 3 vertices specified on its outer face. To *substitute B into G* we do the following. Subdivide every edge of G. For each vertex v in G, delete v from the subdivided graph and identify $d(v)$ vertices on the outer face of a copy of B with the neighbors of v in the subdivided graph.

substitute B into G

Now we consider the result of substituting B into G, as in Definition 2.

Lemma 3. *Let G be a plane graph; denote by n_2 and n_3 the numbers of vertices with degree 2 and 3 in G. Let B be a plane graph with n_B vertices and e_B edges, and with 3 vertices specified on its outer face. Form G' by substituting B into G. Now $e(G') = (n_2 + n_3)e_B$ and $n(G') = n_2(n_B - 1) + n_3(n_B - 3/2)$. Further, if G has no cycle of length ℓ or shorter, and B has no cycle of length ℓ, then G' has no cycle of length ℓ.*

Proof. Each vertex in G gives rise to an edge-disjoint copy of B in G'; thus $e(G') = (n_2 + n_3)e_B$. Each vertex of degree 2 in G contributes $n_B - 1$ vertices to G', since exactly two of its vertices lie in two copies of B in G' (and all others vertices lie in one copy of B). Similarly, each vertex of degree 3 in G contributes $n_B - 3/2$ vertices to G'. Finally, assume G and B satisfy the hypotheses on the lengths of their cycles. Now consider a cycle C' in G'. If C' is contained entirely in one copy of B, then C' has length not equal to ℓ. If C' visits two or more copies of B, then C' maps to a cycle C in G with length no longer than the length of C'. Since each cycle in G has length longer than ℓ, we are done. $\square$

Now suppose that we plan to substitute some plane graph B into a dense planar graph of girth $\ell + 1$. Which B should we choose? Since B must not contain any ℓ-cycle, a natural choice is a triangulation of order $\ell - 1$. Indeed, every such B yields a graph that attains the bound in Conjecture 1. This is Corollary 5, which follows from our next lemma.

Lemma 4. *Let G be a dense graph of girth $\ell + 1$. Form G' by substituting into G a plane graph B with 3 vertices specified on its outer face. Now $e(G') = \frac{e_B(\ell-1)}{(n_B-1)(\ell-1)-2}\left(n(G') - \frac{2(\ell+1)}{\ell-1}\right)$, where $e_B = e(B)$ and $n_B = n(B)$.*

Proof. Let G be a dense graph of girth $\ell + 1$ on n vertices, and let n_2 and n_3 denote, respectively, its numbers of vertices with degree 2 and 3. Recall from Lemma 2 (with $g = \ell + 1$) that $n = (5\ell - 5)\frac{k}{2} - \ell + 3$ for some even integer k, that $n_3 = 10k - 8$, and that $n_2 = n - n_3$. Lemma 3 implies that $e(G') = (n_2 + n_3)e_B = ne_B$ and that $n(G') = n_2(n_B - 1) + n_3(n_B - 3/2) = (n - n_3)(n_B - 1) + n_3(n_B - 3/2) = n(n_B - 1) - n_3/2$. Now we show that $e(G') = \frac{e_B(\ell-1)}{(n_B-1)(\ell-1)-2}(n(G') - \frac{2(\ell+1)}{\ell-1})$. The final equality comes from substituting for n_3 and simplifying (using that $n = (5\ell - 5)\frac{k}{2} - \ell + 3$).

$$
\begin{aligned}
\frac{e(G')}{n(G') - \frac{2(\ell+1)}{\ell-1}} &= \frac{ne_B(\ell-1)}{(n(n_B-1) - n_3/2)(\ell-1) - 2(\ell+1)} \\
&= \frac{e_B(\ell-1)}{(n_B-1)(\ell-1) - \frac{n_3(\ell-1)+4(\ell+1)}{2n}} \\
&= \frac{e_B(\ell-1)}{(n_B-1)(\ell-1) - 2}.
\end{aligned}
$$

$\square$

Corollary 5. *The bound in Conjecture 1 holds with equality for each graph formed by substituting a triangulation on $\ell - 1$ vertices into a dense graph of girth $\ell + 1$.*

Proof. This follows from the above lemma when B is a plane triangulation on $\ell-1$ vertices, so $n_B = \ell - 1$ and $e_B = 3(\ell-1)-6 = 3\ell - 9$. We get

$$\begin{aligned}\frac{e_B(\ell-1)}{(n_B-1)(\ell-1)-2} &= \frac{3(\ell-3)(\ell-1)}{(\ell-2)(\ell-1)-2} \\ &= \frac{3(\ell-3)(\ell-1)}{\ell^2-3\ell+2-2} \\ &= \frac{3(\ell-1)}{\ell}.\end{aligned}$$ □

To beat the bound of Conjecture 1, it will suffice to instead substitute into a dense graph of girth $\ell+1$ any triangulation with order larger than $\ell-1$, as long as it has each cycle of length at most $\ell-1$. This is because the conjectured average degree is less than 6, and is attained by substituting a triangulation of order $\ell-1$, as shown in Corollary 5. However, the average degree of a triangulation tends to 6 (from below) as its order grows. For each $\ell \in \{3, \ldots, 10\}$, every triangulation on ℓ vertices is Hamiltonian, i.e., it contains an ℓ-cycle. But for each $\ell \geq 11$, there exists a triangulation on ℓ vertices with no ℓ-cycle; this is a consequence of Lemma 6, which we prove next. (In fact, much more is true, as we show in Section 3.)

Lemma 6. *For every integer $t \geq 5$, there exist a plane triangulation with $3t-4$ vertices and each cycle of length at most $2t$, and a plane triangulation with $3t-3$ vertices and each cycle of length at most $2t+1$.*

Proof. We start with a plane triangulation on t vertices. First we add into the interior of each face a new vertex, making it adjacent to each vertex on the face. Let A denote the set of vertices in the original triangulation, and let B denote the set of added vertices. Since $|A| = t$ and $|B| = 2t-4$, the resulting graph G_1 has order $3t-4$. Further, B is an independent set. Thus, on every cycle C, at least half of the vertices must be from A. Hence, C has length at most $2|A| = 2t$.

Now we obtain G_2 by adding a single vertex inside some face of G_1. It is easy to check that G_2 is a $(3t-3)$-vertex triangulation with each cycle of length at most $2t+1$. □

We have already outlined the proof of our main result. We let B be a plane triangulation with no ℓ-cycle, and with order at least ℓ, as guaranteed by Lemma 6. We simply substitute B into a dense graph of girth $\ell+1$. For completeness, we include more details in the proof of Theorem 7.

Theorem 7. *For each $\ell \geq 11$, Conjecture 1 is false. In particular, whenever k is positive if $\ell \geq 11$ and ℓ is odd then, $ex_{\mathcal{P}}(n, C_\ell) \geq \frac{9(\ell-5)(\ell-1)}{(3\ell-13)(\ell-1)-4}\left(n - \frac{2(\ell+1)}{\ell-1}\right)$ for $n = ((5\ell-5)\frac{k}{2} - \ell + 3)(\frac{3(\ell-1)}{2} - 5) - (5k-4)$ and if $\ell \geq 11$ and ℓ is even, then $ex_{\mathcal{P}}(n, C_\ell) \geq \frac{3(3\ell-16)(\ell-1)}{(3\ell-14)(\ell-1)-4}\left(n - \frac{2(\ell+1)}{\ell-1}\right)$ for $n = ((5\ell-5)\frac{k}{2} - \ell + 3)(3(\frac{\ell}{2} - 1) - 4) - (5k-4)$.*

Proof. Let $a_1 := \frac{9(\ell-5)(\ell-1)}{(3\ell-13)(\ell-1)-4}$ and $a_2 := \frac{3(3\ell-16)(\ell-1)}{(3\ell-14)(\ell-1)-4}$. Since $\ell \geq 11$, easy algebra implies that $a_i > \frac{3(\ell-1)}{\ell}$, for each $i \in \{1,2\}$. Thus, $a_i(n - \frac{2(\ell+1)}{\ell-1}) > \frac{3(\ell-1)}{\ell}\left(n - \frac{2(\ell+1)}{\ell-1}\right) = \frac{3(\ell-1)}{\ell}n - \frac{6(\ell+1)}{\ell-1}$ for each $i \in \{1,2\}$. So it suffices to show that $\text{ex}_{\mathcal{P}}(n, C_\ell) \geq a_1(n - \frac{2(\ell+1)}{\ell-1})$ when $\ell \geq 11$ and ℓ is odd; and that $\text{ex}_{\mathcal{P}}(n, C_\ell) \geq a_2(n - \frac{2(\ell+1)}{\ell-1})$ when $\ell \geq 11$ and ℓ is even (for the claimed values of n). Let G be a dense graph of girth $\ell+1$. Recall that $n(G) = (5\ell-5)\frac{k}{2} - \ell + 3$ for some even integer k, and that G has $10k-8$ vertices of degree 3; let $n_3 := 10k-8$.

When $\ell \geq 11$ and ℓ is odd, let $t_1 := \frac{\ell-1}{2}$ and $n_{B_1} := 3t_1 - 4 = \frac{3(\ell-1)}{2} - 4$. We have $t_1 \geq 5$; so by Lemma 6, there exists a plane triangulation B_1 with n_{B_1} vertices and with each cycle of length at most $2t_1 = \ell - 1$. By Euler's formula, $e_{B_1} = e(B_1) = 3(3t_1 - 4) - 6 = 9t_1 - 18 = 9(\frac{\ell-1}{2} - 2)$. Form G_1' by substituting B_1 into G. Lemma 3 implies that G' is a plane graph with no cycle of length ℓ, and that $n(G_1') = n(G)(n_{B_1} - 1) - n_3/2 = ((5\ell - 5)\frac{k}{2} - \ell + 3)(\frac{3(\ell-1)}{2} - 5) - (5k - 4)$. By Lemma 4, we have

$$\begin{aligned} e(G_1') &= \frac{e_{B_1}(\ell-1)}{(n_{B_1}-1)(\ell-1)-2}\left(n(G_1') - \frac{2(\ell+1)}{\ell-1}\right) \\ &= \frac{9(\ell-5)(\ell-1)}{(3\ell-13)(\ell-1)-4}\left(n(G_1') - \frac{2(\ell+1)}{\ell-1}\right) \\ &= a_1\left(n(G_1') - \frac{2(\ell+1)}{\ell-1}\right). \end{aligned}$$

Hence, if $\ell \geq 11$ and ℓ is odd, then whenever k is positive and even and $n = ((5\ell - 5)\frac{k}{2} - \ell + 3)(\frac{3(\ell-1)}{2} - 5) - (5k - 4)$, we have $\mathrm{ex}_{\mathcal{P}}(n, C_\ell) \geq a_1\left(n - \frac{2(\ell+1)}{\ell-1}\right) > \frac{3(\ell-1)}{\ell}n - \frac{6(\ell+1)}{\ell}$.

Now suppose $\ell \geq 11$ and ℓ is even. Let $t_2 := \frac{\ell}{2} - 1$ and $n_{B_2} := 3t_2 - 3 = \frac{3\ell}{2} - 6$. Form G_2' by substituting B_2 into G, where B_2 is a plane triangulation with n_{B_2} vertices and each cycle of B_2 has length at most $2t_2 + 1 = \ell - 1$. (The existence of B_2 is guaranteed by Lemma 6.) By Euler's formula, $e_{B_2} = e(B_2) = \frac{9\ell}{2} - 24$. Similarly, it follows from Lemma 3 that G_2' is a plane graph with no cycle of length ℓ, and that $n(G_2') = n(G)(n_{B_2} - 1) - n_3/2 = ((5\ell - 5)\frac{k}{2} - \ell + 3)(\frac{3\ell}{2} - 7) - (5k - 4)$. Lemma 4 implies that

$$\begin{aligned} e(G_2') &= \frac{e_{B_2}(\ell-1)}{(n_{B_2}-1)(\ell-1)-2}\left(n(G_2') - \frac{2(\ell+1)}{\ell-1}\right) \\ &= \frac{3(3\ell-16)(\ell-1)}{(3\ell-14)(\ell-1)-4}\left(n(G_2') - \frac{2(\ell+1)}{\ell-1}\right) \\ &= a_2\left(n(G_2') - \frac{2(\ell+1)}{\ell-1}\right) > \frac{3(\ell-1)}{\ell}n(G_2') - \frac{6(\ell+1)}{\ell}. \end{aligned}$$

This completes our proof. □

Now for each $\ell \geq 11$, we extend the construction in Theorem 7 to all sufficiently large n (which will prove the first sentence of Theorem 1). Our general idea is to build a counterexample with order n', larger than n, and delete vertices to get a counterexample of order precisely n. To see that this works, note that we can substitute different gadgets for different vertices in a sparse planar graph of girth $\ell + 1$. As long as each gadget has more than ℓ vertices, we will beat the bound in Conjecture 1. In fact, we still beat the bound if a bounded number of gadgets have exactly ℓ vertices, and all other gadgets have more vertices (this is only needed in the case that $\ell \in \{11, 12\}$, since that is when the gadget has precisely ℓ vertices). So we follow the construction in Theorem 7, and then repeatedly remove vertices of degree 3 (that lie in B in Lemma 6). We can remove up to $t - 4$ of these from each gadget. And the increase to the order of G' when we increase k in Theorem 7 is less than $(5g - 10)(3t - 5)$. So it suffices that the number of vertices in the sparse planar graph G is greater than $\lceil (5g - 10)(3t - 5)/(t - 4) \rceil \leq 50(g - 2)$. This proves the first sentence of Theorem 1.

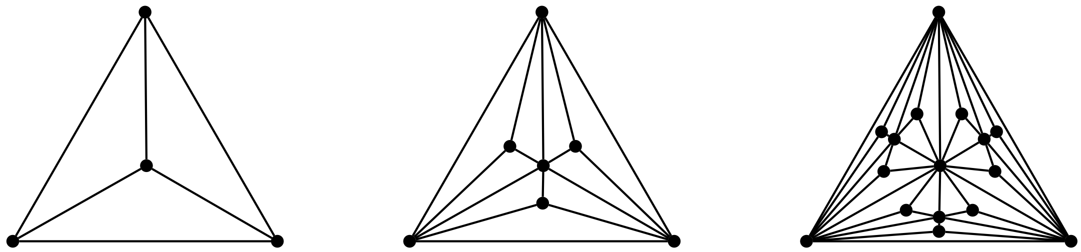

Figure 2: Triangulations T_1, T_2, and T_3.

3 Denser Constructions and a Revised Conjecture

In this short section, we construct counterexamples to Conjecture 1 that are asymptotically much denser than those in the previous section. We also propose two revised versions of Conjecture 1.

By iterating the idea in Lemma 6, Moon and Moser [9] constructed planar triangulations where the length of the longest cycle is sublinear in the order. These triangulations will serve as the gadgets in our denser constructions.

Theorem 8 ([9])**.** *For each positive integer k there exists a 3-connected plane triangulation G_k with $n(G_k) = \frac{3^{k+1}+5}{2}$ and with longest cycle of length less than $\frac{7}{2}n(G_k)^{\log_3 2}$.*

Corollary 9. *There exists a positive real D_1 such that for all integers $\ell \geq 6$ there exists a plane triangulation G_ℓ with $n(G_\ell) \geq D_1 \ell^{\lg_2 3}$ such that G_ℓ has no cycle of length at least ℓ.*

Chen and Yu [2] showed that Theorem 8 is essentially best possible.

Theorem 10 ([2])**.** *There exists a positive real D_2 such that every 3-connected n-vertex planar graph contains a cycle of length at least $D_2 n^{\log_3 2}$.*

We briefly sketch the Moon–Moser construction, which proves Theorem 8. For a more detailed analysis, we recommend Section 2 of [2]. Start with a planar drawing of K_4, which we call T_1. To form T_{i+1} from T_i, add a new vertex v_f inside each face f (other than the outer face), making v_f adjacent to each of the three vertices on the boundary of f, see Figure 2. It is each to check that the order of T_i is $3 + (1 + 3 + \ldots + 3^{i-1}) \approx \frac{3^i}{2}$.

To bound the length of the longest cycle in T_i, we note that the vertices added when forming T_j from T_{j-1} form an independent set, for each j. Thus, for any cycle in T_i, at most half of the vertices were added at the final step. Of those added earlier, at most half were added at the penultimate step, etc. So the length of a longest cycle grows roughly by a factor of 2 at each step (while the order of T_i grows roughly by a factor of 3).

To prove the second statement of Theorem 1, we substitute into a sparse planar graph of girth $\ell+1$ a gadget with no cycle of length ℓ, as guaranteed by Corollary 9. We suspect this construction is extremal. So we conclude with the following two conjectures, which are each best possible.

Conjecture 2. *Fix $\ell \geq 7$, let G be a dense graph of girth $\ell + 1$, and let B be a n-vertex planar triangulation with no ℓ-cycle, where B is chosen to maximize n. If G' is formed by substituting B into G and $n' := |V(G')|$, then $ex_{\mathcal{P}}(n', C_l) = |E(G')|$.*

Proving Conjecture 2 seems plausible for some small values of ℓ. But proving it in general seems difficult. So we also pose the following weaker conjecture. Note that Conjecture 3 would be immediately implied by Conjecture 2 (together with Theorem 10).

Conjecture 3. *There exists a constant D such that for all ℓ and for all sufficiently large n we have $ex_{\mathcal{P}}(n, C_\ell) \leq \frac{3(D\ell^{\lg_2 3}-1)}{D\ell^{\lg_2 3}}n$.*

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