Hadamard Diagonalizable Graphs
of Order at Most 36

Jane Breen

Ontario Tech University
Oshawa, ON, Canada
jane.breen@ontariotechu.ca

Steve Butler[*]

Iowa State University
Ames, IA, U.S.A.
butler@iastate.edu

Melissa Fuentes

University of Delaware
Newark, DE, U.S.A.
melfue@udel.edu

Bernard Lidický[†]

Iowa State University
Ames, IA, U.S.A.
lidicky@iastate.edu

Michael Phillips

University of Colorado, Denver
Denver, CO, U.S.A.
michael.2.phillips@ucdenver.edu

Alex W. N. Riasanovksy[‡]

Iowa State University
Ames, IA, U.S.A.
awnr@iastate.edu

Sung-Yell Song

Iowa State University
Ames, IA, U.S.A.
sysong @iastate.edu

Ralihe R. Villagrán

Centro de Investigación y de Estudios Avanzados del IPN
Mexico City, Mexico
rvillagran@math.cinvestav.mx

Cedar Wiseman

University of Wyoming
Laramie, WY, U.S.A.
cwisema3@uwyo.edu

Xiaohong Zhang

University of Waterloo
Waterloo, ON, Canada
xiaohong.zhang@uwaterloo.ca

Abstract

If the Laplacian matrix of a graph has a full set of orthogonal eigenvectors with entries ± 1, then the matrix formed by taking the columns as the eigenvectors is a Hadamard matrix and the graph is said to be Hadamard diagonalizable.

In this article, we prove that if $n = 8k + 4$ the only possible Hadamard diagonalizable graphs are K_n, $K_{n/2,n/2}$, $2K_{n/2}$, and nK_1, and we develop a computational method for determining all graphs diagonalized by a given Hadamard matrix of any order. Using these two tools, we determine and present all Hadamard diagonalizable graphs up to order 36. Note that it is not even known how many Hadamard matrices there are of order 36.

Mathematics Subject Classifications: 05C50, 15B34, 05B20, 05C76, 05C85

1 Introduction

A real Hadamard matrix is an $n \times n$ matrix H with entries in ± 1 with the property that $H^T H = nI$; in other words, the columns of H are orthogonal. These matrices have been extensively studied, and it is known that a necessary condition for the existence of such a matrix is that $n = 1, 2$, or is a multiple of 4. A well-known and still open problem concerns the question of whether this is sufficient.

Conjecture 1. Hadamard matrices exist for all orders n of the form $n = 4k$.

Examples of Hadamard matrices of order 2^k for $k \geqslant 0$ were constructed by Sylvester in 1867. Defining $H_0 = [1]$, we have

$$H_{k+1} = \left[\begin{array}{cc} H_k & H_k \\ H_k & -H_k \end{array} \right],$$

for $k \geqslant 0$. For example,

$$H_1 = \left[\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right], \quad H_2 = \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array} \right].$$

Two Hadamard matrices are said to be *equivalent* if we can produce one from the other by some combination of the following operations: permuting rows, permuting columns, negating some subset of rows, negating some subset of columns. A *normalized* Hadamard matrix is one which has every entry in the first row and column equal to $+1$. It is easily seen that every Hadamard matrix is equivalent to a normalized Hadamard matrix, by negating combinations of rows and columns.

A graph G is defined in terms of a set of vertices $V(G)$, and a set of edges $E(G)$ which consist of pairs of vertices. The vertices u and v are said to be *adjacent* if there is an edge $\{u, v\} \in E(G)$. The degree of a vertex u, denoted $\deg(u)$, is the number of vertices adjacent to u. A graph G is said to be *regular* if all the degrees of the vertices of G are equal.

Given a graph G, the Laplacian matrix L is defined entrywise by

$$L_{uv} = \begin{cases} \deg(u) & \text{if } u = v, \\ -1 & \text{if } u \text{ adjacent to } v, \\ 0 & \text{otherwise}, \end{cases}$$

where the notation L_{uv} refers to the (u, v) entry of the matrix L. It is common to consider what features of the graph G may be determined via the eigenvalues of the Laplacian. Note that for any graph G, the row-sums of the Laplacian matrix are zero; hence it is immediate that $\lambda = 0$ is an eigenvalue of L, with an eigenvector proportional to the all-ones vector.

In this article, we are interested in graphs whose Laplacian matrix can be diagonalized by a Hadamard matrix; that is, there exists a Hadamard matrix H such that

$$\frac{1}{n} H^T L H = \Lambda,$$

or equivalently,

$$L = \frac{1}{n} H \Lambda H^T,$$

where Λ is a diagonal matrix consisting of the eigenvalues of L, and noting that $H^{-1} = \frac{1}{n} H^T$. If this is the case, we refer to the graph as a Hadamard diagonalizable graph. Clearly, this class of graphs corresponds to graphs for which there exists a full set of ± 1 orthogonal eigenvectors of the Laplacian matrix.

Various properties of Hadamard diagonalizable graphs and a partial characterisation of Hadamard diagonalizable graphs were explored by Barik, Fallat, and Kirkland [1]. A special type of Hadamard matrices, called balancedly splittable Hadamard matrices, was introduced and studied by Kharaghani and Suda [6]; in particular, its connection to Hadamard digonalizable strongly regular graphs was made. Johnston, Kirkland, Plosker, Storey, and Zhang [4] showed that a graph is diagonalizable by a Sylvester's matrix if and only if it is a cubelike graph (a Cayley graph over $\mathbb{Z}_2^d$). In the same paper, Johnston et al. explored the use of Hadamard diagonalizable graphs in quantum information transfer, where a quantum spin network is represented by a graph and quantum information can transfer between spins. An important notion in quantum information transfer is perfect state transfer. Kay [5] showed that a necessary condition for perfect state transfer between vertices j and k of a graph G, is that for a real orthogonal matrix Q which diagonalizes $L(G)$, the corresponding entries in its j-th row and k-th row are either equal to or are the negative of each other. Hadamard diagonalizable graphs certainly satisfy this condition for any pair of vertices, and therefore these graphs are good candidates to admit perfect state transfer. A characterization of when a Hadamard diagonalizable graph admits perfect state transfer was given in terms of its eigenvalues and the normalized diagonalization Hadamard matrix in [4]. Chan, Fallat, Kirkland, Lin, Nasserasr, and Plosker [2] studied complex Hadamard diagonalizable graphs (matrices H with $H^* H = nI$, where the entries can be any complex number of modulus 1 rather than ± 1). Properties and constructions of such graphs were considered, as well as when such a graph admits interesting quantum information transfer phenomena.

Most graphs are not Hadamard diagonalizable. For example, they must have order $n = 1, 2$ or $4k$ (as Hadamard matrices only exist for these orders), but this is not sufficient. The following conditions are well-known.

Order	H. matrices	H. graphs
4	1	4
8	1	10
12	1	4
16	5	50
20	3	4
24	60	26
28	487	4
32	13,710,027	10,196
36	(unknown)	4

Table 1: The order, number of non-equivalent Hadamard matrices (H. matrices), and the number of Hadamard diagonalizable graphs (H. graphs).

Proposition 2 ([1, 4]). *Let G be a Hadamard diagonalizable graph. Then G is regular; moreover, all eigenvalues must be even integers.*

The proof that the graph is regular is given in Theorem 1.3 of [1]; we will give an alternate proof that the eigenvalues must be even integers in Section 3.

We will also make use of the following result.

Proposition 3 ([1]). *A graph G is Hadamard diagonalizable if and only if G^c (the complement of G) is Hadamard diagonalizable.*

This follows immediately by noting that the eigenspaces for the Laplacian matrix of a graph and its complement are the same (although the eigenvalues are different).

Previous research into Hadamard diagonalizable graphs has characterized Hadamard diagonalizable graphs up through order $n = 12$ (see [1]), as well as all Hadamard diagonalizable graphs for the Sylvester construction for Hadamard matrices of order 2^k [4]. The goal of this current paper is to develop further tools to determine the Hadamard diagonalizable graphs of a given order, and to then list all Hadamard diagonalizable graphs up through order $n = 36$. We prove that for $n = 8k + 4$, there are only four Hadamard diagonalizable graphs of order n in Section 2, and we develop computational tools to search for all possible Hadamard diagonalizable graphs of small order in Section 3. Information about the Hadamard diagonalizable graphs is given in Section 4. Concluding comments will be given in Section 5.

In Table 1 we summarize the number of Hadamard diagonalizable graphs as well as the number of inequivalent Hadamard matrices of the indicated order.

2 Hadamard diagonalizable graphs of order $n = 8k + 4$

We show that for order $n = 8k + 4$ there are at most four possible graphs which are Hadamard diagonalizable. We start with the following graph characterization property.

Lemma 4. *Suppose G is a connected graph on n vertices. Then G is a complete graph or a complete bipartite graph if and only if the following condition holds: for any four distinct vertices $\{u, v, w, x\} \subseteq V(G)$ if $uv, vw, wx \in E(G)$ then $xu \in E(G)$.*

Proof. Assume that G satisfies the stated condition for all distinct vertices $u, v, w, x \in V(G)$.

If G is acyclic, then G contains no path on three edges, so G is a star—that is, the complete bipartite graph $K_{1,n-1}$.

Now suppose that G is not acyclic. Then the girth of G (the length of a shortest cycle in G) is either 3 or 4.

Suppose the girth of G is 3, and let U be a maximal clique. Then $|U| \geqslant 3$. Suppose for contradiction that $uv \in E(G)$ such that $u \in U$ and $v \notin U$. Since $v \notin U$, there exists $x \in U$ such that $xv \notin E(G)$. Since $|U| \geqslant 3$, there exists $y \in U \setminus \{u, x\}$. Then $xyuv$ is a path of length 3. By assumption, xv are adjacent, which is a contradiction. So it must be the case that no other vertices in $V(G) \setminus U$ are connected to a vertex in U; since G is connected, we can conclude that G is a complete graph.

Now suppose the girth of G is 4, and let U be a maximal induced complete bipartite subgraph of G, with bipartition $U = U_1 \cup U_2$ such that $|U_1|, |U_2| \geqslant 2$. By symmetry of U_1 and U_2, suppose for contradiction $uv \in E(G)$ where $u \in U_1$ and $v \notin U$. If v was adjacent to any vertex $z \in U_2$, there would be a triangle uvz, violating the girth condition. By the maximality of U_2, $v \notin U_2$ because there exists $x \in U_1$ such that $xv \notin E(G)$. Pick any $w \in U_2$. The path $vuwx$ of length 3 implies that $xv \in E(G)$, which is a contradiction. So it must be the case that no other vertices in G are connected to a vertex in U; and since G is connected we can conclude that G is a complete bipartite graph.

The reverse implication holds by inspection. $\qquad \square$

We can now use this characterization of graphs to establish the possible Hadamard diagonalizable graphs of order $n = 8k + 4$.

Theorem 5. *Let G be a graph of order n. If $n = 8k + 4$ and G is Hadamard diagonalizable, then $G \in \{K_n, K_{n/2,n/2}, nK_1, 2K_{n/2}\}$.*

Proof. Suppose for the sake of contradiction that $n = 8k + 4$, G is a Hadamard diagonalizable graph of order n, and $G \notin \{K_n, K_{n/2,n/2}, nK_1, 2K_{n/2}\}$. Let L be the Laplacian matrix of G. Then there exists a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where λ_k is an eigenvalue of L and λ_k is an even integer, for all $k = 1, \ldots, n$ (see Proposition 2) and

$$L = \frac{1}{n} H \Lambda H^T = \frac{1}{n} \sum_{k=1}^{n} \lambda_k h_k h_k^T,$$

for some $n \times n$ Hadamard matrix H, with columns $h_1, h_2, \ldots, h_k$. For any $i, j \in \{1, \ldots, n\}$,

$$L_{ij} = \frac{1}{n} \sum_{k=1}^{n} \lambda_k (h_k)_i (h_k)_j$$

where the notation $(h_k)_i$ refers to the i-th entry of the vector h_k.

If G is not connected, then the complement G^c is connected. By Lemma 4, then, we have that G or G^c contains a path of length 3 whose endpoints are not adjacent (note that since G must be regular, the only possible connected complete bipartite graph is $K_{n/2,n/2}$). Without loss of generality, we assume $uvwx$ is a path of length 3 in G. Since $L_{uv} = L_{vw} = L_{wx} = -1$ and $L_{ux} = 0$, we have

$$
\begin{aligned}
-3n &= n(L_{uv} + L_{vw} + L_{wx} + L_{ux}) \\
&= \sum_{k=1}^{n} \lambda_k ((h_k)_u (h_k)_v + (h_k)_v (h_k)_w + (h_k)_w (h_k)_x + (h_k)_u (h_k)_x) \\
&= \sum_{k=1}^{n} \lambda_k ((h_k)_u + (h_k)_w)((h_k)_v + (h_k)_x).
\end{aligned}
$$

Since each λ_k is even and each $h_{ij} \in \{-1, 1\}$, it follows that each term in the sum is divisible by 8, meaning that 8 divides the right hand side. This implies that n is a multiple of 8. But that contradicts the assumption that $n = 8k + 4$, concluding the proof. $\qquad\square$

The preceding result shows that if a graph is Hadamard diagonalizable of order $n = 8k + 4$ it must be one of the graphs mentioned. We now must argue that all four of these graphs are realizable, and that they are realizable by any normalized Hadamard matrix H.

Proposition 6. *If H is a normalized Hadamard matrix of order $n \geqslant 4$, then the graphs K_n, $K_{n/2,n/2}$, nK_1, and $2K_{n/2}$ are Hadamard diagonalizable by H.*

Proof. Let H be a normalized Hadamard matrix of order $n \geqslant 4$. Thus H is a Hadamard matrix where h_1 is the all 1s vector and h_2 has exactly $\frac{n}{2}$ 1s and $\frac{n}{2}$ -1s. It suffices to show how to write L as a linear combination of the projection matrices $h_k h_k^T$ for the graphs K_n and $2K_{n/2}$ (since this will have the Laplacian with the correct eigenvalues).

For $G = K_n$ we have

$$
L = \sum_{k=1}^{n} h_k h_k^T - h_1 h_1^T,
$$

since the sum becomes nI and $h_1 h_1^T$ is the all-ones matrix which we denote by J.

For $G = 2K_{n/2}$ we have

$$
L = \frac{1}{2} \sum_{k=1}^{n} h_k h_k^T - \frac{1}{2} h_1 h_1^T - \frac{1}{2} h_2 h_2^T,
$$

since the sum becomes $n2I$ and the last two terms combine to give a matrix which is permutationally equivalent to the matrix $-\begin{pmatrix} J & O \\ O & J \end{pmatrix}$ (note that this permutation similarity is equivalent to a relabelling of the vertices so that those in one clique correspond to the $+1$s in h_2 and the vertices in the other clique correspond to the -1s). $\qquad\square$

Remark 7. This result also follows from Observations 1 and 2 in [1] (in which it is shown that K_n and $K_{n/2,n/2}$ are Hadamard diagonalizable) and the fact that the complement of a Hadamard diagonalizable graph is also Hadamard diagonalizable. We note that the statement of Proposition 6 is a slightly stronger statement than saying that the four graphs are Hadamard diagonalizable, as it makes it explicit that no matter the H we are using, as long as it is normalized, these four graphs are diagonalized by H. In particular this says that any normalized Hadamard matrix will always diagonalize at least four graphs.

3 Finding all graphs diagonalizable by a given Hadamard matrix

In this section, we describe a procedure to search for and produce Hadamard diagonalizable graphs. In particular, given a Hadamard matrix H, we give an algorithm by which all graphs which are diagonalized by H are produced. We assume that H is a normalized Hadamard matrix, since every graph which is Hadamard diagonalizable is also diagonalized by a normalized Hadamard matrix (see [1, Lemma 4]). We note that two inequivalent normalized Hadamard matrices may produce the same graph via this procedure.

3.1 An algorithmic procedure

Proposition 8. *Let G be a Hadamard diagonalizable graph with its Laplacian matrix L. Let H be a normalized Hadamard matrix diagonalizing L. Let Λ be the diagonal matrix with its diagonal entries $\lambda_1 = 0, \lambda_2, \ldots, \lambda_n$, the eigenvalues of L corresponding to the columns of H as their associated eigenvectors. Then the entries $L_{12}, \ldots, L_{1n}$ uniquely determine $\lambda_2, \ldots, \lambda_n$.*

Proof. Suppose that G is a Hadamard diagonalizable graph, and let H be a normalized Hadamard matrix such that $L = \frac{1}{n} H \Lambda H^T$. It follows that the first row of LH is equal to $\begin{bmatrix} 0 & \lambda_2 & \cdots & \lambda_n \end{bmatrix}$. Hence

$$\lambda_j = \sum_{u \sim 1} (1 - H_{uj}),$$

a quantity determined precisely by the vertices of G adjacent to vertex 1; i.e. determined by the entries $L_{12}, \ldots, L_{1n}$ of L and H_{uj} of H. $\square$

The preceding can be used to give a new proof that all Laplacian eigenvalues of a Hadamard diagonalizable graph are even integers (see Proposition 2, originally proven in [1]).

Proof that the eigenvalues are even integers. We have

$$\lambda_j = \sum_{u \sim 1} (1 - H_{uj}),$$

and each entry in the sum is either 0 or 2; hence λ_j is even. $\square$

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1
\end{bmatrix}$$

Table 2: Hadamard matrix `had.16.1`.

Suppose we are given an $n \times n$ normalized Hadamard matrix H and wish to find all graphs which are Hadamard diagonalizable by H. Using Proposition 8, we can narrow our search space down to size 2^{n-1} by looking at all possible $\{0, -1\}$ assignments to $L_{12}, \ldots, L_{1n}$, and rewriting all of the off-diagonal entries of L as linear combinations of $L_{12}, \ldots, L_{1n}$. This rewrite can be done because each entry is some linear combination of the $\lambda_2, \ldots, \lambda_n$, while the proof of Proposition 8 shows that each of the λ_i is a linear combination of $L_{12}, \ldots, L_{1n}$. Then every assignment will produce a matrix via these linear combinations, though not every assignment will correspond to a graph, as there might be other entries $L_{ij} \notin \{0, -1\}$. For any assignment of the values $\{0, -1\}$ to the 'variables' $L_{12}, \ldots, L_{1n}$, the goal will be to determine if the linear combinations that arise elsewhere in the matrix (off the diagonal) are all equal to either 0 or -1; if so, that assignment produces a graph. The search could also turn up isomorphic copies of the same graph. Note that many distinct entries of the Laplacian may be expressed using the same linear combination of the variables $L_{12}, \ldots, L_{1n}$.

To illustrate this, we carry this procedure out for the Hadamard matrix `had.16.1` from Sloane [8], see Table 2, to produce an auxiliary matrix determining the linear combinations. For the 120 entries above the diagonal of a possible Laplacian matrix (by symmetry the entries below the diagonal will be equal) there were 27 distinct linear combinations produced. The auxiliary matrix is given in Table 3 where the $(i, j)^{th}$ entry corresponds to the coefficient of L_{1j} in the ith linear combination. For notational convenience, we have labeled the sixteen rows and columns of L using hexadecimal symbols $\{0, 1, \cdots, 9, A, B, \cdots, F\}$ to more easily indicate which entries of L correspond to the ith linear combination in the accompanying table.

Looking at the auxiliary matrix in Table 3, the identity matrix induced on the first 15 rows is a reflection that the linear combination for an entry from the first row is trivial.

Let us view an assignment of $L_{12}, \ldots, L_{1n}$ to values of 0 or -1 as selecting some subset of the columns (so if L_{1j} is -1 take the jth column; if it is 0 do not take the column). Then

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & -\tfrac{1}{2} \\
0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\tfrac{1}{2} & \tfrac{1}{2} \\
0 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & -\tfrac{1}{2} & 0 & 0 \\
0 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & -\tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} \\
0 & 0 & 0 & 0 & 0 & -\tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} \\
0 & 0 & 0 & \tfrac{1}{2} & -\tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 \\
0 & 0 & 0 & -\tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 \\
0 & \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & -\tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & -\tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & \tfrac{1}{2} & -\tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & -\tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Row	Entries	Row	Entries	Row	Entries
1	$01, 23, 45, 67, 89, AB, CD, EF$	10	$0A, 1B, 28, 39$	19	$27, 36, AF, BE$
2	$02, 13, 8A, 9B$	11	$0B, 1A, 29, 38$	20	$2C, 3D, 4A, 5B$
3	$03, 12, 8B, 9A$	12	$0C, 1D, 48, 59$	21	$2D, 3C, 4B, 5A$
4	$04, 15, 8C, 9D$	13	$0D, 1C, 49, 58$	22	$2E, 3F, 6A, 7B$
5	$05, 14, 8D, 9C$	14	$0E, 1F, 68, 79$	23	$2F, 3E, 6B, 7A$
6	$06, 17, 8E, 9F$	15	$0F, 1E, 69, 78$	24	$46, 57, CE, DF$
7	$07, 16, 8F, 9E$	16	$24, 35, AC, BD$	25	$47, 56, CF, DE$
8	$08, 19, 2A, 3B, 4C, 5D, 6E, 7F$	17	$25, 34, AD, BC$	26	$4E, 5F, 6C, 7D$
9	$09, 18, 2B, 3A, 4D, 5C, 6F, 7E$	18	$26, 37, AE, BF$	27	$4F, 5E, 6D, 7C$

Table 3: The auxiliary matrix for the Hadamard matrix `had.16.1` where each row corresponds with a distinct linear combination appearing in L in terms of the off-diagonal entries in the first row. In the accompanying table, we indicate for each row which entries L_{ij} correspond to this linear combination, with i, j in hexadecimal.

this will produce a Laplacian matrix for a graph *if and only if* the sum of the columns produce a vector with entries in $\{0,1\}$, since this is only the case when the off-diagonal entries will be 0 and -1.

To run through all possible assignments of $L_{12}, \ldots, L_{1n}$, we need to consider all subsets of columns of the auxiliary matrix. However, to reduce the search space, we proceed via a tree-like exploration of the space, where at each step we decide to either add or not add a particular column. After we add a new column to our subset, we then do the following check: if for each entry there is a possibility that the sum of some combination of the remaining columns can result in the value being 0 or 1, proceed; if not then we 'prune the tree' and don't explore any further on that branch.

For example, if we take the columns 2, 3, and 11 in the matrix in Table 3, then the last entry in the sum of these column vectors will be $1/2$. The last entries in the remaining columns which we could add to our subset are all equal to 0, so no matter which combination of columns $12, 13, 14, 15$ we take we can never change that value from $1/2$, and so there is no need to explore that part of the space. To get the most out of this, it is useful to first pre-sort the columns so that such conflicts will arise early.

If we get down to a leaf in the tree and the resulting combination of columns is a 0-1 vector, then we have found a Hadamard diagonalizable graph. To produce the graph we find where the 1s are located and the corresponding Laplacian entries to which they correspond. These corresponding Laplacian entries represent the edges in the graph. As an example if we take the sum of the first three columns in Table 3 then this will produce a 1 in rows $1, 2, 3, 24$, and 25 of the resulting vector. So this will be the graph on the vertex set with vertices $\{0, 1, \ldots, F\}$ and with edges

$$\underbrace{01, 23, 45, 67, 89, AB, CD, EF}_{\text{row 1}}, \underbrace{02, 13, 8A, 9B}_{\text{row 2}}, \underbrace{03, 12, 8B, 9A}_{\text{row 3}}, \underbrace{46, 57, CE, DF}_{\text{row 24}}, \underbrace{47, 56, CF, DE}_{\text{row 25}}$$

which becomes the graph $4K_4$ (cliques on the vertices $0, 1, 2, 3$; and $4, 5, 6, 7$; and $8, 9, A, B$; and C, D, E, F). As graphs are found they are tested to see whether they have been seen before, and we only keep those graphs which have not been seen before; this can be done, for example, by using canonical labeling methods.

The procedure outlined here was implemented in both `SageMath` and `C++` with all computations done using integer variables. The only external call needed is to determine which graphs are discovered up to isomorphism. The program can be downloaded at http://lidicky.name/pub/hadamard/.

3.2 Equivalency of Hadamard matrices

We know that if G is Hadamard diagonalizable, then it is Hadamard diagonalizable by some normalized Hadamard matrix. However, given a normalized Hadamard matrix H, there are many other normalized Hadamard matrices which are equivalent to H. It is not the case that if G is diagonalized by H that it is also diagonalizable by any H' equivalent

to H. For example, consider the standard normalized Hadamard matrix of order 4:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

This Hadamard diagonalizes the complete graph K_4. However, an equivalent Hadamard matrix is obtained by negating the second row:

$$H' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Note that every Laplacian matrix has 0 as an eigenvalue, with an eigenvector proportional to the all-ones vector. Since the columns of a Hadamard matrix that diagonalize a graph represent the eigenvectors of that graph's Laplacian, and this matrix H' has no constant column, it is clear that H' does not diagonalize any connected graph.

We suggest, however, that it may be possible to show the following:

Question 9. Let H_1 and H_2 be equivalent normalized Hadamard matrices. Is it true that G is Hadamard diagonalizable by H_1 if and only if G (up to some relabeling) is Hadamard diagonalizable by H_2?

If true, this would significantly shorten the computational time. We note, in addition, that of the four operations by which an equivalent Hadamard matrix is produced, three of them preserve the graphs diagonalized by that Hadamard matrix. Permuting columns of H corresponds to a permutation of the eigenvectors of L; permuting rows corresponds simply to a relabelling of the vertices of the graph. Negating columns does not change the eigenspaces, it simply scales our representative eigenvectors of L.

4   Hadamard diagonalizable graphs of small order

In this section, we present complete lists of all Hadamard diagonalizable graphs for all orders $n \in \{1, 2, 4, 8, 12, 16, 20, 24, 28, 32, 36\}$, obtained using the theoretical tools of Section 2 and the computational tools of Section 3. Hadamard diagonalizable graphs are characterized in the literature up to order 12 in Barik, Fallat, and Kirkland [1], but we present these as well for completeness. The most significant contributions here on the computational side are for orders $n = 16, 24, 32$.

We will use the following notations:

- G^c is the graph complement of G.

- $G + H$ is the disjoint union of the graphs G and H. In particular, we denote by kG the graph consisting of k disjoint copies of G.

- $G\square H$ is the Cartesian product of G and H. That is $V(G\square H) = V(G) \times V(H)$ and $(u, x), (v, y) \in V(G\square H)$ are adjacent if and only if $u = v$ and $xy \in E(H)$ or $x = y$ and $uv \in E(G)$.

- $G \vee H$ is the join of G and H. The join is obtained from $G + H$ by adding all edges uv, where $u \in V(G)$ and $v \in V(H)$.

- $G \circ H$ is the *lexicographic product* of G with H; that is, the graph formed by replacing each vertex of G with a copy of H, and adding all possible edges between the vertices in the copies of H corresponding to adjacent vertices in G (a form of a blow-up). This product is sometimes referred to as *graph composition*. It has also occasionally appeared in the past as a *wreath product*, due to its connection with wreath products in group theory (see [10]).

- $H_{n,n}$ is the graph $K_{n,n}$ minus a perfect matching (note that $H_{4,4} = Q_3$ is the cube graph on eight vertices);

- CP_{2n} is the cocktail party graph on $2n$ vertices formed by taking the complete graph and removing a perfect matching, so $CP_{2n} = K_{2,2,\ldots,2}$.

- Let $\mathcal{G}$ be a finite group, and let S be some subset of the elements of $\mathcal{G}$. Then $\mathcal{G}(S)$ is a *Cayley graph* with vertices representing elements of the group $\mathcal{G}$, and an edge between u and v whenever $u - v \in S$.

There are many ways to write some of the graphs in what follows, and in the interest of future theoretical research in this area (that is, a pursuit of theoretical characterizations of Hadamard diagonalizable graphs), we will often give several isomorphic representations of the same graph. In particular, many can be written as a Cartesian or lexicographic product of two graphs. This is pursued in earnest for the graphs of order 24, as we suspect (based on these and the preliminary data available for orders 40 and 56) that in a manner similar to orders $8k + 4, k \geqslant 0$, there are at most 26 distinct graphs which are Hadamard diagonalizable of order $16k + 8$ (see more discussion in Question 11).

It is shown in [1] that $G + G \cong 2K_1 \square G$ and $G \vee G \cong K_2 \circ G$ are Hadamard diagonalizable graphs if G is Hadamard diagonalizable, and that the Cartesian product of two Hadamard diagonalizable graphs is also Hadamard diagonalizable. It is also shown in [4] that the direct product of two Hadamard diagonalizable graphs is also Hadamard diagonalizable. We now show that the lexicographic product of two Hadamard diagonalizable graphs is also Hadamard diagonalizable, giving further methods to construct Hadamard diagonalizable graphs.

Lemma 10. *Let G_1 and G_2 be Hadamard diagonalizable. Then $G_1 \circ G_2$ is Hadamard diagonalizable.*

Proof. Assume that the graph G_1 on m vertices is diagonalized by a normalized Hadamard matrix H_1, and the graph G_2 on n vertices is diagonalized by a normalized Hadamard matrix H_2. Since Hadamard diagonalizable graphs are regular, a Hadamard matrix H

diagonalizes the Laplacian of a graph $L(G)$ if and only if it also diagonalizes the adjacency matrix, which we denote $A(G)$. We show that the adjacency matrix of the lexicographic product of G_1 and G_2, $A(G_1 \circ G_2)$, is diagonalizable by the Hadamard matrix $H_1 \otimes H_2$, where $\otimes$ denotes the Kronecker product (or tensor product) of two matrices.

Assume that

$$H_1^{-1} A(G_1) H_1 = \Lambda_1, \quad \text{and} \quad H_2^{-1} A(G_2) H_2 = \Lambda_2.$$

The adjacency matrix of the lexicographic product can be written

$$A(G_1 \circ G_2) = I_m \otimes A(G_2) + A(G_1) \otimes J_n,$$

where I_m is the identity of order m and J_n is the $n \times n$ matrix of all ones.

For any normalized Hadamard matrix H of size n, $H^{-1} J_n H = n e_1 e_1^T = n E_{1,1}$, where e_1 is the vector with a 1 in the first position and zeros elsewhere, and $E_{1,1}$ is a matrix of all-zeros except a 1 in the $(1,1)$ position. Using this and the fact that the Kronecker product is bilinear and satisfies the properties, $(AB) \otimes (CD) = (A \otimes C)(B \otimes D)$ and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, we have

$$
\begin{aligned}
(H_1 \otimes H_2)^{-1} A(G_1 \circ G_2)(H_1 \otimes H_2) &= (H_1 \otimes H_2)^{-1}(I_m \otimes A(G_2) + A(G_1) \otimes J_n)(H_1 \otimes H_2) \\
&= (H_1^{-1} I_m H_1) \otimes (H_2^{-1} A(G_2) H_2) \\
&\qquad + (H_1^{-1} A(G_1) H_1) \otimes (H_2^{-1} J_n H_2) \\
&= (I_m \otimes \Lambda_2) + (n\Lambda_1 \otimes E_{1,1}),
\end{aligned}
$$

which is a diagonal matrix. $\qquad\square$

Note that Lemma 1.7 of [2] indicates that the tensor product of a matrix diagonalizable by H_1 with a matrix diagonalizable by H_2 is Hadamard diagonalizable, and so is any linear combination of such matrices, so the latter half of the proof above follows from this result, once it is observed that the adjacency matrix is of the given form, that I_m and $A(G_1)$ are diagonalizable by H_1, and that $A(G_2)$ and J_n are diagonalizable by H_2.

We also note that for any given Hadamard diagonalizable graphs G_1 and G_2 with corresponding diagonalizing Hadamard matrices H_1 and H_2, respectively, it is interesting to see that the Hadamard matrix $H_1 \otimes H_2$ diagonalizes both $G_1 \square G_2$ and $G_1 \circ G_2$. As two graphs $G_1 \square G_2$ and $G_1 \circ G_2$ are nonisomorphic, in general, they may admit different Hadamard matrices as their diagonalizing matrices as well. See for example, $K_2 \square K_{6,6}$ and $K_2 \circ K_{6,6}$ in Table 6 below.

4.1 Order 1

The graph K_1 is Hadamard diagonalizable.

4.2 Order 2

Both graphs K_2 and $2K_1$ are Hadamard diagonalizable by the unique normalized Hadamard matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

4.3 Orders 4, 12, 20, 28 and 36

From the results of Section 2, the only Hadamard diagonalizable graphs are K_n, $K_{n/2,n/2}$, $2K_{n/2}$, and nK_1 for $n \in \{4, 12, 20, 28, 36\}$.

4.4 Order 8

There is a unique (up to equivalence) normalized Hadamard matrix of order 8, and it is the Sylvester construction. The graphs diagonalizable by a Sylvester Hadamard matrix have been characterized in [4], and for order 8 consist of all Cayley graphs for $\mathbb{Z}_2^3$. We can also represent them as follows: Note that all of them can be expressed in terms of

Graph	Graph complement
$K_2 \circ K_4 \cong K_8$	$2K_1 \square 4K_1 \cong 2(4K_1)$
$K_2 \circ K_{2,2}$	$K_2 \square 4K_1 \cong 2K_1 \square 2K_2 \cong 2(2K_2)$
$K_2 \circ 2K_2$	$K_2 \square 2K_2 \cong 2K_1 \square K_{2,2} \cong 2K_{2,2}$
$K_2 \circ 4K_1 \cong K_{4,4}$	$2K_1 \square K_4 \cong 2K_4$
$K_2 \square K_4$	$K_2 \square K_{2,2} \cong (K_2)^{\square 3} \cong Q_3$

Table 4: Hadamard diagonalizable graphs of order 8.

products of the Hadamard diagonalizable graphs of orders 2 and 4.

4.5 Order 16

There are five non-equivalent normalized Hadamard matrices of order 16. We will follow Sloane [8] and denote them by `had.16.j` for $j \in \{0, 1, 2, 3, 4\}$; note that `had.16.0` is the Sylvester construction. We use the computational tools of Section 3 to produce all non-isomorphic graphs diagonalized by one (or more) of these Hadamard matrices.

There are a total of 50 non-isomorphic Hadamard diagonalizable graphs on 16 vertices, all of which are Cayley graphs. The graphs are given in Table 5. Many of these graphs, not all, can be identified as the products of smaller Hadamard diagonalizable graphs, products of order 2 and order 8, products of order 4 and identical or another graph of order 4, etc. Graphs come in pairs (namely the graph and its complement) and we sometimes only present one of the graphs (to get the other take the complement, for which a simple product notation is not available). We also provide the Cayley expression for some graphs, including the strongly regular graphs, Shrikhande graph, and its cospectral mate, the $(2, 4)$-Hamming graph $K_4 \square K_4$.

The column indicating 'Family' presents information regarding the graphs which are diagonalized by the same Hadamard matrices—that is, if G_1 and G_2 are in the same family, then a normalized Hadamard matrix `had.16.j` diagonalizing the Laplacian of G_1 will also diagonalize the Laplacian of G_2. We now list which graphs are associated with which Hadamard matrices as follows:

- 46 graphs for `had.16.0` are from families A, B, C, D

- 50 graphs for `had.16.1` are from families A, B, C, D, E

- 48 graphs for `had.16.2` are from families A, B, C, E

- 24 graphs for `had.16.3` are from families A, B

- 10 graphs for `had.16.4` are from family A

Family	Graph	Graph complement
A	K_{16}	$16K_1$
A	$K_{8,8}$	$2K_8$
A	$2K_{4,4}$	$K_2 \circ 2K_4$
A	$4K_4$	$K_{4,4,4,4}$
A	$(K_2 \square K_4) \circ (2K_1)$	$H_{4,4} \circ K_2$
B	$8K_2$	$K_8 \circ 2K_1$
B	$4K_{2,2}$	$K_4 \circ (2K_2)$
B	$K_2 \circ 4K_2$	$2K_1 \circ (K_4 \circ 2K_1)$
B	$K_2 \circ 2K_{2,2}$	$2K_1 \circ (K_2 \circ 2K_2)$
B	$(K_2 \square K_4) \circ K_2$	$H_{4,4} \circ K_2^c$
B	$K_2 \square (K_4 \circ 2K_1)$	
B	$K_2 \square K_8$	
C	$K_{4,4} \square K_2$	
C	$2(K_4 \square K_2)$	
C	$2(K_{2,2} \circ K_2)$	
C	$K_{2,2} \square K_{2,2}$	
C	$K_{2,2} \circ K_4$	
C	$2H_{4,4}$	
C	$\mathbb{Z}_2^4(\{(0,0,0,1),(0,0,1,0),(0,1,0,0),(1,0,0,0),(1,1,1,1)\})$	
C	$\mathbb{Z}_2^4(\{(0,0,0,1),(0,0,1,0),(0,1,0,0),(1,0,0,0),(0,0,1,1),(1,1,0,1)\})$	
C	$\mathbb{Z}_2^4(\{(0,0,0,1),(0,0,1,0),(0,1,0,0),(1,0,0,0),(0,0,1,1),(1,1,0,0)\})$ ((2,4)-Hamming)	
C	$\mathbb{Z}_2^4(\{(0,0,0,1),(0,0,1,0),(0,1,0,0),(1,0,0,0),(0,0,1,1),(0,1,0,1),(1,1,1,0)\})$	
D	$\mathbb{Z}_2^4(\{(0,0,1,1),(0,1,0,0),(0,1,0,1),(0,1,1,0),(0,1,1,1),(1,0,0,0),(1,1,0,0)\})$	
E	$\mathbb{Z}_4^2(\{(0,1),(0,-1),(1,0),(-1,0),(1,1),(-1,-1)\})$ (Shrikhande)	
E	$\mathbb{Z}_4^2(\{(0,1),(0,-1),(1,0),(-1,0),(1,1),(-1,-1),(2,2)\})$	

Table 5: Hadamard diagonalizable graphs of order 16.

4.6 Order 24

There are 60 non-equivalent normalized Hadamard matrices of order 24. We will follow Sloane [8] and denote them by `had.24.`j for $j \in \{1, 2, \ldots, 60\}$.

There are a total of 26 Hadamard diagonalizable graphs on 24 vertices, all of which are Cayley graphs. The graphs are given in Table 6. Graphs are presented in complementary pairs, with multiple representations for each. The graph $12K_2$ as a subgraph of G will be denoted by M (a matching) and the graph $G - M$ refers to the graph obtained by the removal of the edges of a matching from G. Again, we indicate equivalence classes in the column 'Family', where graphs from the same family are diagonalized by the same

Hadamard matrices. Note that we use the notation $G \times H$ when $G \circ H$ is isomorphic to $G \square H$.

Family	Graph	Graph complement	
A	$K_2 \circ K_{12} \cong K_{24}$	$2K_1 \times 12K_1 \cong 24K_1$	
A	$K_2 \circ 12K_1 \cong K_{12,12}$	$2K_1 \times K_{12} \cong 2K_{12}$	
B	$K_2 \circ K_{6,6} \cong K_{6,6,6,6}$	$2K_1 \times 2K_6 \cong 4K_6$	
B	$K_2 \circ 2K_6 \cong K_{2,2} \circ K_6$	$2K_1 \times K_{6,6} \cong 2K_{6,6}$	
B	$(K_2)^{\square 3} \circ K_3 \cong Q_3 \circ K_3$	$(K_4 \square K_2) \circ 3K_1$	$*$
C	$K_2 \square 12K_1 \cong 12K_2$	$K_2 \circ K_{12} - M \cong CP_{24}$	
C	$K_2 \square K_{12}$	$K_2 \circ 12K_1 - M \cong H_{12,12}$	
C	$K_{6,6} \circ K_2$	$2K_6 \circ 2K_1 \cong 2K_{12} - M$	
D	$K_2 \square K_{6,6}$	$K_2 \circ 2K_6 - M \cong K_{2,2} \circ K_6 - M$	
D	$K_2 \square 2K_6$	$K_2 \circ K_{6,6} - M$	
D	$K_2 \circ (K_6 \square K_2)$	$2(H_{6,6}) \cong 2K_{6,6} - M$	$*$
D	$(K_4 \square K_2) \circ 3K_1 - M$	$Q_3 \circ K_3 + M$	$*$
D	$(K_4 \square K_2) \circ 3K_1 + M$	$Q_3 \circ K_3 - M$	$*$

Table 6: Hadamard diagonalizable graphs of order 24.

We now list which graphs are associated with which Hadamard matrices as follows:

- 26 graphs for `had.24.j` for $1 \leqslant j \leqslant 7$ are from families A, B, C, D

- 10 graphs for `had.24.8` are graphs from families A, C

- 10 graphs for `had.24.j` for $9 \leqslant j \leqslant 59$ are from families A, B

- 4 graphs for `had.24.60` are from family A

One interesting thing to note is that the graph $2K_1 \circ (K_6 \circ 2K_1)$ (i.e. two disjoint copies of the cocktail party graph CP_{12}) is Hadamard diagonalizable by a Hadamard matrix of order 24, but the cocktail party graph $CP_{12} \cong K_6 \circ 2K_1$ is *not* diagonalized by the unique Hadamard matrix of order 12. In fact, those graphs in the table that are marked by an asterisk can be expressed as products of smaller graphs but some of their factors are not necessarily Hadamard diagonalizable ones. However, all remaining graphs are coming as the products of Hadamard diagonalizable graphs of smaller orders as expressed in the product notation.

4.7 Order 32

The calculation for order 32 is much more involved, as the search space for each individual matrix grows substantially. In addition, the number of Hadamard matrices of order 32 is far greater—there are 13,710,027 non-equivalent normalized Hadamard matrices of order 32. To run the computation, a program was written in `C++` and used on `nauty` [7] (for

graph isomorphism testing) and `parallel` [9] (to speed up the computation). The calculation was performed on a server maintained by the Department of Applied Mathematics at Charles University in Prague. The calculation took 179,736,390 seconds of CPU time, which was about 2 months of real time due to parallel processing. If the answer to Question 9 were positive, the calculation would take only 2 days. The source code, inputs, and outputs can be downloaded from http://lidicky.name/pub/hadamard/; this includes all Hadamard matrices stored as strings, all Hadamard diagonalizable graphs stored as `graph6-strings`, and an additional file that can be used to determine which graphs are associated with which matrix.

There are a total of 10,196 different (i.e. non-isomorphic) Hadamard diagonalizable graphs, and unlike smaller orders, many of them are not Cayley graphs. We can partition the Hadamard matrices according to which graphs they diagonalize: the result is 53,420 different equivalence classes. In Figure 1 we mark the distribution of these equivalence classes in log-log scale; each point is an equivalence class.
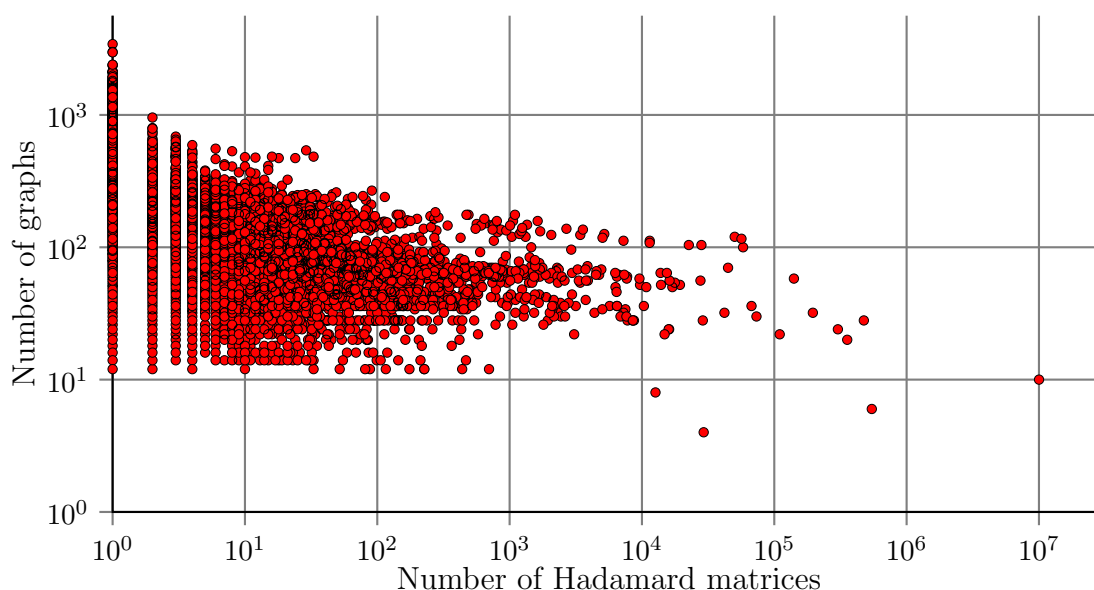

Figure 1: The sizes of the equivalence classes of Hadamard matrices based on which graphs they diagonalize.

Here are a few additional notes about the order 32 Hadamard diagonalizable graphs and the equivalence classes; much remains to be explored.

- The equivalence class corresponding with the fewest number of graphs is associated with only four graphs (K_{32}, $K_{16,16}$, $2K_{16}$, $32K_1$); there are 29,270 Hadamard matrices in this group. These Hadamard matrices *only* diagonalize these four graphs—in particular, the four graphs listed here are the only ones which are universally Hadamard diagonalizable for all Hadamard matrices of order 32.

- The equivalence class corresponding with the greatest number of graphs is associated with 3,430 graphs, and it has a unique Hadamard matrix in the class. This

Hadamard matrix is represented pictorially in Figure 2(a).

- There are 26,064 different equivalence classes which consist of a single Hadamard matrix. The number of graphs associated with these equivalence classes range from 12 at the low end, up to 3,430 at the high end. All Hadamard matrices which diagonalize over 956 graphs are in equivalence classes of size 1; on the other hand there does exist an equivalence class associated with 956 Hadamard diagonalizable graphs which has two non-equivalent Hadamard matrices in the class.

- The largest equivalence class consists of 10,012,656 Hadamard matrices (out of a possible 13,710,027). There are ten graphs associated with this equivalence class: $(32K_1,\ 4K_8,\ 2K_{16},\ K_{32},\ 2K_{8,8},\ K_{16,16},\ K_{8,8,8,8},\ K_2 \circ (2K_8),\ H_{4,4} \circ K_4,\ (K_4 \square K_2) \circ (4K_1))$.

- There are 970 Hadamard diagonalizable graphs for which each graph is associated with a *unique* equivalence class; moreover for 966 of these graphs the equivalence class has size 1. These 966 graphs are spread among 13 different Hadamard matrices; 92 of these graphs are associated with the Hadamard matrix in Figure 2(a) and 224 of these graphs are associated with the Hadamard matrix in Figure 2(b).
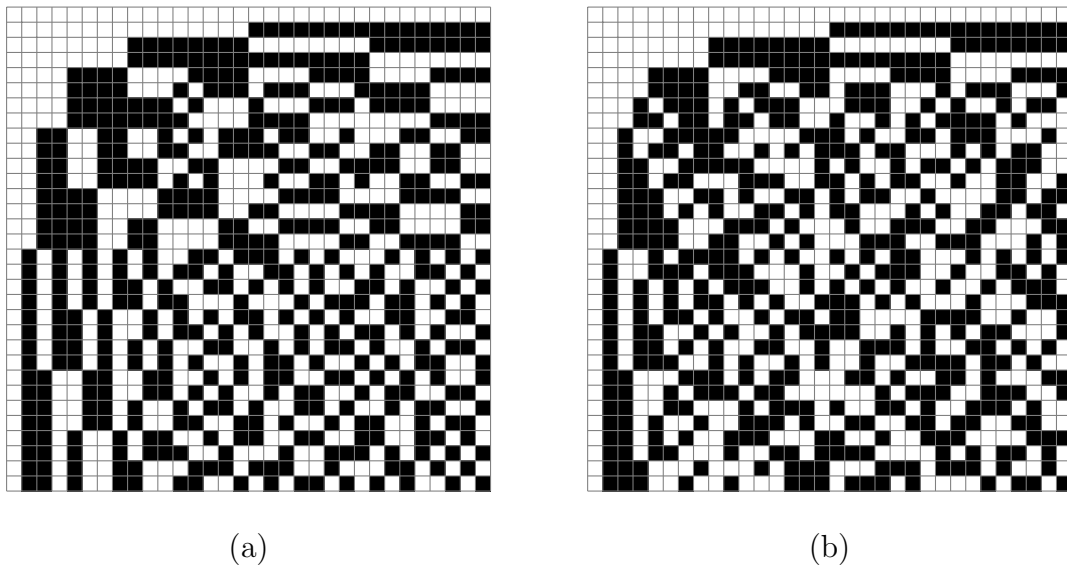

(a)   (b)

Figure 2: Two Hadamard matrices of order 32 presented pictorially with white cells corresponding with 1 and black cells with -1. The Hadamard matrix in (a) diagonalizes $3,430$ graphs (92 unique to this matrix). The Hadamard matrix in (b) diagonalized $1,684$ graphs (224 unique to this matrix).

Among the 10,196 Hadamard diagonalizable graphs of order 32 we have the following data. (Recall that both a graph and its complement are always diagonalized by the same Hadamard matrix.)

- The graphs are regular, and the degrees of the graphs, denoted $d(G)$, are distributed as follows (we only give information up through degree 15, the rest follow by symmetry):

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\|\{G : d(G)=k\}\|$	1	1	1	2	4	6	16	29	56	101	208	343	584	877	1241	1628

- The clique number of a graph, denoted $\nu(G)$, is the size of the largest complete subgraph and its distribution for this class of graphs is as follows:

k	1	2	3	4	5	6	7	8	9	10	16	32
$\|\{G : \nu(G)=k\}\|$	1	53	43	4115	1205	1847	443	2435	1	8	44	1

  Note that taking complements sends cliques to independent sets and so this also gives information about the sizes of maximal independent sets in graphs.

- There are 10,142 graphs which have girth 3, 51 graphs which have girth 4, 1 graph which has girth 6, and 2 graphs which have no cycles.

- There are 54 disconnected graphs; among the remaining 10,142 connected graphs the diameter of the graph, denoted $\mathrm{diam}(G)$, which is the maximal distance between two vertices is distributed as follows:

k	1	2	3	4	5
$\|\{G : \mathrm{diam}(G)=k\}\|$	1	9001	1128	11	1

- By Proposition 2 we know that the eigenvalues for the Laplacian of these graphs consist of even integers. There are 1,228 distinct spectra which are achieved. For 518 of these graphs the spectrum is unique among these graphs (e.g. no other graph from among this list has the same spectrum); the remaining 9,678 graphs each have one or more cospectral mates in the list. The largest cospectral family is for the spectra $\{0^{(1)}, 12^{(10)}, 16^{(15)}, 20^{(6)}\}$ and $\{0^{(1)}, 12^{(6)}, 16^{(15)}, 20^{(10)}\}$ (here exponents represent multiplicity); each family having 528 distinct graphs with that spectra.

  The algebraic connectivity of a graph is the second smallest eigenvalue (counting multiplicity) of the Laplacian matrix of a graph. The algebraic connectivity, $\alpha(G)$, for the graphs are as follows:

k	0	2	4	6	8	10	12	14
$\|\{G : \alpha(G)=k\}\|$	54	56	398	604	2241	1771	3231	822

k	16	18	20	22	24	26	28	30	32
$\|\{G : \alpha(G)=k\}\|$	774	122	88	17	12	2	2	1	1

- 4,130 of the graphs are vertex-transitive; 45 of the graphs are edge-transitive; 38 of the graphs are distance-regular; 32 of the graphs are cographs; and 6 of the graphs are chordal (namely those which are unions of cliques).

5 Concluding remarks

The obstacles to moving forward with larger Hadamard matrices are the size of the computations for any individual Hadamard matrix, combined with a lack of a classification of all Hadamard matrices of order 36 or above. However, we can run the computation on some known Hadamard matrices of higher orders and we summarize the computation results in Table 5.

Hadamard matrix	Number of H. graphs
`had.40.tpal`	26
`had.40.ttoncheviv`	26
`had.40.twill`	26
`had.48.pal`	4
`had.56.tpal2`	26
`had.56.twll`	26

Table 7: Some Hadamard matrices from Sloane [8] and the number of graphs for which that matrix Hadamard diagonalizes the graph.

For the three Hadamard matrices of order 40 the 26 graphs are the same; similarly, for the two Hadamard matrices of order 56. The data, combined with what we know for order 24, suggest the following question.

Question 11. For $n = 16k + 8, k = 1, 2, \ldots$, there are at most 26 distinct graphs which are Hadamard diagonalizable for some Hadamard matrix of order n.

An affirmative answer to this question would come from finding an argument similar to the one used for order $n = 8k + 4$. A negative answer to this question would likely come from computations on additional Hadamard matrices, say of order 40 or order 56, to find additional graphs. (The existing code can readily handle Hadamard matrices of this order.)

We have seen that there exist Hadamard diagonalizable graphs of order $8k + 4$ for all $k < 250$ except for the 13 values of k, namely, $k = 83, 89, 111, 125, 141, 155, 173, 179, 209,$ $221, 239, 243$ and 245 for each of which it is not known whether there exists a Hadamard matrix of order $8k + 4$ as of 2018. Thus we see the existence of Hadamard diagonalizable graphs of various orders. For instance, we know that there are many Hadamard diagonalizable graphs of order 48, as 48 is factored as 2×24 and 4×12 and there are 26 Hadamard diagonalizable graphs of order 24, and 4 for each of order 4 and 12. On the other hand, through the computational search, the Hadamard matrix of order 48 from Sloane [8] has few Hadamard diagonalizable graphs. When we reran the computation using a Hadamard matrix generated by `SageMath` there were 762 distinct Hadamard diagonalizable graphs. Given the lack of classification for Hadamard matrices of order 48, it is not clear how to determine all Hadamard diagonalizable graphs of order 48.

Finally, as a by-product, we have the following interesting observation. We have seen that the Shrikhande graph, say S, shown in Table 6 is the Cayley graph of $\mathbb{Z}_4^2$ with connecting set $\{\pm(0,1), \pm(1,0), \pm(1,1)\}$. Both S and the $(2,4)$-Hamming graph $H(2,4) = K_4 \square K_4$ are strongly regular graph with parameters $(v, k, \lambda, \mu) = (16, 6, 2, 2)$. They are known as co-spectral graphs. Now as both of them are Hadamard diagonalizable graphs, their Cartesian and lexicographic products and powers are all Hadamard diagonalizable graphs.

The Hamming graph $H(d, q)$ is isomorphic to the Cartesian product of d copies of K_q (i.e., $H(d, q) \cong K_q^{\square d}$), which is a distance-regular graph of diameter d.[1] The Cartesian product of l copies of S and one copy of Hamming graph $H(d, 4)$ is known as a Doob graph $D(l, d)$ of diameter $2l + d$. The Doob graph $D(l, d)$, the Hamming graph $H(2l + d, 4)$, the Cartesian product of l copies of $H(2, 4)$ with $H(d, 4)$ are cospectral. As a consequence, we state this as the following:

Corollary 12. *The Hamming graphs $H(d, 4)$, $d \geqslant 1$, and Doob graphs $D(l, d)$, $l, d \geqslant 1$ are all Hadamard diagonalizable graphs.*

Acknowledgements

References

[1] S. Barik, S. Fallat and S. Kirkland, On Hadamard diagonalizable graphs, *Linear Algebra Appl.* **435** (2011), 1885–1902.

[2] A. Chan, S. Fallat, S. Kirkland, J. C.-H. Lin, S. Nasserasr and S. Plosker, Complex Hadamard diagonalizable graphs, arXiv:2001.00251v1 [math.CO].

[3] G. Coutinho and C. Godsil, Perfect state transfer in products and covers of graphs. *Linear and Multilinear Algebra*, **64** (2016), 235–246.

[4] N. Johnston, S. Kirkland, S. Plosker, R. Storey and X. Zhang, Perfect quantum state transfer using Hadamard diagonalizable graphs, *Linear Algebra Appl.* **531** (2017), 375–398.

[5] A. Kay, Basics of perfect communication through quantum networks, *Phys. Rev.* A **84** (2011), 022337.

---

[1] Here a graph G is distance-regular if for any choice of $h, i, j \geqslant 0$ and any $u, v \in V(G)$ with $d(u, v) = h$, the number of vertices $w \in V(G)$ such that $d(u, w) = i$ and $d(v, w) = j$ is independent of the choice of u and v but depends only on the choice of h, i and j.

[6] H. Kharaghani and S. Suda, Balancedly splittable Hadamard matrices. *Discrete Mathematics*, **342** (2019), 546–561.

[7] B. D. McKay and A. Piperno, Practical graph isomorphism II, *Journal of Symbolic Computation*, **60** (2014), 94–112.

[8] N. Sloane, *A Library of Hadamard Matrices*, http://neilsloane.com/hadamard/

[9] O. Tange, *GNU Parallel - The Command-Line Power Tool*, *The USENIX Magazine*, February (2011) 42–47.

[10] B. Weisfeiler (Ed.): *On Construction and Identification of Graphs.* Berlin: Springer-Verlag (1976).