High-order Chebyshev-based Nyström Methods for Electromagnetics

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***Abstract*—Boundary element methods (BEM) have been successfully applied towards solving a broad array of complicated electromagnetic problems. Most BEM approaches rely on flat triangular discretizations and discretization via the Method of Moments (MoM) and low-order basis functions. Although more complicated from an implementation standpoint, it has been shown that high-order methods based on curvilinear patch mesh discretizations can significantly outperform low-order MoM in both accuracy and computational efficiency. In this work, we review a new high-order Nyström method based on using Chebyshev basis functions with curvilinear elements that we have recently developed, present a few scattering examples, and discuss related on-going and future work.**

***Index Terms*—integral equations, spectral methods, Nyström method, scattering**

I. Introduction

Boundary integral equations (BIEs) are a powerful class of methods for solving many different electromagnetic problems, including antennas [1], radar scattering [2], and most recently nanophotonics [3]. Unlike volumetric methods such as the Finite Difference (FD) method and the Finite Element Method (FEM), which require generating complicated volume meshes, boundary integral methods only require meshing of surface boundaries between different material domains. This surface meshing can also result in significantly reduced numbers of unknowns over volumetric approaches, especially for problems with large volume to surface area ratios. However, despite their clear potential for high performance and accuracy, integral equation methods are often overlooked due to difficulties involving numerical evaluation of the integrals involving the weakly singular and hypersingular Green's function kernels.

The most well-known approach for discretization of integral equations in electromagnetics is the Method of Moments (MoM). Although various high-order MoM approaches have been demonstrated, the majority of implementations rely on the well-known, first-order Rao-Wilton-Glisson (RWG) basis functions introduced in the seminal paper by the authors the basis functions are named after [4]. Although MoM methods are convenient due to the ability to reduce the hypersingular integrals to weakly singular by integrating by parts and moving a gradient to the testing function, the required double integrals due to Galerkin testing can greatly increase the computational effort required for applying the operator to a given density (or building the system matrix for a direct matrix implementation). The Nyström method offers an alternative approach for discretizing integral equations by directly using a numerical quadrature rule and testing on the same set of points used by the quadrature [5]–[9]. Due to this "point-matching" testing approach, double surface integrals are not required, leading to potentially more efficient evaluation. High-order convergence can also be readily obtained via Nyström techniques without significant increases in computational costs by employing high-order quadrature rules.

Unfortunately, dealing with the singular integral operators in the Nyström approach is far from trivial and require careful consideration, which has limited their application and popularity compared to MoM. Locally Corrected Nyström (LCN) are perhaps the most well-known approaches in the computational electromagnetics community for dealing with these singular kernels. LCN methods typically work by numerically computing a set of quadrature weights which absorb the kernel and its singularity by solving least-squares problems in the far-field [6], [10]. Although LCN methods can be highly effective, the numerical determination of quadrature weights can lead to increased error and it can be challenging or expensive to increase the solution accuracy beyond a certain extent.

In this work, we will review a new, different approach that we have developed for accurate, high-order discretization of integral equations for Maxwell's equations, recently published in [11]. The approach was first demonstrated by [12] for the 3D Helmholtz scalar case. Since the underlying mathematics and specific implementation details of this Chebyshev-based approach are covered in [11], in this work, we will summarize the method at a high level, present some new scattering results, and discuss recent applications that we are working towards.

II. The Chebyshev-Based Nyström Method

A. *Meshing*

The Chebyshev-based method relies on representing the geometry of the scattering obstacles by a set of nonoverlapping

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curvilinear quadrilateral patches. The primary advantage of using curvilinear patches over flat triangular discretizations is the ability to represent complex curvature accurately with very coarse meshes. In order for high-order methods to exhibit the most computational benefit over first-order methods such as RWG MoM, it is necessary to use large patch sizes to reduce the total number of unknowns—high-order methods are capable of accurately representing the densities on large patches due to the high expansion order, whereas they are not needed for small, sub-wavelength patches over which the density is not expected to vary considerably. Typically very fine flat triangular discretizations are required to represent a geometry accurately, which highlights the importance of curvilinear discretizations when used in conjunction with high-order integration techniques [13], [14].

B. Quadrature

The unknown densities on each patch are expanded in terms of Chebyshev polynomials. Traditionally, when discretizing integral equations (and often times when discussing acceleration techniques such as the Fast Multipole Method [15]–[18]), interactions between nearby and far away elements are considered differently in order to maintain overall accuracy while maximizing computational efficiency. For the non-singular far interactions, we use Fejér's first quadrature—which is a high-order accurate open integration rule over the Chebyshev nodes—to evaluate the integrals with high degree of accuracy.

If a target point is on the integration source patch itself or nearby, then the resulting integral kernels are either singular or nearly singular. In order to deal with these integrals accurately, we precompute the integral of each kernel against each Chebyshev polynomial on each patch for each target point. By leveraging the discrete orthogonality property of Chebyshev polynomials, this allows efficient expansion of the density into the corresponding Chebyshev coefficients. Due to linearity, the integral of each kernel against the density can now be rapidly found by multiplying each coefficient against the corresponding precomputated integral and summing them up.

The issue of the singularity still remains for the precomputation integrals against the Chebyshev basis. Although there exist various effective approaches for dealing with such singular integrals [6], [10], [19]–[21], we employ simple yet effective technique relying on a change of variables which maps the unit UV square back onto itself but clusters points on and around the singular target point (see Fig. 1) and in doing so results in a Jacobian which vanishes at the singularity and cancels it [11], [12]. It should be noted that this technique only works for weakly singular integrands. Hypersingular integrands must be reduced to a weakly singular form prior to applying the change of variables singularity cancellation approach. In the case of dielectric objects, this can be readily accomplished by employing the N-Müller formulation [11], [22], [23]. For the EFIE operator, the gradient of the Green's function with respect to the target coordinate can be pulled outside of the integral, resulting in the integrand becoming weakly singular. This requires numerical differentiation of the resulting integral on the surface after the integration, which can be readily accomplished by Chebyshev differentiation, since the unknowns are already on the Chebyshev nodes [24].

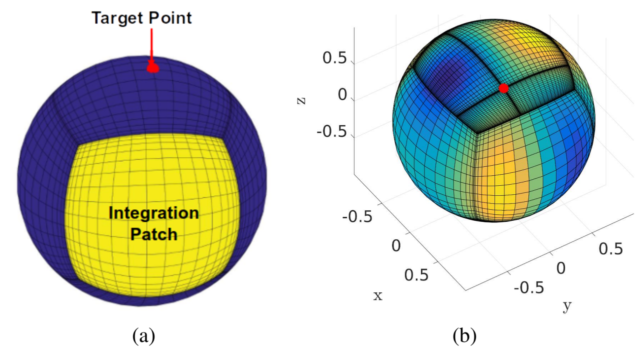

Fig. 1. (a) Two-dimensional Chebyshev grids used to evaluate the integral operators when the target points are sufficiently far away from the integration patch. (b) In the case of target points on-grid, or near the integration patch, an auxiliary clustered grid is used to precompute the integral operators applied to Chebyshev polynomials.

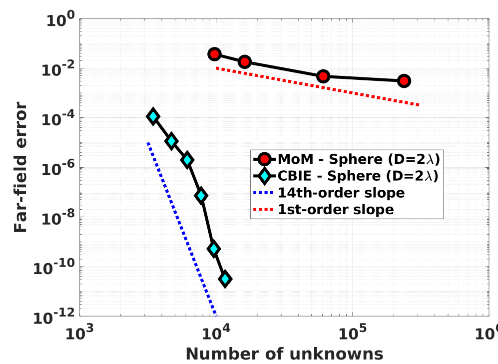

Fig. 2. Convergence in the far field for the Chebyshev-based integral equation method compared to a first-order MoM using a commercially available implementation.

III. Scattering from Large and Complex Objects

Given that BIEs reduce the problem from solving for the fields in a volume to finding the currents on the surface of material interfaces, the advantages of these methods over volumetric techniques, such as finite differences and finite elements, are further highlighted when the ratio of scattering volume to surface is high. In Fig. 3 we show the results of dielectric scattering by a large sphere of diameter $20\lambda_e$ (=$28.3\lambda_i$), where λ_e and λ_i are the wavelengths in the exterior and interior domains. In this case, we used a discretization of 600 patches, each consisting of Chebyshev grids of 18×18 points resulting in $194,400$ discretization points and $777,600$ unknowns. The resulting pointwise error in the current densities (compared to the Mie solution) is less than 0.1% for this configuration.

Although most boundary integral methods can be formulated by assuming a very general representation of the geometry of the scattering objects, for example a triangulation or a set of curved patches, obtaining such representation from a CAD design is usually nontrivial. Typically, for low-order MoM discretizations, a flat triangulation of the scatterer in question is all that is needed, making the process geometrically robust. In fact, one of the biggest hurdles in applying high-order methods in realistic settings is representing complex scatterers using curved patches. However, recent advances in quadrilateral meshing have allowed for easier CAD processing of complex geometries that can be converted to untrimmed, quadrilateral NURBS surfaces [25]. In Fig. 4 we show PEC scattering by a complex geometry using the Chebyshev-based method. The geometry was obtained from a CAD design freely available [26] and processed onto nonoverlapping quadrilateral curvilinear patches using Rhino3D [25].

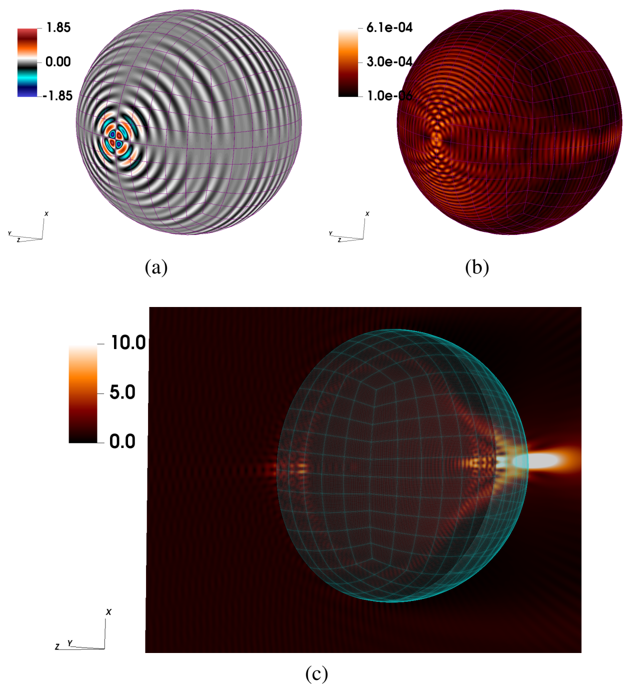

Fig. 3. Scattering by a dielectric sphere of diameter $28.3\lambda_i$. (a) Real part of the magnetic current density (x-component). (b) Pointwise error using 600 patches of 18×18 points. (c) Absolute value of the electric field.

IV. Research Directions

Integral equation methods have proven to be highly-effective in the two-dimensional case for the simulation [27] and optimization of nanophotonic devices [3]. The advantages of using BIE in this context—reducing the problem to boundary integrals, the design parameters translating to parametrizing design constraints directly, high-order accuracy, and reduction in simulation times—all translate to the three-dimensional case. Hence, our current efforts are directed towards applying the high-order methods described in previous sections to the simulation and optimization of nanophotonic devices in the three dimensional context.

Some progress has already been made in this direction, in particular, the effective simulation of three-dimensional

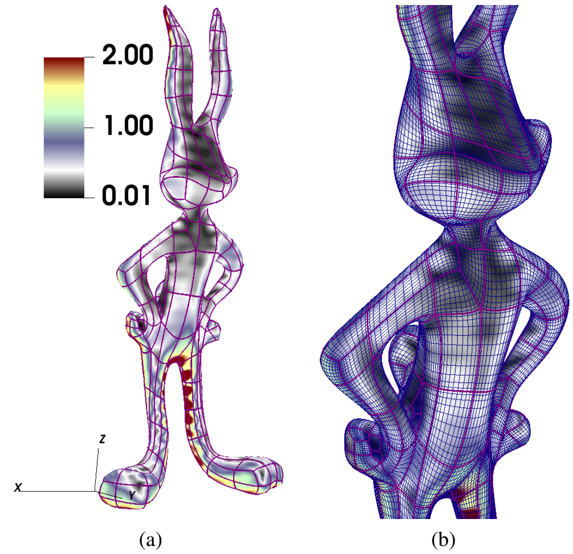

Fig. 4. PEC scattering by a complex geometry represented by quadrilateral NURBS patches. (a) Absolute value of the electric currents induced by plane wave scattering. (b) Close-up of the 402 patch distribution and the 12×12 Chebyshev grids used for this example.

waveguides using BIEs in conjunction with the windowed Green function (WGF) method has been demonstrated in [24], [28]. We have interfaced the Chebyshev BIE representations, together with the WGF method, and with CAD libraries, allowing for highly-complex nanophotonic device prototypes to be simulated. In Fig. 5 we show the simulation a complex branching silicon waveguide structure buried in silicon oxide. For this case, the structure spans about 30 interior wavelengths and was discretized using $308,896$ unknowns.

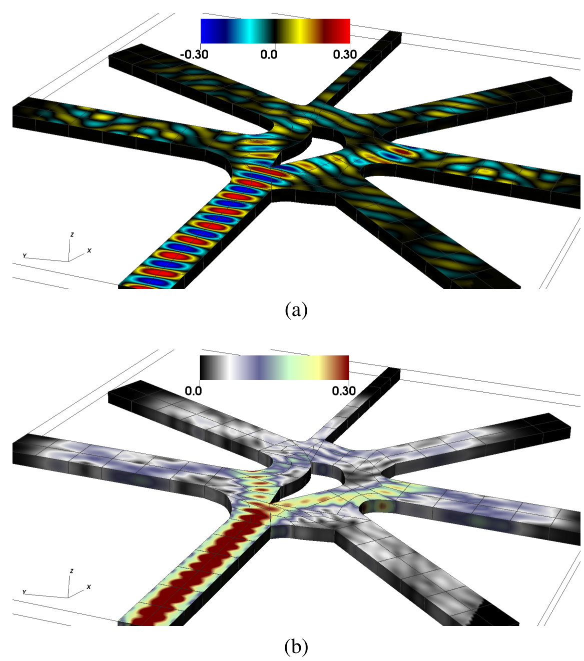

Fig. 5. Scattering by a complex dielectric waveguide structure. (a) Real part and (b) absolute value of the magnetic currents.

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