H-Matrix Accelerated Direct Matrix Solver using Chebyshev-based Nyström Boundary Integral Equation Method

Jin Hu[#], Emrah Sever[*], Omid Babazadeh[$], Reza Gholami[^], Vladimir Okhmatovski[$], Constantine Sideris[#]
[#]Department of Electrical and Computer Engineering, University of Southern California, USA
[*]Department of Electronics Engineering, Gebze Technical University, Turkey
[$]Department of Electrical and Computer Engineering, University of Manitoba, Canada
[^]Mentor Graphics, USA
jinhu@usc.edu, emrahsever@gtu.edu.tr, babazado@myumanitoba.ca, r.gholami@utoronto.ca, Vladimir.Okhmatovski@umanitoba.ca, csideris@usc.edu

2022 IEEE/MTT-S International Microwave Symposium - IMS 2022 | 978-1-6654-9613-1/22/$31.00 ©2022 IEEE | DOI: 10.1109/IMS37962.2022.9865659

***Abstract*—This work introduces a fast direct computational framework based on the high-order Chebyshev Nyström Boundary Integral Equation (CBIE) method accelerated by hierarchical matrices (H-matrix). The recently introduced CBIE offers substantially faster matrix fill than the popular Locally Corrected Nystrom (LCN) approach. Combination of the methods results in a first-in-class fast direct computational framework which is error-controllable, fast, and insensitive to poor conditioning of the pertinent matrix equations. Matrix fill, factorization, and solution times are reported for a metallic sphere, and more complicated examples such as scattering from a curvilinear NURBS-based airplane CAD model and a spiral antenna are presented to demonstrate the solver's versatility.**

***Keywords*—computational electromagnetics, Nyström Method, H-matrices, fast solver, high-order accuracy.**

I. Introduction

Boundary integral equation (BIE) methods are a popular class of numerical methods for solving scattering problems since they only require discretization of the surfaces and explicitly satisfy the radiation conditions. This often makes them considerably more efficient than their volumetric counterparts, such as the Finite Difference and Finite Element methods, for problems with large volume to surface area ratios. Low order Method of Moments (MoM) methods using Rao-Wilton-Glisson (RWG) basis functions and flat triangular meshes are the most common discretization and solution approaches of BIEs today due to their ease of implementation. However, such implementations have trouble achieving high solution accuracy due to difficulties in dealing with the singular nature of the kernels involved and accurately representing the surface current densities and curved geometries. Another major drawback of BIE methods is that they result in dense matrices which require $O(N^3)$ and $O(N^2)$ compute time and storage, respectively, when their pertinent matrix equations are solved directly without acceleration techniques.

The Locally Corrected Nyström (LCN) was first proposed in 1998 by [1] as an alternative to MoM. It was extended in [2] to construct high-order convergent solutions of electromagnetic problems. The original Nyström Method (NM) can only be used for the solution of integral equations with smooth kernels. When the kernel is singular or nearly singular, the LCN approach can be used to compute custom integration weights for the singular and nearly singular kernels. This computation of weights is very expensive and requires substantial computational time when the size of the scatterer increases compared to the wavelength. Recently, [3], [4] and [5] proposed a new Chebyshev-based Nyström Boundary Integral Equation (CBIE) method for acoustic and electromagnetic scattering problems respectively, which uses Chebyshev polynomials to approximate the current on the surface and uses a change of variables method to cancel the singularities in the kernels up to a certain order. This method is significantly faster than LCN since it does not require the evaluation of custom quadrature rules for the singular and nearly singular kernels. Although the CBIE accelerates matrix fill times, computing the full matrix still requires $O(N^2)$ operations and memory and direct factorization of the matrix requires $O(N^3)$ operations. This becomes prohibitive for moderate to large problem sizes. The H-matrix method has previously been used successfully to accelerate MoM approaches for many scattering problems [6] and has also recently been used to accelerate the LCN method [7] . In this work, we present the first direct solver based on the CBIE method accelerated using H-matrices to speed up the direct solution of scattering problems. For the sake of simplicity, we only consider scattering from perfect electrical eonductor (PEC) objects using the magnetic field integral equation (MFIE) formulation, although the same approach can be applied to any integral formulation for modelling metallic and/or dielectric objects, including the EFIE, CFIE, PCMHWT, and Müller formulations. We consider plane wave scattering from a sphere and a curvilinear NURBS-based B2 aircraft and excitation of a spiral antenna to show the versatility of the approach for simulating many different problems.

II. NM Discretization of the MFIE based on Chebyshev Polynomials

The classical MFIE for a PEC object is given by

$$\mathbf{J}/2 + \mathcal{K}\mathbf{J} = \mathbf{n} \times \mathbf{H}^{\text{inc}} \tag{1}$$

where $\mathbf{J}$ is the vector surface current, $\mathbf{n}$ is the normal vector on the surface, $\mathbf{H}^{\text{inc}}$ is the incident magnetic field and the integral operator $\mathcal{K}$ has the following representation

$$\mathcal{K}[\mathbf{J}](\mathbf{r}) = \mathbf{n}(\mathbf{r}) \times \int_S \mathbf{J}(\mathbf{r}') \times \nabla G(\mathbf{r} - \mathbf{r}') dS'. \tag{2}$$

To apply the NM proposed in [4], the surface is first discretised into a certain number of curvilinear patches (e.g., Figs. 2a and

3) followed by mapping them to a unit square in a local uv-coordinate system. The surface current on the p^{th} patch is represented in terms of tangential covariant basis vectors as

$$\mathbf{J}^p(u,v) = \frac{J^{p,u}(u,v)\boldsymbol{a}_u^p(u,v) + J^{p,v}(u,v)\boldsymbol{a}_v^p(u,v)}{\sqrt{|G^p(u,v)|}} \tag{3}$$

where $\boldsymbol{a}_a^p$ $(a = u, v)$ are the unitary vectors and $\sqrt{|G^p(u,v)|}$ is the Jacobian of the surface element. By representing surface S in (2) as a sum over the patches, substituting (3) into (1), and testing the resultant vector equation by the $\sqrt{|G^q|}\boldsymbol{a}^{p,a}$, where $\boldsymbol{a}^{p,a}$ are the contravariant basis vectors, the following system of linear algebraic equations is obtained:

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{uv} \\ \mathbf{K}_{vu} & \mathbf{K}_{vv} \end{bmatrix} \begin{bmatrix} \mathbf{J}^u \\ \mathbf{J}^v \end{bmatrix} = \begin{bmatrix} -\boldsymbol{a}_v^p \cdot \mathbf{H}^{p,\text{inc}} \\ \boldsymbol{a}_u^p \cdot \mathbf{H}^{p,\text{inc}} \end{bmatrix}. \tag{4}$$

In order to get high order accuracy from the solution of (4), each matrix entry must be considered as

$$K_{ba}^{qp} = K_{ba}^{qp(\text{far})} + K_{ba}^{qp(\text{self})} + K_{ba}^{qp(\text{near})}; \quad b, a = u, v \tag{5}$$

where "far" implies the distance between the observation and source patches is far enough such that the integrand in (2) is a smooth function, "self" means that the source and observation patches are the same leading to a singular integrand, and "near" implies the source and observation points are different but still nearby, leading to a nearly singular integrand. The "far" interactions can be handled accurately using standard Fejer quadrature. However, in order to accurately compute the "self" and "near" interactions, the integrals are made regular by using a change-of-variables whose Jacobian cancels the kernel singularity up to a controllable order [4]. This technique enables rapid computation of these interactions with high accuracy, unlike the LCN, which suffers from very slow matrix fill times. When iterative solvers are used to solve the system, the method can efficiently be used to solve large-scale scattering problems, especially if accelerated using FFT or fast multipole method (FMM)-based approaches. However, iterative solvers may have unpredictable convergence or stagnate for many realistic scenarios featuring dense materials, electrically large sizes, resonant structures, multiscale discretization, or low-frequency instabilities. In such situations, direct matrix solutions become a must as they remain largely insensitive to deteriorated conditioning of the matrix. However, their high build and factorization costs necessitate the use of acceleration methods. In order to accelerate the CBIE method, we use H-matrices and the Adaptive Cross Approximation (ACA) method for compression of their rank-deficient blocks. Here, an individual H-matrix is constructed for each of the four $\mathbf{K}_{ab}$ $(a, b = \{u, v\})$ blocks in (4). Detail of the H-matrix construction and block factorization are presented in the following section.

III. H-MATRIX IMPLEMENTATION

The block-wise H-matrices must first be created from the CBIE impedance matrix. A hierarchical cluster tree which consists of multiple levels with leaf nodes of decreasing size is generated through recursive partitioning of the geometry based on the Euclidean distances between pairs of source (quadrature) points. After cluster tree creation, each of the four required H-matrices is built by decomposing the impedance matrix into subblocks based on the interaction between observer clusters and source clusters determined by a prescribed admissibility criteria. The inadmissible (nearby) blocks are stored in a dense full-matrix format, and the admissible blocks are approximated by the ACA algorithm and stored in a compressed low-rank matrix format [7]. After creating the four H-matrices, the system can be solved using fast direct H-matrix block H-LU decomposition followed by a block H-substitution applied to the H-matrix structure. The block H-LU decomposition can be computed as

$$\mathbf{K}^{\boldsymbol{H}} = \begin{bmatrix} \mathbf{K}_{11}^{\boldsymbol{H}} & \mathbf{K}_{12}^{\boldsymbol{H}} \\ \mathbf{K}_{21}^{\boldsymbol{H}} & \mathbf{K}_{22}^{\boldsymbol{H}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}_{11} & \\ \boldsymbol{L}_{21} & \boldsymbol{L}_{22} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{U}_{11} & \boldsymbol{U}_{12} \\ & \boldsymbol{U}_{22} \end{bmatrix} = \boldsymbol{L} \cdot \boldsymbol{U} \tag{6}$$

where $\mathbf{K}_{\boldsymbol{H}}$ represents the H-matrix corresponding to the system matrix in (4). In order to solve for the surface current densities of (4) using the constructed H-matrix system, the solution of the lower triangle system is first obtained as

$$\begin{bmatrix} \boldsymbol{L}_{11} & \\ \boldsymbol{L}_{21} & \boldsymbol{L}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix} \tag{7}$$

with right-hand side $[\boldsymbol{b}_1\ \boldsymbol{b}_2]^T = \left[-\boldsymbol{a}_v^p \cdot \mathbf{H}^{p,\text{inc}}\ \boldsymbol{a}_u^p \cdot \mathbf{H}^{p,\text{inc}}\right]^T$. Next, the solution of the system (4) is obtained as

$$\begin{bmatrix} \boldsymbol{U}_{11} & \boldsymbol{U}_{12} \\ & \boldsymbol{U}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{J}^u \\ \mathbf{J}^v \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix} \tag{8}$$

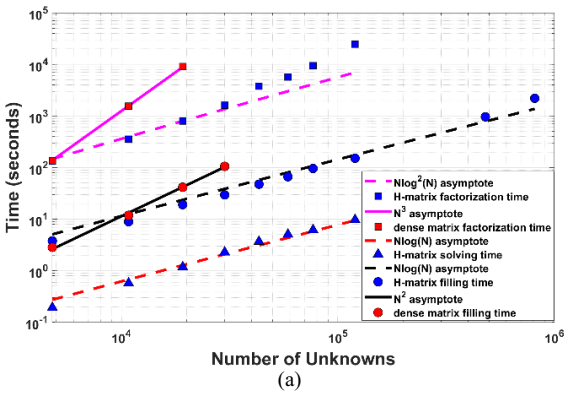
for the surface current. Once the surface current is calculated, the near and far fields can be calculated at any point.

It should be noted that increases in the condition number in the H-matrix fast direct solution scheme proposed in this work may affect the compressibility of the matrix and/or accuracy of the computed currents. The scheme, however, remains robust even in scenarios with poorly conditioned matrices.

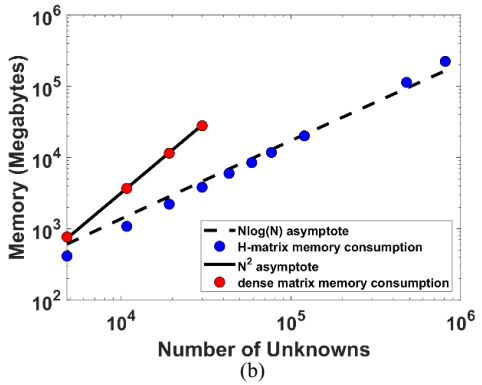
IV. NUMERICAL RESULTS

The first example we present is scattering from a PEC sphere centered at the origin with radius 3λ illuminated by a plane wave propagating in $+\hat{z}$ direction and polarized in $+\hat{x}$ direction with wavelength $\lambda = 1\,\text{m}$ and unit amplitude. The number of patches is varied from 24 to 4056 with 10x10 discretization points per patch, which results in the number of unknowns ranging from 4800 to 811,200. The Compression Ratio (CR) is defined as one minus the ratio of the storage requirement (memory size required) of the compressed H-matrix to that of the full dense matrix and is a useful metric for assessing the efficacy of the method for accelerating a problem. A maximum CR of 98.8% was achieved for the case with 811,200 unknowns. The performance of our proposed accelerated solver in terms of time and memory consumption is compared against the unaccelerated dense matrix case and shown in Fig. 1. Simulations were performed on a server with dual Xeon Gold 6154 CPUs (36 cores). The CPU time required to fill the H-Matrix and the corresponding full dense matrix scales as $O(\text{N}\log\text{N})$ and $O(\text{N}^2)$ respectively. The time to factor the H-Matrix and the corresponding dense matrix approximately scales as $O(\text{N}\log^2\text{N})$ and $O(\text{N}^3)$ respectively. Although the factorization time for the H-Matrix deviates from $O(\text{N}\log^2\text{N})$ scaling, significant reduction of CPU time can still be observed in comparison with the $O(\text{N}^3)$ factorization time

of the dense matrix system. The time taken for backsubstitution after the LU factorization has completed scales as $O(\mathrm{NlogN})$. Finally, Fig. 1b shows that the memory consumption for the H-Matrix case scales approximately as $O(\mathrm{Nlog\,N})$ and that of the corresponding dense matrix case scales as $O(\mathrm{N}^2)$.

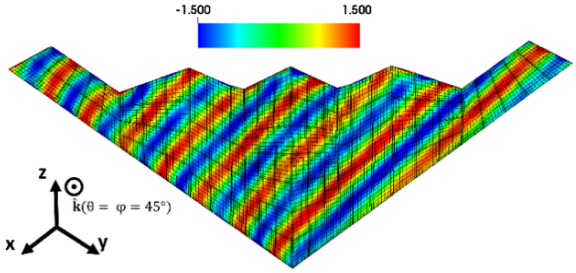


(a)

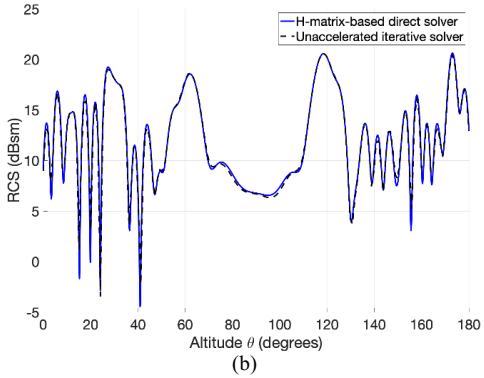


(b)

Fig. 1. Performance comparison of H-matrix direct solver vs. dense matrix direct solver. (a) CPU time complexity. (b) Memory consumption.



(a)



(b)

Fig. 2. (a) Re(J_x) on the surface of the B2 aircraft. (b) RCS for φ=0° comparing H-Matrix direct solver vs. unaccelerted iterative solver.

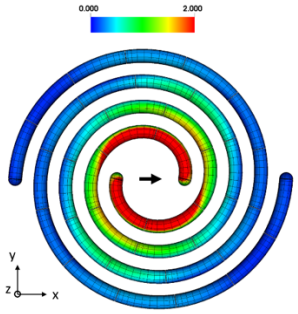


Fig. 3. The magnitude of the current density **|J|** on the surface of the antenna.

In order to show the capability of our proposed solver for handling objects with more complex geometrical features, we consider scattering of a plane wave from a B2 aircraft model with 16λ wingspan. Fig. 2(a) shows the propagation direction $\hat{\mathrm{k}}$ of the incident field and the real part of the x-component of the induced current density **J** on the surface. Fig. 2(b) plots the radar cross-section (RCS) versus θ for the $\varphi = 0°$ plane, computed both by using the proposed solver as well as an unaccelerated iterative solver for comparison using 51,600 unknowns. The H-matrix achieved a 88.2% CR. The two solutions match closely. Finally, we consider excitation of a 2λ diameter metallic spiral antenna by a unit amplitude x-polarized dipole source located at the origin. The spiral was discretized with 11,520 unknowns and achieved a 72% CR. Fig. 3 depicts the magnitude of the solved current density **|J|** on the antenna's surface.

V. Conclusion

An accelerated direct solver for the recently introduced CBIE method was presented using H-matrices. Results were reported on the performance of the approach in terms of matrix fill, factorization, and solve time, as well as memory required and compared favorably against the unaccelerated dense matrix method. The matrix fill time is very fast owing to the use of the CBIE method, requiring only seconds to minutes to complete even for problems approaching 1 million unknowns. Finally, scattering from a NURBS-based airplane CAD model and a metallic spiral antenna is shown to demonstrate the ease of handling complex geometries. Future work involves improving the H-matrix factorization time via parallelization and GPU acceleration.

Acknowledgment

The authors are grateful for support by the Air Force Office of Scientific Research (FA9550-20-1-0087) and by the National Science Foundation (CCF-1849965, CCF-2047433).

References

[1] L.F. Canino, et. al., "Numerical solution of the Helmholtz equation in 2D and 3D using a high-order Nyström discretization," *J. Comp. Phys.,* vol. 146, no. 2, pp. 627-663, 1998.

[2] S. D. Gedney, "On deriving a locally corrected Nyström scheme from a quadrature sampled moment method," *IEEE Trans. Antennas Propag.,* vol. 51, no. 9, pp. 2402-2412, 2003.

[3] O. P. Bruno and E. Garza, “A Chebyshev-based rectangular-polar integral solver for scattering by geometries described by non-overlapping patches,” *J. Comp. Phys.,* vol. 421, p. 109740, 2020.
[4] J. Hu, E. Garza, and C. Sideris, “A Chebyshev-based high-order-ccurate integral equation solver for Maxwell’s equations,” *IEEE Trans. Antennas Propag.,* vol. 69, no. 9, pp. 5790–5800, 2021.
[5] E. Garza, J. Hu, and C. Sideris. "High-order Chebyshev-based Nyström Methods for Electromagnetics," *International Applied Computational Electromagnetics Society Symposium (ACES)*. IEEE, 2021
[6] R. Gholami, et. al., “H-matrix arithmetic for fast direct and iterative method of moment solution of surface-volume-surface EFIE for 3-D radiation problems,” *Prog. Electromagn. Res. B*, vol. 82, pp. 189-210, 2018.
[7] R. Gholami, et. al., “Fast direct error-controlled solution of scattering problems via $\mathscr{H}$-matrix acceleration of locally corrected Nyström method,” *EuCAP 2021*, pp. 1–5.