

The replica symmetric formula for the SK model revisited ^{EP}

Cite as: J. Math. Phys. **63**, 073302 (2022); <https://doi.org/10.1063/5.0073807>

Submitted: 04 October 2021 • Accepted: 24 June 2022 • Published Online: 14 July 2022

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Submitted: 4 October 2021 • Accepted: 24 June 2022 •

Published Online: 14 July 2022



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ABSTRACT

We provide a simple extension of Bolthausen's Morita-type proof of the replica symmetric formula [E. Bolthausen, "A Morita type proof of the replica-symmetric formula for SK," in *Statistical Mechanics of Classical and Disordered Systems*, Springer Proceedings in Mathematics and Statistics (Springer, Cham., 2018), pp. 63–93; arXiv:1809.07972] for the Sherrington–Kirkpatrick model and prove the replica symmetry for all (β, h) that satisfy $\beta^2 \text{Esech}^2(\beta\sqrt{q}Z + h) \leq 1$, where $q = E \tanh^2(\beta\sqrt{q}Z + h)$. Compared to the work of Bolthausen ["A Morita type proof of the replica-symmetric formula for SK," in *Statistical Mechanics of Classical and Disordered Systems*, Springer Proceedings in Mathematics and Statistics (Springer, Cham., 2018), pp. 63–93; arXiv:1809.07972], the key of the argument is to apply the conditional second moment method to a suitably reduced partition function.

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I. INTRODUCTION

We study systems of N spins σ_i , $i \in \{1, \dots, N\}$, with values in $\{-1, 1\}$ and with the Hamiltonian $H_N : \{-1, 1\}^N \rightarrow \mathbb{R}$ defined by

$$H_N(\sigma) = \frac{\beta}{\sqrt{2}} \sum_{1 \leq i < j \leq N} g_{ij} \sigma_i \sigma_j + h \sum_{i=1}^N \sigma_i. \quad (1.1)$$

The interactions $\{g_{ij}\}$ are i.i.d. centered Gaussians of variance $1/N$ for $i \neq j$, and we set $g_{ii} \equiv 0$. $\beta \geq 0$ denotes the inverse temperature, and $h > 0$ denotes the external field strength.

Equation (1.1) corresponds to the Sherrington–Kirkpatrick (SK) spin glass model,¹ and we are interested in its free energy f_N at high temperature, where

$$f_N = \frac{1}{N} \log Z_N, \quad Z_N = \sum_{\sigma \in \{-1, 1\}^N} e^{H_N(\sigma)}. \quad (1.2)$$

The mathematical understanding of the SK model has required substantial efforts until the famous Parisi formula^{2,3} was rigorously established by Guerra⁴ and Talagrand.⁵ Later, Panchenko^{6,7} gave another proof based on the ultrametricity of generic models. For a thorough introduction to the topic, we refer to Refs. 8–11.

Despite the validity of the Parisi formula, it is an interesting question to prove the replica symmetry of the SK model at high temperature, as predicted by de Almeida and Thouless.¹² Replica symmetry is expected for all (β, h) that satisfy

$$\beta^2 E \text{sech}^4(\beta\sqrt{q}Z + h) < 1, \quad (1.3)$$

where q denotes the unique solution of the self-consistent equation,

$$q = E \tanh^2(\beta\sqrt{q}Z + h). \quad (1.4)$$

In both cases, $Z \sim \mathcal{N}(0, 1)$ denotes a standard Gaussian and E denotes the expectation over Z .

In the special case that the external field in the direction of σ_i is a centered Gaussian random variable, $h_i = hg_i$, for i.i.d. $g_i \sim \mathcal{N}(0, 1)$ (independent of g_{ij}), replica symmetry has recently been shown in Ref. 13 for all (β, h) that satisfy (1.3) [in which case h in (1.3) and (1.4) is replaced by hZ' for some $Z' \sim \mathcal{N}(0, 1)$ independent of Z]; see also Ref. 14 for previous results in this case. For Hamiltonians H_N as in (1.1) (or, more generally, Hamiltonians with a non-centered random external field), however, replica symmetry is to date only known above the AT line up to a bounded region in the (β, h) -phase diagram. This was analyzed in Ref. 15. Like Ref. 13, this analysis is based on the Parisi variational problem and we refer to Ref. 15 for the details. For previously obtained results based on the Parisi formula, see also Ref. 11, Chap. 13.

In this article, instead of analyzing the high temperature regime in view of the Parisi variational problem, we give a simple extension of Bolthausen's argument¹⁶ and prove the replica symmetric formula for all (β, h) that satisfy

$$\beta^2 E \operatorname{sech}^2(\beta\sqrt{q}Z + h) \leq 1. \quad (1.5)$$

Although (1.5) is clearly stronger than condition (1.3), it already covers a fairly large region of the high temperature regime; see Fig. 1 for a schematic. It improves upon the inverse temperature range from Ref. 16, where β was assumed to be sufficiently small.

Theorem 1.1. Assume that (β, h) satisfies (1.5). Then,

$$\lim_{N \rightarrow \infty} \mathbb{E} \frac{1}{N} \log Z_N = \log 2 + E \log \cosh(\beta\sqrt{q}Z + h) + \frac{\beta^2}{4} (1 - q)^2. \quad (1.6)$$

Remarks.

- (1) From Ref. 17, it is well-known that $\lim_{N \rightarrow \infty} \mathbb{E} \frac{1}{N} \log Z_N$ exists and that almost surely $\lim_{N \rightarrow \infty} \frac{1}{N} \log Z_N = \lim_{N \rightarrow \infty} \mathbb{E} \frac{1}{N} \log Z_N$.
- (2) It follows from the results of Ref. 4 that the right-hand side of (1.6) provides an upper bound to the free energy $\lim_{N \rightarrow \infty} \mathbb{E} \frac{1}{N} \log Z_N$ for all inverse temperatures and external fields (β, h) . To establish Theorem 1.1, it is, therefore, sufficient to prove that the right-hand side of (1.6) provides a lower bound to $\lim_{N \rightarrow \infty} \mathbb{E} \frac{1}{N} \log Z_N$.

We conclude the Introduction with a quick heuristic outline of the main argument. To this end, consider first the case $h = 0$ where the critical temperature corresponds to $\beta = 1$. In this case, it is straightforward (see, e.g., Ref. 18, Chap. 1, Sec. 3) to see that

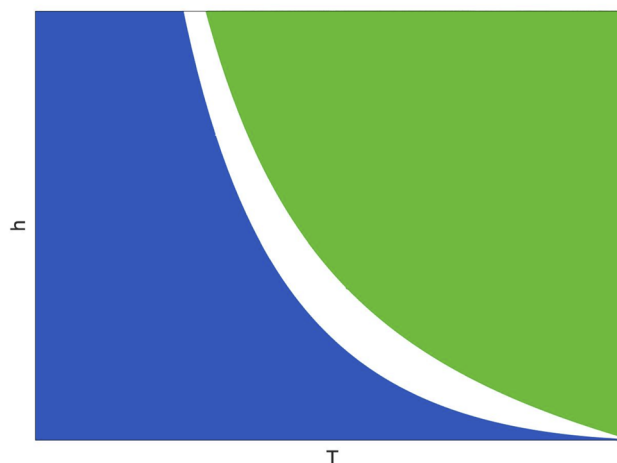


FIG. 1. Schematic of the (T, h) phase diagram, where $T = \frac{1}{\beta}$ denotes the temperature. In the blue region, whose boundary corresponds to the AT line (1.3), the SK model is known to be replica symmetry breaking. The boundary of the green region corresponds to condition (1.5). Theorem 1.1 proves the replica symmetry in the green region.

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} Z_N = \log 2 + \frac{\beta^2}{4}, \quad \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} Z_N^2 = 2 \log 2 + \frac{\beta^2}{2}$$

for all $\beta < 1$. The replica symmetric formula, thus, follows from the second moment method using the Gaussian concentration of the free energy. In fact, for $h = 0$, the fluctuations of $Z_N / \mathbb{E} Z_N$ have also been known for a long time.¹⁹

Clearly, it would be desirable to extend this simple argument to the case $h > 0$, but a direct application of the second moment method does not work here. However, as suggested in Ref. 16, one may hope to obtain a model similar to the case $h = 0$ by centering the spins around suitable magnetizations and viewing Z_N , up to normalization, as an average over the corresponding coin-tossing measure. To center the spins correctly, recall that at high temperature, one expects the TAP equations²⁰ to hold, that is,

$$m_i \approx \tanh \left(h + \beta \sum_{j \neq i} \tilde{g}_{ij} m_j - \beta^2 (1 - q) m_i \right) \quad (\text{for } i = 1, \dots, N), \quad (1.7)$$

where $\tilde{g}_{ij} = (g_{ij} + g_{ji}) / \sqrt{2}$ and $m_i = Z_N^{-1} \sum_{\sigma} \sigma_i e^{H_N(\sigma)}$. The validity of (1.7) is known for sufficiently small β (see Refs. 10 and 21 and, more recently, Ref. 22; see also Ref. 23 on the TAP equations for generic models, valid at all temperatures) and expected to be true under (1.3). In Refs. 16 and 24, Bolthausen provided an iterative construction $(\mathbf{m}^{(s)})_{s \in \mathbb{N}}$ of the solution to (1.7) that converges (in a suitable sense) in the full high temperature regime (1.3). The main result of Ref. 16 is a novel proof of (1.6) for β small enough based on a conditional second moment argument, given the approximate solutions $(\mathbf{m}^{(s)})_{s \in \mathbb{N}}$. It has remained an open question, however, if the approach can be extended to the region (1.3).

In this article, while we are not able to resolve this question for all (β, h) satisfying (1.3), we improve the range of (β, h) to (1.5) as follows: The author of Refs. 16 and 24 showed, roughly speaking, that $\mathbf{m}^{(k+1)} \approx \sum_{s=1}^k \gamma_s \phi^{(s)}$ for certain orthonormal vectors $\phi^{(s)} \in \mathbb{R}^N$ and deterministic numbers γ_s ($\approx \langle \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle$ with high probability), where $\langle \mathbf{x}, \mathbf{y} \rangle = N^{-1} \sum_{i=1}^N x_i y_i$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$. One also has $\mathbf{g} = \mathbf{g}^{(k+1)} + \sum_{s=1}^k \rho^{(s)}$ for the interaction $\mathbf{g} = (g_{ij})_{1 \leq i, j \leq N}$, where $\rho^{(s)} \in \mathbb{R}^{N \times N}$ are measurable with respect to $(\mathbf{m}^{(s)})_{s \leq k+1}$ and where the modified interaction $\mathbf{g}^{(k+1)}$ is Gaussian, conditionally on $(\mathbf{m}^{(s)})_{s \leq k+1}$, with the property that $\mathbf{g}^{(k+1)} \sum_{s=1}^k \gamma_s \phi^{(s)} = 0$. Up to negligible errors, one obtains with $\widehat{\sigma} = \sigma - \mathbf{m}^{(k+1)}$ that

$$\begin{aligned} \frac{1}{N} \log Z_N &\approx \log 2 + E \log \cosh(\beta \sqrt{q} Z + h) \\ &+ \frac{1}{N} \log \sum_{\sigma \in \{-1, 1\}^N} p_{\text{free}}(\sigma) \exp \left[\frac{N\beta}{\sqrt{2}} \langle \widehat{\sigma}, \mathbf{g}^{(k)} \widehat{\sigma} \rangle + N \mathcal{O} \left(\max_s |\gamma_s - \langle \sigma, \phi^{(s)} \rangle| \right) \right], \end{aligned} \quad (1.8)$$

where p_{free} denotes the product measure for which $\sum_{\sigma} p_{\text{free}}(\sigma) \sigma = \mathbf{m}^{(k+1)}$. A simple observation is now that we can ignore the error $N \mathcal{O}(\max_s |\gamma_s - \langle \sigma, \phi^{(s)} \rangle|)$ in (1.8) by restricting the modified partition function to those σ with $\max_s |\gamma_s - \langle \sigma, \phi^{(s)} \rangle| \approx 0$. Note that the probability of the complement of this set is small under p_{free} because $\gamma_s \approx \langle \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle$. This yields a simple lower bound on $\frac{1}{N} \log Z_N$, and we can apply the conditional second moment argument to the restricted partition function. We show that its first conditional moment equals $\beta^2 (1 - q)^2 / 4$ (up to negligible errors) in the full high temperature regime (1.3). To dominate its second moment by the square of the first, on the other hand, we need to impose the stronger condition (1.5).

Note that imposing similar orthogonality restrictions on the partition function has been proved useful before for obtaining lower bounds on the free energy, such as in the TAP analysis of the spherical SK model.²⁵

Although (1.5) already covers a comparably large region of the high temperature phase, as schematically shown in Fig. 1, it remains an open question whether the second moment argument can be extended to the full high temperature regime (1.3); see also the related comments in Ref. 16, Sec. 6.

This paper is structured as follows. In Secs. III and IV, we set up the notation and recall Bolthausen's iterative construction of magnetization.^{16,24} In Sec. IV, we define the reduced partition function and compute its first and second moments. In Sec. V, we apply the conditional second moment method to prove Theorem 1.1.

II. NOTATION

In this section, we introduce the basic notation and conventions. We closely follow Ref. 16.

We usually denote vectors in $\mathbb{R}^N$ by boldface or greek letters. If $\mathbf{x} \in \mathbb{R}^N$ and $g : \mathbb{R} \rightarrow \mathbb{R}$, we define $g(\mathbf{x})$ in the component-wise sense. By $\langle \cdot, \cdot \rangle : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$, we denote the normalized inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{N} \sum_{i=1}^N x_i y_i,$$

and by $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$, we denote the induced norm. We also normalize the tensor product $\mathbf{x} \otimes \mathbf{y} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ so that for all $\mathbf{z} \in \mathbb{R}^N$,

$$(\mathbf{x} \otimes \mathbf{y})(\mathbf{z}) = \langle \mathbf{y}, \mathbf{z} \rangle \mathbf{x}.$$

Given a matrix $\mathbf{A} \in \mathbb{R}^{N \times N}$, $\mathbf{A}^T \in \mathbb{R}^{N \times N}$ denotes its transpose and $\bar{\mathbf{A}} \in \mathbb{R}^{N \times N}$ denotes its symmetrization,

$$\bar{\mathbf{A}} = \frac{\mathbf{A} + \mathbf{A}^T}{\sqrt{2}}.$$

We mostly use the letters Z, Z', Z_1, Z_2 , etc. to denote standard Gaussian random variables independent of the disorder $\{g_{ij}\}$ and independent of one another. When we average over such Gaussians, we denote the corresponding expectation by E to distinguish it from the expectation $\mathbb{E}$ with respect to the disorder $\{g_{ij}\}$. Unless specified otherwise, we consider all Gaussians to be centered.

Finally, given two sequences of random variables $(X_N)_{N \geq 1}, (Y_N)_{N \geq 1}$ that may depend on parameters such as β and h , we say that

$$X_N \simeq Y_N$$

if and only if there exist positive constants $c, C > 0$, which may depend on the parameters, but which are independent of N , such that for every $t > 0$, we have

$$\mathbb{P}(|X_N - Y_N| > t) \leq Ce^{-ct^2N}.$$

III. BOLTHAUSEN'S CONSTRUCTION OF THE LOCAL MAGNETIZATIONS

In this section, we recall Bolthausen's iterative construction of the solution to the TAP equations^{16,24} and list the properties that we will need for the Proof of Theorem 1.1. We follow here the conventions of Ref. 16, and we refer to Ref. 16, Secs. 2, 4, and 5 for the proofs of the following statements.

First of all, we define three sequences $(\alpha_k)_{k \in \mathbb{N}}, (\gamma_k)_{k \in \mathbb{N}}$, and $(\Gamma_k)_{k \in \mathbb{N}}$. Set

$$\alpha_1 = \sqrt{q}\gamma_1, \quad \gamma_1 = E \tanh(\beta\sqrt{q}Z + h), \quad \Gamma_1^2 = \gamma_1^2,$$

where here and in the following, q denotes the unique solution of (1.4). Then, we define $\psi : [0, q] \rightarrow [0, q]$ by

$$\psi(t) = E \tanh(\beta\sqrt{t}Z + \beta\sqrt{q-t}Z' + h) \tanh(\beta\sqrt{t}Z + \beta\sqrt{q-t}Z'' + h),$$

and set recursively

$$\alpha_k = \psi(\alpha_{k-1}), \quad \gamma_k = \frac{\alpha_k - \Gamma_{k-1}^2}{\sqrt{q - \Gamma_{k-1}^2}}, \quad \Gamma_k^2 = \sum_{j=1}^k \gamma_j^2.$$

The following lemma collects important properties of $(\alpha_k)_{k \in \mathbb{N}}, (\gamma_k)_{k \in \mathbb{N}}$, and $(\Gamma_k)_{k \in \mathbb{N}}$.

Lemma 3.1. (Ref. 24, Lemma 2.2, Corollary 2.3, Lemma 2.4 and Ref. 16, Lemma 2)

- (1) ψ is strictly increasing and convex in $[0, q]$ with $0 < \psi(0) < \psi(q) = q$. If (1.3) is satisfied, then q is the unique fixed point of ψ in $[0, q]$.
- (2) The sequence $(\alpha_k)_{k \in \mathbb{N}}$ is increasing and $\alpha_k > 0$ for every $k \in \mathbb{N}$. If (1.3) is satisfied, then $\lim_{k \rightarrow \infty} \alpha_k = q$, and if (1.3) is satisfied with a strict inequality, the convergence is exponentially fast.
- (3) For all $k \geq 2$, we have that $0 < \Gamma_{k-1}^2 < \alpha_k < q$ and that $0 < \gamma_k < \sqrt{q - \Gamma_{k-1}^2}$. If (1.3) is true, then $\lim_{k \rightarrow \infty} \Gamma_k^2 = q$ and, as a consequence, $\lim_{k \rightarrow \infty} \gamma_k = 0$.

Next, we recall Bolthausen's modified interaction matrix. We define

$$\mathbf{g}^{(1)} = \mathbf{g}, \quad \phi^{(1)} = \mathbf{1} \in \mathbb{R}^N, \quad \mathbf{m}^{(1)} = \sqrt{q}\mathbf{1} \in \mathbb{R}^N.$$

Assuming that $\mathbf{g}^{(s)}, \phi^{(s)}, \mathbf{m}^{(s)}$ are defined for $1 \leq s \leq k$, we set

$$\zeta^{(s)} = \bar{\mathbf{g}}^{(s)} \phi^{(s)}$$

and we define the σ -algebra $\mathcal{G}_k$ through

$$\mathcal{G}_k = \sigma(\mathbf{g}^{(s)} \phi^{(s)}, (\mathbf{g}^{(s)})^T \phi^{(s)} : 1 \leq s \leq k).$$

Expectations with respect to $\mathcal{G}_k$ are denoted by $\mathbb{E}_k$. Furthermore, we set

$$\begin{aligned} \mathbf{h}^{(k+1)} &= h1 + \beta \sum_{s=1}^{k-1} \gamma_s \zeta^{(s)} + \beta \sqrt{q - \Gamma_{k-1}^2} \zeta^{(k)}, \quad \mathbf{m}^{(k+1)} = \tanh(\mathbf{h}^{(k+1)}), \\ \phi^{(k+1)} &= \frac{\mathbf{m}^{(k+1)} - \sum_{s=1}^k \langle \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle \phi^{(s)}}{\|\mathbf{m}^{(k+1)} - \sum_{s=1}^k \langle \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle \phi^{(s)}\|}, \end{aligned} \quad (3.1)$$

and we note that $\phi^{(k+1)}$ is $\mathbb{P}$ -a.s well-defined for all k if $k < N$ (Ref. 16, Lemma 5). Finally, the modified interaction matrix $\mathbf{g}^{(k+1)}$ is defined by

$$\mathbf{g}^{(k+1)} = \mathbf{g}^{(k)} - \rho^{(k)},$$

where

$$\rho^{(k)} = \mathbf{g}^{(k)} \phi^{(k)} \otimes \phi^{(k)} + \phi^{(k)} \otimes (\mathbf{g}^{(k)})^T \phi^{(k)} - \langle \mathbf{g}^{(k)} \phi^{(s)}, \phi^{(k)} \rangle \phi^{(k)} \otimes \phi^{(k)}.$$

In particular, this means that $\bar{\mathbf{g}}^{(k+1)}$ is equal to

$$\bar{\mathbf{g}}^{(k+1)} = \bar{\mathbf{g}}^{(k)} - \bar{\rho}^{(k)}, \quad \bar{\rho}^{(k)} = \zeta^{(k)} \otimes \phi^{(k)} + \phi^{(k)} \otimes \zeta^{(k)} - \langle \zeta^{(k)}, \phi^{(k)} \rangle \phi^{(k)} \otimes \phi^{(k)}.$$

It is clear that $(\phi^{(s)})_{s=1}^k$ forms an orthonormal sequence of vectors in $\mathbb{R}^N$, and we denote by $\mathbf{P}^{(k)}$ and $\mathbf{Q}^{(k)}$ the corresponding orthogonal projections in $\mathbb{R}^N$, that is,

$$\mathbf{P}^{(k)} = \sum_{s=1}^k \phi^{(s)} \otimes \phi^{(s)} = (P_{ij}^{(k)})_{1 \leq i,j \leq N}, \quad \mathbf{Q}^{(k)} = 1 - \mathbf{P}^{(k)} = (Q_{ij}^{(k)})_{1 \leq i,j \leq N}.$$

By Ref. 16, Lemma 3, $\mathbf{m}^{(k)}$ and $\phi^{(k)}$ are $\mathcal{G}_{k-1}$ -measurable for all $k \in \mathbb{N}$, and we also have that

$$\mathbf{g}^{(k)} \phi^{(s)} = (\mathbf{g}^{(k)})^T \phi^{(s)} = \bar{\mathbf{g}}^{(k)} \phi^{(s)} = 0, \quad \forall s < k.$$

Proposition 3.2. (Ref. 16, Proposition 4)

- (1) Conditionally on $\mathcal{G}_{k-2}$, $\mathbf{g}^{(k)}$ and $\mathbf{g}^{(k-1)}$ are Gaussian with conditional covariance, given $\mathcal{G}_{k-2}$, equal to

$$\mathbb{E}_{k-2} g_{ij}^{(k)} g_{st}^{(k)} = \frac{1}{N} Q_{is}^{(k-1)} Q_{jt}^{(k-1)}.$$

- (2) Conditionally on $\mathcal{G}_{k-2}$, $\mathbf{g}^{(k)}$ is independent of $\mathcal{G}_{k-1}$. In particular, conditionally on $\mathcal{G}_{k-1}$, $\mathbf{g}^{(k)}$ is Gaussian with the same covariance as in (1).
 (3) Conditionally on $\mathcal{G}_{k-1}$, the random variables $\zeta^{(k)}$ are Gaussian with

$$\mathbb{E}_{k-1} \zeta_i^{(k)} \zeta_j^{(k)} = Q_{ij}^{(k-1)} + \frac{1}{N} \phi_i^{(k)} \phi_j^{(k)}.$$

The main result of Ref. 24 is summarized in the following proposition.

Proposition 3.3. (Ref. 24, Proposition 2.5 and Ref. 16, Proposition 6) For every $k \in \mathbb{N}$ and $s < k$, one has

$$\langle \mathbf{m}^{(k)}, \phi^{(s)} \rangle \simeq \gamma_s, \quad \langle \mathbf{m}^{(k)}, \mathbf{m}^{(s)} \rangle \simeq \alpha_s, \quad \langle \mathbf{m}^{(k)}, \mathbf{m}^{(k)} \rangle \simeq q.$$

The next lemma collects a few auxiliary results that are helpful in the sequel.

Lemma 3.4. [Ref. 16, Lemmas 11, 14, and 15(b)]

- (1) For every $k \in \mathbb{N}$, $\langle \phi^{(k)}, \zeta^{(k)} \rangle = \sqrt{2} \langle \phi^{(k)}, \mathbf{g}^{(k)} \phi^{(k)} \rangle$ is unconditionally Gaussian with variance $2/N$.
 (2) For every Lipschitz continuous $f: \mathbb{R} \rightarrow \mathbb{R}$ with $|f(x)| \leq C(1 + |x|)$ for some $C > 0$, one has for all $k \geq 2$,

$$\lim_{N \rightarrow \infty} \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N f(h_i^{(k+1)}) - \mathbb{E} f(\beta \sqrt{q} Z + h) \right| = 0.$$

- (3) For every $k \in \mathbb{N}$ and $t > 0$, it holds true that

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\left\{ \|\zeta^{(k)}\| \geq 1 + t \right\} \right) = 0.$$

IV. CONDITIONAL MOMENTS OF REDUCED PARTITION FUNCTION

Using Bolthausen's magnetizations, we compute in this section the first two conditional moments of a suitably reduced partition function onto a subset $S \subset \{-1, 1\}^N$ defined below. This will suffice to establish Theorem 1.1, as explained in Sec. V.

Let $\varepsilon > 0$ and $k \in \mathbb{N}$ be fixed. We define the set $S_{\varepsilon,k} \subset \{-1, 1\}^N$ through

$$S_{\varepsilon,k} = \left\{ \sigma \in \{-1, 1\}^N : |\langle \sigma - \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle| \leq \varepsilon/k, \quad \forall 1 \leq s \leq k \right\} \quad (4.1)$$

with $\phi^{(s)}, \gamma_s$ from Sec. III. We define the reduced partition function $Z_N^{(k+1)}(S_{\varepsilon,k})$ by

$$\begin{aligned} Z_N^{(k+1)}(S_{\varepsilon,k}) &= \sum_{\sigma \in S_{\varepsilon,k}} p_{\text{free}}(\sigma) \exp\left(\frac{N\beta}{\sqrt{2}} \langle \sigma, \mathbf{g}^{(k+1)} \rangle\right) \\ &= \sum_{\sigma \in S_{\varepsilon,k}} p_{\text{free}}(\sigma) \exp\left(\frac{N\beta}{2} \langle \sigma, \tilde{\mathbf{g}}^{(k+1)} \rangle\right), \end{aligned} \quad (4.2)$$

where $p_{\text{free}} : \{-1, 1\}^N \rightarrow (0, 1)$ denotes the coin-tossing measure,

$$p_{\text{free}}(\sigma) = \prod_{i=1}^N \frac{1}{2} \frac{\exp(h_i^{(k+1)} \sigma_i)}{\cosh(h_i^{(k+1)})}. \quad (4.3)$$

The following lemma records that $p_{\text{free}}(S_{\varepsilon,k}^c)$ is exponentially small in N .

Lemma 4.1. *Let $\varepsilon > 0$, $k \in \mathbb{N}$, and let $S_{\varepsilon,k}$ and p_{free} be defined as in (4.1) and (4.3), respectively. Then, there exist $c, C > 0$, independent of N and ε , such that*

$$p_{\text{free}}(S_{\varepsilon,k}) \geq 1 - C e^{-cN\varepsilon^2}. \quad (4.4)$$

Proof. By a standard union bound, we have that

$$\begin{aligned} p_{\text{free}}(S_{\varepsilon,k}^c) &\leq k \max_{s=1, \dots, k} p_{\text{free}}\left(\left\{ \sigma \in \{-1, 1\}^N : \langle \sigma, \phi^{(s)} \rangle - \langle \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle > \frac{\varepsilon}{k} \right\}\right) \\ &\quad + k \max_{s=1, \dots, k} p_{\text{free}}\left(\left\{ \sigma \in \{-1, 1\}^N : \langle \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle - \langle \sigma, \phi^{(s)} \rangle > \frac{\varepsilon}{k} \right\}\right), \end{aligned}$$

which implies that

$$\begin{aligned} p_{\text{free}}(S_{\varepsilon,k}^c) &\leq k \max_{s=1, \dots, k} \inf_{\lambda \geq 0} \exp\left[-N\left(\frac{\lambda\varepsilon}{k} - \frac{1}{N} \sum_{i=1}^N \log \frac{\cosh(h_i^{(k+1)} + \lambda\phi_i^{(s)})}{\cosh(h_i^{(k+1)})} + \lambda \langle \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle\right)\right] \\ &\quad + k \max_{s=1, \dots, k} \inf_{\lambda \geq 0} \exp\left[-N\left(\frac{\lambda\varepsilon}{k} - \frac{1}{N} \sum_{i=1}^N \log \frac{\cosh(h_i^{(k+1)} - \lambda\phi_i^{(s)})}{\cosh(h_i^{(k+1)})} - \lambda \langle \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle\right)\right]. \end{aligned}$$

Using the pointwise bound $\log \cosh(x+y) \leq \log \cosh(x) + y \tanh(x) + \frac{y^2}{2}$ for $x, y \in \mathbb{R}$, $\langle \phi^{(s)}, \phi^{(s)} \rangle = 1$, and the identity $\tanh(\mathbf{h}^{(k+1)}) = \mathbf{m}^{(k+1)}$, we obtain

$$p_{\text{free}}(S_{\varepsilon,k}^c) \leq 2k \inf_{\lambda \geq 0} \exp\left[-\frac{N\lambda\varepsilon}{k} + \frac{N\lambda^2}{2}\right] = 2ke^{-N\varepsilon^2/(2k^2)}.$$

This concludes (4.4) for $c = c_k = 1/(2k^2)$, $C = C_k = 2k$. □

We note that the constants $c, C > 0$ in (4.4) are independent of the realization of the disorder $\{g_{ij}\}$. Thus, *a.s.* in the disorder (so that $\phi^{(s)}, s = 1, \dots, k$, and, hence, $\tilde{\mathbf{g}}^{(k+1)}$ and p_{free} are well-defined), $S_{\varepsilon,k} \neq \emptyset$ for N large enough.

The next lemma determines the first conditional moment of the reduced partition function $Z_N^{(k+1)}(S_{\varepsilon,k})$ and is valid in the full high temperature regime (1.3).

Lemma 4.2. *Let $\varepsilon > 0$, $k \in \mathbb{N}$, and let $S_{\varepsilon,k}$, $Z_N^{(k+1)}(S_{\varepsilon,k})$, and p_{free} be as in (4.1)–(4.3), respectively. Assume that (β, h) satisfy the AT condition (1.3). Then,*

$$\lim_{\varepsilon \rightarrow 0} \lim_{k \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E} \left| \frac{1}{N} \log \mathbb{E}_k Z_N^{(k+1)}(S_{\varepsilon,k}) - \frac{\beta^2}{4} (1-q)^2 \right| = 0. \quad (4.5)$$

Proof. By Proposition 3.2, we have that

$$\begin{aligned} \mathbb{E}_k Z_N^{(k+1)}(S_{\varepsilon,k}) &= \sum_{\sigma \in S_{\varepsilon,k}} p_{\text{free}}(\sigma) \exp \left(\frac{\beta^2 N^2}{4} \mathbb{E}_k \langle \sigma, \mathbf{g}^{(k+1)} \sigma \rangle^2 \right) \\ &= \sum_{\sigma \in S_{\varepsilon,k}} p_{\text{free}}(\sigma) \exp \left[\frac{\beta^2 N}{4} \left(1 - \sum_{s=1}^k \langle \sigma, \phi^{(s)} \rangle^2 \right)^2 \right]. \end{aligned}$$

Centering around $\mathbf{m}^{(k+1)}$ yields

$$1 - \sum_{s=1}^k \langle \sigma, \phi^{(s)} \rangle^2 = 1 - \sum_{s=1}^k \langle \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle^2 - 2 \sum_{s=1}^k \langle \sigma - \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle \langle \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle - \sum_{s=1}^k \langle \sigma - \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle^2$$

so that

$$\sup_{\sigma \in S_{\varepsilon,k}} \left| \left(1 - \sum_{s=1}^k \langle \sigma, \phi^{(s)} \rangle^2 \right) - \left(1 - \sum_{s=1}^k \langle \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle^2 \right) \right| \leq C \sup_{\sigma \in S_{\varepsilon,k}} \sum_{s=1}^k |\langle \sigma - \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle| \leq C\varepsilon$$

for some $C > 0$ independent of N and k . This implies with Lemma 4.1 that

$$\left| \frac{1}{N} \log \mathbb{E}_k Z_N^{(k+1)}(S_{\varepsilon,k}) - \frac{\beta^2}{4} \left(1 - \sum_{s=1}^k \langle \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle^2 \right)^2 \right| \leq C\beta^2 \varepsilon + \frac{C}{N} |\log(1 - Ce^{-cN\varepsilon^2})|.$$

Moreover, by Proposition 3.3, we have that $\lim_{N \rightarrow \infty} \sum_{s=1}^k \langle \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle^2 = \Gamma_k^2$ in $L^p(d\mathbb{P})$ for any $p \in [1; \infty)$ so that

$$\limsup_{N \rightarrow \infty} \mathbb{E} \left| \frac{1}{N} \log \mathbb{E}_k Z_N^{(k+1)}(S_{\varepsilon,k}) - \frac{\beta^2}{4} (1 - \Gamma_k^2)^2 \right| \leq C\beta^2 \varepsilon.$$

Finally, since $\lim_{k \rightarrow \infty} \Gamma_k^2 = q$ under the AT condition (1.3), by Lemma 3.1, we let $N \rightarrow \infty$, then $k \rightarrow \infty$, and then $\varepsilon \rightarrow 0$, which implies that

$$\lim_{\varepsilon \rightarrow 0} \lim_{k \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E} \left| \frac{1}{N} \log \mathbb{E}_k Z_N^{(k+1)}(S_{\varepsilon,k}) - \frac{\beta^2}{4} (1-q)^2 \right| = 0.$$

□

The following lemma computes the second conditional moment of $Z_N^{(k+1)}(S_{\varepsilon,k})$ under the stronger high temperature condition (1.5).

Lemma 4.3. *Let $\varepsilon > 0$, $k \in \mathbb{N}$, and let $S_{\varepsilon,k}$, $Z_N^{(k+1)}(S_{\varepsilon,k})$, and p_{free} be as in (4.1)–(4.3), respectively. Assume that (β, h) satisfy condition (1.5). Then,*

$$\lim_{\varepsilon \rightarrow 0} \lim_{k \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E} \left| \frac{1}{N} \log \mathbb{E}_k \left[\left(Z_N^{(k+1)}(S_{\varepsilon,k}) \right)^2 \right] - \frac{\beta^2}{2} (1-q)^2 \right| = 0. \quad (4.6)$$

Proof. Proceeding as in the previous proposition, we compute

$$\begin{aligned}\mathbb{E}_k \left[\left(Z_N^{(k+1)}(S_{\varepsilon,k}) \right)^2 \right] &= \sum_{\sigma, \tau \in S_{\varepsilon,k}} p_{\text{free}}(\sigma) p_{\text{free}}(\tau) \exp \left(\frac{\beta^2 N^2}{4} \mathbb{E}_k \left(\langle \sigma, \mathbf{g}^{(k+1)} \rangle + \langle \tau, \mathbf{g}^{(k+1)} \rangle \right)^2 \right) \\ &= \sum_{\sigma, \tau \in S_{\varepsilon,k}} p_{\text{free}}(\sigma) p_{\text{free}}(\tau) \exp \left[\frac{\beta^2 N}{4} \left(1 - \sum_{s=1}^k \langle \sigma, \phi^{(s)} \rangle^2 \right)^2 + \frac{\beta^2 N}{4} \left(1 - \sum_{s=1}^k \langle \tau, \phi^{(s)} \rangle^2 \right)^2 \right] \\ &\quad \times \exp \left[\frac{\beta^2 N}{2} \left(\langle \sigma, \tau \rangle - \sum_{s=1}^k \langle \sigma, \phi^{(s)} \rangle \langle \phi^{(s)} \tau \rangle \right)^2 \right] \\ &= \sum_{\sigma, \tau \in S_{\varepsilon,k}} p_{\text{free}}(\sigma) p_{\text{free}}(\tau) \exp \left[\frac{\beta^2 N}{2} \langle \sigma, \mathbf{Q}^{(k)} \tau \rangle^2 \right] \\ &\quad \times \exp \left[\frac{\beta^2 N}{2} \left(1 - \sum_{s=1}^k \langle \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle^2 \right)^2 + N \mathcal{O}(\beta^2 \varepsilon) \right].\end{aligned}$$

Arguing as in the previous lemma, we, therefore, see that it is enough to show that

$$\mathbb{E} \left[\frac{1}{N} \log \sum_{\sigma, \tau \in S_{\varepsilon,k}} p_{\text{free}}(\sigma) p_{\text{free}}(\tau) \exp \left[\frac{\beta^2 N}{2} \langle \sigma, \mathbf{Q}^{(k)} \tau \rangle^2 \right] \right]$$

vanishes when $N \rightarrow \infty$. To this end, recall that by the definition of $S_{\varepsilon,k}$, we have that

$$\sup_{\tau \in S_{\varepsilon,k}} \left| \|\mathbf{Q}^{(k)} \tau\|^2 - \left(1 - \langle \mathbf{m}^{(k+1)}, \mathbf{P}^{(k)} \mathbf{m}^{(k+1)} \rangle \right) \right| \leq C\varepsilon \quad (4.7)$$

for some $C > 0$ independent of N and k . For fixed $\tau \in S_{\varepsilon,k}$, we, then, have

$$\begin{aligned}p_{\text{free}}(S_{\varepsilon,k}) &\leq \sum_{\sigma \in S_{\varepsilon,k}} p_{\text{free}}(\sigma) \exp \left[\frac{\beta^2 N}{2} \langle \sigma, \mathbf{Q}^{(k)} \tau \rangle^2 \right] \\ &\leq p_{\text{free}}(S_{\varepsilon,k}) + \int_0^1 dt N \beta^2 t e^{\frac{N}{2} \beta^2 t^2} p_{\text{free}} \left(\left\{ \sigma \in \{-1, 1\}^N : |\langle \sigma, \mathbf{Q}^{(k)} \tau \rangle| > t \right\} \right).\end{aligned}$$

Setting $\lambda = t/(1-q)$ and using that $\log \cosh(x+y) \leq \log \cosh(x) + y \tanh(x) + y^2/2$ for $x, y \in \mathbb{R}$, we can estimate the tail probability in the integral by

$$\begin{aligned}&p_{\text{free}} \left(\left\{ \sigma \in \{-1, 1\}^N : |\langle \sigma, \mathbf{Q}^{(k)} \tau \rangle| > t \right\} \right) \\ &\leq \exp \left[-N\lambda t + N\lambda \langle \mathbf{m}^{(k+1)}, \mathbf{Q}^{(k)} \tau \rangle + \frac{N\lambda^2}{2} \|\mathbf{Q}^{(k)} \tau\|^2 \right] \\ &\quad + \exp \left[-N\lambda t - N\lambda \langle \mathbf{m}^{(k+1)}, \mathbf{Q}^{(k)} \tau \rangle + \frac{N\lambda^2}{2} \|\mathbf{Q}^{(k)} \tau\|^2 \right] \\ &\leq 2 \exp \left[-\frac{Nt^2}{1-q} + \frac{Nt^2}{2} \frac{\|\mathbf{Q}^{(k)} \tau\|^2}{(1-q)^2} \right] \exp \left[\frac{Nt}{1-q} \|\mathbf{Q}^{(k)} \mathbf{m}^{(k+1)}\| \right]\end{aligned}$$

so that

$$\begin{aligned}p_{\text{free}}(S_{\varepsilon,k}) &\leq \sum_{\sigma \in S_{\varepsilon,k}} p_{\text{free}}(\sigma) \exp \left[\frac{\beta^2 N}{2} \langle \sigma, \mathbf{Q}^{(k)} \tau \rangle^2 \right] \\ &\leq p_{\text{free}}(S_{\varepsilon,k}) + 2 \int_0^1 dt N \beta^2 t e^{\frac{N}{2} \beta^2 t^2} \left(\beta^2 (1-q)^{-2} + \frac{\|\mathbf{Q}^{(k)} \tau\|^2}{(1-q)} \right) e^{\frac{Nt}{1-q} \|\mathbf{Q}^{(k)} \mathbf{m}^{(k+1)}\|}.\end{aligned}$$

In particular, by (4.7) and because

$$\lim_{N \rightarrow \infty} \langle \mathbf{m}^{(k+1)}, \mathbf{P}^{(k)} \mathbf{m}^{(k+1)} \rangle = \Gamma_k^2, \quad \lim_{N \rightarrow \infty} \|\mathbf{Q}^{(k)} \mathbf{m}^{(k+1)}\|^2 = q - \Gamma_k^2,$$

in $L^p(d\mathbb{P})$ for $p \in [1; \infty)$, we obtain under condition (1.5), i.e., $\beta^2(1 - q) \leq 1$, that

$$\lim_{\varepsilon \rightarrow 0} \lim_{k \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E} \left| \frac{1}{N} \log \sum_{\sigma, \tau \in S_{\varepsilon, k}} p_{\text{free}}(\sigma) p_{\text{free}}(\tau) \exp \left[\frac{\beta^2 N}{2} \langle \sigma, \mathbf{Q}^{(k)} \tau \rangle^2 \right] \right| = 0,$$

which implies (4.6). $\square$

Remark. The key difficulty in the Proof of Lemma 4.3 is to obtain a strong concentration bound on the overlap $(\sigma, \tau) \mapsto \langle \sigma, \tau \rangle$ under the product measure $p_{\text{free}}^{\otimes 2}$, restricted to $S_{\varepsilon, k} \times S_{\varepsilon, k} \subset \{-1, 1\}^N \times \{-1, 1\}^N$. In our proof, the condition $\beta^2(1 - q) \leq 1$ emerges due to the restrictions on one of the spin variables, say, $\tau \in S_{\varepsilon, k}$. In principle, one may be able to find the optimal temperature condition by taking into account the restrictions on the other spin variable and using standard large deviation variational estimates, but an exact solution seems difficult. We hope to get back to this point in future work.

V. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1 based on Lemmas 4.2 and 4.3. Before we start, let us first re-center the Hamiltonian H_N appropriately, as outlined in the Introduction.

Using the notation of Sec. III, we have that

$$\frac{H_N(\sigma)}{N} = \frac{\beta}{2} \langle \sigma, \tilde{\mathbf{g}} \sigma \rangle + \langle h, \sigma \rangle = \frac{\beta}{2} \langle \sigma, \tilde{\mathbf{g}}^{(k+1)} \sigma \rangle + \frac{\beta}{2} \sum_{s=1}^k \langle \sigma, \tilde{\rho}^{(s)} \sigma \rangle + \langle h, \sigma \rangle.$$

In contrast to Ref. 16, instead of centering the spins σ around $\mathbf{m}^{(k+1)}$, we center the spins in $\langle \sigma, \tilde{\rho}^{(s)} \sigma \rangle$ around $\gamma_s \phi^{(s)}$ in order to produce the right cavity field $\mathbf{h}^{(k+1)}$. Note that the remaining term $\langle \sigma, \tilde{\mathbf{g}}^{(k+1)} \sigma \rangle$ contains automatically centered spins around $\sum_{s=1}^k \gamma_s \phi^{(s)}$ (which approximately equals $\mathbf{m}^{(k+1)}$), as $\tilde{\mathbf{g}}^{(k+1)} \phi^{(s)} = 0$ for $s < k + 1$. We, thus, write

$$\begin{aligned} \langle \sigma, \tilde{\rho}^{(s)} \sigma \rangle &= 2\gamma_s \langle \sigma, \tilde{\rho}^{(s)} \phi^{(s)} \rangle + \langle \sigma - \gamma_s \phi^{(s)}, \tilde{\rho}^{(s)} (\sigma - \gamma_s \phi^{(s)}) \rangle - \gamma_s^2 \langle \phi^{(s)}, \tilde{\rho}^{(s)} \phi^{(s)} \rangle \\ &= 2\gamma_s \langle \sigma, \zeta^{(s)} \rangle + \langle \tilde{\sigma}^{(s)}, \tilde{\rho}^{(s)} \tilde{\sigma}^{(s)} \rangle - \gamma_s^2 \langle \phi^{(s)}, \zeta^{(s)} \rangle, \end{aligned}$$

which follows from $\tilde{\rho}^{(s)} \phi^{(s)} = \zeta^{(s)}$ and where we set $\tilde{\sigma}^{(s)} = \sigma - \gamma_s \phi^{(s)}$. Hence,

$$\frac{H_N(\sigma)}{N} = \frac{\beta}{2} \langle \sigma, \tilde{\mathbf{g}}^{(k+1)} \sigma \rangle + \langle \mathbf{h}^{(k+1)}, \sigma \rangle + \frac{\beta}{2} \sum_{s=1}^k \langle \tilde{\sigma}^{(s)}, \tilde{\rho}^{(s)} \tilde{\sigma}^{(s)} \rangle - \frac{\beta}{2} \sum_{s=1}^k \gamma_s^2 \langle \phi^{(s)}, \zeta^{(s)} \rangle + \beta \left(\gamma_k - \sqrt{q - \Gamma_{k-1}^2} \right) \langle \sigma, \zeta^{(k)} \rangle.$$

Since an exact evaluation of the free energy seems rather involved, let us note here that for configurations $\sigma \in S_{\varepsilon, k}$ as defined in (4.1), instead, we have approximately $H_N(\sigma)/N \approx \frac{\beta}{2} \langle \sigma, \tilde{\mathbf{g}}^{(k+1)} \sigma \rangle + \langle \mathbf{h}^{(k+1)}, \sigma \rangle$. Indeed, we find that

$$\langle \tilde{\sigma}^{(s)}, \tilde{\rho}^{(s)} \tilde{\sigma}^{(s)} \rangle = 2 \langle \sigma - \gamma_s \phi^{(s)}, \zeta^{(s)} \rangle (\langle \phi^{(s)}, \sigma \rangle - \gamma_s) - \langle \phi^{(s)}, \zeta^{(s)} \rangle (\langle \phi^{(s)}, \sigma \rangle - \gamma_s)^2$$

with $\gamma_s \simeq \langle \mathbf{m}^{(k+1)}, \phi^{(s)} \rangle$. Similarly, recall that $\langle \phi^{(s)}, \zeta^{(s)} \rangle \sim \mathcal{N}(0, 2/N)$ for each s and

$$\left| \left(\gamma_k - \sqrt{q - \Gamma_k^2} \right) \sup_{\sigma \in \{-1, 1\}^N} \langle \sigma, \zeta^{(k)} \rangle \right| \leq \left(\gamma_k + \sqrt{q - \Gamma_k^2} \right) \|\zeta^{(k)}\|$$

with $\lim_{k \rightarrow \infty} |q - \Gamma_k^2| = \lim_{k \rightarrow \infty} \gamma_k = 0$ under (1.3) by Lemmas 3.1 and 3.4.

Thus, we obtain the simple lower bound

$$\begin{aligned} \frac{1}{N} \log Z_N &= \frac{1}{N} \log \sum_{\sigma \in \{-1, 1\}^N} e^{H_N(\sigma)} \\ &= \log 2 + \frac{1}{N} \sum_{i=1}^N \log \cosh(h_i^{(k+1)}) + \frac{1}{N} \log \sum_{\sigma \in \{-1, 1\}^N} p_{\text{free}}(\sigma) e^{H_N(\sigma) - N \langle \mathbf{h}^{(k+1)}, \sigma \rangle} \\ &\geq \log 2 + \frac{1}{N} \sum_{i=1}^N \log \cosh(h_i^{(k+1)}) + \frac{1}{N} \log \sum_{\sigma \in S_{\varepsilon, k}} p_{\text{free}}(\sigma) e^{H_N(\sigma) - N \langle \mathbf{h}^{(k+1)}, \sigma \rangle} \end{aligned}$$

so that

$$\begin{aligned} \frac{1}{N} \log Z_N &\geq \log 2 + \frac{1}{N} \sum_{i=1}^N \log \cosh(h_i^{(k+1)}) + \frac{1}{N} \log Z_N^{(k+1)}(S_{\varepsilon,k}) \\ &\quad - C \frac{\beta \varepsilon}{k} \sum_{s=1}^k \|\zeta^{(s)}\| - C \beta \sum_{s=1}^k \|\zeta^{(s)}\| \left| \langle \phi^{(s)}, \mathbf{m}^{(k+1)} \rangle - \gamma_s \right| \\ &\quad - \frac{\beta}{2} \sum_{s=1}^k \gamma_s^2 \langle \phi^{(s)}, \zeta^{(s)} \rangle - \beta \gamma_k \|\zeta^{(k)}\| - \beta \sqrt{q - \Gamma_k^2} \|\zeta^{(k)}\|. \end{aligned} \quad (5.1)$$

We have now all necessary preparations for the Proof of Theorem 1.1.

Proof of Theorem 1.1. Up to minor modifications, we follow Ref. 16, Sec. 3 and we also abbreviate $RS(\beta, h) = \log 2 + E \log \cosh(\beta \sqrt{q} Z + h) + \beta^2(1 - q)^2/4$.

By the Paley–Zygmund inequality, we have that

$$\mathbb{P}_k \left(Z_N^{(k+1)}(S_{\varepsilon,k}) \geq \mathbb{E}_k Z_N^{(k+1)}(S_{\varepsilon,k})/2 \right) \geq \frac{\left(\mathbb{E}_k (Z_N^{(k+1)}(S_{\varepsilon,k}))^2 \right)}{4 \mathbb{E}_k \left[(Z_N^{(k+1)}(S_{\varepsilon,k}))^2 \right]}.$$

Given $\delta_1 > 0$, Lemmas 4.2 and 4.3 imply that

$$\mathbb{P} \left(\frac{2}{N} \log \mathbb{E}_k Z_N^{(k+1)}(S_{\varepsilon,k}) \geq \frac{1}{N} \log 4 \mathbb{E}_k (Z_N^2(S_{\varepsilon,k})) - \delta_1 \right) \geq \frac{1}{2}$$

if we choose $\varepsilon > 0$ sufficiently small, k sufficiently large, and $N \geq N_1(\varepsilon, k) \in \mathbb{N}$ sufficiently large. This also implies that

$$\mathbb{P} \left(\mathbb{P}_k \left(\frac{1}{N} \log Z_N^{(k+1)}(S_{\varepsilon,k}) \geq \frac{1}{N} \log \mathbb{E}_k Z_N^{(k+1)}(S_{\varepsilon,k}) - \frac{\log 2}{N} \right) \geq e^{-\delta_1 N} \right) \geq \frac{1}{2}.$$

On the other hand, applying Lemma 3.4 (3), we note that

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\frac{1}{k} \sum_{s=1}^k \|\zeta^{(s)}\| > 2 \right) \leq \lim_{N \rightarrow \infty} \sum_{s=1}^k \mathbb{P} \left(\|\zeta^{(s)}\| > 2 \right) = 0$$

and that

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\|\zeta^{(k+1)}\| > 2 \right) = 0.$$

Moreover, Lemma 3.4 and the fact that $\langle \phi^{(s)}, \mathbf{m}^{(k+1)} \rangle \simeq \gamma_s$ by Proposition 3.3 imply that

$$\lim_{N \rightarrow \infty} \left(C \beta \sum_{s=1}^k \|\zeta^{(s)}\| \left| \langle \phi^{(s)}, \mathbf{m}^{(k+1)} \rangle - \gamma_s \right| + \frac{\beta}{2} \sum_{s=1}^k \gamma_s^2 \langle \phi^{(s)}, \zeta^{(s)} \rangle \right) = 0$$

and

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \log \cosh(h_i^{(k+1)}) = E \log \cosh(\beta \sqrt{q} Z + h)$$

in probability. Now, the lower bound (5.1) implies that

$$\mathbb{P} \left(\mathbb{P}_k \left(\frac{1}{N} \log Z_N \geq \log 2 + \frac{1}{N} \sum_{i=1}^N \log \cosh(\beta \sqrt{q} Z + h) + \frac{1}{N} \log \mathbb{E}_k Z_N^{(k+1)}(S_{\varepsilon,k}) - \frac{\log 2}{N} - \varepsilon_{\varepsilon,k} \right) \geq e^{-\delta_1 N} \right) \geq \frac{1}{2},$$

where we defined the error $\mathcal{E}_{\varepsilon,k}$ by

$$\begin{aligned}\mathcal{E}_{\varepsilon,k} = & \frac{C\beta\varepsilon}{k} \sum_{s=1}^k \|\zeta^{(s)}\| + C\beta \sum_{s=1}^k \|\zeta^{(s)}\| \left| \langle \phi^{(s)}, \mathbf{m}^{(k+1)} \rangle - \gamma_s \right| \\ & + \frac{\beta}{2} \sum_{s=1}^k \gamma_s^2 \langle \phi^{(s)}, \zeta^{(s)} \rangle + \beta \gamma_k \|\zeta^{(k)}\| + \beta \sqrt{q - \Gamma_k^2} \|\zeta^{(k)}\|.\end{aligned}$$

Given $\delta_2 > 0$, we may choose $\varepsilon > 0$ sufficiently small, $k \in \mathbb{N}$ sufficiently large, and $N \geq N_2(k, \varepsilon) \in \mathbb{N}$ sufficiently large such that

$$\mathbb{P}\left(|\mathcal{E}_{\varepsilon,k}| \leq \frac{\delta_2}{4}, \left| \frac{1}{N} \sum_{i=1}^N \log \cosh(h_i^{(k+1)}) - E \log \cosh(\beta \sqrt{q} Z + h) \right| \leq \frac{\delta_2}{4}\right) \geq \frac{7}{8}$$

and, by Lemma 4.2, also such that

$$\mathbb{P}\left(\frac{1}{N} \log \mathbb{E}_k Z_N^{(k+1)}(S_{\varepsilon,k}) \geq \frac{\beta^2}{4} (1-q)^2 - \frac{\delta_2}{4}\right) \geq \frac{7}{8}.$$

Combining the above observations, we find that

$$\mathbb{P}\left(\mathbb{P}_k\left(\frac{1}{N} \log Z_N \geq RS(\beta, h) - \delta_2\right) \geq e^{-\delta_1 N}\right) \geq \frac{1}{4}$$

for all $N \geq \max(N_1(\varepsilon, k), N_2(\varepsilon, k), 4 \log 2 / \delta_2)$. This implies that

$$\mathbb{P}\left(\frac{1}{N} \log Z_N \geq RS(\beta, h) - \delta_2\right) \geq \frac{1}{4} e^{-\delta_1 N}.$$

By Gaussian concentration of the free energy, i.e.,

$$\mathbb{P}\left(\left|\frac{1}{N} \log Z_N - \frac{1}{N} \mathbb{E} \log Z_N\right| \leq \delta_3\right) \geq 1 - e^{-N\delta_3^2/\beta^2},$$

we may choose $\delta_1 < \delta_3^2/(2\beta^2)$ to conclude for large enough N that

$$\mathbb{E} \frac{1}{N} \log Z_N \geq RS(\beta, h) - \delta_2 - \delta_3.$$

Since $\delta_2, \delta_3 > 0$ were arbitrary, this shows that the right-hand side of (1.6) is a lower bound to $\lim_{N \rightarrow \infty} \mathbb{E} \frac{1}{N} \log Z_N$, and by Remark (2), this proves Theorem 1.1. $\square$

ACKNOWLEDGMENTS

The work of H.-T.Y. was partially supported by the NSF under Grant No. DMS-1855509 and a Simons Investigator Award. The work of C.B. was partly funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – GZ 2047/1, Projekt-ID 390685813

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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