INEQUALITIES OF RIESZ-SOBOLEV TYPE FOR COMPACT CONNECTED ABELIAN GROUPS

MICHAEL CHRIST AND MARINA ILIOPOULOU

Abstract. An analogue of the Riesz-Sobolev convolution inequality is formulated and proved for arbitrary compact connected Abelian groups. Maximizers are characterized, and a quantitative stability theorem is proved, under natural hypotheses. A corresponding stability theorem for sets whose sumset has nearly minimal measure is also proved, sharpening recent results of other authors. For the special case of the group $\mathbb{R}/\mathbb{Z}$, a continuous deformation of sets is developed, under which a scaled Riesz-Sobolev functional is shown to be nondecreasing.

Contents

1. Introduction 1
2. Alternative formulation of (1.2) 8
3. Refinement of a related inequality 8
4. On the Riesz-Sobolev inequality for G 12
5. Two key principles 16
6. A link between Riesz-Sobolev and sumset inequalities 17
7. Two reductions 18
8. Relaxation 23
9. The perturbative Riesz-Sobolev regime 25
10. The perturbative regime for sumsets 32
11. A special case on $\mathbb{T}$ 37
12. When one set is nearly rank one Bohr 42
13. Stability of the Riesz-Sobolev inequality 45
14. Cases of equality in the Riesz-Sobolev inequality 47
15. Stability in the relaxed framework 48
16. A flow of subsets of $\mathbb{T}$ 49
17. Concluding steps for $\mathbb{T}$ 53
References 54

1. Introduction

Let G be a compact connected Abelian topological group, equipped with Haar measure μ. Throughout this paper, the measure μ is assumed to be complete. We say that μ is normalized to mean that $\mu(G) = 1$. By a measurable subset of G we will always mean a

The first author was supported in part by NSF grants DMS-1363324 and DMS-1901413. Both authors were supported in part by the Mathematical Sciences Research Institute, which in turn was also supported in part by the National Science Foundation under Grant No. 1440140. Much of this work was carried out while the second author held a postdoctoral position at UC Berkeley.

μ–measurable subset. μ_* denotes the associated inner measure. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, equipped with Lebesgue measure m, with $m(\mathbb{T}) = 1$.

1.1. **Riesz-Sobolev-type inequality.** The Riesz-Sobolev inequality for $\mathbb{R}^d$ states that for any three Lebesgue measurable subsets $A, B, C \subset \mathbb{R}^d$,

$$\int_C \mathbf{1}_A * \mathbf{1}_B \le \int_{C^\star} \mathbf{1}_{A^\star} * \mathbf{1}_{B^\star}. \tag{1.1}$$

Here, the symmetrizations $A^\star, B^\star, C^\star$ are the closed balls centered at $0 \in \mathbb{R}^d$ whose Lebesgue measures are equal to the Lebesgue measures of A, B, C, respectively. Integration is with respect to Lebesgue measure. $\mathbf{1}_A$ denotes the indicator function of A. See for instance [22].

Our first result is one of several formulations of a Riesz-Sobolev–type inequality for G. Convolution on G is defined by $f * g(x) = \int_G f(x-y)g(y)\,d\mu(y)$. Assuming μ to be normalized, to any measurable set $A \subset G$ we associate the set $A^\star \subset \mathbb{T}$, which is defined to be the closed interval centered at 0 satisfying $m(A^\star) = \mu(A)$. In contrast to the Euclidean case, $A^\star$ is a subset of $\mathbb{T}$, rather than of G.

Theorem 1.1. *Let G be a compact connected Abelian topological group, equipped with normalized Haar measure. For any measurable subsets $A, B, C \subset G$,*

$$\int_C \mathbf{1}_A * \mathbf{1}_B \; d\mu \le \int_{C^\star} \mathbf{1}_{A^\star} * \mathbf{1}_{B^\star} \; dm. \tag{1.2}$$

As is the case for $\mathbb{R}^d$, the inequality for indicator functions implies the generalization

$$\langle f * g, h \rangle_G \le \langle f^\star * g^\star, h^\star \rangle_{\mathbb{T}} \tag{1.3}$$

for arbitrary nonnegative measurable functions defined on G, with the pairing $\langle \varphi, \psi \rangle_G = \int_G \varphi\psi\, d\mu$ of real-valued functions, and with the natural extension of the definition of symmetrization $f^\star$ from indicator functions to general nonnegative functions. Thus if $\mathbb{T}$ is identified with $(-\frac{1}{2}, \frac{1}{2})$ up to a null set by identifying each equivalence class in $\mathbb{R}/\mathbb{Z}$ with its unique representative in this domain, then $f^\star$ is even, is nonincreasing on $[0, \frac{1}{2}]$, and is equimeasurable with f. Theorem 1.5 further extends (1.3).

For $G = \mathbb{T}$, Theorem 1.1 was proved by Baernstein [3],[4], and was stated by Luttinger [23]. For general compact connected Abelian groups, inequality (1.2) is closely related to an inequality of Tao [25], of which an equivalent formulation is

$$\int_G \max(\mathbf{1}_A * \mathbf{1}_B - \tau, 0)\, d\mu \le \int_{\mathbb{T}} \max(\mathbf{1}_{A^\star} * \mathbf{1}_{B^\star} - \tau, 0)\, dm \tag{1.4}$$

for every parameter $\tau \in [0, \min(\mu(A), \mu(B))]$. This inequality is the basis for our proof of (1.2), and is discussed in §3.

1.2. **Characterization of maximizers.** The primary subject of this paper is the inverse problem of quantitatively characterizing triples (A, B, C) that maximize, or nearly maximize, the functional $\int_C \mathbf{1}_A * \mathbf{1}_B\, d\mu$ among all sets of specified Haar measures. Roughly speaking, we show that such A, B, C are rank one Bohr sets (or are well approximated by such). We describe all maximizers in Theorem 1.2 below, after defining the notions relevant to its statement. Near-maximizers are studied in the next subsection.

Definition 1.1. Two measurable sets $A, A' \subset G$ are equivalent if $\mu(A \,\Delta\, A') = 0$. Likewise, two ordered triples $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{E}' = (E'_1, E'_2, E'_3)$ are equivalent if E_j is equivalent to E'_j for each $j \in \{1, 2, 3\}$.

Definition 1.2. For $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\|x\|_{\mathbb{T}} = |y|$ where $y \in [-\frac{1}{2}, \frac{1}{2}]$ is congruent to x modulo 1.

Definition 1.3. A rank one Bohr set $\mathcal{B} \subset G$ is a set of the form

$$\mathcal{B} = \mathcal{B}(\phi, \rho, c) = \{x \in G : \|\phi(x) - c\|_{\mathbb{T}} \le \rho\}, \tag{1.5}$$

where $\phi : G \to \mathbb{T}$ is a continuous homomorphism, $c \in \mathbb{T}$, and $\rho \in [0, \frac{1}{2}]$.

By a homomorphism $\phi : G \to \mathbb{T}$, we will always mean a continuous homomorphism.

Definition 1.4. Two rank one Bohr subsets $\mathcal{B}_1, \mathcal{B}_2$ of G are parallel if they can be represented as $\mathcal{B}_j = \mathcal{B}(\phi_j, c_j, \rho_j)$ with $\phi_1 = \phi_2$. An ordered triple $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ of rank one Bohr subsets of G is parallel if these three sets are pairwise parallel. An ordered triple $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ of Bohr sets $\mathcal{B}_j = \mathcal{B}(\phi_j, c_j, \rho_j)$ is compatibly centered if $c_3 = c_1 + c_2$.

Burchard [6] characterized cases of equality in the Riesz-Sobolev inequality for Euclidean space of arbitrary dimension. As was discussed in [6], if $\mu(C) > \mu(A) + \mu(B)$ then no characterization of cases of equality is possible for the Riesz-Sobolev inequality (1.1), beyond the necessary and sufficient condition that $\mathbf{1}_A * \mathbf{1}_B$ should vanish μ–almost everywhere on the complement of C. This motivates the following definition.

Definition 1.5. Let (E_1, E_2, E_3) be an ordered triple of measurable subsets of G. (E_1, E_2, E_3) is admissible if $0 < \mu(E_i) < 1$ for each $i \in \{1, 2, 3\}$, $\mu(E_1) + \mu(E_2) + \mu(E_3) < 2$, and $\mu(E_k) \le \mu(E_i) + \mu(E_j)$ for each permutation (i, j, k) of $(1, 2, 3)$.

Admissibility of a triple of sets is a property only of the associated triple of measures, so we will often write instead that $(\mu(E_1), \mu(E_2), \mu(E_3))$ is admissible.

The condition that $\mu(E_k) \le \mu(E_i) + \mu(E_j)$ for every permutation (i, j, k) of $(1, 2, 3)$ can be equivalently formulated as the condition

$$|\mu(E_i) - \mu(E_j)| \le \mu(E_k) \le \mu(E_i) + \mu(E_j) \tag{1.6}$$

for any single permutation.

Theorem 1.2 (Uniqueness of maximizers up to symmetries)**.** *Let G be a compact connected Abelian topological group equipped with Haar measure μ satisfying $\mu(G) = 1$. Let (A, B, C) be an admissible triple of measurable subsets of G. Then $\int_C \mathbf{1}_A * \mathbf{1}_B \, d\mu = \int_{C^*} \mathbf{1}_{A^\star} * \mathbf{1}_{B^\star} \, dm$ if and only if (A, B, C) is equivalent to a compatibly centered parallel ordered triple of rank one Bohr sets.*

A stronger result is formulated below in Theorem 1.3.

1.3. **Stability in the Riesz-Sobolev inequality for G.**

Near-maximizers for the Riesz-Sobolev inequality will be studied for triples of sets that satisfy a strict admissibility condition.

Definition 1.6. Let (E_1, E_2, E_3) be an ordered triple of measurable subsets of G. (E_1, E_2, E_3) is strictly admissible if it is admissible and $\mu(E_k) < \mu(E_i) + \mu(E_j)$ for every permutation (i, j, k) of $(1, 2, 3)$.

For any $\eta > 0$, (E_1, E_2, E_3) is η–strictly admissible if it is admissible and

$$\mu(E_k) \le \mu(E_i) + \mu(E_j) - \eta \max(\mu(E_1), \mu(E_2), \mu(E_3)) \tag{1.7}$$

for every permutation (i, j, k) of $(1, 2, 3)$.

Strict and η-strict admissibility are each equivalent to conditions analogous to (1.6). Simple consequences of η–strict admissibility are

$$\mu(E_i) \ge |\mu(E_j) - \mu(E_k)| + \eta \max(\mu(E_1), \mu(E_2), \mu(E_3)), \tag{1.8}$$

$$\mu(E_i) \ge \eta \mu(E_j) \tag{1.9}$$

for every permutation (i, j, k) of $(1, 2, 3)$.

Definition 1.7. The ordered triple (A, B, C) of measurable subsets of G is η–bounded if it satisfies

$$\mu(A) + \mu(B) + \mu(C) \le (2 - \eta)\mu(G), \tag{1.10}$$

$$\min(\mu(A), \mu(B), \mu(C)) \ge \eta \mu(G). \tag{1.11}$$

If (A, B, C) is η–strictly admissible and satisfies (1.10) then

$$\max(\mu(A), \mu(B), \mu(C)) \le \tfrac{2-\eta}{2+\eta} \le 1 - \tfrac{\eta}{2}.$$

Indeed, suppose that $\mu(C)$ is largest. Since $\mu(A) + \mu(B) \ge (1 + \eta)\mu(C)$, $(2 + \eta)\mu(C) \le \mu(A) + \mu(B) + \mu(C) \le 2 - \eta$. □

For every strictly admissible triple (A, B, C), there exists $\eta > 0$ for which (A, B, C) is η-strictly admissible and η-bounded. Therefore, the stability Theorem 1.3 below is a statement regarding every strictly admissible triple, quantitatively involving the corresponding η.

Theorem 1.3 (Stability)**.** *For each $\eta > 0$ there exist $\delta_0 > 0$ and $\mathbf{C} < \infty$ with the following property. Let G be a compact connected Abelian topological group equipped with Haar measure μ satisfying $\mu(G) = 1$. Let (A, B, C) be an η–strictly admissible and η-bounded ordered triple of measurable subsets of G. Let $0 \le \delta \le \delta_0$. If $\int_C \mathbf{1}_A * \mathbf{1}_B \, d\mu \ge \int_{C^\star} \mathbf{1}_{A^\star} * \mathbf{1}_{B^\star} \, dm - \delta$ then there exists a compatibly centered parallel ordered triple $(\mathcal{B}_A, \mathcal{B}_B, \mathcal{B}_C)$ of rank one Bohr sets satisfying*

$$\mu(A \,\Delta\, \mathcal{B}_A) \le \mathbf{C}\delta^{1/2} \tag{1.12}$$

and likewise for $\mu(B \,\Delta\, \mathcal{B}_B)$ and $\mu(C \,\Delta\, \mathcal{B}_C)$.

One part of the definition of η–boundedness is a lower bound (1.11) for the Haar measures of A, B, C. For the special case $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, Theorem 17.1 establishes a stronger variant of Theorem 1.3 that is appropriately uniform without any such lower bound hypothesis.

1.4. **Sumset inequalities.** The Riesz-Sobolev-type inequality (1.2) and the closely related inequality (1.4) are intimately connected with inequalities for sumsets. In fact, our proofs of both Theorems 1.2 and 1.3 rely on inverse theory for a sumset inequality.

More precisely, continue to assume that G is compact, connected, and Abelian. Kneser's inequality [21] states that for all measurable subsets $A, B \subset G$, the inner measure of the sumset $A + B$ satisfies

$$\mu_*(A + B) \ge \min(\mu(A) + \mu(B), \mu(G)). \tag{1.13}$$

A mildly stronger formulation is[1]

$$\mu(A +_0 B) \ge \min(\mu(A) + \mu(B), \mu(G)) \tag{1.14}$$

[1] (1.14) follows from (1.13) for $G = \mathbb{T}^d$ by a simple argument involving points of density, since $A + B = A +_0 B$ if every point of each of A, B is a point of density. For general groups G, (1.14) follows from the special case of $\mathbb{T}^d$ by approximating by elements of the algebra generated by Bohr sets. Alternatively, a stronger form of (1.14) is proved in [25].

where $A +_0 B$ is the open set

$$A +_0 B := \{x : \mathbf{1}_A * \mathbf{1}_B(x) > 0\}.$$

Indeed, $\mu_*(A + B) \geq \mu_*(A +_0 B) = \mu(A +_0 B)$. The Riesz-Sobolev-type inequality (1.2) directly implies the sumset inequality (1.14) by choosing $C = A +_0 B$.

Kneser [21] characterized cases of equality in (1.13). If $A, B \subset G$ are measurable sets that satisfy $\mu(A) + \mu(B) < \mu(G)$ and $\min(\mu(A), \mu(B)) > 0$, then $\mu_*(A + B) = \mu(A) + \mu(B)$ if and only if there exists a pair of parallel rank one Bohr sets satisfying $A \subset \mathcal{B}_A$, $B \subset \mathcal{B}_B$, and $\mu(\mathcal{B}_A \setminus A) = \mu(\mathcal{B}_B \setminus B) = 0$. For compact Abelian groups that are not necessarily connected, matters are more complicated; see for instance [19].

The hypothesis in the inverse Theorem 1.2 for the Riesz-Sobolev inequality is significantly weaker than that of Kneser's inverse theorem in one sense. While it is given in the latter theorem that $a + b$ lies in a set of specified inner measure for all pairs $(a, b) \in A \times B$, in the former theorem this is given only for a set of pairs whose product measure is at least a specified fraction of the measure of $A \times B$. Moreover, this set of pairs is not specified. The primary focus of this paper is this type of variant.

Tao [25] and Griesmer [20] have proved associated stability, or quantitative uniqueness, theorems. Most relevant to our considerations is this result from [20]: For every $\varepsilon, \eta > 0$ there exists $\delta > 0$ such that if $A, B \subset G$ are measurable sets satisfying the auxiliary hypotheses $\mu(A) \geq \eta\mu(G)$, $\mu(B) \geq \eta\mu(G)$, $\mu(A) + \mu(B) \leq (1 - \eta)\mu(G)$ and the main hypothesis $\mu_*(A + B) \leq \mu(A) + \mu(B) + \delta\mu(G)$, then there exists a pair of parallel rank one Bohr sets $(\mathcal{B}_A, \mathcal{B}_B)$ satisfying $A \subset \mathcal{B}_A$, $B \subset \mathcal{B}_B$, and

$$\mu(\mathcal{B}_A \setminus A) + \mu(\mathcal{B}_B \setminus B) < \varepsilon\mu(G).$$

Our proofs of uniqueness and stability theorems for the Riesz-Sobolev inequality (1.2) on G rely directly on this stability theorem for sumsets; our methods do not provide any new insight into its proof.

Although it is not necessary for our analysis of the Riesz-Sobolev inequality, we prove the following more quantitative stability theorem for sumsets, as it may be of independent interest.

Theorem 1.4. *For each $\eta, \eta' > 0$ there exist $\delta_0 > 0$ and $\mathbf{C} < \infty$ with the following property. Let G be any compact connected Abelian topological group equipped with normalized Haar measure μ. Let $A, B \subset G$ be a pair of measurable sets satisfying $\min(\mu(A), \mu(B)) \geq \eta'$ and $\mu(A) + \mu(B) \leq 1 - \eta$. If $\mu(A +_0 B) \leq \mu(A) + \mu(B) + \delta \min(\mu(A), \mu(B))$ and $\delta \leq \delta_0$, then there exists a pair of parallel rank one Bohr sets $(\mathcal{B}_A, \mathcal{B}_B)$ such that $A \subset \mathcal{B}_A$, $B \subset \mathcal{B}_B$, and*

$$\mu(\mathcal{B}_A \setminus A) + \mu(\mathcal{B}_B \setminus B) \leq \mathbf{C}\delta \min(\mu(A), \mu(B)). \tag{1.15}$$

This differs from statements in earlier results of this type for general compact Abelian groups in that the right-hand side of (1.15) is proportional to δ, rather than being $o_\delta(1)$. For $G = \mathbb{T}$, Candela and de Roton [9] have proved a theorem of this type in which the relationship between $m(\mathcal{B}_A \setminus A)$ and $m_*(A + B) - m(A) - m(B)$ is made quite precise, for an interesting range of parameters. We believe that their theorem extends to arbitrary compact connected Abelian groups, with the same type of relationship between parameters as in [9].

1.5. **Relaxation.** The next two theorems generalize Theorems 1.1 and 1.3 from indicator functions of sets to functions taking values in $[0, 1]$. Theorem 1.5 is used in the proof of Theorem 1.3.

Theorem 1.5. *Let G be a compact connected Abelian topological group equipped with Haar measure μ satisfying $\mu(G)=1$. For any measurable functions $f,g,h:G\to[0,1]$,*

$$\langle f*g,h\rangle_G \le \langle \mathbf{1}_{A^\star}*\mathbf{1}_{B^\star},\mathbf{1}_{C^\star}\rangle_{\mathbb{T}} \tag{1.16}$$

where $A^\star,B^\star,C^\star\subset\mathbb{T}$ are intervals centered at 0 satisfying

$$\big(m(A^\star),m(B^\star),m(C^\star)\big)=\big(\textstyle\int_G f\,d\mu,\ \int_G g\,d\mu,\ \int_G h\,d\mu\big).$$

Inequality (1.16) is known for $\mathbb{R}^d$. See for instance [14] for an application of such a relaxed symmetrization inequality and associated inverse inequality in the Euclidean context.

Theorem 1.6. *For each $\eta>0$ there exists $\mathbf{C}<\infty$ with the following property. Let G be a compact connected Abelian topological group equipped with Haar measure μ satisfying $\mu(G)=1$. Let $f,g,h:G\to[0,1]$ be measurable. Let $(A^\star,B^\star,C^\star)\subset\mathbb{T}$ be intervals centered at 0 with Lebesgue measures $(\int f\,d\mu,\int g\,d\mu,\int h\,d\mu)$. Let*

$$\mathcal{D}=\langle \mathbf{1}_{A^\star}*\mathbf{1}_{B^\star},\mathbf{1}_{C^\star}\rangle_{\mathbb{T}}-\langle f*g,h\rangle_G. \tag{1.17}$$

Suppose that $(A^\star,B^\star,C^\star)$ is η–strictly admissible and η–bounded. If $\mathcal{D}$ is sufficiently small as a function of η alone then there exists a compatibly centered parallel triple $(\mathcal{B}_f,\mathcal{B}_g,\mathcal{B}_h)$ of rank one Bohr subsets of G satisfying

$$\|f-\mathbf{1}_{\mathcal{B}_f}\|_{L^1(G,\mu)}\le \mathbf{C}\mathcal{D}^{1/2} \tag{1.18}$$

and likewise for $(g,\mathbf{1}_{\mathcal{B}_g})$ and $(h,\mathbf{1}_{\mathcal{B}_h})$.

All results in this paper are concerned with Abelian groups. Significant progress concerning sumset inequalities for nonabelian groups, and concerning associated inverse theorems, has been made by Jing and Tran [16] and by Jing, Tran, and Zhang [15].

1.6. **Organization of the paper.** In §2 we state an alternative formulation of our Riesz-Sobolev inequality within the admissible regime.

In §3 we review an inequality of Tao [25], stating several equivalent reformulations and establishing a refinement. This refinement is used in §4 to prove the Riesz-Sobolev–type inequality of Theorem 1.1. The defect $\mathcal{D}(A,B,C)=\langle \mathbf{1}_{A^\star}*\mathbf{1}_{B^\star},\mathbf{1}_{C^\star}\rangle-\langle \mathbf{1}_A*\mathbf{1}_B,\mathbf{1}_C\rangle$, and a related defect $\mathcal{D}'(A,B,\tau)$, in terms of which much of our analysis is naturally phrased, are introduced in §4.

In §5 we discuss two key principles, submodularity and complementation. At the heart of our analysis of stability for the Riesz-Sobolev-type inequality (1.2) is a connection, developed in [11] for $G=\mathbb{R}$, between (near) equality in the Riesz-Sobolev equality and (near) equality in the sumset inequality for certain associated sets. This connection only applies directly in the case in which two of the three sets A,B,C have equal measures. §6 reviews this connection and adapts it to general connected compact Abelian groups. §7 begins a reduction of the general case to the special case of two sets of equal measures. This reduction proceeds in a different way than the corresponding reduction in [11] for the Euclidean case.

§9 establishes the conclusion of Theorem 1.3 in its quantitative form, for the perturbative regime in which (A,B,C) is assumed to be within a certain threshold distance of a compatibly centered parallel triple of rank one Bohr sets. §10 digresses to establish Theorem 1.4, concerning quantitative stability for Kneser's inequality. The main step in its proof deals with the perturbative regime, in which A,B are assumed to be moderately close to a pair of parallel rank one Bohr sets, and the relationship between smallness of the defect and closeness to such a Bohr pair is made more precise, without hypotheses of strict admissibility and without any lower bounds on the measures of A, B.

Theorem 1.5 and Theorem 1.6, concerning relaxed variants of the Riesz-Sobolev-type inequality and its companion inverse stability theorem, are proved in §8 and §15, respectively.

§11 and §12 analyze the special case in which the defect $\mathcal{D}(A,B,C)$ is small and one of the three sets is well approximated by a rank one Bohr set. §11 treats the sub-subcase in which $G = \mathbb{T}$ and C is an interval. In §12, we reduce matters from general groups G to $\mathbb{T}$. The situation that arises on $\mathbb{T}$ in this way belongs to the more general framework of Theorems 1.5 and 1.6. That framework comes into play at this juncture. The proof of Theorem 1.3 is completed in §13.

In §14 we use Theorem 1.3 to prove Theorem 1.2.

Another thread is taken up in §16 and §17, which are concerned with the important group $G = \mathbb{T}$. This thread is founded on the monotonicity of a normalized version of the functional $\int_C \mathbf{1}_A * \mathbf{1}_B \, dm$ under a certain continuous deformation of A, B, C. This deformation is developed in §16. As an application, in §17 we establish Theorem 17.1, a refinement for $G = \mathbb{T}$ of Theorem 1.3 which in an appropriate sense eliminates the dependence of the conclusion on a lower bound for $\min(\mu(A), \mu(B), \mu(C))$.

One could alternatively bypass the analysis in §11 of the situation in which $G = \mathbb{T}$ and C is an interval, by invoking the theory for $\mathbb{T}$ established in §17.

1.7. **Notation.** For the sake of economy, we will often refer to (1.2) as the Riesz-Sobolev inequality, or the Riesz-Sobolev inequality for G. Throughout the remainder of the paper, G denotes a compact connected Abelian topological group equipped with a complete Haar measure μ that is normalized in the sense that $\mu(G) = 1$. This is a hypothesis of all lemmas and propositions, though it is not included in their statements. It is implicitly asserted that all constants in upper and lower bounds in theorems, propositions, lemmas, and inequalities are independent of G, except when the special case $G = \mathbb{T}$ is explicitly indicated.

m denotes Lebesgue measure for $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. $m(E)$ is alternatively denoted by $|E|$ in some parts of the discussion. C with no subscript is used to denote a subset of G, rather than a constant. c, c', and $\mathbf{C}$ denote unspecified positive finite constants, whose values may change freely from one occurrence to the next.

It will be convenient in the analysis of the functional $\langle \mathbf{1}_{E_1} * \mathbf{1}_{E_2}, \mathbf{1}_{E_3} \rangle$ to be able to freely interchange the sets E_j. For that purpose, we work with a more symmetric variant. For measurable functions $f_j : G \to [0,\infty)$,

$$\mathcal{T}_G(f_1,f_2,f_3) = \iint_{x+y+z=0} f_1(x) f_2(y) f_3(z) \, d\lambda(x,y,z) \tag{1.19}$$

where λ is the measure on $\{(x,y,z) \in G^3 : x+y+z=0\}$ defined by pulling back the measure $\mu \times \mu$ on $G \times G$ via the mapping $(x,y,z) \mapsto (x,y)$. This definition of λ is invariant with respect to permutation of the three coordinates. Equivalently,

$$\mathcal{T}_G(\mathbf{f}) = \mathcal{T}_G(f_1,f_2,f_3) = \iint_{G^2} f_1(x) f_2(y) f_3(-x-y) \, d\mu(x) \, d\mu(y).$$

For a three-tuple $\mathbf{E} = (E_j : j \in \{1,2,3\})$ of sets, we write $\mathcal{T}_G(\mathbf{E}) = \mathcal{T}_G(\mathbf{f})$ with $f_j = \mathbf{1}_{E_j}$.

We sometimes work simultaneously on a general group G and on $\mathbb{T}$, and write $\mathcal{T}_G$ and/or $\mathcal{T}_\mathbb{T}$ to distinguish between the functionals associated to the two groups. Defining

$$\overline{\mathcal{D}}(A,B,C) := \mathcal{T}_\mathbb{T}(A^\star, B^\star, C^\star) - \mathcal{T}_G(A,B,C),$$

one has $\overline{\mathcal{D}}(A,B,C) = \mathcal{D}(A,B,-C)$.

The authors are grateful to Rupert Frank, who kindly called their attention to a slip in the proof of Theorem 1.6 in a preliminary draft.

2. Alternative formulation of (1.2)

In the parameter range of primary interest, (1.2) can be restated with an alternative expression for the right-hand side. This expression will become relevant to our analysis in Lemma 4.1.

Theorem 2.1. *For any compact connected Abelian topological group G and any measurable subsets $A, B, C \subset G$ satisfying*

$$\begin{cases} |\mu(A) - \mu(B)| \le \mu(C) \le \mu(A) + \mu(B), \\ \qquad \mu(A) + \mu(B) + \mu(C) \le 2, \end{cases} \tag{2.1}$$

one has

$$\int_C \mathbf{1}_A * \mathbf{1}_B \, d\mu \le \tfrac{1}{2}(ab + bc + ca) - \tfrac{1}{4}(a^2 + b^2 + c^2) = ab - \tfrac{1}{4}(a + b - c)^2 \tag{2.2}$$

where $(a, b, c) = (\mu(A), \mu(B), \mu(C))$.

The conclusion (2.2) can also be stated

$$\int_C \mathbf{1}_A * \mathbf{1}_B \, d\mu \le \mu(A)\mu(B) - \tfrac{1}{4}(\mu(A) + \mu(B) - \mu(C))^2 \tag{2.3}$$

where τ is defined by $\mu(C) = \mu(A) + \mu(B) - 2\tau$.

Both hypotheses (2.1) are invariant under permutations of (A, B, C). Likewise, the modified form $\int_{-C} \mathbf{1}_A * \mathbf{1}_B \, d\mu$, where $-C = \{-x : x \in C\}$, is invariant under permutations of (A, B, C).

Equality holds in (2.2), under the indicated hypotheses on $(\mu(A), \mu(B), \mu(C))$, when $G = \mathbb{T}$ and $(A, B, C) = (A^\star, B^\star, C^\star)$. Thus (2.2) is a direct restatement of (1.2) in this parameter regime.

If the hypothesis (2.1) is violated, then (1.2) is easily verified directly, using the trivial upper bound

$$\int_C \mathbf{1}_A * \mathbf{1}_B \, d\mu \le \min(\mu(A)\mu(B), \mu(B)\mu(C), \mu(C)\mu(A)) \tag{2.4}$$

which follows from $\int_C \mathbf{1}_A * \mathbf{1}_B \, d\mu \le \int_G \mathbf{1}_A * \mathbf{1}_B \, d\mu = \mu(A)\mu(B)$ and permutation invariance. In this paper we will focus primarily on the regime in which the hypotheses (2.1) hold.

3. Refinement of a related inequality

In this section we review an inequality of Tao [25], discuss multiple equivalent reformulations, and formulate and prove a sharper inequality, from which the Riesz-Sobolev inequality (1.2) for G will subsequently be derived. In its original formulation, the inequality of Tao was stated in the following terms:

Theorem 3.1 (Tao [25])**.** *For any compact connected Abelian topological group G with normalized Haar measure μ, for any measurable $A, B \subset G$,*

$$\int_G \min(\mathbf{1}_A * \mathbf{1}_B, \tau) \, d\mu \ge \tau \min(\mu(A) + \mu(B) - \tau, 1) \quad \forall\, 0 \le \tau \le \max(\mu(A), \mu(B)). \tag{3.1}$$

In [25] the inequality (3.1) is stated for $0 \le \tau \le \min(\mu(A), \mu(B))$. However, it also holds trivially in the range $\min(\mu(A), \mu(B)) \le \tau \le \max(\mu(A), \mu(B))$, in the sense that for arbitrary A, B equality holds when τ is equal to the minimum or maximum, while for τ in the open interval $\big(\min(\mu(A), \mu(B)), \max(\mu(A), \mu(B))\big)$, the left-hand side is equal to $\mu(A) \cdot \mu(B)$ and (3.1) holds with strict inequality. (3.1) also holds with equality whenever $\mu(A) + \mu(B) \ge 1 + \tau$, for in that case,

$$\mathbf{1}_A * \mathbf{1}_B(x) = \mu(A \cap (x - B)) \ge \mu(A) + \mu(B) - 1 \ge \tau$$

for every $x \in G$, so both the left– and right–hand sides are equal to τ. (3.1) never holds when $\tau > \max(\mu(A), \mu(B))$.

If $G = \mathbb{T}$ and $A, B \subset \mathbb{T}$ are intervals centered at 0 then equality holds in (3.1) whenever $\tau \le \min(\mu(A), \mu(B))$. Therefore this inequality can be equivalently restated as

$$\int_G \min(\mathbf{1}_A * \mathbf{1}_B, \tau)\, d\mu \ge \int_{\mathbb{T}} \min(\mathbf{1}_{A^\star} * \mathbf{1}_{B^\star}, \tau)\, dm \quad \forall\, 0 \le \tau \le \min(\mu(A), \mu(B)). \tag{3.2}$$

By virtue of the identities

$$\int_G \mathbf{1}_A * \mathbf{1}_B \, d\mu = \mu(A) \cdot \mu(B) \tag{3.3}$$

and $\max(f, g) + \min(f, g) = f + g$, (3.2) can in turn be equivalently reformulated as

$$\int_G \max(\mathbf{1}_A * \mathbf{1}_B - \tau, 0)\, d\mu \le (\mu(A) - \tau)(\mu(B) - \tau) \qquad \forall\, \tau \in [\mu(A) + \mu(B) - 1,\, \min(\mu(A), \mu(B))] \tag{3.4}$$

with $\int_G \max(\mathbf{1}_A * \mathbf{1}_B - \tau, 0)\, d\mu = \mu(A)\mu(B) - \tau$ for all $\tau \in [0, \mu(A) + \mu(B) - 1]$.

This can be rephrased as follows.

Theorem 3.2 (Tao [25])**.** *Let G be an Abelian connected compact topological group, equipped with Haar probability measure μ. For all measurable subsets $A, B \subset G$ and for every $0 \le \tau \le \min(\mu(A), \mu(B))$,*

$$\int_G \max(\mathbf{1}_A * \mathbf{1}_B - \tau, 0)\, d\mu \le \int_{\mathbb{T}} \max(\mathbf{1}_{A^\star} * \mathbf{1}_{B^\star} - \tau, 0)\, dm. \tag{3.5}$$

Thus, inequalities (3.1) through (3.5) are equivalent in the sense that any one of them follows from any other by simple manipulations augmented by the above discussion of the cases in which $\min(\mu(A), \mu(B)) \le \tau \le \max(\mu(A), \mu(B))$ or $\tau \le \mu(A) + \mu(B) - 1$.

The inequalities (3.1) through (3.5) can be further reformulated in terms of superlevel sets and associated distribution functions, and these reformulations will be essential to our analysis. The following notation (3.6) will be used throughout the paper.

Definition 3.1. For measurable sets $A, B \subset G$ and for $t \ge 0$, the associated superlevel set is

$$S_{A,B}(t) = \{x \in G : \mathbf{1}_A * \mathbf{1}_B(x) > t\}. \tag{3.6}$$

Superlevel sets appear in fundamental formulae for the functionals of interest here:

$$\int_{S_{A,B}(\tau)} \mathbf{1}_A * \mathbf{1}_B \, d\mu = \tau \mu(S_{A,B}(\tau)) + \int_\tau^\infty \mu(S_{A,B}(t))\, dt, \tag{3.7}$$

$$\int_G \max(\mathbf{1}_A * \mathbf{1}_B - \tau, 0)\, d\mu = \int_\tau^\infty \mu(S_{A,B}(t))\, dt. \tag{3.8}$$

By (3.8), Tao's inequality (3.4) can be equivalently written as

$$\int_\tau^\infty \mu(S_{A,B}(t))\,dt \le (\mu(A)-\tau)(\mu(B)-\tau) \qquad \forall\,\tau\in[\mu(A)+\mu(B)-1,\ \min(\mu(A),\mu(B))]. \tag{3.9}$$

This final reformulation of Tao's inequality implies a sharpening of itself, which we formulate as Theorem 3.3. This refinement will be the basis of our proof of Theorem 1.1.

Theorem 3.3. *Let G be a compact connected Abelian topological group, equipped with normalized Haar measure μ. Suppose that*

$$0\le\tau\le\min(\mu(A),\mu(B)) \tag{3.10}$$

and that

$$\mu(A)+\mu(B)+\mu(S_{A,B}(\tau))\le 2. \tag{3.11}$$

Let $\sigma=\frac{1}{2}(\mu(A)+\mu(B)-\mu(S_{A,B}(\tau)))$. Define

$$h=\begin{cases}(\sigma-\tau)^2 & \text{if } \sigma\le\min(\mu(A),\mu(B))\\ (\min(\mu(A),\mu(B))-\tau)^2 & \text{if } \sigma>\min(\mu(A),\mu(B)).\end{cases} \tag{3.12}$$

Then

$$\int_\tau^\infty \mu(S_{A,B}(t))\,dt \le (\mu(A)-\tau)(\mu(B)-\tau)-h, \tag{3.13}$$

In particular, if $\mu(A)+\mu(B)\le 1+\tau$ then

$$\int_G \max(\mathbf{1}_A*\mathbf{1}_B-\tau,0)\,d\mu \le \int_{\mathbb{T}} \max(\mathbf{1}_{A^\star}*\mathbf{1}_{B^\star}-\tau,0)\,dm - h. \tag{3.14}$$

The conclusion (3.14) can be equivalently written as

$$\int_\tau^\infty \mu(S_{A,B}(t))\,dt \le \int_\tau^\infty m(S_{A^\star,B^\star}(t))\,dt - h, \tag{3.15}$$

where $0\le h=h(\mu(A),\mu(B),\tau,\mu(S_{A,B}(\tau))$. The improvement relative to Theorem 3.2 lies in the presence of the nonpositive term $-h$ on the right-hand side. This term depends on the sets A,B, rather than only on their Haar measures, through its dependence on the measure of the superlevel set $S_{A,B}(\tau)$. On the other hand, Theorem 3.3 has the extra hypothesis (3.11), which has no direct counterpart in Theorem 3.2.

The form of the right-hand side of (3.13) is unnatural when $\mu(A)+\mu(B)>1+\tau$, in the sense that $\int_\tau^\infty m(S_{A^\star,B^\star}(t))\,dt=\mu(A)\mu(B)-\tau$ is strictly smaller than $(\mu(A)-\tau)(\mu(B)-\tau)$ for such values of τ.

A corresponding refinement of Theorem 1.1 is formulated below as Theorem 4.3.

Proof of Theorem 3.3. Write $S(t)=S_{A,B}(t)$ to simplify notation. We seek to apply (3.9) to $\int_\sigma^\infty \mu(S(t))\,dt$. This inequality is applicable if $\mu(A)+\mu(B)-1\le\sigma\le\min(\mu(A),\mu(B))$. The first of these two inequalities is $\mu(A)+\mu(B)-1\le\frac{1}{2}(\mu(A)+\mu(B)-\mu(S(\tau)))$, which is equivalent to $\mu(A)+\mu(B)+\mu(S(\tau))\le 2$, which is indeed a hypothesis of Theorem 3.3. However, the second inequality, $\sigma\le\min(\mu(A),\mu(B))$, need not hold under the hypotheses of Theorem 3.3, in general. The proof is consequently organized into cases.

If $\sigma \le \tau$ then indeed $\sigma \le \min(\mu(A), \mu(B))$, so (3.9) may be applied to obtain

$$\begin{aligned}\int_\tau^\infty \mu(S(t))\,dt &= \int_\sigma^\infty \mu(S(t))\,dt - \int_\sigma^\tau \mu(S(t))\,dt \\ &\le (\mu(A)-\sigma)(\mu(B)-\sigma) - (\tau-\sigma)\mu(S(\tau)) \\ &= (\mu(A)-\tau)(\mu(B)-\tau) - (\sigma-\tau)^2\end{aligned}$$

as follows by expanding $\tau = \sigma - (\tau - \sigma)$ in the product $(\mu(A)-\tau)(\mu(B)-\tau)$ and invoking the relation $\mu(A)+\mu(B) = 2\sigma + \mu(S(\tau))$.

If $\tau \le \sigma$ and if σ does satisfy $\sigma \le \min(\mu(A), \mu(B))$, then again by (3.9),

$$\begin{aligned}\int_\tau^\infty \mu(S(t))\,dt &= \int_\sigma^\infty \mu(S(t))\,dt + \int_\tau^\sigma \mu(S(t))\,dt \\ &\le (\mu(A)-\sigma)(\mu(B)-\sigma) + (\sigma-\tau)\mu(S(\tau))\end{aligned}$$

which we have already stated to be equal to $(\mu(A)-\tau)(\mu(B)-\tau) - (\sigma-\tau)^2$.

If on the other hand $\sigma \ge \min(\mu(A), \mu(B))$ then by permutation invariance, we may assume without loss of generality that $\mu(A) \le \mu(B)$. Thus $\frac{1}{2}(\mu(A)+\mu(B)-\mu(S(\tau))) = \sigma \ge \mu(A)$, so $\mu(S(\tau)) \le \mu(B) - \mu(A)$. Since $\mathbf{1}_A * \mathbf{1}_B \le \mu(A)$,

$$\int_\tau^\infty \mu(S(t))\,dt = \int_\tau^{\mu(A)} \mu(S(t))\,dt \le (\mu(A)-\tau)\mu(S(\tau))$$

since the integrand is a nonincreasing function of t. The right-hand side is

$$\le (\mu(A)-\tau)(\mu(B)-\mu(A)) = (\mu(A)-\tau)(\mu(B)-\tau) - (\mu(A)-\tau)^2,$$

as required. □

Corollary 3.4. *Let G be a compact connected Abelian topological group, equipped with Haar measure μ satisfying $\mu(G) = 1$. Let $A, B \subset G$ be measurable sets. Suppose that*

$$\mu(A) + \mu(B) - 1 < t < \min(\mu(A), \mu(B)). \tag{3.16}$$

If (A, B, t) achieves equality in (3.1) *(equivalently in any or all of* (3.2), (3.4), (3.5)*), then*

$$\mu(S_{A,B}(t)) = \mu(A) + \mu(B) - 2t. \tag{3.17}$$

We remark that $\mu(A)+\mu(B)-2\tau$ is not an extremal value for $\mu(S_{A,B}(\tau))$ for any single value of τ; $\mu(S_{A,B}(\tau))$ can in general be either larger, or smaller.

Proof. If $\mu(A) + \mu(B) + \mu(S_{A,B}(t)) \le 2$ then all hypotheses of Theorem 3.3 are satisfied, and (3.17) follows from its conclusion since t is strictly less than $\min(\mu(A), \mu(B))$.

We claim that $\mu(S_{A,B}(t)) \le 1 - t$, whence $\mu(A) + \mu(B) + \mu(S_{A,B}(t)) \le 1 + t + 1 - t = 2$, completing the proof of the corollary. Suppose to the contrary that $\mu(S_{A,B}(t)) > 1 - t$. Define $\tau \in (0, t)$ by $\mu(A) + \mu(B) = 1 + \tau$.

For every $x \in G$, $\mathbf{1}_A * \mathbf{1}_B(x) = \mu\big(A \cap (x - B)\big) \ge \mu(A) + \mu(B) - 1 = \tau$. Thus, for every $r \in [0, \tau)$, $S_{A,B}(r) = G$, so

$$\mu(S_{A,B}(r)) = 1 \text{ for every } r \in [0, \tau). \tag{3.18}$$

For any $r \in [\tau, t]$, $S_{A,B}(r) \supset S_{A,B}(t)$, so

$$\mu(S_{A,B}(r)) \ge \mu(S_{A,B}(t)) > 1 - t \text{ for every } r \in [\tau, t]. \tag{3.19}$$

The assumption that (A, B, t) satisfies equality in (3.4) means that

$$\int_G \min(1_A * 1_B, t)\,d\mu = t(\mu(A) + \mu(B) - t) = t(1 + \tau - t).$$

Substituting
$$\int_G \min(1_A * 1_B, t)\, d\mu = \int_0^t \mu(S_{A,B}(r))\, dr$$
in the left-hand side and invoking (3.18) and (3.19) gives
$$t(1+\tau-t) = \int_0^t \mu(S_{A,B}(r))\, dr = \int_0^\tau \mu(S_{A,B}(r))\, dr + \int_\tau^t \mu(S_{A,B}(r))\, dr$$
$$> \int_0^\tau 1 + \int_\tau^t (1-t)\, dr = \tau + (t-\tau)(1-t) = t(1+\tau-t),$$
which is a contradiction. Therefore $\mu(S_{A,B}(t)) \le 1-t$, and the proof of the corollary is complete. □

4. On the Riesz-Sobolev inequality for G

In this section we derive the Riesz-Sobolev inequality (1.2) for G from Theorem 3.3. The sharpened form (3.13) of (3.4) for $\sigma \le \min(\mu(A), \mu(B))$ is exactly what is needed in this derivation. We introduce defects $\mathcal{D}(A,B,C)$ and $\mathcal{D}'(A,B,\tau)$ for the functionals $\int_C \mathbf{1}_A * \mathbf{1}_B\, d\mu$ and $\int_G \max(\mathbf{1}_A * \mathbf{1}_B - \tau, 0)\, d\mu$, respectively. We discuss approximation of the set C in the functional $\int_C \mathbf{1}_A * \mathbf{1}_B\, d\mu$ by superlevel sets $S_{A,B}(t)$, under the assumption that $\mathcal{D}(A,B,C)$ is small. We also discuss majorization of $\mathcal{D}(A,B,C)$ by $\mathcal{D}'(A,B,\tau)$ and vice versa, under appropriate hypotheses linking $\mu(C)$ to τ.

The defects $\mathcal{D}(A,B,C)$ and $\mathcal{D}'(A,B,\tau)$ are defined as follows.

Definition 4.1.
$$\mathcal{D}(A,B,C) = \int_{C^\star} \mathbf{1}_{A^\star} * \mathbf{1}_{B^\star}\, dm - \int_C \mathbf{1}_A * \mathbf{1}_B\, d\mu. \tag{4.1}$$
$$\mathcal{D}'(A,B,\tau) = \int_{\mathbb{T}} \max(\mathbf{1}_{A^\star} * \mathbf{1}_{B^\star} - \tau, 0)\, dm - \int_G \max(\mathbf{1}_A * \mathbf{1}_B - \tau, 0)\, d\mu. \tag{4.2}$$

Theorem 1.1 states that $\mathcal{D}(A,B,C) \ge 0$ for any ordered triple, while inequality (3.5) states that $\mathcal{D}'(A,B,\tau) \ge 0$ for all $\tau \in [0, \min(\mu(A), \mu(B))]$. These defects can usefully be expressed in terms of distribution functions $\mu(S_{A,B}(t))$, as discussed in §3.

The following quantity arises throughout our analysis.

Definition 4.2. To sets $A, B, C \subset G$ satisfying $\mu(C) \le \mu(A) + \mu(B)$ is associated
$$\tau_C = \tfrac{1}{2}(\mu(A) + \mu(B) - \mu(C)). \tag{4.3}$$

This quantity satisfies $m(S_{A^\star,B^\star}(\tau_C)) = m(C^\star) = \mu(C)$; it represents the parameter τ for which $C^\star$ equals the superlevel set $S_{A^\star,B^\star}(\tau)$, provided that $(\mu(A), \mu(B), \mu(C))$ is admissible.

Lemma 4.1. *Suppose that $A, B \subset G$ and $\tau \in [0,1]$ satisfy*
$$0 \le \tau \le \min(\mu(A), \mu(B)),$$
$$\mu(A) + \mu(B) + \mu(S_{A,B}(\tau)) \le 2.$$
Then
$$\tau\mu(S_{A,B}(\tau)) + \int_\tau^\infty \mu(S_{A,B}(\alpha))\, d\alpha \le \mu(A)\mu(B) - \tfrac{1}{4}(\mu(A) - \mu(B) - \mu(S_{A,B}(\tau)))^2. \tag{4.4}$$

That is, $(A,B,C) = (A,B,S_{A,B}(\tau))$ satisfies (2.3) (and thus (1.2), under some additional hypotheses).

Proof. Define $\sigma = \frac{1}{2}(\mu(A)+\mu(B)-\mu(S_{A,B}(\tau)))$. Equivalently, $\mu(S_{A,B}(\tau)) = \mu(A)+\mu(B)-2\sigma$. Calculate

$$\begin{aligned}(\mu(A)-\tau)(\mu(B)-\tau) - \big(\mu(A)\mu(B)-\sigma^2\big) &= -\tau(\mu(A)+\mu(B)) + \tau^2 + \sigma^2 \\ &= -\tau(\mu(A)+\mu(B)-2\sigma) + (\sigma-\tau)^2 \\ &= -\tau\mu(S(\tau)) + (\sigma-\tau)^2.\end{aligned}$$

Thus

$$\tau\mu(S_{A,B}(\tau)) = -(\mu(A)-\tau)(\mu(B)-\tau) + \big(\mu(A)\mu(B)-\sigma^2\big) + (\sigma-\tau)^2. \tag{4.5}$$

Note that (A,B,τ) satisfies the hypotheses of Theorem 3.3. Applying Theorem 3.3 to the second term on the left-hand side of (4.4) and then invoking (4.5) gives the desired upper bound

$$\tau\mu(S_{A,B}(\tau)) + (\mu(A)-\tau)(\mu(B)-\tau) - (\sigma-\tau)^2 = \mu(A)\mu(B) - \sigma^2.$$

□

Proof of Theorem 1.1. Let $A,B,C \subset G$. Consider first the case in which $\mu(A)+\mu(B)+\mu(C) \ge 2$. Define t by $\mu(A)+\mu(B) = 1+t$; note that $t \ge 0$. Then $\mathbf{1}_A * \mathbf{1}_B(x) \ge t$ for every $x \in G$. Indeed,

$$\mathbf{1}_A * \mathbf{1}_B(x) = \mu(A \cap (x-B)) \ge \mu(A) + \mu(x-B) - \mu(G) = \mu(A)+\mu(B)-1 = t.$$

Therefore

$$\int_C \mathbf{1}_A * \mathbf{1}_B \, d\mu \le \int_G \mathbf{1}_A * \mathbf{1}_B \, d\mu - t\mu(G \setminus C) = \mu(A)\mu(B) - t(1-\mu(C)).$$

On the other hand, $\mathbf{1}_{A^\star} * \mathbf{1}_{B^\star} \equiv t$ on $\mathbb{T} \setminus C^\star$, and so the same calculation gives

$$\int_{C^\star} \mathbf{1}_{A^\star} * \mathbf{1}_{B^\star} \, dm = m(A^\star)m(B^\star) - t(1-m(C^\star)) = \mu(A)\mu(B) - t(1-\mu(C)).$$

Thus the stated conclusion holds in this case.

If $\mu(C) \le |\mu(A)-\mu(B)|$ then, while $\mathbf{1}_A * \mathbf{1}_B \le \min(\mu(A),\mu(B))$ on C, it also holds that $\mathbf{1}_{A^\star} * \mathbf{1}_{B^\star} \equiv \min(m(A^\star), m(B^\star))$ on $C^\star$. Therefore (1.2) holds. If $\mu(C) \ge \mu(A)+\mu(B)$ then either $\mu(A) \le |\mu(B)-\mu(C)|$ or $\mu(B) \le |\mu(A)-\mu(C)|$. (1.2) thus follows by permutation invariance from the case in which $\mu(C) \le |\mu(A)-\mu(B)|$.

Assume henceforth that $\mu(A)+\mu(B)+\mu(C) < 2$, and that $|\mu(A)-\mu(B)| < \mu(C) < \mu(A)+\mu(B)$.

If there exists $t \in [0,1]$ for which the superlevel set $S = S_{A,B}(t)$ satisfies $\mu(S) = \mu(C)$, then the desired inequality (1.2) holds for (A,B,C). More precisely, $\int_C \mathbf{1}_A*\mathbf{1}_B \le \int_S \mathbf{1}_A*\mathbf{1}_B$. The parameter t satisfies $t \le \min(\mu(A),\mu(B))$, since $\|\mathbf{1}_A * \mathbf{1}_B\|_{C^0} \le \min(\mu(A),\mu(B))$ and $\mu(C) > 0$. It also satisfies $\mu(A)+\mu(B) \le 1+t$. Indeed, if $\mu(A)+\mu(B) > 1+t$ then $\mathbf{1}_A * \mathbf{1}_B(x) > t$ for every $x \in G$ as noted above, so $S = S_{A,B}(t) = G$, so $\mu(C) = \mu(S) = 1$, forcing $\mu(A)+\mu(B)+\mu(C) = \mu(A)+\mu(B)+1 > 2+t \ge 2$ and thereby contradicting the assumption that $\mu(A)+\mu(B)+\mu(C) < 2$.

Thus the hypotheses of Lemma 4.1 are satisfied by A,B,t and $S_{A,B}(t)$. Applying that lemma to $S_{A,B}(t)$ gives the desired upper bound for $\int_{S_{A,B}(t)} \mathbf{1}_A * \mathbf{1}_B$, hence for $\int_C \mathbf{1}_A * \mathbf{1}_B$.

It remains to reduce the general case to that in which there exists $t \in [0,1]$ satisfying $\mu(S_{A,B}(t)) = \mu(C)$, under the hypotheses $\mu(A)+\mu(B)+\mu(C) < 2$ and $|\mu(A)-\mu(B)| < \mu(C) < \mu(A)+\mu(B)$. We may also assume the auxiliary condition

$$\mu(\{x : \mathbf{1}_A * \mathbf{1}_B(x) > 0\}) \ge \mu(C). \tag{4.6}$$

Indeed, if this fails, set $\tilde{C} = C \cap \{x : \mathbf{1}_A * \mathbf{1}_B(x) > 0\}$. The value of the integral $\int_C \mathbf{1}_A * \mathbf{1}_B \, d\mu$ is unchanged when C is replaced by $\tilde{C}$. If $\mu(\tilde{C}) < |\mu(A) - \mu(B)|$ then we have already observed that

$$\int_{\tilde{C}} \mathbf{1}_A * \mathbf{1}_B \, d\mu \le \int_{\tilde{C}^\star} \mathbf{1}_{A^\star} * \mathbf{1}_{B^\star} \, dm$$

(that is, $(A, B, \tilde{C})$ satisfies (1.2)). Since $\mu(\tilde{C}) \le \mu(C)$, the right-hand side is in turn majorized by $\int_{C^\star} \mathbf{1}_{A^\star} * \mathbf{1}_{B^\star} \, dm$, so (1.2) holds for (A, B, C). If $\mu(\tilde{C}) \ge |\mu(A) - \mu(B)|$ then it suffices to prove that $(A, B, \tilde{C})$ satisfies (1.2). Thus matters are reduced to the case in which (A, B, C) satisfies (4.6).

Given (4.6), a sufficient condition for the existence of t satisfying $\mu(C) = \mu(S_{A,B}(t))$ is that all level sets of $\mathbf{1}_A * \mathbf{1}_B$ should be null sets, that is, for every $r > 0$, $\mu(\{x : \mathbf{1}_A * \mathbf{1}_B(x) = r\}) = 0$. Moreover, because $(A, B, C) \mapsto \int_C \mathbf{1}_A * \mathbf{1}_B \, d\mu$ is continuous in the sense that

$$\int_{C_n} \mathbf{1}_{A_n} * \mathbf{1}_{B_n} \, d\mu \to \int_C \mathbf{1}_A * \mathbf{1}_B \, d\mu \text{ if } \mu(A_n \,\Delta\, A) + \mu(B_n \,\Delta\, B) + \mu(C_n \,\Delta\, C) \to 0,$$

it would suffice to construct (A_n, B_n, C_n), converging to (A, B, C) in this sense, such that all level sets of $\mathbf{1}_{A_n} * \mathbf{1}_{B_n}$ are μ–null.

Such a construction does not necessarily exist in G, but it does in the auxiliary group $\tilde{G} = G \times \mathbb{T}$ with normalized Haar measure $\tilde{\mu}$. Consider a sequence of triples $(\alpha_n, \beta_n, \gamma_n)$ of Lebesgue measurable subsets of $\mathbb{T}$ satisfying $\mu(\alpha_n) \to 1$ as $n \to \infty$ and likewise for $\mu(\beta_n), \mu(\gamma_n)$, such that all level sets of $\mathbf{1}_{\alpha_n} * \mathbf{1}_{\beta_n}$ on $\mathbb{T}$ are Lebesgue null sets. The existence of such sequences can be proved in various ways.

Consider $(\tilde{A}, \tilde{B}, \tilde{C}) = (A \times \alpha_n, B \times \beta_n, C \times \gamma_n)$. Then $\mathbf{1}_{\tilde{A}_n} * \mathbf{1}_{\tilde{B}_n}$ is the product function $G \times \mathbb{T} \ni (x, y) \mapsto (\mathbf{1}_A * \mathbf{1}_B(x)) \cdot (\mathbf{1}_{\alpha_n} * \mathbf{1}_{\beta_n}(y))$, so

$$\int_{\tilde{C}_n} \mathbf{1}_{\tilde{A}_n} * \mathbf{1}_{\tilde{B}_n} \, d\tilde{\mu} = \Big(\int_{C_n} \mathbf{1}_A * \mathbf{1}_B \, d\mu \Big) \cdot \Big(\int_{\gamma_n} \mathbf{1}_{\alpha_n} * \mathbf{1}_{\beta_n} \, dm \Big)$$

converges to $\int_C \mathbf{1}_A * \mathbf{1}_B \, d\mu$ as $n \to \infty$. Moreover, all level sets of $\mathbf{1}_{\tilde{A}_n} * \mathbf{1}_{\tilde{B}_n}$ are null sets; this is a simple consequence of Fubini's theorem and the corresponding property of $\mathbf{1}_{\alpha_n} * \mathbf{1}_{\beta_n}$. Therefore the conclusion of Theorem 1.1, or equivalently that of Theorem 2.1 (whose hypotheses are satisfied by $(\tilde{A}_n, \tilde{B}_n, \tilde{C}_n)$ for large n), holds for $(\tilde{A}_n, \tilde{B}_n, \tilde{C}_n)$ for all sufficiently large n. Since $\tilde{\mu}(A_n) = \mu(A) m(\alpha_n) \to \mu(A)$ and likewise for $\tilde{B}_n, \tilde{C}_n$, it follows from passage to the limit that the conclusion also holds for (A, B, C). $\square$

We next formulate several results, Lemma 4.2 through Corollary 4.6, whose proofs are direct adaptations of proofs of corresponding results in [11]. Those proofs are therefore omitted.

The next lemma states that if (A, B, C) nearly maximizes the Riesz-Sobolev functional $\int_C \mathbf{1}_A * \mathbf{1}_B \, d\mu$, then C nearly coincides with a superlevel set $S_{A,B}(\tau)$ (as long as (A, B, C) is appropriately admissible).

Lemma 4.2. [11] *Let $A, B, C \subset G$ be measurable sets with $\mu(A), \mu(B), \mu(C) > 0$. Suppose that*

$$\big|\, \mu(A) - \mu(B) \,\big| + 2\mathcal{D}(A, B, C)^{1/2} < \mu(C) < \mu(A) + \mu(B) - 2\mathcal{D}(A, B, C)^{1/2} \tag{4.7}$$

$$\mu(A) + \mu(B) + \mu(C) < 2 - 2\mathcal{D}(A, B, C)^{1/2}. \tag{4.8}$$

Define τ by $\mu(C) = \mu(A) + \mu(B) - 2\tau$. Then the superlevel set $S_{A,B}(\tau)$ satisfies

$$\mu(S_{A,B}(\tau) \triangle C) \le 4\mathcal{D}(A,B,C)^{1/2} \tag{4.9}$$

$$\big|\mu(S_{A,B}(\tau)) - \mu(C)\big| \le 2\mathcal{D}(A,B,C)^{1/2} \tag{4.10}$$

$$\mathcal{D}(A,B,S_{A,B}(\tau)) \le \mathcal{D}(A,B,C). \tag{4.11}$$

The next result sharpens Theorem 1.1 in the same way that Theorem 3.3 sharpens Theorem 3.1. It is simply a restatement of the conclusion (4.9) in alternative terms.

Theorem 4.3. *Let $A, B, C \subset G$ be measurable sets with positive Haar measures. Suppose that*

$$\big|\,\mu(A) - \mu(B)\,\big| + 2\mathcal{D}(A,B,C)^{1/2} < \mu(C) < \mu(A) + \mu(B) - 2\mathcal{D}(A,B,C)^{1/2}, \tag{4.12}$$

$$\mu(A) + \mu(B) + \mu(C) < 2 - 2\mathcal{D}(A,B,C)^{1/2}. \tag{4.13}$$

Then

$$\int_C \mathbf{1}_A * \mathbf{1}_B \, d\mu \le \int_{C^\star} \mathbf{1}_{A^\star} * \mathbf{1}_{B^\star} \, dm - \tfrac{1}{16}\mu\big(C \,\Delta\, S_{A,B}(\tau_C)\big)^2 \tag{4.14}$$

where $\tau_C = \frac{1}{2}(\mu(A) + \mu(B) - \mu(C))$.

The next two lemmas relate the two defects $\mathcal{D}, \mathcal{D}'$ to one another.

Lemma 4.4. [11] *Let A, B be measurable subsets of G of positive Haar measures, and suppose that $\tau \in [0, \min(\mu(A), \mu(B))]$ and $\mu(A) + \mu(B) < 1 + \tau$. Then*

$$\mathcal{D}(A,B,S_{A,B}(\tau)) \le \mathcal{D}'(A,B,\tau).$$

Lemma 4.5. [11] *Let $A, B, C \subset G$ be measurable sets with positive Haar measures. Let $\tau_C = \frac{1}{2}(\mu(A) + \mu(B) - \mu(C))$. If*

$$\big|\,\mu(A) - \mu(B)\,\big| + 2\mathcal{D}(A,B,C)^{1/2} < \mu(C) < \mu(A) + \mu(B) - 2\mathcal{D}(A,B,C)^{1/2} \tag{4.15}$$

and $\mu(A) + \mu(B) + \mu(C) \le 2 - 2\mathcal{D}(A,B,C)^{1/2}$ then

$$\mathcal{D}'(A,B,\tau_C) \le 2\mathcal{D}(A,B,C). \tag{4.16}$$

Corollary 4.6. [11] *Let G be a compact connected Abelian topological group, equipped with normalized Haar measure μ. Let $A, B \subset G$ be measurable sets with positive Haar measures. Let $\tau \in [0, \min(\mu(A), \mu(B))]$, and suppose that $\mu(A) + \mu(B) \le 1 + \tau$ and*

$$\big|\,\mu(A) - \mu(B)\,\big| \le \mu(S_{A,B}(\tau)),$$
$$\mu(A) + \mu(B) + \mu(S_{A,B}(\tau)) \le 2.$$

Then

$$\big|\mu(S_{A,B}(\tau)) - (\mu(A) + \mu(B) - 2\tau)\big| \le 2\mathcal{D}'(A,B,\tau)^{1/2}. \tag{4.17}$$

Proof. The hypotheses of Theorem 3.3 are satisfied. The hypothesis $\big|\,\mu(A) - \mu(B)\,\big| \le \mu(S_{A,B}(\tau))$ of the corollary is equivalent to $\sigma \le \min(\mu(A), \mu(B))$, where σ is defined by $\mu(S_{A,B}(\tau)) = \mu(A) + \mu(B) - 2\sigma$. Thus, (4.17) holds by being a restatement of the conclusion of Theorem 3.3 for σ in this range. □

5. Two key principles

In analyzing near-maximizers (A,B,C) of the Riesz-Sobolev functional, we have found it to be useful to transform (A,B,C) in several different ways. Two of these are based on the principles of submodularity and complementation, which are developed in this section as Proposition 5.1 and Lemma 5.5, respectively. A third is the transformation of (A,B,C) to a triple (A,B,τ), based on the relationship between $\mathcal{D}(A,B,S_{A,B}(\tau))$ and $\mathcal{D}'(A,B,\tau)$ explored in §4. A fourth is the flow $(A,B,C)\mapsto(A(t),B(t),C(t))$ introduced in §16. A fifth arises when $C\subset G$ is a rank one Bohr set or is well approximated by such a set, and relates $\int_C \mathbf{1}_A*\mathbf{1}_B\,d\mu$ to a relaxed version of this functional for associated data on $\mathbb{T}$. This connection is developed in §12.

At certain stages of the analysis we will pass from a triple (A,B,C) to a related triple (A',B',C') with certain more advantageous properties, or from (A,B,τ) to (A',B',τ'). We want to do this without sacrificing smallness of $\mathcal{D}(A,B,C)$ or of $\mathcal{D}'(A,B,\tau)$, respectively. Two principles that make this possible are submodularity and complementation.

Let G be a compact connected Abelian group G, with normalized Haar measure μ.

Proposition 5.1 (Submodularity)**.** (Tao [25]) *Let A,B_1,B_2 be measurable subsets of G, and let $\tau\in[0,\min(\mu(A),\mu(B_1\cap B_2))]$ with $\mu(A)+\mu(B_1\cup B_2)-\tau\le 1$. Then*

$$\mathcal{D}'(A,B_1\cap B_2,\tau)+\mathcal{D}'(A,B_1\cup B_2,\tau)\le\mathcal{D}'(A,B_1,\tau)+\mathcal{D}'(A,B_2,\tau)$$

and the above four quantities $\mathcal{D}'$ are all nonnegative.

Lemma 5.2. *Suppose that each of A,B,C has Haar measure strictly >0 and strictly <1. (A,B,C) is admissible and satisfies $\mu(A)+\mu(B)+\mu(C)\le 2$ if and only if $(G\setminus A,G\setminus B,C)$ is admissible and satisfies $\mu(G\setminus A)+\mu(G\setminus B)+\mu(C)\le 2$.*

Proof. The relation $\mu(C)\le\mu(G\setminus A)+\mu(G\setminus B)$ is equivalent to $\mu(A)+\mu(B)+\mu(C)\le 2$, and by symmetry $\mu(C)\le\mu(A)+\mu(B)$ is equivalent to $\mu(G\setminus A)+\mu(G\setminus B)+\mu(C)\le 2$.

The relation $\mu(G\setminus A)\le\mu(G\setminus B)+\mu(C)$ is equivalent to $\mu(B)\le\mu(A)+\mu(C)$, and interchanging A,B in this equivalence yields the equivalence of the remaining two relations. □

Lemma 5.3. *For each $\eta>0$ there exists $\eta'>0$ with the following property. Suppose that each of A,B,C has Haar measure strictly >0 and strictly <1, and that (A,B,C) is η–strictly admissible and η-bounded. Then $(G\setminus A,G\setminus B,C)$ is η'–strictly admissible and η'-bounded.*

This is proved in the same way as Lemma 5.2. □

Lemma 5.4. *Suppose that each of A,B has Haar measure strictly >0, that $\mu(A)+\mu(B)<1$, and that $A+B$ is measurable. Then*

$$\mu_*(A+\tilde B)-\mu(A)-\mu(\tilde B)\le\mu(A+B)-\mu(A)-\mu(B) \tag{5.1}$$

where $\tilde B=-\big(G\setminus(A+B)\big)$.

Proof. It holds that $(G\setminus(A+B))-A\subset G\setminus B$. Indeed, let $x\in A$ and $z\notin A+B$. If $y=z-x$ belongs to B then $x+y=z$, whence $z\in A+B$, a contradiction.

Therefore $\mu_*\big(A-G\setminus(A+B)\big)\le 1-\mu(B)$ and consequently

$$\begin{aligned}\mu_*\big(A-G\setminus(A+B)\big)-\mu(A)-\mu(G\setminus(A+B))\\ \le 1-\mu(B)-\mu(A)-[1-\mu(A+B)]\\ =\mu(A+B)-\mu(A)-\mu(B).\end{aligned}$$

□

Lemma 5.5 (Complementation)**.** *If (A,B,C) is admissible and $\mu(A)+\mu(B)+\mu(C)\le 2$ then*

$$\mathcal{D}(A,B,C)=\mathcal{D}(G\setminus A,\,G\setminus B,\,C). \tag{5.2}$$

Proof. Writing $\mathbf{1}_{G\setminus A}=1-\mathbf{1}_A$ and likewise for B, then expanding the integrand, gives

$$\begin{aligned}\int_C \mathbf{1}_{G\setminus A}*\mathbf{1}_{G\setminus B}\,d\mu &= \int_C (1-\mu(A)-\mu(B)+\mathbf{1}_A*\mathbf{1}_B)\,d\mu\\ &= \int_C \mathbf{1}_A*\mathbf{1}_B\,d\mu+\mu(C)(1-\mu(A)-\mu(B)).\end{aligned}$$

Since

$$(G\setminus A)^\star=\{\tfrac12-x : x\in\mathbb{T}\setminus A^\star\}$$

and likewise for $(G\setminus B)^\star$, and since $A^\star,B^\star,C^\star$ are symmetric under $x\mapsto -x$, the same calculation gives

$$\int_{C^\star}\mathbf{1}_{(G\setminus A)^\star}*\mathbf{1}_{(G\setminus B)^\star}\,dm=\int_{C^\star}\mathbf{1}_{A^\star}*\mathbf{1}_{B^\star}\,dm+\mu(C)(1-\mu(A)-\mu(B))$$

since $m(A^\star)=\mu(A)$ and likewise for B. Subtracting these two relations gives $\mathcal{D}(A,B,C)=\mathcal{D}(G\setminus A,\,G\setminus B,\,C)$. □

6. A link between Riesz-Sobolev and sumset inequalities

At the heart of our analysis of inverse theorems for the Riesz-Sobolev inequality (1.2) lies Lemma 6.1. It states that if (A,B,C) is nearly a maximizer for the functional $\int_C \mathbf{1}_A*\mathbf{1}_B\,d\mu$, then a certain associated superlevel set $S=S_{A,B}(\beta)$ has small sumset in the sense that $\mu(S-S)$ is nearly equal to $2\mu(S)$. Invoking inverse theorems of Tao [25] and of Griesmer [20] for sumsets yields the conclusion that S is nearly a rank one Bohr set. The same conclusion then follows for C since, by Lemma 4.2, $C\,\Delta\,S$ has small Haar measure.

However, the proof of Lemma 6.1 requires the very restrictive hypothesis that $\mu(A)=\mu(B)$. In an analysis of the Riesz-Sobolev equality for $\mathbb{R}^1$ in [11], this hypothesis was removed in a subsequent step, by a method that does not apply to compact groups G. In the present paper we will accomplish this removal for compact connected Abelian groups by an unrelated and somewhat lengthy alternative method based in part on ideas of Tao [25]. This necessitates the reductions carried out in §7.

Lemma 6.1. [11] *Let (A,B,C) be an η–strictly admissible ordered triple of measurable subsets of G with positive Haar measures. Suppose that*

$$\mu(A)=\mu(B)\le\tfrac12, \tag{6.1}$$

$$\mu(C)\le\mu(A)-4\mathcal{D}(A,B,C)^{1/2}, \tag{6.2}$$

$$\mathcal{D}(A,B,C)^{1/2}<\tfrac{1}{28}\eta\mu(A). \tag{6.3}$$

Let $\beta=\frac12\big(\mu(A)+\mu(B)-\mu(C)\big)$. Then

$$\mu\big(S_{A,B}(\beta)-S_{A,B}(\beta)\big)\le 2\mu(S_{A,B}(\beta))+12\mathcal{D}(A,B,C)^{1/2}. \tag{6.4}$$

The proof of this lemma is essentially identical to the proof of the corresponding result in [11], so it is not included here. □

Under certain hypotheses, it can be concluded that the set $S_{A,B}(\beta)$ above is nearly a rank one Bohr set.

Corollary 6.2. *For each $\varepsilon, \eta > 0$ there exists $\rho > 0$ with the following property. Let (A, B, C) be an η–strictly admissible η–bounded ordered triple of measurable subsets of G satisfying the hypotheses* (6.1) *and* (6.2) *of Lemma 6.1. If $\mathcal{D}(A, B, C) \le \rho$ then there exists a rank one Bohr set $\mathcal{B} \subset G$ satisfying*

$$\mu(C \bigtriangleup \mathcal{B}) \le \varepsilon. \tag{6.5}$$

If $\mu(C) \le \mu(A) = \mu(B) \le \frac{1}{2} - \eta$ and $\mathcal{D}(A, B, C) = 0$, then there exists a rank one Bohr set $\mathcal{B} \subset G$ satisfying

$$\mu(C \bigtriangleup \mathcal{B}) = 0. \tag{6.6}$$

Proof. Let $\varepsilon > 0$. By (4.9), $\mu(S_{A,B}(\beta) \,\Delta\, C) \le 4\mathcal{D}(A, B, C)^{1/2}$. Moreover, if $\mathcal{D}(A, B, C)$ is sufficiently small as a function of η, then the conclusion (6.4) of Lemma 6.1 states that $S_{A,B}(\beta)$ satisfies a strong form of the hypothesis of the theorems of Tao [25] and Griesmer [20] discussed in §1. The conclusion of those theorems is the existence of a rank one Bohr set satisfying $\mu(\mathcal{B} \,\Delta\, S_{A,B}(\beta)) \le \varepsilon$, where $\varepsilon \to 0$ as $\mathcal{D}(A, B, C) \to 0$ with η fixed. Therefore

$$\mu(\mathcal{B} \,\Delta\, C) \le \mu(\mathcal{B} \,\Delta\, S_{A,B}(\beta)) + \mu(S_{A,B}(\beta) \,\Delta\, C) \le \varepsilon + 4\mathcal{D}(A, B, C)^{1/2}.$$

If $\varepsilon = 0$ and the measures of A, B, C satisfy the indicated hypotheses, then by (4.9) and Lemma 6.1, $S = S_{A,B}(\beta)$ satisfies $\mu(S \,\Delta\, C) = 0$ and $\mu(S - S) \le 2\mu(S)$. Therefore $\mu(S) = \mu(C) \le \frac{1}{2} - \eta$, and S achieves equality in Kneser's inequality. Thus, by Kneser's inverse theorem, there exists a rank one Bohr set $\mathcal{B}$ satisfying $\mu(\mathcal{B}) = \mu(S)$ and $\mu(\mathcal{B} \backslash S) = 0$. Thus $\mu(\mathcal{B} \,\Delta\, C) = 0$ also. □

7. Two reductions

This section is devoted to two auxiliary results, whose purpose is to reduce the analysis of triples that nearly saturate the Riesz-Sobolev inequality to triples that satisfy the hypotheses of Corollary 6.2. In particular, we show that if (A, B, C) nearly maximizes the Riesz-Sobolev functional among triples of sets with specified Haar measures, then there exists a closely related near maximizing triple $(\tilde{A}, \tilde{B}, \tilde{C})$ satisfying supplementary properties, including the hypotheses of Corollary 6.2. Those properties will subsequently be used to deduce that $(\tilde{A}, \tilde{B}, \tilde{C})$ is nearly a compatibly centered parallel triple of rank one Bohr sets. From that we will deduce the same property for (A, B, C). This will be achieved by ultimately applying this reasoning to a short chain of triples (A_n, B_n, C_n), with (A_n, B_n, C_n) constructed recursively from $(A_{n-1}, B_{n-1}, C_{n-1})$ beginning with $(A_0, B_0, C_0) = (A, B, C)$, and with conclusions propagated in reverse from (A_n, B_n, C_n) to $(A_{n-1}, B_{n-1}, C_{n-1})$,

Lemma 7.1. *Let (A, B, C) be an η-strictly admissible and η–bounded triple of μ-measurable subsets of G, satisfying*

$$\mu(C) \le \mu(A) \le \mu(B),$$
$$\mu(A) \le \tfrac{1}{2},$$
$$\mathcal{D}(A, B, C)^{1/2} \le \tfrac{1}{400}\eta^2 \mu(B).$$

Define τ by $\mu(C) = \mu(A) + \mu(B) - 2\tau$. Then there exists a measurable set $B' \subseteq G$ with $\mu(A) = \mu(B')$ such that

$$(A, B', S_{A,B'}(\tau)) \text{ is } \eta/2\text{–strictly admissible and } \eta^2/2\text{–bounded,}$$

$$\mathcal{D}(A, B', S_{A,B'}(\tau)) \le \frac{1}{\eta}\mathcal{D}(A, B, C).$$

Moreover, if $\mu(C) \le (1 - \frac{\eta}{50})\mu(B)$ *then*

$$\mu(S_{A,B'}(\tau)) \le \mu(A) - 4\mathcal{D}(A, B', S_{A,B'}(\tau))^{1/2}. \tag{7.1}$$

Proof. The set B' is constructed via an iterative process, in the course of which B is recursively replaced by successively smaller sets B_j, finally arriving at a set B' with the same Haar measure as A. The quantity $\mathcal{D}'(A, B_j, \tau)$ is controlled by induction on j, yielding control of $\mathcal{D}'(A, B', \tau)$.

Before starting this process, recall that C is essentially equal to $S_{A,B}(\tau)$ in the sense that

$$\big|\mu\big(S_{A,B}(\tau)\big) - \mu(C)\big| \le 2\mathcal{D}(A, B, C)^{1/2}, \tag{7.2}$$

$$\mathcal{D}(A, B, S_{A,B}(\tau)) \le \mathcal{D}(A, B, C), \tag{7.3}$$

$$\mathcal{D}'(A, B, \tau) \le 2\mathcal{D}(A, B, C), \tag{7.4}$$

with these inequalities justified by Lemmas 4.2 and 4.5.

The following lemma will be useful.

Lemma 7.2. *Let* B *be a measurable subset of* G*. For any* $t \in [\mu(B)^2, \mu(B)]$*, there exists* $x_t \in B$ *satisfying* $\mu\big(B \cap (x_t + B)\big) = t$*.*

This is a direct consequence of the connectivity of G, since $x \mapsto \mu(B \cap (B + x))$ is a continuous function from G to $\mathbb{R}$. □

Iteratively invoking Lemma 7.2, a nested sequence of subsets of B will be constructed; the last set in the sequence will be the desired B'. The properties of this sequence are described in the following Claim, the proof of which is postponed until after the proof of Lemma 7.1.

Claim 7.1. There exists a nested sequence $B =: B_0 \supseteq B_1 \supseteq B_2 \supseteq \ldots \supseteq B_J$ of subsets of G, with

$$\mu(B_J) = \mu(A),$$

$$\mathcal{D}'(A, B_j, \tau) \le 2\mathcal{D}'(A, B_{j-1}, \tau) \text{ for each } j \le J, \tag{7.5}$$

$$2^J \le \frac{2}{\eta^2}.$$

It follows that

$$\mathcal{D}'\big(A, B', \tau\big) \le 2^J \cdot 2\mathcal{D}(A, B, C) \le \frac{4}{\eta^2}\mathcal{D}(A, B, C),$$

whence

$$\mathcal{D}'\big(A, B', \tau\big)^{1/2} \le \tfrac{1}{200}\eta\mu(B) \tag{7.6}$$

by the hypothesis on $\mathcal{D}(A, B, C)$.

We claim that (A, B', τ) satisfies the hypotheses of Corollary 4.6. Firstly, $\tau = \frac{1}{2}(\mu(A) + \mu(B) - \mu(C)) \le \min(\mu(A), \mu(B')) = \mu(A)$ is equivalent to $\mu(B) \le \mu(A) + \mu(C)$, which holds since (A, B, C) is admissible. Secondly, the superlevel set $S_{A,B'}(\tau)$ satisfies $|\mu(A) - \mu(B')| \le \mu(S_{A,B'}(\tau))$, since $\mu(A) - \mu(B') = 0$. Thirdly, $\mu(A) + \mu(B') \le 1 + \tau = 1 + \frac{1}{2}(\mu(A) + \mu(B) - \mu(C))$ is equivalent to $\mu(A) + \mu(C) + (2\mu(B') - \mu(B)) \le 2$, which holds since $\mu(A) + \mu(B) + \mu(C) \le 2$ and $\mu(B') \le \mu(B)$. Fourthly, $\mu(A) + \mu(B') + \mu(S_{A,B'}(\tau)) \le 2$, as $\mu(A) = \mu(B') \le \frac{1}{2}$.

Invoking Corollary 4.6 for the triple $(A, B', S_{A,B'}(\tau))$ gives

$$|\mu(S_{A,B'}(\tau)) - (\mu(A) + \mu(B') - 2\tau)| \le 2\mathcal{D}'(A, B', \tau)^{1/2}. \tag{7.7}$$

Since $\mu(A)+\mu(B')-2\tau=\mu(A)+\mu(C)-\mu(B)$ is $\geq \eta\mu(B)\geq \eta\mu(A)$ by the η–strict admissibility hypothesis, while also $2\tau\geq\mu(B)$, it follows from (7.6) that $(A,B',S_{A,B'}(\tau))$ is $\eta/2$-strictly admissible and satisfies the estimates $\mu(A)+\mu(B')+\mu(S_{A,B'}(\tau))\leq 2-\frac{1}{2}\eta$ and $\min(\mu(A),\mu(B'),\mu(S_{A,B'}(\tau)))\geq\eta^2/2$.

Moreover, if $\mu(C)\leq\mu(B)-\frac{1}{50}\eta\mu(B)$ then

$$\begin{aligned}\mu(S_{A,B'}(\tau)) &\leq \mu(A)+\mu(B')-2\tau+2\mathcal{D}'(A,B',\tau)^{1/2}\\ &= \mu(A)+\mu(C)-\mu(B)+2\mathcal{D}'(A,B',\tau)^{1/2}\\ &\leq \mu(A)-\big(\mu(B)-\mu(C)\big)+\tfrac{1}{100}\eta\mu(B)\\ &\leq \mu(A)-\tfrac{1}{50}\eta\mu(B)+\tfrac{1}{100}\eta\mu(B).\end{aligned}$$

Therefore $\mu(S_{A,B'}(\tau))\leq\mu(A)-4\mathcal{D}(A,B',\tau)^{1/2}$, establishing together with Lemma 4.4 the final assertion of Lemma 7.1. □

Proof of Claim 7.1. The sets B_j will be constructed by an iterative use of Lemma 7.2, in such a way that Proposition 5.1 can be invoked to control each $\mathcal{D}'(A,B_j,\tau)$. More precisely, for each $j=1,\ldots,J$, define

$$B_j:=B_{j-1}\cap(x_j+B_j),$$

with $x_j\in G$ chosen to ensure that

$$\mu(B_j)=\mu(B_{j-1})-b_j \tag{7.8}$$

for appropriate quantities $b_j\in[0,\mu(B_{j-1})-\mu(B_{j-1})^2]$ that will be specified later (such x_j exists by Lemma 7.2), where J is defined as the smallest non-negative integer such that $\mu(B_J)=\mu(A)$. (The quantities b_j will be such that such J will exist.)

Now, the $b_j\in[0,\mu(B_{j-1})-\mu(B_{j-1})^2]$ are chosen so that $\mu(B_j)\geq\mu(A)$ for all j, i.e.

$$b_j\leq\mu(B_{j-1})-\mu(A), \tag{7.9}$$

and so that Proposition 5.1 can be applied for (A,B_j,τ) and $\big(A,B_{j-1}\cup(x_j+B_{j-1}),\tau\big)$, to deduce (7.5). To that end, for each j the estimate

$$\mu(A)+\mu\big(B_{j-1}\cup(x_j+B_{j-1})\big)\leq 1+\tau$$

should hold, i.e. $\mu(A)+\mu(B_{j-1})+b_j\leq 1+\tau$ for all j. By (7.8), this is equivalent to

$$\begin{cases}\mu(A)+\mu(B)+b_1\leq 1+\tau \text{ (for } j=1),\\ \mu(A)+\mu(B)-(b_1+b_2+\ldots+b_{j-1})+b_j\leq 1+\tau \text{ for } j\geq 2,\end{cases}$$

that is

$$\begin{cases}b_1\leq d,\\ b_j-(b_1+b_2+\ldots+b_{j-1})\leq d \text{ for } j\geq 2,\end{cases} \tag{7.10}$$

where $d:=\frac{1}{2}\big(2-\mu(A)-\mu(B)-\mu(C))\big)$.

Therefore, it suffices to find $b_j\in[0,\mu(B_{j-1})-\mu(B_{j-1})^2]$ that satisfy (7.9), that are small enough for (7.10) to hold, but also large enough for $\mu(B_J)=\mu(A)$ to hold for some J with $2^J\leq\frac{2}{\eta^2}$.

Observe that, if not for the condition $b_j\in[0,\mu(B_{j-1})-\mu(B_{j-1})^2]$, the quantities $b_j=2^jd$ for all $j=1,\ldots,J-1$ and $b_J=\mu(B_{J-1})-\mu(A)$, where J is the smallest positive integer with $\mu(B)-d-2d-\ldots-2^Jd<\mu(A)$, would work as they satisfy (7.9) and (7.10), while also $2^J\leq\frac{2}{\eta^2}$.

In order to achieve the additional condition $b_j \in [0, \mu(B_{j-1}) - \mu(B_{j-1})^2]$, more care needs to be taken. For simplicity, once B_{j-1} has been defined, denote

$$m_j := \min\big(\mu(B_{j-1}) - \mu(B_{j-1})^2, \mu(B_{j-1}) - \mu(A)\big).$$

Define

$$b_j := 2^j d \text{ for all } j = 1, \dots, J_1 - 1,$$

where J_1 is the smallest non-negative integer j such that $2^j d > m_j$. Observe that the so far defined b_j satisfy the required conditions.

If $2^{J_1} d \le \mu(B_{J_1-1}) - \mu(B_{J_1-1})^2$, then $2^{J_1} d > \mu(B_{J_1-1}) - \mu(A)$, so $\mu(B_{J_1-1}) - \mu(A) \in [0, \mu(B_{J_1-1}) - \mu(B_{J_1-1})^2]$. In this case, define $b_{J_1} := \mu(B_{J_1-1}) - \mu(A)$ and terminate the process. The b_j satisfy all the required conditions.

Otherwise, $2^{J_1} d > \mu(B_{J_1-1}) - \mu(B_{J_1-1})^2$. Define

$$b_j := m_j \text{ for all } J = J_1 + 1, \dots, \bar{J}_2 - 1,$$

where $\bar{J}_2$ is the smallest integer larger than J_1 with $2^{J_1} d \le m_{\bar{J}_2}$, having terminated the process at the smallest j along the way for which $m_j = 0$, if such a j exists. Observe that the so far defined b_j satisfy the required conditions.

If the process has not been terminated, define

$$b_j := 2^{J_1 + j - \bar{J}_2} d \text{ for all } j = \bar{J}_2, \dots, J_2 - 1,$$

where J_2 is the smallest integer j larger than $\bar{J}_2$ with $2^{J_1+j-\bar{J}_2} d > m_j$. The so far defined b_j satisfy the required conditions.

Now, working as above, if $2^{J_1+J_2-\bar{J}_2} d \le \mu(B_{J_1-1}) - \mu(B_{J_1-1})^2$ define $b_{J_2} := \mu(B_{J_2-1}) - \mu(A)$ and terminate the process. Otherwise, define

$$b_j := m_j \text{ for all } J = J_2 + 1, \dots, \bar{J}_3 - 1,$$

where $\bar{J}_3$ is the smallest integer larger than J_2 with $2^{J_1+J_2-\bar{J}_2} d \le m_2$, having terminated the process at the smallest j along the way for which $m_j = 0$, if such a j exists. Continuing this way, one definitely finds $J \in \mathbb{N}$ with $\mu(B_J) = \mu(A)$; that is when the process terminates. The b_j satisfy (7.9) and (7.10). Therefore, it remains to show that $2^J \le \frac{2}{\eta^2}$.

Indeed, $b_1 + \ldots + b_J = \mu(B) - \mu(A)$. Now, let $\mathcal{M}$ be the set of j for which $b_j = m_j$, and $\mathcal{M}' := \{1, \dots, J\} \setminus \mathcal{M}$. On the one hand,

$$\sum_{j \in \mathcal{M}'} b_j = d + 2d + 2^2 d + \ldots + 2^{m'} d \ge 2^{m'} d,$$

where $m' = \#\mathcal{M}'$. Therefore, $2^{m'} d \le \mu(B) - \mu(A)$, so

$$2^{m'} \le \tfrac{1}{\eta}.$$

On the other hand, m equals at most the number of consecutive intervals of the form $[c^2, c]$ needed to cover $[\mu(A), \mu(B)]$ (with the right-most interval being $[\mu(B)^2, \mu(B)]$). This in turn equals the smallest positive integer k with $\mu(B)^{2^k} \le \mu(A)$. Since $\mu(B)^{2^{k-1}} \ge \mu(A)$, it follows that

$$2^m \le 2^{\frac{\ln\left(\frac{1}{\mu(A)}\right)}{\ln\left(\frac{1}{\mu(B)}\right)}} \le 2^{\frac{\ln\left(\frac{1}{\eta}\right)}{\ln 2}} \le \tfrac{2}{\eta}.$$

So, $2^J = 2^{m+m'} \le \frac{2}{\eta^2}$.

□

The next lemma will be used to deduce properties of more general triples from properties of triples that satisfy the hypotheses of Lemma 7.1.

Lemma 7.3. *Let (A,B,C) be η-strictly admissible and η–bounded and satisfy*

$$\mu(C) \le \mu(A) \le \mu(B),$$
$$\mu(A) \le \tfrac{1}{2},$$
$$\mathcal{D}(A,B,C)^{1/2} \le \tfrac{1}{800}\eta\mu(B).$$

Define τ by $\mu(B) = \mu(A) + \mu(C) - 2\tau$. If $\mu(C) > (1-\frac{\eta}{50})\mu(B)$ then there exist measurable sets $C' \subseteq C$ and $A' \subseteq A$ that satisfy

$$\begin{cases} (S_{C',A}(\tau), C', A) \text{ is } \eta/4\text{–strictly admissible and } \eta/4\text{–bounded} \\ \mathcal{D}(S_{C',A}(\tau), C', A) \le 16\mathcal{D}(C,B,A) \\ \mu(C') = \mu(A') = \mu(C) - \frac{1}{10}\eta\mu(B), \end{cases}$$

while

$$\begin{cases} (S_{C',A'}(\tau), C', A') \text{ is } \eta/2\text{–strictly admissible and } \eta/2\text{–bounded} \\ \mathcal{D}(S_{C',A'}(\tau), C', A') \le 16\mathcal{D}(C,B,A) \\ \mu(S_{A',C'}(\tau)) \le (1-\frac{\eta/2}{50})\mu(C'). \end{cases}$$

Proof. Define $\tau = \frac{1}{2}(\mu(A)+\mu(C)-\mu(B))$. Then $\tau \ge \frac{1}{2}\eta\mu(B) \ge \frac{1}{2}\eta^2$ by the η–strict admissibility hypothesis, while $\tau \le \frac{1}{2}\mu(C) \le \frac{1}{4}$ since $\mu(B) \ge \mu(A)$.

Since (A,B,C) is η-strictly admissible and $\mathcal{D}(A,B,C)$ is small relative to $\eta\mu(B)$, Lemma 4.2 gives

$$\big|\mu\big(S_{C,A}(\tau)\big) - \mu(B)\big| \le 2\mathcal{D}(A,B,C)^{1/2}, \tag{7.11}$$

whence $(C, A, S_{C,A}(\tau))$ is $\frac{1}{2}\eta$-strictly admissible. Lemma 4.2 also gives

$$\mathcal{D}(C,A,S_{C,A}(\tau)) \le \mathcal{D}(A,B,C). \tag{7.12}$$

By Lemma 4.5,

$$\mathcal{D}'(C,A,\tau) \le 2\mathcal{D}(A,B,C).$$

Now, there exist $x_C, x_A \in G$ such that $C' := C \cap (x_C + C)$ and $A' := A \cap (x_A + A)$ satisfy

$$\begin{cases} \mu(C') = \mu(C) - \frac{\eta}{10}\mu(B) \ \in [\mu(C)^2, \mu(C)] \\ \mu(A') = \mu(C') \ \in [\mu(A)^2, \mu(A)]. \end{cases}$$

(Observe that $\mu(C) - \frac{\eta}{10}\mu(B) \ge \mu(A)^2$ $(\ge \mu(C)^2)$ because $\mu(A) \le \frac{1}{2}$, thus $\mu(A)^2 \le \frac{1}{2}\mu(A) \le \frac{1}{2}\mu(B)$; combining this with the lower bound assumption on $\mu(C)$, one obtains $\mu(C) - \mu(A)^2 \ge (1 - \frac{\eta}{50} - \frac{1}{2})\mu(B) \ge \frac{\eta}{10}\mu(B)$.)

It holds that

$$0 \le \tau \le \mu(C') = \min\big\{\mu(C'), \mu(A)\big\} = \min\big\{\mu(C'), \mu(A')\big\}$$

and

$$\mu(C') + \mu(A \cup A') - \tau \le \mu(A) + \mu(C \cup C') - \tau < 1$$

(as $2\frac{\eta}{10}\mu(B) < 2 - (\mu(A)+\mu(B)+\mu(C))$). Therefore,

$$0 \le \mathcal{D}'(C',A',\tau) \le 2\mathcal{D}'(C',A,\tau) \le 4\mathcal{D}'(C,A,\tau) \le 8\mathcal{D}(A,B,C)$$

by the submodularity principle, Proposition 5.1.

We apply Corollary 4.6 to the triple (A', C', τ). Its hypotheses are satisfied. First, $0 \le \tau \le \min(\mu(C'), \mu(A')) = \mu(C')$; also, $\mu(A') + \mu(C') < 1 + \tau$ holds, since $\mu(C') = \mu(A') \le \mu(A) \le \frac{1}{2}$ while $\tau > 0$. Second, $\mu(S_{A',C'}(\tau)) \ge 0 = |\mu(A') - \mu(C')|$. Third, $\mu(A') + \mu(C') + \mu(S_{A',C'}(\tau)) \le 2$ because $\mu(A') = \mu(C') \le \mu(A) \le \frac{1}{2}$ while $\mu(S_{A',C'}(\tau)) \le 1$. Therefore the Corollary may be applied to obtain

$$|\mu(S_{C',A'}(\tau)) - (\mu(A') + \mu(C') - 2\tau)| \le 2\mathcal{D}'(C', A', \tau)^{\frac{1}{2}} \le \tfrac{\eta}{100}\mu(B). \tag{7.13}$$

We next show that $(S_{C',A'}(\tau), C', A')$ is $\frac{\eta}{2}$-strictly admissible. Inserting the definition of τ into (7.13) gives

$$\begin{aligned}
\mu(S_{C',A'}(\tau)) &\le \mu(B) - \big(\mu(A) - \mu(A')\big) - \big(\mu(C) - \mu(C')\big) + \tfrac{\eta}{100}\mu(B) \\
&\le \mu(C) + \tfrac{\eta}{50}\mu(B) - 2 \cdot \tfrac{\eta}{10}\mu(B) + \tfrac{\eta}{100}\mu(B) \\
&\le \mu(C') - \tfrac{\eta}{50}\mu(B) \\
&\le (1 - \tfrac{\eta}{50})\mu(C').
\end{aligned}$$

Note that the last of the three conclusions stated for $(A', C', S_{A',B'}(\tau))$ has been verified.

On the other hand,

$$\begin{aligned}
\mu(S_{C',A'}(\tau)) &\ge \mu(B) - \big(\mu(A) - \mu(A')\big) - \big(\mu(C) - \mu(C')\big) - \tfrac{\eta}{100}\mu(B) \\
&\ge \mu(B) - \big(\tfrac{\eta}{10}\mu(B) + \tfrac{\eta}{100}\mu(B)\big) - \tfrac{\eta}{10}\mu(B) - \tfrac{\eta}{100}\mu(B) \\
&\ge \mu(B) - \tfrac{\eta}{4}\mu(B) \\
&\ge \mu(C') - \tfrac{\eta}{4}\mu(B) \\
&> \big(1 - \tfrac{\eta}{50} - \tfrac{\eta}{4}\big)\,\mu(B) \\
&> \tfrac{\eta}{2}\mu(B).
\end{aligned} \tag{7.14}$$

Since $\mu(A') = \mu(C')$ and $\mu(B) \ge \max(\mu(A'), \mu(C'), \mu(S_{A',C'}(\tau)))$, the triple $(A', C', S_{A',C'}(\tau))$ is $\eta/2$–strictly admissible.

We claim next that the intermediate triple $(S_{C',A}(\tau), C', A)$ is $\frac{\eta}{4}$-strictly admissible. Indeed, since $A' \subseteq A$ and $C' \subseteq C$,

$$\mu(S_{C',A'}(\tau)) \le \mu(S_{C',A}(\tau)) \le \mu(S_{C,A}(\tau)),$$

whence, by (7.11) and one of the inequalities in (7.14),

$$\mu(B) - \tfrac{\eta}{4}\mu(B) \le \mu(S_{C',A}(\tau)) \le \mu(B) + 2\mathcal{D}(A, B, C)^{1/2} \le \mu(B) + \tfrac{\eta}{400}\mu(B).$$

Therefore, $\frac{\eta}{4}$-strict admissibility follows from the η–strict admissibility of (A, B, C) and the inequalities $|\mu(C') - \mu(C)| \le \frac{\eta}{10}\mu(B)$ and $|\mu(A) - \mu(B)| \le \frac{\eta}{50}\mu(B)$.

Finally, the $\eta/2$–boundedness of $(A', C', S_{A',C'}(\tau))$ and $\eta/4$–boundedness of $(A, C', S_{A,C'}(\tau))$ follow from estimates shown above. □

8. Relaxation

For function $g_j : G \to [0, 1]$, define $g_j^{\star\star} : \mathbb{T} \to [0, \infty)$ to be the indicator function of the interval centered at 0 whose Lebesgue measure is equal to $\int_G g_j \, d\mu$. Define $\mathbf{g}^{\star\star} = (g_1^{\star\star}, g_2^{\star\star}, g_3^{\star\star})$. Assuming that g_j takes values in $[0, 1]$ for each index j, we say that $\mathbf{g}$ is η–strictly admissible if the triple $(\int_G g_j \, d\mu : 1 \le j \le 3)$ is η–strictly admissible.

With these notations, Theorem 1.5 can be equivalently stated as the inequality

$$\mathcal{T}_G(\mathbf{g}) \le \mathcal{T}_{\mathbb{T}}(\mathbf{g}^{\star\star}) \quad \text{for all functions } g_j : G \to [0, 1]. \tag{8.1}$$

Notation 8.1. For any ordered triple $\mathbf{E}$ of measurable subsets of G, define

$$\overline{\mathcal{D}}(\mathbf{E}) = \mathcal{T}_G(\mathbf{E}^\star) - \mathcal{T}_{\mathbb{T}}(\mathbf{E}). \tag{8.2}$$

More generally, for $g : G \to [0,1]$, define

$$\overline{\mathcal{D}}(\mathbf{g}) = \mathcal{T}_{\mathbb{T}}(\mathbf{g}^{\star\star}) - \mathcal{T}_G(\mathbf{g}), \tag{8.3}$$

and for $\mathbf{g} = (g_j : j \in \{1,2,3\})$, define $\mathbf{g}^{\star\star} = (g_j^\star : j \in \{1,2,3\})$.

Then

$$\mathcal{D}(A,B,C) = \overline{\mathcal{D}}(A,B,-C)$$

for any ordered triple (A,B,C) of measurable subsets of G. That is,

$$\langle \mathbf{1}_{A^\star} * \mathbf{1}_{B^\star}, \mathbf{1}_{C^\star}\rangle_{\mathbb{T}} - \langle \mathbf{1}_A * \mathbf{1}_B, \mathbf{1}_C\rangle_G = \langle \mathbf{1}_{A^\star} * \mathbf{1}_{B^\star}, \mathbf{1}_{(-C)^\star}\rangle_{\mathbb{T}} - \langle \mathbf{1}_A * \mathbf{1}_B, \mathbf{1}_{-C}\rangle_G.$$

Theorem 1.5 can again be restated as $\overline{\mathcal{D}}(\mathbf{E}) \ge 0$ for every triple $\mathbf{E}$.

The function $h : \mathbb{T} \to [0,\infty)$ is said to be symmetric if $h(-x) = h(x)$ for all $x \in \mathbb{T}$. If h is symmetric, h is said to be nonincreasing if its restriction to $[0, \frac{1}{2}] \subset \mathbb{T}$ is nonincreasing, under the usual identification of $\mathbb{T}$ with $[-\frac{1}{2}, \frac{1}{2}]$.

Lemma 8.1. *Let $f_1, f_2, f_3 : \mathbb{T} \to \mathbb{R}$ be symmetric, nonincreasing functions satisfying $0 \le f_1, f_2, f_3 \le 1$. Let $I \subset \mathbb{T}$ be the interval centered at 0 of length $|I| = \int_{\mathbb{T}} f_1\,dm$. Then*

$$\mathcal{T}_{\mathbb{T}}(f_1, f_2, f_3) \le \mathcal{T}_{\mathbb{T}}(\mathbf{1}_I, f_2, f_3).$$

Proof. Defining F by $f_1 = \mathbf{1}_I + F$, one has

$$F \le 0 \text{ on } I,\ F \ge 0 \text{ on } \mathbb{T} \setminus I \text{ and } \int_{\mathbb{T}} F\,dm = 0. \tag{8.4}$$

Since

$$\mathcal{T}_{\mathbb{T}}(f_1, f_2, f_3) = \langle f_1, f_2 * f_3\rangle_{\mathbb{T}} = \langle \mathbf{1}_I, f_2 * f_3\rangle_{\mathbb{T}} + \langle F, f_2 * f_3\rangle_{\mathbb{T}},$$

it suffices to show that $\langle F, f_2 * f_3\rangle_{\mathbb{T}} \le 0$. Now, since f_2, f_3 are symmetric, non-increasing and non-negative, each can be approximated by a superposition of indicator functions of intervals centered at 0. Therefore, it suffices to show that $\langle F, \mathbf{1}_J * \mathbf{1}_K\rangle_{\mathbb{T}} \le 0$ for all intervals J, K centered at 0. This is in fact trivially true, due to (8.4) and the fact that $\mathbf{1}_J * \mathbf{1}_K$ is symmetric, non-increasing and non-negative. Indeed,

$$\begin{aligned}
\langle F, \mathbf{1}_J * \mathbf{1}_K\rangle_{\mathbb{T}} &= \int_I \mathbf{1}_J * \mathbf{1}_K \cdot F\,dm + \int_{\mathbb{T}\setminus I} \mathbf{1}_J * \mathbf{1}_K \cdot F\,dm \\
&\le \int_I \left(\inf_I \mathbf{1}_J * \mathbf{1}_K\right) F\,dm + \int_{\mathbb{T}\setminus I} \left(\sup_{\mathbb{T}\setminus I} \mathbf{1}_J * \mathbf{1}_K\right) F\,dm \\
&= c\int_I F\,dm + c\int_{\mathbb{T}\setminus I} F\,dm = c\int F\,dm = 0,
\end{aligned}$$

where $c := \mathbf{1}_J * \mathbf{1}_K\left(\frac{m(I)}{2}\right)$. □

Proof of Theorem 1.5. By expressing each of f, g, h as a superposition of indicator functions and invoking Theorem 1.1, we deduce that

$$\langle f * g, h\rangle_G \le \langle f^\star * g^\star, h^\star\rangle_{\mathbb{T}}. \tag{8.5}$$

Express $h^\star$ as a superposition $\int_0^1 \mathbf{1}_{D(t)}\,dt$ where each $D(t) \subset \mathbb{T}$ is an interval centered at 0. According to Lemma 8.1,

$$\langle f^\star, g^\star, \mathbf{1}_D\rangle_{\mathbb{T}} \le \langle \mathbf{1}_{A^\star} * \mathbf{1}_{B^\star}, \mathbf{1}_D\rangle_{\mathbb{T}} \tag{8.6}$$

for any interval D centered at 0. Integrating with respect to $t \in [0,1]$ yields

$$\langle f^\star * g^\star, h^\star\rangle_{\mathbb{T}} \le \langle \mathbf{1}_{A^\star} * \mathbf{1}_{B^\star}, h^\star\rangle_{\mathbb{T}}. \tag{8.7}$$

A repetition of this reasoning gives

$$\langle \mathbf{1}_{A^\star} * \mathbf{1}_{B^\star}, h^\star\rangle_{\mathbb{T}} \le \langle \mathbf{1}_{A^\star} * \mathbf{1}_{B^\star}, \mathbf{1}_{C^\star}\rangle_{\mathbb{T}}. \tag{8.8}$$

□

9. The perturbative Riesz-Sobolev regime

This section is dedicated to the proof of the following strengthening lemma, which will be used in the proof of our stability theorem for general G, as well as in our independent treatment of $\mathbb{T}$. Roughly speaking, it states that, if three (appropriately admissible) sets E_1, E_2, E_3 are approximated moderately well by rank one Bohr sets, then smallness of the defect $\mathcal{D}(\mathbf{E}) = \mathcal{T}_{\mathbb{T}}(\mathbf{E}^\star) - \mathcal{T}_G(\mathbf{E})$ implies that E_1, E_2, E_3 are more closely approximated by rank one Bohr sets. The analysis is adapted from [12].

Lemma 9.1. *For each $\eta, \eta' > 0$ there exist $\delta_0 > 0$ and $\mathbf{C} < \infty$ with the following property. Let $\mathbf{E} = (E_1, E_2, E_3)$ be an η–strictly admissible triple of measurable subsets of G satisfying*

$$\mu(E_1) + \mu(E_2) + \mu(E_3) \le 2 - \eta'. \tag{9.1}$$

Suppose that there exists a compatibly centered parallel ordered triple $\mathbf{B} = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ of rank one Bohr sets $\mathcal{B}_j \subset G$ satisfying $\mu(\mathcal{B}_j) = \mu(E_j)$ and

$$\max_j \mu(E_j \,\Delta\, \mathcal{B}_j) \le \delta_0 \max_k \mu(E_k). \tag{9.2}$$

Then there exists $\mathbf{y}$ satisfying $y_1 + y_2 = y_3$ such that

$$\max_j \mu(E_j \,\Delta\, (\mathcal{B}_j + y_j)) \le \mathbf{C}\mathcal{D}(\mathbf{E})^{1/2}. \tag{9.3}$$

Since $0 < \mu(\mathcal{B}_j) < 1 = \mu(G)$, the homomorphism ϕ does not vanish identically.

Definition 9.1. An ordered triple $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ of rank one Bohr sets is $\mathcal{T}_G$-compatibly centered if $(\mathcal{B}_1, \mathcal{B}_2, -\mathcal{B}_3)$ is compatibly centered.

All of our discussion of the Riesz-Sobolev inequality can be rephrased in terms of $\mathcal{T}_G$ since

$$\mathcal{T}_G(\mathbf{E}) = \langle \mathbf{1}_{E_1} * \mathbf{1}_{E_2}, \mathbf{1}_{-E_3}\rangle \tag{9.4}$$

and $\mu(-E_3) = \mu(E_3)$. Theorem 1.1 thus states that

$$\mathcal{T}_G(\mathbf{E}) \le \mathcal{T}_{\mathbb{T}}(\mathbf{E}^\star) \tag{9.5}$$

for all triples $\mathbf{E}$ of measurable subsets of G. Another equivalent formulation is $\mathcal{T}_G(\mathbf{E}) \le \mathcal{T}_G(\mathbf{B})$ for any $\mathcal{T}_G$–compatibly centered ordered triple $\mathbf{B}$ of parallel rank one Bohr sets satisfying $\mu(E_j) = \mu(\mathcal{B}_j)$ for each $j \in \{1,2,3\}$; the right-hand side equals $\mathcal{T}_{\mathbb{T}}(\mathbf{E}^\star)$ for any such triple $\mathbf{B}$.

Lemma 9.1 can thus be equivalently formulated as follows.

Lemma 9.2. *For each $\eta, \eta' > 0$ there exist $\delta_0 > 0$ and $\mathbf{C} < \infty$ with the following property. Let $\mathbf{E} = (E_1, E_2, E_3)$ be an η–strictly admissible triple of measurable subsets of G satisfying*

$$\mu(E_1) + \mu(E_2) + \mu(E_3) \le 2 - \eta'. \tag{9.6}$$

Suppose that there exists a $\mathcal{T}_G$-compatibly centered parallel ordered triple $\mathbf{B} = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ *of rank one Bohr sets* $\mathcal{B}_j \subset G$ *satisfying* $\mu(\mathcal{B}_j) = \mu(E_j)$ *and*

$$\max_j \mu(E_j \,\Delta\, \mathcal{B}_j) \le \delta_0 \max_k \mu(E_k). \tag{9.7}$$

Then there exists $\mathbf{y}$ *satisfying* $y_1 + y_2 + y_3 = 0$ *such that*

$$\max_j \mu(E_j \,\Delta\, (\mathcal{B}_j + y_j)) \le \mathbf{C}\overline{\mathcal{D}}(\mathbf{E})^{1/2}. \tag{9.8}$$

Remark 9.2. Suppose that $\mathbf{E}, \mathbf{B}$ satisfy the hypotheses, and that $\overline{\mathcal{D}}(\mathbf{E})$ vanishes. Then the conclusion is not only that $\mathbf{E}$ is equivalent to some ordered triple of Bohr sets, but that it is equivalent to a translate of $\mathbf{B}$. A consequence is that for any $\eta > 0$, there exists $\varepsilon > 0$ with this property: If B, B' are rank one Bohr sets satisfying $\eta \le \mu(B) = \mu(B') \le 1 - \eta$, and if $\mu(B \,\Delta\, B') < \varepsilon$, then $\mu(B \,\Delta\, B') = 0$. There is no surprise in this consequence, but its relationship to the lemma is worthy of note. To deduce it, assume without loss of generality that B, B' are centered at 0, that is, $B = \{x : \|\phi(x)\|_{\mathbb{T}} \le r\}$ for some homomorphism ϕ and $2r \in [\eta, 1-\eta]$, and likewise for B' with respect to a homomorphism ϕ'. Set $\mathbf{B} = (B, B, B)$ and $\mathbf{E} = (B', B', B')$. The hypotheses of Lemma 9.1 are satisfied, if ε is sufficiently small. Moreover, $\overline{\mathcal{D}}(\mathbf{E}) = 0$; any $\mathcal{T}$–compatibly centered parallel family of rank one Bohr sets saturates the Riesz-Sobolev inequality. The conclusion of the lemma is that B' differs from some translate of B by a μ–null set. □

We will prove Lemma 9.2 in the more general relaxed framework, in which indicator functions of sets are replaced by functions taking values in $[0, 1]$. In the remainder of §9, we study triples $\mathbf{g} = (g_j : j \in \{1, 2, 3\})$ with $g_j : G \to [0, 1]$.

For functions $g : G \to [0, 1]$, define $g^{\star\star} : \mathbb{T} \to [0, \infty)$ to be the indicator function of the interval centered at $0 \in \mathbb{T}$ whose Lebesgue measure is equal to $\int_G g \, d\mu$. For triples $\mathbf{g}$, define $\mathbf{g}^{\star\star} = (g_1^{\star\star}, g_2^{\star\star}, g_3^{\star\star})$. Recall the notation $\overline{\mathcal{D}}(\mathbf{g}) = \mathcal{T}_{\mathbb{T}}(\mathbf{g}^{\star\star}) - \mathcal{T}_G(\mathbf{g})$ introduced in (8.3). Assuming that g_j takes values in $[0, 1]$ for each index j, we say that $\mathbf{g}$ is η–strictly admissible if the triple $(\int_G g_j \, d\mu : 1 \le j \le 3)$ of positive scalars is η–strictly admissible.

The next lemma generalizes Lemma 9.2 to the relaxed framework. The remainder of this section will be devoted to its proof.

Lemma 9.3. *For each* $\eta, \eta' > 0$ *there exist* $\delta_0 > 0$ *and* $\mathbf{C} < \infty$ *with the following property. Let* $\mathbf{g}$ *be an* η*–strictly admissible triple of measurable functions* $g_j : G \to [0, 1]$ *satisfying*

$$\sum_{j=1}^{3} \int g_j \, d\mu \le 2 - \eta'. \tag{9.9}$$

Suppose that there exists a $\mathcal{T}_G$-compatibly centered parallel ordered triple $\mathbf{B} = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ *of rank one Bohr sets* $\mathcal{B}_j \subset G$ *satisfying* $\mu(\mathcal{B}_j) = \int g_j \, d\mu$ *and*

$$\max_j \|g_j - \mathbf{1}_{\mathcal{B}_j}\|_{L^1(G)} \le \delta_0 \max_k \int g_k \, d\mu. \tag{9.10}$$

Then there exists $\mathbf{y} \in G^3$ *satisfying* $y_1 + y_2 + y_3 = 0$ *such that*

$$\max_j \|g_j - \mathbf{1}_{\mathcal{B}_j + y_j}\|_{L^1(G)} \le \mathbf{C}\overline{\mathcal{D}}(\mathbf{g})^{1/2}. \tag{9.11}$$

Define the orbit $\mathcal{O}(\mathbf{A})$ of the triple $\mathbf{A}$ of subsets of G to be the set of all triples $\mathbf{A} + \mathbf{y} = (A_j + y_j : j \in \{1, 2, 3\})$ with $\mathbf{y} \in G^3$ satisfying $y_1 + y_2 + y_3 = 0$. For $g_j : G \to [0, 1]$ and

$\mathbf{B} = (\mathcal{B}_j : 1 \le j \le 3)$ satisfying $\mu(\mathcal{B}_j) = \int g_j \, d\mu$, define

$$\text{distance}\,(\mathbf{g}, \mathcal{O}(\mathbf{B})) = \inf_{\mathbf{y}} \max_{j\in\{1,2,3\}} \|g_j - \mathbf{1}_{\mathcal{B}_j + y_j}\|_{L^1(G)}, \tag{9.12}$$

with the infimum taken over all $\mathbf{y} \in G^3$ satisfying $y_1 + y_2 + y_3 = 0$. With these definitions, Lemma 9.3 states that if $\mathbf{B}, \mathbf{g}$ satisfy its hypotheses then

$$\text{distance}\,(\mathbf{g}, \mathcal{O}(\mathbf{B})) \le \mathbf{C}\overline{\mathcal{D}}(\mathbf{g})^{1/2}. \tag{9.13}$$

We use c to denote a strictly positive constant that depends only on η, but whose value is permitted to change from one occurrence to the next. We write $\langle f, g\rangle = \int_G fg \, d\mu$ for functions $f, g : G \to \mathbb{R}$.

Proof of Lemma 9.3. Set

$$\delta = \text{distance}\,(\mathbf{g}, \mathcal{O}(\mathbf{B})). \tag{9.14}$$

Choose $\mathbf{z}$ satisfying $z_1 + z_2 + z_3 = 0$ so that

$$\max_j \|g_j - \mathbf{1}_{\mathcal{B}_j + z_j}\|_{L^1(G)} = \delta. \tag{9.15}$$

Such a minimizing $\mathbf{z}$ must exist, since $\|g_j - \mathbf{1}_{\mathcal{B}_j + z_j}\|_{L^1(G)}$ is a continuous function of $\mathbf{z}$ with compact domain. If $\delta = 0$ then the conclusion of the lemma certainly holds, so we may assume for the remainder of the proof that $\delta > 0$.

The hypotheses and conclusion of the lemma are invariant under translation of each g_j by $u_j \in G$, with $\sum_j u_j = 0$. By means of such a transformation, we may assume without loss of generality that $\mathcal{B}_j = \{x \in G : \|\phi(x)\|_{\mathbb{T}} \le r_j\}$, with $\phi : G \to \mathbb{T}$ a continuous homomorphism independent of j, and each $z_j = 0$. Here, $0 < r_j = \frac{1}{2}\mu(\mathcal{B}_j) \le \frac{1}{2}(1 - \tilde{\eta})$ with $\tilde{\eta} = \tilde{\eta}(\eta, \eta') > 0$.

Define functions f_j by

$$g_j = \mathbf{1}_{\mathcal{B}_j} + f_j. \tag{9.16}$$

These functions take values in $[-1, 1]$, and satisfy $\int_G f_j \, d\mu = 0$. Moreover, $\max_{k\in\{1,2,3\}} \|f_k\|_{L^1} = \delta$ by (9.15), $f_k \le 0$ in $\mathcal{B}_k$, and $f_k \ge 0$ in $G \setminus \mathcal{B}_k$.

Regard ϕ as a (discontinuous) mapping from G to $(-\frac{1}{2}, \frac{1}{2}]$ by identifying $\mathbb{T}$ with $(-\frac{1}{2}, \frac{1}{2}]$ in the usual way. For each $k \in \{1, 2, 3\}$, write $\{1, 2, 3\} = \{i, j, k\}$ and define

$$K_k(x) = \mathbf{1}_{\mathcal{B}_i} * \mathbf{1}_{\mathcal{B}_j}(x) \quad \text{for } x \in G.$$

K_k is continuous and nonnegative. There exists $\gamma_k > 0$ such that $K_k(x) > \gamma_k$ if $|\phi(x)| < \frac{1}{2}\mu(\mathcal{B}_k)$, $K_k(x) < \gamma_k$ if $|\phi(x)| > \frac{1}{2}\mu(\mathcal{B}_k)$, and $K_k(x) = \gamma_k$ when $|\phi(x)| = \frac{1}{2}\mu(\mathcal{B}_k)$. The η–strict admissibility hypothesis implies that there exists a small positive constant $c > 0$, depending only on η, such that

$$\begin{cases} |K_k(x) - \gamma_k| = \big|\,|\phi(x)| - \frac{1}{2}\mu(B_k)\,\big| & \text{whenever } \big|\,|\phi(x)| - \frac{1}{2}\mu(B_k)\,\big| \le c\mu(B_k), \\ |K_k(x) - \gamma_k| \ge c\mu(B_k) \text{ otherwise.} \end{cases} \tag{9.17}$$

Let λ be a large positive constant, to be chosen below. There exist a decomposition

$$f_j = f_j^\dagger + \tilde{f}_j \tag{9.18}$$

and consequently an expansion $g_j = \mathbf{1}_{\mathcal{B}_j} + f_j^\dagger + \tilde f_j$, with the following properties:

$$\int f_j^\dagger \, d\mu = \int \tilde f_j \, d\mu = 0 \tag{9.19}$$

$$\tilde f_j, f_j^\dagger \ge 0 \text{ on } G \setminus \mathcal{B}_j \tag{9.20}$$

$$\tilde f_j, f_j^\dagger \le 0 \text{ on } \mathcal{B}_j \tag{9.21}$$

$$\text{If } \big|\, |\phi(x)| - \tfrac12 \mu(\mathcal{B}_j) \,\big| \ge \lambda\delta \text{ then } f_j^\dagger(x) = 0. \tag{9.22}$$

$$\|\tilde f_j\|_{L^1} \le 2 \int_{\big|\, |\phi(x)| - \frac12 \mu(\mathcal{B}_j) \,\big| \ge \lambda\delta} |f_j(x)| \, d\mu(x). \tag{9.23}$$

To achieve this, set $\tilde f_j(x) = f_j(x)$ whenever $\big|\, |\phi(x)| - \frac12 \mu(\mathcal{B}_j) \,\big| \ge \lambda\delta$. We do not simply set $\tilde f_j(x) \equiv 0$ otherwise (even though such an $\tilde f_j$ clearly satisfies the desired condition (9.23) above), because the vanishing condition $\int \tilde f_j \, d\mu = 0$ will be essential below. Instead, for $x \in G$ satisfying $\big|\, |\phi(x)| - \frac12 \mu(\mathcal{B}_j) \,\big| < \lambda\delta$, we define $\tilde f_j(x) = f_j(x) \mathbf{1}_S(x)$ with the set S chosen as follows.

If $\int_{\big|\, |\phi(x)| - \frac12 \mu(\mathcal{B}_j) \,\big| \ge \lambda\delta} f_j \, d\mu \ge 0$, then $S \subset \mathcal{B}_j$, and S is chosen so that $\int \tilde f_j \, d\mu = 0$. Such a subset exists because $\int f_j \, d\mu = 0$, $f_j \ge 0$ on $G \setminus \mathcal{B}_j$ and ≤ 0 on $\mathcal{B}_j$, and μ is nonatomic. For our purpose, any such set S suffices.

If $\int_{\big|\, |\phi(x)| - \frac12 \mu(\mathcal{B}_j) \,\big| \ge \lambda\delta} f_j \, d\mu < 0$, then instead choose $S \subset G \setminus \mathcal{B}_j$ to ensure that $\int \tilde f_j \, d\mu = 0$. In both cases, define $f_j^\dagger = f_j - \tilde f_j$. The resulting functions $\tilde f_j, f_j^\dagger$ enjoy all of the required properties.

Set $g_j^\dagger = \mathbf{1}_{\mathcal{B}_j} + f_j^\dagger$. These functions satisfy $g_j = g_j^\dagger + \tilde f_j$, $0 \le g_j^\dagger \le 1$, $-1 \le \tilde f_j, f_j^\dagger \le 1$, and (since $\int \tilde f_j = 0$) $\int g_j^\dagger = \int g_j$.

Define

$$\tilde\delta = \max_j \|\tilde f_j\|_{L^1(G)} \le \delta. \tag{9.24}$$

$\mathcal{T} = \mathcal{T}_G$ satisfies

$$|\mathcal{T}(h_1, h_2, h_3)| \le \|h_1\|_{L^1} \|h_2\|_{L^1} \|h_3\|_{L^\infty} \tag{9.25}$$

for arbitrary functions, and is invariant under permutation of (h_1, h_2, h_3). Using the assumption that $\|g_j\|_{L^\infty} \le 1$, and for each k writing $\{1,2,3\} = \{i,j,k\}$ in some arbitrary manner, it follows that

$$\begin{aligned}
\mathcal{T}(\mathbf{g}) &= \mathcal{T}(g_1^\dagger + \tilde f_1, g_2^\dagger + \tilde f_2, g_3^\dagger + \tilde f_3) \\
&= \mathcal{T}(\mathbf{g}^\dagger) + \sum_{k=1}^3 \mathcal{T}(g_i^\dagger, g_j^\dagger, \tilde f_k) + O(\tilde\delta^2) \\
&= \mathcal{T}(\mathbf{g}^\dagger) + \sum_{k=1}^3 \mathcal{T}(\mathbf{1}_{\mathcal{B}_i}, \mathbf{1}_{\mathcal{B}_j}, \tilde f_k) + O(\tilde\delta \cdot \delta) \\
&= \mathcal{T}(\mathbf{g}^\dagger) + \sum_{k=1}^3 \langle \mathcal{K}_k, \tilde f_k \rangle + O(\tilde\delta \cdot \delta).
\end{aligned}$$

The constant implicit in the $O(\tilde\delta \cdot \delta)$ term is independent of the parameter λ.

Since $\int \tilde f_k\,d\mu = 0$, $\langle \mathcal{K}_k, \tilde f_k\rangle = \langle \mathcal{K}_k - \gamma_k, \tilde f_k\rangle$. On the complement of $\mathcal{B}_k$, $\tilde f_k \ge 0$ and $K_k - \gamma_k \le 0$; on $\mathcal{B}_k$, both signs are reversed. Therefore

$$\langle K_k, \tilde f_k\rangle = \int (K_k - \gamma_k)\tilde f_k\,d\mu = -\int |K_k - \gamma_k|\cdot|\tilde f_k|\,d\mu \le -c\lambda\delta\|\tilde f_k\|_{L^1}$$

according to the properties (9.17) of $\mathcal{K}_k$ and the relation $\lambda\delta \le c\mu(\mathcal{B}_k)$, which holds, for any particular choice of large constant λ, by the smallness hypothesis on $\delta/\mu(\mathcal{B}_k)$. Therefore in all,

$$\mathcal{T}(\mathbf{g}) \le \mathcal{T}(\mathbf{g}^\dagger) - c\lambda\delta\cdot\tilde\delta + O(\tilde\delta\cdot\delta)$$

with both c and the implicit constant in the remainder term $O(\delta^2)$ independent of the parameter λ, but with $\tilde\delta$ dependent on λ. Choosing λ sufficiently large gives

$$\mathcal{T}(\mathbf{g}) \le \mathcal{T}(\mathbf{g}^\dagger) - c\lambda\delta\cdot\tilde\delta \le \min\big(\mathcal{T}(\mathbf{g}^\dagger), \mathcal{T}_{\mathbb{T}}(\mathbf{g}^{\star\star}) - c\lambda\delta\cdot\tilde\delta\big), \tag{9.26}$$

with $c > 0$ independent of λ, and λ independent of $\mathbf{g}$. We have used the bound $\mathcal{T}(\mathbf{g}^\dagger) \le \mathcal{T}_{\mathbb{T}}((\mathbf{g}^\dagger)^{\star\star})$ of Theorem 1.5, and the identity $(\mathbf{g}^\dagger)^{\star\star} = \mathbf{g}^{\star\star}$.

There are now two cases, depending on the magnitude of $\tilde\delta/\delta$. If $\tilde\delta \ge \frac12\delta$ then $\mathcal{T}(\mathbf{g}) \le \mathcal{T}_{\mathbb{T}}(\mathbf{g}^{\star\star}) - \frac12 c\delta^2$. This is the desired conclusion of Lemma 9.3.

In the second case, $\tilde\delta \le \frac12\delta$. From the triangle inequality in the form

$$\max_j \|f_j^\dagger\|_{L^1} = \max_j\big(\|f_j\|_{L^1} - \|\tilde f_j\|_{L^1}\big) \ge \delta - \tilde\delta \ge \tfrac12\delta,$$

it follows that

$$\max_j \|g_j^\dagger - \mathbf{1}_{\mathcal{B}_j}\|_{L^1} = \max_j \|f_j^\dagger\|_{L^1} \ge \tfrac12\delta.$$

In this case, we use the alternative bound $\mathcal{T}(\mathbf{g}) \le \mathcal{T}(\mathbf{g}^\dagger)$ from (9.26). Thus it suffices to prove that

$$\mathcal{T}(\mathbf{g}^\dagger) \le \mathcal{T}_{\mathbb{T}}(\mathbf{g}^{\star\star}) - c\max_j \|g_j^\dagger - \mathbf{1}_{\mathcal{B}_j}\|_{L^1}^2,$$

that is, to establish the conclusion of Lemma 9.3 for $\mathbf{g}^\dagger$.

The modified triple $\mathbf{g}^\dagger$ satisfies all hypotheses of the lemma, and enjoys the supplementary property that $g_j^\dagger - \mathbf{1}_{\mathcal{B}_j} \equiv 0$ whenever $\big|\,|\phi(x)| - \frac12\mu(\mathcal{B}_j)\,\big| \ge \lambda\delta$.

Moreover,

$$\tfrac12\,\text{distance}\,(\mathbf{g}, \mathcal{O}(\mathbf{B})) \le \text{distance}\,(\mathbf{g}^\dagger, \mathcal{O}(\mathbf{B})) \le \tfrac32\,\text{distance}\,(\mathbf{g}, \mathcal{O}(\mathbf{B})) \tag{9.27}$$

by the triangle inequality for $L^1(G)$ norms, since $\tilde\delta \le \frac12\delta$. Therefore we have reduced matters to proving Lemma 9.3 under the supplementary hypothesis that for every $j \in \{1,2,3\}$,

$$g_j - \mathbf{1}_{\mathcal{B}_j} \equiv 0 \text{ whenever } \big|\,|\phi(x)| - \tfrac12\mu(\mathcal{B}_j)\,\big| \ge C_0\delta. \tag{9.28}$$

Here C_0 is some universal constant that is not at our disposal, but is dictated by our choice of λ. For the remainder of the proof of Lemma 9.3 we drop the superscripts $\dagger$, denoting by $\mathbf{g}$ an ordered triple of functions that satisfies the hypotheses of the lemma, as well as (9.28) for δ and $\mathbf{B}$ such that $\max_j \|g_j - \mathbf{1}_{\mathcal{B}_j}\|_1 \sim \delta$. Redefine $f_j = g_j - \mathbf{1}_{\mathcal{B}_j}$.

The perturbative term f_j satisfies (9.28), that is, is supported where $\big|\,|\phi(x)| - \frac12\mu(\mathcal{B}_j)\,\big| \le C_0\delta$. We claim that if ε_0 is a sufficiently small constant multiple of $\eta\max_k\mu(\mathcal{B}_k)$, and if $0 < \delta \le \varepsilon_0$, then this restriction on the support of f_j ensures that

$$\mathcal{T}(f_1, f_2, f_3) = 0. \tag{9.29}$$

Indeed, $f_1 * f_2$ is supported where ϕ differs by at most $2C_0\delta$ from some quantity $(\pm\frac{1}{2}\mu(\mathcal{B}_1)\pm\frac{1}{2}\mu(\mathcal{B}_2))$, while f_3 is supported where ϕ differs by at most $C_0\delta$ from $\pm\frac{1}{2}\mu(\mathcal{B}_3)$. The upper bound on $\mu(\mathcal{B}_1)+\mu(\mathcal{B}_2)+\mu(\mathcal{B}_3)$ and the η–strict admissibility of $\mathbf{B}$ ensure that

$$\eta \max_j \mu(\mathcal{B}_j) \le \big| \pm\mu(\mathcal{B}_1) \pm \mu(\mathcal{B}_2) \pm \mu(\mathcal{B}_3)\big| \le 2-\eta'$$

for all eight choices of signs, yielding (9.29) by the triangle inequality since $\delta \le \varepsilon_0$ is assumed to be small relative to $\eta \max_k \mu(\mathcal{B}_k)$.

For any $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{T}^3$ satisfying $y_1+y_2+y_3 = 0$, these constructions can be applied to the triple $\mathbf{g}^{\mathbf{y}}$ defined by replacing $g_j(x)$ by the translated function $g_j^{y_j}(x) = g_j(x-y_j)$. Then $\mathcal{T}(\mathbf{g}) = \mathcal{T}(\mathbf{g}^{\mathbf{y}})$, and $\int g_j^{y_j}\,d\mu = \int g_j\,d\mu$. Assume that $|\phi(y_j)| = O(\delta)$ for all three indices j. Then

$$\max_j \|g_j^{y_j} - \mathbf{1}_{\mathcal{B}_j}\|_{L^1} \le \max_j \|g_j^{y_j} - \mathbf{1}_{\mathcal{B}_j^{y_j}}\|_{L^1} + \max_j \|\mathbf{1}_{\mathcal{B}_j^{y_j}} - \mathbf{1}_{\mathcal{B}_j}\|_{L^1} = O(\delta).$$

On the other hand,

$$\max_j \|g_j^{y_j} - \mathbf{1}_{\mathcal{B}_j}\|_{L^1} \ge \text{distance}\,(\mathbf{g}^{\mathbf{y}}, \mathcal{O}(\mathbf{B})) = \text{distance}\,(\mathbf{g}, \mathcal{O}(\mathbf{B})) \ge c\delta$$

by $\mathbf{y}$–translation invariance of the orbit and translation invariance of μ. Each translated function $g_j^{y_j} - \mathbf{1}_{\mathcal{B}_j}$ remains supported in $\{x : \big||\phi(x)| - \mu(E_j)/2\big|) \le O(\delta)\}$.

Each $f_j = g_j - \mathbf{1}_{\mathcal{B}_j}$ has a unique additive decomposition $f_j = f_j^+ + f_j^-$, with $f_j^\pm$ supported where $\big|\phi(x) \mp \frac{1}{2}\mu(B_j)\big| = O(\delta)$, respectively. It will be advantageous to work instead with $\mathbf{g}^{\mathbf{y}}$, with $\mathbf{y}$ chosen so that the summands corresponding to $g_j^{y_j} - \mathbf{1}_{\mathcal{B}_j}$ satisfy certain vanishing properties which the summands $f_j^\pm$ potentially lack. In particular, define functions $f_{j,y_j}^\pm$ by first setting $f_{j,y_j} = g_j^{y_j} - \mathbf{1}_{\mathcal{B}_j}$, and then expressing $f_{j,y_j} = f_{j,y_j}^+ + f_{j,y_j}^-$, with $f_{j,y_j}^\pm$ supported where $|\phi(x) \mp \frac{1}{2}\mu(\mathcal{B}_j)| = O(\delta)$.

Lemma 9.4. *For each index j, there exists $y_j \in G$ satisfying $|\phi(y_j)| \le C_0\delta$ and*

$$\int f_{j,y_j}^+\,d\mu = \int f_{j,y_j}^-\,d\mu = 0. \tag{9.30}$$

Proof. $f_{j,y}^+$ is that portion of $g_j^y - \mathbf{1}_{\mathcal{B}_j}$ that is supported where $|\phi(x) - \frac{1}{2}\mu(\mathcal{B}_j)|$ is small. Since $|\phi(y)| \le C_0\delta$ and $g_j(x) = \mathbf{1}_{\mathcal{B}_j}(x)$ wherever $|\phi(x) - \frac{1}{2}\mu(\mathcal{B}_j)| > C_0\delta$, $f_{j,y}^+$ is supported where $|\phi(x) - \frac{1}{2}\mu(\mathcal{B}_j)| \le 2C_0\delta$.

Consider the function that maps $z \in [-C_0\delta, C_0\delta]$ to

$$\int f_{j,y}^+(x)\,d\mu(x) = \int_{|\phi(x)-\frac{1}{2}\mu(\mathcal{B}_j)|\le 2C_0\delta} \big(g_j^y - \mathbf{1}_{\mathcal{B}_j}\big)(x)\,d\mu(x),$$

with $y = y(z)$ satisfying $\phi(y) = z$. While y is not uniquely determined by z via this equation, the integral nonetheless depends only on z. Indeed, the contribution of the term $\mathbf{1}_{\mathcal{B}_j}$ to the integral does not involve y. Substituting $x = u + y$ allows us to rewrite the contribution of $g_j^y(x) = g_j(x-y)$ as

$$\int_{|\phi(x)-\frac{1}{2}\mu(\mathcal{B}_j)|\le 2C_0\delta} g_j(x-y)\,d\mu(x) = \int_{|\phi(u)+z-\frac{1}{2}\mu(\mathcal{B}_j)|\le 2C_0\delta} g_j(u)\,d\mu(u)$$

which likewise depends on z alone.

This function of z is nonnegative when $z = C_0\delta$. Indeed, if $\phi(x) \in [\frac{1}{2}\mu(\mathcal{B}_j) - 2C_0\delta, \frac{1}{2}\mu(\mathcal{B}_j)]$ then $g_j(x-y) = 1$, since $\phi_j(x-y) = \phi_j(x) - C_0\delta \le \frac{1}{2}\mu(\mathcal{B}_j) - C_0\delta$ and (by virtue of the reduction to the case $g_j = g_j^\dagger$ made above) $g_j(u) \equiv 1$ when $\frac{1}{2}\mu(\mathcal{B}_j) - O(\delta) \le \phi(u) \le \frac{1}{2}\mu(\mathcal{B}_j) - C_0\delta$. Thus $g_j^y(x) - \mathbf{1}_{\mathcal{B}_j}(x) = 1 - 1 = 0$ for these values of x. On the other hand, if $\phi(x) \in [\frac{1}{2}\mu(\mathcal{B}_j), \frac{1}{2}\mu(\mathcal{B}_j) + 2C_0\delta]$ then $\mathbf{1}_{\mathcal{B}_j}(x) = 0$, so $g_j^y(x) - \mathbf{1}_{\mathcal{B}_j}(x) \ge 0$.

The same reasoning shows that this function of z is nonpositive when $z = -C_0\delta$. Therefore we may apply the Intermediate Value Theorem on $[-C_0\delta, C_0\delta]$ to conclude that there exists y_j with $\phi(y_j) = z \in [-C_0\delta, C_0\delta]$ satisfying $\int f_{j,y_j}^+ \, d\mu = 0$.

It follows at once that $\int f_{j,y_j}^- \, d\mu = \int f_{j,y_j} \, d\mu - \int f_{j,y_j}^+ \, d\mu = 0$. □

Choose y_1, y_2 to ensure (9.30) for $j = 1, 2$, but then define y_3 by $y_1 + y_2 + y_3 = 0$. With such a choice of $\mathbf{y}$ fixed henceforth, simplify notation by suppressing y_j and writing again $g_j, f_j, f_j^\pm$, continuing to use the notation $\mathbf{g}$ for this modified triple. The quantities $\mathcal{T}(\mathbf{g})$ and distance$(\mathbf{g}, \mathcal{O}(\mathbf{B}))$ are unchanged.

The functions $f_3^\pm$ need not have vanishing integrals. Nonetheless,

$$\int f_i^\pm * f_j^\pm \, d\mu = 0 \ \text{ for any distinct indices } i \ne j \in \{1,2,3\}, \tag{9.31}$$

for all four possible choices of $\pm$ signs, since $\int (f_i^\pm * f_j^\pm) \, d\mu = \int f_i^\pm \, d\mu \cdot \int f_j^\pm \, d\mu$ and at least one of the two indices i, j must belong to $\{1, 2\}$.

Expand $\mathcal{T}(\mathbf{g}) = \mathcal{T}(\mathbf{1}_{\mathcal{B}_j} + f_j : j \in \{1,2,3\})$ into eight terms, using the multilinearity of $\mathcal{T}$. The simplest term is $\mathcal{T}(f_1, f_2, f_3)$. Provided that δ is sufficiently small relative to $\max_j \mu(\mathcal{B}_j)$, with constant of proportionality depending on η, η', this term vanishes for the modified triple $\mathbf{g}$, just as it was shown in (9.29) to vanish for the original triple.

The vanishing of $\mathcal{T}(f_1, f_2, f_3)$ simplifies the expansion of $\mathcal{T}(\mathbf{g})$ to

$$\mathcal{T}(\mathbf{g}) = \mathcal{T}(\mathbf{B}) + \sum_{k=1}^{3} \langle K_k, f_k \rangle + \sum_{i<j} \langle \mathbf{1}_{\mathcal{B}_l}, \, f_i * f_j \rangle. \tag{9.32}$$

In the final sum, $i < j \in \{1,2,3\}$ and l is defined by $\{1,2,3\} = \{i,j,l\}$.

We next discuss the terms

$$\langle K_k, f_k \rangle = -\int |f_k(x)| \, \big| |\phi(x)| - \tfrac{1}{2}\mu(\mathcal{B}_k) \big| \, d\mu(x) \le 0. \tag{9.33}$$

There exist an absolute constant $c_0 > 0$ and $n \in \{1,2,3\}$ such that $\|f_n\|_{L^1} \ge c_0\delta$. Let $c_1 = \frac{1}{8}c_0$. Because $\|f_n\|_{L^\infty} \le 1$ and

$$\mu(\{x \in G : \big| |\phi(x)| - \tfrac{1}{2}\mu(\mathcal{B}_n) \big| \le c_1\delta\}) = 4c_1\delta,$$

necessarily

$$\int_{\big| |\phi(x)| - \frac{1}{2}\mu(\mathcal{B}_n) \big| \ge c_1\delta} |f_n| \, d\mu \ge \|f_n\|_{L^1} - 4c_1\delta \ge \tfrac{1}{2}c_0\delta.$$

Therefore

$$\langle K_n, f_n \rangle \le -\int_{\big| |\phi(x)| - \frac{1}{2}\mu(\mathcal{B}_n) \big| \ge c_1\delta} |f_n(x)| \cdot \big| |\phi(x)| - \tfrac{1}{2}\mu(\mathcal{B}_n) \big| \, d\mu(x) \le -c_1\delta \cdot \tfrac{1}{2}c_0\delta, \tag{9.34}$$

which is comparable to $\max_j \|f_j\|_{L^1}^2$ and therefore to distance$(\mathbf{g}, \mathcal{O}(\mathbf{B}))^2$. Thus

$$\sum_k \langle K_k, f_k \rangle \le -c'\delta^2. \tag{9.35}$$

To complete the proof, we next show that

$$\langle \mathbf{1}_{\mathcal{B}_l}, f_i * f_j \rangle = 0 \ \text{ for any three distinct indices } i, j, l. \tag{9.36}$$

For any of the four possible choices of $\pm$ signs, the support of the convolution $f_i^\pm * f_j^\pm$ is contained in the sum of the supports of the two factors, hence consists of points x at which $\phi(x) = \frac{1}{2}(\pm\mu(B_i) \pm \mu(B_j)) + O(\delta)$. On the other hand, $\mathcal{B}_l$ is the set of x satisfying $|\phi(x)| \le \frac{1}{2}\mu(\mathcal{B}_l)$, and the η–strict admissibility hypothesis says that

$$| \pm \mu(\mathcal{B}_l) \pm \mu(\mathcal{B}_i) \pm \mu(\mathcal{B}_j)| \ge c\eta \max_k \mu(\mathcal{B}_k).$$

A hypothesis of Lemma 9.3 is that δ is small relative to $\eta \max_k \mu(\mathcal{B}_k)$. Therefore for any choice of $\pm$ signs, the support of $f_i^\pm * f_j^\pm$ is either entirely contained in $\mathcal{B}_l$, or entirely contained in its complement. Therefore in the integral

$$\langle \mathbf{1}_{\mathcal{B}_l}, f_i^\pm * f_j^\pm \rangle = \int \mathbf{1}_{\mathcal{B}_l} \cdot (f_i^\pm * f_j^\pm)\, d\mu,$$

the factor $\mathbf{1}_{\mathcal{B}_1}$ is constant. Since $\int f_i^\pm * f_j^\pm \, d\mu = 0$ by (9.31), this integral vanishes. Summing over all four possible choices of signs gives (9.36).

Inserting these results into the expansion (9.32), we conclude that when the supplementary hypothesis (9.28) is satisfied, $\mathcal{T}(\mathbf{g}) \le \mathcal{T}(\mathbf{g}^{\star\star}) - c\delta^2$, that is,

$$\mathcal{T}(\mathbf{g}) \le \mathcal{T}(\mathbf{g}^{\star\star}) - c\, \mathrm{distance}\,(\mathbf{E}, \mathcal{O}(\mathbf{B}))^2, \tag{9.37}$$

as was to be shown. □

10. The perturbative regime for sumsets

In this section we digress from the proof of the stability Theorem 1.3 to prove Theorem 1.4, the quantitative stability result for the inequality $\mu_*(A+B) \ge \min(\mu(A) + \mu(B), \mu(G))$. The core tool, Proposition 10.2, has flavor similar to that of the perturbative Lemma 9.1.

We begin with a small lemma needed in the analysis.

Lemma 10.1. *Let K be a compact Abelian group with Haar measure ν. Let $A, B \subset K$ be compact. Suppose that $B \ne \emptyset$ and that $\nu(A) > \frac{1}{2}\nu(K)$. Then*

$$\nu(A+B) \ge \min(\nu(B) + \tfrac{1}{2}\nu(A), \nu(K)). \tag{10.1}$$

K is not assumed to be connected. The conclusion is false in general, without the hypothesis that $\nu(A) > \frac{1}{2}\mu(K)$. It fails, for instance, if there exists a subgroup H of K satisfying $\nu(H) = \frac{1}{2}\nu(K)$ and $A = B = H$.

Proof. According to a theorem of Kneser [21], either $\nu(A+B) \ge \nu(A)+\nu(B)$ or there exists a subgroup H of K of positive Haar measure satisfying $A+B+H = A+B$ and $\nu(A+B) = \nu(A+H) + \nu(B+H) - \nu(H)$. In the first case, the conclusion of the lemma holds. In the second case, if $\nu(H) = \nu(K)$ then $\nu(A+H)+\nu(B+H)-\nu(H) = \nu(K)+\nu(K)-\nu(K) = \nu(K)$ and again the conclusion holds.

Now, suppose that $\nu(H) < \nu(K)$. $A+H$ is a union of cosets of H. It cannot be a single coset, for $\nu(H) < \nu(K)$ implies $\nu(H) \le \frac{1}{2}\nu(K) < \nu(A)$. Therefore $A+H$ is a union of at least two cosets of H, so $\nu(A+H) \ge 2\nu(H)$, so

$$\nu(A+H) - \nu(H) \ge \tfrac{1}{2}\nu(A+H) \ge \tfrac{1}{2}\nu(A).$$

Thus

$$\nu(A+B) = \nu(A+H) - \nu(H) + \nu(B+H) \ge \tfrac{1}{2}\nu(A) + \nu(B),$$

as was to be shown. □

Let G be a compact Abelian group with Haar measure μ, satisfying $\mu(G) = 1$. Let $|\mathcal{A}|$ denote the Lebesgue measure of any set $\mathcal{A} \subset \mathbb{T}$.

Proposition 10.2. *There exists $\delta_0 > 0$ with the following property. Let $A, B \subset G$ be compact sets of positive measures satisfying*

$$\mu(A) + \mu(B) \le 1 - 200\delta_0 \min(\mu(A), \mu(B)).$$

Suppose that

$$\|\phi(x)\|_{\mathbb{T}} \le \tfrac{1}{2}\mu(A) + \delta_0 \min(\mu(A), \mu(B)) \quad \text{for all } x \in A,$$
$$\|\phi(x)\|_{\mathbb{T}} \le \tfrac{1}{2}\mu(B) + \delta_0 \min(\mu(A), \mu(B)) \quad \text{for all } x \in B,$$
$$\mu(A + B) \le \mu(A) + \mu(B) + \delta \min(\mu(A), \mu(B)) \text{ for some } 0 < \delta \le \delta_0.$$

Then $\phi(A)$ is contained in some interval in $\mathbb{T}$ of length $\mu(A) + 100\delta \min(\mu(A), \mu(B))$. Likewise, $\phi(B)$ is contained in some interval of length $\mu(B) + 100\delta \min(\mu(A), \mu(B))$.

Define $\mathcal{A} = \phi(A)$ and $\mathcal{B} = \phi(B)$ in $\mathbb{T}$. For $t \in \mathbb{T}$ define

$$A_t = \{x \in A : \phi(x) = t\} \subset A \subset G.$$

A_t will be regarded sometimes as a subset of a coset of $K = \text{Kernel}(\phi)$, and sometimes as a subset of K itself (by translating by any appropriate element of G). Likewise define $B_t \subset B$.

Let ν be Haar measure on $H = \text{Kernel}(\phi)$, normalized to satisfy $\nu(H) = 1$.

Each slice $\phi^{-1}(\{t\}) \subset G$ is a coset of H. By translation, ν also defines a measure on each such coset, which will also be denoted by ν. Thus we may write $\nu(A_t)$, even though there is no canonical identification of A_t with a subset of H.

The hypotheses allow us to regard ϕ as a mapping from $A + B$ to $\mathbb{R}$, rather than to $\mathbb{T}$. Indeed, denoting $\eta := \delta_0 \min(\mu(A), \mu(B))$, each element of $\phi(a) \in \phi(A)$ is represented by some element $\tilde{\phi}(a) \in [-\frac{1}{2}\mu(A) - \eta, \frac{1}{2}\mu(A) + \eta]$, and correspondingly for $\phi(B)$. Therefore, for any $a \in A$ and $b \in B$, $\phi(a + b)$ is represented by some element $\tilde{\phi}(a + b) \in (-\frac{1}{2}, \frac{1}{2})$. These satisfy $\tilde{\phi}(a + b) = \tilde{\phi}(a) + \tilde{\phi}(b)$, where addition on the right-hand side is performed in $\mathbb{R}$ rather than in $\mathbb{T}$. These three mappings, all denoted by the common symbol $\tilde{\phi}$, are measure-preserving bijections.

Proof of Proposition 10.2. Define $\rho_A, \rho_B \in [0, 1)$ by

$$1 - \rho_A = \sup_t \nu(A_t) \text{ and } 1 - \rho_B = \sup_s \nu(B_s). \tag{10.2}$$

The hypothesis $\|\phi(x)\|_{\mathbb{T}} \le \frac{1}{2}\mu(A) + \delta_0 \min(\mu(A), \mu(B))$ for $x \in A$ implies that $|\mathcal{A}| \le (1+2\delta_0)\mu(A)$. On the other hand, $\mu(A) \le (1-\rho_A)|\mathcal{A}|$. Therefore $\rho_A \le 1-(1+2\delta_0)^{-1}$. Thus if δ_0 is sufficiently small then $\rho_A < \frac{1}{4}$. Likewise $\rho_B < \frac{1}{4}$. Therefore $\min(1 - 2\rho_A, \rho_B) = \rho_B$. This relation will be used momentarily.

Let $\varepsilon \in (0, \rho_A)$ be sufficiently small so that $1 - \rho_A - \epsilon > \frac{3}{4}$, and choose $\tau \in \mathcal{A}$ satisfying

$$\nu(A_\tau) > 1 - \rho_A - \varepsilon. \tag{10.3}$$

Set

$$A_- = \{a \in A : \phi(a) < \tau\} \text{ and } A_+ = \{a \in A : \phi(a) > \tau\}$$

(where $\mathcal{A}$ is seen as a subset of $\mathbb{R}$ rather than of $\mathbb{T}$).

Regarding $\mathcal{B}$ as a subset of $\mathbb{R}$, let $b_-, b_+ \in \mathbb{R}$ be its minimum and maximum elements, respectively.

Now

$$A + B \supset (A_\tau + B) + (A_- + B_{b_-}) + (A_+ + B_{b_+})$$

and these three sets are pairwise disjoint. Therefore

$$\mu(A + B) \ge \mu(A_\tau + B) + \mu(A_- + B_{b_-}) + \mu(A_+ + B_{b_+}).$$

$A_- + B_{b_-}$ contains a translate of A_-, so $\mu(A_- + B_{b_-}) \ge \mu(A_-)$. Likewise $\mu(A_+ + B_{b_+}) \ge \mu(A_+)$. Therefore

$$\mu(A + B) \ge \mu(A_\tau + B) + \mu(A). \tag{10.4}$$

One application of (10.4) is the relation

$$\max(\rho_A, \rho_B) \le \delta. \tag{10.5}$$

To prepare for its proof recall that according to Lemma 10.1,

$$\nu(A_\tau + B_t) \ge \min\left(\tfrac{1}{2}\nu(A_\tau) + \nu(B_t), 1\right) \ge \min\left(\nu(B_t) + \tfrac{3}{8}, 1\right)$$

for any $t \in \phi(B)$, since $\nu(A_\tau) > \frac{3}{4}$. Therefore

$$\begin{aligned}\mu(A_\tau + B) &= \int_{\mathcal{B}} \nu(A_\tau + B_t)\, dt \\ &\ge \int_{\mathcal{B}} \min(\nu(B_t) + \tfrac{3}{8}, 1)\, dt = \mu(B) + \int_{\mathcal{B}} \big[\min(\tfrac{3}{8}, 1 - \nu(B_t))\big]\, dt \\ &\ge \mu(B) + \int_{\mathcal{B}} \big[\min(\tfrac{3}{8}, \rho_B)\big]\, dt = \mu(B) + \rho_B|\mathcal{B}|\end{aligned}$$

since $\rho_B < \frac{1}{4}$. Since $|\mathcal{B}| \ge \mu(B)$, inserting this bound into (10.4) gives

$$\mu(A + B) \ge \mu(A) + \mu(B) + \rho_B\mu(B).$$

Since $\mu(A + B) \le \mu(A) + \mu(B) + \delta\mu(B)$, we may conclude that $\rho_B \le \delta$. The roles of A, B can be interchanged, so $\rho_A \le \delta$ also.

Let $\mathcal{A}' = \{t \in \mathcal{A} : \nu(A_t) > \frac{1}{2}\}$. Likewise define $\mathcal{B}' \subset \mathcal{B}$.

We claim that

$$\mu(A + B) \ge \mu(A) + \mu(B) + (\tfrac{1}{2} - \rho_A)|\mathcal{B} \setminus \mathcal{B}'|. \tag{10.6}$$

The proof will use the fact that for any subsets S, T of a compact group H satisfying $\mu(S) + \mu(T) > \mu(H)$, the associated sumset $S + T$ is all of H. Connectivity of H is not required for this conclusion; it is valid for the kernel H of ϕ. Indeed, for any $z \in H$ it holds that $\{z - x : x \in S\} \cap T \ne \emptyset$, since the intersection of these sets has measure equal to $\mu(S) + \mu(T) - \mu(H) > 0$. To prove the claim, majorize

$$\mu(A_\tau + B) \ge \int_{\mathcal{B}} \nu(A_\tau + B_t)\, dt. \tag{10.7}$$

One has $\nu(A_\tau + B_t) \ge \nu(B_t)$ for all t. Moreover, if $\nu(B_t) \le \frac{1}{2}$ then

$$\nu(A_\tau + B_t) \ge \nu(A_\tau) \ge 1 - \rho_A - \varepsilon \ge \nu(B_t) + \tfrac{1}{2} - \rho_A - \varepsilon.$$

Therefore

$$\mu(A_\tau + B) \ge \int_{\mathcal{B}} \nu(B_t)\, dt + \int_{\mathcal{B} \setminus \mathcal{B}'} (\tfrac{1}{2} - \rho_A - \varepsilon)\, dt = \mu(B) + (\tfrac{1}{2} - \rho_A - \varepsilon)|\mathcal{B} \setminus \mathcal{B}'|. \tag{10.8}$$

Letting $\varepsilon \to 0$ and combining this with (10.4) gives (10.6). □

From (10.6) together with the hypothesis $\mu(A+B) \le \mu(A) + \mu(B) + \delta \min(\mu(A), \mu(B))$ and the bound $\max(\rho_A, \rho_B) \le \delta$ we deduce that

$$|\mathcal{B} \setminus \mathcal{B}'| \le (2 + O(\delta))\delta \min(\mu(A), \mu(B)). \tag{10.9}$$

Since the roles of A, B can be freely interchanged in this reasoning, $|\mathcal{A} \setminus \mathcal{A}'|$ satisfies the same inequality.

For every $s \in \mathcal{A}'$ and $t \in \mathcal{B}'$, $\nu(A_s + B_t) \ge \min(\nu(A_s) + \nu(B_t), 1) = 1$ since $\nu(A_s) > \frac{1}{2}$ and likewise $\nu(B_t) > \frac{1}{2}$. Therefore $\nu((A+B)_x) = 1$ for every $x \in \mathcal{A}' + \mathcal{B}'$. Therefore $|\mathcal{A}' + \mathcal{B}'| \le \mu(A+B)$, and consequently

$$\begin{aligned} |\mathcal{A}' + \mathcal{B}'| &\le \mu(A) + \mu(B) + \delta \min(\mu(A), \mu(B)) \\ &\le |\mathcal{A}| + |\mathcal{B}| + \delta \min(\mu(A), \mu(B)) \\ &< |\mathcal{A}'| + |\mathcal{B}'| + 6\delta \min(\mu(A), \mu(B)). \end{aligned}$$

On the other hand,

$$\begin{aligned} |\mathcal{A}'| &\ge |\mathcal{A}| - (2 + O(\delta))\delta \min(\mu(A), \mu(B)) \\ &\ge \mu(A) - (2 + O(\delta))\delta \min(\mu(A), \mu(B)) \\ &> \mu(A) - 3\delta \min(\mu(A), \mu(B)) \end{aligned}$$

and likewise for $|\mathcal{B}'|$.

A straightforward adaptation to $\mathbb{R}$ (see [11]) of a theorem of Freĭman states that if $S, S' \subset \mathbb{R}$ are nonempty Lebesgue measurable sets satisfying $|S + S'|_* < |S| + |S'| + \min(|S|, |S'|)$, then S is contained in an interval of length $\le |S + S'| - |S'|$. Regarding $\mathcal{A}', \mathcal{B}'$ as subsets of $\mathbb{R}$, as we may, this result allows us to conclude that if δ_0 is less than some absolute constant, then $\mathcal{A}'$ is contained in an interval I of length $\le |\mathcal{A}'| + (6 + O(\delta))\delta \min(\mu(A), \mu(B))$. Similarly, $\mathcal{B}'$ is contained in an interval J of length $\le |\mathcal{B}'| + (6 + O(\delta))\delta \min(\mu(A), \mu(B))$.

The following claim completes the proof of Proposition 10.2.

Claim 10.1. The full sets $\mathcal{A} = \phi(A)$ and $\mathcal{B} = \phi(B)$ are contained in intervals of lengths $\mu(A) + 100\delta \min(\mu(A), \mu(B))$ and $\mu(B) + 100\delta \min(\mu(A), \mu(B))$, respectively.

The reasoning in the following proof of this claim will be used again below.

Proof of Claim 10.1. Suppose that some point $z \in \mathcal{A}$ were to lie to the left of the left endpoint of I by a distance $\ge C_1\delta \min(\mu(A), \mu(B))$. If $y \in \mathcal{B}'$ lies within distance $C_1\delta \min(\mu(A), \mu(B))$ of the left endpoint of J, then $A_z + B_y$ lies outside $I + J$. The set of all $y \in \mathcal{B}'$ with this property has Lebesgue measure

$$\ge |\mathcal{B}'| - \big(|J| - C_1\delta \min(\mu(A), \mu(B))\big) \ge (C_1 - 6)\delta \min(\mu(A), \mu(B)).$$

The sum of A_z with the union of all such B_y therefore has Haar measure $\ge \frac{1}{2}(C_1 - 6)\delta \min(\mu(A), \mu(B))$. This sumset is disjoint from $\phi^{-1}(\mathcal{A}' + \mathcal{B}') = \phi^{-1}(\mathcal{A}') + \phi^{-1}(\mathcal{B}')$. Therefore

$$\begin{aligned} \mu(A+B) &\ge \mu(\phi^{-1}(\mathcal{A}')) + \mu(\phi^{-1}(\mathcal{B}')) + \tfrac{1}{2}(C_1 - 6)\delta \min(\mu(A), \mu(B)) \\ &\ge (\mu(A) - |\mathcal{A} \setminus \mathcal{A}'|) + (\mu(B) - |\mathcal{B} \setminus \mathcal{B}'|) + \tfrac{1}{2}(C_1 - 6)\delta \min(\mu(A), \mu(B)) \\ &\ge \mu(A) + \mu(B) + \tfrac{1}{2}(C_1 - 18)\delta \min(\mu(A), \mu(B)). \end{aligned}$$

Choosing $C_1 = 21$ yields a contradiction for all sufficiently small δ.

Thus $\mathcal{A} = \phi(A)$ is contained in an interval of length less than

$$|I| + 40\delta \min(\mu(A), \mu(B)) \le |\mathcal{A}'| + 46\delta \min(\mu(A), \mu(B)) \le \mu(A) + 100\delta \min(\mu(A), \mu(B)).$$

Likewise for $\mathcal{B}$. □

The conclusions of Proposition 10.2 hold if A, B satisfy the same hypotheses but are merely assumed to be measurable, rather than compact, except that the constant 200 is replaced by a sufficiently large finite constant $\mathbf{C}$. To prove this, choose compact subsets A', B' of A, B whose Haar measures are nearly those of A, B respectively, and invoke Proposition 10.2 to obtain parallel rank one Bohr sets $\mathcal{B}_{A'} \supset A'$ and $\mathcal{B}_{B'} \supset B'$ satisfying $\mu(\mathcal{B}_{A'}) \le \mu(A) + \mathbf{C}\delta \min(\mu(A), \mu(B))$ with the corresponding bound for $\mu(\mathcal{B}_{B'})$. Then repeat the reasoning in the proof of the claim above to deduce that there exist slightly larger parallel rank one Bohr sets, associated to the same homomorphism ϕ, which contain all of A, B respectively, and whose measures satisfy the required upper bounds with a larger constant factor $\mathbf{C}$. □

With a small modifications, the proof of Proposition 10.2 establishes an extension: Setting $M = \min(\mu(A), \mu(B))$ to simplify notation, the hypotheses that $\|\phi(x)\|_{\mathbb{T}} \le \frac{1}{2}\mu(A) + \delta_0 M$ for all $x \in A$ and analogously for B can be relaxed to

$$\begin{cases} \|\phi(x)\|_{\mathbb{T}} \le \frac{1}{2}\mu(A) + \delta_0 M \quad \forall\, x \in A \text{ outside a set of Haar measure} \le \delta_0 M \\ \|\phi(x)\|_{\mathbb{T}} \le \frac{1}{2}\mu(B) + \delta_0 M \quad \forall\, x \in B \text{ outside a set of Haar measure} \le \delta_0 M, \end{cases} \tag{10.10}$$

to conclude that, provided that δ_0 is sufficiently small, $\phi(A)$ is contained in some interval in $\mathbb{T}$ of length $\mu(A) + \mathbf{C}\delta_0 \min(\mu(A), \mu(B))$, and likewise for $\phi(B)$.

To prove this, define A', B' to be the subsets of A, B, respectively, specified by these inequalities. We will use the proof of Claim 10.1 to control A, B in terms of A', B', demonstrating that A, B satisfy the hypotheses of Proposition 10.2 with δ_0 replaced by ε_0, where ε_0 depends only on δ_0 and tends to zero as $\delta_0 \to 0$. Thus the extension will be proved.

The only change to the reasoning in the proof of the claim is that we may no longer conclude that, in the notation of that discussion, $A_z + B_y$ is disjoint from

$$I + J = \{x \in \mathbb{T} : \|x\|_{\mathbb{T}} \le \tfrac{1}{2}(\mu(A) + \mu(B)) + O(\delta_0)M\}$$

whenever $\phi(z) \in [\frac{1}{2}\mu(A) + 2\mathbf{C}\delta_0 M, \frac{1}{2}]$ and $\phi(y) \in [\frac{1}{2}\mu(B) - \mathbf{C}\delta_0 M, \frac{1}{2}\mu(B)]$. Under the hypotheses of this extension, it is not permissible to regard $\phi(A), \phi(B)$ as subsets of $\mathbb{R}$, and the desired disjointness could fail due to periodicity.

Instead, we claim that if C_1 is a sufficiently large constant and δ_0 is sufficiently small then for any $x \in A$ satisfying

$$\tfrac{1}{2}\mu(A) + C_1\delta_0 M \le |\phi(x)| \le \tfrac{1}{2},$$

the set of all $y \in B'$ satisfying $\phi(x+y) \notin I + J$ has Haar measure $\ge C_2\delta_0 M$, where C_2 depends on C_1 but not on δ_0, and $C_2 \to \infty$ as $C_1 \to \infty$. We may assume without loss of generality that $\phi(x) \in [0, \frac{1}{2}]$ by replacing (A, B) by $(-A, -B)$ if necessary. The two desired conditions for y are that $w = \phi(y)$ should satisfy

$$w \ge \tfrac{1}{2}\mu(B) + \tfrac{1}{2}\mu(A) - \phi(x) + O(\delta_0 M)$$

and

$$w \le 1 - \tfrac{1}{2}\mu(A) - \tfrac{1}{2}\mu(B) - \phi(x) - O(\delta_0 M).$$

The set of all $w \in [-\frac{1}{2}\mu(B), \frac{1}{2}\mu(B)]$ that satisfy both inequalities has Lebesgue measure $\ge \mathbf{C}_1\delta_0 M$ provided that δ_0 is sufficiently small. The inverse image under ϕ of this set of elements w has Haar measure $\ge \mathbf{C}_1\delta_0 M$. The complement of the intersection with B of this inverse image has Haar measure $O(\delta_0 M)$, with the constant in the $O(\cdot)$ notation independent of the choice of $\mathbf{C}_1$. The result therefore follows.

This completes the proof of the extension of Proposition 10.2. □

Proof of Theorem 1.4. Let $\eta > 0$. Let $A, B \subset G$ satisfy $\min(\mu(A), \mu(B)) \geq \eta$ and $\mu(A) + \mu(B) \leq 1 - \eta$. Suppose that $\mu_*(A + B) \leq \mu(A) + \mu(B) + \delta \min(\mu(A), \mu(B))$. By the same reasoning as the one used to extend the statement of Proposition 10.2 to measurable sets, it suffices to treat the case in which A, B are compact.

If δ is sufficiently small, as a function of η alone, then the theorems of Tao [25] and/or Griesmer [20] can be applied. The conclusion is that there exist parallel rank one Bohr sets $\mathcal{B}_A, \mathcal{B}_B$ such that

$$\mu(A \,\Delta\, \mathcal{B}_A) \leq \varepsilon(\delta) \min(\mu(A), \mu(B))$$

and likewise for $\mu(B \,\Delta\, \mathcal{B}_B)$. The quantity $\varepsilon(\delta)$ tends to 0 as $\delta \to 0$, provided that η remains fixed.

The reasoning in the proof of the claim above now shows that the full sets A, B are contained in parallel rank one Bohr sets $\mathcal{B}_A^\sharp, \mathcal{B}_B^\sharp$, respectively, satisfying

$$\mu(\mathcal{B}_A^\sharp) \leq \mu(A) + \varepsilon^\sharp \min(\mu(A), \mu(B))$$

where $\varepsilon^\sharp \to 0$ as $\delta \to 0$. Likewise for $B, \mathcal{B}_B^\sharp$.

This is not the desired conclusion, since it includes no quantitative bound for the dependence of $\varepsilon^\sharp$ on δ. However, since $\varepsilon^\sharp \to 0$ as $\delta \to 0$, it follows that if δ is sufficiently small then the pair (A, B) satisfies the hypotheses of Proposition 10.2. Invoking that proposition completes the proof of the theorem. □

11. A special case on $\mathbb{T}$

In this section, we discuss our functionals for $G = \mathbb{T}$, in the special situation in which one of the sets is an interval. In particular, our next result ensures that, if (A, B, C) satisfies near equality in the Riesz-Sobolev inequality on $\mathbb{T}$, and C is an interval, then A and B are nearly intervals. This will allow us in §12 to understand near equality in the Riesz-Sobolev inequality on general G, when one of the sets is nearly Bohr.

When discussing the special case $G = \mathbb{T}$, we will often use $|E|$, rather than $m(E)$, to denote the Lebesgue measure of E.

Proposition 11.1. *Let $\eta > 0$. There exists a constant $\mathbf{C} < \infty$, depending only on η, with the following property. Let (A, B, C) be an η–strictly admissible and η–bounded triple of measurable subsets of $\mathbb{T}$. Suppose that C is an interval with center x_C. Then*

$$\inf_{x+y=x_C} \big(|A \,\Delta\, (A^\star + x)| + |B \,\Delta\, (B^\star + y)|\big) \leq \mathbf{C}\mathcal{D}(A, B, C)^{1/2}. \tag{11.1}$$

We outline here a proof based on a method relying on reflection symmetry and a two-point inequality of Baernstein and Taylor [5]. This technique does not otherwise appear in this paper. It is also used by O'Neill [24] to analyze the corresponding issue for the sphere S^d, $d \geq 2$.

Proof. The proof will consist of three steps.

Step 1. *If $\mathcal{D}(A, B, C) = 0$ and C is an interval, then A, B differ from intervals by Lebesgue null sets, and these three intervals are compatibly centered.*

Assume without loss of generality that C is centered at 0. Thus $C = C^*$. By the complementation principle described in §5, it may also be assumed that $m(A) \leq \frac{1}{2}$, $m(B) \leq \frac{1}{2}$.

Identify $\mathbb{T}$ with the unit circle in $\mathbb{C} \leftrightarrow \mathbb{R}^2$ via the mapping $x \mapsto e^{2\pi i(x+\frac{\pi}{2})}$. For each $x = (x_1, x_2) \in \mathbb{T}$ let $R(x) = (x_1, -x_2)$ be the reflection of x about the horizontal axis. To any $E \subset \mathbb{T}$ associate $E^\sharp \subset \mathbb{T}$, defined as follows. For each pair of points $\{x, R(x)\}$ with $x = (x_1, x_2)$ with $x_2 \neq 0$, let $x_+ = (x_1, |x_2|)$ and $x_- = (x_1, -|x_2|)$. If both $x_+, x_- \in E$ then both $x_+, x_- \in E^\sharp$; if neither belongs to E then neither belongs to $E^\sharp$; and if exactly one belongs to E then $x_+ \in E^\sharp$ and $x_- \notin E^\sharp$. If $x_2 = 0$ then $x \in E^\sharp$ if and only if $x \in E$.

Define $\mathbb{T}_+ = \{x = (x_1, x_2) \in \mathbb{T} : x_2 > 0\}$. For $y \in \mathbb{T}$ define $R_y E = (E + y)^\sharp$, where addition is in the additive group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Assume without loss of generality that the interval C is centered at 0. The following hold: for any measurable sets $A, B \subset \mathbb{T}$,

$$\text{(a)} \qquad m(A^\sharp) = m(A),\ m(B^\sharp) = m(B),$$

$$\text{(b)} \qquad m(A^\sharp \,\Delta\, B^\sharp) \le m(A \,\Delta\, B),$$

$$\text{(c)} \qquad \langle \mathbf{1}_{A^\sharp} * \mathbf{1}_C, \mathbf{1}_{B^\sharp} \rangle \ge \langle \mathbf{1}_A * \mathbf{1}_C, \mathbf{1}_B \rangle.$$

Consequently the above conclusions hold with $A^\sharp, B^\sharp$ replaced by $R_y A, R_y B$, respectively, for any $y \in \mathbb{T}$. (a) and (b) are direct consequences of the definition of the $\sharp$ operation, while (c) is an almost equally direct consequence [5].

Observe that

$$\text{(d)} \qquad \langle \mathbf{1}_{A^\sharp} * \mathbf{1}_C, \mathbf{1}_{B^\sharp} \rangle \ > \ \langle \mathbf{1}_A * \mathbf{1}_C, \mathbf{1}_B \rangle$$

if the set of all points $(x_+, y_+) \in \mathbb{T}_+^2$ satisfying $x_+ \in A$, $x_- \notin A$, $y_+ \notin B$, $y_- \in B$ and $\|x_+ - y_+\|_{\mathbb{T}} < \frac{1}{2} m(C)$ and $\|x_+ - y_-\|_{\mathbb{T}} > \frac{1}{2} m(C)$ has positive Lebesgue measure in $\mathbb{T}^2$. The same holds if the set of all points $(x_+, y_+) \in \mathbb{T}_+^2$ satisfying $x_+ \notin A$, $x_- \in A$, $y_+ \in B$, $y_- \notin B$ and the above two inequalities has positive Lebesgue measure in $\mathbb{T}^2$.

Moreover, if $A \subset \mathbb{T}$ is a finite union of closed intervals then there exists a finite sequence $y_1, \dots, y_N$ of elements of $\mathbb{T}$ such that

$$\text{(e)} \qquad R_{y_N} R_{y_{N-1}} \cdots R_1 A = A^\star.$$

This is elementary, and its proof is left to the reader.

If $A \subset \mathbb{T}$ is Lebesgue measurable then there exists an infinite sequence $y_n \in \mathbb{T}$ such that

$$\text{(f)} \qquad \lim_{N \to \infty} m\big(R_{y_N} R_{y_{N_1}} \cdots R_1 A \,\Delta\, A^\star\big) = 0;$$

(f) follows by combining (e) with the contraction property (b).

Consider any pair of measurable sets $A, B \subset \mathbb{T}$ that satisfy $\langle \mathbf{1}_A * \mathbf{1}_C, \mathbf{1}_B \rangle = \langle \mathbf{1}_{A^\star} * \mathbf{1}_C, \mathbf{1}_{B^\star} \rangle$. Choose a sequence (y_n) such that the sets defined recursively by $A_0 = A$ and $A_n = R_{y_n} A_{n-1}$ for $n \ge 1$ satisfy

$$m(A_n \,\Delta\, A^\star) \to 0.$$

Define B_n recursively by $B_0 = B$ and $B_n = R_{y_n} B_{n-1}$ for $n \ge 1$. Then $\langle \mathbf{1}_{A_n} * \mathbf{1}_C, \mathbf{1}_{B_n} \rangle = \langle \mathbf{1}_{A^\star} * \mathbf{1}_C, \mathbf{1}_{B^\star} \rangle$ for every n. Choose n_ν so that the sequence $\mathbf{1}_{B_{n_\nu}}$ converges weakly in $L^2(\mathbb{T})$ to some $h \in L^2(\mathbb{T})$, with $0 \le h \le 1$, $\int h \, dm = m(B)$. Denoting (A_{ν_n}, B_{ν_n}) by (A_n, B_n) for simplicity, the above implies

$$\langle \mathbf{1}_{A_n} * \mathbf{1}_C, \mathbf{1}_{B_n} \rangle \to \langle \mathbf{1}_{A^\star} * \mathbf{1}_C, h \rangle.$$

From this and the admissibility hypothesis it follows that $h = \mathbf{1}_{B^\star}$. Thus $\mathbf{1}_{B_n} \to \mathbf{1}_{B^\star}$ weakly. Since $m(\mathbb{T})$ is finite, this forces

$$m(B_n \,\Delta\, B^\star) \to 0.$$

By (a), $A_n^\star = A^\star$ and $B_n^\star = B^*$ for all $n \in \mathbb{N}$; therefore, for all $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ for which

$$m(A_n \,\Delta\, A_n^\star) < \epsilon \text{ and } m(B_n \,\Delta\, B_n^\star) < \epsilon,$$

while also

$$\langle \mathbf{1}_{A_N} * \mathbf{1}_C, \mathbf{1}_{B_N} \rangle = \langle \mathbf{1}_{A_N^*} * \mathbf{1}_{C^*}, \mathbf{1}_{B_N^*} \rangle$$

by (c). Therefore, fixing ε to be sufficiently small as a function of η alone, then the perturbative theory of Lemma 9.1 can be applied, implying that

$$\text{there exists } y_N \text{ such that } A_N = A^\star + y_N \text{ and } B_N = B^\star + y_N. \tag{11.2}$$

Denote by $\mathcal{R} : \mathbb{T} \to \mathbb{T}$ the reflection $\mathcal{R}(x_1, x_2) = (x_1, -x_2)$. Consider any measurable $A, B \subset \mathbb{T}$ such that the triple (A, B, C) (for our fixed C) satisifes the hypotheses of the proposition.

Claim 11.1. If $A^\sharp = A^\star$ and $B^\sharp = B^\star$, and if $\langle \mathbf{1}_A * \mathbf{1}_C, \mathbf{1}_B \rangle = \langle \mathbf{1}_{A^\star} * \mathbf{1}_C, \mathbf{1}_{B^\star} \rangle$, then either $(A, B) = (A^\star, B^\star)$ or $(A, B) = (\mathcal{R}A^\star, \mathcal{R}B^\star)$.

The strict admissibility hypothesis guarantees that there exists $\varepsilon > 0$ such that for any $x \in A^\star$ there exists $y = b(x) \in B^\star$ such that whenever $x', y' \in \mathbb{T}$ satisfy $\|x - x'\|_{\mathbb{T}} \le \varepsilon$ and $\|y - y'\|_{\mathbb{T}} \le \varepsilon$, one has

$$\|x'_+ - y'_+\|_{\mathbb{T}} < \tfrac{1}{2} m(C) \quad \text{but} \quad \|x'_+ - y'_-\|_{\mathbb{T}} > \tfrac{1}{2} m(C). \tag{11.3}$$

Indeed, identifying $\mathbb{T}$ with $[-\frac{1}{2}, \frac{1}{2})$, it suffices to prove the above for $x \in A^\star$ with $x \ge 0$. The η-strict admissibility and η-boundedness of (A, B, C) ensure that $\bar{p} := \frac{1}{2} m(A) - \frac{1}{2} m(C)$ has the property

$$-\tfrac{1}{2} m(B) + \tfrac{\eta^2}{2} \le \bar{p} \le \tfrac{1}{2} m(B) - \tfrac{\eta^2}{2}$$

(in particular $\bar{p} \in B^*$), while the right endpoint $\frac{1}{2} m(A)$ of A^* satisfies

$$\left|\tfrac{1}{2} m(A) - \bar{p}_-\right|_{\mathbb{T}} = \left|\tfrac{1}{2} m(A) - \left(\tfrac{1}{2} - \tfrac{1}{2} m(A) + \tfrac{1}{2} m(C)\right)\right| = \tfrac{1}{2} - m(A) \ge \tfrac{1}{2} m(C),$$

as $m(A) \le \frac{1}{2}$.

Therefore, if $\bar{p} \le 0$, define $b(x) := \bar{p} - \frac{\eta^2}{4}$ for all $x \in A^\star$. If $\bar{x} > 0$, define $b\left(t \frac{m(A)}{2}\right) := \bar{p} - \frac{\eta^2}{4}$ for every element $t \frac{m(A)}{2}$ of $A^\star$, for all $0 < t \le 1$.

Denoting by N_x the ϵ-neighbourhood on $\mathbb{T}$ of any $x \in \mathbb{T}$, it follows by the above and (d) that, for any $x \in A^\star$,

$$\begin{aligned} &\text{either } m\left(\{x' \in A^\star \cap N_x : x'_+ \notin A, x'_- \in A\}\right) = 0 \\ &\qquad \text{or } m\left(\{y' \in B^\star \cap N_{b(x)} : y'_+ \in B, y'_- \notin B\}\right) = 0 \end{aligned} \tag{11.4}$$

and

$$\begin{aligned} &\text{either } m\left(\{x' \in A^\star \cap N_x : x'_+ \in A, x'_- \notin A\}\right) = 0 \\ &\qquad \text{or } m\left(\{y' \in B^\star \cap N_{b(x)} : y'_+ \notin B, y'_- \in B\}\right) = 0. \end{aligned} \tag{11.5}$$

The second conclusion of (11.4) and the second conclusion of (11.5) cannot simultaneously hold, therefore the first conclusion of either (11.4) or (11.5) holds. That is,

$$\text{for every } x \in A^\star, \text{ either } m(\mathcal{R}A \cap N_x) = 0 \text{ or } m(A \cap N_x) = 0.$$

Now assume that, for some $x \in A^\star$ with $N_x \subset A^\star$, it holds that $m(\mathcal{R}A \cap N_x) = 0$. It will be shown that

$$m(\mathcal{R}A \cap N_y) = 0 \text{ for all } y \in A^\star \text{ with } \|x - y\|_{\mathbb{T}} < \epsilon \text{ and } N_y \subset A^\star$$

(and therefore, by the connectivity of $A^\star$, $m(\mathcal{R}A \cap A^\star) = 0$, i.e. $A = A^\star$ up to a Lebesgue null set).

Indeed, let $y \in A^\star$ as above, and suppose that $m(\mathcal{R}A \cap N_y) > 0$. Due to the fact that $A^\sharp = A^\star$, it holds that $m(\mathcal{R}A \cap N_z) + m(A \cap N_z) = m(A^\star \cap N_z)$ for every $z \in A^\star$. Therefore, $m(\mathcal{R}A \cap N_y) = m(A \cap N_x) = \epsilon$. Since the sets $\mathcal{R}A$ and A share at most two points (as $m(A) \le \frac{1}{2}$), it follows that

$$m\big((\mathcal{R}A \cap N_y) \cup (A \cap N_x)\big) = 2\epsilon.$$

This is a contradiction, as the set $(\mathcal{R}A \cap N_y) \cup (A \cap N_x)$ is contained in the arc $N := N_y \cup N_x$ of $\mathbb{T}$, of length $< \frac{\epsilon}{2} + \epsilon + \frac{\epsilon}{2} < 2\epsilon$.

Therefore, if x as above exists, then $A = A^\star$ up to a Lebesgue null set. In a similar manner it can be shown that if there exists $x \in A^\star$ with $m(A \cap N_x) = 0$ and $N_x \subset A^\star$, then $\mathcal{R}(A) = A^\star$ up to a Lebesgue null set.

Thus, either $A = A^*$ or $A = \mathcal{R}A^\star$ up to a Lebesgue null set. Without loss of generality, it is assumed that the former holds (the functional $\langle \mathbf{1}_A * \mathbf{1}_C, \mathbf{1}_B \rangle$ is invariant under simultaneous translations of A and B). Then, the fact that $\mathcal{D}(A, B, C) = 0$ means that

$$\langle \mathbf{1}_{A^\star} * \mathbf{1}_C, \mathbf{1}_{-B} \rangle = \langle \mathbf{1}_{A^\star} * \mathbf{1}_C, \mathbf{1}_{-B^\star} \rangle;$$

since $A^\star, C$ are intervals, the above implies that $|B \,\Delta\, B^\star| = 0$.

It has thus been shown that either $(A, B) = (A^\star, B^\star)$ or $(A, B) = (\mathcal{R}A^\star, \mathcal{R}B^\star)$ (up to Lebesgue null sets). This completes the proof of the claim. □

We are now in a position to complete the proof for Step 1. Return to the sequence of pairs (A_n, B_n), for $n = 1, \ldots, N$. By (11.2), $(A_N, B_N) = (A^\star + y_N, B^\star + y_N)$ up to Lebesgue null sets. Now, $(A_N, B_N) = ((A_{N-1} - y)^\sharp, (B_{N-1} - y)^\sharp)$ for some $y \in \mathbb{T}$. Therefore, either (A_{N-1}, B_{N-1}) equals either $(A^\star + y_N + y, B^\star + y_N + y)$ or $(\mathcal{R}A^\star + y_N + y, \mathcal{R}B^\star + y_N + y)$, up to Lebesgue null sets. Repeating this reasoning recursively for $n = N-2, N-3, \ldots$, we get a similar conclusion for (A, B).

This completes the discussion of Step 1.

Step 2. *For any $\varepsilon > 0$, there exists $\delta = \delta(\eta) > 0$ such that if $\mathcal{D}(A, B, C)^{1/2} \le \delta$ then*

$$\inf_{x+y=x_C} \big(|A \,\Delta\, (A^\star + x)| + |B \,\Delta\, (B^\star + y)|\big) \le \varepsilon. \tag{11.6}$$

We argue by contradiction. If the conclusion fails to hold then there exists an η–strictly admissible η–bounded sequence (A_k, B_k, C_k) such that $\lim_{k\to\infty} \mathcal{D}(A_k, B_k, C_k) = 0$, but

$$\inf_{x+y=x_{C_k}} \big(|A_k \,\Delta\, (A_k^\star + x)| + |B_k \,\Delta\, (B_k^\star + y)|\big) \ge \varepsilon. \tag{11.7}$$

It may be assumed without loss of generality that each $x_{C_k} = 0$.

By passing to subsequences we may assume that $\mathbf{1}_{A_k}, \mathbf{1}_{B_k}$ converge weakly in $L^2(\mathbb{T})$ to $f, g \in L^2(\mathbb{T})$, respectively. Then $0 \le f, g \le 1$, $\lim_{k\to\infty} |A_k|$ exists and is equal to $\int_{\mathbb{T}} f$, and likewise $|B_k| \to \int_{\mathbb{T}} g$. Moreover, after a further diagonal argument, $\mathbf{1}_{C_k} \to \mathbf{1}_C$ weakly for some interval C centered at 0.

Because the C_k are intervals, a simple compactness argument shows that $\mathbf{1}_{A_k} * \mathbf{1}_{C_k}$ converges strongly in $L^2(\mathbb{T})$. Therefore $\lim_{k\to\infty} \mathcal{T}(A_k, B_k, C_k) = \mathcal{T}(A^\star, B^\star, C)$ where $A^\star, B^\star$ denote here the intervals centered at 0 of lengths $\int_{\mathbb{T}} f, \int_{\mathbb{T}} g$, respectively. By continuity, the limiting triple $(A^\star, B^\star, C)$ satisfies $\mathcal{D}(A^\star, B^\star, C) = 0$.

By Lemma 8.1,

$$\int_C f * g \le \int_C f * \mathbf{1}_{B^\star} \le \langle f^\star, \mathbf{1}_{B^\star} * \mathbf{1}_C \rangle.$$

The triple $(\int_{\mathbb{T}} f^\star, |B^\star|, |C|)$ is η–strictly admissible. Because $0 \le f^\star \le 1$ and $\int_{\mathbb{T}} f^\star = |A^\star|$, η–strict admissibility ensures that $\mathbf{1}_{B^\star} * \mathbf{1}_C$, which is is symmetric and nonincreasing, is also strictly decreasing with derivative identically equal to -1 in $\{x : |\, x - |A^\star|/2\,| \le r\}$ for some $r > 0$ which depends only on η. Therefore $\langle f^\star, \mathbf{1}_{B^\star} * \mathbf{1}_C\rangle = \langle \mathbf{1}_{A^\star}, \mathbf{1}_{B^\star} * \mathbf{1}_C\rangle$ if and only if $f^\star = \mathbf{1}_{A^\star}$ almost everywhere. Thus $f^\star$ is the indicator function of a set. Since f has the same distribution function as $f^\star$, we conclude that $f = \mathbf{1}_A$ for some $A \subset \mathbb{T}$. Likewise, $g = \mathbf{1}_B$ for some set B.

Thus $\mathbf{1}_{A_k} \to \mathbf{1}_A$ and $\mathbf{1}_{B_k} \to \mathbf{1}_B$ weakly in $L^2(\mathbb{T})$. Therefore $|A_k \,\Delta\, A| + |B_k \,\Delta\, B| \to 0$, and $\mathcal{D}(A, B, C) = 0$. Step 1 now applies, allowing us to conclude that A and B differ from intervals by Lebesgue null sets, and that the centers of A, satisfy $x_A + x_B = 0$. This contradicts (11.7), completing Step 2. □

Step 3. Let (A, B, C) be a triple satisfying the hypotheses of the proposition. Let δ_0 be the constant appearing in the statement of the perturbative Lemma 9.1. By Step 2, there exists $\delta = \delta(\eta) > 0$, such that if $\mathcal{D}(A, B, C)^{1/2} \le \delta \max(|A|, |B|, |C|)$ then

$$\inf_{x+y=x_C} \big(|A \,\Delta\, (A^\star + x)| + |B \,\Delta\, (B^\star + y)| \le \delta_0 \eta \le \delta_0 \max(|A|, |B|, |C|),$$

where the last inequality is due to the η-boundedness of (A, B, C). Therefore, by Lemma 9.1, there exist $x', y', z' \in \mathbb{T}$ with $x' + y' = z'$ such that

$$|A \,\Delta\, (A^\star + x')|, |B \,\Delta\, (B^\star + y')|, |C \,\Delta\, (C^\star + z')| \le \mathbf{C}\mathcal{D}(A, B, C)^{1/2}, \tag{11.8}$$

for some $\mathbf{C} > 0$ depending only on η. This would be the desired result if $z' = x_C$, something which does not necessarily follow from Lemma 9.1. However, it can be proved that z' is very close to x_C; so close that, perturbing z' to become x_C and perturbing x' by the same amount, the truth of (11.8) is not violated, up to multiplication by constant factors.

More precisely, it holds that $\|z' - x_C\|_{\mathbb{T}} \le \mathbf{C}\mathcal{D}(A, B, C)^{1/2}$. Indeed, first observe that $C \cap (C^\star + z') \ne \emptyset$, as otherwise (11.8) would imply

$$\mathcal{D}(A, B, C)^{1/2} \ge \tfrac{1}{\mathbf{C}}|C \,\Delta\, (C^\star + z')| = \tfrac{2}{\mathbf{C}}|C| \ge \tfrac{2\eta}{\mathbf{C}} \max(|A|, |B|, |C|)$$

by the η-strict admissibility of (A, B, C), a contradiction for δ sufficiently small. Thus, since $C, C^\star + z'$ are intervals centered at x_C, z', respectively, it holds that $\|z' - x_C\|_{\mathbb{T}} = \frac{1}{2}|C \,\Delta\, (C^\star + z')| \le \mathbf{C}\mathcal{D}(A, B, C)^{1/2}$.

Therefore, $\bar{x}' := x' + (x_C - z')$ satisfies $\bar{x}' + y' = x_C$ and

$$\begin{aligned}\big|A \,\Delta\, (A^\star + \bar{x}')\big| &\le \big|A \,\Delta\, (A^\star + x')\big| + \big|(A^\star + x') \,\Delta\, (A^\star + \bar{x}')\big| \\ &\le \mathbf{C}\mathcal{D}(A, B, C)^{1/2} + \|x' - \bar{x}'\|_{\mathbb{T}} \\ &\le 2\mathbf{C}\mathcal{D}(A, B, C)^{1/2};\end{aligned}$$

likewise for B. Therefore, the triple (A, B, C) satisfies (11.1) with constant depending only on η.

As long as the quantity δ in the argument above is chosen sufficiently small, the complementary situation in which $\mathcal{D}(A, B, C)^{1/2} > \delta \max(|A|, |B|, |C|)$ also leads to (11.1) with constant $\mathbf{C} = 2\delta^{-1}$, simply because, for all $x \in \mathbb{T}$,

$$|A \,\Delta\, (A^\star + x)| \le 2|A| \le 2 \max(|A|, |B|, |C|).$$

Likewise for B. □

12. When one set is nearly rank one Bohr

The aim of this section is to establish for general groups G that if (A,B,C) is a strictly admissible triple with $\mathcal{D}(A,B,C)$ small, if (A,B,C) satisfies appropriate auxiliary hypotheses, and if one of the three sets A,B,C is nearly a rank one Bohr set, then the other two are also nearly rank one Bohr sets (parallel to the first, with the triple compatibly centered).

Proposition 12.1. *Let G be a compact connected Abelian topological group with normalized Haar measure μ. For any $\eta,\eta'>0$, there exist $c=c(\eta,\eta')>0$ and $\mathbf{C}=\mathbf{C}(\eta,\eta',c)<\infty$ such that the following holds. Let (A,B,C) be an η-strictly admissible triple of μ-measurable subsets of G, with $\min(\mu(A),\mu(B),\mu(C))\ge\eta$ and $\mu(A)+\mu(B)+\mu(C)\le 2-\eta'$. If there exists a rank one Bohr set $\mathcal{B}$ with*

$$\mu(C\,\Delta\,\mathcal{B})\le c(\eta,\eta')\max\big(\mu(A),\mu(B),\mu(C)\big),$$

then there exists a compatibly centered parallel ordered triple $(\mathcal{B}_A,\mathcal{B}_B,\mathcal{B}_C)$ of rank one Bohr sets satisfying

$$\mu(A\,\Delta\,\mathcal{B}_A)\le\mathbf{C}\mathcal{D}(A,B,C)^{1/2}, \tag{12.1}$$

and likewise for $\mu(B\,\Delta\,\mathcal{B}_B)$ and $\mu(C\,\Delta\,\mathcal{B}_C)$.

Proof. Let $\eta,\eta'>0$ and (A,B,C) be as in the statement of the proposition. We may assume that (A,B,C) satisfy the supplementary hypothesis

$$\mathcal{D}(A,B,C)<c(\eta,\eta')\max\big(\mu(A),\mu(B),\mu(C)\big)^2 \tag{12.2}$$

for a small constant $c(\eta,\eta')$. Indeed, otherwise

$$\mu(A\,\Delta\,\mathcal{B}_A)\le\mathbf{C}(\eta,\eta')\mathcal{D}(A,B,C)^{1/2}$$

holds trivially for any rank one Bohr set $\mathcal{B}_A$ with $\mu(\mathcal{B}_A)=\mu(A)$; likewise for B and C.

First, consider the case in which C is a rank one Bohr set. That is, $C=\phi^{-1}(C^\star)+x$, for some continuous homomorphism $\phi:G\to\mathbb{T}$ and some $x\in G$. We assume without loss of generality that $C=\phi^{-1}(C^\star)$. Define $\phi_*:L^1(G)\to L^1(\mathbb{T})$ by

$$\int_E\phi_*(f)\,dm=\int_{\phi^{-1}(E)}f\,d\mu\ \text{ for all measurable } E\subset\mathbb{T}.$$

Then

$$\phi_*(\mathbf{1}_A*\mathbf{1}_B)=\phi_*(\mathbf{1}_A)*\phi_*(\mathbf{1}_B),$$

and consequently

$$\int_C\mathbf{1}_A*\mathbf{1}_B\,d\mu=\mathcal{T}_G(\mathbf{1}_A,\mathbf{1}_B,\mathbf{1}_C)=\mathcal{T}_{\mathbb{T}}(\phi_*(\mathbf{1}_A),\phi_*(\mathbf{1}_B),\mathbf{1}_{C^\star})=\mathcal{T}_{\mathbb{T}}(f,g,\mathbf{1}_{C^\star}),$$

where the functions

$$f:=\phi_*(\mathbf{1}_A)\text{ and }g:=\phi_*(\mathbf{1}_B)$$

from G to $[0,\infty)$ satisfy

$$0\le f,g\le 1,\ \textstyle\int_{\mathbb{T}}f\,dm=\mu(A),\ \int_{\mathbb{T}}g\,dm=\mu(B).$$

Thus, by the Riesz-Sobolev inequality on $\mathbb{T}$,

$$\mathcal{T}_G(\mathbf{1}_A,\mathbf{1}_B,\mathbf{1}_C)=\mathcal{T}_{\mathbb{T}}(f,g,\mathbf{1}_{C^\star})\le\mathcal{T}_{\mathbb{T}}(f^\star,g^\star,\mathbf{1}_{C^\star}).$$

Applying Lemma 8.1 to the functions $f^\star, g^\star, \mathbf{1}_{C^\star}$ gives

$$\begin{aligned} \mathcal{T}_G(\mathbf{1}_A, \mathbf{1}_B, \mathbf{1}_C) &\le \mathcal{T}_{\mathbb{T}}(f^\star, g^\star, \mathbf{1}_{C^\star}) \\ &\le \max\{\mathcal{T}_{\mathbb{T}}(f^\star, \mathbf{1}_{B^\star}, \mathbf{1}_{C^\star}), \mathcal{T}_{\mathbb{T}}(\mathbf{1}_{A^\star}, g^\star, \mathbf{1}_{C^\star})\} \\ &\le \mathcal{T}_{\mathbb{T}}(\mathbf{1}_{A^\star}, \mathbf{1}_{B^\star}, \mathbf{1}_{C^\star}). \end{aligned} \tag{12.3}$$

Moreover, since $f^\star$, $g^\star$ are non-increasing functions with $0 \le f^\star, g^\star \le 1$, $\int_{\mathbb{T}} f\,dm = m(A^\star)$ and $\int_{\mathbb{T}} g\,dm = m(B^\star)$, the following holds.

Claim 12.1. There exists $\mathbf{C} < \infty$, depending only on η, such that

$$\|f^\star - \mathbf{1}_{A^\star}\|_{L^1(\mathbb{T})} + \|g^\star - \mathbf{1}_{B^\star}\|_{L^1(\mathbb{T})} \le \mathbf{C}\mathcal{D}(A,B,C)^{1/2}. \tag{12.4}$$

Proof. By (12.3) and because $\int_{\mathbb{T}}(\mathbf{1}_{A^\star} - f^\star)\,dm = 0$,

$$\mathcal{D}(A,B,C) \ge \int_{\mathbb{T}} (\mathbf{1}_{A^\star} - f^\star)\cdot(\mathbf{1}_{B^\star} * \mathbf{1}_{C^\star})\,dm = \int_{\mathbb{T}} (\mathbf{1}_{A^\star} - f^\star)\cdot(\mathbf{1}_{B^\star} * \mathbf{1}_{C^\star} - \gamma)\,dm$$

for any constant γ, and in particular for $\gamma = \mathbf{1}_{B^\star} * \mathbf{1}_{C^\star}\left(\frac{\mu(A)}{2}\right)$. The function $K(x) = \mathbf{1}_{B^\star} * \mathbf{1}_{C^\star} - \gamma$ is nonnegative on $A^\star$ and nonpositive on $\mathbb{T} \setminus A^\star$, as is $\mathbf{1}_{A^\star} - f^\star$, so

$$\mathcal{D}(A,B,C) \ge \int_{\mathbb{T}} |\mathbf{1}_{A^\star} - f^\star| \cdot |K|\,dm. \tag{12.5}$$

Let $a = \mu(A)/2$. Obtaining a lower bound for the right-hand side would be simpler if $|K|$ enjoyed a strictly positive lower bound, but $K(a) = 0$. K does satisfy $|K(x)| = |x - a|$ for $x \in [0, \frac{1}{2}]$ with $|x - a| \le \frac{1}{2}\min(\mu(B) + \mu(C) - \mu(A),\ \mu(A) - |\mu(B) - \mu(C)|)$, and the η–strict admissibility hypothesis ensures that this holds whenever $|x - a| \le \frac{1}{2}\eta\max(\mu(A), \mu(B), \mu(C))$. Since $\mathbf{1}_{B^\star} * \mathbf{1}_{C^\star}$ is nonincreasing, we find that, for $x \in [0, \frac{1}{2}]$,

$$|K(x)| \ge \begin{cases} |x - a| & \text{if } |x - a| \le \frac{\eta}{2}\max(\mu(A), \mu(B), \mu(C)) \\ \frac{\eta}{2}\max(\mu(A), \mu(B), \mu(C)) & \text{otherwise.} \end{cases}$$

It is elementary that if $0 \le \psi \le 1$ then $\int_{\mathbb{R}} |x|\psi(x)\,dx \ge \frac{1}{4}\|\psi\|^2_{L^1(\mathbb{R})}$. Therefore from the lower bound for K and the upper bound $\|\mathbf{1}_{A^\star} - f^\star\|_{C^0} \le 1$ it follows that

$$\int_{\mathbb{T}} |\mathbf{1}_{A^\star} - f^\star| \cdot |K|\,dm \ge c\min\Big(\|\mathbf{1}_{A^\star} - f^\star\|_{L^1(\mathbb{T})},\ \eta\max(\mu(A), \mu(B), \mu(C))\Big) \cdot \|\mathbf{1}_{A^\star} - f^\star\|_{L^1(\mathbb{T})}$$

for a certain absolute constant $c > 0$. Now $\|\mathbf{1}_{A^\star} - f^\star\|_{L^1(\mathbb{T})} \le 2\mu(A)$, so, provided that $\eta \le 1$, this implies that

$$\int_{\mathbb{T}} |\mathbf{1}_{A^\star} - f^\star| \cdot |K|\,dm \ge c\|\mathbf{1}_{A^\star} - f^\star\|^2_{L^1(\mathbb{T})},$$

for a constant $c > 0$ that only depends on η. The indicated conclusion for $\mathbf{1}_{A^\star} - f^\star$ follows directly from this and (12.5). The same holds for $\mathbf{1}_{B^\star} - g^\star$ since the roles of A, B can be interchanged. □

Since $f, f^\star$ have identical distribution functions and likewise for $g, g^\star$, there exist $\tilde{A}, \tilde{B} \subset \mathbb{T}$ satisfying $\|f - \mathbf{1}_{\tilde{A}}\|_{L^1(\mathbb{T})} = \|f^\star - \mathbf{1}_{A^\star}\|_{L^1(\mathbb{T})}$ and $\|g - \mathbf{1}_{\tilde{B}}\|_{L^1(\mathbb{T})} = \|g^\star - \mathbf{1}_{B^\star}\|_{L^1(\mathbb{T})}$, with $m(\tilde{A}) = \mu(A) = \int_{\mathbb{T}} f\,dm$ and $m(\tilde{B}) = \mu(B) = \int_{\mathbb{T}} g\,dm$.

Therefore, if $c(\eta,\eta')$ is sufficiently small, the triple $(\tilde A,\tilde B,C^*)$ is $\frac{\eta}{2}$-strictly admissible, $\min(\eta,\eta')$-bounded and $m(\tilde A)+m(\tilde B)+m(C^*)\le 2-\frac{\eta'}{2}$. Since C^* is an interval, Proposition 11.1 states that there exists $\bar x\in\mathbb{T}$ satisfying

$$m(\tilde A\,\Delta\,(A^\star+\bar x))+m(\tilde B\,\Delta\,(B^\star-\bar x))\le \mathbf{C}\mathcal{D}(\tilde A,\tilde B,C^*)^{1/2}, \tag{12.6}$$

for a constant $\mathbf{C}$ depending only on η,η'. Now

$$\mathcal{D}(\tilde A,\tilde B,C^\star)\le \mathbf{C}\mathcal{D}(A,B,C)^{1/2}\max(m(A),m(B),m(C)). \tag{12.7}$$

Indeed, since $m(\tilde A)=m(A)$ and $m(\tilde B)=m(B)$, it follows that $\tilde A^\star=A^\star$ and $\tilde B^\star=B^\star$, so

$$\mathcal{T}_{\mathbb{T}}(\tilde A^\star,\tilde B^\star,C^\star)=\mathcal{T}_{\mathbb{T}}(A^\star,B^\star,C^\star),$$

while

$$\begin{aligned}\mathcal{T}_{\mathbb{T}}(\tilde A,\tilde B,C^\star)&=\mathcal{T}_{\mathbb{T}}\big(f+(\mathbf{1}_{\tilde A}-f),g+(\mathbf{1}_{\tilde B}-g),\mathbf{1}_{C^\star}\big)\\ &\ge \mathcal{T}_{\mathbb{T}}(f,g,\mathbf{1}_{C^\star})\\ &\quad-(\|\mathbf{1}_{\tilde A}-f\|_{L^1(\mathbb{T})}+\|\mathbf{1}_{\tilde B}-g\|_{L^1(\mathbb{T})})m(C^\star)+\|\mathbf{1}_{\tilde A}-f\|_{L^1(\mathbb{T})}\|\mathbf{1}_{\tilde B}-g\|_{L^1(\mathbb{T})}\\ &\ge \mathcal{T}_G(A,B,C)-\mathbf{C}\mathcal{D}(A,B,C)^{1/2}\max(m(A),m(B),m(C))\end{aligned}$$

by Claim 12.1. Thus, (12.7) follows by (12.2).

The homomorphism ϕ preserves measure in the sense that $\mu(\phi^{-1}(E))=m(E)$ for any measurable $E\subset\mathbb{T}$. Therefore, since $f=\phi_*(\mathbf{1}_A)$,

$$\mu\big(A\,\Delta\,\phi^{-1}(\tilde A)\big)=\|f-\mathbf{1}_{\tilde A}\|_{L^1(\mathbb{T})}\le \mathbf{C}\mathcal{D}(A,B,C)^{1/2}. \tag{12.8}$$

Moreover, (12.6) and (12.7) together with this property of ϕ yield

$$\mu\big(\phi^{-1}(\tilde A)\,\Delta\,\phi^{-1}(A^\star+\bar x)\big)\le \mathbf{C}\mathcal{D}(A,B,C)^{1/4}\max(\mu(A),\mu(B),\mu(C))^{1/2}.$$

In all,

$$\begin{aligned}\mu(A\,\Delta\,\mathcal{B}_A)&\le \mathbf{C}\mathcal{D}(A,B,C)^{1/4}\max(\mu(A),\mu(B),\mu(C))^{1/2}\\ &\le \mathbf{C}c(\eta,\eta')^{1/4}\max(\mu(A),\mu(B),\mu(C))\end{aligned}$$

with $\mathcal{B}_A=\phi^{-1}(A^\star)+x$ for some $x\in G$, and likewise for B, with x replaced by $-x$. The last inequality above is due to (12.2), and it ensures that, as long as $c(\eta,\eta')$ is sufficiently small, the perturbative Lemma 9.1 can be applied, yielding the desired conclusion for (A,B,C). The analysis of the case in which the set C coincides with a rank one Bohr set is now complete.

Suppose next that

$$\mu(C\,\Delta\,\bar C)\le c(\eta,\eta')\max\big(\mu(A),\mu(B),\mu(C)\big),$$

where $\bar C=\phi^{-1}(C^*)$ for some continuous homomorphism $\phi:G\to\mathbb{T}$.

If $c(\eta,\eta')$ is sufficiently small, then the triple $(A,B,\bar C)$ is $\frac{\eta}{2}$-strictly admissible and satisfies $\mu(A)+\mu(B)+\mu(\bar C)\le 2-\frac{\eta}{2}$, while, by (12.2),

$$\begin{aligned}\mathcal{D}(A,B,\bar C)&\le \mathbf{C}c(\eta,\eta')\max\big(\mu(A),\mu(B),\mu(C)\big)^2\\ &\le \mathbf{C}c(\eta,\eta')\big(1+c(\eta,\eta')\big)^2\max\big(\mu(A),\mu(B),\mu(\bar C)\big)^2.\end{aligned}$$

Therefore, since $\bar C$ is a rank one Bohr subset of G, if $c(\eta,\eta')$ is sufficiently small then the partial result proved above can be applied to $(A,B,\bar C)$, ensuring that there exists a compatibly centered parallel ordered triple $(\mathcal{B}_A, \mathcal{B}_B, \mathcal{B}_{\bar{C}})$ of rank one Bohr sets, such that

$$\begin{aligned}\mu(A \,\Delta\, \mathcal{B}_A) &\le \mathbf{C}(\eta,\eta')\mathcal{D}(A,B,\bar{C})^{1/2}\\ &\le \mathbf{C}(\eta,\eta')c(\eta,\eta')\max\big(\mu(A),\mu(B),\mu(C)\big),\end{aligned}$$

and likewise for B and $\bar{C}$. Now, this further implies that

$$\begin{aligned}\mu(C \,\Delta\, \mathcal{B}_{\bar{C}}) &\le \mu(C \,\Delta\, \bar{C}) + \mu(\bar{C} \,\Delta\, \mathcal{B}_{\bar{C}})\\ &\le \mathbf{C}(\eta,\eta')c(\eta,\eta')\max\big(\mu(A),\mu(B),\mu(C)\big).\end{aligned}$$

Therefore, if $c(\eta,\eta')$ is sufficiently small then the triple (A,B,C) satisfies the hypotheses of the perturbative Lemma 9.1, the conclusion of which implies the desired estimate for (A,B,C).

□

13. Stability of the Riesz-Sobolev inequality

In this section we complete the proof of Theorem 1.3. This proof consists of five main steps. Firstly, given $\mathbf{E}$ with small defect $\mathcal{D}(\mathbf{E})$ for the Riesz-Sobolev functional, an associated triple $\mathbf{E}'$ is constructed, also with small defect but with altered Haar measures $\mu(E_j')$ satisfying a supplementary condition. Secondly, under this supplementary condition, small Riesz-Sobolev defect for $\mathbf{E}'$ implies that E_3' nearly saturates Kneser's sumset inequality. Thirdly, the inverse theorems of Griesmer and/or Tao imply that any saturator E_3' nearly coincides with a rank one Bohr set. Fourthly, this conclusion for E_3' implies that the given triple $\mathbf{E}$ nearly coincides with a parallel compatibly centered triple of rank one Bohr sets, with $o_{\mathcal{D}(\mathbf{E})}(1)$ control. In the fifth step, this crude bound is refined to $O(\mathcal{D}(\mathbf{E})^{1/2})$. All of the ingredients have been developed in preceding sections. Here, we link them together.

Proof of Theorem 1.3. Let $\eta > 0$. Let $\delta_0 > 0$ be a sufficiently small positive constant, which will depend only on η. Let (A,B,C) be an η–strictly admissible η–bounded ordered triple of measurable subsets of G satisfying

$$\mathcal{D}(A,B,C) \le \delta_0. \tag{13.1}$$

In this discussion, $\mathbf{C}_\eta$ will denote positive constants that depend only on η, not on (A,B,C). $\mathbf{C}_\eta$ is allowed to change in value from one occurrence to the next.

Assume without loss of generality that $\mu(C) \le \mu(A) \le \mu(B)$. The proof is organized into three cases, reflecting the analysis in §7.

Case 1: $\mu(A) \le \frac{1}{2}$ *and* $\mu(C) \le (1 - \frac{\eta}{50})\mu(B)$.

In this case, the lower bound assumption $\min(\mu(A),\mu(B),\mu(C)) \ge \eta$ implies that, for δ_0 sufficiently small, (A,B,C) satisfies the hypotheses of Lemma 7.1. Define τ by $\mu(C) = \mu(A) + \mu(B) - 2\tau$, and define $C' = S_{A,B'}(\tau)$. According to Lemma 7.1, there exists a measurable set $B' \subset G$ such that the triple (A,B',C') also nearly saturates the Riesz-Sobolev inequality, in the sense that

$$\mathcal{D}(A,B',C') \le \eta^{-1}\mathcal{D}(A,B,C) \le \delta_0\eta^{-1}, \tag{13.2}$$

satisfies the key supplementary condition

$$\mu(A) = \mu(B'), \tag{13.3}$$

and satisfies the technical conditions

$$(A, B', C') \text{ is } \eta/2\text{–strictly admissible and } \eta^2/2\text{–bounded,}$$
$$\mu(C') \le \mu(A) - 4\mathcal{D}(A, B', C')^{1/2}.$$

Therefore, if δ_0 is sufficiently small then the triple (A, B', C') satisfies the hypotheses of Lemma 4.2, whose conclusion is that C' nearly coincides with a superlevel set:

$$\mu(C' \,\Delta\, S_{A,B'}(\beta)) \le 4\mathcal{D}(A, B', C')^{1/2} \le 4(\delta_0/\eta)^{1/2} \tag{13.4}$$

with $\beta = \frac{1}{2}\big(\mu(A) + \mu(B') - \mu(C')\big)$. Moreover, (A, B', C') satisfies the hypotheses of the key Lemma 6.1 (in particular, $\mu(A) = \mu(B')$), whose conclusion is that the superlevel set $S_{A,B'}(\beta)$ has small difference set:

$$\begin{aligned}\mu\big(S_{A,B'}(\beta) - S_{A,B'}(\beta)\big) &\le 2\mu(S_{A,B'}(\beta)) + 12\mathcal{D}(A, B', S_{A,B'}(\beta))^{1/2} \\ &\le 2\mu(S_{A,B'}(\beta)) + 12(\delta_0/\eta)^{1/2}.\end{aligned}$$

So long as δ_0 is appropriately small, $S_{A,B'}(\beta)$ satisfies the hypotheses of Corollary 6.2, whose proof relied on the stability theorems of Tao [25] and/or Griesmer [20] for Kneser's inequality. Its conclusion is that there exists a rank one Bohr set $\mathcal{B}_\beta$ satisfying $\mu(\mathcal{B}_\beta \,\Delta\, S_{A,B'}(\beta)) \le \mathbf{C}_\eta \delta_0$. Combining this with (13.4) yields

$$\mu(\mathcal{B}_\beta \,\Delta\, C') \le \mathbf{C}_\eta \delta_0.$$

Therefore for sufficiently small δ_0, the triple (A, B', C') satisfies the hypotheses of Proposition 12.1, with parameters that depend only on η; C' nearly coincides with a rank one Bohr set, and $\mathcal{D}(A, B', C')$ is small. The proposition states that A and B' consequently also nearly coincide with rank one Bohr sets; in particular, there exists a rank one Bohr set $\mathcal{B}'_A$ satisfying

$$\mu(\mathcal{B}'_A \,\Delta\, A) \le \mathbf{C}_\eta \mathcal{D}(A, B', C')^{1/2} \le \mathbf{C}_\eta (\delta_0/\eta)^{1/2}$$

for some finite constant $\mathbf{C}_\eta$. The last inequality is (13.2).

With this control of A we return to the originally given triple (A, B, C). For sufficiently small δ_0, the η-strictly admissible, η-bounded triple (A, B, C) satisfies the hypotheses of Proposition 12.1, since A is now known to nearly coincide with a rank one Bohr set. The proposition states that there exists a compatibly centered parallel ordered triple $(\mathcal{B}_A, \mathcal{B}_B, \mathcal{B}_C)$ of rank one Bohr sets satisfying

$$\mu(A \,\Delta\, \mathcal{B}_A) + \mu(B \,\Delta\, \mathcal{B}_B) + \mu(C \,\Delta\, \mathcal{B}_C) \le \mathbf{C}_\eta \mathcal{D}(A, B, C)^{1/2}.$$

This completes the proof in Case 1. □

Case 2: *$\mu(A) \le \frac{1}{2}$ and $\mu(C) > (1 - \frac{\eta}{50})\mu(B)$.*

In this case, η–strict admissibility and η–boundedness together with sufficient smallness of δ_0 ensure that (A, B, C) satisfies the hypotheses of Lemma 7.3. Therefore, with τ defined by $\mu(C) = \mu(A) + \mu(B) - 2\tau$, there exist measurable sets $C' \subset C$ and $A' \subset A$ that satisfy

$$\begin{cases} (S_{C',A}(\tau), C', A) \text{ is } \eta/4\text{–strictly admissible and } \eta/4\text{–bounded} \\ \mathcal{D}(S_{C',A}(\tau), C', A) \le 16\mathcal{D}(C, B, A) \\ \mu(C') = \mu(A') = \mu(C) - \frac{1}{10}\eta\mu(B), \end{cases}$$

while

$$\begin{cases} (S_{C',A'}(\tau), C', A') \text{ is } \eta/2\text{–strictly admissible and } \eta/2\text{–bounded} \\ \mathcal{D}(S_{C',A'}(\tau), C', A') \le 16\mathcal{D}(C,B,A) \\ \mu(S_{A',C'}(\tau)) \le (1 - \frac{\eta/2}{50})\mu(C'). \end{cases}$$

The triple $(S_{A',C'}(\tau), C', A')$ falls into Case 1 above, with parameters that depend only on η. Therefore, if δ_0 is sufficiently small then there exists a rank one Bohr set $\mathcal{B}_{C'}$ satisfying

$$\mu(C' \,\Delta\, \mathcal{B}_{C'}) \le \mathbf{C}_\eta \mathcal{D}(S_{A',C'}(\tau), A', C')^{1/2} \le \mathbf{C}_\eta \delta_0^{1/2}.$$

Setting $F := S_{C',A}(\tau)$, the $\eta/4$-strict admissibility and $\eta/4$-boundedness of the triple (F, C', A) ensure that, for sufficiently small δ_0, (F, C', A) satisfies the hypotheses of Proposition 12.1. Therefore there exists a rank one Bohr set $\mathcal{B}_A$ satisfying

$$\mu(\mathcal{B}_A \,\Delta\, A) \le \mathbf{C}_\eta \mathcal{D}(F, C', A)^{1/2} \le \mathbf{C}_\eta \delta_0^{1/2}.$$

By η–admissibility and η–boundeness, (A, B, C) satisfies the hypotheses of Proposition 12.1 provided that δ_0 is sufficiently small. Therefore there exists a compatibly centered parallel ordered triple $(\mathcal{B}'_A, \mathcal{B}_B, \mathcal{B}_C)$ of rank one Bohr sets satisfying

$$\mu(A \,\Delta\, \mathcal{B}'_A) + \mu(B \,\Delta\, \mathcal{B}_B) + \mu(C \,\Delta\, \mathcal{B}_C) \le \mathbf{C}_\eta \mathcal{D}(A, B, C)^{1/2}.$$

□

Case 3: $\mu(A) > \frac{1}{2}$.

As discussed in §5, the triple $(C, G \setminus A, G \setminus B)$ is $\frac{\eta}{4}$-strictly admissible and $\frac{\eta}{4}$-bounded. Moreover, since $\frac{1}{2} < \mu(A) \le \mu(B)$, $\mu(G \setminus A) < \frac{1}{2}$ and $\mu(G \setminus B) < \frac{1}{2}$. Therefore, $(C, G \setminus A, G \setminus B)$ falls in the range of one of the two cases already analyzed above. Thus there exists a compatibly centered parallel ordered triple $(\mathcal{B}_C, \mathcal{B}_{G\setminus A}, \mathcal{B}_{G\setminus B})$ of rank one Bohr sets satisfying

$$\mu\big((G \setminus A) \,\Delta\, \mathcal{B}_{G\setminus A}\big) \le \mathbf{C}_\eta \mathcal{D}(C, G \setminus A, G \setminus B)^{1/2} = \mathbf{C}_\eta \mathcal{D}(A, B, C)^{1/2} \le \mathbf{C}_\eta \delta^{1/2}$$

and likewise for $\mu\big((G \setminus B) \,\Delta\, \mathcal{B}_{G\setminus B}\big)$ and for $\mu(C \,\Delta\, \mathcal{B}_C)$. The equality of $\mathcal{D}(C, G \setminus A, G \setminus B)^{1/2}$ with $\mathcal{D}(A, B, C)^{1/2}$ was established in Lemma 5.5.

For any measurable subsets E_1, E_2 of G, $\mu(E_1 \,\Delta\, E_2) = \mu\big((G \setminus E_1) \,\Delta\, (G \setminus E_2)\big)$. Therefore the compatibly centered parallel ordered triple $(\mathcal{B}_A, \mathcal{B}_B, \mathcal{B}_C)$ of rank one Bohr sets with $\mathcal{B}_A := G \setminus \mathcal{B}_{G\setminus A}$, $\mathcal{B}_B := G \setminus \mathcal{B}_{G\setminus B}$ satisfies

$$\mu(A \,\Delta\, \mathcal{B}_A) + \mu(B \,\Delta\, \mathcal{B}_B) + \mu(C \,\Delta\, \mathcal{B}_C) \le \mathbf{C}_\eta \mathcal{D}(A, B, C)^{1/2}.$$

The proof of Theorem 1.3 is complete. □

14. Cases of equality in the Riesz-Sobolev inequality

Theorem 1.2 states that if $\mathcal{T}_G(\mathbf{E}) = \mathcal{T}_{\mathbb{T}}(\mathbf{E}^\star)$, and if $\mathbf{E}$ is admissible, then there exists a $\mathcal{T}_G$–compatibly centered ordered triple of parallel rank one Bohr sets satisfying $\mu(E_j \,\Delta\, \mathcal{B}_j) = 0$ for every $j \in \{1, 2, 3\}$.

There are two cases in the proof. If $\mathbf{E}$ is strictly admissible, then there exists $\eta > 0$ such that $\mathbf{E}$ is η–strictly admissible and η–bounded. Therefore $\mathbf{E}$ satisfies the hypotheses of Theorem 1.3, the quantitative stability theorem, with $\delta = 0$. That theorem, whose proof has been completed above, gives the required conclusion.

If $\mathbf{E}$ is admissible but not strictly admissible, then after appropriate permutation of the three indices, $\mu(E_1)+\mu(E_2)=\mu(E_3)<1$, and

$$\langle \mathbf{1}_{E_1}*\mathbf{1}_{E_2},\mathbf{1}_{-E_3}\rangle = \mu(E_1)\mu(E_2) = \langle \mathbf{1}_{E_1}*\mathbf{1}_{E_2},\mathbf{1}_{G}\rangle.$$

Therefore $\mathbf{1}_{E_1}*\mathbf{1}_{E_2}=0$ μ–almost everywhere on the complement of $-E_3$, that is, $E_1+_0E_2$ is contained in the union of $-E_3$ with a nullset. Thus $\mu(E_1+_0E_2)\le\mu(E_3)$. The converse inequality holds by Kneser's theorem, so $\mu(\Delta(E_1+_0E_2,-E_3)=0$. It is a corollary of more quantitative results of Griesmer [20] and Tao [25] that equality of $\mu(E_1+_0E_2)$ with $\mu(E_1)+\mu(E_2)$ implies existence of a parallel pair of rank one Bohr sets satisfying $\mu(E_j\,\Delta\,\mathcal{B}_j)=0$ for $j=1,2$. Set $\mathcal{B}_3=\mathcal{B}_1+\mathcal{B}_2$. Then $(\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3)$ is an ordered triple of rank one Bohr sets with all required properties. □

15. Stability in the relaxed framework

Theorem 1.6, a stability theorem for the Riesz-Sobolev inequality in the situation in which indicator functions of sets are replaced by functions taking values in $[0,1]$, follows from slight modification of the proof of Theorem 1.3.

Proof of Theorem 1.6. Let f,g,h be as in the statement of the theorem. To simplify notation, set

$$M=\max\big(\textstyle\int f\,d\mu,\int g\,d\mu,\int h\,d\mu\big).$$

With the notation of §12,

$$\langle f*g,h\rangle_G\le\langle f^\star*g^\star,h^\star\rangle_{\mathbb{T}}\le\langle \mathbf{1}_{A^\star}*\mathbf{1}_{B^\star},h^\star\rangle_{\mathbb{T}}\le\langle \mathbf{1}_{A^\star}*\mathbf{1}_{B^\star},\mathbf{1}_{C^\star}\rangle_{\mathbb{T}}.$$

As shown in §12, this implies that

$$\|h^\star-\mathbf{1}_{C^\star}\|_{L^1(\mathbb{T})}\le\mathbf{C}\mathcal{D}^{1/2}.$$

Since h has the same distribution function as $h^\star$, there exists a set $C\subset G$ satisfying

$$\|h-\mathbf{1}_C\|_{L^1(G,\mu)}=\|h^\star-\mathbf{1}_{C^\star}\|_{L^1(\mathbb{T})}\le\mathbf{C}\mathcal{D}^{1/2}. \tag{15.1}$$

The same reasoning applies to f and to g, yielding corresponding sets $A,B\subset\mathbb{T}$, respectively. Now

$$\big|\,\langle f*g,h\rangle_G-\langle\mathbf{1}_A*\mathbf{1}_B,\mathbf{1}_C\rangle_G\,\big|\le\mathbf{C}M\mathcal{D}^{1/2},$$

so, for $\mathcal{D}$ sufficiently small as a function of η alone,

$$\begin{aligned}\langle\mathbf{1}_A*\mathbf{1}_B,\mathbf{1}_C\rangle_G&\ge\langle\mathbf{1}_{A^\star}*\mathbf{1}_{B^\star},\mathbf{1}_{C^\star}\rangle_{\mathbb{T}}-\mathbf{C}M\mathcal{D}^{1/2}\\&=\langle\mathbf{1}_{A^\star}*\mathbf{1}_{B^\star},\mathbf{1}_{C^\star}\rangle_{\mathbb{T}}-\mathbf{C}\max(\mu(A),\mu(B),\mu(C))\mathcal{D}^{1/2}\\&\ge\langle\mathbf{1}_{A^\star}*\mathbf{1}_{B^\star},\mathbf{1}_{C^\star}\rangle_{\mathbb{T}}-\mathbf{C}M\mathcal{D}^{1/2}\end{aligned}$$

with the convention that the constant $\mathbf{C}\in(0,\infty)$ may change from one occurrence to the next. In the final line we have used the fact that $\max(\mu(A),\mu(B),\mu(C))$ is comparable to M.

Therefore according to Theorem 1.3, there exists a compatibly centered parallel triple $(\tilde A,\tilde B,\tilde C)$ of rank one Bohr subsets of G such that

$$|A\,\Delta\,\tilde A|\le\mathbf{C}M^{1/2}\mathcal{D}^{1/4},$$

with the same bound for $|B\,\Delta\,\tilde B|$ and $|C\,\Delta\,\tilde C|$. In combination with (15.1) and the corresponding results for f,g, this gives

$$\max\big(\|f-\mathbf{1}_{\tilde A}\|_{L^1},\,\|g-\mathbf{1}_{\tilde B}\|_{L^1},\,\|h-\mathbf{1}_{\tilde C}\|_{L^1}\big)\le\mathbf{C}M^{1/2}\mathcal{D}^{1/4}.$$

It is given in the hypotheses of Theorem 1.6 that $\mathcal{D}/M^2$ is less than some small absolute constant that is at our disposal, but no lower bound is given. Therefore this conclusion is weaker than the desired bound $\mathbf{C}\mathcal{D}^{1/2}$. However, any bound of the form $o_{\mathcal{D}/M^2}(1) \cdot M$ is sufficient to place us in the perturbative context of Lemma 9.3, which gives the desired bound, completing the proof of Theorem 1.6. □

16. A flow of subsets of $\mathbb{T}$

This section and the next develop an alternative approach which, as it now stands, applies directly only for $G = \mathbb{T}$, but which yields slightly superior results for that group; the bounds remain appropriately uniform as the measures of A, B, C tend to zero. It is based on monotonicity of the functional $(A, B, C) \mapsto \mathcal{T}_{\mathbb{T}}(A, B, C)$ under a certain continuous one-parameter deformation. Such a monotonicity phenomenon is well-known for $G = \mathbb{R}$ [12]. The variant developed here, which applies to $\mathbb{T}$, is less effective than the classical version for $\mathbb{R}$, but is nonetheless useful. In the present section we develop the deformation and its basic properties for Kneser's inequality and for the Riesz-Sobolev inequality. In the following section we apply it to establish an improved stability theorem for $\mathbb{T}$.

In the present section and in §17, the Lebesgue measure of a subset $E \subset \mathbb{T}$ is denoted by $|E|$. All integrals over $\mathbb{T}$ are formed with respect to Lebesgue measure.

Let $\mathcal{L}(\mathbb{T})$ be the class of all equivalence classes of Lebesgue measurable sets $E \subset \mathbb{T}$ with $|E| > 0$, and E equivalent to E' if and only if $|E \,\Delta\, E'| = 0$. Assuming that $|E| > 0$, define

$$T_E = -\ln(|E|) > 0. \tag{16.1}$$

In the next theorem, E and E_j denote arbitrary equivalence classes of Lebesgue measurable subsets of $\mathbb{T}$. For equivalence class A, B, the notation $A \subset B$ means of course that any two representatives of these classes satisfy $|B \setminus A| = 0$.

Recall that

$$A +_0 B = \{x : \mathbf{1}_A * \mathbf{1}_B(x) > 0\}. \tag{16.2}$$

The inequality $|A + B|_* \ge \min(|A| + |B|, 1)$ for all measurable $A, B \subset G$ implies that

$$|A +_0 B| \ge \min(|A| + |B|, 1) \ \text{ for all measurable } A, B \subset \mathbb{T}. \tag{16.3}$$

Indeed, given $A, B \subset \mathbb{T}$, denote by $A^\dagger \subset A$ and $B^\dagger \subset B$ the sets of Lebesgue points of A, B, respectively. From the fact that almost every point is a Lebesgue point, it follows easily that

$$A^\dagger +_0 B^\dagger = A^\dagger + B^\dagger.$$

Therefore, since $A^\dagger \subset A$ and $B^\dagger \subset B$, it follows that

$$|A +_0 B| \ge |A^\dagger +_0 B^\dagger| = |A^\dagger + B^\dagger| \ge \min(|A^\dagger| + |B^\dagger|, 1) = \min(|A| + |B|, 1),$$

establishing (16.3) for A, B.

Theorem 16.1. *There exists a flow $(t, E) \mapsto E(t)$ of elements of $\mathcal{L}(\mathbb{T})$, defined for $t \in [0, T_E]$, having the following properties.*

1. *$E(0) = E$ and $E(T_E) = \mathbb{T}$.*
2. *$E(s) \subset E(t)$ whenever $s \le t$.*
3. *$|E(t)| = e^t |E|$ for all $t \in [0, T_E]$.*
4. *$|E(s) \,\Delta\, E(t)| \to 0$ as $s \to t$.*
5. *If $E \subset \tilde{E}$ then $E(s) \subset \tilde{E}(s)$ for all $s \in [0, T_{\tilde{E}}]$.*

(6) *$e^{-t}|E_1(t)\,\Delta\,E_2(t)| \le e^{-s}|E_1(s)\,\Delta\,E_2(s)|$ for all E_1, E_2 and every $0 \le s \le t \le \min(T_{E_1}, T_{E_2})$.*
(7) *If $0 \le s \le t \le T_E$ then $E(t) = (E(s))(t-s)$*
(8) *If E is the rank one Bohr set $\{x : \|\phi(x)\| \le r\}$ associated to a nonconstant homomorphism $\phi : \mathbb{T} \to \mathbb{T}$ then $E(t) = \{x : \|\phi(x)\| \le e^t r\}$.*
(9) *$(E+y)(t) = E(t) + y$ for every $E \in \mathcal{L}(\mathbb{T})$, $y \in \mathbb{T}$, and $t \le T_E$. Likewise, $(-E)(t) = -E(t)$.*
(10) *The function $t \mapsto e^{-t}|E_1(t) +_0 E_2(t)|$ is nonincreasing on $[0, \min(T_{E_1}, T_{E_2})]$.*
(11) *The function $t \mapsto e^{-2t}\mathcal{T}(E_1(t), E_2(t), E_3(t))$ is nondecreasing on $[0, \tau]$ provided that $\tau \le \min_{j\in\{1,2,3\}}(T_{E_j})$ and $\sum_{j=1}^{3} |E_j(\tau)| \le 2$.*

Each conclusion is to be interpreted in terms of equivalence classes of measurable sets. Thus, for instance, $A \subset B$ means $|B \setminus A| = 0$.

As mentioned earlier, a flow with variants of these properties is known for $\mathbb{R}$. See for instance a discussion in [12]. Such a flow acting on a dense class of sets, namely finite unions of intervals, is discussed in [22]. That it extends to arbitrary sets has been known to experts [7], though it seems not to have been extensively discussed in the literature.

The flow for $\mathbb{R}$ [12] preserves Lebesgue measures, whereas that of Theorem 16.1 does not. There exists no flow for $\mathbb{T}$ that mimics all properties of the flow for $\mathbb{R}$. Indeed, a rank one Bohr set $E \subset \mathbb{T}$ is a union of small intervals centered at the elements of a finite cyclic subgroup H of $\mathbb{T}$, or at elements of a coset of H. E satisfies $|E+E| = 2|E|$ if $|E| < \frac{1}{2}$, so E realizes equality in the sumset inequality. There is no way to continuously deform one such set E to another, through sets satisfying $|E(t) + E(t)| = 2|E(t)|$ with $|E(t)|$ independent of t, if the two sets in question are associated to subgroups H having different numbers of elements.

The flow of Theorem 16.1 lacks another key property of its analogue for $\mathbb{R}$, a lack which may appear to severely limit its utility, although we will show in the next section that it is nonetheless a valuable tool. The functionals $e^{-2t}\mathcal{T}(E_1(t), E_2(t), E_3(t))$ and $e^{-t}|E_1(t) +_0 E_2(t)|$ are only defined with desired monotonicity properties for $t \le T$, for a certain terminal time T. The defect is that the sets $E_j(T)$ reached at the terminal time need not possess any particular structure (such as $E_j(T) = E_j(T)^\star$ up to translation, or $E_j(T) = \mathbb{T}$). In contrast, the corresponding flow for $\mathbb{R}$ deforms each set E_j to its symmetrization $E_j^\star$.

Proof. The proof is nearly identical in many respects to that of a corresponding result for $\mathbb{R}$ proved in [12], with the exception of the conclusion concerning $|E_1(t) +_0 E_2(t)|$. We will provide only a sketch which deals with those points at which differences arise.

One begins by defining $t \mapsto E(t)$ in the special case in which E is a finite union of closed intervals. One verifies the stated properties in that case, then uses these properties to show that the flow extends to $\mathcal{L}(\mathbb{T})$ via uniform continuity with respect to the metric $\rho(E, E') = |E\,\Delta\,E'|$.[2]

Let $E = \cup_j I_j$ (a finite union), where $I_j \subset \mathbb{T}$ is a closed arc of length $|I_j|$ with center c_j, and these closed arcs are pairwise disjoint. Define $E(t) = \cup_j I_j(t)$, where $I_j(t)$ is the arc with center c_j and length $e^t|I_j|$, for all $0 \le t \le T_1$, where T_1 is the smallest t for which some pair of arcs $I_i(t), I_j(t)$ intersect. Any two arcs that do intersect share only an endpoint (or two endpoints, in the case in which the union has length 1). Thus $E(T_1)$ may be expressed in a unique way as a disjoint union of finitely many closed arcs, with certain centers. The

[2] The flow of Theorem 16.1 acts on equivalence classes of sets. Its restriction to finite unions of closed intervals agrees with the preliminary flow defined on finite unions of intervals, up to Lebesgue null sets.

number of such arcs is strictly smaller than the number of arcs comprising the initial set E. Repeat the first step for this new collection of arcs, stopping at the first time $T_2 > T_1$ at which intersection occurs. Again reorganize $E(T_2)$ as a union of finitely many pairwise disjoint closed arcs, and repeat until a single arc remains. This occurs, because the number of arcs is reduced with each iteration, and it is not possible for the number of arcs to exceed 1 if the measure of their union equals 1. Continue until $|E(t)| = 1$.

We claim that if E_j is a finite union of N_j pairwise disjoint closed arcs for each index $j \in \{1, 2, 3\}$, and if $\tau > 0$ is sufficiently small that $E_j(t)$ is defined for $t \in [0, \tau]$ and is a union of exactly N_j pairwise disjoint closed arcs for every $t \in [0, \tau)$ for each index j, then $e^{-2t}\mathcal{T}(\mathbf{E}(t))$ is a nondecreasing function of $t \in [0, \tau]$. It suffices to prove this for $t \in [0, T_1]$.

Write $\mathbf{1}_{E_j(t)} = \sum_{n=1}^{N_j} \mathbf{1}_{I_{j,n}(t)}$ with the natural notations. Then $|I_{j,n}(t)| = e^t|I_{j,n}(0)|$ for all indices j, n. By linearity of $\mathcal{T}$, it suffices to show that $t \mapsto e^{-2t}\mathcal{T}(\mathbf{I}(t))$ is a nondecreasing function for any triple $\mathbf{I}(t) = (I_j(t) : j \in \{1, 2, 3\})$ of intervals, with centers c_j of $I_j(t)$ independent of t and with lengths $|I_j(t)| = e^t|I_j(0)|$. By translation-invariance, we may assume that $c_1 = c_2 = 0$. By reflecting about 0 if necessary, we may assume that the center $\bar{c}_3 := -c_3$ of $-I_3$ satisfies $e^t\bar{c}_3 \in [0, \frac{1}{2}]$.

Set $l_j = |I_j(0)|/2$. Now

$$\mathcal{T}(\mathbf{I}(t)) = \iint_{\mathbb{T}^2} \mathbf{1}_{\|x\| \le e^t l_1} \mathbf{1}_{\|y\| \le e^t l_2} \mathbf{1}_{\|x+y-\bar{c}_3\| \le e^t l_3} \, dx \, dy.$$

Define $K(x) = \mathbf{1}_{\tilde{I}_1} * \mathbf{1}_{\tilde{I}_2}(x)$ for $x \in \mathbb{R}$, where $\tilde{I}_j = [-\frac{1}{2}l_j, \frac{1}{2}l_j] \subset \mathbb{R}$. Then, since $|I_1(t)| + |I_2(t)| < 1$ (as $t < T_1$), $\mathcal{T}(\mathbf{I}(t))$ can be expressed as

$$\mathcal{T}(\mathbf{I}(t)) = \int_{\mathbb{R}} e^t(K(e^{-t}u) + K(e^{-t}(u-1)))\mathbf{1}_{|u-\bar{c}_3| \le e^t l_3}(u) \, du.$$

Splitting this as a sum of two integrals and substituting $u = e^t x$ in one and $u = e^t y + 1$ in the other gives

$$e^{-2t}\mathcal{T}(\mathbf{I}(t)) = \int_{\mathbb{R}} K(x)\mathbf{1}_{-I_3-\bar{c}_3}(x - e^{-t}\bar{c}_3) \, dx + \int_{\mathbb{R}} K(y)\mathbf{1}_{-I_3-\bar{c}_3}(y + e^{-t}(1-\bar{c}_3)) \, dy.$$

Because K is nonnegative, even, and is nonincreasing on $[0, \infty)$, each of the two integrals above represents a nondecreasing function of t for any interval I_3. This completes the proof of monotonicity.

The conclusions of Theorem 16.1 now follow in the same way as in [12], with the exception of monotonicity of $e^{-t}|E_1(t) +_0 E_2(t)|$, which was not discussed there. Set $E_3 = -(E_1 +_0 E_2)$. Then $\mathcal{T}(E_1, E_2, E_3) = |E_1||E_2|$. We have shown above that $t \mapsto e^{-2t}\mathcal{T}(E_1(t), E_2(t), E_3(t))$ is a nondecreasing function of t. In particular,

$$e^{-2t}\mathcal{T}(E_1(t), E_2(t), E_3(t)) \ge \mathcal{T}(E_1(0), E_2(0), E_3(0)) = \mathcal{T}(E_1, E_2, E_3) = |E_1| \cdot |E_2|.$$

But

$$\int_{\mathbb{T}} \mathbf{1}_{E_1(t)} * \mathbf{1}_{E_2(t)} \le |E_1(t)| \cdot |E_2(t)| = e^{2t}|E_1| \cdot |E_2|.$$

Therefore

$$\int_{-E_3(t)} \mathbf{1}_{E_1(t)} * \mathbf{1}_{E_2(t)} = \int_{\mathbb{T}} \mathbf{1}_{E_1(t)} * \mathbf{1}_{E_2(t)},$$

forcing $\{x : \mathbf{1}_{E_1(t)} * \mathbf{1}_{E_2(t)}(x) > 0\} \subset -E_3(t)$ up to a Lebesgue null set. Therefore

$$e^{-t}|E_1(t) +_0 E_2(t)| \le e^{-t}|E_3(t)| = |E_1 +_0 E_2|.$$

If $0 \le s \le t$ then $E_j(t) = (E_j(s))(t-s)$, so the general relation

$$e^{-s}|E_1(s) +_0 E_2(s)| \le e^{-t}|E_1(t) +_0 E_2(t)|$$

follows from the case $s = 0$. □

Remark 16.1. An equivalent formulation of the monotonicity of $e^{-2t}\mathcal{T}(E_1(t), E_2(t), E_3(t))$ is that $t \mapsto e^{-2t}\mathcal{D}(E_1(t), E_2(t), E_3(t))$ is nonincreasing on $[0, \tau]$, provided that $\tau \le \min_{j\in\{1,2,3\}} T_{E_j}$ and $\sum_{j=1}^3 |E_j(\tau)| \le 2$. The monotonicity will be invoked in this form.

Indeed, for $t \in [0, \tau]$,

$$\begin{aligned} e^{-2t}\mathcal{D}(E_1(t),&E_2(t), E_3(t)) \\ &= e^{-2t}\mathcal{T}\big(E_1(t)^\star, E_2(t)^\star, (-E_3(t))^\star\big) - e^{-2t}\mathcal{T}\big(E_1(t), E_2(t), -E_3(t)\big) \\ &= e^{-2t}\mathcal{T}\big(E_1^\star(t), E_2^\star(t), (-E_3)^\star(t)\big) - e^{-2t}\mathcal{T}\big(E_1(t), E_2(t), (-E_3)(t)\big) \\ &= \mathcal{T}\big(E_1^\star, E_2^\star, (-E_3)^\star\big) - e^{-2t}\mathcal{T}\big(E_1(t), E_2(t), (-E_3)(t)\big). \end{aligned}$$

Now $e^{-2t}\mathcal{T}\big(E_1(t), E_2(t), (-E_3)(t)\big)$ is nondecreasing by the final conclusion of Theorem 16.1; its hypotheses are satisfied since $|(-E_3)(\tau)| = |E_3(\tau)|$ and $T_{-E_3} = T_{E_3}$.

The following remark, which will not be used in this paper but which may nonetheless be of interest, also follows in the same way as in [12].

Proposition 16.2. *Let $E \subset \mathbb{R}^1$ be a Lebesgue measurable set with finite measure. For each $t \in (0, T_E]$, $E(t)$ equals a union of intervals, up to a Lebesgue null set.*

That is, there exists a countable family of pairwise disjoint intervals $I_n(t)$ such that $|E(t)\,\Delta\,\bigcup_n I_n(t)| = 0$.

The next lemma makes it possible to propagate control of a triple $\mathbf{E}(t)$ backwards in time, with respect to the flow $t \mapsto \mathbf{E}(t)$, in the analysis of inequality (1.2) for $\mathbb{T}$.

Lemma 16.3 (Time reversal)**.** *For each $\eta, \eta' > 0$ there exist $\delta_1 > 0$ and $\mathbf{C} < \infty$ with the following property. Let $\mathbf{E}$ be an η–strictly admissible ordered triple of measurable subsets of $\mathbb{T}$, satisfying $\sum_j |E_j| \le 2 - \eta'$. Let $0 < t \le \min_{1\le j\le 3} T_{E_j}$ with $e^t - 1 \le \delta_1$. Suppose that there exists $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{T}^3$ satisfying $y_1 + y_2 = y_3$ such that*

$$|E_j(t)\,\Delta\,(E_j(t)^\star + y_j)| \le \delta_1 \max_j |E_j(t)| \qquad \forall\, j \in \{1, 2, 3\}. \tag{16.4}$$

Then there exists $\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{T}^3$ satisfying $z_1 + z_2 = z_3$ such that

$$|E_j\,\Delta\,(E_j^\star + z_j)| \le \mathbf{C}\mathcal{D}(\mathbf{E})^{1/2} \qquad \forall\, j \in \{1, 2, 3\}. \tag{16.5}$$

Proof. Requiring $\delta_1 \le 1$, as we may, yields

$$\begin{aligned} |E_j\,\Delta\,(E_j^\star + y_j)| &\le |E_j\,\Delta\,E_j(t)| + |E_j(t)\,\Delta\,(E_j(t)^\star + y_j)| + |(E_j(t)^\star + y_j)\,\Delta\,(E_j^\star + y_j)| \\ &\le (e^t - 1)|E_j| + \delta_1 e^t \max_k |E_k| + (e^t - 1)|E_j| \\ &\le (2(e^t - 1) + \delta_1) \max_k |E_k| \\ &= O(\delta_1 \max_k |E_k|). \end{aligned}$$

Therefore, if δ_1 is sufficiently small, then $\mathbf{E}$ satisfies the hypotheses of Lemma 9.1. The conclusion of that lemma is the desired inequality (16.5). □

17. Concluding steps for $\mathbb{T}$

In this section we exploit the properties of the flow developed in §16 to establish an improvement of Theorem 1.3 for the case $G = \mathbb{T}$. This improvement lies in the absence of any lower bound for $\min(m(A), m(B), m(C))$. That no lower bound is needed, is to be expected after the work of Bilu [1] on the sumset inequality.

Theorem 17.1. *For each $\eta > 0$ there exist $\delta_0 > 0$ and $\mathbf{C} < \infty$ with the following property. Let (A, B, C) be an η–strictly admissible ordered triple of Lebesgue measurable subsets of $\mathbb{T}$ satisfying $m(A) + m(B) + m(C) \le 2 - \eta$. Let $\delta \le \delta_0$. If*

$$\int_C \mathbf{1}_A * \mathbf{1}_B \, dm \ge \int_{C^\star} \mathbf{1}_{A^\star} * \mathbf{1}_{B^\star} \, dm - \delta \max(m(A), m(B), m(C))^2 \tag{17.1}$$

then there exists a compatibly centered parallel ordered triple $(\mathcal{B}_A, \mathcal{B}_B, \mathcal{B}_C)$ of rank one Bohr subsets of $\mathbb{T}$ satisfying

$$m(A \,\Delta\, \mathcal{B}_A) \le \mathbf{C}\delta^{1/2} \max(m(A), m(B), m(C)) \tag{17.2}$$

and likewise for $m(B \,\Delta\, \mathcal{B}_B)$ and $m(C \,\Delta\, \mathcal{B}_C)$.

Proof. By Theorem 1.3, the desired conclusion holds for all triples (A, B, C) that additionally satisfy $\min(m(A), m(B), m(C)) \ge \frac{1}{3}\eta^2$.

Now, let (A, B, C) be a triple satisfying the hypotheses of the theorem, but with

$$\min(m(A), m(B), m(C)) < \tfrac{1}{3}\eta^2.$$

Set $\mathbf{E} = (E_1, E_2, E_3) = (A, B, C)$ and consider the flowed triples $\mathbf{E}(t)$ for $0 \le t \le T$, with T chosen so that

$$\min_{j=1,2,3} m(E_j(t)) = \tfrac{1}{3}\eta^2.$$

That is, $\frac{1}{3}\eta^2 = e^T m$, for $m := \min_{j=1,2,3} m(E_j)$.

For all $t \in [0, T]$, the triple $\mathbf{E}(t)$ is η-strictly admissible. Setting $M := \max_{j=1,2,3} m(E_j)$, the η-strict admissibility of $\mathbf{E}$ ensures that

$$\max_{j=1,2,3} m(E_j(T)) = e^T M \le e^T m \eta^{-1} = \tfrac{1}{3}\eta,$$

whence

$$\sum_{j=1}^{3} m(E_j(t)) \le \eta \le 2 - \eta \text{ for all } t \in [0, T].$$

Moreover, the assumption $\mathcal{D}(\mathbf{E}) \le \delta M^2$ together with the monotonicity of the Riesz-Sobolev functional under the flow discussed in §16 imply that

$$\mathcal{D}(\mathbf{E}(t)) \le e^{2t}\mathcal{D}(\mathbf{E}) \le e^{2t}\delta M^2 = \delta \max_{j=1,2,3} m(E_j(t))^2 \text{ for all } t \in [0, T].$$

The triple $\mathbf{E}(T)$ enjoys the additional property that it is η^2-bounded, and therefore satisfies the hypotheses of Theorem 1.3 with parameters depending only on η. It follows that, provided that δ_0 is sufficiently small as a function of η alone, there exists a compatibly centered parallel ordered triple $\mathbf{B} := (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ of rank one Bohr sets with

$$m\big(\mathcal{B}_j \,\Delta\, E_j(T)\big) \le \mathbf{C}\delta^{1/2} \max_{j=1,2,3} m(E_j(T)).$$

Assuming again that δ_0 is sufficiently small as a function of η, the time reversal Lemma 16.3 can be applied in a straightforward series of reverse time steps to conclude that there exists a compatibly centered triple $(\mathcal{B}_1', \mathcal{B}_2', \mathcal{B}_3')$ of rank one Bohr sets such that

$$m(\mathcal{B}_j' \,\Delta\, E_j) \le \mathbf{C}\mathcal{D}(\mathbf{E})^{1/2}$$

for each $j \in \{1,2,3\}$. □

References

[1] Y. Bilu, *The $(\alpha + 2\beta)$-inequality on a torus*, J. London Math. Soc. (2) 57 (1998), no. 3, 513–528
[2] H. J. Brascamp, E. Lieb and J. M. Luttinger, *A general rearrangement inequality for multiple integrals*, J. Analysis 17 (1974), 227–237
[3] A. Baernstein, *Convolution and rearrangement on the circle*, Complex Variables Theory Appl. 12 (1989), no. 1-4, 33–37
[4] ———, *Correction to: "Convolution and rearrangement on the circle"* Complex Variables Theory Appl. 26 (1995), no. 4, 381–382
[5] A. Baernstein II and B. A. Taylor, *Spherical rearrangements, subharmonic functions, and $*$-functions in n-space*, Duke Math. J. 43 (1976), no. 2, 245–268
[6] A. Burchard, *Cases of equality in the Riesz rearrangement inequality*, Ann. of Math. (2) 143 (1996), no. 3, 499–527
[7] ———, personal communication.
[8] P. Candela, D. González-Sánchez, and A. de Roton, A Plünnecke-Ruzsa inequality in compact abelian groups. Rev. Mat. Iberoam. 35 (2019), no. 7, 2169–2186.
[9] P. Candela and A. de Roton, *On sets with small sumset in the circle*, Q. J. Math. 70 (2019), no. 1, 49–69.
[10] M. Christ, *Near-extremizers of Young's Inequality for Euclidean groups*, Rev. Mat. Iberoam. 35 (2019), no. 7, 1925–1972
[11] ———, *Near equality in the Riesz-Sobolev inequality*, Acta Math. Sin. (Engl. Ser.) 35 (2019), no. 6, 783–814
[12] ———, *A sharpened Riesz-Sobolev inequality*, preprint, arXiv:1706.02007 math.CA
[13] C. Draghici, *An inequality of rearrangements on the unit circle*, Proc. Roy. Soc. Edinburgh Sect. A 137 (2007), no. 4, 757–769
[14] R. L. Frank and E. H. Lieb, *Proof of spherical flocking based on quantitative rearrangement inequalities*, preprint, arXiv:1909.04595 math.CA
[15] Y. Jing, C.-M. Tran, and R. Zhang, *A nonabelian Brunn-Minkowski inequality*, preprint, arXiv:2011.07702 math.GR
[16] Y. Jing and C.-M. Tran, *Minimal and nearly minimal measure expansions in connected unimodular groups*, preprint, arXiv:2006.01824 math.CO
[17] R. Friedberg and J. M. Luttinger, *Rearrangement inequality for periodic functions*, Arch. Ration. Mech. Analysis 61 (1976), 35–44
[18] B. Green and I. Z. Ruzsa, *Freiman's theorem in an arbitrary abelian group*, J. Lond. Math. Soc. (2) 75 (2007), no. 1, 163-175
[19] J. T. Griesmer, *An inverse theorem: when the measure of the sumset is the sum of the measures in a locally compact abelian group*, Trans. Amer. Math. Soc. 366 (2014), no. 4, 1797–1827
[20] ———, *Semicontinuity of structure for small sumsets in compact abelian groups*, Discrete Anal. 2019, Paper No. 18.
[21] M. Kneser, *Summenmengen in lokalkompakten abelschen Gruppen*, Math. Z. 66 (1956), 88–110
[22] E. Lieb and M. Loss, *Analysis*, Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001
[23] J. M. Luttinger, *Generalized isoperimetric inequalities. III.* J. Mathematical Phys. 14 (1973), 1448–1450
[24] K. O'Neill, *A sharpened rearrangement inequality for convolution on the sphere*, preprint, arXiv:1810.06813 math.CA
[25] T. Tao, *An inverse theorem for an inequality of Kneser*, Published in Russian in Tr. Mat. Inst. Steklova 303 (2018), 209–238. Proc. Steklov Inst. Math. 303 (2018), no. 1, 193–219.
[26] T. Tao and V. Vu, *Additive Combinatorics*, Cambridge Studies in Advanced Mathematics, 105. Cambridge University Press, Cambridge, 2006

Michael Christ, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA
Email address: mchrist@berkeley.edu

Marina Iliopoulou, School of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury, CT2 7PE, UK
Email address: m.iliopoulou@kent.ac.uk