A SYMMETRIZATION INEQUALITY SHORN OF SYMMETRY

MICHAEL CHRIST AND DOMINIQUE MALDAGUE

Abstract. An inequality of Brascamp-Lieb-Luttinger and of Rogers states that among subsets of Euclidean space $\mathbb{R}^d$ of specified Lebesgue measures, (tuples of) balls centered at the origin are maximizers of certain functionals defined by multidimensional integrals. For $d > 1$, this inequality only applies to functionals invariant under a diagonal action of $\mathrm{Sl}(d)$. We investigate functionals of this type, and their maximizers, in perhaps the simplest situation in which $\mathrm{Sl}(d)$ invariance does not hold. Assuming a more limited symmetry encompassing dilations but not rotations, we show under natural hypotheses that maximizers exist, and moreover, that there exist distinguished maximizers whose structure reflects this limited symmetry. For small perturbations of the $\mathrm{Sl}(d)$–invariant framework we show that these distinguished maximizers are strongly convex sets with infinitely differentiable boundaries. It is shown that in the absence of partial symmetry, maximizers fail to exist for certain arbitrarily small perturbations of $\mathrm{Sl}(d)$–invariant structures.

1. Introduction

Let J be a finite index set, and for each $j \in J$ let $L_j : \mathbb{R}^D \to \mathbb{R}^{d_j}$ be a surjective linear mapping. Writing $\mathbf{f} = (f_j : j \in J)$, consider the functional $\mathbf{f} \mapsto \Lambda(\mathbf{f})$ defined by

$$\Lambda(\mathbf{f}) = \int_{\mathbb{R}^D} \prod_{j \in J} f_j(L_j(\mathbf{x}))\, d\mathbf{x}. \tag{1.1}$$

The functions $f_j : \mathbb{R}^{d_j} \to [0, \infty]$ are assumed to be nonnegative and Lebesgue measurable. The theory of Hölder-Brascamp-Lieb inequalities [20], [4], [5], [9], [6], [1], [18], [2], [3] is concerned with inequalities $\Lambda(\mathbf{f}) \le A \prod_{j \in J} \|f_j\|_{L^{p_j}(\mathbb{R}^{d_j})}$. It includes a necessary and sufficient condition on the data D, J, d_j, L_j, p_j for there to exist $A < \infty$ for which such an inequality holds for all $\mathbf{f}$, it provides an expression of sorts for the optimal constant A, it includes algorithms for computing certain elements of the theory, it has discrete variants which are closely connected with Hilbert's tenth problem (over $\mathbb{Q}$), it includes a characterization of maximizing tuples $\mathbf{f}$ under certain auxiliary hypotheses, and the optimal constant $\sup_{\mathbf{f}} \Lambda(\mathbf{f}) / \prod_{j \in J} \|f_j\|_{L^{p_j}}$ has been shown to be a Hölder continuous function of $\mathcal{L} = (L_j : j \in J)$ within an appropriate domain and under appropriate hypotheses.

One of the foundational instances of this theory concerns the Riesz-Sobolev functional

$$(f_1, f_2, f_3) \mapsto \langle f_1 * f_2, f_3 \rangle = \int_{\mathbb{R}^d \times \mathbb{R}^d} f_1(x) f_2(y) f_3(x+y)\, dx\, dy \tag{1.2}$$

defined by pairing the convolution $f_1 * f_2$ with f_3. The Riesz-Sobolev inequality extends the conclusion beyond the Hölder-Brascamp-Lieb theory through the symmetrization inequality

$$\langle f_1 * f_2, f_3 \rangle \le \langle f_1^\star * f_2^\star, f_3^\star \rangle, \tag{1.3}$$

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where $f^\star : \mathbb{R}^d \to [0,\infty)$ is (up to redefinition on Lebesgue null sets) the unique function that is radially symmetric, is a nonincreasing function of $|x|$, and is equimeasurable with f. The general inequality is a direct consequence of the special case in which each function f_j is the indicator function $\mathbf{1}_{E_j}$ of a set.

In this paper, we are concerned with functionals (1.1), acting only on tuples of indicator functions of sets. We abuse notation systematically by writing $\Lambda(\mathbf{E})$ for $\Lambda(\mathbf{f})$ where $f_j = \mathbf{1}_{E_j}$ and $\mathbf{E} = (E_j : j \in J)$, assuming always that each $E_j \subset \mathbb{R}^{d_j}$ is a Lebesgue measurable subset of $\mathbb{R}^{d_j}$ with finite Lebesgue measure. More accurately, each E_j is an equivalence class of sets, with E equivalent to E' if and only if $|E \,\Delta\, E'| = 0$.

To $E \subset \mathbb{R}^d$ is associated its symmetrization $E^\star \subset \mathbb{R}^d$, defined to be the closed ball whose Lebesgue measure equals that of E if $|E| > 0$, and to be the empty set if $|E| = 0$. Define $\mathbf{E}^\star = (E_j^\star : j \in J)$. Rogers [23], [24] and Brascamp-Lieb-Luttinger [7] have extended[1] the Riesz-Sobolev symmetrization inequality to

$$\Lambda(\mathbf{E}) \le \Lambda(\mathbf{E}^\star), \tag{1.4}$$

under certain natural hypotheses. Firstly, it is assumed that $d_j = d$ is independent of the index $j \in J$. Secondly, $D/d = m \in \mathbb{N}$. If $d = 1$ then (1.4) holds under these hypotheses. If $d > 1$ then (1.4) holds under an additional symmetry hypothesis, under which there exists an identification of $\mathbb{R}^D = \mathbb{R}^{md}$ with $(\mathbb{R}^d)^m$ so that the diagonal action of $\mathrm{Sl}(d)$ on $(\mathbb{R}^d)^m$ is a symmetry of Λ, in the sense that

$$\Lambda(\mathbf{f}) = \Lambda(\mathbf{f} \circ T) \;\text{ for every } T \in \mathrm{Sl}(d), \tag{1.5}$$

where $\mathbf{f} \circ T = (f_j \circ T : j \in J)$. There is also a natural translation action of the additive group $\mathbb{R}^{md}$ by $\mathbf{y} \mapsto \big(\mathbf{f} \mapsto (f_j + L_j(\mathbf{y}) : j \in J)\big)$, under which Λ is invariant.

The inequality (1.4) for indicator functions can be read in two ways: as a statement of monotonicity of Λ under the mapping $\mathbf{E} \mapsto \mathbf{E}^\star = (E_j^\star : j \in J)$, or alternatively as a formula for the functional

$$\Theta(\mathbf{e}) = \sup_{|E_j| = e_j} \Lambda(\mathbf{E}) \tag{1.6}$$

where the supremum is taken over all tuples of measurable sets of the specified Lebesgue measures. In particular, (1.4) states that maximizers of Θ exist, and that among these maximizers are tuples of balls centered at the origin of the specified measures. Consequently, according to the symmetry hypothesis, tuples of homothetic ellipsoids whose centers belong to the orbit of $0 \in (\mathbb{R}^d)^J$ under the group of translation symmetries are also maximizers. This orbit is the set of all $|J|$–tuples $(L_j(\mathbf{v}) : j \in J)$, where $\mathbf{v}$ ranges over $\mathbb{R}^D$.

Uniqueness theorems [8], [14], [12], [15] state that these are the only maximizers, under certain additional hypotheses, of which the primary one is known as admissibility [8]. These uniqueness theorems for indicator functions do not have simple extensions to general nonnegative functions, yet they can sometimes be used to analyze uniqueness and stability questions for functionals of general nonnegative functions [10], [16], [11].

In this paper, we take up the question of whether any part of this theory for indicator functions survives in the absence of the Rogers-Brascamp-Lieb-Luttinger symmetry hypothesis. In general, ellipsoids are not maximizers, as this example reveals: Let $J = \{0, 1, 2 \ldots, D\}$. Let $d_j = 1$ for every $j \ne 0$ and $d_0 = D - 1$. For $1 \le j \le d$ define $L_j(x_1, x_2, \ldots, x_d) = x_j$. Let $L_0 : \mathbb{R}^D \to \mathbb{R}^{D-1}$ be a generic surjective linear mapping. Let $E_j \subset \mathbb{R}^1$ be the interval of length 1 centered at 0 for each $j \in \{1, 2, \ldots, D\}$. Let E_0 remain

[1]The treatment of Rogers [24] for $d > 1$ may be incomplete.

unspecified as yet. $\Lambda(\mathbf{E})$ is equal to $\int_{E_0} K$ where $K : \mathbb{R}^{D-1} \to [0,\infty)$ and $K(y)$ is the one-dimensional measure of the slice $\{\mathbf{x} : L_0(\mathbf{x}) = y\}$ of the unit cube in $\mathbb{R}^D$. For $|E_0|$ in a suitable parameter range, maximizing sets E_0 are superlevel sets $\{y : K(y) \ge r\}$ of K, with r a function of $|E_0|$. These superlevel sets are convex polytopes.

We study the equidimensional case in which $d_j = d$ for every index j. We consider the simplest equidimensional situation not subsumed by existing theory: $d = 2$, $D = 2d = 4$, and the index set J is $\mathcal{I} = \{1, 2, 3, 4\}$. We impose a partial symmetry hypothesis, discussed below. Our first two main conclusions concerning this situation are that there is a suitable generalization of the concept of admissibility, and that maximizing tuples $\mathbf{E}$ exist. This raises the question of the nature of such maximizers. In the subcase in which the tuple $\mathcal{L}$ of mappings L_j is a small perturbation of a tuple for which the symmetry hypothesis holds, we also show that for any partially symmetrized maximizer $\mathbf{E}$, each component set E_j is strongly convex with C^∞ boundary. Finally, we analyze a family of perturbed structures for which the partial symmetry is overtly broken in a specific way, and show that maximizers $\mathbf{E}$ exist for these structures if and only if they are equivalent via certain changes of coordinates in $\mathbb{R}^4$ to structures with the partial symmetry. Generically, such changes of coordinates do not exist. Thus the partial symmetry condition is not wholly artificial.

Our partial symmetry hypothesis is most transparently expressed in coordinates. For $\mathbb{R}^4$, we use coordinates $(\mathbf{x};\mathbf{y}) = (x_1, x_2; y_1, y_2)$. We assume that each target space $\mathbb{R}^2$ is equipped with coordinates with respect to which the linear mapping $L_j : \mathbb{R}^4 \to \mathbb{R}^2$ takes the form

$$L_j(\mathbf{x},\mathbf{y}) = \big(L_j^1(\mathbf{x}), L_j^2(\mathbf{y})\big) \tag{1.7}$$

with $L_j^i : \mathbb{R}^2 \to \mathbb{R}^1$ a surjective linear mapping. The perturbed structures of our nonexistence examples take the form $L_j(\mathbf{x},\mathbf{y}) = \big(L_j^1(\mathbf{x}), L_j^2(\mathbf{x},\mathbf{y})\big)$.

Structures of the form (1.7) enjoy two types of symmetries. Firstly, there is a translation action of $\mathbb{R}^4$ on $(\mathbb{R}^2)^4$ defined by

$$(x_j : j \in \mathcal{I}) \mapsto (x_j + L_j(\mathbf{w})) : j \in \mathcal{I})$$

for $\mathbf{w} \in \mathbb{R}^4$. Secondly, there are dilation actions of $\mathbb{R}^+$ on $\mathbb{R}^2$ and on $(\mathbb{R}^2)^4$, defined by

$$D_t(x,y) = (tx, t^{-1}y) \tag{1.8}$$

and $D_t(z_j : j \in \mathcal{I}) = (D_t z_j : j \in \mathcal{I})$.

In the fully symmetric case, Steiner symmetrization [21], [7] and rotational symmetry combine to provide a powerful tool. Our partial symmetry hypothesis allows Steiner symmetrization with respect to the horizontal and vertical axes, but not with respect to arbitrary directions in $\mathbb{R}^2$. This limited symmetrization is a useful tool, but certainly a less powerful one.

An essential element in the theory of maximizers in the fully symmetric situation is the notion of admissibility. In the Riesz-Sobolev inequality, if $|E_3|^{1/d} > |E_1|^{1/d} + |E_2|^{1/d}$ then maximizing configurations are those in which the sumset $E_1 + E_2$ has measure $\le |E_3|$ and is contained in E_3. Thus maximizers exist, but have little structure and are not a natural topic of discussion. Admissibility for this inequality is the condition that $|E_k|^{1/d} \le |E_i|^{1/d} + |E_j|^{1/d}$ for all permutations (i,j,k) of $(1,2,3)$. We formulate a suitable definition of admissibility for our context, and combine Steiner symmetrization with the translation and dilation symmetries to develop a compactness argument which establishes the existence of maximizers in the admissible regime.

We study in more detail those maximizers $\mathbf{E}$ that are Steiner symmetric with respect to both the horizontal and vertical axes and show that (under a certain auxiliary hypothesis of genericity) each component set E_j is strictly convex with C^∞ boundary. This is a type of regularity theorem for a coupled system of free boundary problems. The 4–tuple $\mathbf{E}$ satisfies a generalized Euler-Lagrange relation, formulated in Proposition 9.2, which states (formally) that the boundary of each E_i is a level set of a certain function K_i defined in terms of the other three sets E_j. A bootstrapping argument is used to establish C^∞ regularity along with strong convexity.

2. Notation, hypotheses, and preliminaries

Throughout the paper we write $\mathcal{I} = \{1,2,3,4\}$. All sets $E_i \subset \mathbb{R}^d$ are assumed to be Lebesgue measurable and to have finite Lebesgue measures, unless otherwise indicated.

Consider functionals of the form

$$\Lambda_{\mathcal{L}}(\mathbf{E}) = \int_{\mathbb{R}^4} \prod_{i=1}^{4} 1_{E_i}(L_i(x_1,x_2,y_1,y_2))dx_1dy_1dx_2dy_2 \tag{2.1}$$

where $\mathbf{E} = (E_i : i \in \mathcal{I})$ is a 4–tuple of Lebesgue measurable subsets of $\mathbb{R}^2$ and $\mathcal{L} = (L_i : i \in \mathcal{I})$ is a collection of linear maps from $\mathbb{R}^4 \to \mathbb{R}^2$. The following structural hypothesis on the maps L_i will be in force throughout this paper: For each $i \in \mathcal{I}$, we require that L_i can be expressed in the form

$$L_i(x_1,x_2,y_1,y_2) = (L_i^1(x_1,x_2), L_i^2(y_1,y_2)) \tag{2.2}$$

where $L_i^1 : \mathbb{R}^2 \to \mathbb{R}$ and $L_i^2 : \mathbb{R}^2 \to \mathbb{R}$ are linear and surjective. We refer to (2.2) as the partial symmetry hypothesis.

Definition 2.1. *A tuple $\mathcal{L}^0 = (L_j^0 : j \in \mathcal{I})$ is said to satisfy the Rogers-Brascamp-Lieb-Luttinger symmetry hypothesis if satisfies* (2.2) *and*

$$L_i^1 = L_i^2 \quad \text{for each } i \in \mathcal{I}. \tag{2.3}$$

We say more succinctly that $\mathcal{L}^0$ satisfies the full symmetry hypothesis.

This implies the presence of a large symmetry group. Define $T(\mathbf{E}) = (T(E_j) : j \in \mathcal{I})$. Then (2.2) and (2.3) imply that

$$\Lambda(T(\mathbf{E})) = \Lambda(\mathbf{E}) \ \text{ for every } \mathbf{E} \text{ and } T \in \mathrm{Sl}(2).$$

The following notion of nondegeneracy is equivalent to Definition 2.3 of [12] when $\mathcal{L}$ satisfies the full symmetry hypothesis.

Definition 2.2. *A family $\mathcal{L} = (L_i : i \in \mathcal{I})$ of linear mappings $L_j : \mathbb{R}^4 \to \mathbb{R}^2$ that satifies* (2.2) *is nondegenerate if for any $i \neq j \in \mathcal{I}$, the mappings $\mathbf{x} \mapsto (L_i^1(\mathbf{x}), L_j^1(\mathbf{x}))$ and $\mathbf{y} \mapsto (L_i^2(\mathbf{y}), L_j^2(\mathbf{y}))$ are bijective linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.*

Notation 2.3. *The Lebesgue measure preserving dilations $D_t : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by*

$$D_t(x,y) = (tx, t^{-1}y)$$

for $t \in \mathbb{R}^+$. We also write

$$D_t\mathbf{E} = D_t(E_j : j \in \mathcal{I}) = (D_tE_j : j \in \mathcal{I}).$$

These dilations are symmetries of $\Lambda_{\mathcal{L}}$ in the sense that

$$\Lambda_{\mathcal{L}}(D_t\mathbf{E}) = \Lambda_{\mathcal{L}}(\mathbf{E}) \tag{2.4}$$

for all 4–tuples $\mathbf{E}$ of sets of Lebesgue measurable subsets of $\mathbb{R}^2$. $\Lambda_{\mathcal{L}}$ also enjoys a translation symmetry. For any $\mathbf{v} \in \mathbb{R}^4$, $\Lambda(E_j : j \in \mathcal{I}) = \Lambda(E_j + L_j(\mathbf{v}) : j \in \mathcal{I})$. This follows by making a change of variables $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, \mathbf{y}) - \mathbf{v}$ in the integral defining $\Lambda(\mathbf{E})$.

Notation 2.4. *$|E|$ denotes the Lebesgue measure of a subset of Euclidean space $\mathbb{R}^d$ of any dimension d. $|\mathbf{E}|$ denotes $(|E_1|, |E_2|, |E_3|, |E_4|) \in [0,\infty]^4$ where each E_i is a Lebesgue measurable subset of $\mathbb{R}^2$.*

Notation 2.5. *For $\mathbf{e} = (e_i : i \in \mathcal{I}) \in (0,\infty)^4$,*

$$\Theta(\mathbf{e}) := \sup_{\mathbf{E}:|\mathbf{E}|=\mathbf{e}} \Lambda_{\mathcal{L}}(\mathbf{E}).$$

Lemma 2.1. *Θ satisfies a triangle inequality*

$$\Theta(\mathbf{e} + \mathbf{e}') \geq \Theta(\mathbf{e}) + \Theta(\mathbf{e}'). \tag{2.5}$$

Proof. Consider any $\mathbf{E}, \mathbf{E}'$ satisfying $|\mathbf{E}| = \mathbf{e}$ and $|\mathbf{E}'| = \mathbf{e}'$ such that all of the component sets E_j, E_j' are bounded. Choose a vector $\mathbf{v} \in \mathbb{R}^4$ that does not belong to the nullspace of any of the four mappings L_j. For large $r \in \mathbb{R}^+$ consider the 4–tuple $\mathbf{E}^{(r)}$ of sets defined by $E_j^{(r)} = E_j \cup (E_j' + rL_j(\mathbf{v}))$. For sufficiently large r, $E_j' + rL_j(\mathbf{v})$ is disjoint from E_j, so $|E_j^{(r)}| = e_j + e_j'$. Since $\mathbf{1}_{\tilde{E}_j} = \mathbf{1}_{E_j} + \mathbf{1}_{E_j' + rL_j(\mathbf{v})}$, $\Lambda(\tilde{\mathbf{E}}) \geq \Lambda(\mathbf{E}) + \Lambda(E_j' + rL_j(\mathbf{v}) : j \in \mathcal{I})$. Indeed, $\Lambda(\tilde{\mathbf{E}})$ is the sum of the two terms on the right-hand side of this last inequality, plus $2^4 - 2$ other terms, each of which is nonnegative. By the translation invariance of Λ, this is equal to

$$\Lambda(\mathbf{E}) + \Lambda(E_j' + rL_j(\mathbf{v}) : j \in \mathcal{I}) = \Lambda(\mathbf{E}) + \Lambda(\mathbf{E}').$$

Upon taking the supremum over all tuples $\mathbf{E}, \mathbf{E}'$ with bounded component sets, the triangle inequality follows. □

The vertical Steiner symmetrization $\mathbf{E}^\sharp = (E_j^\sharp : j \in \mathcal{I})$ is defined as follows. For $E \subset \mathbb{R}^2$ with finite Lebesgue measure, $E^\sharp \subset \mathbb{R}^2$ is

$$E^\sharp = \{(x,y) : |y| \leq \tfrac{1}{2}|\{t \in \mathbb{R} : (x,t) \in E|\} \tag{2.6}$$

if $|\{t \in \mathbb{R} : (x,t) \in E\}| > 0$, and otherwise $\{y : (x,y) \in E^\sharp\}$ is empty. Then $|E^\sharp| = |E|$, and the intersection of $E^\sharp$ with any vertical line has the same one-dimensional Lebesgue measure as the intersection of E with that same vertical line. The horizontal Steiner symmetrizations $E^\flat$ and $\mathbf{E}^\flat$ are defined by interchanging the roles of the horizontal and vertical axes. Define

$$E^\dagger = (E^\sharp)^\flat \quad \text{and} \quad \mathbf{E}^\dagger = (\mathbf{E}^\sharp)^\flat. \tag{2.7}$$

It is elementary that

$$E^\dagger = (E^\dagger)^\sharp = (E^\dagger)^\flat \tag{2.8}$$

up to Lebesgue null sets.

In general, $(E^\sharp)^\flat$ and $(E^\flat)^\sharp$ need not be equal, or even closely related. Consider for instance the situation in which $E \subset \mathbb{R}^2$ is a rectangle centered at the origin, with sides of lengths 1 and $\varepsilon \ll 1$, with long axis making angles of $\pi/4$ with the positive horizontal and vertical axes. However, $E^\dagger = (E^\dagger)^\sharp = (E^\dagger)^\flat$.

Definition 2.6. *A Lebesgue measurable set $E \subset \mathbb{R}^2$ satisfying $|E| < \infty$ is symmetrized if $E = E^\sharp = E^\flat$ up to Lebesgue null sets. A tuple $\mathbf{E} = (E_j : j \in \mathcal{I})$ is symmetrized if each set E_j is symmetrized.*

Lemma 2.2. *Λ satisfies*

$$\Lambda(\mathbf{E}) \le \Lambda(\mathbf{E}^\dagger) \tag{2.9}$$

for all tuples of sets of finite Lebesgue measure.

Proof. Under the partial symmetry hypothesis,

$$\Lambda(\mathbf{E}) \le \Lambda(\mathbf{E}^\sharp) \ \text{ and } \ \Lambda(\mathbf{E}) \le \Lambda(\mathbf{E}^\flat) \tag{2.10}$$

for arbitrary $\mathbf{E}$. These inequalities are proved in [7], under the full Rogers-Brascamp-Lieb-Luttinger symmetry hypothesis of Definition 2.1, but only the partial symmetry hypothesis is needed in their proofs since only Steiner symmetrizations in horizontal and vertical directions are employed. (2.9) follows from (2.10) since

$$\Lambda(\mathbf{E}) \le \Lambda(\mathbf{E}^\sharp) \le \Lambda((\mathbf{E}^\sharp)^\flat) = \Lambda(\mathbf{E}^\dagger).$$

□

According to Lemma 2.2, if a maximizing tuple $\mathbf{E}$ exists, then there exists a symmetrized maximizing tuple.

3. Admissibility

We regard $(0,\infty)^4$ as being partially ordered.

Notation 3.1. $\mathbf{e} \le \mathbf{e}'$ *means that $e_j \le e'_j$ for all four indices $j \in \mathcal{I}$.* $\mathbf{e} < \mathbf{e}'$ *means that* $\mathbf{e} \le \mathbf{e}'$ *and $e_j < e'_j$ for at least one index $j \in \mathcal{I}$.*

Definition 3.2. *$(\mathcal{L}, \mathbf{e})$ is admissible if there exists no $\mathbf{e}' < \mathbf{e}$ satisfying $\Theta(\mathbf{e}') = \Theta(\mathbf{e})$.*

We will sometimes write "$\mathbf{e}$ is admissible" instead.

$\mathbf{E}$ is said to be a maximizer if $\Lambda(\mathbf{E}) = \Theta(|\mathbf{E}|)$. $\Theta(\mathbf{e})$ is said to be attained if there exists a maximizer with $|\mathbf{E}| = \mathbf{e}$.

Lemma 3.1. *Θ is locally Lipschitz continuous. More precisely, there exists $C < \infty$ depending only on $\mathcal{L}$ such that for any $\mathbf{e}, \mathbf{e}' \in (0,\infty)^4$,*

$$\big|\,\Theta(\mathbf{e}') - \Theta(\mathbf{e})\,\big| \le C \max_{k\in\mathcal{I}} (e_k + e'_k) \max_{j\in\mathcal{I}} |e_j - e'_j|. \tag{3.1}$$

Proof. The mapping $\mathbf{E} \mapsto \Lambda_{\mathcal{L}}(\mathbf{E})$ is locally Lipschitz in the sense that

$$|\Lambda(\mathbf{E}) - \Lambda(\mathbf{E}')| \le C(\max_{i\in\mathcal{I}} |E_i| + \max_{j\in\mathcal{I}} |E'_j| \max_{k\in\mathcal{I}} |E_k \,\Delta\, E'_k| \tag{3.2}$$

for arbitrary 4-tuples of Lebesgue measurable subsets of $\mathbb{R}^2$. This constant C depends only on $\mathcal{L}$.

Given $\mathbf{e}$ and $\delta > 0$, choose $\mathbf{E} = (E_j : j \in \mathcal{I})$ satisfying $|\mathbf{E}| = \mathbf{e}$ and $\Lambda(\mathbf{E}) \ge \Theta(\mathbf{e}) - \delta$. From the sets E_j, construct sets $E'_j \subset \mathbb{R}^2$ satisfying $|E'_j| = e'_j$ with $|E'_j \,\Delta\, E_j| = |e'_j - e_j|$. It follows from (3.2) that

$$\Lambda(\mathbf{E}') \ge \Lambda(\mathbf{E}) - C \max_{k\in\mathcal{I}} e_k \cdot \max_{j\in\mathcal{I}} |e_j - e'_j|,$$

where $C < \infty$ depends only on $\mathcal{L}$. By letting $\delta \to 0$ we conclude that

$$\Theta(\mathbf{e}') \ge \Theta(\mathbf{e}) - C \max_{k\in\mathcal{I}} (e_k + e'_k) \max_{j\in\mathcal{I}} |e_j - e'_j|.$$

□

The mapping $\mathcal{L} \mapsto \Lambda_{\mathcal{L}}(\mathbf{E})$ is not continuous in $\mathcal{L}$ *uniformly in* $\mathbf{E}$.

Proposition 3.2. *For any $t > 0$ there exists an admissible $\mathbf{e} \in (0,\infty)^4$ satisfying $\Theta(\mathbf{e}) = t$. For any $\mathbf{e} \in (0,\infty)^4$ there exists an admissible $\mathbf{e}' \le \mathbf{e}$ satisfying $\Theta(\mathbf{e}') = \Theta(\mathbf{e})$.*

Proof. $\Theta(r\mathbf{e}) = \Theta(re_1, re_2, re_3, re_4) = r^2\Theta(\mathbf{e})$ for any $r \in (0,\infty)$ and $\mathbf{e} \in (0,\infty)^4$. Therefore for any $t \in (0,\infty)$ there exists $\tilde{\mathbf{e}}$ satisfying $\Theta(\tilde{\mathbf{e}}) = t$.

Let $t > 0$. Choose A so that there exists $\mathbf{E}$ satisfying $\Lambda_{\mathcal{L}}(\mathbf{E}) = t$ with $|E_j| \le A$ for each j. Let S be the set of all $\mathbf{e}$ satisfying $\Theta(\mathbf{e}) = t$ and $e_j \le A$ for each j. The choice of A ensures that $S \ne \emptyset$. It is a consequence of the continuity of Θ that S is closed. Since the nondegeneracy hypothesis ensures that

$$\Lambda(\mathbf{E}) \le C|E_i| \cdot |E_j| \text{ for any } i \ne j \in \mathcal{I}$$

where $C < \infty$ depends only on $\mathcal{L}$, it follows that $\inf_{\mathbf{e}\in S} \min_{i\in\mathcal{I}} e_i$ is strictly positive. Thus S is a compact subset of the open upper quadrant.

Let $\bar{e}_1 = \min_{\mathbf{e}\in S} e_1$. Define

$$\bar{e}_2 = \min_{\substack{\mathbf{e}\in S \\ e_1=\bar{e}_1}} e_2$$

and iterate this process to define $\bar{e}_3$ and then $\bar{e}_4$. Because S is compact, these quantities $\bar{e}_i$ exist. Because S is closed, $\bar{\mathbf{e}} = (\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4)$ lies in S. The construction guarantees that there exists no $\mathbf{e} \in S$ satisfying $\mathbf{e} < \bar{\mathbf{e}}$. □

To any $\mathcal{L}$, any ordered tuple (E_j, E_k, E_l), and any index i such that $\{i, j, k, l\} = \mathcal{I}$ is associated a unique function $K_i : \mathbb{R}^2 \to [0,\infty)$ characterized by the relation

$$\Lambda(E_1, E_2, E_3, E_4) = \langle \mathbf{1}_{E_i}, K_i \rangle = \int K_i \mathbf{1}_{E_i} \quad \text{for every } E_i \subset \mathbb{R}^2. \tag{3.3}$$

If $\mathbf{E}^0$ is a 4–tuple of balls in $\mathbb{R}^2$ centered at the origin, and if $\mathcal{L} = \mathcal{L}^0$ satisfies the full symmetry hypothesis of Definition 2.1 then the associated quantities K_i^0 are radially symmetric for each $i \in \mathcal{I}$.

Definition 3.3. *Let $\mathcal{L}^0$ satisfy the full symmetry hypothesis of Definition 2.1. Let $\mathbf{E}^0$ be a 4–tuple of balls in $\mathbb{R}^2$ centered at the origin and let $\mathbf{e} = |\mathbf{E}^0|$. $(\mathcal{L}^0, \mathbf{e})$ is strictly admissible if for each $i \in \mathcal{I}$, $K_i^0 > 0$ in some neighorhood of ∂E_i^0, and $\frac{d}{du}K_i^0(u^-, 0) < 0$, where $u \in (0,\infty)$ is defined by the property that $(u, 0)$ belongs to the boundary of $E_i^0 \subset \mathbb{R}^2$.*

The notation $\frac{d}{du}K_i^0(u^-, 0)$ denotes the one-sided derivative

$$\lim_{h\to 0^-} h^{-1}\big(K_i^0(u+h, 0) - K_i^0(u, 0)\big).$$

It is shown in [15] that if $\mathcal{L}^0$ is nondegenerate[2] and satisfies the full symmetry hypothesis of Definition 2.1, and if $(\mathcal{L}^0, \mathbf{e})$ is strictly admissible, then every maximizer $\mathbf{E}$ satisfying $|\mathbf{E}| = \mathbf{e}$ for $\Lambda_{\mathcal{L}^0}$ is in the orbit of a 4–tuple of balls centered at the origin in $\mathbb{R}^2$ under the symmetry group generated by translations and by the diagonal action of Sl(2). Thus each E_i is an ellipse, these ellipses are homothetic, and the the 4–tuple of their centers belongs to the orbit of $(0, 0, 0, 0) \in (\mathbb{R}^2)^4$ under the translation symmetry group.

[2]This statement is proved for $d \ge 2$ in [15]. The corresponding statement for $d = 1$, with a supplementary genericity hypothesis, is proved in [12].

K_i can be written in the form

$$K_i(u) = \int_{\mathbb{R}^2} \prod_{j\neq i} \mathbf{1}_{E_j}(\ell_{j,i}(u,v))\,dv \tag{3.4}$$

where $\ell_{j,i} : \mathbb{R}^4 \to \mathbb{R}^2$ are surjective linear maps of the form

$$\ell_{j,i}(u,v) = \big(\ell_{j,i,1}(u_1,v_1), \ell_{j,i,2}(u_2,v_2)\big) \tag{3.5}$$

with $\ell_{j,i,m} : \mathbb{R}^2 \to \mathbb{R}^1$ linear and surjective. Moreover, $\ell_{j,i} - \ell^0_{j,i} = O(\|\mathcal{L} - \mathcal{L}^0\|)$ and $\ell^0_i = (\ell^0_{j,i} : j \in \mathcal{I} \setminus \{i\})$ commute with the diagonal action of Sl(2). $\{\ell_{j,i} : j \neq i\}$ is nondegenerate in the sense that for any two distinct indices $j,k \neq i$, the linear mapping $(u,v) \mapsto (\ell_{j,i}(u,v), \ell_{k,i}(u,v))$ from $\mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}^2 \times \mathbb{R}^2$ is nonsingular.

An extra condition which had not previously appeared in the theory for the fully symmetric framework arises naturally in our analysis. Formulation of this condition requires some additional notation. Let $\mathcal{L}^0$ be nondegenerate and satisfy the full symmetry hypothesis of Definition 2.1. Let $(\mathcal{L}^0, \mathbf{e})$ be strictly admissible. Let $\mathbf{E}^0 = (E^0_j : j \in \mathcal{I})$ be a tuple of balls centered at the origin of size $|\mathbf{E}^0| = \mathbf{e}$. For any $i \neq j \in \mathcal{I}$, and any $w \in \mathbb{R}^2$, define the sets $\tilde{E}_j(w) \subset \mathbb{R}^2$ by

$$\mathbf{1}_{E^0_j}(\ell_{j,i}(w,v)) = \mathbf{1}_{\tilde{E}_j(w)}(v).$$

These sets are balls, whose centers are $\mathbb{R}^2$–valued linear functions of $w \in \mathbb{R}^2$; the mappings from $\mathbb{R}^2 \ni w$ to their centers are radially symmetric functions.

Write $\mathcal{I} = \{i,j,k,l\}$, with i playing the same role as in (3.4) and (3.5). Let $w = (u,0) \in \partial E^0_i$ with $u > 0$. The Lebesgue measure of $\tilde{E}_j(u,0) \cap \tilde{E}_k(u,0) \cap \tilde{E}_l(u,0) \subset \mathbb{R}^2$ equals $K^0_i(u,0)$, which is strictly positive by the strict admissibility hypothesis. The centers of these closed three balls $\tilde{E}_j(u,0), \tilde{E}_i(u,0), \tilde{E}_l(u,0)$ lie on the horizontal axis in $\mathbb{R}^2$. Strict admissibility implies that none of these three balls is contained in the interiors of the other two.

Definition 3.4. *Let $\mathcal{L}^0$ be nondegenerate and satisfy the full symmetry hypothesis of Definition 2.1, and let $(\mathcal{L}^0, \mathbf{e})$ be strictly admissible. Let $\mathbf{E}^0$ be a 4–tuple of balls centered at 0 satisfying $|E^0_j| = e_j$, let $u \in (0,\infty)$ be defined by $(u,0) \in \partial E^0_i$, and for $w \in \mathbb{R}^2$ let $\tilde{E}_j(w)$ be associated to E_j as above. $(\mathcal{L}^0, \mathbf{e})$ is generic if one of the following two mutually exclusive cases holds:*
(i) After some permutation of (j,k,l), $\tilde{E}_j(u,0) \cap \tilde{E}_k(u,0)$ is contained in the interior of $\tilde{E}_l(u,0)$, and the boundary of $\tilde{E}_j(u,0) \cap \tilde{E}_k(u,0)$ consists of a subarc of the boundary of $\tilde{E}_j(u,0)$ and a subarc of the boundary of $\tilde{E}_k(u,0)$, meeting transversely at two points.
(ii) The threefold intersection $\tilde{E}_j(u,0) \cap \tilde{E}_k(u,0) \cap \tilde{E}_l(u,0)$ is a connected, simply connected domain whose boundary is a piecewise C^∞ curve consisting of 4 subarcs of circles, with two of these arcs contained in the boundary of one of the three closed balls, exactly one of the arcs contained in the boundary of another of the three closed balls, the final arc contained in the boundary of the remaining closed ball, and with arcs meeting transversely where they intersect on the boundary.

Excluded by this definition of genericity is that case in which, after permutation of (j,k,l), $\tilde{E}_l(u)$ contains $\tilde{E}_j(u) \cap \tilde{E}_k(u)$, but the interior of $\tilde{E}_l(u)$ does not contain this intersection. In this situation, the boundary of the three-fold intersection consists of one subarc of $\partial\tilde{E}_j(u)$ and one subarc of $\partial\tilde{E}_k(u)$, meeting transversely, but the points at which these subarcs meet also belong to $\partial\tilde{E}_l(u)$. This situation is unstable, giving rise to either case (i) or (ii) upon arbitrary small perturbation. We expect this instability to lead to failure of the boundary of components E_i of maximizing tuples $\mathbf{E}$ to be C^∞, for general perturbations $\mathcal{L}$ of $\mathcal{L}^0$.

The following example shows that the property that $\mathbf{e}$ is generic does not follow from $\mathbf{e}$ being strictly admissible: Let $\mathbf{E}$ be a 4-tuple of subsets of $\mathbb{R}^2$ and consider the functional

$$\int_{\mathbb{R}^4} 1_{E_1}(x_1, y_1)1_{E_2}(x_2, y_2)1_{E_3}(x_1 + x_2, y_1 + y_2)1_{E_4}(x_1 - x_2, y_1 - y_2)d\mathbf{z} \tag{3.6}$$

written in coordinates $\mathbf{z} = (x_1, x_2, y_1, y_2)$. The data $\mathbf{e} = (1, 1, r, 1)$ is strictly admissible for the above functional if $1 < r < 2$. Because the functional satisfies the symmetry hypotheses of the Rogers-Brascamp-Lieb-Luttinger inequality, the sets (B, B, B_r, B), where B is the radius 1 ball centered at the origin and B_r is the radius r ball centered at the origin, extremize the functional restricted to tuples of sets of size $|\mathbf{E}| = (1, 1, r, 1)$. Using the notation from the previous definition with $i = 1$, we have $u = 1$, $\tilde{E}_1 = \tilde{E}_4 = B$ and $\tilde{E}_3 = B_r$. The intersection

$$B \cap (B_r - (1, 0)) \cap (B + (1, 0))$$

varies in type, satisfying case (i) or (ii) from Definition 3.4, and sometimes neither, for different values of $1 < r < 2$, as shown in Figure 1 below.

4. Main results

The main results of this paper are as follows.

Theorem 4.1 (Existence of maximizers). *Let $\mathcal{L}$ be nondegenerate and satisfy the partial symmetry hypothesis* (2.2)*. For each $\mathbf{e} \in (0, \infty)^4$ there exists $\mathbf{E}$ satisfying $|\mathbf{E}| = \mathbf{e}$ and $\Lambda(\mathbf{E}) = \Theta(\mathbf{e})$.*

It will suffice to prove Theorem 4.1 in the admissible case. For we have shown that if $(\mathcal{L}, \mathbf{e})$ is not admissible, then there exists an admissible $\mathbf{e}' \le \mathbf{e}$ satisfying $\Theta(\mathbf{e}') = \Theta(\mathbf{e})$. If there exists $\mathbf{E}'$ satisfying $|\mathbf{E}'| = \mathbf{e}'$ and $\Lambda(\mathbf{E}') = \Theta(\mathbf{e}') = \Theta(\mathbf{e})$, then any tuple $\mathbf{E}$ satisfying $E_j \supset E'_j$ for each $j \in \mathcal{I}$ and $|\mathbf{E}| = \mathbf{e}$ is a maximizer for $(\mathcal{L}, \mathbf{e})$.

For admissible $(\mathcal{L}, \mathbf{e})$ a stronger result will be proved.

Theorem 4.2 (Qualitative stability of maximizers). *Let $\mathbf{e} \in (\mathbb{R}^+)^4$ be admissible. Let $\mathbf{E}^{(n)}$ be a sequence of tuples satisfying $|\mathbf{E}^{(n)}| = \mathbf{e}$, $\lim_{n\to\infty} \Lambda(\mathbf{E}^{(n)}) = \Theta(\mathbf{e})$, and $(\mathbf{E}^{(n)})^\dagger = \mathbf{E}^{(n)}$. Then there exist a subsequence of indices n_k, real numbers $\lambda_k > 0$, and a tuple $\mathbf{E}$ such that for each index $i \in \mathcal{I}$,*

$$\lim_{k\to\infty} |E_i \,\Delta\, D_{\lambda_k}(E_i^{n_k})| = 0. \tag{4.1}$$

Theorem 4.1 for admissible $(\mathcal{L}, \mathbf{e})$ is a direct consequence of Theorem 4.2. Indeed, let $\mathbf{e}$ be admissible. According to the definition of $\Theta(\mathbf{e})$ as a supremum and by virtue of the symmetrization inequality $\Lambda(\mathbf{E}) \le \Lambda(\mathbf{E}^\dagger)$, there exists a sequence $(\mathbf{E}_\nu : \nu \in \mathbb{N})$ of tuples satisfying $\mathbf{E}_\nu = \mathbf{E}_\nu^\dagger$, $|\mathbf{E}_\nu| = \mathbf{e}$, and $\lim_{\nu\to\infty} \Lambda(\mathbf{E}_\nu) = \Theta(\mathbf{e})$. Since Λ and the relation $|\mathbf{E}| = \mathbf{e}$ are invariant under the group of dilations D_t, the continuity of Λ implies that the tuple $\mathbf{E} = (E_i : i \in \mathcal{I})$ whose existence is guaranteed by Theorem 4.2 satisfies $\Lambda(\mathbf{E}) = \lim_{\nu\to\infty} \Lambda(\mathbf{E}_\nu) = \Theta(\mathbf{e})$. Theorem 4.2 is proved in §7.

In a perturbative regime we obtain structural information about (partially) symmetrized maximizers $\mathbf{E}$.

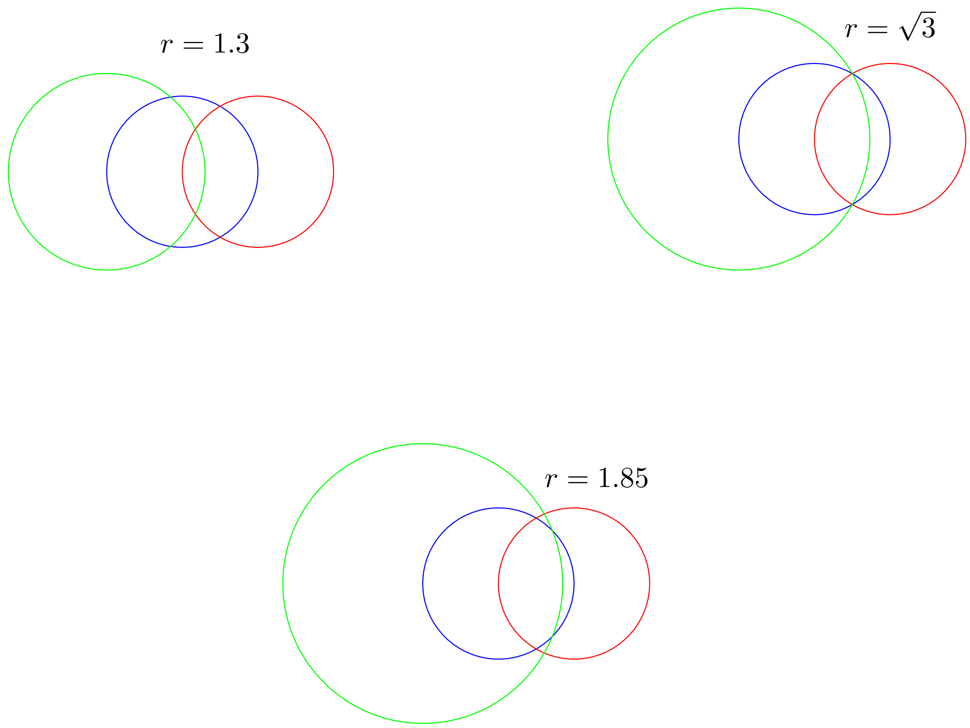

FIGURE 1. The balls that form $B \cap (B_r - (1,0)) \cap (B + (1,0))$ for different values of r. The intersection is of type (i) for $r = 1.3$, type (ii) for $r = 1.85$, and neither of type (i) nor of type (ii) for $r = \sqrt{3}$.

Theorem 4.3 (Convexity and regularity)**.** *Let $\mathcal{L}^0$ satisfy the full symmetry hypothesis of Definition 2.1. Suppose that $(\mathcal{L}^0, \mathbf{e})$ is strictly admissible and generic. There exists $\delta > 0$ with the following property. Let $\mathcal{L}$ satisfy the limited symmetry hypothesis and satisfy $\|\mathcal{L} - \mathcal{L}^0\| < \delta$. Let $\mathbf{E}$ satisfy $|\mathbf{E}| = \mathbf{e}$ and be a maximizer for $\Lambda_{\mathcal{L}}$. Suppose that $\mathbf{E}^\dagger = \mathbf{E}$. Then for each $j \in \mathcal{I}$, E_j is a strongly convex set with C^∞ boundary.*

A C^2 domain is said to be strongly convex if it is convex, and its boundary has nonzero curvature at every point. Theorem 4.3 is proved in §10.

A final result indicates that the partial symmetry hypothesis (2.2) is not entirely artificial.

Theorem 4.4 (Nonexistence of maximizers)**.** *Let $\mathcal{L}^0$ satisfy the full symmetry hypothesis (2.3). There exist nondegenerate $\mathcal{L}$ arbitrarily close to $\mathcal{L}^0$ such that for any $\mathbf{e}$ such that $(\mathcal{L}^0, \mathbf{e})$ is strictly admissible, there exist no maximizers $\mathbf{E}$ for $\Lambda_{\mathcal{L}}$ satisfying $|\mathbf{E}| = \mathbf{e}$.*

Proposition 11.1, formulated below, states more specifically that for tuples of mappings $\mathcal{L}$ of the form $L_j(\mathbf{x}, \mathbf{y}) = \big(L_j^1(\mathbf{x}), L_j^2(\mathbf{x}, \mathbf{y})\big)$, maximizers $\mathbf{E}$ cannot exist unless $L_j^2(\mathbf{x}, \mathbf{y})$ takes a special form which makes the functional $\Lambda_{\mathcal{L}}$ equivalent, in a natural way, to $\Lambda_{\tilde{\mathcal{L}}}$ where $\tilde{\mathcal{L}}$ satisfies (2.2). It is proved in Section 11.

5. Conjectures

In this paper we explore a rather specific situation in the hope of building insight into what is true, and what might be proved, in a broader framework. It is natural to venture various conjectures in this regard. In all of these conjectures, we assume that $(\mathcal{L}, \mathbf{e})$ satisfies appropriate nondegeneracy and admissibility hypotheses, which remain to be given precise formulations.

Conjecture 1. *For generic and admissible* $(\mathcal{L}, \mathbf{e})$*, maximizers* $\mathbf{E}$ *are not tuples of ellipsoids.*

It is natural to envision a computer aided proof of this conjecture, using Taylor expansions to verify that no tuples of ellipsoids can satisfy the generalized Euler-Lagrange relation (see Proposition 9.2) that is satisfied by maximizers.

Conjecture 2. *For generic* $(\mathcal{L}, \mathbf{e})$ *satisfying the partial symmetry hypothesis* (2.2)*, any maximizer* $\mathbf{E}$ *is a translate of a symmetrized maximizer.*

This conclusion will be established in a sequel for the case in which $\mathcal{L}$ is a small perturbation of a fully symmetric $\mathcal{L}^0$ and $(\mathcal{L}^0, \mathbf{e})$ is strictly admissible. Thus symmetrized maximizers do play a central role in the subject, justifying the attention accorded them in this paper.

Conjecture 3. *Let* $(\mathcal{L}^0, \mathbf{e})$ *be nondegenerate and strictly admissible. For generic admissible* $(\mathcal{L}, \mathbf{e})$ *satisfying* (2.2) *with* $\mathcal{L}$ *sufficiently close to* $\mathcal{L}^0$*, symmetrized maximizers of* $\Lambda_{\mathcal{L}}$ *are unique up to measure-preserving dilations of* $\mathbb{R}^2$*.*

Conjecture 4. *Under the partial symmetry hypothesis* (2.2)*, the conclusion that the component sets* E_i *of any maximizer* $\mathbf{E}$ *are convex, holds with suitable strict admissibility and nondegeneracy hypotheses on* $(\mathcal{L}, \mathbf{e})$*, without any hypothesis that* $\mathcal{L}$ *is a small perturbation of a tuple of mappings that possesses* $\mathrm{Sl}(d)$ *invariance.*

Conjecture 5. *The results of this paper concerning existence and convexity of maximizers have analogues for generic nondegenerate data* $\mathcal{L}$ *without partial symmetry.*

Proposition 11.1 demonstrates that the requirement that $\mathcal{L}$ be generic cannot be entirely omitted. However, the construction on which the Proposition is based requires structural properties not shared by generic $\mathcal{L}$, and we do not regard these examples as indicative of the state of affairs for generic data.

Conjecture 6. *Consider small perturbations* $\mathcal{L}$*, satisfying* (2.2)*, of tuples* $\mathcal{L}^0$ *that satisfy* (2.3)*. If the genericity hypothesis on* $(\mathcal{L}^0, \mathbf{e})$ *is omitted then maximizers need not have* C^∞ *boundaries.*

Question 7. *For generic* $(\mathcal{L}, \mathbf{e})$ *satisfying suitable nondegeneracy hypotheses, do maximizers exist?*

Question 8. *To what extent are maximizers* $\mathbf{E}$ *unique up to translation, in the absence of partial symmetry, for generic* $(\mathcal{L}, \mathbf{e})$ *satisfying suitable nondegeneracy and admissibility hypotheses?*

6. Compatibility

Definition 6.1. *For tuples of sets* $\mathbf{E} = (E_j : j \in \mathcal{I})$*,*

$$\lambda(\mathbf{E}) = \sup_R \min_{j \in \mathcal{I}} \frac{|E_j \cap R|}{|E_j| + |R|} \tag{6.1}$$

with the supremum taken over all rectangles $R \subset \mathbb{R}^2$ centered at 0 with sides parallel to the coordinate axes.

If $E = E^\dagger$, and if R is a rectangle centered at the origin with sides parallel to the coordinate axes, then

$$|(R+y) \cap E| \le |R \cap E| \text{ for every } y \in \mathbb{R}^2. \tag{6.2}$$

Lemma 6.1 (Compatibility)**.** *Let K be a compact subset of $(\mathbb{R}^+)^4$. For each $\varepsilon > 0$ there exists $\delta > 0$ with the following property. Let $\mathbf{e} \in K$, and let $\mathbf{E}$ be a 4–tuple of subsets of $\mathbb{R}^2$ satisfying $|\mathbf{E}| = \mathbf{e}$ and $\mathbf{E} = \mathbf{E}^\sharp = \mathbf{E}^\flat$. If $\lambda(\mathbf{E}) < \delta$ then $\Lambda(\mathbf{E}) < \varepsilon$.*

Sublemma 6.2. *For $j \in \mathcal{I}$ let $R_j = I_j \times I'_j \subset \mathbb{R}^2$ be a rectangle with sides parallel to the axes. Then for any permutation (i,j,k,l) of $(1,2,3,4)$,*

$$\Lambda(\mathbf{1}_{R_j} : j \in \mathcal{I}) \le C|I'_k| \cdot |I'_l| \cdot |I'_i|^{-1}|I'_j|^{-1}|R_i| \cdot |R_j|. \tag{6.3}$$

Proof. $\Lambda(\mathbf{1}_{R_j} : j \in \mathcal{I})$ is majorized by the Lebesgue measure of the set of all $(\mathbf{x},\mathbf{y}) = (x_1,x_2,y_1,y_2) \in \mathbb{R}^4$ for which $L^1_i(\mathbf{x}) \in I_i$, $L^1_j(\mathbf{x}) \in I_j$, $L^2_k(\mathbf{y}) \in I'_k$, and $L^2_l(\mathbf{y}) \in I'_l$. The mappings $\mathbf{x} \mapsto (L^1_i(\mathbf{x}), L^1_j(\mathbf{x}))$ and $\mathbf{y} \mapsto (L^2_k(\mathbf{y}), L^2_l(\mathbf{y}))$ are bijective linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. Thus $\Lambda(\mathbf{1}_{R_j} : j \in \mathcal{I})$ is bounded by a constant, which depends only on $(L_n : n \in \mathcal{I})$, multiplied by $|I_i| \cdot |I_j| \cdot |I'_k| \cdot |I'_l|$. □

An analogous conclusion holds if the roles of I_k and I'_k are reversed.

In the following discussion, $\mathbf{k} = (k_1,k_2,k_3,k_4) \in \mathbb{Z}^4$.

Sublemma 6.3. *For each $j \in \mathcal{I}$ let $\{R^{(j)}_k : k \in \mathbb{Z}\}$ be a family of rectangles in $\mathbb{R}^2$ of the form $R^{(j)}_k = I^{(j)}_k \times J^{(j)}_k$ with $J^{(j)}_k$ of length 2^k. Suppose that $\sum_{k\in\mathbb{Z}} |R^{(j)}_k| < \infty$ for each $j \in \mathcal{I}$. There exists $C < \infty$ such that for any set $S \subset \mathbb{Z}^4$,*

$$\sum_{\mathbf{k}\in S} \Lambda(R^{(1)}_{k_1}, R^{(2)}_{k_2}, R^{(3)}_{k_3}, R^{(4)}_{k_4}) \le C \sup_{\mathbf{k}\in S} 2^{-(\max_i k_i - \min_j k_j)/2} \cdot \sup_{\mathbf{k}\in S} \max_{n\in\mathcal{I}} |R^{(n)}_{k_n}| \cdot \max_{j\in\mathcal{I}} \sum_{k\in\mathbb{Z}} |R^{(j)}_k|.$$

Proof. By dilating we may assume without loss of generality that

$$\sup_{j\in\mathcal{I}} \sum_{k\in\mathbb{Z}} |R^{(j)}_k| = 1.$$

It suffices to treat the summation over all $\mathbf{k} \in S$ that satisfy

$$k_4 \le k_3 \le k_2 \le k_1. \tag{6.4}$$

The same reasoning will apply with arbitrary permutations of the indices $1,2,3,4$. For the rest of the proof, we assume that every $\mathbf{k} \in S$ satisfies (6.4). Set $\rho = \sup_{\mathbf{k}\in S} 2^{(k_4+k_3-k_2-k_1)/2}$.

Individual summands satisfy

$$\begin{aligned}\Lambda(R^{(1)}_{k_1}, R^{(2)}_{k_2}, R^{(3)}_{k_3}, R^{(4)}_{k_4}) &\le C 2^{k_3+k_4-k_1-k_2} |R^{(1)}_{k_1}| \cdot |R^{(2)}_{k_2}| \\ &\le C\rho 2^{-(k_1-k_2)/2} 2^{(k_3+k_4-2k_2)/2} |R^{(1)}_{k_1}| \cdot |R^{(2)}_{k_2}|.\end{aligned}$$

Summation over all $k_3, k_4 \le k_2$ yields an upper bound

$$\rho \sum_{k_2 \le k_1} 2^{-(k_1-k_2)/2} |R^{(1)}_{k_1}| \cdot |R^{(2)}_{k_2}| \le C\rho \big(\sum_{k_1} |R^{(1)}_{k_1}|^2\big)^{1/2} \cdot \big(\sum_{k_2} |R^{(2)}_{k_2}|^2\big)^{1/2}$$

with the first sum taken over all (k_1,k_2) satisfying $k_2 \le k_1$ for which there exist k_3,k_4 for which $\mathbf{k}\in S$, the second sum over all k_1 for which there exist k_2,k_3,k_4 for which $\mathbf{k}\in S$, and so on. Now

$$\big(\sum_{k_1} |R^{(1)}_{k_1}|^2\big)^{1/2} \le \sup_k |R^{(1)}_k|^{1/2}\cdot \big(\sum_{k_1} |R^{(1)}_{k_1}|\big)^{1/2} \le \sup_k |R^{(1)}_k|^{1/2},$$

with a corresponding majorization for $\big(\sum_{k_2} |R^{(2)}_{k_2}|^2\big)^{1/2}$. □

Proof of Lemma 6.1. Define $E_j^+ = \{(x,y)\in E_j : x>0\}$ and $E_j^- = \{(x,y)\in E_j : x<0\}$. We will analyze $\Lambda(E_j^+ : j\in\mathcal{I})$; the same reasoning will apply equally well to $\Lambda(E_j^\pm : j\in\mathcal{I})$ with all possible choices of $\pm$ signs.

To E_j associate rectangles $R^{(j)}_k \subset \mathbb{R}^2$ with sides parallel to the coordinate axes, defined as follows: Express E_j^+, up to a Lebesgue null set, as

$$E_j^+ = \{(x,y) : x>0 \text{ and } |y| < f_j(x)\}$$

where $f_j : (0,\infty)\to[0,\infty)$ is nonincreasing and right continuous. Define

$$R^{(j)}_k = \{x\in\mathbb{R}^+ : 2^k \ge f_j(x) > 2^{k-1}\}\times[-2^k,2^k].$$

Then $E_j^+ \subset \cup_{k=-\infty}^{\infty} R^{(j)}_k$, so $|E_j| \le 2\sum_k |R^{(j)}_k|$. On the other hand, $|R^{(j)}_k\cap E_j| \ge \frac12 |R^{(j)}_k|$, so

$$\sum_{k\in\mathbb{Z}} |R^{(j)}_k| \le 2|E_j^+| = |E_j|.$$

Express

$$\Lambda(E_j^+ : j\in\mathcal{I}) = \sum_{\mathbf{k}\in\mathbb{Z}^4} \Lambda(E_j^+\cap R^{(j)}_{k_j} : j\in\mathcal{I}) \le \sum_{\mathbf{k}\in\mathbb{Z}^4} \Lambda(R^{(j)}_{k_j} : j\in\mathcal{I}).$$

Let $\eta=\eta(\delta)>0$ be a small parameter which will be chosen below to depend only on δ, and will tend to zero as $\delta\to0$. Introduce

$$\begin{aligned} S_1 &= \{\mathbf{k}\in S : \max_{j\in\mathcal{I}} |R^{(j)}_{k_j}| \le \eta\}\\ S_2 &= \{\mathbf{k}\in S\setminus S_1 : \max_{i\in\mathcal{I}} k_i - \min_{j\in\mathcal{I}} k_j > \log_2(1/\eta)\}\\ S' &= S\setminus(S_1\cup S_2). \end{aligned}$$

By Sublemma 6.3,

$$\Lambda(E_j^+ : j\in\mathcal{I}) \le C\eta^{1/2} + C\sum_{\mathbf{k}\in S'} \Lambda(R^{(j)}_{k_j} : j\in\mathcal{I}).$$

Matters are thus reduced to the sum over $\mathbf{k}\in S'$.

As above, by partitioning S' into finitely many subsets, we may assume for the remainder of the proof that $|R^{(1)}_{k_1}| \ge |R^{(2)}_{k_2}| \ge |R^{(3)}_{k_3}| \ge |R^{(4)}_{k_4}|$ for each $\mathbf{k}\in S'$. Since each $R^{(j)}_{k_j}$ is a rectangle with sides parallel to the coordinate axes with vertical side of length 2^{k_j}, and since $\max_{i,j\in\mathcal{I}} 2^{k_i}/2^{k_j} \le \eta^{-1}$, (6.3) gives

$$\Lambda(R^{(j)}_{k_j} : j\in\mathcal{I}) \le C2^{k_1}2^{k_3}2^{-k_4}2^{-k_2}|R^{(4)}_{k_4}|\cdot|R^{(2)}_{k_2}| \le C\eta^{-2}\frac{|R^{(4)}_{k_4}|}{|R^{(1)}_{k_1}|}\cdot|R^{(1)}_{k_1}|\cdot|R^{(2)}_{k_2}|$$

Since $\mathbf{k} \notin S_2$, $k_3 \le k_2 + \log_2(1/\eta)$, and summing over such k_3 yields the bound

$$C\eta^{-1} 2^{k_1} 2^{-k_4} |R^{(4)}_{k_4}| \cdot |R^{(2)}_{k_2}|.$$

Summing over k_2 gives

$$C\eta^{-1} 2^{k_1} 2^{-k_4} |R^{(4)}_{k_4}|.$$

Again since $\mathbf{k} \notin S_2$, $-k_4 \le \log_2(1/\eta) - k_1$. Summing over all remaining indices which in addition satisfy $\min_{i\in\mathcal{I}} |R^{(i)}_{k_i}| \le \eta^3 \max_{j\in\mathcal{I}} |R^{(j)}_{k_j}|$ gives the bound

$$\begin{aligned} C\eta^{-1} \sum_{k_1} \sum_{2^{k_1-k_4} \le 1/\eta} & 2^{k_1-k_4} \left(|R^{(4)}_{k_4}| / |R^{(1)}_{k_1}| \right) |R^{(1)}_{k_1}| \\ & \le C\eta^2 \sum_{k_1} \sum_{2^{k_1-k_4} \le 1/\eta} 2^{k_1-k_4} |R^{(1)}_{k_1}| \\ & \le C\eta \sum_{k_1} |R^{(1)}_{k_1}| = C\eta. \end{aligned}$$

Therefore it remains to consider indices $\mathbf{k} \in S'$ which satisfy $\min_{i\in\mathcal{I}} |R^{(i)}_{k_i}| > \eta^3 \max_{j\in\mathcal{I}} |R^{(j)}_{k_j}|$.

Assume that $\lambda(\mathbf{E}) < \delta$, with $\delta > 0$ small. Let $\tilde{S}$ be the set of all $\mathbf{k} \in S$ that remain untreated, and for which $\Lambda(R^{(j)}_{k_j} : j \in \mathcal{I}) \ne 0$. To complete the proof, it suffices to show that if $\eta(\delta) \to 0$ sufficiently slowly as $\delta \to 0$, then the hypothesis that $\lambda(\mathbf{E}) < \delta$ forces $\tilde{S}$ to be empty.

Consider any $\mathbf{k} \in \tilde{S}$. The associated four rectangles $R^{(j)}_{k_j}$ have sides parallel to the coordinate axes, have vertical sides of comparable lengths — meaning that the ratios of any two of these lengths are bounded above by a function of η alone — and have comparable Lebesgue measures, in the same sense of comparability. Therefore the lengths of their horizontal sides are likewise comparable. Therefore there exists a single rectangle $R \subset \mathbb{R}^2$, with sides parallel to the coordinate axes, such that $R^{(j)}_{k_j}$ is contained in a translate $R + \mathbf{v}_j$ and has Lebesgue measure comparable to $|R|$, for each $j \in \mathcal{I}$. Since $|R^{(j)}_{k_j} \cap E_j| \ge \frac{1}{2} |R^{(j)}_{k_j}|$, and since $|E_j|$ is comparable to 1, the ratio $\frac{|E_j \cap R^{(j)}_{k_j}|}{|E_j| + |R^{(j)}_{k_j}|}$ is comparable to $|R^{(j)}_{k_j}|$. Therefore we find that for each $j \in \mathcal{I}$,

$$|R| \le C(\eta) |R^{(j)}_{k_j}| \le C(\eta) \frac{|E_j \cap R^{(j)}_{k_j}|}{|E_j| + |R^{(j)}_{k_j}|} \le C(\eta) \frac{|E_j \cap R|}{|E_j| + |R|},$$

where $C(\eta) < \infty$ depends only on η and we used (6.2) in the last inequality. By the definition of $\lambda(\mathbf{E})$, we have

$$|R| \le C(\eta) \lambda(\mathbf{E}) \le C(\eta)\delta.$$

Therefore if $C(\eta) \cdot \delta < \eta$ then we conclude that $\mathbf{k} \in S_1$, whence $\mathbf{k} \notin \tilde{S}$. Thus $\tilde{S}$ would be empty.

For each $\eta > 0$, the inequality $C(\eta)\delta < \eta$ holds for all sufficiently small $\delta > 0$. Therefore there exists a function $\delta \mapsto \eta(\delta)$ satisfying both $\lim_{\delta\to 0} \eta(\delta) = 0$, and $C(\eta(\delta)) \cdot \delta < \eta(\delta)$ for every $\delta > 0$. $\square$

Corollary 6.4. *For each compact set $K \subset (\mathbb{R}^+)^4$ and each $\delta > 0$ there exists $\rho > 0$ with the following property. If $|\mathbf{E}| = \mathbf{e} \in K$, if $\mathbf{E} = \mathbf{E}^\dagger$, and if $\Lambda(\mathbf{E}) \geq \delta$ then there exists $t \in \mathbb{R}^+$ such that for each $j \in \mathcal{I}$,*

$$[-\rho, \rho] \times [-\rho, \rho] \subset D_t(E_j). \tag{6.5}$$

Proof. This is a direct consequence of Lemma 6.1 and the definition of $\lambda(\mathbf{E})$. Choose a rectangle $R \subset \mathbb{R}^2$, with sides parallel to the coordinate axes and centered at the origin, that maximizes the ratio defining $\lambda(\mathbf{E})$, up to a factor of 2. The lower bound for $\Lambda(\mathbf{E})$ implies a lower bound for the Lebesgue measure of R. An appropriate dilation gives $\rho = c|R|^{1/2}$ where $c > 0$ is a constant. □

7. Precompactness

In this section we apply the results of Section 6 to establish Theorem 4.2, concerning the precompactness of symmetrized maximizing sequences for admissible $(\mathcal{L}, \mathbf{e})$ up to the dilation and translation symmetries of Λ introduced above. A pivotal issue is how the admissibility of $(\mathcal{L}, \mathbf{e})$ comes into play in the proof. As was implicitly shown in the comment following Theorem 4.1, precompactness cannot hold if $(\mathcal{L}, \mathbf{e})$ is not admissible, for if $\mathbf{e}' < \mathbf{e}$ with $e'_j < e_j$, and if $\mathbf{E}' = (E'_i : i \in \mathcal{I})$ satisfies $\Lambda(\mathbf{E}') = \Theta(\mathbf{e}') = \Theta(\mathbf{e})$ then any tuple $(E_i : i \in \mathcal{I})$ with $E_i = E'_i$ for every $i \neq j$, $E_j \supset E'_j$, and $|E_j| = e_j$ satisfies $\Lambda(\mathbf{E}) = \Theta(\mathbf{e})$ and $|\mathbf{E}| = \mathbf{e}$.

Proof of Theorem 4.2. Suppose that $|\mathbf{E}^{(n)}| = \mathbf{e}$ for each $n \in \mathbb{N}$, that each $\mathbf{E}^{(n)}$ is symmetrized, and that $\Lambda(\mathbf{E}^{(n)}) \to \Theta(\mathbf{e})$ as $n \to \infty$. Write $\mathbf{E}^{(n)} = (E_j^n : j \in \mathcal{I})$. By invoking Corollary 6.4 and replacing each $\mathbf{E}^{(n)}$ with a suitable dilate, we may assume that there exists a cube $\tilde{Q}$ with positive sidelength, centered at the origin, that is contained in E_i^n for every n and every $i \in \mathcal{I}$.

By the Banach-Alaoglu theorem, there exists a subsequence n_k of indices and functions $g_i \in L^2(\mathbb{R}^2)$ such that for every $i \in \mathcal{I}$, $1_{E_i^{n_k}}$ converges weakly in L^2 to g_i as $k \to \infty$. By replacing $\mathbf{E}^{(n)}$ by a subsequence, we may assume henceforth that the full sequence of indicator functions $\mathbf{1}_{E_i^n}$ converges weakly in L^2.

The set E_i^n intersected with $(0, \infty) \times [0, \infty)$ is the region under the graph of a nonnegative, nonincreasing function $f_{i,n}$. For any $s > 0$, $sf_{i,n}(s) \leq \frac{1}{4}|E_i^n| = \frac{1}{4}e_i$. A simple consequence of the Helly selection theorem is that the weak limit of $(\mathbf{1}_{E_i^n} : n \in \mathbb{N})$ is the indicator function of a region

$$E_i = \{(u, v) : |v| \leq f_i(u)\}$$

where f_i is even, the restriction of f_i to $(0, \infty)$ is nonincreasing, and $uf_i(u) \leq e_i/4$ for every $u > 0$. Thus $g_i = \mathbf{1}_{E_i}$.

Set

$$\mathbf{E} = (E_1, E_2, E_3, E_4),$$
$$\mathbf{E}^{(n)} \cap \mathbf{E} = (E_i^n \cap E_i : i \in \mathcal{I}),$$
$$\mathbf{E}^{(n)} \setminus \mathbf{E} = (E_i^n \setminus E_i : i \in \mathcal{I}).$$

Lemma 7.1.

$$\lim_{n\to\infty} \Lambda(\mathbf{E}^{(n)}) = \lim_{n\to\infty} [\Lambda(\mathbf{E}^{(n)} \cap \mathbf{E}) + \Lambda(\mathbf{E}^{(n)} \setminus \mathbf{E})]. \tag{7.1}$$

Proof of Lemma 7.1. Expressing the indicator function of E_i^n as the sum of the indicator functions of $E_i^n \cap E_i$ and of $E_i^n \setminus E_i$, and then invoking the multilinearity of Λ, produces an expansion of $\Lambda(\mathbf{E}^{(n)})$ as a sum of 2^4 terms, of which two are the main terms $\Lambda(\mathbf{E}^{(n)} \cap \mathbf{E})$ and $\Lambda(\mathbf{E}^{(n)} \setminus \mathbf{E})$. Each of the remaining 14 terms takes the form $\Lambda(E_1^n \cap E_1, E_2^n \setminus E_2, F_3^n, F_4^n)$ with F_j^n equal either to $E_j^n \cap E_j$ or to $E_j^n \setminus E_j$, up to permutation of the indices $1, 2, 3, 4$. Moreover,

$$\Lambda(E_1^n \cap E_1, E_2^n \setminus E_2, F_3^n, F_4^n) \le \Lambda(E_1^n \cap E_1, E_2^n \setminus E_2, E_3^n, E_4^n).$$

Thus in order to prove (7.1), it will suffice to show that

$$\Lambda(E_1^n \cap E_1, E_2^n \setminus E_2, E_3^n, E_4^n) \to 0$$

as $n \to \infty$, provided that the same reasoning applies with the indices permuted, as it indeed will.

To analyze $\Lambda(E_1^n \cap E_1, E_2^n \setminus E_2, E_3^n, E_4^n)$, let $\varepsilon > 0$. For $R > 0$, let $Q_R = [-R, R]^2$. For $N, M > 0$, we have the upper bound

$$\begin{aligned}\Lambda(E_1^n \cap E_1, E_2^n \setminus E_2, E_3^n, E_4^n) &\le \Lambda(E_1 \cap Q_M, E_2^n \setminus (E_2 \cup Q_N), E_3^n, E_4^n) \\ &\quad + \Lambda(E_1 \cap Q_M, (E_2^n \setminus E_2) \cap Q_N, E_3^n, E_4^n) + C|E_1 \setminus Q_M| e_2 \\ &\le \Lambda(E_1 \cap Q_M, E_2^n \setminus Q_N, E_3^n, E_4^n) \\ &\quad + \Lambda(E_1, (E_2^n \setminus E_2) \cap Q_N, E_3^n, E_4^n) + C|E_1 \setminus Q_M| e_2.\end{aligned}$$

We claim that for any $M < \infty$,

$$\Lambda(E_1 \cap Q_M, E_2^n \setminus Q_N, E_3^n, E_4^n) \le \rho_{M,\mathbf{e}}(N) \tag{7.2}$$

where the function $\rho_{M,\mathbf{e}}(N)$ depends only on $\mathcal{L}, \mathbf{e}, M, N$ and $\rho_{M,\mathbf{e}}(N) \to 0$ as $N \to \infty$ while $M, \mathbf{e}, \mathcal{L}$ remain fixed. Indeed, define α so that the intersection of E_2^n with $\{N\} \times \mathbb{R}$, which is an interval, has length 2α. Then $(-N, N) \times (-\alpha, \alpha) \subset E_2^n$, so $\alpha \le 4N^{-1} e_2$. Likewise, defining β so that the intersection of E_2^n with $\mathbb{R} \times \{N\}$ has length 2β, one has $\beta \le 4N^{-1} e_2$. Therefore if N is sufficiently large, $E_2^n \setminus Q_N$ is contained in the union of $\mathbb{R} \times (-\alpha, \alpha)$ with $(-\beta, \beta) \times \mathbb{R}$. Define $E_{2,h}^n$ to be the former portion of E_2^n, and $E_{2,v}^n$ to be the latter. Consider $\Lambda(E_1 \cap Q_M, E_{2,h}^n \setminus Q_N, E_3^n, E_4^n)$, which is majorized by $\Lambda(E_1 \cap Q_M, \tilde{E}_2^n, E_3^n, E_4^n)$, where $\tilde{E}_2^n = (E_{2,h}^n)^\flat$, is the horizontal Steiner symmetrization of $E_{2,h}^n$.

We will apply Lemma 6.1, which asserts that $\Lambda(\mathbf{E})$ is small if the quantity $\lambda(\mathbf{E})$ defined in (6.1) is small. Consider any rectangle $R \subset \mathbb{R}^2$ with sides parallel to the coordinate axes and centered at 0. In evaluating $\lambda(\mathbf{E})$, clearly only rectangles R whose vertical sides have length $\lesssim \alpha$ need be considered. If R does have vertical length $\lesssim \alpha$ then R has small measure unless its horizontal side has length $\gtrsim \alpha^{-1} e_2 \gtrsim N/4$. However, in this case $|R \cap E_1| \le |R \cap Q_M| \lesssim \alpha M \lesssim M N^{-1} e_2$. Therefore $\lambda(E_1 \cap Q_M, \tilde{E}_2^n, E_3^n, E_4^n)$ becomes arbitrarily small as N becomes arbitrarily large. Therefore by Lemma 6.1, the same goes for $\Lambda(E_1 \cap Q_M, E_{2,h}^n, E_3^n, E_4^n)$. The same reasoning applies to $\Lambda(E_1 \cap Q_M, E_{2,v}^n, E_3^n, E_4^n)$.

Choose M sufficiently large that $|E_1 \setminus Q_M| < \varepsilon$. Then choose N large enough so that $\rho_{M,\mathbf{e}}(N) < \varepsilon$. Finally, the weak convergence of E_2^n implies that

$$|(E_2^n \setminus E_2) \cap Q_N| = |E_2^n \cap (Q_N \setminus E_2)| \to |E_2 \cap (Q_N \setminus E_2)| = 0 \ \text{ as } n \to \infty.$$

Thus $\limsup_{n \to \infty} \Lambda(E_1^n \cap E_1, E_2^n \setminus E_2, E_3^n, E_4^n) \le 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this proves that $\Lambda(E_1^n \cap E_1, E_2^n \setminus E_2, E_3^n, E_4^n) \to 0$. The same reasoning, with natural changes in notation, proves that the other cross terms in the expansion of $\Lambda(\mathbf{E}^{(n)})$ also have limit zero. This completes the proof of Lemma 7.1. □

We claim next that

$$\lim_{n\to\infty} \Lambda((\mathbf{E}^{(n)} \setminus \mathbf{E})) = 0. \tag{7.3}$$

It suffices to show that $\lim_{n\to\infty} \Lambda((\mathbf{E}^{(n)} \setminus \mathbf{E})^\dagger) = 0$. To prove this, note that since $|E_i^n \cap E_i| \to |E_i|$ by the weak convergence,

$$\lim_{n\to\infty} |(\mathbf{E}^{(n)} \setminus \mathbf{E})^\dagger| = (e_1 - |E_1|, e_2 - |E_2|, e_3 - |E_3|, e_4 - |E_4|).$$

If $\limsup_{n\to\infty} \Lambda((\mathbf{E}^{(n)} \setminus \mathbf{E})^\dagger) > 0$ then after passing to a subsequence of indices n that realizes the limit supremum, we may invoke Corollary 6.4 to conclude that there exist a sequence of dilations D_{η_k} and $\delta > 0$ such that the cube $[-\delta, \delta]^2 = Q_\delta$ is contained in $D_{\eta_k}((E_i^k \setminus E_i)^\dagger)$ for all i, k.

By a change of variables that preserves the partially symmetric multilinear structure of Λ (permitted by Definition 2.2), we may assume without loss of generality that $L_1(x_1, y_1, x_2, y_2) = (x_1, y_1)$ and $L_2(x_1, y_1, x_2, y_2) = (x_2, y_2)$. For each k, let $N_k > 0$ be large enough so $|E_1^{n_k} \setminus Q_{N_k}| < \frac{1}{k}$ and $|D_{\eta_k}((E_1^{n_k} \setminus E_1)^\dagger) \setminus Q_{N_k}| < \frac{1}{k}$. Also let $v_1^k \in \mathbb{R}^2$ be large enough so that $Q_{N_k} \cap (Q_{N_k} + v_1^k) = \emptyset$. Then

$$\begin{aligned}
\Lambda(\mathbf{E}^{(n_k)} \cap \mathbf{E}) + \Lambda(\mathbf{E}^{(n_k)} \setminus \mathbf{E}) &\le \Lambda(E_1^{n_k} \cap E_1 \cap Q_{N_k}, E_2^{n_k} \cap E_2, E_3^{n_k} \cap E_3, E_4^{n_k} \cap E_4) \\
&\quad + \Lambda(D_{\eta_k}(E_1^{n_k} \setminus E_1)^\dagger \cap Q_{N_k}, D_{\eta_k}(E_2^{n_k} \setminus E_2)^\dagger, D_{\eta_k}(E_3^{n_k} \setminus E_3)^\dagger, D_{\eta_k}(E_4^{n_k} \setminus E_4)^\dagger) \\
&\qquad + Ce_2 k^{-1} \\
&= \Lambda(E_1^{n_k} \cap E_1 \cap Q_{N_k} + v_1^k, E_2^{n_k} \cap E_2, E_3^{n_k} \cap E_3 + v_3^k, E_4^{n_k} \cap E_4 + v_4^k) \\
&\quad + \Lambda(D_{\eta_k}(E_1^{n_k} \setminus E_1)^\dagger \cap Q_{N_k}, D_{\eta_k}(E_2^{n_k} \setminus E_2)^\dagger, D_{\eta_k}(E_3^{n_k} \setminus E_3)^\dagger, D_{\eta_k}(E_4^{n_k} \setminus E_4)^\dagger) \\
&\qquad + Ce_2 k^{-1}
\end{aligned}$$

for certain $v_3^k, v_4^k \in \mathbb{R}^2$ determined by v_1^k and $\mathcal{L}$.

The two sets $\left(E_1^{n_k} \cap E_1 \cap Q_{N_k}\right) + v_1^k$ and $[D_{\eta_k}(E_1^{n_k} \setminus E_1)^\dagger] \cap Q_{n_k}$ are disjoint for each k. Let $F_1^{n_k}$ be the union of these two sets. Define $F_j^{n_k}$ for $j = 2, 3, 4$ as follows. $F_2^{n_k}$ is the union of $E_2^{n_k} \cap E_2$ with $D_{\eta_k}(E_2^{n_k} \setminus E_2)^\dagger$. For $j = 3, 4$, $F_j^{n_k}$ is the union of $(E_j^{n_k} \cap E_j) + v_j^k$ with $D_{\eta_k}(E_j^{n_k} \setminus E_j)^\dagger$. Then $|F_j^{n_k}| \le e_j$ for each $j \in \mathcal{I}$.

$$\begin{aligned}
&\Lambda(E_1^{n_k} \cap E_1 \cap Q_{N_k} + v_1^k, E_2^{n_k} \cap E_2, E_3^{n_k} \cap E_3 + v_3^k, E_4^{n_k} \cap E_4 + v_4^k) \\
&\qquad\qquad \le \Lambda(E_1^{n_k} \cap E_1 \cap Q_{N_k} + v_1^k, F_2^{n_k}, F_3^{n_k}, F_4^{n_k})
\end{aligned}$$

since $E_2^{n_k} \cap E_2 \subset F_2$ and similarly for the indices $j = 3$ and $j = 4$. Likewise,

$$\begin{aligned}
&\Lambda(D_{\eta_k}(E_1^{n_k} \setminus E_1)^\dagger \cap Q_{N_k}, D_{\eta_k}(E_2^{n_k} \setminus E_2)^\dagger, D_{\eta_k}(E_3^{n_k} \setminus E_3)^\dagger, D_{\eta_k}(E_4^{n_k} \setminus E_4)^\dagger) \\
&\qquad\qquad \le \Lambda(D_{\eta_k}(E_1^{n_k} \setminus E_1)^\dagger \cap Q_{N_k}, F_2^{n_k}, F_3^{n_k}, F_4^{n_k}).
\end{aligned}$$

Thus we have shown that

$$\begin{aligned}
\Lambda(\mathbf{E}^{(n_k)} \cap \mathbf{E}) + \Lambda(\mathbf{E}^{(n_k)} \setminus \mathbf{E}) &\le \Lambda(E_1^{n_k} \cap E_1 \cap Q_{N_k} + v_1^k, F_2^{n_k}, F_3^{n_k}, F_4^{n_k}) \\
&\quad + \Lambda(D_{\eta_k}(E_1^{n_k} \setminus E_1)^\dagger \cap Q_{N_k}, F_2^{n_k}, F_3^{n_k}, F_4^{n_k}) + o(1) \\
&= \Lambda(F_1^{n_k}, F_2^{n_k}, F_3^{n_k}, F_4^{n_k}) + o(1)
\end{aligned}$$

with the equality holding because $F_1^{n_k}$ is the disjoint union of $E_1^{n_k} \cap E_1$ with $D_{\eta_k}(E_1^{n_k} \setminus E_1)^\dagger \cap Q_{N_k}$.

Thus by Lemma 7.1,

$$\Theta(\mathbf{e}) = \lim_{k\to\infty} \Lambda(\mathbf{E}^{(n_k)}) \le \limsup_{k\to\infty} \Theta(e_1, |F_2^{n_k}|, e_3, e_4).$$

There exists a cube $\tilde{Q}$ centered at $0 \in \mathbb{R}^2$, of positive sidelength, that is contained in E_2 and in $E_2^{n_k}$ for every k. Therefore for every sufficiently large k,

$$|(E_2^{n_k} \cap E_2) \cup D_{\eta_k}(E_2^{n_k} \setminus E_2)^*| \le e_2 - |\tilde{Q} \cap Q_\delta| < e_2.$$

By letting $k \to \infty$ one deduces that

$$\Theta(e_1, e_2, e_3, e_4) \le \Theta(e_1, e_2 - |\tilde{Q} \cap Q_\delta|, e_3, e_4).$$

This contradicts the definition of admissibility of $\mathbf{e}$. Therefore (7.3) must hold.

Inserting (7.3) into (7.1), we find that $\Lambda(\mathbf{E}^{(n)} \cap \mathbf{E}) \to \Theta(\mathbf{e})$ as $n \to \infty$. Since $|E_j^n| \le e_j$ for every n and every j by assumption, the same holds for the subsets $E_j^n \cap E_j$. If there were to exist $j \in \mathcal{I}$ and a subsequence satisfying $\limsup_{k\to\infty} |E_j^{n_k} \cap E_j| < e_j$, then we would conclude that $\Theta_{\mathcal{L}}(\mathbf{e}') = \Theta_{\mathcal{L}}(\mathbf{e})$ for some $\mathbf{e}' < \mathbf{e}$, contradicting the admissibility of $\mathbf{e}$. Therefore $\liminf_{n\to\infty} |E_i^n \cap E_i| = e_i$ for each $i \in \mathcal{I}$. Since $e_i = |E_i^n|$ and $|E_i| \le e_i$, this forces $\lim_{n\to\infty} |E_i^n \,\Delta\, E_i| = 0$. Therefore $|E_i| = e_i$ for each $i \in \mathcal{I}$, and $\Lambda(\mathbf{E}) = \Theta_{\mathcal{L}}(\mathbf{e})$. That completes the proof of Theorem 4.2. □

8. Continuity of Θ with respect to $\mathcal{L}$

In the next lemma, $\mathcal{L}^0$ is not assumed to satisfy the full symmetry hypothesis (2.3), even though this notation is reserved for that special case in nearly all of this paper.

Lemma 8.1. *Let $\mathcal{L}^0$ and $\mathcal{L}_\nu$ be nondegenerate and satisfy* (2.2). *Let $((\mathcal{L}_\nu, \mathbf{e}_\nu) : \nu \in \mathbb{N})$ be a sequence of data such that $(\mathcal{L}_\nu, \mathbf{e}_\nu) \to (\mathcal{L}^0, \mathbf{e}^0)$ as $\nu \to \infty$. Then*

$$\lim_{\nu\to\infty} \Theta_{\mathcal{L}_\nu}(\mathbf{e}_\nu) = \Theta_{\mathcal{L}^0}(\mathbf{e}^0). \tag{8.1}$$

Proof. There exists a symmetrized maximizing configuration $\mathbf{E}^0$ for $(\mathcal{L}^0, \mathbf{e}^0)$. Modifying each component E_i^0 appropriately yields a sequence $\mathbf{E}_\nu$ of symmetrized 4–tuples satisfying $|\mathbf{E}_\nu| = \mathbf{e}_\nu$. Then $|E_{\nu,i} \,\Delta\, E_i^0| \to 0$ as $\nu \to \infty$ for each $i \in \mathcal{I}$. It follows that $\Lambda_{\mathcal{L}_\nu}(\mathbf{E}_\nu) \to \Lambda_{\mathcal{L}^0}(\mathbf{E}^0) = \Theta_{\mathcal{L}^0}(\mathbf{e}^0)$. Therefore

$$\limsup_{\nu\to\infty} \Theta_{\mathcal{L}_\nu}(\mathbf{e}_\nu) \ge \Theta_{\mathcal{L}^0}(\mathbf{e}^0). \tag{8.2}$$

However, no converse inequality follows with comparable ease, because the mapping $(\mathcal{L}, \mathbf{E}) \mapsto \Lambda_{\mathcal{L}}(\mathbf{E})$ fails to be continuous in any sufficiently uniform sense with respect to $\mathbf{E}$.

To prove the converse, pass to a subsequence to ensure that $\Theta_{\mathcal{L}_\nu}(\mathbf{e}_\nu)$ converges to $\Theta_{\mathcal{L}^0}(\mathbf{e}^0)$ as $\nu \to \infty$. Let $(\mathbf{E}_\nu)$ satisfy $|\mathbf{E}_\nu| = \mathbf{e}_\nu$ and $\limsup_{\nu\to\infty} \Lambda_{\mathcal{L}_\nu}(\mathbf{E}_\nu) = \limsup_{\nu\to\infty} \Theta_{\mathcal{L}_\nu}(\mathbf{e}_\nu)$, and let each $\mathbf{E}_\nu$ satisfy $\mathbf{E}_\nu = \mathbf{E}_\nu^\dagger$. Write $\mathbf{E}_\nu = (E_{\nu,i} : i \in \mathcal{I})$. By replacing $\mathbf{E}_\nu$ by $D_{t_\nu}\mathbf{E}_\nu$ for an appropriately chosen sequence of parameters $t_\nu \in (0,\infty)$, we may assume that there exists $\rho > 0$ such that $[-\rho,\rho]^2 \subset E_{\nu,i}$ for each $i \in \mathcal{I}$.

By repeating the reasoning in the proof of Theorem 4.2 we conclude that after passing to a subsequence, there exists $\mathbf{E}^{0,\sharp} = (E_i^{0,\sharp} : i \in \mathcal{I})$, satisfying $\mathbf{E}^{0,\sharp} = (\mathbf{E}^{0,\sharp})^\dagger$, such that for each $i \in \mathcal{I}$, $E_{\nu,i}$ may be expressed as a disjoint union $E_{\nu,i}^\sharp \cup E_{\nu,i}^\flat$, satisfying $E_{\nu,i}^\sharp = (E_{\nu,i}^\sharp)^\dagger$ and $E_{\nu,i}^\flat = (E_{\nu,i}^\flat)^\dagger$, with $|E_{\nu,i}^\sharp \,\Delta\, E_{i,1}^0| \to 0$, and

$$\lim_{\nu\to\infty} \big(\Lambda_{\mathcal{L}_\nu}(\mathbf{E}_\nu^\sharp) + \Lambda_{\mathcal{L}_\nu}(\mathbf{E}_\nu^\flat)\big) = \lim_{\nu\to\infty} \Theta_{\mathcal{L}_\nu}(\mathbf{e}_\nu),$$

where $\mathbf{E}^\sharp_\nu = (E^\sharp_{\nu,i} : i \in \mathcal{I})$ and analogously for $\mathbf{E}^\flat_\nu$. The cross terms that arise in the proof of Theorem 4.2 contribute zero in the limit $\nu \to \infty$, because the bounds in Lemma 6.1 are uniform in ν since $\mathcal{L}_\nu \to \mathcal{L}^0$.

Since $|E^\sharp_{\nu,i} \,\Delta\, E^{0,\sharp}_i| \to 0$ and $\mathcal{L}_\nu \to \mathcal{L}^0$ as $\nu \to \infty$,

$$\lim_{\nu\to\infty} \Lambda_{\mathcal{L}_\nu}(\mathbf{E}^\sharp_\nu) = \Lambda_{\mathcal{L}^0}(\mathbf{E}^{0,\sharp}). \tag{8.3}$$

Therefore setting $\mathbf{e}_{\nu,1} = \mathbf{e}^\flat_\nu$ and $\mathbf{e}^0_1 = \mathbf{e}^{0,\sharp}$,

$$\limsup_{\nu\to\infty} \Theta_{\mathcal{L}_\nu}(\mathbf{e}_\nu) \le \Theta_{\mathcal{L}^0}(\mathbf{e}^0_1) + \limsup_{\nu\to\infty} \Theta_{\mathcal{L}_\nu}(\mathbf{e}^\flat_{\nu,1}) \tag{8.4}$$

and

$$\mathbf{e}^0 = \mathbf{e}^0_1 + \lim_{\nu\to\infty} \mathbf{e}_{\nu,1} \tag{8.5}$$

with addition and limits defined componentwise for 4–tuples. If $\lim_{\nu\to\infty} \Theta_{\mathcal{L}_\nu}(\mathbf{e}_{\nu,1}) = 0$ then the proof is complete.

Write $\mathbf{e}_{\nu,1} = (\mathbf{e}_{\nu,1,i} : i \in \mathcal{I})$. Observe that $\min_{i\in\mathcal{I}} |E^0_i| \ge 4\rho^2$ since $E_{\nu,i} \supset [-\rho,\rho]^2$ for every ν. Therefore for every sufficiently large ν, $e_{\nu,1,i} \le e^0_i - 3\rho^2$.

Pass from the full sequence to a subsequence of indices ν, along which $\limsup_{\nu\to\infty} \Theta_{\mathcal{L}_\nu}(\mathbf{e}_{\nu,1})$ is achieved in the limit. Apply the above construction to obtain a partition of $\mathbf{E}_{\nu,1}$ in terms of $\mathbf{E}^\sharp_{\nu,1}$ and $\mathbf{E}^\flat_{\nu,1}$ and a limiting set $\mathbf{E}^{0,\sharp}_2 = \mathbf{E}^0_2$. Conclude in the same way that

$$\lim_{\nu\to\infty} \Theta_{\mathcal{L}_\nu}(\mathbf{e}_\nu) \le \Theta_{\mathcal{L}^0}(\mathbf{e}^0_1) + \Theta_{\mathcal{L}^0}(\mathbf{e}^0_2) + \limsup_{\nu\to\infty} \Theta_{\mathcal{L}_\nu}(\mathbf{e}_{\nu,2}) \tag{8.6}$$

where $\mathbf{e}_{\nu,2} = \mathbf{e}^\flat_{\nu,1}$, with

$$\lim_{\nu\to\infty} \big(\mathbf{e}^0_1 + \mathbf{e}^2_0 + \mathbf{e}_{\nu,2}\big) = \mathbf{e}^0. \tag{8.7}$$

As in the initial step, each component $e^0_{2,i}$ of the tuple $\mathbf{e}^0_2$ is minorized by a positive quantity, which in turn is minorized by a positive function of $\lim_{\nu\to\infty} \Theta_{\mathcal{L}_\nu}(\mathbf{e}_{\nu,2}))$.

Iterate this process. It may halt after finitely many steps, in which case it produces a finite sequence $\mathbf{e}^0_k$ satisfying $\sum_k \mathbf{e}^0_k \le \mathbf{e}^0$ and $\limsup_{\nu\to\infty} \Theta_{\mathcal{L}^0}(\mathbf{e}_\nu) = \sum_k \Theta_{\mathcal{L}^0}(\mathbf{e}^0_k)$. Obviously $\sum_k \Theta_{\mathcal{L}^0}(\mathbf{e}^0_k) \le \Theta_{\mathcal{L}^0}(\mathbf{e}^0)$, completing the proof.

If the process fails to halt after finitely many steps then it produces infinite sequences $\mathbf{e}^0_k$ and $\mathbf{e}_{\nu,k}$. Necessarily

$$\lim_{k\to\infty} \limsup_{\nu\to\infty} \Theta_{\mathcal{L}_\nu}(\mathbf{e}_{\nu,k}) = 0. \tag{8.8}$$

Indeed, at each step there exists $\rho_k > 0$ for which $[-\rho_k,\rho_k]^2 \subset E^{0,\sharp}_i$ for each $i \in \mathcal{I}$, and ρ_k is bounded below by a strictly positive quantity if $\limsup_{\nu\to\infty} \Theta_{\mathcal{L}_\nu}(\mathbf{e}_{\nu,k})$ is bounded away from zero. If $\limsup_{\nu\to\infty} \Theta_{\mathcal{L}_\nu}(\mathbf{e}_{\nu,k})$ did not tend to zero then each component of $\mathbf{e}^0_k$ would be bounded away from zero uniformly in k, contradicting the relation $\sum_k \mathbf{e}^0_k \le \mathbf{e}^0$.

Therefore

$$\limsup_{\nu\to\infty} \Theta_{\mathcal{L}_\nu}(\mathbf{e}_\nu) \le \sum_{k=0}^{\infty} \Theta_{\mathcal{L}^0}(\mathbf{e}^0_k) \tag{8.9}$$

with $\sum_k \mathbf{e}^0_k \le \mathbf{e}^0$. This last inequality implies that $\sum_{k=0}^\infty \Theta_{\mathcal{L}^0}(\mathbf{e}^0_k) \le \Theta_{\mathcal{L}^0}(\mathbf{e}^0)$, once more completing the proof. □

Corollary 8.2. *Let $\mathbf{e}',\mathbf{e}''\in(0,\infty)^4$. Let $\mathcal{L}$ be nondegenerate and satisfy the partial symmetry hypothesis* (2.2)*. If $(\mathcal{L},\mathbf{e}'+\mathbf{e}'')$ is admissible then*

$$\Theta_{\mathcal{L}}(\mathbf{e}')+\Theta_{\mathcal{L}}(\mathbf{e}'')<\Theta_{\mathcal{L}}(\mathbf{e}'+\mathbf{e}''). \tag{8.10}$$

Proof. First suppose that $\mathbf{e}',\mathbf{e}''$ are admissible. Let $\mathbf{E}',\mathbf{E}''$ be symmetrized maximizing tuples satisfying $|\mathbf{E}'|=\mathbf{e}'$ and $|\mathbf{E}''|=\mathbf{e}''$. As in the proof of Theorem 4.2, there exists $c>0$ such that for any $\varepsilon>0$ there exists $\mathbf{v}\in\mathbb{R}^4$ such that the tuple $\mathbf{E}=(E_i'\cup(E_i''+\mathcal{L}_i(\mathbf{v})):i\in\mathcal{I})$ satisfies $\Lambda(\mathbf{E})\ge\Lambda(\mathbf{E}')+\Lambda(\mathbf{E}'')-\varepsilon$ and $|E_1'\cup(E_1''+\mathcal{L}_1(\mathbf{v}))|\le e_1'+e_1''-c$. Thus $\Theta(e_1'+e_1''-c,e_2'+e_2'',\dots)\ge\Theta(\mathbf{e}'+\mathbf{e}'')$, contradicting the admissibility of $\mathbf{e}'+\mathbf{e}''$. □

Lemma 8.3. *Let $(\mathcal{L}^0,\mathbf{e}^0)$ and $(\mathcal{L}_\nu,\mathbf{e}_\nu)$ satisfy the hypotheses of Lemma 8.1. Suppose that $(\mathcal{L}^0,\mathbf{e}^0)$ satisfies the full symmetry hypothesis of Definition 2.1, and is strictly admissible. Suppose that $|\mathbf{E}_\nu|=\mathbf{e}_\nu$, that each $\mathbf{E}_\nu$ is symmetrized, and that $\Lambda_{\mathcal{L}_\nu}(\mathbf{E}_\nu)\to\Theta_{\mathcal{L}^0}(\mathbf{e}^0)$ as $\nu\to\infty$. Then there exists a sequence t_ν such that*

$$|(D_{t_\nu}\mathbf{E}_\nu)\,\Delta\,\mathbf{E}^0|\to 0 \text{ as } \nu\to\infty. \tag{8.11}$$

Proof. Choose t_ν so that there exists $\rho>0$ for which $[-\rho,\rho]^2\subset D_{t_\nu}\mathbf{E}_\nu$ for all sufficiently large ν. Assume henceforth that ν is indeed large, and replace $\mathbf{E}_\nu$ by $D_{t_\nu}\mathbf{E}_\nu$.

Apply the proof of Theorem 4.2. The quantity $\mathbf{e}_1^0$ constructed in that proof must be equal to $\mathbf{e}^0$, for otherwise a contradiction would be reached immediately using Corollary 8.2. Therefore $\mathbf{e}_{\nu,1}\to 0$. □

While the two notions of strict admissibility on the one hand, and admissibility on the other, are defined somewhat differently, they are related. If $\mathcal{L}^0$ satisfies the full symmetry hypothesis of Definition 2.1, and if $(\mathcal{L}^0,\mathbf{e}^0)$ is strictly admissible, then $(\mathcal{L}^0,\mathbf{e}^0)$ is admissible in the sense of Definition 3.2.

Lemma 8.4. *Let $(\mathcal{L}^0,\mathbf{e}^0)$ be nondegenerate, satisfy the full symmetry hypothesis* (2.3)*, and be strictly admissible. Then $(\mathcal{L},\mathbf{e})$ is admissible whenever $\mathcal{L}$ satisfies the partial symmetry hypothesis* (2.2) *and $(\mathcal{L},\mathbf{e})$ is sufficiently close to $(\mathcal{L}^0,\mathbf{e}^0)$.*

Proof. Consider any sequence $(\mathcal{L}_\nu,\mathbf{e}_\nu)$ that satisfies the hypotheses and tends to $(\mathcal{L}^0,\mathbf{e}^0)$ as $\nu\to\infty$. There exists $\mathbf{e}_\nu'\le\mathbf{e}_\nu$ such that $(\mathcal{L}_\nu,\mathbf{e}_\nu')$ is admissible and satisfies $\Theta_{\mathcal{L}_\nu}(\mathbf{e}_\nu')=\Theta_{\mathcal{L}_\nu}(\mathbf{e}_\nu)$.

In this situation, $\mathbf{e}_\nu'\to\mathbf{e}^0$. Indeed, suppose instead that after passing to a subsequence, $\mathbf{e}_\nu'\to\mathbf{e}'<\mathbf{e}^0$. Each component of $\mathbf{e}'$ is strictly positive, since $|\Theta_{\mathcal{L}_\nu}(\mathbf{e}_\nu')|$ is uniformly minorized by a positive quantity, and is majorized by a constant multiple of the product of the two smallest components of $\mathbf{e}_\nu'$, uniformly in ν. By Lemma 8.1,

$$\Theta_{\mathcal{L}^0}(\mathbf{e})=\lim_{\nu\to\infty}\Theta_{\mathcal{L}_\nu}(\mathbf{e}_\nu')=\lim_{\nu\to\infty}\Theta_{\mathcal{L}_\nu}(\mathbf{e}_\nu)=\Theta_{\mathcal{L}^0}(\mathbf{e}^0).$$

This forces $\mathbf{e}'=\mathbf{e}^0$, that is, $\mathbf{e}_\nu'\to\mathbf{e}^0$. Indeed, the function $\tilde{\mathbf{e}}\mapsto\Theta_{\mathcal{L}^0}(\tilde{\mathbf{e}})$ is nondecreasing. Since $(\mathcal{L}^0,\mathbf{e}^0)$ is strictly admissible, this function is strictly increasing in a neighborhood of $\mathbf{e}^0$. That is, if $\mathbf{e}'<\mathbf{e}''$ and both are close to $\mathbf{e}^0$, then $\Theta_{\mathcal{L}^0}(\mathbf{e}')<\Theta_{\mathcal{L}^0}(\mathbf{e}'')$. Therefore if $\mathbf{e}'<\mathbf{e}^0$ then $\Theta_{\mathcal{L}^0}(\mathbf{e}')<\Theta_{\mathcal{L}^0}(\mathbf{e}^0)$, contradicting the conclusion of the preceding paragraph.

Let $\mathbf{E}_\nu$ satisfy $|\mathbf{E}_\nu|=\mathbf{e}_\nu'$, $\mathbf{E}_\nu=\mathbf{E}_\nu^\dagger$, and $\Lambda_{\mathcal{L}_\nu}(\mathbf{E}_\nu)=\Theta_{\mathcal{L}_\nu}(\mathbf{e}_\nu')=\Theta_{\mathcal{L}_\nu}(\mathbf{e}_\nu)$. By replacing $\mathbf{E}_\nu$ by $D_{t_\nu}(\mathbf{E}_\nu)$ for appropriate parameters $t_\nu\in(0,\infty)$ we may assume that the associated component sets satisfy $|E_{\nu,i}\,\Delta\,E_{\nu,i}^0|\to 0$ as $\nu\to\infty$ for each $i\in\mathcal{I}$.

It follows that the associated functions $K_i^0,K_{\nu,i}$ satisfy $K_{\nu,i}\to K_i^0$ in $C^0(\mathbb{R}^2)$ norm. A consequence of strict admissibility is that K_i^0 is bounded below by a strictly positive quantity in a neighborhood of the closure of E_i^0. Therefore $K_{\nu,i}$ is also bounded below by a strictly positive quantity in a subset of $\mathbb{R}^2 \setminus E_{\nu,i}$ whose Lebesgue measure is also bounded below by a strictly positive quantity, uniformly in ν. Write $\mathbf{e}'_\nu = (e'_{\nu,i} : i \in \mathcal{I})$. Let $r > e'_{\nu,i}$ be close to $e'_{\nu,i}$. Define $\tilde{\mathbf{e}} = (\tilde{e}_k : k \in \mathcal{I})$ by $\tilde{e}_j = e'_{\nu,j}$ for $j \neq i$ and $\tilde{e}_i = r$. Construct $\tilde{\mathbf{E}}$ satisfying $|\tilde{\mathbf{E}}| = \tilde{\mathbf{e}}$ with $\tilde{E}_j = E_{\nu,j}$ for $j \neq i$, $\tilde{E}_i \supset E_{\nu,i}$, and $K_{\nu,i}$ strictly positive on $\tilde{E}_i \setminus E_{\nu,i}$. Then

$$\Theta_{\mathcal{L}_\nu}(\tilde{\mathbf{e}}) \geq \Lambda_{\mathcal{L}_\nu}(\tilde{\mathbf{E}}) = \int_{\tilde{E}_i} K_{\nu,i} > \int_{E_{\nu,i}} K_{\nu,i} = \Lambda_{\mathcal{L}_\nu}(\mathbf{E}_\nu) = \Theta_{\mathcal{L}_\nu}(\mathbf{e}'_\nu).$$

This holds for every $r > e'_{\nu,i}$, for each $i \in \mathcal{I}$. Since $\mathbf{e}'_\nu < \mathbf{e}_\nu$, this forces $\Theta_{\mathcal{L}_\nu}(\mathbf{e}'_\nu) < \Theta_{\mathcal{L}_\nu}(\mathbf{e}_\nu)$, which is a contradiction. Therefore $(\mathcal{L}_\nu, \mathbf{e}_\nu)$ is admissible for every sufficiently large ν. □

9. Nonpositive first variation

If $\mathbf{E}$ maximizes $\Lambda_{\mathcal{L}}$ among tuples of sets with prescribed measures $\mathbf{e}$, then $\Lambda_{\mathcal{L}}(\mathbf{E}') \leq \Lambda_{\mathcal{L}}(\mathbf{E})$ whenever there exists $i \in \mathcal{I}$ for which $E'_j = E_j$ for every $j \neq i$ and $|E'_i| = |E_i|$. If $(\mathcal{L}, \mathbf{e})$ is admissible then this inequality has an interpretation that will be exploited in the proof of Theorem 4.3.

Definition 9.1. *E is a superlevel set of a function $K \geq 0$ if either there exist $\tilde{E}$ satisfying $|\tilde{E} \,\Delta\, E| = 0$ and $t > 0$ such that*

$$\{x : K(x) > t\} \subset \tilde{E} \subset \{x : K(x) \geq t\}, \tag{9.1}$$

or

$$\big|E \,\Delta\, \{x : K(x) > 0\}\big| = 0. \tag{9.2}$$

The following assertion is immediate.

Lemma 9.1. *Let $0 \leq K \in L^1(\mathbb{R}^d)$, and let $m \in (0, \infty)$. Assume that $|\{x : K(x) > 0\}| \geq m$. Let $E \subset \mathbb{R}^d$ satisfy $|E| = m$. Then $\int_E K = \sup_{|A|=m} \int_A K$ if and only if E is a superlevel set of K.*

Proposition 9.2. *Let $(\mathcal{L}, \mathbf{e})$ be admissible and nondegenerate. Suppose that $|\mathbf{E}| = \mathbf{e}$ and that $\mathbf{E}$ is a maximizer, that is, that $\Lambda_{\mathcal{L}}(\mathbf{E}) = \Theta_{\mathcal{L}}(\mathbf{e})$. For each $k \in \mathcal{I}$, let K_k be associated to $\mathcal{L}, \mathbf{E}$ as in* (3.3),(3.4)*. Then for each $i \in \mathcal{I}$, E_i is a superlevel set of K_i.*

Thus under the nondegeneracy and admissibility hypotheses, if $\mathbf{E}$ is a maximizer then the tuple $\mathbf{E}$ is a solution of a coupled system of free boundary problems.

Proof. The nondegeneracy hypothesis ensures that $K_i : \mathbb{R}^2 \to [0, \infty)$ is continuous, and that $K_i(u) \to 0$ as $|x| \to \infty$. Since $\mathbf{E}$ is a maximizing tuple, any tuple $\mathbf{E}'$ satisfying $E'_j = E_j$ for all $j \neq i$ and $|E'_i| = |E_i|$, satisfies $\Lambda(\mathbf{E}') \leq \Lambda(\mathbf{E})$. This conclusion can be equivalently reformulated as

$$\int_{E'_i} K_i \leq \int_{E_i} K_i \quad \text{for every set } E'_i \text{ satisfying } |E'_i| = |E_i|. \tag{9.3}$$

If $|\{u : K_i(u) > 0\}| < |E_i|$ then $\tilde{E}_i = E_i \cap \{u : K_i(u) > 0\}$ satisfies $|\tilde{E}_i| < |E_i|$ and $\int_{\tilde{E}_i} K_i = \int_{E_i} K_i$. Thus if $\tilde{E}_j = E_j$ for $j \neq i$ then $\Lambda(\tilde{\mathbf{E}}) = \Lambda(\mathbf{E})$ contradicting the admissibility of $(\mathcal{L}, \mathbf{e})$. Therefore $|\{u : K_i(u) > 0\}| \geq |E_i|$. It follows from Lemma 9.1 that E_i is a superlevel set of K_i. □

10. Convexity and regularity of maximizers

In this section we prove Theorem 4.3, concerning the qualitative properties of maximizing tuples, by exploiting the Euler-Lagrange relation of Proposition 9.2.

The set of all 4 tuples $\mathcal{L} = (L_j : j \in \mathcal{I})$ of linear mappings $L_j : \mathbb{R}^4 \to \mathbb{R}^2$ is a finite-dimensional vector space. Choose any norm $\|\cdot\|$ for this space, and fix it for the remainder of the analysis. This allows us to quantify the difference between $\mathcal{L}$ and $\mathcal{L}^0$ as $\|\mathcal{L} - \mathcal{L}^0\|$.

Throughout section 10, $\mathcal{L}^0$ is assumed to be nondegenerate and to satisfy the full symmetry hypothesis (2.3), and $(\mathcal{L}^0, \mathbf{e}^0)$ is assumed to be strictly admissible. It is assumed that $\mathcal{L}$ is nondegenerate and satisfies the partial symmetry hypothesis (2.2), that $(\mathcal{L}, \mathbf{e})$ is admissible, and that $(\mathcal{L}, \mathbf{e})$ is a small perturbation of $(\mathcal{L}^0, \mathbf{e}^0)$.

Let $\mathbf{E} = \{E_j : j \in \mathcal{I}\}$ be fixed and satisfy $|E_j| = e_j$. Let the functions K_i be defined in terms of the sets E_j by (3.3),(3.4). We will prove Theorem 4.3 via a bootstrapping argument, in which properties of $\{E_i : i \in \mathcal{I}\}$ are used to deduce information concerning $\{K_i : i \in \mathcal{I}\}$, leading in turn to stronger properties of $\{E_i : i \in \mathcal{I}\}$. Several iterations are required to reach desired conclusions. The most involved step is Lemma 10.3 below, which establishes smallness of $K_i - K_i^0$ in the Lipschitz norm, leading to the conclusion that E_i is a Lipschitz domain whose boundary is close to that of E_i^0 in the Lipschitz sense.

As was proved in [15], since $(\mathcal{L}^0, \mathbf{e}^0)$ is strictly admissible, if $\mathbf{E}^0$ satisfies $|\mathbf{E}^0| = \mathbf{e}^0$ then $\mathbf{E}^0$ maximizes Λ^0 if and only if $\mathbf{E}^0$ is a 4–tuple of homothetic ellipses, whose centers form a 4–tuple that belongs to the orbit of $(0,0,0,0)$ under the translation symmetry group. Among these tuples are those with sides parallel to the axes and with each E_j^0 centered at $0 \in \mathbb{R}^2$. These special maximizers of Λ^0 naturally play a distinguished role in the analysis.

According to Theorem 4.2, if $(\mathcal{L}^0, \mathbf{e})$ is strictly admissible, if $(\mathcal{L}, \mathbf{e})$ is admissible and $\mathcal{L}$ is sufficiently close to $\mathcal{L}^0$, and if $\mathbf{E}$ is a maximizer for $\Lambda_{\mathcal{L}}$ satisfying $\mathbf{E} = \mathbf{E}^\dagger$ and $|\mathbf{E}| = \mathbf{e}$, then there exists t such that $D_t\mathbf{E}$ is close to a special maximizer $\mathbf{E}^0$ of Λ^0, in the sense that

$$|(D_t E_j) \,\Delta\, E_j^0| < \varepsilon \text{ for each } j \in \mathcal{I},$$

where $\varepsilon \to 0$ as $\|\mathcal{L} - \mathcal{L}^0\| \to 0$. We will prove the conclusions of Theorem 4.3 for a dilate D_t of $\mathbf{E}$ chosen so that for each $j \in \mathcal{I}$, $|(D_t E_j) \,\Delta\, E_j^0| < \varepsilon$ where E_j^0 is a ball centered at the origin. Assume henceforth that $\mathbf{E}^0$ is a 4–tuple of balls centered at the origin.

By $o_\delta(1)$ we mean a quantity that depends on $\mathcal{L}^0$ and on $\mathbf{e}$, but tends to 0 as $\delta \to 0$ provided that $(\mathcal{L}^0, \mathbf{e})$ remains fixed, or more generally, remains inside a compact subset of the set of all $(\mathcal{L}^0, \mathbf{e})$ that satisfy our hypotheses.

Lemma 10.1. *Let $\mathcal{L}^0$ be nondegenerate and satisfy the full symmetry hypothesis of Definition 2.1. Let $(\mathcal{L}^0, \mathbf{e})$ be strictly admissible and generic. Let $\mathbf{E}^0$ be the 4–tuple of balls in $\mathbb{R}^2$ centered at the origin satisfying $|\mathbf{E}^0| = \mathbf{e}$. For each $i \in \mathcal{I}$ there exists a neighborhood of ∂E_i^0 in which K_i^0 is C^∞ and has nowhere vanishing gradient.*

Proof. Since E_j^0 are balls centered at the origin for all $j \neq i$, and since the diagonal action of $O(2)$ on $(\mathbb{R}^2)^4$ defines a symmetry of $\Lambda_{\mathcal{L}^0}$, K_i^0 is a radially symmetric function. Thus it suffices to analyze its gradient at the unique point in ∂E_i^0 of the form $(\bar{u}, 0)$ with $\bar{u} > 0$.

Consider u in a small neighborhood of $\bar{u}$. $K_i^0(u, 0)$ is the Lebesgue measure of

$$\bigcap_{i \neq j \in \mathcal{I}} \tilde{E}_j^0(u, 0) = \bigcap_{i \neq j \in \mathcal{I}} (\tilde{E}_j^0 + (\rho_j u, 0))$$

where the three real coefficients ρ_j are determined by the mappings $\ell_{j,i}$ defined in (3.4) and are pairwise distinct, and $\tilde{E}_j^0 = \tilde{E}_j^0(0,0)$ is a closed ball centered at the origin. Write

$\{j \in \mathcal{I} : j \neq i\}$ as $\{j,k,l\}$, with the indices labeled so that $\rho_j < \rho_l < \rho_k$. By making a u–dependent translation change of variables in the integral over $\mathbb{R}^2$ that defined $K_i^0(u,0)$ we may reduce to the case in which $\rho_l = 0$. Then $\tilde{E}_l^0(u,0) = \tilde{E}_l^0$ is a closed ball centered at the origin in $\mathbb{R}^2$.

According to the strict admissibility and genericity hypotheses, either $\tilde{E}_j^0(\bar{u},0) \cap \tilde{E}_k^0(\bar{u},0)$ is contained in the interior of $\tilde{E}_l^0$, or the threefold intersection of these three balls is a convex domain bounded by circular arcs of positive lengths that meet transversely, with 2 of these arcs being subarcs of $\tilde{E}_n^0(\bar{u},0)$ for each $n \in \{j,k,l\}$. In either case, it is an elementary consequence of the inequality $\rho_j < 0 = \rho_l < \rho_k$ that in a neighborhood of $\bar{u}$, the volume of the threefold intersection is a C^∞ function of u with strictly negative derivative.[3] □

The tuples $\mathbf{E} = (E_i : i \in \mathcal{I})$ depend on $(\mathcal{L}, \mathbf{e})$, but this dependence is not indicated in our notations.

Lemma 10.2. *Let $\delta_0 > 0$ be a sufficiently small constant, depending only on $\mathcal{L}^0, \mathbf{e}$. Let $\|\mathcal{L} - \mathcal{L}^0\| \le \delta \le \delta_0$. For each $i \in \mathcal{I}$, E_i is a bounded set whose diameter is bounded above, uniformly in $\mathcal{L}$. The Hausdorff distance from ∂E_i to ∂E_i^0 is $o_\delta(1)$.*

Proof. It follows directly from (3.4) and the nondegeneracy hypothesis that K_i is continuous, and that $K_i(u) \to 0$ as $|u| \to \infty$. The same holds for K_i^0. Moreover, $\|K_i\|_{C^0} \le C|E_k| \cdot |E_l|$ for any $k \neq l \in \mathcal{I} \setminus \{i\}$, and

$$\|K_i - K_i^0\|_{C^0} \le C \max_{j \neq i} |E_j \,\Delta\, E_j^0|$$

where $C < \infty$ depends on $\mathcal{L}^0, \mathbf{e}$ and on an upper bound for δ. Therefore $\|K_i - K_i^0\|_{C^0} \le o_\delta(1)$.

The strict admissibility hypothesis implies that each set E_i^0 is a superlevel set $\{x : K_i^0(x) \ge t_i > 0\}$ of K_i^0, and ∇K_i^0 vanishes nowhere on the boundary of E_i^0. Since $\|K_i - K_i^0\|_{C^0}$ is small, and since $\mathbf{E}$ is a maximizing tuple, for each $i \in \mathcal{I}$, $E_i \subset \{u : K_i(u) > t_i - o_\delta(1)\}$ provided that δ is sufficiently small. Therefore the diameters of the sets E_i are majorized by an acceptable constant.

The nonvanishing of ∇K_i^0 in a neighborhood of ∂E_i^0 and the smallness of $\|K_i - K_i^0\|_{C^0}$ together imply smallness of the Hausdorff distance from the boundary of E_i to the boundary of E_i^0. □

Lemma 10.3. *If $\delta > 0$ is sufficiently small then each function K_i is Lipschitz continuous, with Lipschitz constants uniformly bounded above, for all $\mathcal{L}$ satisfying $\|\mathcal{L} - \mathcal{L}^0\| \le \delta$.*

For any $i \in \mathcal{I}$, for each $j \neq i$ define $\tilde{E}_j$ to be the inverse image of E_j under the mapping $v \mapsto \ell_{j,i}(0,v)$. Define $\tilde{E}_j^0$ in the corresponding way, in terms of E_j^0 and $\mathcal{L}^0$. $\tilde{E}_j$ could more properly be denoted by $\tilde{E}_{j,i}$, but the simplified notation will be sufficiently unambiguous for our purpose.

Proof.

$$\begin{aligned} |K_i(u) - K_i(u')| &\le \int_{\mathbb{R}^2} \Big| \prod_{j \neq i} \mathbf{1}_{E_j}(\ell_{j,i}(u,v)) - \prod_{j \neq i} \mathbf{1}_{E_j}(\ell_{j,i}(u',v)) \Big| \, dv \\ &= \int_{\mathbb{R}^2} \Big| \prod_{j \neq i} \mathbf{1}_{E_j}(\ell_{j,i}((u-u'),v)) - \prod_{j \neq i} \mathbf{1}_{E_j}(\ell_{j,i}(0,v)) \Big| \, dv. \end{aligned}$$

[3] See also (10.3) below.

As a function of $v \in \mathbb{R}^2$, $\mathbf{1}_{E_j}(\ell_{j,i}(u-u',v))$ and $\mathbf{1}_{E_j}(\ell_{j,i}(0,v))$ are translates of $\tilde{E}_j$. Moreover, they are translates by quantities whose difference is a linear transformation of $u - u'$. Therefore it suffices to verify that $|(E_j+w)\,\Delta\, E_j| = O(|w|)$ for $w \in \mathbb{R}^2$. Moreover, it suffices to verify this merely for $w \in \mathbb{R}\times\{0\}$, and for $w \in \{0\}\times\mathbb{R}^2$. Since the horizontal and vertical coordinates can be freely interchanged in this theory, it suffices to treat $w = (z,0) \in \mathbb{R}\times\{0\}$. The bound $|(E_j + w)\,\Delta\, E_j| = O(|w|)$ is an immediate consequence of two properties of E_j: the intersection of E_j with any horizontal line in $\mathbb{R}^2$ is an interval, and the diameter of E_j is bounded above by an acceptable constant. □

Lemma 10.4. *For each $i \in \mathcal{I}$,*

$$\big|\,(K_i - K_i^0)(u) - (K_i - K_i^0)(u')\,\big| \le o_\delta(1)\cdot|u-u'| + O(|u-u'|^2)$$

for all u, u' sufficiently close to $\partial(E_i^0)$.

Proof. Fix any index $i \in \mathcal{I}$. For $u \in \mathbb{R}^2$ define

$$\Omega_i(u) = \{v \in \mathbb{R}^2 : \ell_{j,i}(u,v) \in E_j \ \forall j \ne i\} = \bigcap_{j\ne i} \tilde{E}_j(u).$$

Thus $K_i(u) = |\Omega_i(u)|$. Likewise define $\Omega_i^0(u)$ in terms of $\mathcal{L}^0$ and the sets $E_j^0, \tilde{E}_j^0$. For each u in a neighborhood of ∂E_i^0, $\Omega_i^0(u)$ is a bounded connected set, whose boundary is a union of two or four arcs of circles. The genericity hypothesis guarantees that these arcs meet transversely at any points of intersection, and that only two arcs meet at any such point. These are subarcs of translates of the boundaries of the balls $\tilde{E}_j^0$, respectively. The intersection of $\Omega_i(u)$ with any horizontal or vertical line is empty, or is an interval.

Each set $\tilde{E}_j$ is invariant with respect to reflection about both the horizontal and vertical axes. In the first quadrant, the boundary of $\tilde{E}_j$ is a rectifiable curve that can be parametrized by arclength as $s \mapsto (x(s), y(s))$ with

$$\dot{x}(s) \ge 0 \quad\text{and}\quad \dot{y}(s) \le 0. \tag{10.1}$$

This curve is contained in an $o_\delta(1)$–neighborhood of the boundary of $\tilde{E}_j^0$. The sets $\tilde{E}_j(u), \tilde{E}_j^0(\bar{u})$ are translates of $\tilde{E}_j, \tilde{E}_j^0$, respectively.

Let $\delta > 0$ be small, and let $\mathcal{L}$ satisfy $\|\mathcal{L} - \mathcal{L}^0\| \le \delta$. Consider any $i \in \mathcal{I}$, any $\bar{u}$ in an $o_\delta(1)$–neighborhood of $\partial\tilde{E}_j^0$, and any u near $\bar{u}$. Choose a collection of two or four disks $\{Q_\alpha\}$ in $\mathbb{R}^2$ whose radii tend to zero slowly as $\delta \to 0$, centered at the two or four intersection points of the arcs comprising the boundary of $\Omega_i^0(\bar{u})$. The boundary if $\Omega_i(u)$ then consists of two or four arcs of circles in the boundaries of the disks Q_α, together with two or four rectifiable curves, each of which has Hausdorff distance $o_\delta(1)$ to a subarc of one of the circular arcs comprising the boundary of $\Omega_i^0(\bar{u})$, is a translate of a subarc of the boundary of $\tilde{E}_j$ for some $j \in \mathcal{I}$, and has monotonicity properties thereby inherited from (10.1).

For u sufficiently close to $\bar{u}$, the boundaries $\partial\tilde{E}_k(u)$ enjoy the following two properties. For each $k \in \mathcal{I}$ and each α, if $\partial\tilde{E}_k(\bar{u})$ meets ∂Q_α then these intersect at a single point $z(k,\alpha,\bar{u})$. There exists a subarc $\Gamma(u)$ of $\partial\tilde{E}_k(u)$ of arclength $O(|u-\bar{u}|)$ such that the portion of $\partial\tilde{E}_k(u)$ not lying in $\Gamma(u)$ lies at distance $\ge C_0|u-\bar{u}|$ from the boundary of Q_α. Moreover, for all u sufficiently close to $\bar{u}$, $\partial\tilde{E}_k(u)$ meets ∂Q_α at a single point, and this point lies within distance $O(|u-\bar{u}|)$ of $z(k,\alpha,\bar{u})$. These properties are consequences of the monotonicity properties (10.1) of the boundaries of $\tilde{E}_j$ and the assumption that u takes the form $(u_1, 0)$.

For $u \in \mathbb{R}^2$ near $\bar u$, for sufficiently small δ,

$$K_i(u) = \int_{\Omega_i(u)} dv_1 \wedge dv_2 = \int_{\Omega_i(u)\setminus\cup_\alpha Q_\alpha} d(v_1\, dv_2) + \sum_\alpha \int_{Q_\alpha\cap\Omega_i(u)} dv_1 \wedge dv_2.$$

The same reasoning as in the proof of Lemma 10.3 shows that each term $\int_{Q_\alpha\cap\Omega_i(u)} dv_1 \wedge dv_2$ defines a locally Lipschitz function of u, whose Lipschitz norm is $o_\delta(1)$ because the Lebesgue measure of Q_α is $o_\delta(1)$. $K_i^0(u)$ can be analyzed in the same way, producing a corresponding term that is also Lipschitz with norm $o_\delta(1)$.

The main term for $K_i(u)$, $\int_{\Omega_i(u)\setminus\cup_\alpha Q_\alpha} d(v_1\, dv_2)$, can be rewritten via Stokes' theorem. What results is a sum of integrals of the one-form $\omega = v_1\, dv_2$ over finitely many rectifiable arcs $\gamma_\beta(u)$. Each of these arcs is either a subarc of the boundary of a single $\tilde E_k(u)$, or is a subarc of the boundary of some Q_α. Label these arcs so that for each index β, $\gamma_\beta(u)$ is close in the Hausdorff metric to each of $\gamma_\beta(\bar u)$, $\gamma^0_\beta(u)$, and $\gamma^0_\beta(\bar u)$, provided that δ and $|u-\bar u|$ are sufficiently small.

Consider the contribution of an arbitrary $\gamma(u) = \gamma_\beta(u)$ of the former type. Its contribution is $\int_{\gamma(u)} v_1\, dv_2$. We wish to compare this quantity to $\int_{\gamma(\bar u)} v_1\, dv_2$. $\gamma(u)$ is a subarc of the full boundary $\partial\tilde E_k(u)$ for some index $k \in \mathcal{I}\setminus\{i\}$. This full boundary may be expressed as a translate

$$\partial\tilde E_k(u) = \partial\tilde E_k(\bar u) + \ell^\sharp_{k,i}(u-\bar u)$$

for a certain linear mapping $\ell^\sharp_{k,i}: \mathbb{R}^2 \to \mathbb{R}^2$, which differs by $o_\delta(1)$ from the corresponding mapping $\ell^\sharp_{k,i}$ associated to $\mathcal{L}^0$.

Denote the first component of the $\mathbb{R}^2$–valued linear map $\ell^\sharp_{k,i}$ by $\tilde\ell_{k,i}$. The contribution of $\gamma(u)$ is

$$\begin{aligned}\int_{\gamma(u)} v_1\, dv_2 &= \int_{\gamma(u)-\ell^\sharp_{k,i}(u-\bar u)} (v_1 + \tilde\ell_{k,i}(u-\bar u))\, dv_2 \\ &= \int_{\gamma(\bar u)} (v_1 + \tilde\ell_{k,i}(u-\bar u))\, dv_2 + \mathcal{R}(u,\bar u)\end{aligned}$$

where the remainder $\mathcal{R}(u,\bar u)$ is expressed as an integral over the symmetric difference between $\gamma(\bar u)$ and $\gamma(u,\bar u) = \gamma(u) - \ell^\sharp_{k,i}(u-\bar u)$ of an integrand of the form $v_1\, dv_2 + O(|u-\bar u|)$.

Both $\gamma(\bar u)$ and $\gamma(\bar u, u)$ are subarcs of $\partial\tilde E_k(\bar u)$, so their symmetric difference is a union of two or fewer rectifiable arcs. We claim that each of these two or fewer arcs has diameter $O(|u-\bar u|)$. Indeed, if $z \in \partial\tilde E_k(\bar u)$ lies at distance $\ge C_0|u-\bar u|$ from the boundary of Q_α then $z + \ell^\sharp_{k,i}(u-\bar u) \in \partial\tilde E_k(u)$ shares this property with C_0 replaced by $C_0/2$, and conversely, provided that the constant C_0 is chosen to be sufficiently large. Thus the portion of $\gamma(\bar u)\,\Delta\,\gamma(u,\bar u)$ that lies within distance $o_\delta(1)$ of the boundary of Q_α lies entirely within distance $O(|u-\bar u|)$ of ∂Q_α. By the key property of $\partial\tilde E_k(\bar u)$ noted above, this establishes the claim.

The corresponding quantity associated to $K_i(\bar u)$ is simply $\int_{\gamma(\bar u)} v_1\, dv_2$. Subtracting this from $\int_{\gamma(u)} v_1\, dv_2$ yields

$$-\tilde\ell_{k,i}(u-\bar u)\int_{\gamma(\bar u)} dv_2 + \mathcal{R}(u,\bar u).$$

The factor $\int_{\gamma(\bar u)} dv_2$ is independent of u. Moreover, it is the integral of an exact one-form over the curve $\gamma(\bar u)$, and consequently depends only on the two endpoints of this curve.

Analyzing $K_i^0(u)-K_i^0(\bar u)$ in the same way results in a corresponding term, which depends in the same way on the two endpoints of the corresponding elliptical arc $\gamma(\bar u)$. Because the Hausdorff distance between ∂E_k and ∂E_k^0 is $o_\delta(1)$, the difference between these two contributions is therefore $O(|u-\bar u|)\cdot o_\delta(1)$.

To complete discussion of the contribution of γ, it remains to analyze $\mathcal{R}(u,\bar u)-\mathcal{R}^0(u,\bar u)$. Consider the two rectifiable curves that comprise the symmetric difference between $\gamma(\bar u)$ and $\gamma(\bar u,u)$. On each, the function v_1 may be expressed as a constant plus $O(|u-\bar u|)$. Terms that are $O(|u-\bar u|)$ produce contributions that are $O(|u-\bar u|^2)$ since the integrals here are taken over curves whose lengths are $O(|u-\bar u|)$. As above, the constant terms give rise to integrands which are constant multiples of dv_2. Subtracting the corresponding terms for K_i^0 and exploiting exactness of dv_2, we conclude that

$$|\mathcal{R}(u,\bar u)-\mathcal{R}^0(u,\bar u)|\le O(|u-\bar u|)o_\delta(1)+O(|u-\bar u|^2).$$

The analysis of subarcs of the boundaries of the disks Q_α is very slightly simpler, since these arcs are not translated. Their contributions are entirely of the type of the remainders $\mathcal{R}(u,\bar u)$. The same analysis as carried out for $\mathcal{R}(u,\bar u)$ above applies to them. □

Corollary 10.5. *For each $i\in\mathcal{I}$, E_i is a Lipschitz domain, uniformly for all $\mathcal{L}$ sufficiently close to $\mathcal{L}^0$.*

Proof. In a neighborhood of the boundary of E_i^0, K_i^0 is a C^∞ function with nowhere vanishing gradient. Since $\|K_i-K_i^0\|_{\text{Lip}}\le o_\delta(1)$, if δ is sufficiently small then $E_i=\{u:K_i(u)\ge t_i=t_i(\mathcal{L},\mathbf{e})\}$ is a Lipschitz domain whose boundary lies in an $o_\delta(1)$–neighborhood of the boundary of E_i^0. □

Conclusion of proof of Theorem 4.3. Continuing the discussion in the proof of Corollary 10.5, for any $u\in\partial E_i$, for any u' sufficiently near u, $u-u'$ lies in a cone of aperture $o_\delta(1)$ centered around a vector that is tangent to ∂E_i^0 at a point whose distance to u is $o_\delta(1)$. Therefore $\partial\Omega_i(u)$ is a Lipschitz domain whose boundary consists of finitely many Lipschitz arcs $\gamma_\beta(u)$, with each $\gamma_\beta(u)$ being a subarc of the boundary of a translate of $\tilde E_k$ for some $i\ne k\in\mathcal{I}$. Denote the endpoints of $\gamma_\beta(u)$ by $\mathbf{x}_\beta(u),\mathbf{x}'_\beta(u)$. At any intersection of these arcs, only two arcs meet, and any such intersection is transverse in the sense that tangent cones are separated by a positive angle.

Inserting this information into the proof of Lemma 10.4, the disks Q_α can now be dispensed with, yielding the representation

$$K_i(u)=\sum_\beta\int_{\gamma_\beta(u)} v_1\,dv_2 \tag{10.2}$$

for all u in some neighborhood of ∂E_i^0.

From this and the fact that $\gamma_\beta(u)$ is a subarc of a translate of the Lipschitz boundary of $\tilde E_k(\bar u)$ by a linear function of $u-\bar u$ we deduce that $K_i\in C^1$ in a neighborhood of ∂E_i with ∇K_i expressible as

$$\nabla K_i(u)=\sum_\beta\int_{\gamma_\beta(u)} w_{k(\beta)}(u-\bar u)\,dv_1=\sum_\beta w_{k(\beta)}(u-\bar u)\,(y_\beta(u)-y'_\beta(u)) \tag{10.3}$$

where $y_\beta,y'_\beta\in\mathbb{R}$ are the second coordinates of $\mathbf{x}_\beta,\mathbf{x}'_\beta\in\mathbb{R}^2$, respectively, and $w_k(u-\bar u)$ are $\mathbb{R}^2$–valued linear functions of $u-\bar u$ determined by $\mathcal{L}$ and the indices i,k, mediated by the function $(v_1,v_2)\mapsto v_1$. Therefore $|w_{k(\beta)}(u-\bar u)|=O(|u-\bar u|)$. The representation

(10.3) holds for C^1 boundaries, and follows for Lipschitz boundaries from the C^1 case by a limiting argument.

Therefore the superlevel set E_i of K_i is also a C^1 domain. It follows that the endpoints $\mathbf{x}_\beta(u), \mathbf{x}'_\beta(u)$ are themselves C^1 functions of u, since they are translates by affine functions of u of transversely intersecting C^1 arcs. Moreover, $\mathbf{x}_\beta(u) - \mathbf{x}^0_\beta(u)$ and its gradient with respect to u are uniformly $o_\delta(1)$. Likewise for $\mathbf{x}'_\beta(u)$.

Inserting this information into (10.3), we conclude that $K_i \in C^2$. Since $\mathbf{x}_\beta(u) - \mathbf{x}^0_\beta(u)$ and its gradient with respect to u are uniformly $o_\delta(1)$ and likewise for $\mathbf{x}'_\beta(u)$, it follows moreover that $\|K_i - K_i^0\|_{C^2} = 0_\delta(1)$ in a neighborhood of ∂E_i^0. Therefore E_i is strongly convex.

This reasoning can be iterated to conclude that $K_i \in C^\infty$. □

11. Nonexistence of maximizers

In this section we discuss a family of data that do not satisfy the partial symmetry hypothesis (2.2). In exceptional cases these data are reducible via simple skew-shift changes of variables to data that satisfy the full symmetry hypothesis. We show that for all other data of this special type, maximizers $\mathbf{E}$ fail to exist. Thus our partial symmetry hypothesis (2.2) is less artificial than it may appear to be.

Let $L_j^0(\mathbf{x},\mathbf{y}) = (L^0_{j,1}(\mathbf{x}), L^0_{j,2}(\mathbf{y}))$ with $L^0_{j,1} = L^0_{j,2}$, so that $\mathcal{L}^0 = (L^0_j : j \in \mathcal{I})$ satisfies the full symmetry hypothesis. Suppose that $\mathcal{L}^0$ is nondegenerate.

Let $\ell_j : \mathbb{R}^2 \to \mathbb{R}^1$ be linear, and consider $\mathcal{L} = (L_i : i \in \mathcal{I})$ with

$$L_j(\mathbf{x},\mathbf{y}) = (L_{j,1}(\mathbf{x}), L_{j,2}(\mathbf{x},\mathbf{y})) \tag{11.1}$$

of the form

$$L_{j,1} = L^0_{j,1} \text{ and } L_{j,2}(\mathbf{x},\mathbf{y}) = L^0_{j,2}(\mathbf{y}) + \ell_j(\mathbf{x}). \tag{11.2}$$

Define $\ell : \mathbb{R}^2 \to \mathbb{R}^4$ by $\ell(\mathbf{x}) = (\ell_i(\mathbf{x}) : i \in \mathcal{I})$ and $L^0_2 : \mathbb{R}^2 \to \mathbb{R}^4$ by $(L^0_{j,2}(\mathbf{y}) : y \in \mathcal{I})$.

Proposition 11.1 (Nonexistence of maximizers)**.** *Let $\mathcal{L}^0$ be nondegenerate. Let $(\mathcal{L}^0, \mathbf{e})$ be strictly admissible. If the range of ℓ is not contained in the range of L^0_2, then there exists no 4–tuple $\mathbf{E}$ satisfying $|\mathbf{E}| = \mathbf{e}$ and $\Lambda_{\mathcal{L}}(\mathbf{E}) = \Theta_{\mathcal{L}}(\mathbf{e})$.*

Thus maximizers can fail to exist for arbitrarily small perturbations $\mathcal{L}$ of $\mathrm{Sl}(d)$–invariant data $\mathcal{L}^0$.

We believe that this remains true if the full symmetry hypothesis is relaxed to (2.2), but the proof below utilizes a property of maximizers that has as yet been established under only the full symmetry hypothesis, or (as a corollary, using Theorem 4.3) for partially symmetric data that are sufficiently small perturbations of fully symmetric data.

The proof of Proposition 11.1 uses partial symmetrization. Recall the vertical symmetrizations $E^\sharp$ and $\mathbf{E}^\sharp$ introduced in (2.6), with $E^\sharp_x = [-\frac12|E_x|, \frac12|E_x|]$ if $|E_x| > 0$ where $E_x = \{y \in \mathbb{R}^1 : (x,y) \in E\}$. Recall also the dilations D_t introduced in 2.3.

Lemma 11.2. *For any* $\mathbf{E}$*,*

$$\Lambda_{\mathcal{L}}(\mathbf{E}) \le \Lambda_{\mathcal{L}^0}(\mathbf{E}^\sharp). \tag{11.3}$$

Proof. Consider

$$\Lambda_{\mathcal{L}}(\mathbf{E}) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\prod_{j\in\mathcal{I}} \mathbf{1}_{E_j}(L_j(\mathbf{x},\mathbf{y}))\,d\mathbf{y}\,d\mathbf{x}$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\prod_{j\in\mathcal{I}} \mathbf{1}_{E_{j,L_{j,1}(\mathbf{x})}}(L^0_{j,2}(\mathbf{y})+\ell_j(\mathbf{x}))\,d\mathbf{y}\,d\mathbf{x}$$

where $E_{j,z} = \{y : (z,y)\in E_j\}$. For each $\mathbf{x}\in\mathbb{R}^2$, apply the symmetrization inequality of Rogers-Brascamp-Lieb-Luttinger to the inner integral. No symmetry hypothesis comes into play, since each set $E_{j,z}$ is a subset of $\mathbb{R}^1$. Therefore

$$\Lambda_{\mathcal{L}}(\mathbf{E}) \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\prod_{j\in\mathcal{I}} \mathbf{1}_{E^\star_{j,L_{j,1}(\mathbf{x})}}(L^0_{j,2}(\mathbf{y}))\,d\mathbf{y}\,d\mathbf{x}$$

where $E^\star_{j,z}\subset\mathbb{R}^1$ is the usual symmetrization of $E_{j,z}$; it is the closed interval centered at $0\in\mathbb{R}^1$ whose Lebesgue measure equals that of $E_{j,z}$ if this measure is strictly positive, and is empty otherwise. The right-hand side of this last inequality is equal to $\Lambda_{\mathcal{L}^0}(\mathbf{E}^\sharp)$. □

This proof of (11.3) yields supplementary information that is essential to our purpose: If $\mathbf{E}=\mathbf{E}^\sharp$, then $\Lambda_{\mathcal{L}}(\mathbf{E}) = \Lambda_{\mathcal{L}^0}(\mathbf{E})$ if and only if for almost every $\mathbf{x}\in\mathbb{R}^2$,

$$\int_{\mathbb{R}^2}\prod_{j\in\mathcal{I}} \mathbf{1}_{E_{j,L_{j,1}(\mathbf{x})}}(L^0_{j,2}(\mathbf{y})+\ell_j(\mathbf{x}))\,d\mathbf{y} = \int_{\mathbb{R}^2}\prod_{j\in\mathcal{I}} \mathbf{1}_{E_{j,L_{j,1}(\mathbf{x})}}(L^0_{j,2}(\mathbf{y}))\,d\mathbf{y}. \tag{11.4}$$

Upon taking the supremum over all $\mathbf{E}$ satisfying $|\mathbf{E}|=\mathbf{e}$, we conclude from (11.3) that $\Theta_{\mathcal{L}}(\mathbf{e})\le\Theta_{\mathcal{L}^0}(\mathbf{e})$ for any $\mathbf{e}$. More is true:

Proposition 11.3. *Let $\ell:\mathbb{R}^2\to\mathbb{R}^4$ be an arbitrary linear map. Let $\mathcal{L},\mathcal{L}^0$ be as described above. For any $\mathbf{e}\in(0,\infty)^4$,*

$$\Theta_{\mathcal{L}}(\mathbf{e}) = \Theta_{\mathcal{L}^0}(\mathbf{e}). \tag{11.5}$$

Proof. By (11.3), it suffices to show that for any 4–tuple satisfying $\mathbf{E}=\mathbf{E}^\sharp$,

$$\Lambda_{\mathcal{L}}(D_t\mathbf{E}) \to \Lambda_{\mathcal{L}^0}(\mathbf{E}) \ \text{ as } t\to 0. \tag{11.6}$$

To evaluate this limit write

$$\Lambda_{\mathcal{L}}(D_t\mathbf{E}) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\prod_{j\in\mathcal{I}} \mathbf{1}_{D_tE_j}\big(L^0_{j,1}(\mathbf{x}), L^0_{j,2}(\mathbf{y})+\ell_j(\mathbf{x})\big)\,d\mathbf{y}\,d\mathbf{x}$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\prod_{j\in\mathcal{I}} \mathbf{1}_{E_j}\big(t^{-1}L^0_{j,1}(\mathbf{x}), tL^0_{j,2}(\mathbf{y})+t\ell_j(\mathbf{x})\big)\,d\mathbf{y}\,d\mathbf{x}$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\prod_{j\in\mathcal{I}} \mathbf{1}_{E_j}\big(L^0_{j,1}(\mathbf{x}), L^0_{j,2}(\mathbf{y})+t^2\ell_j(\mathbf{x})\big)\,d\mathbf{y}\,d\mathbf{x}$$

by substituting $\mathbf{x}=t\,\mathbf{u}$ and $\mathbf{y}=t^{-1}\,\mathbf{v}$ and then replacing $(\mathbf{u},\mathbf{v})$ by $(\mathbf{x},\mathbf{y})$ to obtain the last line.

For any interval $I\subset\mathbb{R}$ centered at the origin $|I\,\Delta\,(I+t)|\le|t|$. The conclusion (11.5) follows by applying this bound together with routine majorizations to the expression for $\Lambda_{\mathcal{L}}(D_t\mathbf{E})$ in the final line of the chain of identities in the preceding paragraph. □

Proof of Proposition 11.1. Suppose that $\mathbf{E}$ were a maximizer for $\Lambda_{\mathcal{L}}$. Then $\mathbf{E}^\sharp$ is also a maximizer for $\Lambda_{\mathcal{L}}$, by (11.3). Moreover, by (11.5), $\mathbf{E}^\sharp$ is a maximizer for $\Lambda_{\mathcal{L}^0}$. So consider any common maximizer for $\Lambda_{\mathcal{L}}$ and for $\Lambda_{\mathcal{L}^0}$ that satisfies $\mathbf{E} = \mathbf{E}^\sharp$.

Arbitrary maximizers for $\Lambda_{\mathcal{L}^0}$ have been characterized for nondegenerate strictly admissible $(\mathcal{L}^0, \mathbf{e})$ [15]. They have the property that there exists $\eta > 0$, depending on $\mathbf{E}$, such that for all $\mathbf{x} \in \mathbb{R}^2$ satisfying $|\mathbf{x}| \le \eta$, the 4–tuple $(|E_{i,L_{i,1}(\mathbf{x})}| : i \in \mathcal{I})$ of measures of associated one-dimensional sets is strictly admissible for the lower-dimensional form

$$\int_{\mathbb{R}^2} \prod_{j\in\mathcal{I}} \mathbf{1}_{F_j}(L^0_{j,2}(\mathbf{y}))\, d\mathbf{y}, \tag{11.7}$$

where $(F_j : j \in \mathcal{I})$ represents a 4–tuple of subsets of $\mathbb{R}^1$.

Maximizers of (11.7) have been characterized [12] under hypotheses of nondegeneracy, strict admissibility, and genericity. These hypotheses are satisfied by $(L^0_{j,2} : j \in \mathcal{I})$ and $\mathbf{e}$. We conclude that for any $\mathbf{x}$ for which $(|E_{i,L_{i,1}(\mathbf{x})}| : i \in \mathcal{I})$ is a strictly admissible 4–tuple, the vector $\ell(\mathbf{x}) = (\ell_j(\mathbf{x}) : j \in \mathcal{I}) \in \mathbb{R}^4$ takes the form $(L^0_{j,2}(\mathbf{u}) : j \in \mathcal{I})$ for some $\mathbf{u} \in \mathbb{R}^2$; that is, $\ell(\mathbf{x})$ belongs to the range of L^0_2. Since ℓ, L^0_2 are linear mappings, the range of ℓ is contained in the range of L^0_2, as claimed.

Conversely, if the range of $\ell = (\ell_i : i \in \mathcal{I})$ is contained in the range of $L^0_2 = (L^0_{2,i} : i \in \mathcal{I})$, then the theory developed in this paper for data $\mathcal{L}$ satisfying the partial symmetry hypothesis (2.2) can be applied to $\Lambda_{\mathcal{L}}$ after a simple change of variables. It is not necessary to assume that $\mathcal{L}^0$ satisfies the full symmetry hypothesis (2.3). Indeed the hypothesis of inclusion of the ranges implies that there exists a linear mapping $h : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $\ell_i = L^0_{2,i} \circ h$ for every $i \in \mathcal{I}$. Thus

$$\Lambda_{\mathcal{L}}(\mathbf{E}) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \prod_{j\in\mathcal{I}} \mathbf{1}_{E_{j,L_{j,1}(\mathbf{x})}}(L^0_{j,2}(\mathbf{y} + h(\mathbf{x})))\, d\mathbf{y}\, d\mathbf{x}.$$

The linear change of variables $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, \mathbf{y} + h(\mathbf{x}))$ in $\mathbb{R}^4$ transforms this double integral to

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \prod_{j\in\mathcal{I}} \mathbf{1}_{E_{j,L_{j,1}(\mathbf{x})}}(L^0_{j,2}(\mathbf{y}))\, d\mathbf{y}\, d\mathbf{x} = \Lambda_{\mathcal{L}^0}(\mathbf{E}).$$

□

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Michael Christ, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA
E-mail address: `mchrist@berkeley.edu`

Dominique Maldague, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA
E-mail address: `dmal@math.berkeley.edu`