Annals of Finance
https://doi.org/10.1007/s10436-022-00415-w

RESEARCH ARTICLE

Delta-hedging in fractional volatility models

Qi Zhao[1] · Alexandra Chronopoulou[2]

Received: 7 July 2020 / Accepted: 4 October 2022

Abstract
In this paper, we propose a delta-hedging strategy for a long memory stochastic volatility model (LMSV). This is a model in which the volatility is driven by a fractional Ornstein–Uhlenbeck process with long-memory parameter H. We compute the so-called hedging bias, i.e. the difference between the Black–Scholes Delta and the LMSV Delta as a function of H, and we determine when a European-type option is over-hedged or under-hedged.

Keywords Long-memory · Stochastic volatility · Hedging · Hedging bias

JEL Classification C02 · C32 · C65 · G12

1 Introduction

It has been well documented in the literature that the celebrated Black–Scholes model does not explain stylized facts related to the volatility, such as the volatility smile or the volatility persistence, which led to the introduction of stochastic volatility models, Heston (1993), Fouque et al. (2000a, b). In recent works, Chronopoulou and Viens (2010), Comte et al. (2012), Lima (1994), Breidt et al. (1998), there has been empirical evidence that the volatility is highly persistent, which means that even for options with long maturity, there exist pronounced smile effects. Furthermore, a unit root behavior of the conditional variance process is prominent, particularly when the data are of higher frequencies. On the other hand, when maturities are shorter and the data are

Research of A.C. is supported in part by the National Science Foundation DMS 1811859.

Alexandra Chronopoulou
achronop@illinois.edu

Qi Zhao
qiz2@illinois.edu

1 Department of Industrial & Enterprise Systems Engineering, University of Illinois at Urbana-Champaign, Urbana, USA

2 Department of Statistics, University of Illinois at Urbana-Champaign, Champaign, USA

Published online: 09 November 2022

(ultra) high-frequent, the volatility displays an antiperistent or rough behavior that cannot be captured by traditional stochastic volatility models either, Gatheral et al. (2018), Carr and Wu (2016), Fukasawa (2017).

Long memory, or long-range dependence, in financial datasets has been observed in practice long before the use of long memory stochastic volatility models. For example, the authors in Ding et al. (1993), Lima (1994), Breidt et al. (1998) observed that the squared returns of market indexes have the long-memory property, which intuitively means that observations that are far apart are highly correlated. First, a discrete time model was introduced, Harvey (1998), Breidt et al. (1998), under which the log-volatility was modeled as a fractional ARIMA(p,d,q) process (Beran et al. 2013), while Comte and Renault in Comte and Renault (1998), *first* introduced a stochastic volatility model with long-memory in continuous time. More recent works (Chronopoulou and Viens 2010, 2012; Comte et al. 2012; Bezborodov et al. 2019) have also explored the effect of volatility persistence in option pricing.

On the other hand, rough volatility models have a prominent role in the current literature initiated by the work in Gatheral et al. (2018). Specifically, the log-volatility is modeled by a rough (fractional) process where the implied volatility under this model, for very short maturities, is shown to produce very strong skews. Along the same lines, the authors in Bayer et al. (2016) investigate pricing under such models, while the authors in Fukasawa (2017) study the case of small volatility fluctuations.

In this article, we focus on the long-range dependent case and specifically, we work with the *continous time long-memory stochastic volatility (LMSV)* model introduced in Comte and Renault (1998): If S_t is the price process and Y_t is the volatility process, then

$$\begin{cases} dS_t = r\ S_t\ dt + \sigma(Y_t) S_t\ dB_t, \\ dY_t = -\lambda\ Y_t\ dt + \beta\ dW_t^H, \end{cases} \tag{1}$$

where B_t is a standard Brownian motion and W_t^H is a fractional Brownian motion with Hurst index $H \in (1/2, 1]$. The fractional Brownian motion is a process that captures volatility persistence, when $H > 1/2$ and roughness when $H < 1/2$. (See also Sect. 2 for the model definition).

The question of pricing derivatives under the long memory stochastic volatility models has already been addressed in the literature. First, the authors in Comte and Renault (1998) provide a representation for the option price, while the authors in Garnier and Sølna (2017) derive a corrected Black–Scholes formula in a fractional stochastic volatility environment. In Chronopoulou and Viens (2010), Chronopoulou and Viens (2012) a numerical method based on recombining quadrinomial trees is developed to compute European and American-type option prices.

Our goal in this article is to propose a hedging strategy for the long memory stochastic volatility model, when the Hurst parameter is greater than 1/2. It is well known that perfect hedging cannot be achieved in the stochastic volatility framework and this remains the case when the volatility exhibits long memory. However, one can study an imperfect (partial) delta-hedging strategy and its implications in a fractional environment.

In order to do so, first we need to establish the differentiability of the option price with respect to the asset and then compute the derivative with respect to the fluctuations of the underlying, a.k.a. the option's Delta. We also investigate the so-called hedging bias, as introduced in Renault and Touzi (1996). In particular, we look at the distance of the Delta under the long memory volatility model from the Delta under the Black–Scholes model and we determine if our position is under/over-hedged when an (imperfect) Delta hedging strategy is adopted. To this extent, we show that the Black–Scholes implied volatility-hedging bias leads to (i) an under-hedged position for the in-the-money options (in the sense that the Black–Scholes hedging ratio is smaller than the long memory stochastic volatility one), and (ii) an over-hedged poistion for out-of-the money options. Moving one step further, we also investigate the Delta's behavior, when the volatility process is a slow fractional Ornstein–Uhlenbeck process for which we derive a "corrected" Black–Scholes Delta formula for the European call option.

The structure of the paper is as follows: In Sect. 2, we present the general framework and provide a quick overview of the model we consider along with its main properties. In Sect. 3 we introduce the option pricing framework, the proof of differentiability of the option price and the derivation of the hedging biases. In Sect. 4, we derive the approximate formula for the corrected Delta. In Sect. 5, we discuss the practical implications of our method and along with our conclusions. Technical proofs and Lemmas are delegated in the Appendix.

2 Long-memory stochastic volatility model

In this section, we present all the mathematical ingredients required to properly define our model. Under an equivalent martingale measure

$$dS_t = r\ S_t\ dt + \sigma(Y_t) S_t\ dB_t, \tag{2}$$

$$dY_t = -\lambda\ Y_t\ dt + \beta\ dW_t^H, \tag{3}$$

where r is the short-term risk-free rate of interest, $\{B_t; t \geq 0\}$ is a standard Wiener process and $\{W_t^H; t \geq 0\}$ is a standard fractional Brownian motion with $H > 1/2$, both under the martingale measure. We assume that B and W^H are *independent*, although this assumption could be relaxed, in order to account for leverage effects, which will be the topic of future investigations.

2.1 Fractional Brownian motion

Before discussing the properties of the fractional SDE (3), we first introduce some of the properties of the fractional Brownian motion, which is the driving noise of the volatility process.

Definition 1 A fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1]$ is a centered Gaussian process $\{B_t^H; t \in \mathbb{R}_+\}$ whose distribution is defined by its covariance

$$Cov\left(W_t^H, W_s^H\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right),\ t, s \in \mathbb{R}_+,$$

and the fact that its paths are continuous with probability 1.

The covariance of fBm immediately implies that it has H-self-similar increments; for every $c > 0$ the processes $\{W_{ct}^H; t \in \mathbb{R}_+\}$ and $\{c^H W_t^H; t \in \mathbb{R}_+\}$ have the same distribution. The mean square of the increments of fBm computes as

$$\mathbf{E}\left(|W_t^H - W_s^H|^2\right) = |t-s|^{2H}, \tag{4}$$

which directly indicates that the increments are stationary. For $H = \frac{1}{2}$, the process is the standard Brownian motion. However, contrary to standard Brownian motion, fBm is not a semimartingale nor a Markov process when $H \neq \frac{1}{2}$.

The fBm has one additional very important property for certain values of H: its long-range dependence (a.k.a. long-memory). Indeed, when $H \neq \frac{1}{2}$, the increments of fBm over disjoint intervals, $(W_n^H - W_{n-1}^H)$, are not independent; their correlation function is

$$\rho_H(n) = \frac{1}{2}\Big((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\Big).$$

We observe that when $H < \frac{1}{2}$ then $\rho_H(n) < 0$ and the increments over disjoint intervals are negatively correlated. When $H > \frac{1}{2}$ then $\rho_H(n) > 0$ and the increments over disjoint intervals are positively correlated. More specifically in the case that $H > \frac{1}{2}$ the stationary sequence $(W_n^H - W_{n-1}^H)$ exhibits *long-range dependence* (or *long memory*) in the sense that $\sum_{n=1}^{\infty} \rho_H(n) = \infty$, which follows immediately from the asymptotics

$$\rho_H(n) = H(2H-1)n^{2H-2} + o\left(n^{2H-2}\right).$$

When $H < \frac{1}{2}$, then $\sum_{n=1}^{\infty} |\rho_H(n)| < \infty$ and one often says that the process has short memory, although it might be preferable to call it “medium” memory, since exponentially decaying correlations might better describe “short” memory.

The fractional Brownian motion with $H > 1/2$ also admits an integral representation with respect to a standard Brownian motion which is:

$$W_t^H = \int_0^t K_H(t,s)dZ_s. \tag{5}$$

where

$$K_H(t,s) = c_H s^{1/2-H} \int_s^t |u-s|^{H-3/2} u^{H-1/2} du$$

and $\{Z_t\}$ is a standard Brownian motion. More details on fBm can be found in Mishura (2008), Nualart (2006).

2.2 Fractional Ornstein–Uhlenbeck process

The fractional Ornstein–Uhlenbeck process is the fractional analogue of the well-known Ornstein–Uhlenbeck process; it is a continuous-time first-order autoregressive process $Y = \{Y_t; t \geq 0\}$ which is the solution of an one-dimensional homogeneous linear stochastic differential equation driven by a fBm $\{W_t^H; t \in \mathbb{R}_+\}$ with Hurst parameter $H \in [1/2, 1)$. Specifically, it is the unique Gaussian process satisfying the following linear stochastic integral equation

$$Y_t = -\lambda \int_0^t Y_s \, ds + \beta \, W_t^H, \tag{6}$$

where $\lambda > 0$ and β are constant drift and variance parameters, respectively. The solution to equation (6) is

$$Y_t = \exp(-\lambda t) \left(Y_0 + \beta \int_0^t exp(\lambda u) d W_u^H \right)$$

and is almost surely continuous and H-self-similar. The decay of the autocovariance function of $\{Y_t; t \in \mathbb{R}_+\}$ is similar to that of the increments of the fBm, and thus it exhibits long-range dependence. See Cheridito et al. (2003) for more details. In our work for simplicity we take $Y_0 = 0$.

In the remainder of the paper, we will make the following assumptions regarding the function $\sigma(\cdot)$ in (2):

Assumption 1 (i) $\sigma(\cdot)$ is bounded above by a positive number A. This assumption may be further relaxed later.

(ii) For some large enough N, $\sigma(\cdot) \geq |x|^{m/2} e^{\frac{-x^2}{4\Sigma^2}}$ for some $m > 1$, up to a constant for $x \leq -N$. Here $\Sigma^2 = \sup_{t \in [0,T]} var(\int_t^T Y_s ds)$.

(iii) $\sigma(\cdot)$ is sub-linear, that is $\sigma(x_1) + \sigma(x_2) \geq \sigma(x_1 + x_2)$.

3 Delta hedging and hedging bias

3.1 The fractional option pricing formula

To start our discussion, we will review the results about pricing under fractional stochastic volatility model in Comte and Renault (1998). For notational convenience, in this section we denote $\sigma_t = \sigma(Y_t)$.

Let $(\Omega, \mathcal{F}, P)$ be the fundamental probability space and $\{\mathcal{F}_t\}$ the P-augmented filtration generated by the two Brownian motions $\{B_u, B_u^*\}_{u \leq t}$. We focus on a European call option on a given financial asset $\{S_t; t \geq 0\}$, with payoff $\max\{S_T - K, 0\}$ where K

is the strike price, and T the maturity. The filtration $\{\mathcal{F}_t\}$ coincides with the filtration generated by the stock and the volatility process, $\{S_u, \sigma_u\}_{u\leq t}$ or $\{m_u, \sigma_u\}_{u\leq t}$, where we define $m_t = \ln(S_t/K)$. We assume that the market permits continuous and frictionless trading, but is in equilibrium in the sense that no arbitrage profits are available from trading the underlying asset and riskless bonds. We also assume that the interest rate r is constant, although this can be relaxed to incorporate a deterministic, time-dependent process r_t. A zero coupon bond with maturity T is denoted by $D(t, T) = \exp\{-r(T-t)\}$.

In the remainder of the paper, as in Comte and Renault (1998) we will work under the equivalent martingale measure (as it is presented in Sect. 2). For a price process given by

$$dS_t = \mu(t, S_t)\ S_t\ dt + \sigma(Y_t)S_t\ dB_t,$$

the existence of an equivalent martingale measure Q has been established in Comte and Renault (1998). The density process of any probability measure Q equivalent to P and can be written as

$$M_t = \exp\left\{-\int_0^t \lambda_u^S dB_u - \frac{1}{2}\int_0^t \lambda_u^{S^2} du - \int_0^t \lambda_u^\sigma dB_u^* - \frac{1}{2}\int_0^t \lambda_u^{\sigma^2} du\right\},$$

where $\lambda_u^S, \lambda_u^\sigma$ are adapted to the filtration $\mathcal{F}_t$ and standard integrability conditions $\int_0^T \lambda_u^{S^2} du < \infty\ a.s.$ and $\int_0^T \lambda_u^{\sigma^2} du < \infty\ a.s.$. The discounted asset price process $\{S_t D_t, 0 \leq t \leq T\}$ is a Q martingale if and only if

$$\lambda_u^S \sigma_t = \mu(t, S_t) - r.$$

Since the stock is the only traded asset, λ_u^σ is not fixed and as the market is incomplete, the martingale probability Q is not unique. So, for any choice of λ_u^σ, the density process M_t is an equivalent martingale measure. We also have that

$$\tilde{B}_t = B_t + \int_0^T \lambda_u^S du \quad \text{and} \quad \tilde{B^*}_t = B_t^* + \int_0^T \lambda_u^\sigma du$$

where $\tilde{B}_t$ and $\tilde{B^*}_t$ are independent under Q by construction. Therefore, the European Call option price can be expressed as

$$C_t = D(t, T)E^{Q_{\lambda^\sigma}}\left[\max\{S_T - K, 0\}|\mathcal{F}_t\right]$$

In the remainder of the paper, we fix λ^σ and as a result Q_{λ^σ}. However, for simplicity in the notation, we will only denote it by Q.

Since the two noises B and W^H are independent, conditionally on $\mathcal{F}_t$ and the volatility path $\{\sigma_t : t \in [t, T]\}$, the distribution of $\log S_t$ is Normal with mean $\left(r(T-t) - \frac{1}{2}\int_t^T \sigma_s^2 ds\right)$ and variance $\int_t^T \sigma_s^2 ds$. Therefore, based on the results in Comte and Renault (1998), we can compute the European option price by first conditioning on the larger sigma-algebra $\widehat{\mathcal{F}}_t := \mathcal{F}_t \vee \{\sigma_u;\ u \in [t, T]\}$ and then narrowing down the sigma-algebra to $\mathcal{F}_t$ to obtain

$$C_t = S_t \left\{ E^Q \left[\Phi \left(\frac{m_t}{U_{t,T}} + \frac{U_{t,T}}{2} \right) | \mathcal{F}_t \right] - e^{-m_t} E^Q \left[\Phi \left(\frac{m_t}{U_{t,T}} - \frac{U_{t,T}}{2} \right) | \mathcal{F}_t \right] \right\}, \tag{7}$$

where $\Phi(\cdot)$ denotes the cdf of standard normal distribution, and $U_{t,T}$ is defined as

$$U_{t,T} = \sqrt{\int_t^T \sigma_u^2 du}.$$

3.2 Price differentiability with respect to S_t

To investigate whether the option price is differentiable with respect to the underlying asset we use Fourier transform techniques, similar to El Euch and Rosenbaum (2018), and we prove the following theorem:

Theorem 1 *Under Assumption 1, the European Call Option price defined as in (7), C_t, is differentiable with respect to S_t.*

Proof First, assume that we can find an $a > 1$ such that $E^Q(S_t)^a < \infty$. For the existence of this a, we can take $a = 2$. For the reason please refer to Proposition 3 in Appendix.

Then the European Call option price is

$$C_t = E^Q \left[(S_T - K)_+ | \mathcal{F}_t \right]$$

Let $x_t = \ln S_t$, and define

$$g(x) = e^{-ax}(e^x - K)_+.$$

Then, we know that $g \in L^1(R) \cap L^2(R)$. Therefore it has a Fourier transform according to

$$g(x) = \frac{1}{2\pi} \int \hat{g}(b) e^{-ibx} db$$

where for $\hat{g}$ we have:

$$\hat{g}(b) = \frac{\exp\{(1 - a + ib) log(K)\}}{(ib - a)(ib - a - 1)}$$

Thus we know that

$$C_t = E^Q \left[g(x_T) e^{a x_T} | \mathcal{F}_t \right] = \frac{1}{2\pi} \int \hat{g}(b) E^Q [e^{(a-ib)x_T} | \mathcal{F}_t]$$

Thus the only thing we need to show is that the term

$$E^Q\left[e^{(a-ib)(x_T-x_t)}|\mathcal{F}_t\right]$$

only depends on the volatility. We already know that the stock process S_t is a semi-martingale, and we have that

$$x_T - x_t = \int_t^T \left(r - \frac{1}{2}(\sigma(Y_s))^2\right) ds + \int_t^T \sigma(Y_s) dW_s.$$

If we plug-in to the price C_t, we obtain

$$\begin{aligned}
C_t &= \frac{1}{2\pi}\int \hat{g}(b) E^Q\left[\exp\left\{(a-ib)\,x_T\right\}|\mathcal{F}_t\right] db \\
&= \frac{1}{2\pi}\int \hat{g}(b)\exp\left\{(a-ib)\,x_t\right\} E^Q\left[\exp\left\{(a-ib)(x_T-x_t\right\}|\mathcal{F}_t\right] db \\
&= \frac{1}{2\pi}\int \hat{g}(b)\exp\left\{(a-ib)\,x_t\right\} \\
&\quad E^Q\left[\exp\left\{(a-ib)\int_t^T\left(r-\frac{1}{2}\sigma^2(Y_s)\right)ds + \int_t^T \sigma(Y_s)dW_s\right\}\middle|\mathcal{F}_t\right] db
\end{aligned}$$

We can use the fact that the conditional distribution of $x_T - x_t$ under $\mathcal{F}_t$ and the path is a Normal distribution with $\mu = r(T-t) - 1/2 \cdot V$ and variance $\Sigma^2 = V$, where $V = \int_t^T \sigma(Y_s)^2 ds$. Thus, we have:

$$\begin{aligned}
&E^Q\left[\exp\ \left\{(a-ib)\,(x_T-x_t)\right\}\middle|\mathcal{F}_t\right] \\
&\qquad = E^Q\left[\int_R e^{(a-ib)x}\exp\left\{\frac{-(x-\mu)^2}{2\Sigma^2}\right\}\cdot\frac{1}{\sqrt{2\pi}\,\Sigma}dx\middle|\mathcal{F}_t\right]
\end{aligned}$$

which is the characteristic function of $x_T - x_t$. Thus, we have

$$\begin{aligned}
&E^Q\left[\exp\left\{(a-ib)\,(x_T-x_t)\right\}\middle|\mathcal{F}_t\right] \\
&= E^Q\left[\int_R e^{(-ib)x}\exp\left\{\frac{-(x-\mu-a\Sigma^2)^2}{2\Sigma^2} + a\mu + \frac{1}{2}a^2\Sigma^2\right\}\cdot\frac{1}{\sqrt{2\pi}\,\Sigma}dx\middle|\mathcal{F}_t\right]
\end{aligned}$$

Using the characteristic function of a Normally distributed variable, we can re-write the above equation as

$$\begin{aligned}
&E^Q\left[\exp\left\{(a-ib)\,(x_T-x_t)\right\}\middle|\mathcal{F}_t\right] \\
&\qquad = E^Q\left[\exp\left\{a\mu + \frac{1}{2}a^2\Sigma^2\right\} exp\left\{-imb - \frac{1}{2}b^2\Sigma^2\right\}\middle|\mathcal{F}_t\right],
\end{aligned}$$

Where $m = \mu + a\Sigma^2$. Finally, we can write the option price as

$$\begin{aligned} C_t =& \frac{1}{2\pi} \int \hat{g}(b) \exp\{(a - ib)x_t\} \\ & E^Q \left[\exp\left\{a\mu + \frac{1}{2}a^2\Sigma^2\right\} \exp\left\{-imb - \frac{1}{2}b^2\Sigma^2\right\} \middle| \mathcal{F}_t \right] db \end{aligned}$$

Combining all the above, we have

$$\begin{aligned} C_t =& \frac{1}{2\pi} \int \hat{g}(b) \exp\{(a - ib)\log(S_t)\} \\ & E^Q \left[\exp\left\{a\mu + \frac{1}{2}a^2 V\right\} \exp\left\{-imb - \frac{1}{2}b^2 V\right\} \middle| \mathcal{F}_t \right] db, \end{aligned}$$

where $V = \int_0^{T-t} \sigma^2_{(s+t)} ds$. Thus what remains to be checked is whether C_t is differentiable with respect to S_t, and this involves whether it is legitimate to take derivative under the integral. In fact, what we can see here is that if we first take the derivative with S_t inside the integral, we will have the following:

$$\begin{aligned} l(b) = & \hat{g}(b) \frac{ib}{s} \exp\{(a - ib)\log(S_t)\} \\ & \cdot E^Q \left[\exp\left\{a\mu + \frac{1}{2}a^2 V\right\} \exp\left\{-imb - \frac{1}{2}b^2 V\right\} \middle| \mathcal{F}_t \right] \end{aligned}$$

Using the inequality

$$\log(b) - \frac{1}{2}b^2 V \leq \log\left(\frac{1}{\sqrt{V}}\right)$$

we have that

$$E^Q \left[\exp\left\{a\mu + \frac{1}{2}a^2 V\right\} b \cdot \exp\left\{-\frac{1}{2}b^2 V\right\} |\mathcal{F}_t \right] \leq E \left[\exp\left\{a\mu + \frac{1}{2}a^2 V\right\} \frac{1}{\sqrt{V}} |\mathcal{F}_t \right],$$

if we assume that $\frac{1}{\sqrt{V}}$ is square integrable. Then, by an application of the dominated convergence theorem it follows that $\int_R l(b) db$ and as a consequence C is differentiable with respect to S and the derivative can be taken under the integral sign. What is left to prove is the integrability of $\frac{1}{V}$, which is obtained by Lemma 1 in the Appendix. □

3.3 Delta-hedging and implied volatilities

In the case of a standard stochastic volatility model, a natural idea to solve the hedging problem is to follow a delta-sigma hedging strategy with Δ_t^* units of the underlying asset and Σ_t^* units of another option with price $C_t^{(2)}$. Under certain conditions, it has been proven that there exist unique Δ_t^* and Σ_t^* quantities that solve the delta-sigma hedging problem. However, even in this classical framework, it is common for practitioners to mainly focus on the risk associated with the underlying asset fluctuations and consider an imperfect hedging strategy with $\Sigma_t^* = 0$ and

$$\Delta_t = \frac{\partial C_t}{\partial S_t}.$$

According to our discussion in the previous section, we know that under the long memory stochastic volatility model, C_t is differentiable with respect to m_t (and S_t as a consequence), so we can write

$$\Delta_t = \frac{\partial C_t}{\partial S_t} = E^Q \left[\Phi \left(\frac{m_t}{U_{t,T}} + \frac{U_{t,T}}{2} \right) \Big| \mathcal{F}_t \right]$$

On the other hand, if one assumes that the underlying pricing model is the Black–Scholes with constant volatility σ (i.e. $U_{t,T} = \sigma\sqrt{T-t}$), then the corresponding Black–Scholes Delta computes as

$$\Delta_t^{BS} = \frac{\partial C_t}{\partial S_t} = E^Q \left[\Phi \left(\frac{m_t}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2} \right) \Big| \mathcal{F}_t \right]$$

In order to use the Black–Scholes Delta in practice, we need to calculate the implied volatility, that is the value of σ that is obtained by calibrating the Black–Scholes price to realized option prices [see for example Fouque et al. (2000a)], in which the Black–Scholes model is the one that is assumed to be true. Following the ideas of Renault and Touzi (1996), we assume that the Black–Scholes impled volatility is the unique solution to

$$C_t = C_t^{BS} \left(S_t, \sigma_t^i \right),$$

where C_t is the option price under the long-memory stochastic volatility model.

A second notion of implied volatility is the so-called hedging volatility, first introduced in Renault and Touzi (1996), according to which the implied parameter σ_t^h is the volatility that satisfies the following equation

$$\Delta_t = \Delta_t^{BS} \left(S_t, \sigma_t^h \right),$$

which is the volatility parameter that equates the Black–Scholes hedging ratio against the underlying asset price variations to the long memory stochastic volatility one. The hedging implied volatility, σ_t^h computes as

$$\sigma_t^h = \frac{1}{\sqrt{T-t}} \left\{ \left[\Phi^{-1} E^Q \left[\left(\frac{m_t}{U_{t,T}} + \frac{U_{t,T}}{2} \right) \Big| \mathcal{F}_t \right] \right] \right.$$

$$+\sqrt{\left[\Phi^{-1}E^{Q}\left[\left(\frac{m_t}{U_{t,T}}+\frac{U_{t,T}}{2}\right)\Big|\mathcal{F}_t\right]\right]^2-2m_t}\right\}$$

In contrast to the Black–Scholes implied volatility, the hedging volatility is not observed, and it is approximated by the Black–Scholes implied volatility.

3.4 Hedging bias

Following Renault and Touzi (1996), we define hedging bias to be the difference between the Black–Scholes implied volatility-based hedging ratio and the long memory volatility one, i.e.

$$\Delta_t^{BS}(S_t,\sigma)-\Delta_t.$$

In order to quantify the difference of the two volatilities, similar to Renault and Touzi (1996), we have the following theorem:

Theorem 2 *For the long-memory stochastic volatility model* (1), *the following inequality always holds:*

$$\sigma_t^h \geq \sigma_t^i$$

when $m_t \geq 0$.

The proof of this theorem will be the same following Renault and Touzi (1996). And we will list the proof in the appendix.

With Theorem 2, we can say that the implied parameters are different for in-the-money and out-of the money options. The main result for the hedging bias follows immediately from the Theorem and is summarized in the Corollary below.

Corollary 1 *When we have* $\ln\left(\frac{S_t}{KD(t,T)}\right)\geq 0$, *we will always have the following:*

$$\Delta_t \geq \Delta_t^{BS}\left(S_t,\sigma_t^i\right).$$

When $\ln\left(\frac{S_t}{KD(t,T)}\right)<0$,

$$\Delta_t \leq \Delta_t^{BS}\left(S_t,\sigma_t^i\right).$$

When $\ln\left(\frac{S_t}{KD(t,T)}\right)=0$,

$$\Delta_t = \Delta_t^{BS}\left(S_t,\sigma_t^i\right).$$

Proof Same as Comte and Renault (1998), the inequalities follow directly from Theorem 3 and from the fact that the function $\sigma(\cdot)$ is increasing. ◻

The implications of Corollary 1 are that for in-the-money options the use of the Black–Scholes implicit volatility leads to an under-hedged position, in the sense that the hedging ratio is smaller than the partial hedging ratio, while for out-of-the money options the use of the Black–Scholes implicit volatility leads to an over-hedged position. In the case of at-the-money options, the use of the Black–Scholes volatility leads to a perfect partial hedging.

4 Correction to the Black–Scholes delta

In this section, we will consider a special case of our model (1). Specifically, we impose the following conditions:

Assumption 2 (i) The volatility is having the form $\sigma(Y_t) = \bar{\sigma} + F(\delta Y_t)$, where $\bar{\sigma}$ is a constant and Y_t the volatility process as before.
(ii) The function F not only satisfies *Assumption 1*, but also $F(0) = 0$ and $F'(0) = 1$.

Our goal in this section is to derive a correction to the Black–Scholes Delta formula when δ tends to 0. In order to do, we need first an appropriate expansion for the option price. Extending the results in Garnier and Sølna (2017) in the case of Ornstein–Uhlenbeck processes with a drift component, we obtain the following proposition:

Proposition 1 *When δ is small, we have*

$$C_t = Q_t(S_t) + \mathcal{O}(\delta^2),$$

where $Q_t(S_t)$ have the following form:

$$Q_t(S_t) = Q(S_t, t) + \delta\bar{\sigma}\ \phi_t \left(S_t^2 \frac{\partial^2 Q(S_t, t)}{\partial S_t^2} \right)$$

with $Q(S_t, t)$ being the Black–Scholes formula with the constant volatility $\bar{\sigma}$, and $\phi_t = E[\int_t^T Y_s ds \big| \mathcal{F}_t]$.

Proof The proof of this proposition is along the lines of Proposition 3.1 in Garnier and Sølna (2017), if we establish that $\phi_t = E[\int_t^T Y_s ds \big| \mathcal{F}_t]$ is a continuous semimartingale. The proof of the latter is delegated in the Appendix in Lemma 3. □

Since we have established a first order correction to the option price, we can also obtain a first order correction to the option's Delta:

Proposition 2 *When δ is small, we have*

$$\Delta_t(S_t) = \frac{\partial Q_t(S_t)}{\partial S_t} + \mathcal{O}(\delta^2)$$

where

$$\frac{\partial Q_t(S_t)}{\partial S_t} = \frac{\partial Q(S_t,t)}{\partial S_t} + \delta\bar{\sigma}\phi_t\left(2S_t\frac{\partial^2 Q(S_t,t)}{\partial S_t^2} + S_t^2\frac{\partial^3 Q(S_t,t)}{\partial S_t^3}\right)$$

Proof The differentiability of $Q_t(S_t)$ is obtained from the results in Sect. 3. Once we have that the result follows by direct computations. □

5 Discussion and conclusion

5.1 Practical implications

In Sect. 3 of this article we quantified the difference between the Black–Scholes Delta and the Delta under the long memory stochastic volatility model, and we obtained guidelines on whether the option is over-hedged or under-hedged when we believe that the stock behaves according to a fractional model and a Black–Scholes Delta is used for partial hedging.

However, in order to derive the expression for the Delta, we conditioned with respect to the entire volatility path. Therefore, in order to use this method in practice, we need to estimate or filter the underlying volatility process. In our case the underlying process exhibits long-memory which makes the filtering task harder. Furthermore, since we conditioned on the entire volatility path, not only we need to estimate the volatility up to the current time t, but also to predict the volatility values for future times $t < s \leq T$.

Therefore, we are going to divide the problem into two time periods: (i) when $s < t$ and (ii) when $t < s \leq T$, where t is considered the be the current time. The question of filtering long-range dependent processes with only discrete observations available has been studied in the literature. For example, one can use particle filtering techniques as in Chronopoulou and Viens (2010), a sequential Monte Carlo approach as in Chronopoulou and Spiliopoulos (2018), or Bayesian techniques, similar to those developed in Beskos et al. (2015). All these methods are directly tracking the volatility process up to time t. Alternatively, one can use an indirect approach, by first discretizing the continuous time model, secondly applying a discrete semimartingale transformation, Brouste and Kleptsyna (2012), and then apply traditional filtering methods before inverting the transformation.

Once the volatility has been tracked up to time t, then we can make use of the fact that our volatility fractional SDE has an explicit solution, (6), which we can use to estimate the future volatility values.

5.2 Conclusion

In this article, we discussed Delta hedging for a long-memory stochastic volatility model. Using Fourier techniques, we proved that the option price under such a model is differentiable with respect to the stock price, and hence we were able to derive an expression for the option's Delta. Then, we quantified the hedging bias, that is the difference between the Black–Scholes Delta and the Delta under the long memory model and we determined that our position is under-hedged for in-the-money options and over-hedged for out-of-the money options, while the partial Delta hedging is perfect for at-the-money options. Finally, we considered a special case for the volatility process for which we derived a corrected formula to the Black–Scholes Delta.

Funding Funding was provided by Directorate for Mathematical and Physical Sciences (Grant Number 1811859).

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

Proposition 3 *For Eq. (1), if $E[(S_0)^2] < \infty$, there exists a strong solution, which is a semimartingale, and also Ito's lemma can be applied. Also $E[(S_t)^2] < \infty, \quad \forall t \le T$*

Proof The proof will be mainly following the general method of proving the existence of strong solutions of stochastic differential equations. And we will refer to Theorem 8.3 in Le Gall (2016) for reference. Assume that the function $\sigma(\cdot)$ is bounded by constant A, i.e. $|\sigma(\cdot)| \le A$. Then we construct the solution using Picard approximation series. For any $t \in [0, T]$ and for any positive integer n,

$$\begin{aligned} S_t^0 &= S_0 \\ S_t^n &= S_0 + \textstyle\int_0^t \sigma(Y_s) S_s^{n-1} dW_s + \int_0^t r S_s^{n-1} ds \end{aligned}$$

We need to know that (i) S_t^0 is continuous, (ii) the iteration is valid for every step and (iii) S_t^n is continuous for every n. We focus on the proof of (ii). For this a necessary and sufficient condition is

$$E\left[\int_0^T (\sigma(Y_s)S_s^n)^2 ds\right] < \infty, \ \forall n \ge 0.$$

Since $|\sigma(\cdot)| \le A$, the above statement can be deduced by

$$E\left[\int_0^T (S_s^n)^2 ds\right] < \infty, \ \forall n \ge 0.$$

Observe that

$$E\left[\int_0^T (S_s^n)^2 ds\right] \le T E\left[\sup_{0 \le s \le T} (S_s^n)^2\right].$$

For $E[\sup_{0\le s\le T}(S_s^n)^2]$, we know

$$E\left[\sup_{0\le s\le T}(S_s^n)^2\right]\le 2E\left[\sup_{0\le s\le T}\left[\left(\int_0^s\sigma(Y_l)S_l^{n-1}dW_l\right)^2+\left(\int_0^s rS_l^{n-1}dl\right)^2\right]\right]$$

Observe that

$$\begin{aligned}&E\left[\sup_{0\le s\le T}\left[\left(\int_0^s\sigma(Y_l)S_l^{n-1}dW_l\right)^2+\left(\int_0^s rS_l^{n-1}dl\right)^2\right]\right]\\&\quad\le E\left[\sup_{0\le s\le T}\left(\int_0^s\sigma(Y_l)S_l^{n-1}dW_l\right)^2\right]+E\left[\sup_{0\le s\le T}\left(\int_0^s rS_l^{n-1}dl\right)^2\right].\end{aligned}$$

For the second part, we have the following inequality:

$$E\left[\sup_{0\le s\le T}\left(\int_0^s rS_l^{n-1}dl\right)^2\right]\le TE\left[\sup_{0\le s\le T}\int_0^s(rS_l^{n-1})^2dl\right]\le TE\left[\int_0^T(rS_l^{n-1})^2dl\right]$$

For first part, denote $F_s=\int_0^s\sigma(Y_l)S_l^{n-1}dW_l$. This is a continuous martingale if $E[\int_0^T(\sigma(Y_s)S_s^{n-1})^2ds]<\infty$, which is implied by $E[\int_0^T(S_l^{n-1})^2dl]<\infty$. Then, if F_s is a continuous semi-martingale, we can use the Burkholder–Davis–Gundy inequality, to obtain

$$E\left[\sup_{0\le s\le T}(F_s)^2\right]\le C_2E\left[\langle F_s,F_s\rangle_T\right]=C_2E\left[\int_0^T(\sigma(Y_s)S_s^{n-1})^2ds\right]$$

Since $\sigma(Y_s)\le A$, we have $E[\sup_{0\le s\le T}(F_s)^2]<\infty$ to be deduced by

$$E[\int_0^T(S_s^{n-1})^2ds]<\infty.$$

Thus, we obtain $E[\int_0^T(S_s^n)^2ds]<\infty$ if $E[\int_0^T(S_s^{n-1})^2ds]<\infty$. Since we know that $E[\int_0^T(S_s^0)^2ds]=TE[(S_0)^2]<\infty$, by induction we can show that
$E[\int_0^T(\sigma(Y_s)S_s^n)^2ds]<\infty$, $\forall n\ge 0$. Hence we can prove that for every step the iteration is valid.

Last, (iii) holds since it is the sum of a finite variation process and a continuous martingale.
Now, for every $n\ge 1$, take $g_n(t)=E[\sup_{0\le s\le t}(S_s^n-S_s^{n-1})^2]$. Then we will have

$$g_{n+1}(t)\le 2E[\sup_{0\le s\le t}|\int_0^s\sigma(Y_l)(S_l^n-S_l^{n-1})dW_l|^2+\sup_{0\le s\le t}|\int_0^s r(S_l^n-S_l^{n-1})dl|^2].$$

Applying Burkholder–Davis–Gundy inequality, we have

$$E\left[\sup_{0\le s\le t}|\int_0^s \sigma(Y_l)(S_l^n - S_l^{n-1})dW_l|^2\right] \le C_2 E\left[\int_0^t (\sigma(Y_l)S_l^n - S_l^{n-1})^2 dl\right].$$

which leads to

$$g_{n+1}(t) \le 2(C_2A^2 + Tr^2)E[\int_0^t (S_l^n - S_l^{n-1})^2 dl] \le C\int_0^t g_n(l)dl$$

Since $g_1(t)$ is bounded by a constant C_1 on $[0, T]$, we know that $g_n(t) \le C_1C^{n-1}\dfrac{t^{n-1}}{(n-1)!}$. Then, since square root is a concave function, we have

$$\sqrt{g_n(t)} \ge E\left[\sqrt{\sup_{0\le s\le t}(S_s^n - S_s^{n-1})^2}\right] = E\left[\sup_{0\le s\le t}|S_s^n - S_s^{n-1}|\right],$$

which implies

$$E\left[\sum_{n=1}^{\infty}\sup_{0\le s\le T}|S_s^n - S_s^{n-1}|\right] < \infty,$$

$$\sum_{n=1}^{\infty}\sup_{0\le s\le T}|S_s^n - S_s^{n-1}| < \infty \ \ a.s$$

and $S_s^n \to S_s$ uniformly almost surely. Then S_s has continuous sample path. Finally, it is easy to check that S_s is a solution and satisfies

$$dS_t = rS_t dt + \sigma(Y_t)S_t dB_t.$$

The square integrability of S_t is easy to derive. We know that

$$S_t^2 = exp\left[2\left(\int_0^t\left(r - \frac{1}{2}\sigma^2(Y_s)\right)ds + \int_0^t \sigma(Y_s)dB_s\right)\right]$$

And thus $E(S_t^2) < \infty$ is guaranteed by $E\left(exp\left[2\int_0^t \sigma(Y_s)dB_s\right]\right) < \infty$, which is then guaranteed by applying Novikov's criterion to

$$E\left(exp\left[8\int_0^T \sigma^2(Y_s)ds\right]\right) < \infty$$

□

Lemma 1 *With the assumption that for some large enough N,* $\sigma(\cdot) \geq |x|^{m/2} e^{\frac{-x^2}{4\Sigma^2}}$ *for some* $m > 1$*, up to a constant for* $x \leq -N$*,* $\frac{1}{\sqrt{V}}$ *will be square integrable. Here* $\Sigma^2 = \sup_{t\in[0,T]} var(\int_t^T Y_s ds)$.

Proof First from the sublinearity of σ, we know that

$$V \geq \left(\int_t^T \sigma(Y_s) ds\right)^2 \Big/ (T-t) \geq \sigma^2 \left(\int_t^T Y_s ds\right) \Big/ (T-t)$$

This implies that $\frac{1}{V} \leq \frac{T-t}{\sigma^2(\int_t^T Y_s ds)}$. Since Y_s is a Gaussian Process, $Z_t = \int_t^T Y_s ds$ is a Normally distributed random variable, thus we will have that

$$E\left(\frac{1}{V}\right) \leq \int_{-\infty}^{\infty} \frac{T-t}{\sigma^2(x)} \exp\left\{\frac{-(x-\mu)^2}{2\Sigma_V^2}\right\} dx,$$

where $\Sigma_V^2 = var(Z_t) \leq \Sigma^2$. Thus

$$\frac{1}{\sigma^2(x)} exp\left(\frac{-(x-\mu)^2}{2\Sigma_V^2}\right) \leq |x|^{-m}$$

for $x \leq -N$, which concludes that $\frac{1}{\sqrt{V}}$ is square integrable. □

Proof of Theorem 2 The proof will be the same following the proof of proposition 4.1 in Renault and Touzi (1996). First, define the quantity H_t,

$$H_t = \frac{C_t}{S_t} = \Delta_t^{BS}\left(m_t, \sigma_t^h\right) - e^{-x}\left[1 - \Delta_t^{BS}\left(-m_t, \sigma_t^h\right)\right]$$

and let

$$h(v) = e^{m_t}\left(H_t - H^{BS}(m_t, v)\right)$$

be a strictly decreasing function (easily checked by computing the derivative), with $H^{BS}(x, v)$ defined as:

$$H^{BS}(m_t, v) = \Phi\left(\frac{m_t}{v\sqrt{T-t}} + \frac{v\sqrt{T-t}}{2}\right) - e^{-m_t}\Phi\left(\frac{m_t}{v\sqrt{T-t}} - \frac{v\sqrt{T-t}}{2}\right)$$

Combining all the above, we have

$$\begin{aligned} h(v) = e^{m_t}&\left[\Delta_t^{BS}\left(m_t, \sigma_t^h\right) - \Delta_t^{BS}(m_t, v)\right] \\ &+ \left[\Delta_t^{BS}\left(-m_t, \sigma_t^h\right) - \Delta_t^{BS}(-m_t, v)\right] \end{aligned}$$

Since $h(v)$ is decreasing with $h(\sigma_t^i) = 0$, we can deduce the relationship between σ_t^h and σ_t^i is determined by the sign of $h(\sigma_t^h)$, which is given by the sign of

$$\Phi^{-1} E^Q[\Phi\left(\frac{-m_t}{U_{t,T}} + \frac{U_{t,T}}{2}\right)|\mathcal{F}_t] - \sqrt{[\Phi^{-1} E^Q[\Phi\left(\frac{m_t}{U_{t,T}} + \frac{U_{t,T}}{2}\right)|\mathcal{F}_t]]^2 - 2m_t}.$$

From Lemma 2 in the Appendix, we know that the latter is less than or equal to 0. This implies that when $m_t \geq 0$, $h(\sigma_t^h) \leq 0$, which together with the monotonicity of $h(\cdot)$ results in

$$\sigma_t^h \geq \sigma_t^i, \quad \text{when } m_t \geq 0$$

□

Lemma 2

$$\Phi^{-1} E^Q\left[\Phi\left(\frac{-m_t}{U_{t,T}} + \frac{U_{t,T}}{2}\right)\Big|\mathcal{F}_t\right] - \sqrt{\left[\Phi^{-1} E^Q\left[\Phi\left(\frac{m_t}{U_{t,T}} + \frac{U_{t,T}}{2}\right)\Big|\mathcal{F}_t\right]\right]^2 - 2m_t} \leq 0$$

when $m_t \geq 0$.

Proof The proof of this lemma will also be the same following the proof of proposition 4.1 in Renault and Touzi (1996). We first prove the case of $U_{t,T}$ being a constant, i.e $U_{t,T} = C \ a.s$. Then it is obvious that

$$E^Q\left[\Phi\left(\frac{-m_t}{U_{t,T}} + \frac{U_{t,T}}{2}\right)\Big|\mathcal{F}_t\right] = \Phi\left(\frac{-m_t}{U_{t,T}} + \frac{U_{t,T}}{2}\right).$$

Then, everything follows from the fact that

$$\left(\frac{-m_t}{U_{t,T}} + \frac{U_{t,T}}{2}\right)^2 \leq \left(\frac{m_t}{U_{t,T}} + \frac{U_{t,T}}{2}\right)^2 - 2m_t$$

Let $U = \sum_{i=1}^n a_i 1_{A_i}$ be a step function, with $a_i \geq 0$ and $\bigcup A_i = \Omega$. Then, we have

$$E^Q\left[\Phi\left(\frac{-m_t}{U_{t,T}} + \frac{U_{t,T}}{2}\right)\Big|\mathcal{F}_t\right] = \sum_{i=1}^n \Phi\left(\frac{-m_t}{a_i} + \frac{a_i}{2}\right) E^Q\left[1_{A_i}|\mathcal{F}_t\right]$$
$$E^Q\left[\Phi\left(\frac{m_t}{U_{t,T}} + \frac{U_{t,T}}{2}\right)\Big|\mathcal{F}_t\right] = \sum_{i=1}^n \Phi\left(\frac{m_t}{a_i} + \frac{a_i}{2}\right) E^Q\left[1_{A_i}|\mathcal{F}_t\right]$$

Since $\sum_{i=1}^n E^Q[1_{A_i}|\mathcal{F}_t] = 1 \ a.s$, following the same steps as in Renault and Touzi (1996), we obtain the desired result.

For general square integrable positive random variable U, we can use the density argument by using a sequence of step random variable increasingly approximating U, also with the boundedness of Φ, we can use bounded convergence theorem to prove the lemma. □

Lemma 3 $\phi_t = E[\int_t^T Y_s ds | \mathcal{F}_t]$ *is a continuous semimartingale.*

Proof It is easy to observe that ϕ_t is a semimartingale, because of the following decomposition:

$$\phi_t = E\left[\int_0^T Y_s ds \middle| \mathcal{F}_t\right] - \int_0^t Y_s ds$$

So, our main task is to prove continuity of ϕ_t. We start by writing

$$Y_t = \exp\{-\lambda t\}\xi + \sigma \int_0^t \left(\int_s^t \exp\{\lambda(u-t)\} \frac{\partial K(u,s)}{\partial u} du\right) dB_s,$$

which is derived from the solution of the fractional Ornstein–Uhlenbeck process as the follows:

$$\begin{aligned}
Y_t &= \exp\{-\lambda t\}\xi + \sigma \int_0^t \exp\{\lambda(u-t)\} dB_t^H \\
&= \exp\{-\lambda t\}\xi + \sigma B_t^H - \lambda\sigma \int_0^t \exp\{\lambda(u-t)\} B_u^H du \\
&= \exp\{-\lambda t\}\xi + \sigma B_t^H - \lambda\sigma \int_0^t \exp\{\lambda(u-t)\} \left(\int_0^u K(u,s) dB_s\right) du \\
&= \exp\{-\lambda t\}\xi + \sigma B_t^H - \lambda\sigma \int_0^t \left(\int_s^t \exp\{\lambda(u-t)\} K(u,s) du\right) dB_s \\
&= \exp\{-\lambda t\}\xi + \sigma B_t^H - \sigma \int_0^t \left(\int_s^t K(u,s) d\exp\{\lambda(u-t)\}\right) dB_s \\
&= \exp\{-\lambda t\}\xi + \sigma B_t^H + \sigma \int_0^t \left(\int_s^t \exp\{\lambda(u-t)\} \frac{\partial K(u,s)}{\partial u} du\right) dB_s \\
&\quad - \sigma \int_0^t K(t,s) dB_s \\
&= \exp\{-\lambda t\}\xi + \sigma \int_0^t \left(\int_s^t \exp\{\lambda(u-t)\} \frac{\partial K(u,s)}{\partial u} du\right) dB_s
\end{aligned}$$

Here we need to check the interchangeability. According to Theorem 65 [Chapter 6 of Mishura (2008)], what we only need to prove that

$$\left(\int_0^T \exp\{2\lambda u\} K^2(u,s) du\right)^{\frac{1}{2}} \in L^2[0,T].$$

The above calculations also indicate that the pathwise integral and wiener type integral with respect to B^H coincide. Here ξ is the initial value. And $K(u,s)$ is the kernel of

FBM, which is also related with Hurst Number H. Also it is easy to derive that

$$E[\int_0^T Y_s ds \big| \mathcal{F}_t] = \int_0^T E[Y_s \big| \mathcal{F}_t] ds$$

Furthermore observe that

$$E[Y_t \big| \mathcal{F}_{s_1}] = \exp\{-\lambda t\}\, \xi + \sigma \int_0^{s_1 \wedge t} \zeta(t,s) dB_s,$$

where $\zeta(t,s) = \int_s^t exp(\lambda(u-t)) \dfrac{\partial K(u,s)}{\partial u} du < K(t,s)$. Thus we can write:

$$\int_0^T E[Y_t \big| \mathcal{F}_{s_1}] dt = \int_0^T exp(-\lambda t) \xi dt + \int_0^{s_1} \sigma \int_0^t \zeta(t,s) dB_s dt + \int_{s_1}^T \sigma \int_0^{s_1} \zeta(t,s) dB_s dt$$

which boils down to showing that

$$(\int_0^T \zeta^2(t,s) dt)^{\frac{1}{2}} \in L^2[0,T]$$

Since $\zeta(t,s)$ and $K(t,s)$ are both defined only on the set $\{(t,s)|t \geq s\}$

$$\begin{aligned}
\int_0^T \zeta^2(t,s) dt &< \int_0^T K^2(t,s) dt \\
&= \int_s^T c_H s^{1-2H} (\int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du)^2 dt \\
&\leq \int_s^T c_H s^{1-2H} t^{2H-1} (\int_s^t (u-s)^{H-\frac{3}{2}} du)^2 dt \\
&= \int_s^T c_H s^{1-2H} t^{2H-1} (t-s)^{2H-1} dt \\
&\leq c_H s^{1-2H} T^{2H-1} \int_s^T (t-s)^{2H-1} dt \\
&= c_H^2 s^{1-2H} T^{2H-1} (T-s)^{2H}
\end{aligned}$$

The above formula shows that $\tilde{h}(s) = \int_0^T \zeta^2(t,s) dt$ is of the order $1-2H$, which is less than 0 but greater than -1. This indicates that $(\int_0^T \zeta^2(t,s) dt)^{\frac{1}{2}} \in L^2[0,T]$. Therefore, we can apply the stochastic Fubini Theorem. Let

$$\begin{aligned}
Z_{s_1} &= \int_0^T E[Y_t \big| \mathcal{F}_{s_1}] dt \\
&= \int_0^T \exp\{-\lambda t\}\, \xi dt + \sigma \int_0^{s_1} \int_{s_1}^T \zeta(t,s) dt dB_s + \sigma \int_0^{s_1} \int_s^{s_1} \zeta(t,s) dt dB_s
\end{aligned}$$

$$
\begin{aligned}
Z_{s_2} &= \int_0^T E[Y_t|\mathcal{F}_{s_2}]dt \\
&= \int_0^T \exp\{-\lambda t\}\xi dt + \sigma \int_0^{s_2}\int_{s_2}^T \zeta(t,s)dtdB_s + \sigma \int_0^{s_2}\int_s^{s_2} \zeta(t,s)dtdB_s
\end{aligned}
$$

Assuming $s_2 > s_1$, we know that $Zs_2, s_1 := Z_{s_2} - Z_{s_1}$ is a Gaussian random variable with

$$
\begin{aligned}
E|Z_{s_2} - Z_{s_1}|^2 &= \sigma^2 \int_{s_1}^{s_2} \left(\int_s^T \zeta(t,s)dt\right)^2 ds \\
&\le \sigma^2 \int_{s_1}^{s_2}\int_s^T (\zeta(t,s))^2 dtds \\
&\le \sigma^2 \int_{s_1}^{s_2} c_H^2 s^{1-2H} T^{2H-1}(T-s)^{2H} ds \\
&\le \sigma^2 c_H^3 |s_2 - s_1|
\end{aligned}
$$

Therefore, $Z_{s_2} - Z_{s_1} \sim N(0, \sigma^2_{s_2 - s_1})$ with $\sigma^2_{s_2-s_1} \le \sigma^2 c_H^3 |s_2 - s_1|$, which implies that

$$
E|Z_{s_2} - Z_{s_1}|^p \le c|s_2 - s_1|^{\frac{p}{2}} E[|Z|^p]
$$

where $Z \sim N(0,1)$. Setting p larger than 2, we conclude that there is a continuous version of the process by Kolmogorov-Centsov theorem. □

References

Bayer, C., Friz, P., Gatheral, J.: Pricing under rough volatility. Quant Finance **16**(6), 887–904 (2016)

Beran, J., Feng, Y., Ghosh, S., Kulik, R.: Long-Memory Processes: Probabilistic Properties and Statistical Methods. Springer, Berlin (2013)

Beskos, A., Dureau, J., Kalogeropoulos, K.: Bayesian inference for partially observed stochastic differential equations driven by fractional Brownian motion. Biometrika **102**(4), 809–827 (2015)

Bezborodov, V., Di Persio, L., Mishura, Y.: Option pricing with fractional stochastic volatility and discontinuous payoff function of polynomial growth. Methodol Comput Appl Probab **21**(1), 331–366 (2019)

Breidt, F.J., Crato, N., De Lima, P.: The detection and estimation of long-memory in stochastic volatility. J Econom **83**, 325–348 (1998)

Brouste, A., Kleptsyna, M.: Kalman type filter under stationary noises. Syst Control Lett **61**, 1229–1234 (2012)

Carr, P., Wu, L.: Analyzing volatility risk and risk premium in option contracts: a new theory. J Financ Econ **120**(1), 1–20 (2016)

Cheridito, P., Kawaguchi, H., Maejima, M.: Fractional Ornstein–Uhlenbeck processes. Electron J Probab **8**(3), 1–14 (2003)

Chronopoulou, A., Spiliopoulos, K.: Sequential Monte Carlo for fractional stochastic volatility models. Quant Finance Taylor & Francis J **18**(3), 507–517 (2018)

Chronopoulou, A., Viens, F.: Estimation and pricing under long-memory stochastic volatility. Ann Finance **8**, 379–403 (2010)

Chronopoulou, A., Viens, F.: Stochastic volatility and option pricing with long-memory in discrete and continuous time. Quant Finance **12**, 635–649 (2012)

Comte, F., Renault, E.: Long memory in continuous-time stochastic volatility models. Math Finance **8**(4), 291–323 (1998)
Comte, F., Coutin, L., Renault, E.: Affine fractional stochastic volatility models. Ann Finance **8**, 337–352 (2012)
Ding, Z., Granger, W., Engle, R.: A long memory property of stock market returns and a new model. J Empir Finance **1**, 83 (1993)
El Euch, O., Rosenbaum, M., et al.: Perfect hedging in rough heston models. Ann Appl Probab **28**(6), 3813–3856 (2018)
Fouque, J.P., Papanicolaou, G., Sircar, K.R.: Derivatives in Financial Markets with Stochastic Volatility. Cambridge University Press (2000a)
Fouque, J.P., Papanicolaou, G., Sircar, K.R.: Mean-reverting stochastic volatility. Int J Theor Appl Finance **3**, 101–142 (2000b)
Fukasawa, M.: Short-time at-the-money skew and rough fractional volatility. Quant Finance **17**(2), 189–198 (2017)
Garnier, J., Sølna, K.: Correction to Black–Scholes formula due to fraction stochastic volatility. SIAM J Financ Math **8**, 560–588 (2017)
Gatheral, J., Jaisson, T., Rosenbaum, M.: Volatility is rough. Quant Finance **18**(6), 933–949 (2018). https://doi.org/10.1080/14697688.2017.1393551
Harvey, A.C.: Long Memory in Stochastic Volatility. Butterworth-Haineman, Oxford (1998)
Heston, S.: A closed-form solution for options with stochastic volatility with applications to bond and currency options. Rev Financ Stud **6**, 327–343 (1993)
Le Gall, J.F.: Brownian Motion, Martingales, and Stochastic Calculus. Springer (2016)
Lima, P.D.: Crato: Long-range dependence in the conditional variance of stock returns. Econ Lett **45**, 281–285 (1994)
Mishura, Y.S.: Stochastic Calculus for Fractional Brownian Motion and Related Processes. Lecture Notes in Math, Springer, Berlin (2008)
Nualart, D.: The Malliavin Calculus and Related Topics. Probability and Its Applications. Springer, Berlin (2006)
Renault, E., Touzi, N.: Option hedging and implied volatilities in a stochastic volatility model. Math Finance **6**(3), 279–302 (1996)