Detecting initial system-environment correlations in open systems

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Correlations between a system and its environment lead to errors in an open quantum system. Detecting those correlations would be valuable for avoiding and/or correcting those errors. Here we show that we can detect correlations by measuring only the system itself if we know the cause of the interaction between the two, for example, in the case of a dipole-dipole interaction. We investigate the unitary U which is associated with the exchange Hamiltonian and examine the ability to detect initial correlations between a system and its environment for various types of initial states. The states we select are motivated by realistic experimental conditions, and we provide bounds for when we can state with certainty that there are initial system-environment correlations given experimental data.

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I. INTRODUCTION

Entanglement is a uniquely quantum-mechanical property and is responsible for many of the advantages that quantum systems have over their classical counterparts. Its detection and manipulation are therefore of great importance in the search for the implementation of practical quantum technologies [1,2]. When there is entanglement in a bipartite quantum system and one has access to both subsystems, one can perform measurements on both to detect its presence.

However, entanglement and, more generally, correlations can pose a problem when they are unwanted. Unwanted interactions between the system and environment cause noise in the quantum system that leads to errors in the quantum information and/or quantum control of the system. Such errors can cause irreparable damage to the state of the system and must be avoided, suppressed, and/or corrected to achieve accurate control [3]. Like entanglement, more general correlations can be detected if both parts of a bipartite system can be accessed, controlled, and measured. But this is often not the case for errors which are caused by an unwanted interaction with the environment. The reason is that many environments are not readily accessible in the sense that they are not controllable or measurable except, perhaps, for some bulk properties.

In the case that correlations arise between a system and environment or some other inaccessible quantum system, what can be done to detect properties of the correlations to aid in error prevention and control? Fairly recently, important progress was made toward detecting correlations between a system and its environment by making measurements only on the system. For example, if one prepares many different initial states, one can use the method proposed in [4,5] to find whether the effective environment state depends on the preparation of the system [5]. This process requires many preparations and measurements. Another method uses two differently prepared states, the state of interest and a second state that is the result of operating on the original state. Correlations can then be witnessed by comparing the two with distance measures [6–11]. This method does not make any assumptions about the state of the environment or the structure of the system-environment correlations and has been shown to be experimentally successful at detecting initial correlations [12–14].

Yet another recently discovered method can be used to find correlations between a system and its environment by measuring only the system if one makes the experimentally reasonable assumptions that (i) the system-environment state can be repeatedly prepared, (ii) the system can be measured, and (iii) the system-environment interaction is known. It was found that when the system and environment consist of one qubit each, a wide range of correlated initial states can be detected [15]. This method differs from the methods described in [6–11] since we do not need to prepare different initial states. Instead, we may use any initial state of the system. It is similar in the sense that the environment does not need to be accessed. It also requires many fewer preparations and measurements than the method described in Ref. [4]. We also want to emphasize that the assumption of the system-environment interaction is that we know how the system interacts with the environment. For example, here we assume that the interaction is a Heisenberg-type coupling [16]. It is also worth emphasizing that we are measuring only the system and do not assume any access to the environment. Our method is applicable when a system is prepared in the presence of the environment and correlations between the two arise during preparation. It could also be used to detect correlations between the system and environment after the system has been stored for some time.

In this work, we develop this theory for practical use and examine the experimentally relevant Hamiltonian sometimes called the Heisenberg interaction [16]. This Hamiltonian, which essentially has the form $\vec{S}_1 \cdot \vec{S}_2$, is relevant for spin-spin interactions such as those one may find in a system in which the electron spin is correlated with a nuclear spin. We consider various initial states of the spin-1/2 system and spin-1/2

environment ρ_{SE} and ask how correlations between the system and environment might be detected by letting the system and environment evolve under this Hamiltonian for some time t. We then present an analysis of the robustness of our results showing that it is indeed possible to detect correlations by measuring only the system.

II. DETECTING INITIAL CORRELATIONS

Our general method is described as follows. We assume that the combined system and environment state ρ_{SE} evolves unitarily. The evolution of $\rho_S = \mathrm{Tr}_E(\rho_{SE})$ (where Tr_E is the trace over the environmental degrees of freedom) can then be described by

$$\rho_S \to \rho_S{}'(t) = \mathrm{Tr}_E[U_{SE}(t)(\rho_{SE})U_{SE}^\dagger(t)]. \tag{1}$$

If the combined system that evolves as described above is initially uncorrelated, then

$$\rho_{SE} = \rho_S \otimes \rho_E. \tag{2}$$

In either case we describe the initial state of the system by ρ_S and the final state of the system by ρ_S' in Eq. (1). Furthermore, we assume that both ρ_S and ρ_S' can be measured using standard tomographic techniques [17,18]. The presence of initial correlations between the system and environment in ρ_{SE} would mean that this combined state is not in a product state $\rho_{SE} \neq \rho_S \otimes \rho_E$. Thus, given an initial state $\rho_S = \mathrm{Tr}_E[\rho_{SE}]$ and the final state ρ_S' as described in Eq. (1), the presence of initial correlations between ρ_S and ρ_E can be determined by checking whether the same ρ_S' can be obtained by considering the transformation of an uncorrelated state $\tilde{\rho}_{SE} = \rho_S \otimes \tilde{\rho}_E$. If it cannot, then correlations must be present. In our examples below, we take the system to be one qubit and the environment to be an unknown two-state system. The hypothetical arbitrary environment state $\tilde{\rho}_E$ is written as

$$\tilde{\rho}_E = \tfrac{1}{2}(\mathbf{I} + x\mathbf{X} + y\mathbf{Y} + z\mathbf{Z}), \tag{3}$$

where $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ correspond to the Pauli operators and x, y, z are real numbers such that $x^2 + y^2 + z^2 \leqslant 1$. In other words, we want to know whether it is possible to find a valid state $\tilde{\rho}_E$ such that Eq. (1) is satisfied. Since we make no assumptions about the state of the environment, if we are able to obtain the same ρ_S' with this uncorrelated $\tilde{\rho}_{SE}$, then it is possible that our system-environment state of interest ρ_{SE} was uncorrelated all along. Conversely, if we are not able to produce ρ_S' for any realistic $\tilde{\rho}_E$, then the combined state is shown to possess some correlations between the system and environment.

We are not necessarily concerned here with how correlations between the system and environment have arisen; it can be presumed that some previous interaction has produced these correlations. However, for the purposes of this paper we choose various ρ_{SE} whose system-environment correlations are rather experimentally realistic. Each of these could describe a different experimentally prepared initial state that had some initial strong coupling between the system and environment or was not accurately prepared.

Consider two two-state systems interacting via the Heisenberg exchange Hamiltonian with some coupling constant J which determines the interaction strength:

$$H_{\mathrm{ex}} = J(\mathbf{XX} + \mathbf{YY} + \mathbf{ZZ}), \tag{4}$$

which gives rise to the dipole-dipole interaction and is of particular interest because of the pervasiveness of such interactions in experiment. The time evolution of the state corresponds to U, which, for simplification, we express in terms of a parameter $\alpha = Jt$, with J being the coupling constant and t being time (and letting $\hbar = 1$):

$$U(\alpha) = \exp(-iH_{\mathrm{ex}}t) = \begin{pmatrix} e^{-i\alpha} & 0 & 0 & 0 \\ 0 & e^{i\alpha}\cos 2\alpha & -ie^{i\alpha}\sin 2\alpha & 0 \\ 0 & -ie^{i\alpha}\sin 2\alpha & e^{i\alpha}\cos 2\alpha & 0 \\ 0 & 0 & 0 & e^{-i\alpha} \end{pmatrix}. \tag{5}$$

Note that $U(\pi/4)$ is the SWAP operator (times an overall phase $e^{i\pi/4}$ which is irrelevant). Since nothing is assumed about the state of the environment, we can simulate any evolution when the SWAP operator acts on an initial product state of the form $\tilde{\rho}_E = \rho_S'$. In other words, if we suppose the final state of S is ρ_S', then the initial state of the system plus environment ρ_{SE} can be taken to be $\rho_S \otimes \rho_S'$. Also, $U(\pi/2) = i\mathbf{I}$. So this evolution can also always be simulated with a product state since $\rho_S' = \rho_S$.

In this work we will show how to detect initial correlations between the system and environment undergoing this unitary transformation for three different states ρ_{SE}. For each ρ_{SE}, we will compare the state $\tilde{\rho}_S' \equiv \mathrm{Tr}_E[U\rho_S \otimes \tilde{\rho}_E U^\dagger]$, which is produced by the transformation on the uncorrelated $\tilde{\rho}_{SE}$, to ρ_S', which is the result of the transformation of ρ_{SE}, which may or may not be correlated. The difference between these two states is $D \equiv \rho_S' - \tilde{\rho}_S'$, corresponding to each ρ_{SE}. If $D = \mathbf{0}$, we are not able to detect the initial correlations between the system ρ_S and its environment in ρ_{SE}.

Let us emphasize that the difference is between the output of the experiment ρ_S' and the possible states $\tilde{\rho}_S'$ that arise from $\tilde{\rho}_{SE}$ and account for all $\tilde{\rho}_E$ making up a possible initial product state for the combined system and environment. Thus, D can be used to construct a distance d between the state ρ_S' and the set of possible final states generated by $\mathrm{Tr}_E[U\tilde{\rho}_{SE}U^\dagger]$:

$$d \equiv \min_{n^2 \leqslant 1} \left[\sum_{ij} |D_{ij}|^2 \right]^{1/2},$$

where $n^2 = x^2 + y^2 + z^2$.

A. Maximally entangled ρ_{SE}

For the first and motivational example of how initial correlations can be detected, we consider

$$\rho_{SE} = |\Psi\rangle\langle\Psi|, \tag{6}$$

where $|\Psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + i|10\rangle)$. $|\Psi\rangle$ is a maximally entangled state that is locally equivalent to the Bell state $|\Psi^+\rangle$. Clearly, ρ_{SE} contains correlations (since maximal entanglement produces the strongest correlations) between the system and environment.

Here $\rho_S = \mathbf{I}/2$, and we suppose the state ρ_{SE} evolves according to Eq. (5). The final state is $\rho_S' = \frac{1}{2}[1 -$

$\sin(4\alpha)]\,|0\rangle\langle 0| + \frac{1}{2}[1+\sin(4\alpha)]\,|1\rangle\langle 1|$. In matrix form,

$$\rho_S' = \begin{pmatrix} \frac{1}{2}[1-\sin(4\alpha)] & 0 \\ 0 & \frac{1}{2}[1+\sin(4\alpha)] \end{pmatrix}. \tag{7}$$

Choosing α will determine ρ_S'; we can easily see that $\rho_S' = |0\rangle\langle 0|$ when $\alpha = \frac{3\pi}{8}$. In order to detect initial correlations (which we know to be present in a maximally entangled state), we must see whether an uncorrelated $\tilde{\rho}_{SE}$ can create the same evolution. D comparing the state

$$\begin{aligned}\tilde{\rho}_S' &= \mathrm{Tr}_E[U(\alpha)\tilde{\rho}_{SE}U^\dagger(\alpha)] \\ &= \frac{1}{4}\begin{pmatrix} 2+z-z\cos 4\alpha & 2(x-iy)\sin^2 2\alpha \\ 2(x+iy)\sin^2 2\alpha & 2-z+z\cos 4\alpha \end{pmatrix}\end{aligned} \tag{8}$$

and the original ρ_S' is

$$D = \frac{1}{4}\begin{pmatrix} A_{00} & A_- \\ A_+ & -A_{00} \end{pmatrix}, \tag{9}$$

where $A_{00} = z[-1+\cos(4\alpha)] + 2\sin 4\alpha)$ and $A_\pm = -2(x\pm iy)\sin^2(2\alpha)$. Note that D is sometimes, but not always, equal to $\mathbf{0}$, depending on the values of α, x, y, and z.

Recall that x, y, z are parameters that determine the arbitrary environment state in Eq. (3). If, for a value of α, x, y, z can be chosen such that $\tilde{\rho}_{SE}$ can model the transformation and ρ_S' is obtained, the result will give $D = \mathbf{0}$. It is thus evident that for us to be able to detect the initial correlations in ρ_{SE} for a given value of α, there must not be any choice of x, y, z such that $x^2+y^2+z^2 \leqslant 1$ and $D = \mathbf{0}$. If there is such a set of x, y, and z that will allow the uncorrelated $\tilde{\rho}_{SE}$ to undergo the same transformation $\rho_S \to \rho_S'$, then we cannot state that there were initial correlations. If we cannot find such x, y, z, we know that there are initial correlations in ρ_{SE}. This means we can distinguish between ρ_S' and $\tilde{\rho}_S'$ (i.e., D will not be $\mathbf{0}$). Therefore, the task is to try to find $\tilde{\rho}_S'$ with a valid $\tilde{\rho}_E$ such that $D = \mathbf{0}$. For example, if $\alpha = \frac{3\pi}{8}$, then ρ_S' is a pure state (as noted previously) while $\tilde{\rho}_S'$ remains mixed. Recall that α depends on both the coupling constant and the time over which the interaction occurs; these two parameters may be ascertained or varied to obtain a particular α.

For this example, if $\alpha = \frac{3\pi}{8}$, then D is

$$D = \begin{pmatrix} \frac{1}{4}(-2-z) & \frac{1}{4}(-x+iy) \\ \frac{1}{4}(-x-iy) & \frac{1}{4}(2+z) \end{pmatrix}. \tag{10}$$

The only possible combination of x, y, and z for which $D = \mathbf{0}$ is $x = y = 0$, $z = -2$, but this solution does not satisfy the necessary condition that $x^2+y^2+z^2 \leqslant 1$ ($\tilde{\rho}_E$ must be a valid density matrix); therefore, we know that the unitary is effective in detecting initial correlations for $\alpha = \frac{3\pi}{8}$.

Varying α changes the evolution undertaken by the state, and we can find which values of α will not enable us to detect initial correlations. By considering all possible values of α and the associated solution for x, y, z such that D (which is $\frac{\pi}{2}$ periodic) equals $\mathbf{0}$, we can determine the efficacy of our method for those values. Consideration of D in Eq. (9) shows that the solution will always require $x = y = 0$ unless $\sin(2\alpha) = 0$ (i.e., $U = i\mathbf{I}$), and thus, for those α such that $\sin(2\alpha) \neq 0$, we need to consider only z. The relationship between z and α can be seen graphically in Fig. 1.

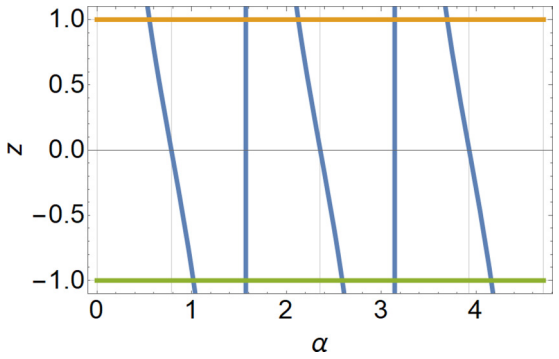

FIG. 1. The blue (vertical) function shows the relationship between α and the solution for z ($x = y = 0$) at which $D = \mathbf{0}$ for ρ_{SE} as in Eq. (6). Both of these quantities are dimensionless. The orange (upper horizontal) and green (lower horizontal) lines represent the restriction on z because $x^2+y^2+z^2 \leqslant 1$. Note the periodic behavior.

Numerically, we can say that if $D = \mathbf{0}$, then

$$z = \frac{-2\sin 4\alpha}{\cos 4\alpha - 1}. \tag{11}$$

By considering the restriction on z, i.e., $-1 \leqslant z \leqslant 1$ (because $x = y = 0$ and $x^2+y^2+z^2 \leqslant 1$), we can calculate bounds on α for which $D \neq \mathbf{0}$:

$$\frac{1}{2}\arctan(-2) + \frac{n\pi}{2} \leqslant \alpha \leqslant \frac{1}{2}\arctan(2) + \frac{n\pi}{2} \tag{12}$$

for any positive or negative integer n, except $\alpha = \frac{\pi}{2} \pm \frac{n\pi}{2}$, for which D is always $\mathbf{0}$. Thus, if we hope to describe the dynamics using an initial product state, α must be between these values so that we may possibly find a z that would make $D = 0$. As expected, $\alpha = \frac{3\pi}{8}$ lies within the range of acceptable values.

B. Pure state mixed with a maximally entangled state

We now consider a mixture of a pure state with a maximally entangled state. This ρ_{SE} is of particular interest because of its connection to an experiment in which there is an attempt to create a pure state, but there are some interactions between the system and environment that produce correlations. Here, too, we will show that evolution by the same unitary U [Eq. (5)] will result in detectable correlations. Thus, we take

$$\rho_{SE} = p\,|01\rangle\langle 01| + \frac{(1-p)}{2}(|01\rangle + i\,|10\rangle)(\langle 01| - i\langle 10|). \tag{13}$$

Here ρ_{SE} may or may not be correlated depending on the value of p, where $0 \leqslant p \leqslant 1$. If we choose $p = 0$, we obtain the entangled state in Eq. (6). The entanglement of this state is dependent on p and can be calculated as described in Appendix A and is graphically depicted in Fig. 2.

One can easily determine that

$$\rho_S = p\,|0\rangle\langle 0| + \frac{1-p}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \tag{14}$$

and

$$\rho_S' = \frac{1}{2}\begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \end{pmatrix}, \tag{15}$$

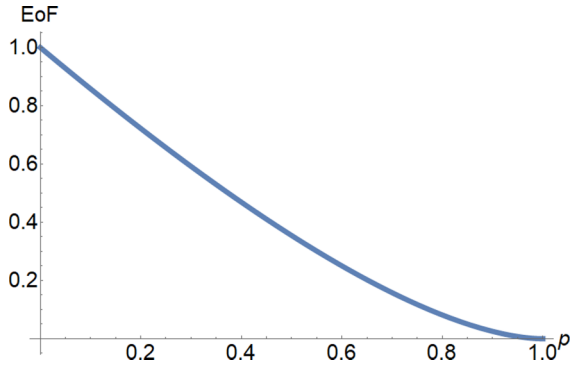

FIG. 2. The entanglement of formation contained in the state ρ_{SE} [Eq. (13)] as a function of p (both quantities dimensionless). When $p = 0$, the maximum amount of entanglement is present.

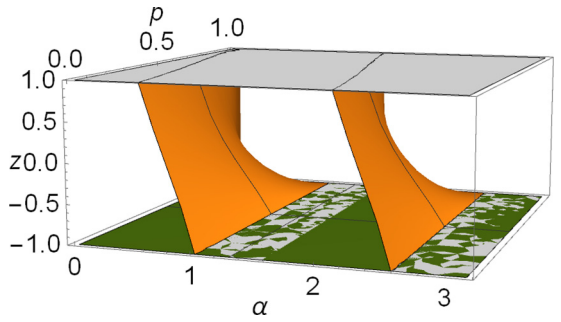

FIG. 3. The orange (curved) surface shows the relationship between p, α, and z (dimensionless) between the bounds $z = -1$ and $z = 1$ (lower and upper horizontal planes, respectively) in the range of $0 \leqslant \alpha \leqslant \pi$ when $x = y = 0$. Note the discontinuity at $p = 1$.

where $\rho_{\pm} = 1 \pm p\cos(4\alpha) \pm (-1 + p)\sin(4\alpha)$. D between this state and an initially uncorrelated state obtained by the evolution of $\tilde{\rho}_{SE} = \rho_S \otimes \tilde{\rho}_E$ is now

$$D = \frac{1}{2}\begin{pmatrix} -B_{00} & B_{-} \\ B_{+} & B_{00} \end{pmatrix}, \tag{16}$$

where $B_{00} = (p + z){S_2}^2 + 2(p - 1)S_2C_2$ and $B_{\pm} = 2(x \pm iy)(-1 + C_4 \pm ipS_4)$. Here $S_i = \sin(i\alpha)$ and $C_i = \cos(i\alpha)$. It is clear that for this matrix to be $\mathbf{0}$, x and y will once again be 0, and the value of z depends on both α (as before) and p.

For example, if $\alpha = \frac{3\pi}{8}$, a value that enabled the detection of initial correlations for ρ_{SE} in Eq. (6),

$$D = \frac{1}{4}\begin{pmatrix} (-2 + p - z) & (ix + y)(i + p) \\ (-ix + y)(-i + p) & (2 - p + z) \end{pmatrix}. \tag{17}$$

For this value of α, it can easily be seen that $D = \mathbf{0}$ when $x = y = 0$ and $z = p - 2$ and, because p ranges between 0 and 1, $-2 \leqslant z \leqslant -1$. z depends linearly on p. However, $z = -1$ is the only physically possible value for z in this range, and thus, D can be made $\mathbf{0}$ only when $p = 1$, which represents a completely uncorrelated ρ_{SE} [see Eq. (13)]. Therefore, this value of α is effective for the detection of initial correlations present in the initial state for a ρ_{SE} of this form.

However, it is ultimately the measurement of the initial and final system states that indicates the presence of initial correlations to the experimenter; therefore, ρ_S' and $\tilde{\rho}_S'$ must be distinguishable. This may be more difficult when there are errors and/or uncertainty in the measurements. Note that when $\alpha = \frac{3\pi}{8}$ is used in Eq. (15), we find that

$$\rho_S' = \frac{1}{2}\begin{pmatrix} 2 - p & 0 \\ 0 & p \end{pmatrix}. \tag{18}$$

This in turn shows that large correlations ($p \approx 0$) are easily distinguishable—as Eq. (18) will be close to a pure state—while small correlations will be more difficult to detect: the mixed state obtained when $p \approx 1$ is close to the maximally mixed state that presents the possibility of no correlations, as shown in the expression for D.

Generally (i.e., for all values of α), the dependence of z on α and p for which $D = \mathbf{0}$ can be described by the following [a simplification of the diagonal terms of Eq. (16)]:

$$z = -2(p - 1)T_2 - p = 2T_2 - p(2T_2 + 1), \tag{19}$$

where $T_2 = \cot(2\alpha)$. This relationship is represented graphically in Fig. 3. Figure 1 is easily derived from the graph in Fig. 3 when $p = 0$.

We once again find the range of values of α for which all correlations between the system and environment of this state can be detected, that is to say, no values of x, y, z can be found such that $D = \mathbf{0}$ unless $p = 1$:

$$\frac{\operatorname{arccot}\left(\frac{-1}{2}\right)}{2} + \frac{n\pi}{2} \leqslant \alpha \leqslant \frac{\operatorname{arccot}\left(\frac{p+1}{1-p}\right)}{2} + \frac{n\pi}{2}, \tag{20}$$

for any positive or negative integer n except (see the discussion concerning U at $\alpha = \frac{\pi}{4}, \frac{\pi}{2}$)

$$\alpha = \frac{\pi}{4} \pm \frac{n\pi}{4}.$$

The behavior at $p = 1$ is best determined by reasoning opposite to the analysis of parameter z in the discussion surrounding Eq. (16): The uncorrelated state is always modeled by the product state with $z = -1$.

Our previously analyzed choice of $\alpha = \frac{3\pi}{8}$ works perfectly fine at detecting all initial correlations, and since, for this state, entanglement is presented for $0 \leqslant p < 1$ (see Fig. 2), the evolution caused by U also indicates the presence of entanglement here. Of course, the bounds imply that there are other α which indicate initial correlations. However, values that lie closer to $\alpha = \frac{\pi}{2}$ for which D is always $\mathbf{0}$ [see the restriction on Eq. (20)] may be less favorable due to the potential for error.

C. Maximally mixed state

We now turn our attention to performing the protocol with the following initial state:

$$\rho_{SE} = p\frac{1}{4}\mathbf{I} + \frac{1 - p}{2}(|01\rangle + i|10\rangle)(\langle 01| - i\langle 10|). \tag{21}$$

The amount of entanglement present in this state is once again dependent on p, as can be seen in Fig. 4, and is 0 for $p \geqslant 2/3$. It would therefore be useful for our protocol to be able to detect correlations within the range of p where entanglement is present but also beyond this range, all the way

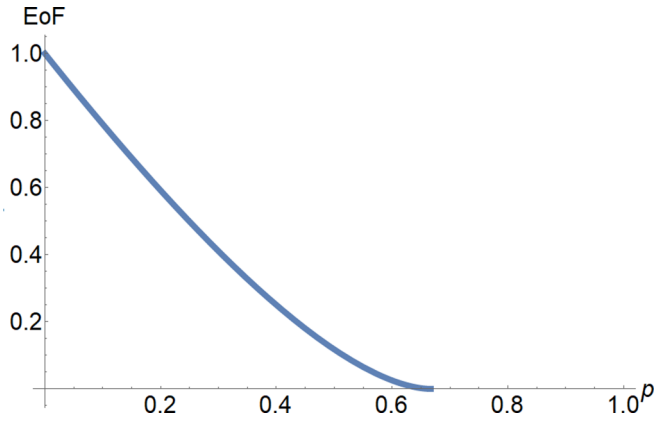

FIG. 4. The graph above shows the entanglement of formation (EOF) vs p (both dimensionless) for the state in Eq. (21). The EOF goes to zero at $p = \frac{2}{3}$.

to (but excepting) $p = 1$, as the state is correlated for all values $0 \leqslant p < 1$.

Conducting an analysis similar to the previous two examples, the final state is

$$\rho_S' = \frac{1}{2}\begin{pmatrix} 1 + (-1+p)S_4 & 0 \\ 0 & 1 - (-1+p)S_4 \end{pmatrix}, \tag{22}$$

where $S_4 = \sin(4\alpha)$ once again, and

$$D = \frac{1}{2}\begin{pmatrix} -C_{00} & -(x-iy){S_2}^2 \\ -(x+iy){S_2}^2 & C_{00} \end{pmatrix}, \tag{23}$$

where $C_{00} = S_2[2(-1+p)C_2 + zS_2]$, $S_2 = \sin(2\alpha)$, and $C_2 = \cos(2\alpha)$. The off-diagonal of Eq. (23) implies that all solutions for $D = \mathbf{0}$ must have $x = y = 0$. The diagonal terms indicate the relation between z, p, and α:

$$z = -2(-1+p)\cot(2\alpha), \tag{24}$$

which is very similar to the corresponding relation in Sec. II B, Eq. (13) (the latter includes an additional component). Equation (24) is graphically represented by Fig. 5.

We are once again tasked with finding a value of α for which we are unable to find a z satisfying $D = \mathbf{0}$, indicating the ability to distinguish between uncorrelated and correlated ρ_{SE}.

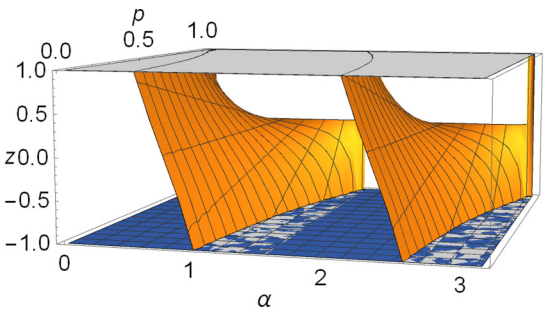

FIG. 5. The orange curved surface shows the relationship between parameters z, p, and α (dimensionless) for D corresponding to the state in Eq. (21) for $x = y = 0$ in the range $0 \leqslant \alpha \leqslant \pi$ between the bounds of $z = -1$ and $z = 1$ (lower and upper horizontal planes, respectively). Note the behavior at $\alpha = \frac{\pi}{2}$ ($U = i\mathbf{I}$).

The value of α that was successful in distinguishing the states in Secs. II A and II B, $\frac{3\pi}{8}$, is no longer successful at detecting correlations here for all values of p. This is evident upon considering D when $\alpha = \frac{3\pi}{8}$:

$$D = \begin{pmatrix} \frac{1}{4}(-2+2p-z) & \frac{1}{4}(-x+iy) \\ \frac{1}{4}(-x-iy) & \frac{1}{4}(2-2p+z) \end{pmatrix}. \tag{25}$$

When $p \geqslant 0.5$, i.e., when only weak correlations are present, a z value can be found which models the transformation with an uncorrelated initial state ($D = \mathbf{0}$, and thus, the protocol fails at detecting the correlations in states where $p \geqslant 0.5$). Can this restriction on p for which correlations are detected be relieved?

By considering the p and z relations for various α, we verify their periodicity and at the same time find that our ability to detect correlations increases as α approaches, but does not reach, $\frac{\pi}{2}$. For example, when $\alpha = \frac{13\pi}{32}$, the limitation on p becomes $p \leqslant \frac{2}{3}$, and for $\alpha = \frac{15\pi}{32}$, it is approximately $p \leqslant 0.90$ for the protocol to detect the correlations.

Approaching $\alpha = \frac{\pi}{2}$ even more, we have $p < 0.95075$ when $\alpha = \frac{31\pi}{64}$ and $p < 0.9754$ for $\alpha = \frac{63\pi}{128}$. These latter values would, in theory, allow us to detect even the smallest system-environment correlations present in ρ_{SE}.

Realistically, detecting slight correlations, however, is difficult because when $\alpha = \frac{\pi}{2}$, no correlations can be detected. Therefore, the protocol for detecting most to all correlations for this state is not very fault tolerant; choosing α close to, but not equal to, $\frac{\pi}{2}$ is difficult to achieve in experiment without error.

It is interesting, however, that for $\alpha = \frac{13\pi}{32} \pm \frac{n\pi}{2}$ (for any positive or negative integer n), the range for which correlations can be detected is nearly $p < \frac{2}{3}$. As mentioned previously and depicted in Fig. 4, this is also the bound on p for which entanglement is present. Performing the protocol with $\alpha = \frac{13\pi}{32}$ could therefore be used as a way to detect entanglement in the state. This choice of α is also more fault tolerant; a small error in this value, likely to occur in an experiment, would not make it equal to $\frac{\pi}{2}$, for which D is always $\mathbf{0}$ and therefore no correlations at all would be detected.

III. ROBUSTNESS OF RESULTS

In order to examine the robustness of the results described in the previous section, we follow the technique described in [19]. We let an uncorrelated state $\tau_S \otimes \rho_E$ evolve unitarily according to Eq. (1), and we obtain the final state τ_S'. τ_S and ρ_E are both arbitrary states, where

$$\tau_S = \tfrac{1}{2}(\mathbf{I} + \vec{n}\cdot\vec{\sigma}) \tag{26}$$

and

$$\rho_E = \tfrac{1}{2}(\mathbf{I} + \vec{m}\cdot\vec{\sigma}). \tag{27}$$

After the state has evolved, we can compare the final and initial states τ_S and τ_S' to their respective counterparts, ρ_S and ρ_S', by varying values for n_1, n_2, and n_3 (m_1, m_2, m_3). If we cannot equate both pairs using one set of values for n_i and m_i, the example is robust to errors.

Evolving $\tau_S \otimes \rho_E$ with the unitary in Eq. (5) and tracing out the environment gives us

$$\tau_S' = \frac{1}{4}\begin{pmatrix} \tau'_{S00+} & \tau'_{S01-} \\ \tau'_{S01+} & \tau'_{S00-} \end{pmatrix}, \tag{28}$$

where

$$\tau'_{S00\pm} = 2\pm[m_3 + n_3 + (-m_3 + n_3)\cos(4\alpha) + (-m_2 n_1 + m_1 n_2)\sin(4\alpha)]$$

and

$$\tau'_{S01\pm} = m_1 \pm i m_2 + n_1 \pm i n_2 + [(-m_1 \mp i m_2 + n_1 \pm i n_2)\cos(4\alpha) + (i m_3 n_1 \mp m_3 n_2 - i m_1 n_3 \pm m_2 n_3)\sin(4\alpha)].$$

Let us consider the first example, i.e., Eq. (6). Comparing the initial state $\rho_S = \frac{\mathbf{I}}{2}$ to Eq. (26), we find that to equate the two, $|\vec{n}|$ must be zero. Here we consider that small experimental errors may be present so that the measurement gives a magnitude of $\vec{n}$ that is small, but nonzero. Then, comparing Eq. (28) to Eq. (7), we find that $m_2 n_1 = 1$. If $|\vec{n}|$ is small, we can take $n_1 = \epsilon$, where ϵ is some small value. Then, $m_2 = \frac{1}{\epsilon}$, which is greater than 1. Since we must have $|\vec{m}| \leqslant 1$ for a valid environment state, this is not an acceptable value for m_2. Thus, the evolution cannot be produced by a product initial state, and the result is robust to small measurement errors which would give a small, but nonzero, value for the magnitude of $\vec{n}$.

By equating Eqs. (14) and (26) in the second example, we find that $n_3 = p$, while n_1, n_2 approach 0 and thus can be considered to have some small value ϵ as long as $|\vec{n}| \leqslant 1$ holds true. From our comparison of ρ_S' and τ_S' [Eqs. (15) and (28)], we see that $m_3 = -p$. Finally, to completely equate the two final states, we must satisfy the following condition:

$$\frac{-m_2 n_1 + m_1 n_2}{2} = -1 + p. \tag{29}$$

Since both n_1 and n_2 approach 0 and we have already seen that neither m_2 nor m_1 is able to have a value of $\frac{1}{\epsilon}$ for some small value ϵ in the first example, the equation (29) is satisfied only when $p = 1$. This value of p is the completely uncorrelated state, and thus, the only value for which we expect to be able to equate ρ_S and τ_S. This example is also robust.

Finally, for the initial third state, we must equate the partial trace over the environment of Eq. (21) with Eq. (26). Once again, we derive that $|\vec{n}|$ must be small. As for the final state, so that Eq. (22) equals Eq. (28), we know that m_3 must also be small. Then we are left to satisfy the same condition as in Eq. (29). Even though our value for m_3 is different, the same reasoning as in the previous case applies, and we are not able to find m_1, m_2 such that $|\vec{m}| \leqslant 1$ unless $p = 1$, the uncorrelated case, proving that the final example is also robust to errors in the initial and final states.

IV. CONCLUSION

We have described a method which enables the detection of correlations between a system and environment using measurements only on the system. It was shown to be effective for systems which evolve under a Heisenberg interaction Hamiltonian [Eq. (5)]. When a two-state system evolves from an initial state ρ_S into a final state ρ_S', correlations present between the system and environment can be detected. This depends, of course, on the time that they evolve, which we described by the parameter α. This may be exemplified experimentally by a dipole-dipole interaction. By varying α, we can tune our protocol for detecting initial correlations.

We have shown how our method detects correlations using three different initial states ρ_{SE} [Eqs. (6), (13), and (21)]. These different initial states describe models for a system interacting with its environment to produce correlations which we were able to vary using a parameter p. For the ρ_{SE} described in Eq. (6) and in Eq. (13), choosing $\alpha = \frac{3\pi}{8}$ enables us to state with certainty when correlations are present, although this is not the only value of α that enables this. For ρ_{SE} as described in Eq. (21), choosing α to be close to, but not equal to, $\frac{\pi}{2}$ enables the detection of large, as well as small, correlations. Finally, a choice of $\alpha = \frac{13\pi}{32}$ for this state enables the detection of entanglement. We have also shown that these results are robust to experimental error for properly chosen α.

Our work is different from earlier methods described in the Introduction in which initial correlations in an open system are detected without assumptions of its evolution but use multiple initial states. While we assume a particular form for the interaction undergone by the system, it is compatible with experimental reality as the experimenter is likely to have knowledge about the type of interactions the system undergoes and, using this method, is able to detect correlations by considering only the state of interest.

Future work is required to describe the distinguishability of outcomes ρ_S' for correlated and uncorrelated initial ρ_{SE}, as experimental measurement error can result in a lack of accuracy in the determination of the system's state. We also have plans to use higher-dimensional environments to show that the methods here are generally applicable. There is a justification for believing this since any state of a two-state system can be taken to any other by using only one two-state ancilla and a joint unitary transformation. As discussed previously, this method can be used for other types of interactions that define the state evolution, although we appreciate our choice of unitary because of its connection to experiment [16].

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APPENDIX: MEASURING ENTANGLEMENT

There are various so-called entanglement measures that can be used to quantify the amount of entanglement present in state ρ. One of these is the entanglement of formation E, which is closely related to the concurrence C of a pure state Φ as follows:

$$E(\Phi) = \mathcal{E}(C(\Phi)), \tag{A1}$$

where $\mathcal{E}$ is defined by

$$\mathcal{E}(C) = h\left(\frac{1+\sqrt{1-C^2}}{2}\right), \tag{A2}$$

$$h(a) = -a\log_2 a - (1-a)\log_2(1-a). \tag{A3}$$

The more general case involves a mixed state ρ, where $\mathcal{E} = E(\rho)$, $C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, and $\{\lambda_i\}$ are the square roots of the eigenvalues of $\bar{\rho}$, which is, in turn, defined by

$$\bar{\rho} = \rho\tilde{\rho}, \tag{A4}$$

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho * (\sigma_y \otimes \sigma_y), \tag{A5}$$

where the asterisk (*) indicates taking the complex conjugate in the standard basis [20].

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