Potential, field, and interactions of multipole spheres: Coated spherical magnets

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Abstract

We show that the energy, force, and torque between two spherically symmetric multipole density distributions are identical to those between two point multipoles, and apply this point-sphere equivalence to coated spherical dipole magnets. We also show that the potential and field of such a distribution are equivalent to those due to point multipoles located at the center of the distribution. We expand the inverse-distance potential in terms of harmonic (Hermite irreducible) tensors, whose properties enable us to express the potential energy, force, and torque for two arbitrary source distributions in a series of point-multipole interactions. This work generalizes recent work on interactions between uniformly magnetized dipole spheres [B. F. Edwards, D. M. Riffe, J.-Y. Ji, and W. A. Booth, Am. J. Phys. **85**, 130 (2017)] to interactions between spherically-symmetric multipole spheres.

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I. INTRODUCTION

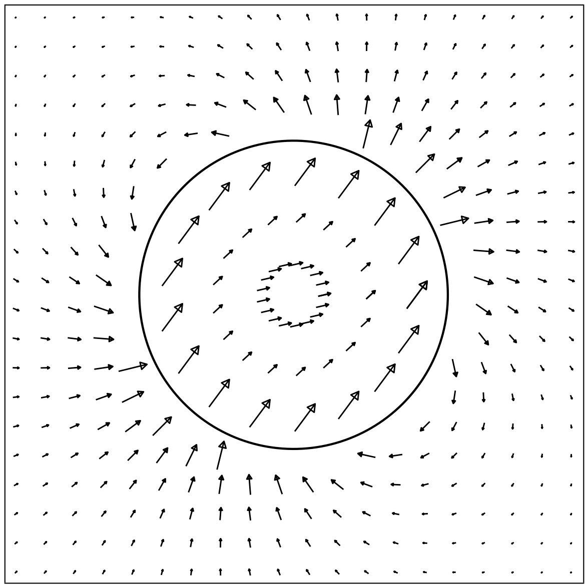
Dipolar magnetic interactions are often used to approximate the interactions between permanent magnets of various shapes [1–8]. In the case of uniformly magnetized spheres, this approximation is exact – the energy, force, and torque between uniformly magnetized spheres are identical to those between point magnetic dipoles, even at short separations. Before this point-sphere equivalence was shown for arbitrary magnet orientations [9], it was shown for magnetizations that are perpendicular to the line through the sphere centers [10], for parallel magnetizations that make an arbitrary angle with this line [11], and for one magnetization parallel to this line and the other in an arbitrary direction [12]. Investigations of the dynamics of pairs of uniformly magnetized spheres [13–18] and chains of such spheres [19–26] exploit this point-sphere equivalence.

Many of these investigations are motivated by interest in collections of small neodymium magnet spheres that are used to build beautiful sculptures, some made from thousands of magnets, including models of molecules, fractals, and Platonic solids [27]. These spheres have spawned a learning community dedicated to sharing photos and tutorials of magnetic sculptures, with YouTube tutorial videos attracting over a hundred million views [28]. These spheres offer engaging hands-on exposure to principles of magnetism, and are used both in and out of the classroom to teach principles of mathematics, physics, chemistry, biology, and engineering [28, 29].

These neodymium magnet spheres carry protective coatings of nickel, copper, and other materials of total thickness up to 0.05 mm [30–32]; such coatings occupy up to 12% of the volume of magnets of diameter 2.5 mm [33]. Coating materials carry lower magnetizations than the neodymium-iron-boron alloy ($Nd_2Fe_{14}B$) that is used for the magnet cores. Thus, coated magnets are non-uniformly magnetized and the point-sphere equivalence for uniformly magnetized spheres does not apply to them.

Such coated magnet spheres do have spherically symmetric dipole density distributions, meaning that their magnetization (the magnetic dipole moment per unit volume of material) depends only on the distance from the magnet center. For these magnets, it is the magnitude of the magnetization that depends on the distance from the center, not the direction of the magnetization, which is the same throughout the sphere. We ignore any changes in the direction of the magnetization in the magnetically soft coating that might result from interactions with nearby magnets.

In this paper, we generalize the point-sphere equivalence to spherically symmetric multipole density distributions and show thereby that this equivalence applies to coated spherical magnets.



(a) Spherically symmetric dipole distribution

(b) Equivalent dipole

Figure 1. Panel (a) shows a spherically symmetric dipole distribution, with arrows inside the sphere denoting the dipole moment density. This distribution is uniform within each of three spherical shells, but varies radially from shell to shell in magnitude and direction. The arrows outside the sphere represent the field produced by this distribution. Panel (b) shows a single equivalent dipole that produces the same field. For coated spherical magnets, there are two shells, the outer shell (the coating) with magnetization in the same direction as the inner shell (the core), but with smaller magnitude. Although this figure depicts the special case of a spherically symmetric dipole distribution, our calculations apply generally for spherically symmetric multipole distributions.

Accordingly, investigations of magnet pair dynamics [13–18] and magnet chain dynamics [19–26] now pertain to coated spherical magnets. Figure 1 shows a spherically symmetric dipole distribution.

The multipole expansion has a long history of use to form reduced physical models, to simplify calculations, and to develop analytical theory. The multipole expansion is often truncated under the assumption that the associated truncation error is within some tolerable limit. Some symmetries offer significant, exact simplifications. The multipole expansion is commonly used to represent the angular dependence of a function. This expansion is often written using either spherical harmonics in spherical-polar coordinates [34–37] or irreducible tensorial Hermite polynomials in Cartesian coordinates [38, 39]. Each term in the series describes progressively finer angular features.

The multipole expansion is especially powerful for expressing potentials that exhibit inverse distance dependence, where long-range interactions are accurately approximated with only a few multipole moments. This fact has been exploited by the highly successful fast multipole method [40, 41]. In kinetic theory, the moment method can be generalized by combining the multipole expansion with polynomials to represent the velocity dependence of the velocity distribution function. This expansion has been used extensively in the general moment method of the kinetic theory of plasmas [42–47].

Potentials that exhibit inverse distance dependence are found in a wide variety of physical interactions such as gravitational [48], Coulomb, and magnetic forces [34, 49]. The multipole expansion arises from a Taylor expansion of the inverse distance between two points in Cartesian coordinates. Below, we develop this expansion using harmonic (totally symmetric, traceless) tensors expressed in terms of successive gradients of the inverse distance, and write the potential as a sum over these tensors with coefficients that are multipole moments of the charge density.

In this work, we consider the interaction between non-overlapping bodies with potentials produced by isotropic multipole moment densities. We establish that the potential and field outside these sources are equivalent to those of a single multipole centered at the point of spherical symmetry. Furthermore, we show that the potential energy of and force and torque between two spherical bodies are equivalent to those of single point-like multipoles at the center of each source. Such point sources can be used to model more realistic and complex systems, and these results can be used to increase pedagogical and computational efficiency. Our derivation is carried out for a general inverse-distance potential, and therefore applies for distributions of magnetic charge density, electric charge density, and mass density.

In Sec. II we write a multipole expansion of the potential and the field outside of an arbitrary source distribution by defining the harmonic tensor and the corresponding multipoles. In Sec. III we calculate the potential energy, force, and torque between two arbitrary source distributions. Defining the multipole density, we calculate the potential and the field outside a spherically symmetric multipole sphere and the potential energy, force, and torque between two non-overlapping spheres. Sec. IV is devoted to discussion.

II. MULTIPOLE EXPANSION FOR THE r^{-1} POTENTIAL

The r^{-1} potential ($r = |\mathbf{r}|$) due to a localized source ρ in a volume V can be written as

$$\phi(\mathbf{r}) = k \int_V \mathrm{d}^3\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{1}$$

where k is the coupling constant. For the electric potential ϕ, ρ is electric charge density and $k = 1/4\pi\epsilon_0$. For a component of the magnetic vector potential, ρ is the corresponding component of the electric current and $k = \mu_0/4\pi$. For the magnetic scalar potential, ρ is the effective magnetic charge density, defined by $\nabla \cdot \mathbf{M}$ with magnetization $\mathbf{M}$, and $k = \mu_0/4\pi$. For the Newtonian gravitational potential, ρ is the mass density and $k = G$ the gravitational constant. In this work, we consider the potential outside the volume ($r > r'$).

For $r > r'$, we use the Taylor expansion

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} - \mathbf{r}' \cdot \nabla \frac{1}{r} + \frac{1}{2}\mathbf{r}'\mathbf{r}' : \nabla\nabla \frac{1}{r} + \cdots = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \mathbf{r}'^{\otimes l} \cdot \nabla^{\otimes l} \frac{1}{r}, \tag{2}$$

where the superscript $\otimes l$ denotes a tensor-product power, for example $\nabla^{\otimes 2} = \nabla\nabla$ which must be distinguished from $\nabla^2 = \nabla \cdot \nabla$. The $\cdot$ between two tensors denotes an l-fold dot product with l being a lower rank. The : is commonly used for the 2-fold dot product. We define an l-th rank tensor P^l (called a harmonic tensor) by

$$\nabla^{\otimes l} \frac{1}{r} = \frac{\delta_l}{r^{2l+1}} \mathsf{P}^l(\mathbf{r}) \tag{3}$$

with

$$\delta_l = (-1)^l (2l-1)!!, \tag{4}$$

where $(2l-1)!! = (2l-1)(2l-3)\cdots(-1)!!$ and $(-1)!! = 1$. Note that $\mathsf{T}^{(l)} = -\nabla^{\otimes l} r^{-1}$ in Eq. (3) of Ref. [38]. In component form, Eq. (3) becomes

$$\partial_{i_1}\partial_{i_2}\cdots\partial_{i_l} \frac{1}{r} = \frac{\delta_l}{r^{2l+1}} P^l_{i_1 i_2 \cdots i_l}, \tag{5}$$

where $\partial_i = \partial/\partial r_i$ with $r_1 = x$, $r_2 = y$, and $r_3 = z$ in Cartesian coordinates. From Eq. (5), we can easily see that P^l is symmetric and traceless for any pair of indices, that is,

$$P^l_{\cdots i \cdots j \cdots} = P^l_{\cdots j \cdots i \cdots}, \tag{6}$$

and

$$\sum_{i=1}^{3} P^l_{\cdots i \cdots i \cdots} = 0. \tag{7}$$

The harmonic tensor can be explicitly written as

$$\mathsf{P}^l(\mathbf{r}) = \frac{l!}{(2l-1)!!}\sum_{m=0}^{[l/2]} \frac{(-1)^m(2l-2m)!}{2^l m!(l-m)!(l-2m)!} r^{2m}\{\mathbf{r}^{\otimes l-2m}\mathsf{I}^{\otimes m}\}_+, \tag{8}$$

where $\mathsf{I} = \sum_{i=1}^3 \mathbf{e}_i\mathbf{e}_i$ is the unit tensor and $\{\}_+$ denotes symmetrization of vectors. The lowest orders are

$$\begin{aligned}
\mathsf{P}^0(\mathbf{r}) &= 1,\\
\mathsf{P}^1(\mathbf{r}) &= \mathbf{r},\\
\mathsf{P}^2(\mathbf{r}) &= \mathbf{rr} - \frac{r^2}{3}\mathsf{I},\\
\mathsf{P}^3(\mathbf{r}) &= \mathbf{rrr} - \frac{3}{5}r^2\{\mathbf{r}\mathsf{I}\}_+.
\end{aligned}$$

For vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\ldots$, the symmetrization is defined by

$$\{\mathbf{abc}\cdots\}_+ = \frac{1}{\text{symmetry factor}}(\text{permutations of } \mathbf{abc}\cdots). \tag{9}$$

For example,

$$\{\mathbf{ab}\}_+ = \frac{1}{2}(\mathbf{ab}+\mathbf{ba}), \tag{10}$$

$$\{\mathbf{abc}\}_+ = \frac{1}{3!}(\mathbf{abc}+\mathbf{bca}+\mathbf{cab}+\mathbf{acb}+\mathbf{bac}+\mathbf{cba}), \tag{11}$$

and so on. Note that the symmetry factor for $\{\mathbf{r}^{\otimes l-2m}\mathsf{I}^{\otimes m}\}_+$ in Eq. (8) is

$$S^l_m = \frac{l!}{(l-2m)!2^m m!}.$$

For example, $S^3_1 = 3$ and

$$\{\mathbf{r}\mathsf{I}\}_+ = \sum_{i=1}^{3}\{\mathbf{r}\mathbf{e}_i\mathbf{e}_i\}_+ = \sum_{i=1}^{3}\frac{1}{3}(\mathbf{r}\mathbf{e}_i\mathbf{e}_i + \mathbf{e}_i\mathbf{r}\mathbf{e}_i + \mathbf{e}_i\mathbf{e}_i\mathbf{r}),$$

which also can be obtained from Eq. (11) by setting $\mathbf{a} = \mathbf{r}$ and $\mathbf{b} = \mathbf{c} = \mathbf{e}_i$ with a summation over i. The harmonic tensors satisfy the following orthogonality relation, Eq. (18) of Ref. [45], for a symmetric traceless tensor $\mathsf{M}^n(r)$ which is independent of $\hat{\mathbf{r}}$

$$\int \mathrm{d}^2\hat{\mathbf{r}}\mathsf{M}^n(r)\cdot\mathsf{P}^n(\mathbf{r})\mathsf{P}^l(\mathbf{r}) = 4\pi r^{2l}\sigma_l\mathsf{M}^n(r)\delta_{ln} \tag{12}$$

with

$$\sigma_l = \frac{l!}{(2l+1)!!}, \tag{13}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r = \mathbf{e}_1 \sin\theta\cos\varphi + \mathbf{e}_2 \sin\theta\sin\varphi + \mathbf{e}_3\cos\theta$ and $\mathrm{d}^2\hat{\mathbf{r}} = \sin\theta \mathrm{d}\theta\mathrm{d}\varphi$ in spherical polar coordinates.

Since every term in $\mathsf{P}^l(\mathbf{r}') - \mathbf{r}'^{\otimes l}$ has at least one I as shown in Eq. (8) and the traceless property (7) means that $\mathsf{I} : \mathsf{P}^l(\mathbf{r}) = 0$, we find that $[\mathsf{P}^l(\mathbf{r}') - \mathbf{r}'^{\otimes l}] \cdot \mathsf{P}^l(\mathbf{r}) = 0$ and similarly $\mathsf{P}^l(\mathbf{r}') \cdot [\mathsf{P}^l(\mathbf{r}) - \mathbf{r}^{\otimes l}] = 0$. Therefore we have

$$\mathbf{r}'^{\otimes l} \cdot \mathsf{P}^l(\mathbf{r}) = \mathsf{P}^l(\mathbf{r}') \cdot \mathsf{P}^l(\mathbf{r}) = \mathsf{P}^l(\mathbf{r}') \cdot \mathbf{r}^{\otimes l}, \tag{14}$$

and rewrite Eq. (2) as

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \mathsf{P}^l(\mathbf{r}') \cdot \nabla^{\otimes l} \frac{1}{r} = \sum_{l=0}^{\infty} \frac{(2l-1)!!}{l!} \frac{1}{r^{2l+1}} \mathsf{P}^l(\mathbf{r}') \cdot \mathsf{P}^l(\mathbf{r}). \tag{15}$$

Note that the orthogonal relation implies

$$\int \mathrm{d}^2\hat{\mathbf{r}}' \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{4\pi}{r} \ (r > r'), \tag{16}$$

which can also be easily obtained by direct integration.

Here we define the lth order multipole moment for distribution $\rho(\mathbf{r})$ as

$$\mathsf{m}^l = \frac{(2l-1)!!}{l!} \int_V \mathrm{d}^3\mathbf{r}' \rho(\mathbf{r}') \mathsf{P}^l(\mathbf{r}'). \tag{17}$$

The lowest orders are explicitly written as follows:

$$\begin{aligned}
\mathsf{m}^0 &= \int_V \mathrm{d}^3\mathbf{r}' \rho(\mathbf{r}'), \\
\mathsf{m}^1 &= \int_V \mathrm{d}^3\mathbf{r}' \rho(\mathbf{r}') \mathbf{r}', \\
\mathsf{m}^2 &= \frac{3}{2} \int_V \mathrm{d}^3\mathbf{r}' \rho(\mathbf{r}') \left(\mathbf{r}'\mathbf{r}' - \frac{1}{3} r'^2 \mathsf{I} \right), \\
\mathsf{m}^3 &= \frac{5}{2} \int_V \mathrm{d}^3\mathbf{r}' \rho(\mathbf{r}') \left(\mathbf{r}'\mathbf{r}'\mathbf{r}' - \frac{3}{5} r'^2 \{\mathbf{r}'\mathsf{I}\}_+ \right).
\end{aligned}$$

Plugging Eq. (15) into Eq. (1) and using the definition (17), we can write

$$\phi(\mathbf{r}) = \sum_{l=0}^{\infty} \phi^l(\mathbf{r}) = k \frac{1}{\delta_l} \mathsf{m}^l \cdot \nabla^{\otimes l} \frac{1}{r} = k \sum_{l=0}^{\infty} \frac{\mathsf{m}^l \cdot \mathsf{P}^l(\mathbf{r})}{r^{2l+1}}, \tag{18}$$

where $\phi^l(\mathbf{r})$ is the potential at $\mathbf{r}$ due to the point multipole m^l located at the origin. Equation (18) states that the potential due to an arbitrary source distribution can be expressed in a series of potentials $\phi^l(\mathbf{r})$. The electric field can be obtained by $\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$ from Eq. (18)

$$\mathbf{E}(\mathbf{r}) = \sum_{l=0}^{\infty} \mathbf{E}^l(\mathbf{r}) = -k \sum_{l=0}^{\infty} \frac{1}{\delta_l} \mathsf{m}^l \cdot \nabla^{\otimes l+1} \frac{1}{r} = k \sum_{l=0}^{\infty} (2l+1) \frac{\mathsf{m}^l \cdot \mathsf{P}^{l+1}(\mathbf{r})}{r^{2l+3}}, \tag{19}$$

where Eqs. (3) and (4) have been used in the last equality.

III. INTERACTIONS BETWEEN MULTIPOLE SPHERES

In the last section we have obtained expressions for the potential and field due to a point multipole. Now we consider the potential and field due to a spherically symmetric distribution of multipoles (see Fig. 1), and the force and torque between two distributions. Here we do not consider self-interactions between multipoles inside the sphere but the external field generated by the spherical distribution.

A. Potential and field due to a multipole sphere

From the multipole moment (17), the multipole moment density M^l is defined by

$$\mathsf{M}^l(\mathbf{r}) = \lim_{\Delta V \to 0} \frac{\Delta \mathsf{m}^l}{\Delta V}, \tag{20}$$

where $\Delta \mathsf{m}^l$ is the multipole moment in a volume element ΔV. The potential due to multipole density localized in a solid sphere S_o centered at the origin is

$$\phi_o^l(\mathbf{r}) = k \int_o \mathrm{d}^3\mathbf{r}' \frac{\mathsf{M}^l(\mathbf{r}') \cdot \mathsf{P}^l(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^{2l+1}}, \tag{21}$$

where $\int_o$ denotes a volume integral over S_o. Using Eq. (3), we write

$$\phi_o^l(\mathbf{r}) = \frac{k}{\delta_l} \nabla^{\otimes l} \cdot \int_o \mathrm{d}^3\mathbf{r}' \mathsf{M}^l(\mathbf{r}') \frac{1}{|\mathbf{r} - \mathbf{r}'|}. \tag{22}$$

For a spherically symmetric distribution $\mathsf{M}^l(r')$ that is independent of $\hat{\mathbf{r}}$, we can write

$$\int \mathrm{d}^3\mathbf{r}' \mathsf{M}^l(r') = \int 4\pi r'^2 \mathrm{d}r' \mathsf{M}^l(r') = \mathsf{m}^l, \tag{23}$$

and Eq. (16) implies

$$\int \mathrm{d}^3\mathbf{r}' \mathsf{M}^l(r') \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \int 4\pi r'^2 \mathrm{d}r' \mathsf{M}^l(r') \frac{1}{r} = \frac{\mathsf{m}^l}{r}. \tag{24}$$

Hence we can simply write Eq. (22) as

$$\phi_o^l(\mathbf{r}) = \frac{k}{\delta_l} \mathsf{m}^l \cdot \nabla^{\otimes l} \frac{1}{r} = k \frac{\mathsf{m}^l \cdot \mathsf{P}^l(\mathbf{r})}{r^{2l+1}}, \tag{25}$$

which is equal to $\phi^l(\mathbf{r})$ in Eq. (18). Thus we have shown that the potential due to a spherically symmetric distribution is equal to the potential due to the point multipole at the center. We further note that, for the dipole moment, not only a uniform dipole density [9] but also any spherically symmetric density with radial variation yields the potential of a point dipole.

The field due to the lth multipole sphere can be obtained from Eq. (25),

$$\mathbf{E}_o^l(\mathbf{r}) = -\nabla\phi_o^l(\mathbf{r}) = -k\frac{\mathsf{m}^l \cdot \nabla^{\otimes l+1}}{\delta_l}\frac{1}{r} = k(2l+1)\frac{\mathsf{m}^l \cdot \mathsf{P}^{l+1}(\mathbf{r})}{r^{2l+3}}, \tag{26}$$

which shows that the field due to a multipole sphere is equal to the field due to the corresponding point multipole. Then the potential and field due to a sphere S_b centered at $\mathbf{r}_b$ are $\phi_b^l(\mathbf{r}) = \phi_o^l(\mathbf{r}-\mathbf{r}_b)$ and $\mathbf{E}_b^l(\mathbf{r}) = \mathbf{E}_o^l(\mathbf{r}-\mathbf{r}_b)$, respectively.

B. Potential energy and force between charge distributions

In the previous section, we have shown that the potential and field due to a spherically symmetric multipole sphere are equal to those due to the corresponding point multipole at the center of the sphere. Now we discuss the energy and force between two spheres.

First we consider a general charge distribution around $\mathbf{r}_a$ and calculate the potential energy due to another distribution around $\mathbf{r}_b$

$$U(\mathbf{r}_a, \mathbf{r}_b) = \int_{V_a} \mathrm{d}^3\mathbf{r}'\rho_a(\mathbf{r}')\phi_b(\mathbf{r}_a + \mathbf{r}'), \tag{27}$$

where the potential is given by Eq. (18). Considering the source at $\mathbf{r}_b$ instead of the origin and using the expansion (15),

$$\begin{aligned}\phi_b(\mathbf{r}_a + \mathbf{r}') &= k\sum_{l=0}^{\infty}\frac{1}{\delta_l}\mathsf{m}_b^l \cdot \nabla_a^{\otimes l}\frac{1}{|\mathbf{r}_{ab} + \mathbf{r}'|} \\ &= k\sum_{l=0}^{\infty}\frac{1}{\delta_l}\mathsf{m}_b^l \cdot \nabla_a^{\otimes l}\sum_{n=0}^{\infty}\frac{1}{n!}\mathsf{P}^n(\mathbf{r}') \cdot \nabla_a^{\otimes n}\frac{1}{r_{ab}},\end{aligned} \tag{28}$$

where ∇_a is the gradient operator with respect to $\mathbf{r}_a$, $\mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b$, and $r_{ab} = |\mathbf{r}_{ab}|$. Plugging the potential into (27), integrating over $\mathbf{r}'$, and using the definition (17), we have

$$U(\mathbf{r}_a, \mathbf{r}_b) = \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} k\frac{(-1)^n}{\delta_n\delta_l}(\mathsf{m}_a^n\mathsf{m}_b^l) \cdot \nabla_a^{\otimes n+l}\frac{1}{r_{ab}} = \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} U_{ab}^{nl}, \tag{29}$$

where U_{ab}^{nl} is the potential energy of two point multipoles m_a^n at $\mathbf{r}_a$ and m_b^l at $\mathbf{r}_b$ and can also be written as, by Eq. (3),

$$U_{ab}^{nl} = k\frac{(-1)^n\delta_{n+l}}{\delta_n\delta_l}(\mathsf{m}_a^n\mathsf{m}_b^l) \cdot \mathsf{P}^{n+l}(\mathbf{r}_{ab}). \tag{30}$$

Thus we have expressed the potential energy of two localized sources as the sum of potential energy of two point multipoles. Here are a few low-order examples:

$$U_{ab}^{0,1} = k m_a \frac{\mathbf{m}_b \cdot \mathbf{r}_{ab}}{r_{ab}^3}, \tag{31}$$

$$U_{ab}^{1,0} = -k m_b \frac{\mathbf{m}_a \cdot \mathbf{r}_{ab}}{r_{ab}^3}, \tag{32}$$

$$U_{ab}^{1,1} = -3 \frac{\mathbf{m}_a \cdot \mathbf{r}_{ab} \mathbf{m}_b \cdot \mathbf{r}_{ab}}{r_{ab}^5} + \frac{\mathbf{m}_a \cdot \mathbf{m}_b}{r_{ab}^3}. \tag{33}$$

Now the force on the distribution around $\mathbf{r}_a$ due to another distribution around $\mathbf{r}_b$

$$\mathbf{F}(\mathbf{r}_a, \mathbf{r}_b) = \int_{V_a} \mathrm{d}^3\mathbf{r}' \rho_a(\mathbf{r}') \mathbf{E}_b(\mathbf{r}_a + \mathbf{r}') \tag{34}$$

can be easily obtained from

$$\mathbf{F}(\mathbf{r}_a, \mathbf{r}_b) = -\nabla_a U(\mathbf{r}_a, \mathbf{r}_b) = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \mathbf{F}_{ab}^{nl}, \tag{35}$$

where $\mathbf{F}_{ab}^{nl}$ is the force between two point multipoles of order n and l. From Eq. (29), we have

$$\mathbf{F}_{ab}^{nl} = -\nabla U_{ab}^{nl} = k \frac{(-1)^n}{\delta_n \delta_l} (\mathsf{m}_a^n \mathsf{m}_b^l) \cdot \nabla_a^{\otimes n+l+1} \frac{1}{r_{ab}}. \tag{36}$$

Or using definition (3), we also write

$$\mathbf{F}_{ab}^{nl} = k \frac{(-1)^n \delta_{n+l+1}}{\delta_n \delta_l r_{ab}^{2n+2l+3}} (\mathsf{m}_a^n \mathsf{m}_b^l) \cdot \mathsf{P}^{n+l+1}(\mathbf{r}_{ab}). \tag{37}$$

C. Energy and force between spheres

Next we consider two multipole spheres S_a and S_b centered at $\mathbf{r}_a$ and $\mathbf{r}_b$ with total moments m_a and m_b of multipole orders n and l, respectively. Since the potential due to a multipole sphere is the potential of the point multipole, the potential energy is obtained by integrating the multipole density over sphere S_a in the multipole potential ϕ_b^l. From Eq. (29), replacing $\mathbf{r}_a$ with $\mathbf{r}_a + \mathbf{r}'$ and m_a with $\int_a \mathrm{d}^3\mathbf{r}' \mathsf{M}_a^n(r')$, we can write the potential energy of the two spheres as

$$U_{S,ab}^{nl} = k \frac{(-1)^n}{\delta_n \delta_l} \int_a \mathrm{d}^3\mathbf{r}' [\mathsf{M}_a^n(r') \mathsf{m}_b^l] \cdot \nabla_a^{\otimes n+l} \frac{1}{|\mathbf{r}_{ab} + \mathbf{r}'|}. \tag{38}$$

With the help of Eq. (24) for $r_{ab} > r'$ (two spheres not overlapped),

$$\int \mathrm{d}^3\mathbf{r}' \mathsf{M}_a^n(r') \frac{1}{|\mathbf{r}_{ab} - \mathbf{r}'|} = \frac{\mathsf{m}_a^l}{r_{ab}}, \tag{39}$$

and Eq. (38) becomes

$$U^{nl}_{S,ab} = k\frac{(-1)^n}{\delta_n\delta_l}\mathsf{m}^n_a\mathsf{m}^l_b \cdot \nabla_a^{\otimes n+l}\frac{1}{r_{ab}} = U^{nl}_{ab}. \tag{40}$$

This equals the energy of two point multipoles in Eq. (29). Therefore the potential energy of a multipole sphere in a field of another multipole sphere is the same as the potential energy of the corresponding point multipoles at the center of each sphere.

The force between the two spheres can be obtained from

$$\mathbf{F}^{nl}_{S,ab} = -\nabla_a U^{nl}_{S,ab}(\mathbf{r}_a). \tag{41}$$

Since $U^{nl}_{S,ab} = U^{nl}_{ab}$, it is obvious that

$$\mathbf{F}^{nl}_{S,ab} = \mathbf{F}^{nl}_{ab}. \tag{42}$$

We can arrive at the same conclusion by starting from Eq. (36). Replacing $\mathbf{r}_a$ with $\mathbf{r}_a + \mathbf{r}'$ and m_a with $\int_a \mathrm{d}^3\mathbf{r}'\mathsf{M}^n_a(r')$, we can write the force between two spheres as

$$\mathbf{F}^{nl}_{S,ab} = \int \mathrm{d}^3\mathbf{r}' k\frac{(-1)^n}{\delta_n\delta_l}[\mathsf{M}^n_a(r')\mathsf{m}^l_b] \cdot \nabla_a^{\otimes n+l+1}\frac{1}{|\mathbf{r}_{ab}+\mathbf{r}'|}. \tag{43}$$

Using Eq. (39) we have

$$\mathbf{F}^{nl}_{S,ab} = k\frac{(-1)^n}{\delta_n\delta_l}\mathsf{m}^n_a\mathsf{m}^l_b \cdot \nabla_a^{\otimes n+l+1}\frac{1}{r_{ab}}. \tag{44}$$

Therefore the force between two spherically symmetric multipole spheres is equal to the force between two point multipoles.

D. Torque between spheres

Similarly, we calculate the torque $\boldsymbol{\tau}(\mathbf{r}_a, \mathbf{r}_b)$ on a charge distribution around $\mathbf{r}_a$ due to another distribution around $\mathbf{r}_b$

$$\boldsymbol{\tau}(\mathbf{r}_a, \mathbf{r}_b) = \int_{V_a} \mathrm{d}^3\mathbf{r}'\rho_a(\mathbf{r}')(\mathbf{r}_a + \mathbf{r}') \times \mathbf{E}_b(\mathbf{r}_a + \mathbf{r}'). \tag{45}$$

The torque can be decomposed into the orbital (orb) and intrinsic (int) parts

$$\boldsymbol{\tau}(\mathbf{r}_a, \mathbf{r}_b) = \boldsymbol{\tau}_{\mathrm{orb}}(\mathbf{r}_a, \mathbf{r}_b) + \boldsymbol{\tau}_{\mathrm{int}}(\mathbf{r}_a, \mathbf{r}_b), \tag{46}$$

defining

$$\boldsymbol{\tau}_{\mathrm{orb}}(\mathbf{r}_a, \mathbf{r}_b) = \mathbf{r}_a \times \int_{V_a} \mathrm{d}^3\mathbf{r}'\rho_a(\mathbf{r}')\mathbf{E}_b(\mathbf{r}_a + \mathbf{r}'). \tag{47}$$

and

$$\boldsymbol{\tau}_{\mathrm{int}}(\mathbf{r}_a, \mathbf{r}_b) = \int_{V_a} \mathrm{d}^3\mathbf{r}' \rho_a(\mathbf{r}')\mathbf{r}' \times \mathbf{E}_b(\mathbf{r}_a + \mathbf{r}'). \tag{48}$$

For the orbital torque, it is obvious from Eqs. (34) and (35) that

$$\boldsymbol{\tau}_{\mathrm{orb}}(\mathbf{r}_a, \mathbf{r}_b) = \mathbf{r}_a \times \mathbf{F}(\mathbf{r}_a, \mathbf{r}_b) = \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} \boldsymbol{\tau}^{nl}_{\mathrm{orb},ab} \tag{49}$$

where

$$\boldsymbol{\tau}^{nl}_{\mathrm{orb},ab} = \mathbf{r}_a \times \mathbf{F}^{nl}_{ab} \tag{50}$$

as expected.

For the intrinsic torque, we use Eq. (26) similarly to Eq. (28)

$$\begin{aligned}\mathbf{E}_b(\mathbf{r}_a + \mathbf{r}') &= -k\sum_{l=0}^{\infty} \frac{1}{\delta_l}\mathsf{m}^l_b \cdot \nabla_a^{\otimes l+1} \frac{1}{|\mathbf{r}_{ab} + \mathbf{r}'|} \\ &= -k\sum_{l=0}^{\infty} \frac{1}{\delta_l}\mathsf{m}^l_b \cdot \nabla_a^{\otimes l+1} \sum_{n=0}^{\infty} \frac{1}{n!}\mathsf{P}^n(\mathbf{r}') \cdot \nabla_a^{\otimes n} \frac{1}{r_{ab}}\end{aligned} \tag{51}$$

where we also have used Eq. (15). Plugging this into Eq. (48) and rearranging one ∇_a operator that has a cross product with $\mathbf{r}'$, we can write

$$\boldsymbol{\tau}_{\mathrm{int}}(\mathbf{r}_a, \mathbf{r}_b) = k\sum_{n=0}^{\infty} \nabla_a \times \int_{V_a} \mathrm{d}^3\mathbf{r}' \rho_a(\mathbf{r}')\mathbf{r}'\mathsf{P}^n(\mathbf{r}') \cdot \nabla_a^{\otimes n} \sum_{l=0}^{\infty} \frac{1}{\delta_l}\mathsf{m}^l_b \cdot \nabla_a^{\otimes l} \frac{1}{n!}\frac{1}{r_{ab}}.$$

Now we use the recurrence relation which can be easily derived from Eq. (3),

$$\mathsf{P}^{n+1}(\mathbf{r}) = \mathbf{r}\mathsf{P}^n(\mathbf{r}) - \frac{r^2}{2n+1}\nabla\mathsf{P}^n(\mathbf{r}), \tag{52}$$

the identity $(\nabla_a \times \nabla_a = 0)$

$$\nabla_a \times \nabla'\mathsf{P}^n(\mathbf{r}') \cdot \nabla_a^{\otimes n} r_{ab}^{-1} = \nabla_a \times \nabla'\mathbf{r}'^{\otimes n} \cdot \nabla_a^{\otimes n} r_{ab}^{-1} = 0, \tag{53}$$

and the definition (17) to derive

$$\boldsymbol{\tau}_{\mathrm{int}}(\mathbf{r}_a, \mathbf{r}_b) = \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} \boldsymbol{\tau}^{n+1,l}_{\mathrm{int},ab}, \tag{54}$$

where $\boldsymbol{\tau}^{n+1,l}_{\mathrm{int},ab}$ is the torque on the $(n+1)$st point multipole at $\mathbf{r}_a$ due to the lth point multipole at $\mathbf{r}_b$

$$\boldsymbol{\tau}^{n+1,l}_{\mathrm{int},ab} = k\frac{(-1)^{n+1}(n+1)}{\delta_l\delta_{n+1}} \nabla_a \times \mathsf{m}^{n+1}_a \cdot^n \nabla_a^{\otimes n}\mathsf{m}^l_b \cdot \nabla_a^{\otimes l} \frac{1}{r_{ab}}. \tag{55}$$

It can also be written as

$$\boldsymbol{\tau}^{n+1,l}_{\text{int},ab} = k\frac{(-1)^n(n+1)!}{\delta_l n!\delta_{n+1}}(\mathsf{m}^{n+1}_a \cdot \nabla^{\otimes n}_a) \times (\mathsf{m}^l_b \cdot \nabla^{\otimes l+1}_a)\frac{1}{r_{ab}}$$
$$= k\frac{(-1)^n(n+1)\delta_{n+l+1}}{\delta_l\delta_{n+1}r^{2n+2l+3}_{ab}}\mathsf{m}^{n+1}_a\mathsf{m}^l_b(\times\cdot^{n+l})\mathsf{P}^{n+l+1}(\mathbf{r}_{ab}). \tag{56}$$

where $(\times\cdot^{n+l})$ denotes an l-fold dot product with m^l_b and an n-fold dot product with m^{n+1}_a followed by a cross product with m^{n+1}_a. Eq. (54) states that the intrinsic torque of two localized sources is the sum of torques between two point multipoles of the source distributions. Here are the lowest-order examples:

$$\boldsymbol{\tau}^{10}_{\text{int},ab} = k\mathsf{m}^1_a \times \mathsf{m}^0_b\frac{\mathbf{r}_{ab}}{r^3_{ab}}, \tag{57}$$
$$\boldsymbol{\tau}^{11}_{\text{int},ab} = k\mathsf{m}^1_a \times \left(\frac{3\mathsf{m}^1_b \cdot \mathbf{r}_{ab}\mathbf{r}_{ab}}{r^5_{ab}} - \frac{\mathsf{m}^1_b}{r^3_{ab}}\right), \tag{58}$$

which reproduce the torques on a dipole, $\mathsf{m}^1_a \times \mathbf{E}(\mathbf{B})$ for the electro(magneto)static case.

Now we consider the intrinsic torque between two multipole spheres. Since the force between two multipole spheres is the same as that of the corresponding point multipoles, it follows from Eq. (50) that the orbital torque between two multipole spheres is equal to the orbital torque between two point multipoles

$$\boldsymbol{\tau}^{nl}_{\text{orb},S,ab} = \mathbf{r}_a \times \mathbf{F}^{nl}_{S,ab} = \mathbf{r}_a \times \mathbf{F}^{nl}_{ab} = \boldsymbol{\tau}^{nl}_{\text{orb},ab}.$$

The intrinsic torque between two multipole spheres is, from Eq. (55),

$$\boldsymbol{\tau}^{n+1,l}_{\text{int},S,ab} = k\frac{(-1)^n(n+1)!}{\delta_l n!\delta_{n+1}}\int \mathrm{d}^3\mathbf{r}'[\mathsf{M}^{n+1}_a(r') \cdot \nabla^{\otimes n}_a] \times (\mathsf{m}^l_b \cdot \nabla^{\otimes l+1}_a)\frac{1}{|\mathbf{r}_{ab} + \mathbf{r}'|}$$

and again with the help of Eq. (24), we can easily see that

$$\boldsymbol{\tau}^{n+1,l}_{\text{int,S},ab} = \boldsymbol{\tau}^{n+1,l}_{\text{int},ab}.$$

Therefore we have shown that the torque between two spherically symmetric multipole spheres is equal to the torque between the corresponding point multipoles located at each center.

IV. DISCUSSION

We have established an equivalency between a spherically symmetric multipole distribution and a single point multipole for any order of multipole, which generalizes the equivalency between a spherical dipole distribution and a point dipole. First, the potential and field produced outside of an isotropic multipole distribution are equivalent to those produced by a single multipole at the center of the distribution. Second, the potential energies of isotropic multipole distribution in the field of others are the same as those of corresponding point multipoles at the center of each isotropic distribution. Finally, the force and torque of one isotropic multipole distribution acting on another is equivalent to those of one point multipole acting on another point multipole.

While the multipole expansion is used to write multipole distributions as the superposition of single point-like multipoles, such as is done in the fast multipole method, we emphasize that this work shows that spherically symmetric multipole moment density distributions can be written as one single corresponding point-like multipole. Moreover, the interaction between spherically symmetric multipole density distributions is the same as the interaction between point-like multipoles. Furthermore, this is not an approximation that is valid only for long-range interactions, but is an exact result for non-overlapping distributions.

The equivalence relationships presented here generalize and relate previously established ones. For example, the subclass of isotropic monopole density distributions provides a familiar special case, often taught to introductory physics students, where Gauss' law can be used to show that the electric field and potential outside an isotropic distribution of electric charge is equivalent to those of a point charge with a charge equal to the net charge of the distribution. Another familiar special case is the equivalence of the magnetic field produced by a uniformly magnetized sphere (hard ferromagnet) and that of a single point magnetic dipole [9, 34]. This work generalizes these cases to any multipole distribution with spherical symmetry.

Jansen made use of the multipole expansion to study the Coulomb interaction between molecules with a high degree of rotational symmetry [38], where the asymptotic properties of such molecules are approximated by values averaged over the azimuthal angle. He established an equivalence theorem between rotationally symmetric charge distributions and an assembly of point charges lying on axis. A special case of his work is an equivalence between isotropic charge distributions and a point charge monopole. Our work extends this special case to isotropic multipole distributions.

This work achieves a significant reduction in the description of interacting isotropic multipole spheres, which may be useful in teaching and research. For example, Edwards, et. al. reduced the interactions between uniformly magnetized spheres to those of single point dipoles [9]. One can construct more complex arrangements of isotropic multipole spheres than uniformly magnetized spheres, such as coated spherical magnets (see introduction), nested magnetized spheres, hollow magnets, complex materials, magnetically interacting astrophysical bodies, etc., where descriptions and numerical simulations of such interacting bodies would be greatly simplified by making use of this reduction.

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