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ORBITAL STABILITY OF SMOOTH SOLITARY WAVES FOR THE DEGASPERIS-PROCESI EQUATION

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ABSTRACT. The Degasperis-Procesi equation is an integrable Camassa-Holm-type model which is an asymptotic approximation for the unidirectional propagation of shallow water waves. This work establishes the orbital stability of localized smooth solitary waves to the Degasperis-Procesi (DP) equation on the real line, extending our previous work on their spectral stability [13]. The main difficulty stems from the fact that the natural energy space is a subspace of L^3, but the translation symmetry for the DP equation gives rise to a conserved quantity equivalent to the L^2-norm, resulting in L^3 higher-order nonlinear terms in the augmented Hamiltonian. But the usual coercivity estimate is in terms of L^2 norm for DP equation, which can not be used to control the L^3 higher order term directly. The remedy is to observe that, given a sufficiently smooth initial condition satisfying some mild constraint, the L^∞ orbital norm of the perturbation is bounded above by a function of its L^2 orbital norm, yielding the higher order control and the orbital stability in the $L^2 \cap L^\infty$ space.

1. INTRODUCTION

Sitting at the intersection of integrable systems and nonlinear hydrodynamic models of shallow water waves, the DP equation [4],

$$m_t + 2ku_x + 3mu_x + um_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

together with the Korteweg-de Vries (KdV) equation [10],

$$u_t + u_{xxx} + uu_x = 0, \tag{1.2}$$

and the Camassa-Holm (CH) equation [1, 7]

$$m_t + 2ku_x + 2mu_x + um_x = 0, \tag{1.3}$$

where $m \triangleq u - u_{xx}$ is the momentum density and $k > 0$ is a parameter related to the critical shallow water speed, has drawn much attention throughout the years. The link between these three equations was established in the same paper where the DP equation was first found: Degasperis and Procesi in 1999 [4] showed that the KdV equation, the CH equation and the DP equation are the only three integrable candidates passing the asymptotic integrability test in a broad family of

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third-order dispersive nonlinear PDEs. Despite strong similarities, they are indeed distinctively different from each other in various ways. See our recent work [13] for more discussion.

For solitons in the DP and CH equation, there are two distinctive scenarios, depending on the value of k.

- In the limiting case of vanishing linear dispersion ($k = 0$), smooth solitons degenerate into peaked solutions, called peakons. The orbital stability of these peakons in the CH and DP equations has been verified respectively in [2] and [14].
- In the case of non-vanishing linear dispersion ($k \neq 0$), while the orbital stability of smooth solitons of the CH equation is well understood by now [3], it is less clear for the DP equation. In fact, we (only) established the spectral stability in our former work [13] and the goal of this paper is to establish orbital stability of smooth solitons of the DP equation.

We first recall the existence result of smooth solitary waves established in [13].

Proposition 1.1 (existence [13]). Given the physical condition $c > 2k > 0$, there exists a unique $c-$speed smooth solitary wave $\phi^c(\xi)$ with its shape depending on c and its maximum height

$$\frac{c-2k}{4} < \Phi_c \triangleq \max_{\xi\in\mathbb{R}}\{\phi^c(\xi)\} < c - 2k.$$

In addition, the function $\phi^c(\xi)$ is even, unimodal and decays monotonically to zero as ξ goes to $\pm\infty$.

The DP equation (1.1), after being applied with the operator $(1-\partial_x^2)^{-1}$, can be rewritten in a weak form in terms of u; that is,

$$\partial_t u + \partial_x\left(\frac{1}{2}u^2 + p*(\frac{3}{2}u^2 + 2ku)\right) = 0, \quad t > 0, \;\; x \in \mathbb{R}, \tag{1.4}$$

where $p(x) = \frac{1}{2}e^{-|x|}$ is the impulse response corresponding to the operator $1 - \partial_x^2$ so that for all $f \in L^2(\mathbb{R})$,

$$(1-\partial_x^2)^{-1} f = p * f.$$

From now on, whenever we mention the DP equation, it is this weak form we refer to. The DP equation can be written as an infinite dimensional Hamiltonian PDE:

$$u_t = J\frac{\delta H}{\delta u}(u), \tag{1.5}$$

where

$$J \triangleq \partial_x(4-\partial_x^2)(1-\partial_x^2)^{-1}, \quad H(u) \triangleq -\frac{1}{6}\int\left(u^3 + 6k\left((4-\partial_x^2)^{-\frac{1}{2}}u\right)^2\right)dx,$$

giving rise to a conserved quantity:

$$S(u) \triangleq \frac{1}{2}\int_{\mathbb{R}} u\cdot(1-\partial_x^2)(4-\partial_x^2)^{-1}u\,dx; \tag{1.6}$$

see [13] for a more detailed discussion. The spatial translation of any solitary wave ψ_c generates a family of solutions, named *the orbit of the solitary wave* ψ_c and denoted as

$$\mathcal{M}_c = \{\phi^c(\cdot + x_0) \mid \; x_0 \in \mathbb{R}\}.$$

As a typical result for nonlinear dispersive PDEs with extra conserved quantities, the solitary wave ϕ^c is not even a critical point of the Hamiltonian. Instead, it is a critical point, but still not a local minimum, of the augmented Hamiltonian

$$Q(u;\lambda) \triangleq H(u)+\lambda S(u) = -\frac{1}{6}\int \left(u^3 + 6k\left((4-\partial_\xi^2)^{-\frac{1}{2}}u\right)^2\right)d\xi + \frac{\lambda}{2}\int_{\mathbb{R}} u\cdot(1-\partial_\xi^2)(4-\partial_\xi^2)^{-1}u\,d\xi.$$

As a result, the best we can hope for is *orbital stability*; that is, a wave starting sufficiently close to the solitary wave ϕ^c remains close to the orbit of the solitary wave up to the time of existence. Indeed, the orbital stability of solitary wave ϕ^c is the main result of this paper.

Theorem 1.1 (Orbital stability)**.** *Assume that $c > 2k > 0$. The solitary wave $\phi^c(x-ct)$ of the DP equation* (1.4) *is orbitally stable in the following sense: for every $\epsilon > 0$, there is $\delta > 0$ such that, for the initial value problem of the DP equation,*

$$\begin{cases} \partial_t u + \partial_x\left(\frac{1}{2}u^2 + p*\left(\frac{3}{2}u^2+2ku\right)\right) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.7}$$

with initial condition satisfying the following properties:

- *(Regularity) There is a positive constant $s > 3/2$ such that $u_0 \in H^s(\mathbb{R})$. In addition,*

$$w_0 \triangleq u_0 - (u_0)_{xx} + \frac{2k}{3} > 0,$$

  is a positive Radon measure in the sense that the mapping $f \mapsto \int_{\mathbb{R}} fw\,dx$ gives a continuous linear functional on the space of compact-supported continuous scalar functions equipped with the canonical limit topology;
- *(Smallness) $\|u_0 - \phi^c\|_{L^2} < \delta$,*

then the solution $u(t,x)$ to the initial value problem (1.7) *is global that*

$$u \in C([0,\infty), H^s(\mathbb{R})) \cap C^1([0,\infty), H^{s-1}(\mathbb{R})),$$

and for any $t \geq 0$,

$$\inf_{x_0\in\mathbb{R}} \|u(t,\cdot)-\phi^c(\cdot - x_0)\|_{L^2} < \epsilon, \quad \inf_{x_0\in\mathbb{R}} \|u(t,\cdot)-\phi^c(\cdot - x_0)\|_{L^\infty} < C\epsilon^{\frac{2}{3}},$$

for some $C > 0$ independent of ϵ.

Remark 1.1. The set of initial profiles satisfying the regularity and smallness conditions in Theorem 1.1 is not empty. It is straightforward to verify that $\phi^c - \phi^c_{xx} > 0$. Therefore, a sufficient condition for the regularity requirements is that $u_0 \in H^s(\mathbb{R})$ with some $s \geq 3$ and

$$\|u_0 - \phi^c\|_{H^3} \leq \frac{2\sqrt{2}}{3}k;$$

Remark 1.2. The global existence of strong solutions was given in [16, 21] for $k = 0$. The proof for the case $k > 0$ is a slight modification of the $k = 0$ one and given in Section 2. There is also a global existence of weak solutions in L^2-space given in [5]. The regularity requirement in Theorem (1.1) can be relaxed to

$$u_0 \in L^2(\mathbb{R}), \quad w = u_0 - (u_0)_{xx} + \frac{2k}{3} \text{ is a positive Radon measure.}$$

The peakon case when $k = 0$ can be seen in [11].

The orbital stability proof follows the framework seminally developed by Grillakis, et.al. [8, 12], with extra work on (cubic) nonlinear estimates. Typically, this framework requires

- The linear operator, corresponding to the second variational derivative of the augmented Hamiltonian, admits certain spectral properties.
- Convexity of the scalar function which maps velocity to the augmented Hamiltonian evaluated at the solitary wave with that specific velocity.

While for the CH and KdV equations, the above two lead to orbital stability in energy space, it is not for the DP equation. We obtained the above for the DP equation [13], but only concluded the spectrum stability applying the framework of [15]. The problem lies in the higher order term, say $\int h^3\,dx$, which can not generically be bounded above solely by the L^2-norm of h. The remedy is to control the L^∞-norm in terms of the L^2-norm by imposing additional (mild) regularity on the initial condition. We also remark that for the case of null-linear dispersion, the uniform L^∞ control is not needed. Instead, the control of a point distance is enough, say the difference between the peak of the peakon and that of the perturbation [16].

2. Well-posedness and *a priori* estimates

The well-posedness of the initial value problem serves as the precondition of any qualitative study of the dynamics.

A local well-posedness result for the Cauchy problem (1.7) with $k = 0$ is obtained in [20] via applying Kato's theorem [9]. With exactly the same argument, we have the following local well-posedness result for the Cauchy problem (1.7) with $k > 0$.

Proposition 2.1 (Uniqueness and local existence of strong solutions). Given the initial profile $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, there exist a maximal time $T = T(u_0) \in (0, \infty]$, independent of the choice of s, and a unique solution u to the Cauchy problem (1.7) such that

$$u = u(\cdot; u_0) \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R})).$$

Moreover, the solution depends continuously on the initial data.

Furthermore, the strong solution is a global one if the initial condition is sufficiently "regular". More specifically, the arguments of [16, 21] lead to the following global existence result.

Proposition 2.2 (Global existence of strong solutions). Given that the initial profile $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$ and $w = u_0 - u_{0,xx} + \frac{2}{3}k$ is a Radon measure of fixed sign, the strong solution to the Cauchy problem (1.7) then exists globally in time; that is,

$$u = u(\cdot; u_0) \in C([0,\infty); H^s(\mathbb{R})) \cap C^1([0,\infty); H^{s-1}(\mathbb{R})),$$

which admits the following additional estimates.

(1) The magnitude of u_x is bounded above by the sum of the magnitude of u and the constant $\frac{2k}{3}$. As a matter of fact, we have, for all $(t, x) \in [0,\infty) \times \mathbb{R}$,

$$|u_x(t,x)| \le |u(t,x) + \frac{2}{3}k|. \tag{2.1}$$

(2) The L^∞ norm of u is bounded. More specifically, we have, for all $t \in [0,\infty)$,

$$\|u(t,\cdot)\|_{L^\infty} \leq \sqrt{2}(1+\sqrt{2})\|u_0\|_{L^2(\mathbb{R})} + \frac{4}{3}k. \tag{2.2}$$

The following *a priori* estimate is useful in higher order nonlinear control.

Proposition 2.3 (*a priori* L^∞-L^2 estimate). Let $\psi \in W^{1,\infty} \cap L^2(R)$ and the initial data $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$ and $w_0 = m_0 + \frac{2k}{3}$ a Radon measure of fixed sign. The difference between the strong solution u to the Cauchy problem (1.7) and the function ψ, denoted as $g(t,x) \triangleq u(t,x) - \psi(x)$, admits the following estimate

$$\|g(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \|g(t,\cdot)\|_{L^2(\mathbb{R})}^{2/3}\bigg(1+\frac{4}{3}k+\sqrt{2}\|g(t,\cdot)\|_{L^2(\mathbb{R})}^{2/3}+2\|\psi\|_{L^\infty(\mathbb{R})}+2\|\psi'\|_{L^\infty(\mathbb{R})}\bigg), \quad \forall t \in [0,\infty). \tag{2.3}$$

Proof. Fix $t \in [0,\infty)$, we denote $\alpha(t) = \|g(t,\cdot)\|_{L^2(\mathbb{R})}^{2/3}$ and assume $\alpha(t) > 0$, due to the fact that the case $\alpha(t) = 0$ makes both sides of (2.3) zero. Fixing $x \in \mathbb{R}$, there exists $k \in \mathbb{Z}$ such that $x \in [k\alpha(t),\, (k+1)\alpha(t))$. By the mean value theorem, there exists $\bar{x} \in [(k-1)\alpha(t),\, k\alpha(t)]$ such that

$$g^2(t,\bar{x}) = \frac{1}{\alpha(t)}\int_{(k-1)\alpha(t)}^{k\alpha(t)} g^2(t,\eta)d\eta \leq \frac{1}{\alpha(t)}\|g(t,\cdot)\|_{L^2(\mathbb{R})}^2 = \alpha(t)^2,$$

which, together with Proposition 2.2 and that $0 \leq x - \bar{x} \leq 2\alpha(t)$, yields

$$\begin{aligned}
g(t,x) &= g(t,\bar{x}) + \int_{\bar{x}}^{x} g_\eta(t,\eta)d\eta \\
&\geq -\alpha(t) - \frac{4\alpha(t)}{3}k - \sqrt{2\alpha(t)}\left\|\Big(|u(t,\cdot)| + |\psi'|\Big)\right\|_{L^2([(k-1)\alpha(t),(k+1)\alpha(t)])} \\
&\geq -\alpha(t) - \frac{4\alpha(t)}{3}k - \sqrt{2\alpha(t)}\left\|\Big(|g(t,\cdot)| + |\psi| + |\psi'|\Big)\right\|_{L^2([(k-1)\alpha(t),(k+1)\alpha(t)])} \\
&\geq -\alpha(t) - \frac{4\alpha(t)}{3}k - \sqrt{2\alpha(t)}\left[\|g(t,\cdot)\|_{L^2(\mathbb{R})} + \sqrt{2\alpha(t)}\Big(\|\psi\|_{L^\infty(\mathbb{R})} + \|\psi'\|_{L^\infty(\mathbb{R})}\Big)\right] \\
&= -\alpha(t)\bigg(1 + \frac{4}{3}k + \sqrt{2}\alpha(t) + 2\|\psi\|_{L^\infty(\mathbb{R})} + 2\|\psi'\|_{L^\infty(\mathbb{R})}\bigg).
\end{aligned} \tag{2.4}$$

We prove by contradiction and suppose that there exists $x_* \in \mathbb{R}$ such that

$$g(t,x_*) > \alpha(t)\bigg(1 + \frac{4}{3}k + \sqrt{2}\alpha(t) + 2\|\psi\|_{L^\infty(\mathbb{R})} + 2\|\psi'\|_{L^\infty(\mathbb{R})}\bigg).$$

Then there exists $k_* \in \mathbb{R}$ with $x_* \in [k_*\alpha(t), (k_*+1)\alpha(t))$ such that by the mean value theorem, there exists $\bar{x}_* \in [(k_*+1)\alpha(t), (k_*+2)\alpha(t)]$ such that, on one hand,

$$g^2(t,\bar{x}_*) = \frac{1}{\alpha(t)}\int_{(k_*+1)\alpha(t)}^{(k_*+2)\alpha(t)} g^2(t,\eta)d\eta \leq \alpha(t)^2.$$

On the other hand, proceeding as in (2.4), we have

$$\begin{aligned}u(t,\bar{x}_*)-\psi(\bar{x}_*)=&u(t,x_*)-\psi(x_*)+\int_{x_*}^{\bar{x}_*}[u_\eta(t,\eta)-\psi'(\eta)]d\eta\\ >&\alpha(t)\bigg(1+\frac{4}{3}k+\sqrt{2}\alpha(t)+2\|\psi\|_{L^\infty(\mathbb{R})}+2\|\psi'\|_{L^\infty(\mathbb{R})}\bigg)-\\ &\frac{4\alpha(t)}{3}k-\sqrt{2\alpha(t)}\bigg\|\bigg(|u(t,\cdot)|+|\psi'|\bigg)\bigg\|_{L^2([k\alpha(t),(k+2)\alpha(t)])}\\ \geq&\alpha(t)\bigg(1+\frac{4}{3}k+\sqrt{2}\alpha(t)+2\|\psi\|_{L^\infty(\mathbb{R})}+2\|\psi'\|_{L^\infty(\mathbb{R})}\bigg)-\\ &\frac{4\alpha(t)}{3}k-\alpha(t)\left(\sqrt{2}\alpha(t)+2\|\psi\|_{L^\infty(\mathbb{R})}+2\|\psi'\|_{L^\infty(\mathbb{R})}\right)\\ =&\alpha(t).\end{aligned}$$

The incompatibility of the above two estimates concludes the proof of the proposition. □

3. Orbital Stability of Degasperis-Procesi Solitons

In this section, we prove Theorem 1.1, based on the framework established by Weinstein[18, 19] and Grillakis *et al.* [8] with major modifications on nonlinear estimates. We also refer to [6, 12, 17] for a more contemporary review of this framework.

The smooth solitary wave ϕ^c of the DP equation is a critical point of the augmented Hamiltonian

$$Q(u;c)=Q_c(u)\triangleq H(u)+cS(u).$$

The scalar function

$$R(c)\triangleq Q(\phi^c,c)\text{ with }R(c)\;:\;(2k,\infty)\longrightarrow\mathbb{R}, \tag{3.1}$$

is shown to be strictly convex in [13, Lemma 4.2].

Lemma 3.1 (Convexity, [13]). *The function R is strictly convex in the sense that*

$$R''(c)=\frac{d}{dc}\bigg(S(\phi^c)\bigg)>0,\quad\forall c>2k. \tag{3.2}$$

It is standard that one only needs to prove Theorem 1.1 under the extra conservation constraint

$$S(u)=S(\phi^c).$$

For convenience, we from now on fix $c>2k$, suppress the supper index of ϕ^c and also introduce a local foliation of a neighborhood of the orbit $\mathcal{M}_c=\{\phi(\cdot+x_0\mid x_0\in\mathbb{R})\}$. More specifically, there exists $\delta_1>0$ such that for any

$$u\in\mathcal{N}_c\triangleq\{v\in L^2(\mathbb{R})\mid\inf_{x_0\in\mathbb{R}}\|v(\cdot)-\phi(\cdot-x_0)\|_{L^2(\mathbb{R})}<\delta_1\},$$

there exists a unique foliation decomposition

$$u=\mathcal{T}(r)\bigg(\phi+h\bigg),$$

where $\mathcal{T}(r)u(\cdot) \triangleq u(\cdot + r)$ is the translation operator and $h \in L^2(\mathbb{R})$ is perpendicular to $\partial_x\phi$; that is $(h, \partial_x\phi) \triangleq \int_{\mathbb{R}} h\partial_x\phi dx = 0$. If the initial data falls in the neighborhood $\mathcal{N}_c$; that is,

$$\|u_0 - \phi\|_{L^2(\mathbb{R})} < \delta_1,$$

there exist a maximal time $T_m > 0$ such that the strong solution u stays within $\mathcal{N}_c$ for $t \in [0, T_m)$; that is,

$$T_m \triangleq \max_{T \geq 0}\{T \mid u(t,\cdot) \in \mathcal{N}_c, \forall t \in [0,T)\} > 0.$$

As a result, for $t \in [0, T_m)$, the strong solution admits the foliation decomposition

$$u(t,x) = \mathcal{T}(r(t))\Big(\phi(x) + h(t,x)\Big),$$

where $(h(t,\cdot), \partial_x\phi) = 0$.

We now introduce the time-invariant quantity

$$\overline{Q_c} \triangleq Q_c(u) - Q_c(\phi),$$

whose expansion in terms of h admits the expression

$$\overline{Q_c} = Q_c(\phi(x) + h(t,x)) - Q_c(\phi) = \frac{1}{2}(L_c h, h) - \frac{1}{6}\int h^3 d\xi, \tag{3.3}$$

where

$$L_c \triangleq \frac{\delta^2 Q_c}{\delta u^2}(\phi) = c - \phi - (3c + 2k)(4 - \partial_x^2)^{-1}, \tag{3.4}$$

and $h(t,\cdot)$ lies in the nonlinear admissible set

$$\mathcal{A} \triangleq \{h \in L^2(\mathbb{R}) \mid S(h + \phi) = S(\phi), (h, \partial_x\phi) = 0\}.$$

We established in [13] the following properties about L_c:

Proposition 3.1. [13] The spectrum set of the operator $L_c : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, denoted as $\sigma(L_c)$, admits the following properties.

(1) The spectrum set $\sigma(L_c)$ lies on the real line; that is, $\sigma(L_c) \subset \mathbb{R}$.
(2) 0 is a simple eigenvalue of L_c with ϕ_x as its eigenfunction.
(3) On the negative axis $(-\infty, 0)$, the spectrum set $\sigma(L_c)$ admits nothing but only one simple eigenvalue, denoted as λ_*, with its corresponding normalized eigenfunction, denoted as ϕ_*.
(4) The set of essential spectrum $\sigma_{ess}(L_c)$ lies on the positive real axis, admitting a positive distance to the origin.

With these properties, it is standard to prove the following

Proposition 3.2. For sufficiently small $h \in \mathcal{A}$, there exist $\alpha, \beta > 0$ such that

$$\frac{1}{2}(L_c h, h) \geq \alpha\|h\|^2_{L^2(\mathbb{R})} - \beta\|h\|^3_{L^2(\mathbb{R})}. \tag{3.5}$$

We now give the proof of Theorem 1.1.

Proof of Theorem 1.1. For convenience, we assume $\delta < 1$. We first derive an upper bound of $|\overline{Q_c}|$ in terms of the L^2 norm of $h_0(x) \triangleq h(0,x)$. We recall from Proposition

1.1 that $\Phi_c = \max_{\xi\in\mathbb{R}}\{\phi(\xi)\} < c-2k$ and obtain the following lower and upper bounds of $(L_c h, h)$,

$$c\|h\|^2_{L^2(\mathbb{R})} \geq (L_c h, h) \geq (c - \Phi_c - \frac{3c+2k}{4})\|h\|^2_{L^2(\mathbb{R})} \geq (-\frac{3}{4}c + \frac{3}{2}k)\|h\|^2_{L^2(\mathbb{R})} \geq -c\|h\|^2_{L^2(\mathbb{R})},$$

which, together with the expansions (3.3) and (3.4), yield

$$|\overline{Q_c}| \leq \frac{1}{2}c\|h_0\|^2_{L^2(\mathbb{R})} + \frac{1}{6}\|h_0\|_{L^\infty(\mathbb{R})} \cdot \|h_0\|^2_{L^2(\mathbb{R})}. \tag{3.6}$$

According to Proposition 2.3, the L^∞ norm of h admits the following estimate

$$\begin{aligned}\|h(t,\cdot)\|_{L^\infty(\mathbb{R})} =& \|\mathcal{T}(r(t))h(t,\cdot)\|_{L^\infty(\mathbb{R})} = \|u(t,\cdot) - \mathcal{T}(r(t))\phi(\cdot)\|_{L^\infty} \\ \leq& \|h(t,\cdot)\|^{2/3}_{L^2(\mathbb{R})}\left(1 + \frac{4}{3}k + \sqrt{2}\|h(t,\cdot)\|^{2/3}_{L^2(\mathbb{R})} + 2\|\phi\|_{L^\infty(\mathbb{R})} + 2\|\phi'\|_{L^\infty(\mathbb{R})}\right).\end{aligned} \tag{3.7}$$

The estimate (3.7) for $t = 0$, plugged into the estimate (3.6), leads to

$$|\overline{Q_c}| \leq \frac{1}{2}c\|h_0\|^2_{L^2(\mathbb{R})} + \gamma\|h_0\|^{8/3}_{L^2(\mathbb{R})} + \frac{\sqrt{2}}{6}\|h_0\|^{10/3}_{L^2(\mathbb{R})} < K\delta^2, \tag{3.8}$$

where $\gamma(c,k) \triangleq \frac{1}{6}\big(1 + \frac{4}{3}k + 2\|\phi\|_{L^\infty(\mathbb{R})} + 2\|\phi'\|_{L^\infty(\mathbb{R})}\big)$ and $K \triangleq \max\{\frac{1}{2}c, \gamma, \frac{\sqrt{2}}{6}\}$.

Similarly, we also derive a lower bound of $|\overline{Q_c}|$ in terms of the L^2 norm of $h_0(x) \triangleq h(0,x)$. We first conclude from the expansion (3.3) and the inequality (3.5) that

$$|\overline{Q_c}| \geq \alpha\|h\|^2_{L^2(\mathbb{R})} - \beta\|h\|^3_{L^2(\mathbb{R})} - \frac{1}{6}\|h\|_{L^\infty(\mathbb{R})} \cdot \|h\|^2_{L^2(\mathbb{R})}. \tag{3.9}$$

which, together with the inequality (3.7), yields that

$$\begin{aligned}|\overline{Q_c}| \geq& \alpha\|h\|^2_{L^2(\mathbb{R})} - \beta\|h\|^3_{L^2(\mathbb{R})} - \frac{1}{6}\|h\|^{8/3}_{L^2(\mathbb{R})}\left(1 + \frac{4}{3}k + \sqrt{2}\|h\|^{2/3}_{L^2(\mathbb{R})} + 2\|\phi\|_{L^\infty(\mathbb{R})} + 2\|\phi'\|_{L^\infty(\mathbb{R})}\right) \\ =& \alpha\|h\|^2_{L^2(\mathbb{R})} - \gamma\|h\|^{\frac{8}{3}}_{L^2(\mathbb{R})} - \beta\|h\|^3_{L^2(\mathbb{R})} - \frac{\sqrt{2}}{6}\|h\|^{\frac{10}{3}}_{L^2(\mathbb{R})},\end{aligned} \tag{3.10}$$

For small $|\overline{Q_c}|$, the function

$$f(r) \triangleq |\overline{Q_c}| - \alpha r^2 + \gamma r^{\frac{8}{3}} + \beta r^3 + \frac{\sqrt{2}}{6}r^{\frac{10}{3}}$$

admits two consecutive positive roots

$$0 < r_1 = \mathcal{O}(|\overline{Q_c}|^{1/2}) < r_2 = \mathcal{O}(1),$$

which, together with the estimate (3.8), shows that

$$r_1 = \mathcal{O}(\delta).$$

As a result, there exists $\delta_0 \in (0,1)$ such that

$$r_1 < \min\{\epsilon, \delta_1, \frac{1}{2}\left(\frac{\alpha}{\gamma}\right)^{3/2}\} < r_2.$$

Furthermore, we conclude from the inequality (3.10) and the continuity of $h(t)$ that if $\|h_0\|_{L^2(\mathbb{R}^2)} \in (0, r_1)$, then $\|h(t,\cdot)\|_{L^2(\mathbb{R}^2)} \in (0, r_1)$ holds globally for $t \in [0,\infty)$. Therefore, for any $\varepsilon > 0$, we can choose $\delta = \delta_0$ such that if

$$\|u_0 - \phi\|_{L^2(\mathbb{R})} = \|h_0\|_{L^2(\mathbb{R})} \leq \delta,$$

then

$$\inf_{r\in\mathbb{R}} \|u(t,\cdot) - \mathcal{T}(r)\phi\|_{L^2(\mathbb{R})} = \|h(t)\|_{L^2(\mathbb{R})} < r_1 < \varepsilon, \quad \forall t \in [0,\infty).$$

Noting that the L^∞ estimate in the theorem follows from the L^2 estimate and Proposition 2.3, we conclude the proof of Theorem 1.1. □

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