A series representation of the discrete fractional Laplace operator of arbitrary order

Tiffany Frugé Jones[a], Evdokiya Georgieva Kostadinova[b], Joshua Lee Padgett[b,c,*], Qin Sheng[b,d]

[a]*Department of Mathematics, The University of Arizona, Tucson, Arizona 85721, USA.*
[b]*Center for Astrophysics, Space Physics and Engineering Research, Baylor University, Waco, Texas 76798, USA.*
[c]*Department of Mathematical Sciences, University of Arkansas, Fayetteville, Arkansas 72701, USA.*
[d]*Department of Mathematics, Baylor University, Waco, Texas 76798, USA.*

Abstract

Although fractional powers of non-negative operators have received much attention in recent years, there is still little known about their behavior if real-valued exponents are greater than one. In this article, we define and study the discrete fractional Laplace operator of arbitrary real-valued positive order. A series representation of the discrete fractional Laplace operator for positive non-integer powers is developed. Its convergence to a series representation of a known case of positive integer powers is proven as the power tends to the integer value. Furthermore, we show that the new representation for arbitrary real-valued positive powers of the discrete Laplace operator is consistent with existing theoretical results.

1. Introduction

Due to its wide array of applications in multi-physical sciences, the construction and approximation of fractional powers of the Laplace operator have been of great interest for nearly a century (cf., e.g., [8, 39, 43, 53, 61] and references therein). Classically, only fractional powers of the order $s \in (0,1)$ are considered, and in this case, one can define the fractional Laplace operator applied to a smooth enough function in a natural way. Specifically, for $d \in \mathbb{N} = \{1,2,3,\ldots\}$, $s \in (0,1)$ let $u\colon \mathbb{R}^d \to \mathbb{R}$ be a smooth function, and for every $\varepsilon \in (0,\infty)$, $x \in \mathbb{R}^d$ let $B_\varepsilon(x)$ be the d-dimensional ball of radius ε centered at x (with respect to the typical topology of $\mathbb{R}^d$). Then for every $x \in \mathbb{R}^d$ we may define the s-order fractional Laplace operator applied to u at x as

$$\big((-\Delta)^s u\big)(x) = c_{d,s} \lim_{\varepsilon \to 0^+} \left[\int_{\mathbb{R}^d \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{d+2s}}\, dy \right], \tag{1.1}$$

where $c_{d,s} \in [0,\infty)$ is a known normalization constant.

It is worth noting that the recent rapid increase in interest in the fractional Laplace operator is also due to the seminal work of Caffarelli and Silvestre [9]. In their work, it was shown that one may study the non-local operator given by (1.1) via the Dirichlet-to-Neumann operator associated

*Corresponding author

Email addresses: tnjones@math.arizona.edu (Tiffany Frugé Jones), eva_kostadinova@baylor.edu (Evdokiya Georgieva Kostadinova), padgett@uark.edu (Joshua Lee Padgett), qin_sheng@baylor.edu (Qin Sheng)

with a particular extension problem posed in $\mathbb{R}^d \times [0,\infty)$ (albeit, one trades the non-locality for a problem which is either singular or degenerate depending upon the value of $s \in (0,1)$). The employed Dirichlet-to-Neumann operator is a particular example of the Poincaré-Stecklov operator (cf., e.g., [37]). For a fixed domain, the Poincaré-Stecklov operator is known to map the boundary values of a harmonic function to the normal derivative values of the same harmonic function on the same boundary. We can summarize the results of Caffarelli and Silvestre (cf., e.g., [9, Eq. (3.1)]) as follows. Let $d \in \mathbb{N}$, $s \in (0,1)$, let $u\colon \mathbb{R}^d \to \mathbb{R}$ be a smooth function, and let $v\colon [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ satisfy for all $x \in \mathbb{R}^d$ that $v(0,x) = u(x)$ and for all $t \in (0,\infty)$, $x \in \mathbb{R}^d$ that

$$\Big(\tfrac{\partial^2}{\partial t^2} v\Big)(t,x) + \tfrac{1-2s}{t}\big(\tfrac{\partial}{\partial t} v\big)(t,x) + (\Delta_x v)(t,x) = 0. \tag{1.2}$$

Then there exists $c \in [0,\infty)$ such that for all $x \in \mathbb{R}^d$ it holds that

$$\big((-\Delta)^s u\big)(x) = c\bigg[\lim_{t\to 0^+} t^{1-2s}\big(\tfrac{\partial}{\partial t} v\big)(t,x)\bigg]. \tag{1.3}$$

Interestingly, the constant $c \in [0,\infty)$ in (1.3) depends only upon the parameter $s \in (0,1)$ and not upon $d \in \mathbb{N}$. More importantly, this demonstrates that one may trade out the highly non-local problem given by (1.1) for the local problem given by (1.2) and (1.3). This technique has also been recently further generalized to cases of arbitrary non-negative operators defined on Banach spaces [5, 21, 44, 45, 50].

While the above formulations (i.e., (1.1), (1.2), and (1.3)) may be used to provide insights into the continuous fractional Laplace operator with order $s \in (0,1)$, they cannot be directly used to provide any insight into the discrete case or the case where $s \in (0,\infty)$. The discrete case is a natural consideration as it arises in the study of numerous physically relevant phenomena (cf., e.g., [31, 32, 51] and references therein) and also in an attempt to numerically approximate (1.1). The consideration of a truly discrete case—that is, the case which is the fractional power of the discrete Laplace operator rather than a direct approximation of (1.1)—was originally studied by Ciaurri et al. [13]. By employing the basic language of semigroups (e.g., a special case of Ciaurri et al. [12, Eq. (1)] combined with, e.g., Padgett [50, Theorem 2.1]) Ciaurri et al. were able to develop the first series representation for the discrete fractional Laplace operator of order $s \in (0,1)$ (cf. Definition 4.10, for clarity). Moreover, it was shown that this formulation did converge to the continuous case via adaptive mesh refinements (cf. Ciaurri et al. [12, Theorems 1.7 and 1.8]). However, it important to note that while this aforementioned convergence was observed, it is the case that the series representation developed by Ciaurri et al. is an *exact* representation and not a numerical approximation.

The consideration of higher-order fractional Laplace operators has recently received an increase in attention in continuous cases (cf., e.g., [11, 19, 22, 54, 62]), but to the authors' knowledge the only study in the discrete case has been carried out by Padgett et al. [51]. Rectifying this aforementioned gap in theory is the primary goal of this article (although the applicability of such derivations in the study of localization will be outlined in Section 2 below). In particular, we develop a series representation of the discrete fractional Laplace operator of order $s \in (0,\infty)$. This development is illustrated in Theorem 1.1, which is a partial description of the main result of this article focused on the case of positive non-integer powers of the discrete Laplace operator.

Theorem 1.1. *Let $m \in \mathbb{N}$, $s \in (m-1,m)$, let $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$, let $\mathbb{R}$ be the real number field, let $\ell^2(\mathbb{Z})$ be the set of all $w\colon \mathbb{Z} \to \mathbb{R}$ which satisfy that $\sum_{k\in\mathbb{Z}}|w(k)|^2 < \infty$, let $-\Delta\colon \ell^2(\mathbb{Z}) \to$*

$\ell^2(\mathbb{Z})$ satisfy for all $w \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ that $(-\Delta w)(n) = 2w(n) - w(n-1) - w(n+1)$, let $u \in \ell^2(\mathbb{Z})$, and let[1] *$v\colon [0,\infty)\times\mathbb{Z} \to \mathbb{R}$ satisfy for all $n \in \mathbb{Z}$ that $v(0,n) = ((-\Delta)^{m-1}u)(n)$ and for all $t \in (0,\infty)$, $n \in \mathbb{Z}$ that*

$$\left(\tfrac{\partial^2}{\partial t^2} v\right)(t,x) + \tfrac{1-2(s-m+1)}{t}\left(\tfrac{\partial}{\partial t} v\right)(t,x) + (\Delta v)(t,x) = 0. \tag{1.4}$$

Then

(i) there exists $c \in [0,\infty)$ such that for all $n \in \mathbb{Z}$ it holds that

$$\big((-\Delta)^{s-m+1}(-\Delta)^{m-1}u\big)(n) = \big((-\Delta)^s u\big)(n) = c\left[\lim_{t\to 0^+} t^{1-2(s-m+1)}\left(\tfrac{\partial}{\partial t} v\right)(t,n)\right] \tag{1.5}$$

and

(ii) there exists $K\colon \mathbb{Z} \to \mathbb{R}$, $C \in [0,\infty)$ such that for all $n \in \mathbb{Z}\backslash\{0\}$ it holds that $|K(n)| \leq C|n|^{-(1+2s)}$ and for all $n \in \mathbb{Z}$ it holds that $K(-n) = K(n)$ and

$$\big((-\Delta)^s u\big)(n) = \sum_{k\in\mathbb{Z}} K(k)\big(u(n) - u(n-k)\big). \tag{1.6}$$

We now provide some clarifying remarks regarding the objects in Theorem 1.1. In Theorem 1.1 we intend to construct an exact series representation of the so-called co-normal derivative of the function $v(0,\cdot)\colon \mathbb{Z} \to \mathbb{R}$. The positive real number $s \in (0,\infty)$ describes the fractional power of the discrete Laplace operator, the positive integer m describes the smallest positive integer that is greater than or equal to $s \in (0,\infty)$, and the set $\ell^2(\mathbb{Z})$ is the standard Hilbert space of square-summable sequences defined on the integers. The operator $-\Delta\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is the standard one-dimensional discrete Laplace operator and is the primary object used in the construction of the desired series representation. The function $v\colon [0,\infty)\times\mathbb{Z} \to \mathbb{R}$ is the solution to the extension problem in (1.4) and the trace of this function will coincide with the function obtained by applying the discrete fractional Laplace operator of order $s \in (0,\infty)$ to some given square-summable function $u\colon \mathbb{Z} \to \mathbb{R}$.

We now provide some clarifying remarks regarding the results in Theorem 1.1. Item (i) of Theorem 1.1 above is a direct consequence of combining Definition 4.11 and Padgett [50, Theorem 2.1] (applied for every $n \in \mathbb{Z}$ with $s \curvearrowleft s - m + 1$, $A \curvearrowleft \Delta$, $u_0 \curvearrowleft ((-\Delta)^{m-1}u)(n)$, $(u(t))_{t\in[0,\infty)} \curvearrowleft (v(t,n))_{t\in[0,\infty)}$ in the notation of Padgett [50, Theorem 2.1]). See the beginning of Section 3 for an explanation of this "applied with" notation (i.e., the symbol "$\curvearrowleft$"). The right-hand side of (1.5) is not considered in detail, herein, as it is an elementary consequence of Padgett [50, Theorem 2.1]. Item (ii) of Theorem 1.1 follows directly from Lemma 5.4 and Lemma 6.1.

The main result of this article is Theorem 6.4 in Section 6 below. This result provides a complete description of the series representation of the discrete fractional Laplace operator. The most surprising implication of Theorem 6.4 is that the formula for the function $K\colon \mathbb{Z} \to \mathbb{R}$ in Theorem 1.1 depends only on the parameter $s \in (0,\infty)$ (cf. Definition 5.1 below). In fact, this function is continuous with respect to the parameter s for all $s \in (0,\infty)\backslash\mathbb{N}$ with the points $s \in \mathbb{N}$

[1] Note that we define integer powers of $-\Delta\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ inductively. That is, we have for all $k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, $w \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ that if $k = 0$ it holds that $((-\Delta)^k w)(n) = -w(n)$ and if $k \in \mathbb{N}$ it holds that $((-\Delta)^k w)(n) = (-\Delta(-\Delta)^{k-1}w)(n)$.

all being removable singularities of the function $K\colon \mathbb{Z}\to\mathbb{R}$. Hence, we may extend the definition of $K\colon \mathbb{Z}\to\mathbb{R}$ to that of an analytic function (cf. (6.20) of Theorem 6.4).

The remainder of this article is organized as follows. In Section 2 we briefly motivate our interest in the development of a series representation for the discrete fractional Laplace operator of arbitrary order. In particular, we focus on its application to the study of the Anderson localization in materials science and its application to transport problems in plasma physics. Next, in Section 3 we recall several basic definitions and properties of sequence spaces and introduce the so-called logarithmic norm. Afterwards, in Section 4 we define the discrete Laplace operator of arbitrary real-valued positive order. We do so by introducing the heat semigroup generated by the discrete Laplace operator and then defining higher-order powers via induction. In Section 5 we define a discrete fractional kernel function and provide a detailed investigation of its various quantitative and qualitative properties. Thereafter, in Section 6 we construct a series representation for real-valued positive powers of the discrete fractional Laplace operator by employing the results from Sections 4 and 5. Finally, in Section 7, a number of useful concluding remarks are provided. Continuing avenues of research based on the results developed in this article are also outlined.

2. Motivation of study

In 1958, P. W. Anderson suggested that the existence of sufficiently large disorder in a semiconductor could lead to spatial localization of electrons [4]. This localization of electrons in space has since been referred to as *Anderson localization.* In an effort to better understand the conditions under which Anderson localization may occur, there have been numerous theoretical and experimental studies of the phenomenon. As such, three definitions of how one can mathematically define Anderson localization have emerged: dynamical localization, statistical localization, and spectral localization. Localization in the dynamical sense is characterized by an exponential decay with respect to time of the wave function which represents the particle of interest. Localization in the statistical sense occurs if the eigenvalues of of the system's associated Hamiltonian are discrete and infinitely close to one another when projected onto a finite-dimensional subspace. These two definitions of localization have often been related to the classical techniques used to study localization; e.g., scaling and perturbation theory (cf., e.g., [2, 7, 20, 33, 36, 52] and the references therein). The third definition—the spectral definition—uses the spectrum of the system's infinite-dimensional Hamiltonian operator to study Anderson localization (cf., e.g., [3, 26, 27]). In particular, this definition states that if the Hamiltonian driving the physical system exhibits absolutely continuous spectrum then the system dynamics will not be localized. This spectral definition will be the one of interest in the ensuing motivation.

We now provide a brief motivation for our interest in the development of a series representation of the discrete fractional Laplace operator of arbitrary order. Let $\mathfrak{C}, s, c, T \in (0,\infty)$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $\ell^2(\mathbb{Z})$ be the set of all $v\colon \mathbb{Z}\to\mathbb{R}$ which satisfy that $\sum_{k\in\mathbb{Z}}|v(k)|^2<\infty$, let $\langle\cdot,\cdot\rangle\colon \ell^2(\mathbb{Z})\times\ell^2(\mathbb{Z})\to\mathbb{R}$ satisfy for all $v,w\in\ell^2(\mathbb{Z})$ that $\langle v,w\rangle=\sum_{n\in\mathbb{Z}}v(n)w(n)$, let $K\colon\mathbb{Z}\to\mathbb{R}$ satisfy for all $n\in\mathbb{Z}$ that $K(-n)=K(n)$ and $|K(n)|\le\mathfrak{C}|n|^{-(1+2s)}$, let $\varepsilon_n\colon\Omega\to[-c/2,c/2]$, $n\in\mathbb{Z}$, be i.i.d. random variables[2], let $\delta_n\in\ell^2(\mathbb{Z})$, $n\in\mathbb{Z}$, satisfy for all $k,n\in\mathbb{Z}$ with $k\neq n$ that $\delta_n(n)=1$ and $\delta_n(k)=0$, and let $u\colon[0,T]\times\mathbb{Z}\times\Omega\to\mathbb{R}$ satisfy for all $t\in[0,T]$, $n\in\mathbb{Z}$ that $u(0,n)=\delta_0(n)$

[2]Note that the expression *i.i.d* is an abbreviation for the expression *independently and identically distributed*

and

$$\left(\tfrac{d}{dt}u\right)(t,n) = \sum_{k\in\mathbb{Z}} K(k)\big(u(t,n)-u(t,n-k)\big) + \sum_{k\in\mathbb{Z}} \varepsilon_k \langle u(t,k),\delta_k\rangle \delta_k. \tag{2.1}$$

The situation above is the mathematical formulation of the physical scenario in which electrons are moving through a disordered lattice via (possibly) long-range interactions. The positive real number $c \in (0,\infty)$ represents the maximum magnitude of disorder which can occur at a given point in the lattice $\mathbb{Z}$. The i.i.d. random variables $\varepsilon_n\colon \Omega \to [-c/2, c/2]$, $n \in \mathbb{Z}$, represent the actual disorder at each point in the lattice $\mathbb{Z}$. Note that the probabilistic nature of this formulation allows for the existence of, e.g., measurement errors. The function $K\colon \mathbb{Z} \to \mathbb{R}$ describes which long-range jumps are observed to occur as electrons traverse through the lattice $\mathbb{Z}$. Observe that this implies that the real number $s \in (0,\infty)$ imposes a decay condition on the probability of long-range jumps. Finally, for every $t \in [0,T]$, $n \in \mathbb{Z}$ it holds that $u(t,n)$ represents the (possibly scaled) probability that the electron will be located at lattice point $n \in \mathbb{Z}$ at time $t \in [0,T]$.

Based on the formulation above, there are two immediate questions of interest.

(I) What are appropriate (or physically-relevant) choices of the kernel function K?

(II) Will Anderson localization occur for a system with the long-range interactions described in (2.1) for all choices of $s, c \in (0,\infty)$?

First, note that there is no unique answer to the question posed in item (I) due to the fact that the construction of mathematical models often depends upon the employed assumptions and individual goals of the scientist constructing them. It has been recently demonstrated that the discrete fractional Laplace operator is well suited to describe long-range interactions observed in various physical systems, including semi-crystalline polymers and dusty plasma (cf., e.g., [30, 32, 51] and the references therein). In fact, it is the case that when $s \in (0,1)$ one observes so-called *superdiffusion* and when $s \in (1,\infty)$ one observes so-called *subdiffusion*. Thus, as a starting point we consider the case where the functions K coincide with the definition of the discrete fractional Laplace operator.

Next, under the assumption that the functions K coincide with the definition of the discrete fractional Laplace operator, we consider the question posed in item (II). It is well-known that if $s = 1$, then the solution u of (2.1) will exhibit Anderson localization for all $c \in (0,\infty)$. What is not known—and an interest which motivates the current study—is whether or not for all $s \in (0,\infty)$ it holds that the solution u of (2.1) will exhibit Anderson localization for all $c \in (0,\infty)$. This question can be studied via the following result which follows immediately from Liaw [38, Corollary 3.2].

Corollary 2.1. *Let $\mathfrak{C}, s, c, T \in (0,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\ell^2(\mathbb{Z})$ be the set of all $v\colon \mathbb{Z} \to \mathbb{R}$ which satisfy that $\sum_{k\in\mathbb{Z}} |v(k)|^2 < \infty$, let $\langle\cdot,\cdot\rangle\colon \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \to \mathbb{R}$ satisfy for all $v, w \in \ell^2(\mathbb{Z})$ that $\langle v, w\rangle = \sum_{n\in\mathbb{Z}} v(n)w(n)$, let $K\colon \mathbb{Z} \to \mathbb{R}$ satisfy for all $n \in \mathbb{Z}$ that $K(-n) = K(n)$ and $|K(n)| \le \mathfrak{C}|n|^{-(1+2s)}$, let $\varepsilon_n\colon \Omega \to [-c/2, c/2]$, $n \in \mathbb{Z}$, be i.i.d. random variables, let $\delta_n \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$, satisfy for all $k, n \in \mathbb{Z}$ with $k \ne n$ that $\delta_n(n) = 1$ and $\delta_n(k) = 0$, and let $H\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ satisfy for all $u\colon \mathbb{Z} \times \Omega \to \mathbb{R}$, $n \in \mathbb{Z}$ with $\mathbb{P}(u \in \ell^2(\mathbb{Z})) = 1$ that*

$$(Hu)(n) = \sum_{k\in\mathbb{Z}} K(k)\big(u(n)-u(n-k)\big) + \sum_{k\in\mathbb{Z}} \varepsilon_k \langle u(k),\delta_k\rangle \delta_k. \tag{2.2}$$

Then if H has purely singular spectrum $\mathbb{P}$-a.s. it holds for all $v \in \ell^2(\mathbb{Z})$ with $\langle v, v\rangle = 1$ that

$$\mathbb{P}\Big(\lim_{m\to\infty} \operatorname{dist}\Big(v, \operatorname{span}\big\{H^k\delta_0\colon k \in \{0,1,\dots,m\}\big\}\Big) = 0\Big) = 1. \tag{2.3}$$

Observe, that Corollary 2.1 shows that in order to study the Anderson localization problem for the operator in (2.2) above via the spectral definition, we must be able to compute the forward orbit of the operator H with respect to the vector δ_0. From a numerical perspective, Corollary 2.1 implies that if one can find some $v \in \ell^2(\mathbb{Z})$ with $\langle v, v \rangle = 1$ such that

$$\mathbb{P}\left(\lim_{m\to\infty} \operatorname{dist}\Big(v, \operatorname{span}\big\{H^k\delta_0 \colon k \in \{0, 1, \ldots, m\}\big\}\Big) > 0 \right) \in (0, 1] \tag{2.4}$$

then it holds that H exhibits absolutely continuous spectrum $\mathbb{P}$-a.s.; i.e., Anderson localization does not occur. Thus, in the case of predicting localization behavior for a system numerically, one would need to be able to compute the action of the operator H *exactly* (or else additional approximations must be introduced). This need is precisely what motivates our current interest in the development of a series representation of the discrete fractional Laplace operator of arbitrary order.

For improved clarity, we close this section with a few important points. The formulation above demonstrates the need to construct an *exact* representation of the *action* of the discrete fractional Laplace operator. We accomplish this goal, herein, via the construction of a series representation of the operator (cf. Theorem 6.4 below). The goal of such constructions is predicated on the assumption that the discrete fractional Laplace operator is a good choice for models corresponding to (2.1). So-called anomalous diffusion has been experimentally observed in various strongly coupled fluids such as ultracold neutral plasma (cf., e.g., Strickler et al. [58]), two-dimensional and quasi-two-dimensional Yukawa liquids (cf., e.g., [25, 40, 41, 49]), and dusty plasmas (cf., e.g., [46, 48, 60]). As both *subdiffusion* and *superdiffusion* have been observed, it is the case that the discrete fractional Laplace operator of arbitrary order is an ideal candidate to model such physical systems, as $s \in (0, 1)$ may be used to model superdiffusion and $s \in (1, \infty)$ may be used to model subdiffusion. Whether or not there exist better models for such physical systems remains an open question in both mathematics and physics. However, we do not consider such issues further in this article.

3. Background

In this section we review several basic concepts regarding sequence spaces and the logarithmic norm. More specifically, in Subsection 3.1 we introduce the standard $\ell^2(\mathbb{Z})$ sequence space and an associated function which we denote the semi-inner product. In particular, Lemma 3.5 demonstrates that the standard $\ell^2(\mathbb{Z})$ inner product coincides with our particular semi-inner product. Afterwards, in Subsection 3.2 we define the logarithmic norm and the so-called upper-right Dini derivative. We then demonstrate a very useful property in Lemma 3.8 regarding the upper-right Dini derivative of $\ell^2(\mathbb{Z})$ norms.

It is worth noting that the contents of this section have been studied in various parts of the scientific literature (although rarely together and in this particular setting). The concept of semi-inner products has been studied extensively in the literature; cf., e.g., [16, 24, 42]. They were originally introduced in an effort to extend standard Hilbert space-type arguments to the more general setting of normed vector spaces. Herein, we employ a slight abuse of notation as Definition 3.4 does not define a semi-inner product in the sense of Lumer (cf., e.g., [42]). However, the object defined in Definition 3.4 does possess many of the desired properties of a semi-inner product and Lemma 3.5 demonstrates that no generality is lost by employing this definition. It is also worth noting that we are not the only authors to employ such notation; cf. e.g., Söderlind [57, Definition 5.1]. Moreover, it is worth mentioning that Lemma 3.8 appears in Jones et al. [28, Lemma 2.4] in a more general setting but we include it below for clarity and completeness.

Throughout this article, $\mathbb{R}$ and $\mathbb{C}$ stand for the usual real and complex number fields, respectively. Further, let $i = \sqrt{-1} \in \mathbb{C}$, let $\mathbb{Z}$ denote the set of integers, let $\mathbb{N} = \{1, 2, 3, \ldots\}$, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and for every $z \in \mathbb{C}$ let $\Re(z) \in \mathbb{R}$ denote the real part of the complex number z. In addition, we briefly mention a particular notation used throughout this article which emphasizes how various outside results are applied. If, for example, a result is referenced which names a particular mathematical object $\mathcal{X}$, then in order to state results about a family of objects, herein, e.g., $\mathcal{Y}_t$, $t \in \mathbb{R}$, we will write "applied for every $t \in \mathbb{R}$ with $\mathcal{X} \curvearrowleft \mathcal{Y}_t$ in the notation of ..." in order to clarify its use. We generalize this approach in the natural way in the case if multiple mathematical objects are involved (cf., e.g., the proof of Lemma 4.7). Moreover, when carrying out mathematical induction on a variable, say $n \in \mathbb{N}_0$, we will use the notation "$\mathbb{N}_0 \ni (n-1) \dashrightarrow n \in \mathbb{N}$" to emphasize and clarify both the inductive set and the inductive variable (cf., e.g., the proof of Lemma 4.8 below).

3.1. Sequence spaces

Definition 3.1 (Set of all sequences)**.** We denote by $\mathbb{S}$ the set of all functions with domain $\mathbb{Z}$ and codomain $\mathbb{R}$.

Definition 3.2 (The $\ell^2(\mathbb{Z})$ Hilbert space)**.** We denote by $\ell^2(\mathbb{Z})$ the set of all $u \in \mathbb{S}$ satisfying that $\sum_{k\in\mathbb{Z}} |u(k)|^2 < \infty$ (cf. Definition 3.1). Furthermore, we denote by $\|\cdot\|_2 \colon \mathbb{S} \to [0, \infty]$ the function which satisfies for all $u \in \mathbb{S}$ that $\|u\|_2^2 = \sum_{k\in\mathbb{Z}} |u(k)|^2$.

Definition 3.3 (The $\ell^2(\mathbb{Z})$ inner product)**.** We denote by $\langle\cdot,\cdot\rangle \colon \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \to \mathbb{R}$ the function which satisfies for all $u, v \in \ell^2(\mathbb{Z})$ that $\langle u, v\rangle = \sum_{k\in\mathbb{Z}} u(k)v(k)$ (cf. Definition 3.2).

Definition 3.4 (Semi-inner product)**.** We denote by $[\cdot,\cdot] \colon \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \to \mathbb{R}$ the function which satisfies for all $u, v \in \ell^2(\mathbb{Z})$ that

$$[u, v] = \left[\lim_{\varepsilon\to 0^+} \frac{\|v + \varepsilon u\|_2 - \|v\|_2}{\varepsilon}\right] \|v\|_2 \tag{3.1}$$

(cf. Definition 3.2).

Lemma 3.5. *Let $u, v \in \ell^2(\mathbb{Z})$ (cf. Definition 3.2). Then*

(i) it holds that $\langle u, u\rangle = \|u\|_2^2$ and

(ii) it holds that $\langle u, v\rangle = [u, v]$

(cf. Definitions 3.3 and 3.4).

Proof of Lemma 3.5. First, observe that item (i) follows immediately from Definitions 3.2 and 3.3. Next, note that item (i), Definition 3.4, and the fact that $\langle\cdot,\cdot\rangle \colon \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \to \mathbb{R}$ is a symmetric bilinear form[3] assure that

$$[u, v] = \left[\lim_{\varepsilon\to 0^+} \frac{\|v + \varepsilon u\|_2 - \|v\|_2}{\varepsilon}\right] \|v\|_2 = \left[\lim_{\varepsilon\to 0^+} \frac{\|v + \varepsilon u\|_2 - \|v\|_2}{\varepsilon} \cdot \frac{\|v + \varepsilon u\|_2 + \|v\|_2}{\|v + \varepsilon u\|_2 + \|v\|_2}\right] \|v\|_2$$

[3]It is well known that $\langle\cdot,\cdot\rangle \colon \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \to \mathbb{R}$ is a symmetric bilinear form as this follows immediately from Definition 3.3.

$$= \left[\lim_{\varepsilon\to 0^+} \frac{\|v+\varepsilon u\|_2^2 - \|v\|_2^2}{\varepsilon\big(\|v+\varepsilon u\|_2 + \|v\|_2\big)}\right]\|v\|_2 = \left[\lim_{\varepsilon\to 0^+} \frac{\langle v+\varepsilon u, v+\varepsilon u\rangle - \langle v, v\rangle}{\varepsilon\big(\|v+\varepsilon u\|_2 + \|v\|_2\big)}\right]\|v\|_2$$
$$= \left[\lim_{\varepsilon\to 0^+} \frac{\langle v, v\rangle + 2\varepsilon\langle u, v\rangle + \varepsilon^2\langle u, u\rangle - \langle v, v\rangle}{\varepsilon\big(\|v+\varepsilon u\|_2 + \|v\|_2\big)}\right]\|v\|_2 = \left[\lim_{\varepsilon\to 0^+} \frac{2\varepsilon\langle u, v\rangle + \varepsilon^2\langle u, u\rangle}{\varepsilon\big(\|v+\varepsilon u\|_2 + \|v\|_2\big)}\right]\|v\|_2$$
$$= \left[\lim_{\varepsilon\to 0^+} \frac{2\langle u, v\rangle + \varepsilon\langle u, u\rangle}{\|v+\varepsilon u\|_2 + \|v\|_2}\right]\|v\|_2 = \left[\frac{2\langle u, v\rangle}{2\|v\|_2}\right]\|v\|_2 = \langle u, v\rangle. \tag{3.2}$$

This establishes item (ii). The proof of Lemma 3.5 is thus complete. □

3.2. The logarithmic norm

Definition 3.6 (Logarithmic norm)**.** We denote by $\mu(A) \in \mathbb{R}$, $A\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, the function which satisfies for all $A\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ that

$$\mu(A) = \sup_{\substack{v\in\ell^2(\mathbb{Z})\\ \|v\|_2\neq 0}} \frac{\langle Av, v\rangle}{\|v\|_2^2} \tag{3.3}$$

(cf. Definitions 3.2 and 3.3).

Definition 3.7 (Upper-right Dini derivative)**.** For every $v\colon [0,\infty) \to \mathbb{R}$ we denote by $D_t^+ v(t) \in [-\infty,\infty]$, $t \in [0,\infty)$, the function which satisfies for all $t \in [0,\infty)$ that

$$D_t^+ v(t) = \limsup_{\varepsilon\to 0^+} \frac{v(t+\varepsilon) - v(t)}{\varepsilon}. \tag{3.4}$$

Lemma 3.8. *It holds for all differentiable $v\colon [0,\infty) \to \ell^2(\mathbb{Z})$, $t \in [0,\infty)$ that*

$$D_t^+ \|v(t)\|_2 = \left[\frac{\langle \frac{d}{dt}v(t), v(t)\rangle}{\|v(t)\|_2^2}\right]\|v(t)\|_2 \tag{3.5}$$

(cf. Definitions 3.2, 3.3, and 3.7).

Proof of Lemma 3.8. Throughout this proof let $v\colon [0,\infty) \to \ell^2(\mathbb{Z})$, let $t \in [0,\infty)$, and assume without loss of generality that $\|v(t)\|_2 \neq 0$ (cf. Definition 3.2). Note that the hypothesis that v is differentiable and Taylor's theorem (cf., e.g., Cartan et al. [10, Theorem 5.6.3]) yield that there exists $\delta_t(\varepsilon) \in \ell^2(\mathbb{Z})$, $\varepsilon \in \mathbb{R}$, such that for all $\varepsilon \in \mathbb{R}$ with $|\varepsilon|$ sufficiently small

(A) it holds that $v(t+\varepsilon) = v(t) + \varepsilon \frac{d}{dt}v(t) + |\varepsilon|\delta_t(\varepsilon)$ and

(B) it holds that $\lim_{\varepsilon\to 0} \delta_t(\varepsilon) = 0$.

Combining items (A) and (B) with Definition 3.7 and item (ii) of Lemma 3.5 hence shows that

$$D_t^+ \|v(t)\|_2 = \limsup_{\varepsilon\to 0^+} \frac{\|v(t) + \varepsilon\frac{d}{dt}v(t) + |\varepsilon|\delta_t(\varepsilon)\|_2 - \|v(t)\|_2}{\varepsilon}$$
$$= \limsup_{\varepsilon\to 0^+} \frac{\|v(t) + \varepsilon\frac{d}{dt}v(t)\|_2 - \|v(t)\|_2}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{\|v(t) + \varepsilon\frac{d}{dt}v(t)\|_2 - \|v(t)\|_2}{\varepsilon} \tag{3.6}$$

$$= \left[\lim_{\varepsilon \to 0^+} \frac{\|v(t) + \varepsilon \frac{d}{dt} v(t)\|_2 - \|v(t)\|_2}{\varepsilon} \right] \frac{\|v(t)\|_2^2}{\|v(t)\|_2^2} = \frac{\langle \frac{d}{dt} v(t), v(t) \rangle}{\|v(t)\|_2^2} \|v(t)\|_2$$

(cf. Definition 3.3). The proof of Lemma 3.8 is thus complete. □

We close Subsection 3.2 with a brief discussion of Definition 3.6. For every $A\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ let $\|A\|_{\mathrm{op}} = \inf\{c \in [0,\infty]\colon \forall\, v \in \ell^2(\mathbb{Z})$ it holds that $\|Av\|_2 \leq c\|v\|_2\}$ (cf. Definition 3.2). Then Definitions 3.4 and 3.6 and the Rayleigh quotient theorem (cf., e.g., [17, Theorem A.26]) imply that for every $A\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ with $\|A\|_{\mathrm{op}} \in [0,\infty)$, $A = A^*$, and nonempty pure point spectrum it holds that

$$\mu(A) = \sup_{\substack{v \in \ell^2(\mathbb{Z}) \\ \|v\|_2 \neq 0}} \frac{\langle Av, v \rangle}{\|v\|_2^2} = \max\{\lambda \in \mathbb{R}\colon \exists\, v \in \ell^2(\mathbb{Z}) \text{ with } \|v\|_2 \neq 0 \text{ and } Av = \lambda v\} \tag{3.7}$$

(e.g., $\mu(A)$ is the maximal eigenvalue of A). This fact will prove quite useful in the proof of Lemma 4.7 in Subsection 4.1.

4. The discrete Laplace operator of arbitrary order

In this section we introduce the discrete fractional Laplace operator and define the notion of real-valued positive powers of this operator. First, in Subsection 4.1 we define the discrete Laplace operator as well as introduce and study its associated discrete heat semigroup. Proposition 4.2 is presented in order to clarify the fact that positive integer powers of the discrete Laplace operator map elements of $\ell^2(\mathbb{Z})$ into $\ell^2(\mathbb{Z})$ (cf. Definition 3.2). The associated discrete heat semigroup is shown to be a strongly continuous contraction semigroup via the tools developed in Subsection 3.2. Note that Definition 4.1 is provided for clarity, as the evaluation of the Gamma function with arguments whose real parts are negative occurs frequently throughout the remainder of this article.

Afterwards, in Subsection 4.2 we provide a series representation of positive integer powers of the discrete Laplace operator. The result in Lemma 4.8 is well-known in the literature, but its proof is included for completeness (cf., e.g., Kelley and Peterson [29, Eq. (2.1)]). The series representation presented in Lemma 4.8 will be a crucial component in proving the main result of this article (cf. Theorem 6.4).

In Subsection 4.3 we define arbitrary real-valued positive powers of the discrete Laplace operator (cf. Definition 4.11). This is accomplished by first defining the case when the positive real-valued power is bounded above by one (cf. Definition 4.10). We then define higher-order positive real-valued powers via an inductive procedure. This definition is shown to be well-defined in $\ell^2(\mathbb{Z})$ in Lemma 4.9; i.e., it is shown that the discrete fractional Laplace operator maps $\ell^2(\mathbb{Z})$ into $\ell^2(\mathbb{Z})$. At this point, we wish to again emphasize that Definition 4.11 is not a direct "discretization" of the pointwise formula for the continuous case (cf., e.g., (1.1) for the case where $s \in (0,1)$), but rather the fractional power of the discrete Laplace operator.

4.1. The discrete Laplace operator and its associated semigroup

Definition 4.1 (Gamma function)**.** Let $X = \{z \in \mathbb{C}\colon \Re(z) \in (0,\infty)\}$ and let $\tilde{\Gamma}\colon X \to \mathbb{C}$ be the function which satisfies for all $z \in X$ that $\tilde{\Gamma}(z) = \int_0^\infty x^{z-1} \exp(-x)\, dx$. Then we denote[4]

[4] Note that $\lfloor \cdot \rfloor\colon \mathbb{Z} \to \mathbb{R}$ satisfies for all $x \in \mathbb{R}$ that $\lfloor x \rfloor = \max\{n \in \mathbb{Z}\colon n \leq x\}$.

by $\Gamma\colon \mathbb{C} \to \mathbb{C}$ the function which satisfies for all $z \in X$ that $\Gamma(z) = \tilde{\Gamma}(z)$, for all $z \in \mathbb{C}$ with $\Re(z) \in (-\infty, 0] \backslash \{\dots, -2, -1, 0\}$ that

$$\Gamma(z) = \frac{\tilde{\Gamma}(z + |\lfloor \Re(z) \rfloor|)}{(z + |\lfloor \Re(z) \rfloor| - 1)(z + |\lfloor \Re(z) \rfloor| - 2) \cdot \ldots \cdot z}, \tag{4.1}$$

and for all $z \in \mathbb{C}$ with $\Re(z) \in \{\dots, -2, -1, 0\}$ that $1/\Gamma(z) = 0$.

Proposition 4.2. *It holds*[5] *for all $s \in \mathbb{N}$, $u \in \ell^2(\mathbb{Z})$ that*

$$\sum_{n \in \mathbb{Z}} \left| \sum_{k=0}^{2s} (-1)^{k-s} \binom{2s}{k} u(n - s + k) \right|^2 < \infty \tag{4.2}$$

(cf. Definitions 3.2 and 4.1).

Proof of Proposition 4.2. Throughout this proof let $s \in \mathbb{N}$, $u \in \ell^2(\mathbb{Z})$. Observe that the fact that $u \in \ell^2(\mathbb{Z})$ and Definition 3.2 ensure that $\sum_{k \in \mathbb{Z}} |u(k)|^2 < \infty$. This, the triangle inequality, the fact that $s \in \mathbb{N}$, and the fact Jensen's inequality implies that for all $r, m \in \mathbb{N}$, $v_1, v_2, \dots, v_m \in [0, \infty)$ it holds that $[\sum_{k=1}^m v_k]^r \le m^{\max\{r-1,0\}} \sum_{k=1}^m v_k^r$ assure that

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \left| \sum_{k=0}^{2s} (-1)^{k-s} \binom{2s}{k} u(n - s + k) \right|^2 &\le \sum_{n \in \mathbb{Z}} \left[\sum_{k=0}^{2s} \left| \binom{2s}{k} u(n - s + k) \right| \right]^2 \\ &\le \sum_{n \in \mathbb{Z}} \left[2s \sum_{k=0}^{2s} \binom{2s}{k} |u(n - s + k)|^2 \right] = 2s \sum_{k=0}^{2s} \binom{2s}{k} \left[\sum_{n \in \mathbb{Z}} |u(n - s + k)|^2 \right] < \infty. \end{aligned} \tag{4.3}$$

The proof of Proposition 4.2 is thus complete. □

Definition 4.3 (Discrete Laplace operator)**.** We denote by $\Delta\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ the function which satisfies for all $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ that

$$(\Delta u)(n) = u(n-1) - 2u(n) + u(n+1) \tag{4.4}$$

(cf. Definition 3.2).

Definition 4.4 (Identity operator)**.** We denote by $\mathbb{I}\colon \mathbb{S} \to \mathbb{S}$ the operator which satisfies for all $u \in \mathbb{S}$, $n \in \mathbb{Z}$ that $(\mathbb{I}u)(n) = u(n)$ (cf. Definition 3.1).

Proposition 4.5. *Let $u \in \ell^2(\mathbb{Z})$ (cf. Definition 3.2). Then it holds for all $z \in [0, \infty)$ that*

$$\sum_{n \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} \exp(-2z) \left[\sum_{j \in \mathbb{N}_0} \frac{z^{2j+n-k}}{\Gamma(1+j)\Gamma(j+n-k+1)} \right] u(k) \right|^2 < \infty \tag{4.5}$$

(cf. Definition 4.1).

[5] Note that for all $n \in \mathbb{N}$, $k \in \{0, 1, \dots, n\}$ it holds that $\binom{n}{k} = \Gamma(1+n)/(\Gamma(1+k)\Gamma(1+n-k))$ (cf. Definition 4.1).

Proof of Proposition 4.5. Throughout this proof let $I_k\colon [0,\infty) \to \mathbb{R}$, $k \in \mathbb{Z}$, satisfy for all $z \in [0,\infty)$, $k \in \mathbb{Z}$ that

$$I_k(z) = \sum_{j\in\mathbb{N}_0} \frac{\left(z/2\right)^{2j+k}}{\Gamma(1+j)\Gamma(j+k+1)} \tag{4.6}$$

(cf. Definition 4.1). Observe that (4.6) and Olver et al. [47, Eq. 10.27.1] (applied for every $k \in \mathbb{Z}$ with $I_k \curvearrowleft I_k$, $z \curvearrowleft 2z$ in the notation of Olver et al. [47, Eq. 10.27.1]) assure that for all $z \in [0,\infty)$ it holds that

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \left[\sum_{j\in\mathbb{N}_0} \frac{z^{2j+k}}{\Gamma(1+j)\Gamma(j+k+1)} \right] &= \sum_{k\in\mathbb{Z}} I_k(2z) \\ &= I_0(2z) + \sum_{k\in\mathbb{N}} \big(I_k(2z) + I_{-k}(2z) \big) = I_0(2z) + 2\sum_{k\in\mathbb{N}} I_k(2z). \end{aligned} \tag{4.7}$$

Combining this, (4.6), the triangle inequality, Minkowski's inequality, and Olver et al. [47, Eq. 10.35.5] (applied with $I_k \curvearrowleft I_k$, $z \curvearrowleft 2z$ in the notation of Olver et al. [47, Eq. 10.35.5]) ensures that for all $z \in [0,\infty)$ it holds that

$$\begin{aligned} &\sum_{n\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} \exp(-2z) \left[\sum_{j\in\mathbb{N}_0} \frac{z^{2j+n-k}}{\Gamma(1+j)\Gamma(j+n-k+1)} \right] u(k) \right|^2 = \sum_{n\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} \exp(-2z) I_k(2z) u(n-k) \right|^2 \\ &\leq \exp(-4z) \left[\sum_{k\in\mathbb{Z}} |I_k(2z)| \left(\sum_{n\in\mathbb{Z}} |u(n-k)|^2 \right)^{1/2} \right]^2 = \exp(-4z) \left[\sum_{k\in\mathbb{Z}} I_k(2z) \|u\|_2 \right]^2 \\ &= \exp(-4z) \left[I_0(2z) + 2\sum_{k\in\mathbb{N}} I_k(2z) \right]^2 \|u\|_2^2 = \exp(-4z) \big[\exp(2z)\big]^2 \|u\|_2^2 = \|u\|_2^2 < \infty. \end{aligned} \tag{4.8}$$

The proof of Proposition 4.5 is thus complete. ☐

Definition 4.6 (Discrete heat semigroup)**.** We denote by $S_z(\Delta)\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, $z \in [0,\infty)$, the function which satisfies for all $z \in [0,\infty)$, $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ that

$$\big(S_z(\Delta)u\big)(n) = \sum_{k\in\mathbb{Z}} \exp(-2z) \left[\sum_{j\in\mathbb{N}_0} \frac{z^{2j+n-k}}{\Gamma(1+j)\Gamma(j+n-k+1)} \right] u(k) \tag{4.9}$$

(cf. Definitions 3.2, 4.1, 4.3, and 4.6).

Lemma 4.7. *Let $u \in \ell^2(\mathbb{Z})$ (cf. Definition 3.2). Then*

(i) it holds that $S_z(\Delta)\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, $z \in [0,\infty)$, is a strongly continuous semigroup,

(ii) it holds for all $z \in [0,\infty)$, $n \in \mathbb{Z}$ that $\frac{d}{dz}(S_z(\Delta)u)(n) = (\Delta S_z(\Delta)u)(n)$,

(iii) it holds that $\mu(\Delta) \in (-\infty, 0)$, and

(iv) it holds for all $z \in [0,\infty)$ that $\|S_z(\Delta)u\|_2 \leq \exp(z\mu(\Delta))\|u\|_2 \leq \|u\|_2$

(cf. Definitions 3.6, 4.3, and 4.6).

Proof of Lemma 4.7. First, note that $S_z(\Delta)\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, $z \in [0, \infty)$, is a strongly continuous semigroup[6] if

(A) it holds that $S_0(\Delta) = \mathbb{I}$,

(B) it holds for all $t, z \in [0, \infty)$ that $S_{t+z}(\Delta) = S_t(\Delta)S_z(\Delta)$, and

(C) it holds for all $v \in \ell^2(\mathbb{Z})$ that $\lim_{z\to 0^+}\|S_z(\Delta)v - v\|_2 = 0$

(cf. Definitions 4.3, 4.4, and 4.6). Observe that Definition 4.6, the fact that for all $v \in \ell^2(\mathbb{Z})$ it holds that $\sup_{n\in\mathbb{Z}}|v(n)| < \infty$, and, e.g., Ciaurri et al. [12, Proposition 1] (applied for every $v \in \ell^2(\mathbb{Z})$ with $f \curvearrowleft v$, $W_t \curvearrowleft S_z(\Delta)$ in the notation of Ciaurri et al. [12, Proposition 1]) show that $S_z(\Delta)\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, $z \in [0, \infty)$ satisfies items (A), (B), and (C). This establishes item (i). Next, note that Definition 4.6 and, e.g., Ciaurri et al. [12, Proposition 2] (applied for every $n \in \mathbb{Z}$ with $f \curvearrowleft u$, $(u(n,t))_{t\in[0,\infty)} \curvearrowleft ((S_z(\Delta)u)(n))_{z\in[0,\infty)}$ in the notation of Ciaurri et al. [12, Proposition 2]) establish item (ii). In addition, observe that Definition 3.4 implies that if for all $v \in \ell^2(\mathbb{Z})$ with $\|v\|_2 \neq 0$ it holds that $\langle \Delta v, v\rangle \in (-\infty, 0)$ then it holds that $\mu(\Delta) \in (-\infty, 0)$. To that end, note that for all $v \in \ell^2(\mathbb{Z})$ it holds that

$$\begin{aligned}\langle \Delta v, v\rangle &= \sum_{k\in\mathbb{Z}}\big[(\Delta v)(k)\big]v(k) = \sum_{k\in\mathbb{Z}}\big(v(k-1) - 2v(k) + v(k+1)\big)v(k)\\ &= \sum_{k\in\mathbb{Z}} v(k-1)v(k) - 2\sum_{k\in\mathbb{Z}} v(k)^2 + \sum_{k\in\mathbb{Z}} v(k+1)v(k)\\ &= \sum_{k\in\mathbb{Z}} v(k-1)v(k) - \sum_{k\in\mathbb{Z}} v(k)^2 - \sum_{k\in\mathbb{Z}} v(k-1)^2 + \sum_{k\in\mathbb{Z}} v(k)v(k-1)\\ &= -\sum_{k\in\mathbb{Z}}\big[v(k-1)^2 - 2v(k-1)v(k) + v(k)^2\big] = -\sum_{k\in\mathbb{Z}}\big[v(k-1) - v(k)\big]^2\end{aligned} \tag{4.10}$$

(cf. Definition 3.3). This demonstrates that for all $v \in \ell^2(\mathbb{Z})$ with $\|v\|_2 \neq 0$ it holds that $\langle \Delta v, v\rangle \in (-\infty, 0)$. This and Definition 3.6 establish item (iii). Moreover, observe that item (i), item (iii), and Jones et al. [28, Lemma 2.8] (applied with $x \curvearrowleft u$, $A \curvearrowleft \Delta$, $(\mathbb{T}_t(A))_{t\in[0,\infty)} \curvearrowleft (S_z(\Delta))_{z\in[0,\infty)}$ in the notation of Jones et al. [28, Lemma 2.8]) hence prove item (iv). The proof of Lemma 4.7 is thus complete. ☐

4.2. Positive integer powers of the discrete Laplace operator

Lemma 4.8. *It holds for all $s \in \mathbb{N}$, $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ that*

$$\big((-\Delta)^s u\big)(n) = \sum_{k=0}^{2s}(-1)^{k-s}\binom{2s}{k}u(n-s+k) \tag{4.11}$$

(cf. Definitions 3.2 and 4.11).

[6] Cf., e.g., Jones et al. [28, Definition 2.6]

Proof of Lemma 4.8. We prove (4.11) by induction on $s \in \mathbb{N}$. For the base case $s = 1$ observe that Definition 4.3 establishes (4.11). This proves (4.11) in the case $s = 1$. For the induction step $\mathbb{N} \ni (s-1) \dashrightarrow s \in \mathbb{N} \cap [2, \infty)$, let $s \in \mathbb{N} \cap [2, \infty)$ and assume for all $\mathfrak{s} \in \{1, 2, \ldots, s-1\}$, $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ that (4.11) holds. Note that the fact that $s \in \mathbb{N} \cap [2, \infty)$ implies that for all $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ it holds that

$$\big((-\Delta)^s u\big)(n) = \big(-\Delta((-\Delta)^{s-1} u)\big)(n). \tag{4.12}$$

This and the induction hypothesis ensure that for all $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ it holds that

$$\begin{aligned}
\big((-\Delta)^s u\big)(n) &= \left(-\Delta\left(\sum_{k=0}^{2(s-1)} (-1)^{k-(s-1)} \tbinom{2(s-1)}{k} u(\cdot - (s-1) + k)\right)\right)(n) \\
&= 2 \sum_{k=0}^{2s-2} (-1)^{k-s+1} \tbinom{2s-2}{k} u(n - s + 1 + k) \\
&\quad - \sum_{k=0}^{2s-2} (-1)^{k-s+1} \tbinom{2s-2}{k} u((n-1) - s + 1 + k) \\
&\quad - \sum_{k=0}^{2s-2} (-1)^{k-s+1} \tbinom{2s-2}{k} u((n+1) - s + 1 + k) \\
&= 2 \sum_{k=0}^{2s-2} (-1)^{k-s+1} \tbinom{2s-2}{k} u(n - s + 1 + k) - \sum_{k=0}^{2s-2} (-1)^{k-s+1} \tbinom{2s-2}{k} u(n - s + k) \\
&\quad - \sum_{k=0}^{2s-2} (-1)^{k-s+1} \tbinom{2s-2}{k} u(n + 2 - s + k) \qquad (4.13) \\
&= 2 \sum_{k=1}^{2s-1} (-1)^{k-s} \tbinom{2s-2}{k-1} u(n - s + k) + \sum_{k=0}^{2s-2} (-1)^{k-s} \tbinom{2s-2}{k} u(n - s + k) \\
&\quad + \sum_{k=2}^{2s} (-1)^{k-s} \tbinom{2s-2}{k-2} u(n - s + k) \\
&= \left[\sum_{k=1}^{2s-1} (-1)^{k-s} \tbinom{2s-2}{k-1} u(n - s + k) + \sum_{k=0}^{2s-2} (-1)^{k-s} \tbinom{2s-2}{k} u(n - s + k)\right] \\
&\quad + \left[\sum_{k=1}^{2s-1} (-1)^{k-s} \tbinom{2s-2}{k-1} u(n - s + k) + \sum_{k=2}^{2s} (-1)^{k-s} \tbinom{2s-2}{k-2} u(n - s + k)\right]
\end{aligned}$$

(cf. Definition 4.3). Combining this and the fact that for all $n \in \mathbb{N}$, $k \in \{1, 2, \ldots, n-1\}$ it holds that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ assures that for all $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ it holds that

$$\begin{aligned}
\big((-\Delta)^s u\big)(n) &= (-1)^{s-1} u(n + s - 1) + (-1)^{-s} u(n - s) + \sum_{k=1}^{2s-2} (-1)^{k-s} \tbinom{2s-1}{k} u(n - s + k) \\
&\quad + (-1)^{1-s} u(n + 1 - s) + (-1)^s u(n + s) \\
&\quad + \sum_{k=2}^{2s-1} (-1)^{k-s} \tbinom{2s-1}{k-1} u(n - s + k) \\
&= (-1)^{s-1} u(n + s - 1) + (-1)^{-s} u(n - s) + (-1)^{1-s} u(n + 1 - s) \\
&\quad + (-1)^s u(n + s) + (-1)^{1-s} \tbinom{2s-1}{1} u(n - s + 1) \\
&\quad + (-1)^{s-1} \tbinom{2s-1}{2s-2} u(n - s + (2s - 1)) \qquad (4.14)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=2}^{2s-2}(-1)^{k-s}\Big[\tbinom{2s-1}{k}+\tbinom{2s-1}{k-1}\Big]u(n-s+k)\\
&=(-1)^{s-1}u(n+s-1)+(-1)^{-s}u(n-s)+(-1)^{1-s}u(n+1-s)\\
&\quad+(-1)^{s}u(n+s)+(-1)^{1-s}\tbinom{2s-1}{1}u(n-s+1)\\
&\quad+(-1)^{s-1}\tbinom{2s-1}{2s-2}u(n+s-1)+\sum_{k=2}^{2s-2}(-1)^{k-s}\tbinom{2s}{k}u(n-s+k).
\end{aligned}
$$

Next, observe that the fact that for all $n\in\mathbb{N}$, $k\in\{1,2,\dots,n-1\}$ it holds that $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$ and the fact that for all $n\in\mathbb{N}$, $k\in\{0,1,\dots,n\}$ it holds that $\binom{n}{k}=\binom{n}{n-k}$ show that

$$
1+\tbinom{2s-1}{2s-2}=\tbinom{2s-1}{2s-1}+\tbinom{2s-1}{2s-2}=\tbinom{2s}{2s-1} \tag{4.15}
$$

and

$$
1+\tbinom{2s-1}{1}=\tbinom{2s-1}{0}+\tbinom{2s-1}{1}=\tbinom{2s}{1}. \tag{4.16}
$$

Combining (4.14), (4.15), and (4.16) hence implies that for all $u\in\ell^2(\mathbb{Z})$, $n\in\mathbb{Z}$ it holds that

$$
\begin{aligned}
\big((-\Delta)^s u\big)(n)&=(-1)^{-s}\tbinom{2s}{0}u(n-s)+(-1)^{s}\tbinom{2s}{2s}u(n+s)+(-1)^{1-s}\tbinom{2s}{1}u(n-s+1)\\
&\quad+(-1)^{s-1}\tbinom{2s}{2s-1}u(n+s-1)+\sum_{k=2}^{2s-2}(-1)^{k-s}\tbinom{2s}{k}u(n-s+k)\\
&=\sum_{k=0}^{2s}(-1)^{k-s}\tbinom{2s}{k}u(n-s+k).
\end{aligned} \tag{4.17}
$$

Induction therefore establishes (4.11). The proof of Lemma 4.8 is thus complete. □

4.3. The discrete fractional Laplace operator of arbitrary order

Lemma 4.9. *Let $m\in\mathbb{N}$, $s\in(m-1,m)$. Then it holds for all $u\in\ell^2(\mathbb{Z})$ that*

$$
\left\|\frac{1}{\Gamma(-(s-m+1))}\int_0^\infty z^{-s+m-2}\big[S_z(\Delta)-\mathbb{I}\big]\big((-\Delta)^{m-1}u\big)\,dz\right\|_2<\infty \tag{4.18}
$$

(cf. Definitions 3.2, 4.1, 4.3, 4.4, and 4.6).

Proof of Lemma 4.9. Throughout this proof let $m\in\mathbb{N}$, $s\in(m-1,m)$. We claim that for all $z\in(0,\infty)$, $u\in\ell^2(\mathbb{Z})$ it holds that

$$
\big\|(S_z(\Delta)-\mathbb{I})u\big\|_2\le\int_0^z\exp\big((z-w)\mu(\Delta)\big)\|\Delta u\|_2\,dw \tag{4.19}
$$

(cf. Definitions 3.2, 3.6, 4.3, 4.4, and 4.6). Note that the fact that $\langle\cdot,\cdot\rangle\colon\ell^2(\mathbb{Z})\times\ell^2(\mathbb{Z})\to\mathbb{R}$ is a symmetric bilinear form and Definition 4.6 ensure that for all $u\in\ell^2(\mathbb{Z})$, $z\in(0,\infty)$ it holds that

$$
\begin{aligned}
\frac{d}{dz}\big\|(S_z(\Delta)-\mathbb{I})u\big\|_2&=\frac{\langle\frac{d}{dz}(S_z(\Delta)-\mathbb{I})u,(S_z(\Delta)-\mathbb{I})u\rangle}{\|(S_z(\Delta)-\mathbb{I})u\|_2}\\
&=\frac{\langle\Delta S_z(\Delta)u,(S_z(\Delta)-\mathbb{I})u\rangle}{\|(S_z(\Delta)-\mathbb{I})u\|_2}=\frac{\langle\Delta(S_z(\Delta)-\mathbb{I})u,(S_z(\Delta)-\mathbb{I})u\rangle}{\|(S_z(\Delta)-\mathbb{I})u\|_2}+\frac{\langle\Delta u,(S_z(\Delta)-\mathbb{I})u\rangle}{\|(S_z(\Delta)-\mathbb{I})u\|_2}
\end{aligned} \tag{4.20}
$$

$$\leq \left[\sup_{v\in\ell^2(\mathbb{Z})} \frac{\langle \Delta v, v\rangle}{\|v\|_2}\right] + \frac{\langle \Delta u, (S_z(\Delta)-\mathbb{I})u\rangle}{\|(S_z(\Delta)-\mathbb{I})u\|_2} = \mu(\Delta) + \frac{\langle \Delta u, (S_z(\Delta)-\mathbb{I})u\rangle}{\|(S_z(\Delta)-\mathbb{I})u\|_2}$$

(cf. Definition 3.3). This, the Cauchy-Swartz inequality, and the fact that item (i) of Lemma 4.7 implies that for all $u \in \ell^2(\mathbb{Z})$ it holds that $\lim_{z\to 0^+} \|(S_z(\Delta)-\mathbb{I})u\|_2 = 0$ assure that for all $z \in (0,\infty)$, $u \in \ell^2(\mathbb{Z})$ it holds that

$$\begin{aligned} \big\|(S_z(\Delta)-\mathbb{I})u\big\|_2 &\leq \int_0^z \exp\!\big((z-w)\mu(\Delta)\big) \frac{\big|\langle \Delta u, (S_w(\Delta)-\mathbb{I})u\rangle\big|}{\|(S_w(\Delta)-\mathbb{I})u\|_2}\,dw \\ &\leq \int_0^z \exp\!\big((z-w)\mu(\Delta)\big) \frac{\|\Delta u\|_2 \|(S_w(\Delta)-\mathbb{I})u\|_2}{\|(S_w(\Delta)-\mathbb{I})u\|_2}\,dw = \int_0^z \exp\!\big((z-w)\mu(\Delta)\big)\|\Delta u\|_2\,dw. \end{aligned} \tag{4.21}$$

Combining this and the fact that Proposition 4.2 (applied with $s \curvearrowleft 1$ in the notation of Proposition 4.2) ensures that for all $u \in \ell^2(\mathbb{Z})$ it holds that $\|\Delta u\|_2 < \infty$ proves (4.19). Next, observe that (4.19), Lemma 4.8, and Jensen's inequality guarantee that for all $u \in \ell^2(\mathbb{Z})$ it holds that

$$\begin{aligned} &\left\|\frac{1}{\Gamma(-(s-m+1))}\int_0^\infty z^{-s+m-2}\big[S_z(\Delta)-\mathbb{I}\big]\big((-\Delta)^{m-1}u\big)\,dz\right\|_2 \\ &\leq \frac{1}{|\Gamma(-(s-m+1))|}\int_0^\infty z^{-s+m-2}\big\|\big[S_z(\Delta)-\mathbb{I}\big]\big((-\Delta)^{m-1}u\big)\big\|_2\,dz \\ &\leq \frac{\|(-\Delta)^m u\|_2}{|\Gamma(-(s-m+1))|}\int_0^\infty z^{-s+m-2}\left[\int_0^z \exp\!\big((z-w)\mu(\Delta)\big)\,dw\right]dz \end{aligned} \tag{4.22}$$

(cf. Definition 4.1). In addition, note that item (iii) of Lemma 4.7 and integration by parts show that

$$\begin{aligned} 0 &\leq \int_0^\infty z^{-s+m-2}\left[\frac{1-\exp\!\big(z\mu(\Delta)\big)}{-\mu(\Delta)}\right]dz \\ &= \int_0^1 z^{-s+m-2}\left[\frac{1-\exp\!\big(z\mu(\Delta)\big)}{-\mu(\Delta)}\right]dz + \int_1^\infty z^{-s+m-2}\left[\frac{1-\exp\!\big(z\mu(\Delta)\big)}{-\mu(\Delta)}\right]dz \\ &\leq \int_0^1 z^{-s+m-2}\left[\frac{1-\exp\!\big(z\mu(\Delta)\big)}{-\mu(\Delta)}\right]dz + \int_1^\infty z^{-s+m-2}\,dz \\ &= \lim_{w\to 0^+}\left[\frac{z^{-s+m-1}}{-s+m-1}\right]\left[\frac{1-\exp\!\big(z\mu(\Delta)\big)}{-\mu(\Delta)}\right]\Bigg|_{z=w}^1 - \int_0^1\left[\frac{z^{-s+m-1}}{-s+m-1}\right]\exp(z\mu(\Delta))\,dz \\ &\quad + \lim_{w\to\infty}\frac{z^{-s+m-1}}{-s+m-1}\Bigg|_{z=1}^w \\ &= \left[\frac{1}{-s+m-1}\right]\left[\frac{1+\mu(\Delta)-\exp\!\big(\mu(\Delta)\big)}{-\mu(\Delta)}\right] - \int_0^1\left[\frac{z^{-s+m-1}}{-s+m-1}\right]\exp(z\mu(\Delta))\,dz \\ &\leq \left[\frac{1}{-s+m-1}\right]\left[\frac{1+\mu(\Delta)-\exp\!\big(\mu(\Delta)\big)}{-\mu(\Delta)}\right] + \int_0^1\left[\frac{z^{-s+m-1}}{s-m+1}\right]dz \\ &= \left[\frac{1}{-s+m-1}\right]\left[\frac{1+\mu(\Delta)-\exp\!\big(\mu(\Delta)\big)}{-\mu(\Delta)}\right] + \frac{1}{(-s+m)(s-m+1)} < \infty. \end{aligned} \tag{4.23}$$

Combining (4.22), (4.23), and the fact that Proposition 4.2 assures that for all $u \in \ell^2(\mathbb{Z})$ it holds that $\|(-\Delta)^m u\|_2 < \infty$ hence proves (4.18). The proof of Lemma 4.9 is thus complete. ☐

Definition 4.10 (Discrete fractional Laplace operator for $s \in (0,1)$)**.** Let $s \in (0,1)$. Then we denote by $(-\Delta)^s \colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ the function which satisfies for all $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ that

$$\big((-\Delta)^s u\big)(n) = \frac{1}{\Gamma(-s)} \int_0^\infty z^{-s-1} \big[S_z(\Delta) - \mathbb{I}\big] u(n)\, dz \tag{4.24}$$

(cf. Definitions 3.2, 4.1, 4.3, 4.4, and 4.6).

Definition 4.11 (Discrete fractional Laplace operator for $s \in (0,\infty)$)**.** Let $s \in (0,\infty)$. Then we denote by $(-\Delta)^s \colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ the function which satisfies for all $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ that

$$\begin{aligned} \big((-\Delta)^s u\big)(n) &= \big((-\Delta)^{s-\lfloor s \rfloor}(-\Delta)^{\lfloor s \rfloor} u\big)(n) \\ &= \begin{cases} \big((-\Delta)^s u\big)(n) & : s \in \mathbb{N} \\ \frac{1}{\Gamma(-(s-\lfloor s \rfloor))} \int_0^\infty z^{-(s-\lfloor s \rfloor)-1} \big[S_z(\Delta) - \mathbb{I}\big] \big((-\Delta)^{\lfloor s \rfloor} u\big)(n)\, dz & : s \in (0,\infty) \backslash \mathbb{N} \end{cases} \end{aligned} \tag{4.25}$$

(cf. Definitions 3.2, 4.1, 4.3, 4.4, 4.6, and 4.10).

5. The discrete fractional kernel

In this section we introduce a kernel function which will allow us to conveniently provide a series representation of (4.25) in Definition 4.11. Proposition 5.2 and Lemma 5.3 are preliminary results which allow us to prove Lemma 5.4—a result which outlines useful properties exhibited by the kernel defined in Definition 5.1. It is worth noting that Proposition 5.2 is a well-known result and that Lemma 5.3 is a generalization of Ciaurri et al. [13, Lemma 9.2 (a)], which was only proven in the case where $s \in (0,1)$. Proposition 5.5 and Lemma 5.7 are the main results of this section and allow us to prove Lemma 6.1 in Subsection 6.1.

Definition 5.1 (Fractional kernel)**.** We denote[7] by $K_s \colon \mathbb{Z} \to \mathbb{R}$, $s \in \mathbb{R}$, the function which satisfies for all $k \in \mathbb{Z}$, $m \in \mathbb{N}$, $s \in (m-1, m)$ that

$$K_s(k) = \frac{-\mathbb{1}_{\mathbb{Z}\backslash\{0\}}(k)\, 4^s \Gamma(1/2 + s)\Gamma(|k| - s)}{\sqrt{\pi}\,\Gamma(-s)\Gamma(|k| + 1 + s)} \tag{5.1}$$

(cf. Definition 4.1).

Proposition 5.2. *It holds for all $a, b \in \mathbb{R}$, $\lambda \in (0,\infty)$ with $0 \leq a < b < \infty$ that*

$$\min\{\lambda, 1\} \leq \frac{b^\lambda - a^\lambda}{b^{\lambda-1}(b-a)} \leq \max\{\lambda, 1\}. \tag{5.2}$$

Proof of Proposition 5.2. First, note that for all $a, b \in \mathbb{R}$ with $0 \leq a < b < \infty$ it holds that

$$0 \leq a/b < 1 \tag{5.3}$$

Observe that (5.3) ensures that for all $\lambda \in (0,1)$, $a, b \in \mathbb{R}$ with $0 \leq a < b < \infty$ it holds that $0 \leq a/b \leq \big(a/b\big)^\lambda < 1$. This assures that for all $\lambda \in (0,1)$, $a, b \in \mathbb{R}$ with $0 \leq a < b < \infty$ it holds that

$$\frac{b^\lambda - a^\lambda}{b^{\lambda-1}(b-a)} = \frac{1 - (a/b)^\lambda}{1 - a/b} \leq 1. \tag{5.4}$$

[7] Let $A \subseteq \mathbb{R}$. Then it holds for all $x \in A$ that $\mathbb{1}_A(x) = 1$ and for all $x \in \mathbb{R}\backslash A$ that $\mathbb{1}_A(x) = 0$.

Combining the mean value theorem and (5.4) hence guarantee that for all $\lambda \in (0,1)$, $a, b \in \mathbb{R}$ with $0 \le a < b < \infty$ it holds that there exists $c \in (a/b, 1)$ such that

$$\frac{b^\lambda - a^\lambda}{b^{\lambda-1}(b-a)} = \frac{1 - (a/b)^\lambda}{1 - a/b} = \lambda c^{\lambda-1} \ge \lambda. \tag{5.5}$$

Next, note that (5.3) demonstrates that for all $\lambda \in [1,\infty)$, $a, b \in \mathbb{R}$ with $0 \le a < b < \infty$ it holds that $0 \le \left(a/b\right)^\lambda \le a/b < 1$. This shows that for all $\lambda \in [1,\infty)$, $a, b \in \mathbb{R}$ with $0 \le a < b < \infty$ it holds that

$$\frac{b^\lambda - a^\lambda}{b^{\lambda-1}(b-a)} = \frac{1 - (a/b)^\lambda}{1 - a/b} \ge 1. \tag{5.6}$$

Combining (5.6) with the mean value theorem therefore proves that for all $\lambda \in [1,\infty)$, $a, b \in \mathbb{R}$ with $0 \le a < b < \infty$ it holds that there exists $d \in (a/b, 1)$ such that

$$\frac{b^\lambda - a^\lambda}{b^{\lambda-1}(b-a)} = \frac{1 - (a/b)^\lambda}{1 - a/b} = \lambda d^{\lambda-1} \le \lambda. \tag{5.7}$$

Combining (5.4), (5.5), (5.6), and (5.7) thus establishes (5.2). The proof of Proposition 5.2 is thus complete. □

Lemma 5.3. *It holds for all $m \in \mathbb{N}$, $s \in (m-1, m)$ that there exists $C \in \mathbb{R}$ such that for all $k \in \mathbb{Z}$ with $|k| \in [m, \infty)$ it holds that*

$$\left| \frac{\Gamma(|k| - s)}{\Gamma(|k| + 1 + s)} - \frac{1}{|k|^{1+2s}} \right| \le \frac{C}{|k|^{2+2s}} \tag{5.8}$$

(cf. Definition 4.1).

Proof of Lemma 5.3. Throughout this proof let $m \in \mathbb{N}$, $s \in (m-1, m)$ and without loss of generality let $k \in \mathbb{Z}$ with $k \in [m, \infty)$. Note that the triangle inequality assures that

$$\begin{aligned} \left| \frac{\Gamma(|k| - s)}{\Gamma(|k| + 1 + s)} - \frac{1}{|k|^{1+2s}} \right| &\le \left| \frac{\Gamma(|k| - s)}{\Gamma(|k| + 1 + s)} - \frac{1}{|k-s|^{1+2s}} \right| + \left| \frac{1}{|k-s|^{1+2s}} - \frac{1}{|k|^{1+2s}} \right| \\ &= \left| \frac{\Gamma(k - s)}{\Gamma(k + 1 + s)} - \frac{1}{(k-s)^{1+2s}} \right| + \left| \frac{1}{(k-s)^{1+2s}} - \frac{1}{k^{1+2s}} \right|. \end{aligned} \tag{5.9}$$

Next, observe that Proposition 5.2 (applied with $\lambda \curvearrowleft 1 + 2s$, $a \curvearrowleft 1/k$, $b \curvearrowleft 1/(m-s)$ in the notation of Proposition 5.2) ensures that

$$\begin{aligned} \left| \frac{1}{(k-s)^{1+2s}} - \frac{1}{k^{1+2s}} \right| &= \left[\frac{\left|(k-s)^{-(1+2s)} - k^{-(1+2s)}\right|}{(k-s)^{-2s}\left[(k-s)^{-1} - k^{-1}\right]} \right] (k-s)^{-2s}\left[(k-s)^{-1} - k^{-1}\right] \\ &\le \frac{\max\{1+2s, 1\}}{(k-s)^{2s}} \left[\frac{1}{k-s} - \frac{1}{k} \right] = \frac{1+2s}{(k-s)^{2s}} \left[\frac{k - (k-s)}{(k-s)k} \right] = \frac{(1+2s)s}{k(k-s)^{1+2s}}. \end{aligned} \tag{5.10}$$

This and the fact that $k \in [m, \infty)$ show that

$$\left| \frac{1}{(k-s)^{1+2s}} - \frac{1}{k^{1+2s}} \right| \leq \frac{(1+2s)s}{(k-s)^{2+2s}} \leq \left[\sup_{k \in [m,\infty)} \frac{k^{2+2s}}{(k-s)^{2+2s}} \right] \frac{(1+2s)s}{k^{2+2s}}. \tag{5.11}$$

In addition, note that, e.g., Tricomi and Erdélyi [59, Eq. (15), page 140] (applied with $z \curvearrowleft k$, $\alpha \curvearrowleft -s$, $\beta \curvearrowleft 1+s$ in the notation of Tricomi and Erdélyi [59, Eq. (15), page 140]) guarantees that

$$\frac{\Gamma(k-s)}{\Gamma(k+1+s)} = \frac{1}{\Gamma(1+2s)} \int_0^\infty \exp(-(k-s)v)\big(1-\exp(-v)\big)^{2s}\,dv. \tag{5.12}$$

This, the fact that for all $r \in [0, \infty)$ it holds that $\int_0^\infty \exp(-rv)v^{2s}\,dv = \Gamma(1+2s)r^{-(1+2s)}$, and Jensen's inequality prove that

$$\begin{aligned} &\left| \frac{\Gamma(k-s)}{\Gamma(k+1+s)} - \frac{1}{(k-s)^{1+2s}} \right| \\ &= \left| \frac{1}{\Gamma(1+2s)} \int_0^\infty \exp(-(k-s)v)\big(1-\exp(-v)\big)^{2s}\,dv - \frac{1}{(k-s)^{1+2s}} \right| \\ &= \left| \frac{1}{\Gamma(1+2s)} \int_0^\infty \exp(-(k-s)v)\Big[\big(1-\exp(-v)\big)^{2s} - v^{2s}\Big]\,dv \right| \\ &\leq \frac{1}{\Gamma(1+2s)} \int_0^\infty \exp(-(k-s)v)v^{2s} \left| \Big(\frac{1-\exp(-v)}{v}\Big)^{2s} - 1 \right| dv. \end{aligned} \tag{5.13}$$

Moreover, observe that Proposition 5.2 (applied with $\lambda \curvearrowleft 2s$, $a \curvearrowleft {}^{(1-\exp(-v))}/_{v}$, $b \curvearrowleft 1$ in the notation of Proposition 5.2) ensures that for all $v \in (0, \infty)$ it holds that

$$1 - \Big(\frac{1-\exp(-v)}{v}\Big)^{2s} \leq \max\{2s, 1\}\left[1 - \frac{1-\exp(-v)}{v}\right]. \tag{5.14}$$

Combining this, (5.13), the fact that for all $v \in (0, \infty)$ it holds that $v - (1 - \exp(-v)) < {}^{v^2}/_{2}$, the fact that for all $r \in [0, \infty)$ it holds that $\int_0^\infty \exp(-rv)v^{1+2s}\,dv = \Gamma(2+2s)r^{-(2+2s)}$, and the fact that Definition 4.1 implies that for all $z \in (0, \infty)$ it holds that $\Gamma(1+z) = z\Gamma(z)$ assures that

$$\begin{aligned} &\left| \frac{\Gamma(k-s)}{\Gamma(k+1+s)} - \frac{1}{(k-s)^{1+2s}} \right| \\ &\leq \frac{\max\{2s,1\}}{\Gamma(1+2s)} \int_0^\infty \exp(-(k-s)v)v^{2s} \left| 1 - \Big(\frac{1-\exp(-v)}{v}\Big) \right| dv \\ &\leq \frac{\max\{s, 1/2\}}{\Gamma(1+2s)} \int_0^\infty \exp(-(k-s)v)v^{1+2s}\,dv = \frac{\max\{s, 1/2\}\Gamma(2+2s)}{\Gamma(1+2s)} \frac{1}{(k-s)^{2+2s}} \\ &= \max\{s, 1/2\}(1+2s)\frac{1}{(k-s)^{2+2s}} \leq \left[\sup_{k \in [m,\infty)} \frac{k^{2+2s}}{(k-s)^{2+2s}} \right] \frac{\max\{s, 1/2\}(1+2s)}{k^{2+2s}}. \end{aligned} \tag{5.15}$$

Combining this, (5.9), (5.11), and the fact that $\sup_{k \in [m,\infty)} k^{2+2s}(k-s)^{-(2+2s)} \in \mathbb{R}$ hence proves (5.8). The proof of Lemma 5.3 is thus complete. □

Lemma 5.4. *Let $m \in \mathbb{N}$, $s \in (m-1, m)$. Then*

(i) it holds for all $k \in \mathbb{Z}$ that $K_s(-k) = K_s(k)$ and

(ii) it holds that there exists $C \in \mathbb{R}$ such that for all $k \in \mathbb{Z}$ with $|k| \in \mathbb{Z}\backslash\{0\}$ it holds that

$$|K_s(k)| \leq \frac{C}{|k|^{1+2s}} \tag{5.16}$$

(cf. Definition 5.1).

Proof of Lemma 5.4. First, note that Definition 5.1 ensures that for all $k \in \mathbb{Z}$ it holds that

$$\begin{aligned} K_s(-k) &= \frac{-\mathbb{1}_{\mathbb{Z}\backslash\{0\}}(-k)\, 4^s \Gamma(1/2 + s)\Gamma(|-k| - s)}{\sqrt{\pi}\Gamma(-s)\Gamma(|-k| + 1 + s)} \\ &= \frac{-\mathbb{1}_{\mathbb{Z}\backslash\{0\}}(k)\, 4^s \Gamma(1/2 + s)\Gamma(|k| - s)}{\sqrt{\pi}\Gamma(-s)\Gamma(|k| + 1 + s)} = K_s(k) \end{aligned} \tag{5.17}$$

(cf. Definitions 4.1 and 5.1). This establishes item (i). Next, observe that for all $k \in \mathbb{Z} \cap (-m, m)$ with $k \neq 0$ it holds that

$$\begin{aligned} |K_s(k)| &\leq \frac{4^s \Gamma(1/2 + s)|\Gamma(|k| - s)|}{\sqrt{\pi}|\Gamma(-s)|\Gamma(|k| + 1 + s)} = \frac{4^s \Gamma(1/2 + s)|\Gamma(|k| - s)|}{\sqrt{\pi}|\Gamma(-s)|\Gamma(|k| + 1 + s)} \cdot \frac{|k|^{1+2s}}{|k|^{1+2s}} \\ &\leq \frac{4^s \Gamma(1/2 + s) m^{1+2s}}{\sqrt{\pi}|\Gamma(-s)|} \left[\sup_{j \in \mathbb{Z} \cap (-m,m)} \frac{|\Gamma(|j| - s)|}{\Gamma(|j| + 1 + s)} \right] \frac{1}{|k|^{1+2s}}. \end{aligned} \tag{5.18}$$

This, the fact that $s \in (0, \infty)\backslash\mathbb{N}$ implies that $\sup_{j \in \mathbb{Z} \cap (-m,m)} |\Gamma(|j|-s)|/\Gamma(|j|+1+s) \in \mathbb{R}$, and Lemma 5.3 establish item (ii). The proof of Lemma 5.4 is thus complete. □

Proposition 5.5. *It holds for all $m \in \mathbb{N}$, $s \in (m-1, m)$, $k \in \mathbb{Z}$ that*

$$K_s(k) = \frac{\mathbb{1}_{\mathbb{Z}\backslash\{0\}}(k)\, (-1)^{k+1} \Gamma(2s + 1)}{\Gamma(1 + s + k)\Gamma(1 + s - k)} \tag{5.19}$$

(cf. Definitions 4.1 and 5.1).

Proof of Proposition 5.5. Throughout this proof let $m \in \mathbb{N}$, $s \in (m-1, m)$ and without loss of generality let $k \in \mathbb{N}$ (cf. item (i) of Lemma 5.4). Observe that Definition 5.1 and the Legendre duplication formula (cf., e.g., Abramowitz and Stegun [1, Eq. (6.1.18), Page 256]) ensure that

$$\begin{aligned} K_s(k) &= \frac{-4^s \Gamma(1/2 + s)\Gamma(k - s)}{\sqrt{\pi}\Gamma(-s)\Gamma(k + 1 + s)} = \frac{-4^s \Gamma(1/2 + s)\Gamma(k - s)}{\sqrt{\pi}\Gamma(-s)\Gamma(k + 1 + s)} \cdot \frac{\Gamma(s)}{\Gamma(s)} \\ &= \frac{-4^s \left[2^{1-2s}\sqrt{\pi}\Gamma(2s)\right]\Gamma(k - s)}{\sqrt{\pi}\Gamma(-s)\Gamma(s)\Gamma(k + 1 + s)} = \frac{-2\Gamma(2s)\Gamma(k - s)}{\Gamma(-s)\Gamma(s)\Gamma(k + 1 + s)}. \end{aligned} \tag{5.20}$$

This and the fact that Definition 4.1 implies that for all $z \in \mathbb{C}$ with $\Re(z) \in \mathbb{R}\backslash\{\ldots, -2, -1, 0\}$ it holds that $z\Gamma(z) = \Gamma(1 + z)$ assure that

$$K_s(k) = \frac{-2\Gamma(2s)\Gamma(k - s)}{\Gamma(-s)\Gamma(s)\Gamma(k + 1 + s)} \cdot \frac{s}{s} = \frac{\Gamma(2s + 1)\Gamma(k - s)}{\Gamma(1 - s)\Gamma(s)\Gamma(k + 1 + s)}. \tag{5.21}$$

Next, note that the Euler reflection formula (cf., e.g., Abramowitz and Stegun [1, Eq. (6.1.17), Page 256]) guarantees that

$$\Gamma(s)\Gamma(1-s) = (-1)^{k+1}\Gamma(k-s)\Gamma(1+s-k). \tag{5.22}$$

Combining (5.21) and (5.22) hence yields (5.19). The proof of Proposition 5.5 is thus complete. □

Proposition 5.6. *Let $s \in (0,\infty)\backslash\mathbb{N}$. Then it holds for all $m \in \mathbb{N}$ that*

$$\frac{\Gamma(m-s)}{2s\Gamma(m+s)} + \sum_{k=1}^{m-1} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} = \frac{-\Gamma(-s)}{2\Gamma(1+s)} \tag{5.23}$$

(cf. Definition 4.1).

Proof of Proposition 5.6. We prove (5.23) by induction on $m \in \mathbb{N}$. For the base case $m = 1$ note that the fact that Definition 4.1 ensures that for all $z \in \mathbb{C}$ with $\mathfrak{R}(z) \in \mathbb{R}\backslash\{\dots,-2,-1,0\}$ it holds that $z\Gamma(z) = \Gamma(z+1)$ guarantees that

$$\frac{\Gamma(1-s)}{2s\Gamma(1+s)} + \sum_{k=1}^{0} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} = \frac{\Gamma(1-s)}{2s\Gamma(1+s)} = \frac{-s\Gamma(-s)}{2s\Gamma(1+s)} = \frac{-\Gamma(-s)}{2\Gamma(1+s)}. \tag{5.24}$$

This establishes (5.23) in the case $m = 1$. For the induction step $\mathbb{N} \ni (m-1) \dashrightarrow m \in \mathbb{N} \cap [2,\infty)$, let $m \in \mathbb{N} \cap [2,\infty)$ and assume for all $\mathfrak{m} \in \{1,2,\dots,m-1\}$ that (5.23) holds. Observe that the induction hypothesis shows that for all $m \in \mathbb{N} \cap [2,\infty)$ it holds that

$$\begin{aligned} &\frac{\Gamma(m-s)}{2s\Gamma(m+s)} + \sum_{k=1}^{m-1} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} \\ &= \left[\frac{\Gamma(m-s)}{2s\Gamma(m+s)} + \frac{\Gamma((m-1)-s)}{\Gamma((m-1)+1+s)}\right] + \sum_{k=1}^{(m-1)-1} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} \\ &= \frac{\Gamma(m-s)}{2s\Gamma(m+s)} + \frac{\Gamma(m-1-s)}{\Gamma(m+s)} + \frac{-\Gamma(m-1-s)}{2s\Gamma(m-1+s)} + \frac{-\Gamma(-s)}{2\Gamma(1+s)}. \end{aligned} \tag{5.25}$$

Next, note that the fact that Definition 4.1 ensures that for all $z \in \mathbb{C}$ with $\mathfrak{R}(z) \in \mathbb{R}\backslash\{\dots,-2,-1, 0\}$ it holds that $z\Gamma(z) = \Gamma(z+1)$ demonstrates that for all $m \in \mathbb{N} \cap [2,\infty)$ it holds that

$$\begin{aligned} &\frac{\Gamma(m-s)}{2s\Gamma(m+s)} + \frac{-\Gamma(m-1-s)}{2s\Gamma(m-1+s)} = \frac{\Gamma(m-s)}{2s\Gamma(m+s)} + \frac{-(m-1+s)\Gamma(m-1-s)}{2s(m-1+s)\Gamma(m-1+s)} \\ &= \frac{\Gamma(m-s) - (m-1+s)\Gamma(m-1-s)}{2s\Gamma(m+s)} = \frac{(m-1-s)\Gamma(m-s) - (m-1+s)\Gamma(m-s)}{2s(m-1-s)\Gamma(m+s)} \\ &= \left[\frac{(m-1-s)-(m-1+s)}{2s(m-1-s)}\right]\frac{\Gamma(m-s)}{\Gamma(m+s)} = \frac{-\Gamma(m-s)}{(m-1-s)\Gamma(m+s)}. \end{aligned} \tag{5.26}$$

Moreover, observe that Definition 4.1 assures that for all $z \in \mathbb{C}$ with $\mathfrak{R}(z) \in \mathbb{R}\backslash\{\dots,-2,-1,0\}$ it holds that $z\Gamma(z) = \Gamma(z+1)$ demonstrates that for all $m \in \mathbb{N} \cap [2,\infty)$ it holds that

$$\frac{-\Gamma(m-s)}{(m-1-s)\Gamma(m+s)} + \frac{\Gamma(m-1-s)}{\Gamma(m+s)} = \frac{-\Gamma(m-s) + (m-1-s)\Gamma(m-1-s)}{(m-1-s)\Gamma(m+s)}$$

$$= \frac{-\Gamma(m-s)+\Gamma(m-s)}{(m-1-s)\Gamma(m+s)} = 0. \tag{5.27}$$

Combining (5.25), (5.26), and (5.27) therefore proves (5.23). The proof of Proposition 5.6 is thus complete. □

Lemma 5.7. *It holds for all $m \in \mathbb{N}$, $s \in (m-1, m)$ that*

$$\sum_{k\in\mathbb{Z}} K_s(k) = \frac{4^s\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(1+s)} \tag{5.28}$$

(cf. Definitions 4.1 and 5.1).

Proof of Lemma 5.7. First, note that item (ii) of Lemma 5.4 ensures that for all $m \in \mathbb{N}$, $s \in (m-1, m)$ it holds that $\sum_{k\in\mathbb{Z}} K_s(k) \in \mathbb{R}$. Next, observe that Definition 5.1 and item (i) of Lemma 5.4 assure that for all $m \in \mathbb{N}$, $s \in (m-1, m)$ it holds that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} K_s(k) &= \frac{-4^s\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(-s)}\left[\sum_{k\in\mathbb{Z}} \frac{\mathbb{1}_{\mathbb{Z}\setminus\{0\}}(k)\Gamma(|k|-s)}{\Gamma(|k|+1+s)}\right] \\ &= \frac{-2\cdot 4^s\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(-s)}\left[\sum_{k\in\mathbb{N}} \frac{\Gamma(k-s)}{\Gamma(k+1+s)}\right].\end{aligned} \tag{5.29}$$

In addition, note that, e.g., Artin [6, Eq. (2.13)] (applied for every $m \in \mathbb{N}$, $s \in (m-1, m)$, $k \in \mathbb{N} \cap [m, \infty)$ with $x \curvearrowleft k-s$, $y \curvearrowleft 1+2s$ in the notation of Artin [6, Eq. (2.13)]) implies that for all $m \in \mathbb{N}$, $s \in (m-1, m)$, $k \in \mathbb{N} \cap [m, \infty)$ it holds that

$$\begin{aligned}\frac{\Gamma(k-s)}{\Gamma(k+1+s)} = \frac{1}{\Gamma(1+2s)}\left[\frac{\Gamma(k-s)\Gamma(1+2s)}{\Gamma(k+1+s)}\right] &= \frac{1}{\Gamma(1+2s)}\int_0^1 (1-z)^{(1+2s)-1} z^{(k-s)-1}\,dz \\ &= \frac{1}{\Gamma(1+2s)}\int_0^1 (1-z)^{2s} z^{k-s-1}\,dz.\end{aligned} \tag{5.30}$$

This and Fubini's theorem guarantee that for all $m \in \mathbb{N}$, $s \in (m-1, m)$ it holds that

$$\begin{aligned}&\sum_{k\in\mathbb{N}} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} = \sum_{k=1}^{m-1} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} + \sum_{k=m}^{\infty} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} \\ &= \sum_{k=1}^{m-1} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} + \sum_{k=m}^{\infty}\left[\frac{1}{\Gamma(1+2s)}\int_0^1 (1-z)^{2s} z^{k-s-1}\,dz\right] \\ &= \sum_{k=1}^{m-1} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} + \frac{1}{\Gamma(1+2s)}\int_0^1 (1-z)^{2s} z^{-s-1}\left[\sum_{k=m}^{\infty} z^k\right] dz \\ &= \sum_{k=1}^{m-1} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} + \frac{1}{\Gamma(1+2s)}\int_0^1 (1-z)^{2s} z^{m-s-1}\left[\sum_{k=0}^{\infty} z^k\right] dz \\ &= \sum_{k=1}^{m-1} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} + \frac{1}{\Gamma(1+2s)}\int_0^1 (1-z)^{2s-1} z^{m-s-1}\,dz.\end{aligned} \tag{5.31}$$

Combining this, the fact that Definition 4.1 ensures that for all $z \in \mathbb{C}$ with $\Re(z) \in \mathbb{R}\backslash\{\ldots, -2, -1, 0\}$ it holds that $z\Gamma(z) = \Gamma(z+1)$, and, e.g., Artin [6, Eq. (2.13)] (applied for every $m \in \mathbb{N}$, $s \in (m-1, m)$ with $x \curvearrowleft m - s$, $y \curvearrowleft 2s$ in the notation of Artin [6, Eq. (2.13)]) demonstrates that for all $m \in \mathbb{N}$, $s \in (m-1, m)$ it holds that

$$\begin{aligned}\sum_{k\in\mathbb{N}} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} &= \sum_{k=1}^{m-1} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} + \frac{1}{\Gamma(1+2s)}\left[\frac{\Gamma(m-s)\Gamma(2s)}{\Gamma(m+s)}\right] \\ &= \sum_{k=1}^{m-1} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} + \frac{1}{2s\Gamma(2s)}\left[\frac{\Gamma(m-s)\Gamma(2s)}{\Gamma(m+s)}\right] \\ &= \sum_{k=1}^{m-1} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} + \frac{\Gamma(m-s)}{2s\Gamma(m+s)}.\end{aligned} \tag{5.32}$$

Combining this and Proposition 5.6 proves that for all $m \in \mathbb{N}$, $s \in (m-1, m)$ it holds that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} K_s(k) &= \frac{-2\cdot 4^s\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(-s)}\left[\sum_{k=1}^{m-1} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} + \frac{\Gamma(m-s)}{2s\Gamma(m+s)}\right] \\ &= \frac{-2\cdot 4^s\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(-s)}\left[\frac{-\Gamma(-s)}{2\Gamma(1+s)}\right] = \frac{4^s\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(1+s)}.\end{aligned} \tag{5.33}$$

The proof of Lemma 5.7 is thus complete. □

6. Series representation for the discrete fractional Laplace operator

In this section we prove the main result of this article. First, in Subsection 6.1 we provide a series representation of the real-valued non-integer powers of the discrete Laplace operator (cf. Lemma 6.1). This representation employs the fractional kernel introduced in Definition 5.1 and its proof hinges upon the results developed in Section 5. It is particularly interesting to note that the representation obtained in Lemma 6.1 coincides with the representation presented in Ciaurri et al. [13, Theorem 1.1] (the case where $m = 1$) and Padgett et al. [51, Theorem 2] (the case where $m = 2$).

In Subsection 6.2 we demonstrate that the representation presented in Lemma 6.1 holds for $s \in \mathbb{N}$ if we consider the limiting values of the discrete kernel function. The main result of the article, Theorem 6.4, follows immediately from the combination of Lemmas 6.1 and 6.3. In particular, Theorem 6.4 demonstrates that all real-valued positive powers of the discrete Laplace operator may be represented with the same series (or, at least, as the limit of this series). Therefore, it is the case that the discrete fractional Laplace operator is, in some sense, a perturbation of the standard positive integer power case.

While it is not the purpose of this article to discuss such issues, we wish to emphasize the importance of the last sentence in the previous paragraph. The fact that the discrete fractional Laplace operator's series representation coincides with the series representation for positive integer powers provides a framework to endow fractional calculus with potentially enlightening physical interpretations. A particularly lacking feature of the fractional calculus is the lack of meaningful physical interpretations in many situations, which has been one of the primary limiting factor in its widespread application. However, Theorem 6.4 allows us to view the fractional powers as "transitional phases" between the positive integer cases, loosely speaking. Thus, we may use the known physical intuition for positive integer powers to provide a deeper understanding of the positive non-integer cases.

6.1. Series representation for positive non-integer order

Lemma 6.1. *It holds for all $m \in \mathbb{N}$, $s \in (m-1, m)$, $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ that*

$$\big((-\Delta)^s u\big)(n) = \sum_{k\in\mathbb{Z}} K_s(n-k)\big(u(n)-u(k)\big) \tag{6.1}$$

(cf. Definitions 3.2, 4.11, and 5.1).

Proof of Lemma 6.1. Throughout this proof let $u \in \ell^2(\mathbb{Z})$, let $v\colon \mathbb{Z} \to \ell^2(\mathbb{Z})$ satisfy for all $n \in \mathbb{Z}$ that $v(n) = (-\Delta u)(n)$, and let $A_s \in \mathbb{R}$, $s \in (0,\infty)$, satisfy for all $s \in (0,\infty)$ that

$$A_s = \frac{4^s\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(1+s)} \tag{6.2}$$

(cf. Definitions 4.1 and 4.3). Observe that Padgett et al. [51, Theorem 1] establishes (6.1) in the case that $m = 1$, $s \in (0,1)$. Next, note that Definition 4.11 and the fact that for all $m \in \mathbb{N}\cap[2,\infty)$, $n \in \mathbb{Z}$ it holds that $((-\Delta)^m u)(n) = ((-\Delta)^{m-1}(-\Delta)u)(n)$ (i.e., we are invoking the fact that standard function composition is associative on its domain) ensure that for all $m \in \mathbb{N}\cap[2,\infty)$, $s \in (m-1,m)$, $n \in \mathbb{Z}$ it holds that

$$\begin{aligned}\big((-\Delta)^s u\big)(n) &= \big((-\Delta)^{s-(m-1)}((-\Delta)^{m-1}u)\big)(n) = \big((-\Delta)^{s-(m-1)}((-\Delta)^{m-2}(-\Delta u))\big)(n)\\ &= \big((-\Delta)^{s-(m-1)}((-\Delta)^{m-2}v)\big)(n) = \big((-\Delta)^{s-1}v\big)(n).\end{aligned} \tag{6.3}$$

We now claim that for all $m \in \mathbb{N}\cap[2,\infty)$, $s \in (m-1,m)$, $n \in \mathbb{N}$ it holds that

$$\big((-\Delta)^s u\big)(n) = \sum_{k\in\mathbb{Z}} K_s(n-k)\big(u(n)-u(k)\big). \tag{6.4}$$

We prove (6.4) by induction on $m \in \mathbb{N}\cap[2,\infty)$. For the base case $m = 2$ observe that Padgett et al. [51, Theorem 2] establishes (6.4). For the induction step $\mathbb{N}\cap[2,\infty) \ni (m-1) \dashrightarrow m \in \mathbb{N}\cap[3,\infty)$, let $m \in \mathbb{N}\cap[3,\infty)$ and assume for all $\mathfrak{m} \in \{2,3,\ldots,m-1\}$, $s \in (\mathfrak{m}-1,\mathfrak{m})$, $n \in \mathbb{Z}$ that (6.4) holds. Observe that the induction hypothesis, (6.2), (6.3), (6.4), and Lemma 5.7 demonstrate that for all $s \in (m-1,m)$, $n \in \mathbb{Z}$ it holds that

$$\begin{aligned}\big((-\Delta)^s u\big)(n) &= \big((-\Delta)^{s-1}v\big)(n) = \textstyle\sum_{k\in\mathbb{Z}} K_{s-1}(n-k)\big(v(n)-v(k)\big)\\ &= v(n)\textstyle\sum_{k\in\mathbb{Z}} K_{s-1}(n-k) - \sum_{k\in\mathbb{Z}} K_{s-1}(n-k)v(k) = A_{s-1}v(n) - \sum_{k\in\mathbb{Z}} K_{s-1}(n-k)v(k).\end{aligned} \tag{6.5}$$

This, Definition 5.1, the fact that for all $n \in \mathbb{Z}$ it holds that $v(n) = (-\Delta u)(n)$, and item (i) of Lemma 5.4 show that for all $s \in (m-1,m)$, $n \in \mathbb{Z}$ it holds that

$$\begin{aligned}\big((-\Delta)^s u\big)(n) &= A_{s-1}v(n) - \textstyle\sum_{k\in\mathbb{Z}} K_{s-1}(n-k)v(k)\\ &= A_{s-1}\big(2u(n)-u(n-1)-u(n+1)\big)\end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{k\in\mathbb{Z}} K_{s-1}(n-k)\big(2u(k)-u(k-1)-u(k+1)\big) \\
&= A_{s-1}\big(2u(n)-u(n-1)-u(n+1)\big) \\
&\quad - \sum_{k\in\mathbb{Z}} \big[2K_{s-1}(k)-K_{s-1}(k-1)-K_{s-1}(k+1)\big]u(n-k) \\
&= A_{s-1}\big(2u(n)-u(n-1)-u(n+1)\big) \\
&\quad - \big[2K_{s-1}(0)-K_{s-1}(-1)-K_{s-1}(1)\big]u(n) \\
&\quad - \big[2K_{s-1}(1)-K_{s-1}(0)-K_{s-1}(2)\big]u(n-1) \\
&\quad - \big[2K_{s-1}(-1)-K_{s-1}(-2)-K_{s-1}(0)\big]u(n+1) \\
&\quad - \sum_{k\in\mathbb{Z}\backslash\{-1,0,1\}} \big[2K_{s-1}(k)-K_{s-1}(k-1)-K_{s-1}(k+1)\big]u(n-k) \\
&= \big[2A_{s-1}+2K_{s-1}(1)\big]u(n) \\
&\quad - \big[A_{s-1}+2K_{s-1}(1)-K_{s-1}(2)\big]\big(u(n-1)+u(n+1)\big) \\
&\quad - \sum_{k\in\mathbb{Z}\backslash\{-1,0,1\}} \big[2K_{s-1}(k)-K_{s-1}(k-1)-K_{s-1}(k+1)\big]u(n-k).
\end{aligned} \tag{6.6}
$$

Next, note that (6.2), the fact that Definition 4.1 implies that for all $z\in\mathbb{C}$ with $\Re(z)\in\mathbb{R}\backslash\{\dots,-2,-1,0\}$ it holds that $\Gamma(z+1)=z\Gamma(z)$, and the fact that $1-s\in(-\infty,0]$ guarantee that for all $s\in(m-1,m)$ it holds that

$$
\begin{aligned}
K_{s-1}(1) &= \frac{-\mathbb{1}_{\mathbb{Z}\backslash\{0\}}(1)\, 4^{s-1}\Gamma(1/2+(s-1))\Gamma(|1|-(s-1))}{\sqrt{\pi}\Gamma(-(s-1))\Gamma(|1|+1+(s-1))} = \frac{-4^{s-1}\Gamma(s-1/2)\Gamma(2-s)}{\sqrt{\pi}\Gamma(1-s)\Gamma(1+s)} \\
&= \frac{-4^{s-1}\Gamma(s-1/2)(1-s)\Gamma(1-s)}{\sqrt{\pi}\Gamma(1-s)s\Gamma(s)} = \frac{4^{s-1}\Gamma(s-1/2)(s-1)}{\sqrt{\pi}s\Gamma(s)}
\end{aligned} \tag{6.7}
$$

and

$$
\begin{aligned}
K_{s-1}(2) &= \frac{-\mathbb{1}_{\mathbb{Z}\backslash\{0\}}(2)\, 4^{s-1}\Gamma(1/2+(s-1))\Gamma(|2|-(s-1))}{\sqrt{\pi}\Gamma(-(s-1))\Gamma(|2|+1+(s-1))} = \frac{-4^{s-1}\Gamma(s-1/2)\Gamma(3-s)}{\sqrt{\pi}\Gamma(1-s)\Gamma(2+s)} \\
&= \frac{-4^{s-1}\Gamma(s-1/2)(2-s)(1-s)\Gamma(1-s)}{\sqrt{\pi}\Gamma(1-s)(1+s)s\Gamma(s)} = \frac{4^{s-1}\Gamma(s-1/2)(2-s)(s-1)}{\sqrt{\pi}(1+s)s\Gamma(s)}.
\end{aligned} \tag{6.8}
$$

Observe that (6.2), (6.7), and the fact that Definition 4.1 implies that for all $z\in\mathbb{C}$ with $\Re(z)\in\mathbb{R}\backslash\{\dots,-2,-1,0\}$ it holds that $\Gamma(z+1)=z\Gamma(z)$ hence ensure that for all $s\in(m-1,m)$ it holds that

$$
\begin{aligned}
2A_{s-1}+2K_{s-1}(1) &= 2\left[\frac{4^{s-1}\Gamma(s-1/2)}{\sqrt{\pi}\Gamma(s)}\right] + 2\left[\frac{4^{s-1}\Gamma(s-1/2)(s-1)}{\sqrt{\pi}s\Gamma(s)}\right] \\
&= \frac{2\cdot 4^{s-1}\Gamma(s-1/2)}{\sqrt{\pi}\Gamma(s)}\left[1+\frac{s-1}{s}\right] = \frac{2\cdot 4^{s-1}\Gamma(s-1/2)}{\sqrt{\pi}\Gamma(s)}\left[\frac{2s-1}{s}\right] \\
&= \frac{4^{s}(s-1/2)\Gamma(s-1/2)}{\sqrt{\pi}s\Gamma(s)} = \frac{4^{s}\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(1+s)} = A_s.
\end{aligned} \tag{6.9}
$$

In addition, observe that Proposition 5.5 and the fact that Definition 4.1 implies that for all $z\in\mathbb{C}$ with $\Re(z)\in\mathbb{R}\backslash\{\dots,-2,-1,0\}$ it holds that $\Gamma(z+1)=z\Gamma(z)$ demonstrate that for all $k\in$

$\mathbb{Z}\backslash\{-1,0,1\}$, $s \in (m-1,m)$ it holds that

$$\begin{aligned}
&2K_{s-1}(k) - K_{s-1}(k-1) - K_{s-1}(k+1) \\
&\quad= \frac{-2(-1)^k\Gamma(2s-1)}{\Gamma(s+k)\Gamma(s-k)} - \frac{(s+k-1)(-1)^k\Gamma(2s-1)}{(s-k)\Gamma(s+k)\Gamma(s-k)} - \frac{(s-k-1)(-1)^k\Gamma(2s-1)}{(s+k)\Gamma(s+k)\Gamma(s-k)} \\
&\quad= \frac{(-1)^k\Gamma(2s-1)}{\Gamma(s+k)\Gamma(s-k)}\left[-2 - \frac{s+k-1}{s-k} - \frac{s-k-1}{s+k}\right] \\
&\quad= \frac{(-1)^k\Gamma(2s-1)}{\Gamma(s+k+1)\Gamma(s+k-1)}\left[2s-4s^2\right] = \frac{(-1)^{k+1}\Gamma(2s+1)}{\Gamma(1+s+k)\Gamma(1+s-k)} = K_s(k).
\end{aligned} \tag{6.10}$$

Moreover, note that Definition 5.1, (6.2), (6.7), (6.8), and the fact that Definition 4.1 implies that for all $z \in \mathbb{C}$ with $\mathfrak{R}(z) \in \mathbb{R}\backslash\{\dots,-2,-1,0\}$ it holds that $\Gamma(z+1) = z\Gamma(z)$ assure that for all $s \in (m-1,m)$ it holds that

$$\begin{aligned}
&A_{s-1} + 2K_{s-1}(1) - K_{s-1}(2) \\
&\quad= \frac{4^{s-1}\Gamma(s-1/2)}{\sqrt{\pi}\Gamma(s)} + 2\left[\frac{4^{s-1}\Gamma(s-1/2)(s-1)}{\sqrt{\pi}s\Gamma(s)}\right] - \frac{4^{s-1}\Gamma(s-1/2)(2-s)(s-1)}{\sqrt{\pi}(1+s)s\Gamma(s)} \\
&\quad= \frac{4^{s-1}\Gamma(s-1/2)}{\sqrt{\pi}\Gamma(s)}\left[1 + 2\left(\frac{s-1}{s}\right) + \frac{(s-2)(s-1)}{s(1+s)}\right] = \frac{4^{s-1}\Gamma(s-1/2)}{\sqrt{\pi}\Gamma(s)}\left[\frac{4(s-1/2)}{s+1}\right] \\
&\quad= \frac{4^s\Gamma(s+1/2)}{\sqrt{\pi}\Gamma(s)(s+1)} = \frac{4^s\Gamma(s+1/2)}{\sqrt{\pi}\Gamma(s)(s+1)} \cdot \frac{s\Gamma(-s)}{s\Gamma(-s)} = \frac{-4^s\Gamma(s+1/2)\Gamma(1-s)}{\sqrt{\pi}\Gamma(-s)\Gamma(2+s)} = K_s(1).
\end{aligned} \tag{6.11}$$

Combining (6.6), (6.9), (6.10), (6.11), and item (i) of Lemma 5.4 therefore yields that for all $s \in (m-1,m)$, $n \in \mathbb{Z}$ it holds that

$$\begin{aligned}
\big((-\Delta)^s u\big)(n) &= \big[2A_{s-1} + 2K_{s-1}(1)\big]u(n) \\
&\quad - \big[A_{s-1} + 2K_{s-1}(1) - K_{s-1}(2)\big]\big(u(n-1)+u(n+1)\big) \\
&\quad - \sum_{k\in\mathbb{Z}\backslash\{-1,0,1\}}\big[2K_{s-1}(k) - K_{s-1}(k-1) - K_{s-1}(k+1)\big]u(n-k) \\
&= A_s u(n) - K_s(1)\big(u(n-1)+u(n+1)\big) - \sum_{k\in\mathbb{Z}\backslash\{-1,0,1\}} K_s(k)u(n-k) \\
&= A_s u(n) - \sum_{k\in\mathbb{Z}\backslash\{0\}} K_s(k)u(n-k) = \sum_{k\in\mathbb{Z}} K_s(n-k)\big(u(n)-u(k)\big).
\end{aligned} \tag{6.12}$$

Induction hence establishes (6.4). The proof of Lemma 6.1 is thus complete. □

6.2. Series representation for arbitrary positive order

Lemma 6.2. *Let $s \in \mathbb{N}$. Then it holds for all $k \in \mathbb{Z}$ that*

$$\lim_{z\to s} K_z(k) = \frac{\mathbb{1}_{\{1,2,\dots,s\}}(k)\,(-1)^{k+1}\Gamma(2s+1)}{\Gamma(1+s+k)\Gamma(1+s-k)} \tag{6.13}$$

(cf. Definitions 4.1 and 5.1).

Proof of Lemma 6.2. First, note that Definition 5.1 and Proposition 5.5 ensure that for all $k \in \mathbb{Z}$ it holds that

$$\lim_{z\to s} K_z(k) = \lim_{z\to s} \frac{\mathbb{1}_{\mathbb{Z}\setminus\{0\}}(k)\,(-1)^{k+1}\Gamma(2s+1)}{\Gamma(1+s+k)\Gamma(1+s-k)} \tag{6.14}$$

(cf. Definitions 4.1 and 5.1). Combining this and the fact that Definition 4.1 ensures that for all $z \in \{\ldots,-2,-1,0\}$ it holds that $1/\Gamma(z) = 0$ establishes (6.13). The proof of Lemma 6.2 is thus complete. □

Lemma 6.3. *It holds for all $s \in \mathbb{N}$, $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ that*

$$\big((-\Delta)^s u\big)(n) = \sum_{k=0}^{2s} (-1)^{k-s}\binom{2s}{k} u(n-s+k) = \lim_{z\to s}\left[\sum_{k\in\mathbb{Z}} K_z(n-k)\big(u(n)-u(k)\big)\right] \tag{6.15}$$

(cf. Definitions 3.2, 4.11, and 5.1).

Proof of Lemma 6.3. First, note that Definition 4.1 and the fact that for all $a \in \mathbb{N}$, $b \in \{0,1,\ldots,a\}$ it holds that $\binom{a}{b} = \binom{a}{a-b}$ assure that for all $s \in \mathbb{N}$, $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ it holds that

$$\begin{aligned}
\sum_{k=0}^{2s} (-1)^{k-s}\binom{2s}{k} u(n-s+k) &= \sum_{k=-s}^{s} (-1)^k \binom{2s}{s+k} u(n-k) \\
&= \sum_{k=-s}^{s} \left[\frac{(-1)^k\Gamma(2s+1)}{\Gamma(1+s+k)\Gamma(1+s-k)}\right] u(n-k) \\
&= \left[\frac{\Gamma(2s+1)}{\Gamma(1+s)\Gamma(1+s)}\right] u(n) - \sum_{k=1}^{s}\left[\frac{(-1)^{k+1}\Gamma(2s+1)}{\Gamma(1+s+k)\Gamma(1+s-k)}\right]\big(u(n-k)+u(n+k)\big)
\end{aligned} \tag{6.16}$$

(cf. Definitions 3.2 and 4.1). Next, observe that Lemma 5.7 ensures that for all $z \in (0,\infty)\setminus\mathbb{N}$, $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ it holds that

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} K_z(n-k)\big(u(n)-u(k)\big) &= u(n)\sum_{k\in\mathbb{Z}} K_z(n-k) - \sum_{k\in\mathbb{Z}} K_z(n-k)u(k) \\
&= \left[\frac{4^z\Gamma(1/2+z)}{\sqrt{\pi}\Gamma(1+z)}\right] u(n) - \sum_{k\in\mathbb{Z}} K_z(n-k)u(k)
\end{aligned} \tag{6.17}$$

(cf. Definition 5.1). This, Definition 4.1, Definition 5.1, items (i) and (ii) of Lemma 5.4, Lemma 6.2, and, e.g., Rudin [55, Theorem 7.17] guarantee that for all $s \in \mathbb{N}$, $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ it holds that

$$\begin{aligned}
\lim_{z\to s}\left[\sum_{k\in\mathbb{Z}} K_s(n-k)\big(u(n)-u(k)\big)\right] &= \left[\lim_{z\to s}\frac{4^z\Gamma(1/2+z)}{\sqrt{\pi}\Gamma(1+z)}\right] u(n) - \lim_{z\to s}\left[\sum_{k\in\mathbb{Z}} K_z(n-k)u(k)\right] \\
&= \left[\frac{4^s\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(1+s)}\right] u(n) - \lim_{z\to s}\left[\sum_{k\in\mathbb{N}} K_z(k)\big(u(n-k)+u(n+k)\big)\right] \\
&= \left[\frac{4^s\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(1+s)}\right] u(n) - \sum_{k\in\mathbb{N}}\left[\lim_{z\to s} K_z(k)\right]\big(u(n-k)+u(n+k)\big)
\end{aligned} \tag{6.18}$$

$$= \left[\frac{4^s\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(1+s)}\right]u(n) - \sum_{k\in\mathbb{N}}\left[\frac{\mathbb{1}_{\{1,2,\ldots,s\}}(k)\,(-1)^{k+1}\Gamma(2s+1)}{\Gamma(1+s+k)\Gamma(1+s-k)}\right]\big(u(n-k)+u(n+k)\big)$$
$$= \left[\frac{4^s\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(1+s)}\right]u(n) - \sum_{k=1}^{s}\left[\frac{(-1)^{k+1}\Gamma(2s+1)}{\Gamma(1+s+k)\Gamma(1+s-k)}\right]\big(u(n-k)+u(n+k)\big).$$

In addition, note that the fact that Definition 4.1 implies that for all $z \in \mathbb{C}$ with $\Re(z) \in \mathbb{R}\backslash\{\ldots,-2,-1,0\}$ it holds that $z\Gamma(z) = \Gamma(1+z)$ and the Legendre duplication formula (cf., e.g., [1, Eq. (6.1.18), Page 256]) demonstrate that for all $s \in \mathbb{N}$ it holds that

$$\frac{4^s\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(1+s)} = \frac{4^s\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(1+s)}\cdot\frac{s\Gamma(s)}{s\Gamma(s)} = \frac{4^s\left[2^{1-2s}\sqrt{\pi}s\Gamma(2s)\right]}{\sqrt{\pi}\Gamma(s)\Gamma(1+s)} = \frac{\Gamma(2s+1)}{\Gamma(1+s)\Gamma(1+s)}. \tag{6.19}$$

Combining this, (6.16), (6.18), Lemma 4.8, and Definition 4.11 proves (6.15). The proof of Lemma 6.3 is thus complete. □

Theorem 6.4. *It holds for all $s \in (0,\infty)$, $u \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ that*

$$\big((-\Delta)^s u\big)(n) = \lim_{z\to s}\left[\sum_{k\in\mathbb{Z}} K_z(n-k)\big(u(n)-u(k)\big)\right] \tag{6.20}$$

(cf. Definitions 3.2, 4.11, and 5.1).

Proof of Theorem 6.4. Note that combining Lemmas 6.1 and 6.3 establishes (6.20). The proof of Theorem 6.4 is thus complete. □

7. Conclusions and future endeavors

7.1. Concluding remarks

In this article we developed novel results regarding real-valued positive fractional powers of the discrete Laplace operator. In particular, we defined a discrete fractional Laplace operator for arbitrary real-valued positive powers (cf. Definition 4.11) and then developed its series representation (cf. Theorem 6.4). This latter task was primarily accomplished through the development of two sets of results. First, we constructed the series representation for positive integer powers of the discrete Laplace operator (cf. Lemmas 4.8 and 6.3). Next, we developed series representations for positive non-integer powers of the discrete Laplace operator (cf. Lemma 6.1). The main result of the article (cf. Theorem 6.4) is obtained by showing that the series representations obtained in each of the previous steps in fact coincide.

The main results developed—i.e., the results of Section 6—required numerous preliminary results from various areas of mathematics. The results in Section 3 are of a functional analysis flavor and allow for a beautiful description of important properties of strongly continuous semigroups. These results were combined with results from discrete harmonic analysis in Section 4 in order to define and study the discrete fractional Laplace operator. Since the presented definition of this operator (cf. Definition 4.11) employs a so-called semigroup language, it was imperative that all novel mathematical objects are determined to be well defined in $\ell^2(\mathbb{Z})$ (cf. Definition 3.2). Finally, Section 5 provides a detailed study of the proposed fractional kernel function (cf. Definition 5.1) which is necessary for the development of the coefficients of the series representations presented in Section 6. Therein, it is shown that the proposed fractional kernel function is well-defined, symmetric, and continuous for all $s \in (0,\infty)\backslash\mathbb{N}$. It is later shown in Lemma 6.2 that the values $s \in \mathbb{N}$ are in fact removable singularities.

As a final remark, we wish to emphasize the importance of the presented results (for a specific physical motivation, see Section 2 above). Due to the rapidly growing interest in problems related to fractional calculus, there is a need to determine the validity of including fractional operators into existing models. The study of the discrete fractional Laplace operator, or its continuous counterpart, for the case when $s \in (0,1)$ is well-understood and often used in physical sciences. In this setting, the operator may be used to model *super-diffusive* phenomena [51]. Moreover, there have been rigorous studies of the operator in this parameter regime which demonstrate that such considerations are well-defined and well-behaved. As such, it is natural to attempt to extend these studies to the case when $s \in (1,\infty)$, as well. The current article has demonstrated that such extensions are indeed well-defined in the discrete case. In addition, Theorem 6.4 shows that one may potentially use the existing understanding of the case of positive integer powers of the discrete Laplace operator to provide some much needed physical intuition to the discrete fractional Laplace operator. However, there are still numerous unanswered questions regarding important properties of these operators and we will outline a few such open problems and research directions in Subsection 7.2 below.

7.2. Related future endeavors

First and foremost, there is a need to continue the analytical work presented herein in order to obtain a better understanding of the discrete fractional Laplace operator. In this article, we have considered the setting where all objects are defined in $\ell^2(\mathbb{Z})$, however, this is not always the appropriate setting for physically relevant problems. As such, we intend to extend our study to the situation where the underlying function spaces have less regularity (e.g., Hölder spaces) and develop standard regularity estimates. We also intend to develop similar series representations for the case of real-valued negative exponents. Such representations are highly important for studying fractional Poisson-like problems, as they provide representations of the solution to these problems. Finally, we hope to develop an understanding of the spectral properties of the discrete fractional Laplace operator. While it is well-known that the discrete Laplace operator (cf. Definition 4.3) has purely absolutely continuous spectra (cf., e.g., Dutkay and Jorgensen [18]), to the authors' knowledge this has not been rigorously proven in the case of the discrete fractional Laplace operator. Demonstrating this to be the case is of utmost importance and will have far-reaching implications in mathematics and physics (for clarification, see the techniques outlined in Liaw [38]).

The proposed discrete fractional Laplace operator is also of interest due to its importance in the physical sciences. An example of interest for future research is transport in turbulent plasmas. It has been experimentally observed that heat and particle transport in turbulent plasmas is *non-local* (i.e., *anomalous*) in nature (cf., e.g., [15, 23, 56]). Comparison between transport models using the fractional Laplace operator and experimental results have demonstrated that electron transport in turbulent fusion plasmas is characterized by fractional exponents in the range $s \in (0.6, 1)$, which indicates super-diffusive behavior [34, 35]. Moreover, a generalized approach to modeling anomalous diffusive transport in turbulent plasmas employs diffusion-type equations where fractional derivatives occur in both space and time (cf., e.g., del Castillo-Negrete et al. [14]). Using the series representations presented herein, we intend to show that the fractional derivative in time can be incorporated into the spatial derivative, which can greatly simplify such equations.

Acknowledgments

The second author acknowledges funding by the National Science Foundation (NSF 1903450) and the Department of Energy (DE-SC0021284). The third author acknowledges funding by the National Science Foundation (NSF 1903450). The fourth author would like to thank the College of Arts and Sciences at Baylor University for partial support through a research leave award.

References

[1] Abramowitz, M., and Stegun, I. A., Eds. *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* Dover Publications, Inc., New York, 1992. Reprint of the 1972 edition.

[2] Aizenman, M., and Molchanov, S. Localization at large disorder and at extreme energies: An elementary derivations. *Communications in Mathematical Physics 157*, 2 (1993), 245–278.

[3] Aizenman, M., Sims, R., and Warzel, S. Absolutely continuous spectra of quantum tree graphs with weak disorder. *Comm. Math. Phys. 264*, 2 (2006), 371–389.

[4] Anderson, P. W. Absence of diffusion in certain random lattices. *Phys. Rev. 109* (Mar 1958), 1492–1505.

[5] Arendt, W., ter Elst, A. F. M., and Warma, M. Fractional powers of sectorial operators via the Dirichlet-to-Neumann operator. *Comm. Partial Differential Equations 43*, 1 (2018), 1–24.

[6] Artin, E. *The gamma function.* Translated by Michael Butler. Athena Series: Selected Topics in Mathematics. Holt, Rinehart and Winston, New York-Toronto-London, 1964.

[7] Brandes, T., and Kettemann, S. *Anderson localization and its ramifications: Disorder, phase coherence, and electron correlations*, vol. 630. Springer Science & Business Media, 2003.

[8] Bucur, C., and Valdinoci, E. *Nonlocal diffusion and applications*, vol. 20 of *Lecture Notes of the Unione Matematica Italiana.* Springer, [Cham]; Unione Matematica Italiana, Bologna, 2016.

[9] Caffarelli, L., and Silvestre, L. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations 32*, 7-9 (2007), 1245–1260.

[10] Cartan, H., Moore, J., Husemoller, D., and Maestro, K. *Differential Calculus on Normed Spaces: A Course in Analysis.* CreateSpace Independent Publishing Platform, 2017.

[11] Chen, Y. K., Lei, Z., and Wei, C. H. Extension problems related to the higher order fractional Laplacian. *Acta Math. Sin. (Engl. Ser.) 34*, 4 (2018), 655–661.

[12] Ciaurri, O., Gillespie, T. A., Roncal, L., Torrea, J. L., and Varona, J. L. Harmonic analysis associated with a discrete Laplacian. *J. Anal. Math. 132* (2017), 109–131.

[13] Ciaurri, O., Roncal, L., Stinga, P. R., Torrea, J. L., and Varona, J. L. Nonlocal discrete diffusion equations and the fractional discrete Laplacian, regularity and applications. *Adv. Math. 330* (2018), 688–738.

[14] del Castillo-Negrete, D., Carreras, B., and Lynch, V. Fractional diffusion in plasma turbulence. *Physics of Plasmas 11*, 8 (2004), 3854–3864.

[15] del Castillo-Negrete, D., Carreras, B., and Lynch, V. Nondiffusive transport in plasma turbulence: a fractional diffusion approach. *Physical review letters 94*, 6 (2005), 065003.

[16] Dragomir, S. S. *Semi-inner products and applications.* Nova Science Publishers, Inc., Hauppauge, NY, 2004.

[17] Driver, B. K. Functional analysis tools with examples.

[18] Dutkay, D. E., and Jorgensen, P. E. T. Spectral theory for discrete Laplacians. *Complex Anal. Oper. Theory 4*, 1 (2010), 1–38.

[19] Felli, V., and Ferrero, A. Unique continuation principles for a higher order fractional Laplace equation. *Nonlinearity 33*, 8 (2020), 4133–4190.

[20] Fröhlich, J., and Spencer, T. Absence of diffusion in the Anderson tight binding model for large disorder or low energy. *Communications in Mathematical Physics 88*, 2 (1983), 151–184.

[21] Galé, J. E., Miana, P. J., and Stinga, P. R. Extension problem and fractional operators: semigroups and wave equations. *J. Evol. Equ. 13*, 2 (2013), 343–368.

[22] García-Ferrero, M. A., and Rüland, A. Strong unique continuation for the higher order fractional Laplacian. *Math. Eng. 1*, 4 (2019), 715–774.

[23] Gentle, K., Bravenec, R., Cima, G., Gasquet, H., Hallock, G., Phillips, P., Ross, D., Rowan, W., Wootton, A., Crowley, T., et al. An experimental counter-example to the local transport paradigm. *Physics of Plasmas 2*, 6 (1995), 2292–2298.

[24] Giles, J. R. Classes of semi-inner-product spaces. *Trans. Amer. Math. Soc. 129* (1967), 436–446.

[25] Hou, L.-J., Piel, A., and Shukla, P. K. Self-diffusion in 2D dusty-plasma liquids: Numerical-simulation results. *Phys. Rev. Lett. 102* (Feb 2009), 085002.

[26] Jakšić, V., and Last, Y. Spectral structure of Anderson type Hamiltonians. *Invent. Math. 141*, 3 (2000), 561–577.

[27] Jakšić, V., and Last, Y. Simplicity of singular spectrum in Anderson-type Hamiltonians. *Duke Math. J. 133*, 1 (2006), 185–204.

[28] Jones, T. F., Padgett, J. L., and Sheng, Q. Intrinsic properties of strongly continuous fractional semigroups in normed vector spaces. *arXiv preprint arXiv:2012.11092* (2020).

[29] Kelley, W. G., and Peterson, A. C. *Difference equations*, second ed. Harcourt/Academic Press, San Diego, CA, 2001. An introduction with applications.

[30] Kostadinova, E. G., Banka, R., Padgett, J. L., Liaw, C. D., Matthews, L. S., and Hyde, T. W. Active turbulence in a dusty plasma monolayer. *arXiv preprint arXiv:2102.09344* (2021).

[31] Kostadinova, E. G., Busse, K., Ellis, N., Padgett, J. L., Liaw, C. D., Matthews, L. S., and Hyde, T. W. Delocalization in infinite disordered two-dimensional lattices of different geometry. *Physical Review B 96*, 23 (2017), 235408.

[32] Kostadinova, E. G., Padgett, J. L., Liaw, C. D., Matthews, L. S., and Hyde, T. W. Numerical study of anomalous diffusion of light in semicrystalline polymer structures. *Phys. Rev. Research 2* (Dec 2020), 043375.

[33] Kramer, B., and MacKinnon, A. Localization: theory and experiment. *Reports on Progress in Physics 56*, 12 (1993), 1469.

[34] Kullberg, A., del Castillo-Negrete, D., Morales, G., and Maggs, J. Isotropic model of fractional transport in two-dimensional bounded domains. *Physical Review E 87*, 5 (2013), 052115.

[35] Kullberg, A., Morales, G., and Maggs, J. Comparison of a radial fractional transport model with tokamak experiments. *Physics of Plasmas 21*, 3 (2014), 032310.

[36] Lagendijk, A., Van Tiggelen, B., and Wiersma, D. S. Fifty years of Anderson localization. *Phys. Today 62*, 8 (2009), 24–29.

[37] Lebedev, V. I., and Agoshkov, V. I. *Poincaré-Steklov operators and their applications in analysis.* Akad. Nauk SSSR, Vychisl. Tsentr, Moscow, 1983.

[38] Liaw, C. Approach to the extended states conjecture. *Journal of Statistical Physics 153*, 6 (2013), 1022–1038.

[39] Lischke, A., Pang, G., Gulian, M., and et al. What is the fractional Laplacian? A comparative review with new results. *J. Comput. Phys. 404* (2020), 109009, 62.

[40] Liu, B., and Goree, J. Superdiffusion in two-dimensional Yukawa liquids. *Phys. Rev. E 75* (Jan 2007), 016405.

[41] Liu, B., and Goree, J. Superdiffusion and non-Gaussian statistics in a driven-dissipative 2D dusty plasma. *Phys. Rev. Lett. 100* (Feb 2008), 055003.

[42] Lumer, G. Semi-inner-product spaces. *Trans. Amer. Math. Soc. 100* (1961), 29–43.

[43] Meerschaert, M. M., and Sikorskii, A. *Stochastic models for fractional calculus*, second ed., vol. 43 of *De Gruyter Studies in Mathematics.* De Gruyter, Berlin, 2019.

[44] Meichsner, J., and Seifert, C. Fractional powers of non-negative operators in Banach spaces via the Dirichlet-to-Neumann operator. *arXiv preprint arXiv:1704.01876* (2017).

[45] Meichsner, J., and Seifert, C. On the harmonic extension approach to fractional powers in Banach spaces. *Fract. Calc. Appl. Anal. 23*, 4 (2020), 1054–1089.

[46] Nunomura, S., Samsonov, D., Zhdanov, S., and Morfill, G. Self-diffusion in a liquid complex plasma. *Phys. Rev. Lett. 96* (Jan 2006), 015003.

[47] Olver, F. W. J., Lozier, D. W., Boisvert, R. F., and Clark, C. W., Eds. *NIST handbook of mathematical functions.* U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010. With 1 CD-ROM (Windows, Macintosh and UNIX).

[48] Ott, T., and Bonitz, M. Anomalous and Fickian diffusion in two-dimensional dusty plasmas. *Contributions to Plasma Physics 49*, 10 (2009), 760–764.

[49] Ott, T., Bonitz, M., Donkó, Z., and Hartmann, P. Superdiffusion in quasi-two-dimensional Yukawa liquids. *Phys. Rev. E 78* (Aug 2008), 026409.

[50] Padgett, J. L. Analysis of an approximation to a fractional extension problem. *BIT 60*, 3 (2020), 715–739.

[51] Padgett, J. L., Kostadinova, E. G., Liaw, C. D., Busse, K., Matthews, L. S., and Hyde, T. W. Anomalous diffusion in one-dimensional disordered systems: a discrete fractional Laplacian method. *J. Phys. A 53*, 13 (2020), 135205, 21.

[52] Pastur, L. A. Spectral properties of disordered systems in the one-body approximation. *Communications in Mathematical Physics 75*, 2 (1980), 179–196.

[53] Pozrikidis, C. *The fractional Laplacian.* CRC Press, Boca Raton, FL, 2016.

[54] Ros-Oton, X., and Serra, J. Local integration by parts and Pohozaev identities for higher order fractional Laplacians. *Discrete Contin. Dyn. Syst. 35*, 5 (2015), 2131–2150.

[55] Rudin, W. *Principles of mathematical analysis*, third ed. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976. International Series in Pure and Applied Mathematics.

[56] Shalchi, A. Magnetic field line random walk in two-dimensional turbulence: Markovian diffusion versus superdiffusion. *Contributions to Plasma Physics 51*, 10 (2011), 920–930.

[57] Söderlind, G. The logarithmic norm. History and modern theory. *BIT 46*, 3 (2006), 631–652.

[58] Strickler, T. S., Langin, T. K., McQuillen, P., Daligault, J., and Killian, T. C. Experimental measurement of self-diffusion in a strongly coupled plasma. *Phys. Rev. X 6* (May 2016), 021021.

[59] Tricomi, F. G., and Erdélyi, A. The asymptotic expansion of a ratio of gamma functions. *Pacific J. Math. 1* (1951), 133–142.

[60] Vaulina, O. S., and Vladimirov, S. V. Diffusion and dynamics of macro-particles in a complex plasma. *Physics of Plasmas 9*, 3 (2002), 835–840.

[61] Vázquez, J. L. The mathematical theories of diffusion: nonlinear and fractional diffusion. In *Nonlocal and nonlinear diffusions and interactions: new methods and directions*, vol. 2186 of *Lecture Notes in Math.* Springer, Cham, 2017, pp. 205–278.

[62] Yang, R. On higher order extensions for the fractional Laplacian. *arXiv preprint arXiv:1302.4413* (2013).