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Two-scale cut-and-projection convergence for quasiperiodic monotone operators

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ABSTRACT

Averaging a certain class of quasiperiodic monotone operators can be simplified to the periodic homogenization setting by mapping the original quasiperiodic structure onto a periodic structure in a higher dimensional space using the cut-and projection method. We characterize the cut-and-projection convergence limit of the nonlinear monotone partial differential operator $-\mathrm{div}\ \sigma\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}, \nabla u_\eta\right)$ for a bounded sequence u_η in $W_0^{1,p}(\Omega)$, where $1 < p < \infty$, and Ω is a bounded open subset in $\mathbb{R}^n$ with Lipschitz boundary. We identify the homogenized problem with a local equation defined on a hyperplane, or a lower dimensional plane in the higher-dimensional space. A new corrector result is established.

1. Introduction

Nonlinear physical phenomena are ubiquitous in modern electronic devices. A few examples are current surge protectors made of varistor ceramics, solid state amplifiers, and integrated circuits. This is one motivation to develop mathematical tools that can be used to analyze the effective properties of polycrystalline quasiperiodic semiconductors. In Braides et al. (2009), it is shown that integral energies F_η where the spatial dependence follows the geometry of a Penrose tiling, or more general quasicrystalline geometries, can be homogenized. More precisely,

$$F_\eta(u) = \int_\Omega f\left(\frac{\boldsymbol{x}}{\eta}, \nabla u(\boldsymbol{x})\right)\, d\boldsymbol{x}\ ,\ u \in W^{1,p}(\Omega) \tag{1}$$

where Ω is an open subset of $\mathbb{R}^2$, and f depends on $\boldsymbol{x}$ through the shape and the orientation of the cell containing $\boldsymbol{x}$ in an a-periodic tiling of the space, Γ-converge in $W^{1,p}(\Omega)$ with respect to the L^p convergence to the functional

$$F_0(\xi) = \liminf_{T\to\infty} \left\{ \frac{1}{T^2} \int_{(0,T)^2} f\left(\boldsymbol{y}, \nabla v(\boldsymbol{y}) + \xi\right)\, d\boldsymbol{y}\ ,\ v \in W_0^{1,p}((0,T)^2) \right\} \tag{2}$$

where ξ is the macroscopic field. This general homogenization result was shown using that f is Besicovitch almost periodic in $\boldsymbol{y}$ and thus a previous result on Besicovitch almost periodic functionals (Braides, 1986) could be applied. Homogenization of interfacial energies on Penrose lattices making use of Γ-convergence for similar functionals to (1) but with the surface integral replaced by a line integral has also been addressed in Braides et al. (2012).

Γ-convergence is a very powerful tool in homogenization theory (Braides, 2002), but two-scale convergence (Nguetseng, 1989; Allaire, 1992) can more easily identify homogenized equations in the periodic setting. A similar tool is the periodic unfolding approach (Cioranescu et al., 2008), in which one first maps the original sequence of functions to a sequence that is defined on $\mathbb{R}^n \times]0,1[^n$, and then takes the usual weak limit in suitable function spaces, using this extended domain. This is similar to the two-scale Fourier transform approach proposed in Wellander (2009).

Due to its simplicity, one might wish to apply two-scale homogenization (or the periodic unfolding and Fourier transform approaches) to quasiperiodic materials or mixtures of materials with rational and irrational periodicity, *e.g.*, see Braides (1991) and Casado-Diaz and Gayte (2002) for a setting in an almost periodic regime, Blanc et al. (2015a,b) for some recent work on quasiperiodic multiscale homogenization setting.

This has been proposed in Guenneau (2001) and Bouchitté et al. (2010) wherein two-scale convergence is applied to the quasiperiodic setting making use of the cut-and-projection method. Indeed, quasiperiodic materials can often be described by periodic structures in higher spatial dimensions cut and projected onto a hyperplane or a lower dimensional (physical) space, typically $\mathbb{R}^2$ (such as for Penrose tilings) and $\mathbb{R}^3$, as proposed by de Bruijn (1981) and generalized in Duneau and Katz (1985). This makes it possible to use standard periodic homogenization tools such as two-scale convergence, to homogenize quasiperiodic materials, illustrated in Fig. 1. We note that

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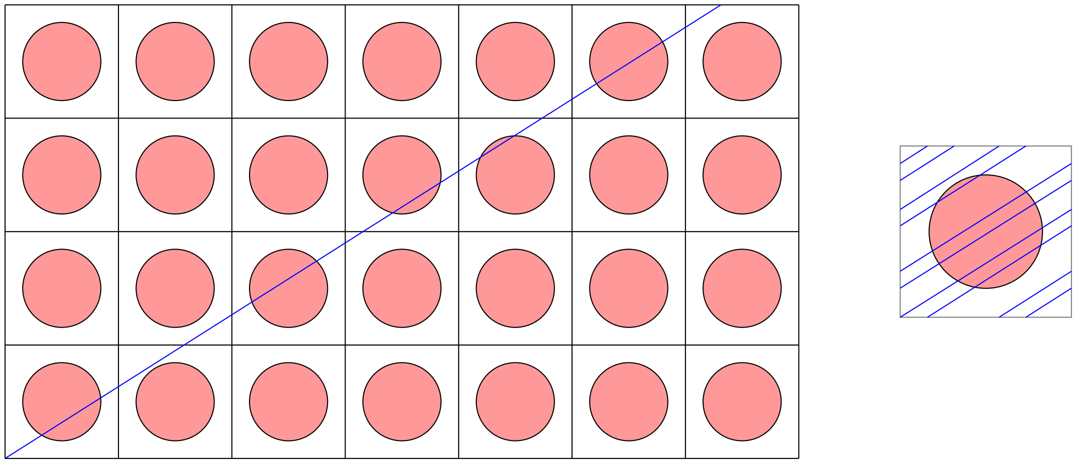

Fig. 1. Principle of cut-and-projection method: the projection of a periodic structure in higher dimensional space R^m ($m = 2$ here) onto a hyperplane (represented by a straight line) or a lower dimensional plane R^n, generates a quasiperiodic structure in R^n ($n = 1$ here), when the slope is irrational. With a rational slope the line is folded back onto a finite number of segments in the periodic cell Y^m, whereas with an irrational slope, as shown here, the set of segments is countable and dense in Y^m (note that we only show 10 segments within the countable set).

in Golden et al. (1990), effective properties of quasiperiodic structures were deduced from the cut-and-projection of checkerboards.

To do that one has to complement existing tools with the cut-and-projection operator, which was done in Bouchitté et al. (2010) in the framework of $W^{1,2}$, making use of Fourier representation of two-scale limits of gradients. Importantly, this was revisited in Wellander et al. (2018, 2019).

However, two-scale convergence can also be applied to nonlinear operators, see Allaire (1992). The approach is based on a generalization of the usual weak convergence in Lebesgue spaces L^p, $1 < p < \infty$, in which one uses oscillating test functions to capture oscillations on the same scale as the test functions in the sequence of functions that are investigated. As a consequence one obtains limit functions that are defined on the product space $\mathbb{R}^n \times]0,1[^n$.

In this paper we extend the two-scale cut-and-projection convergence method to Sobolev spaces $W^{1,p}$, $1 < p < \infty$. We build upon (Wellander et al., 2019) to characterize the limits of nonlinear partial differential operators in this setting. We illustrate the method on a nonlinear electrostatic problem that was previously homogenized using the tool of G-convergence for a larger class of almost periodic functions in Braides et al. (1992). We finally establish a corrector result for the gradients. We note that Kozlov (1979) established a corrector result for the case of almost periodic coefficients, which are the restriction of sufficiently smooth periodic functions of greater number of variables, when the problem is set in the whole space.

1.1. Setup of a nonlinear electrostatic equation in a quasiperiodic structure

Throughout the paper, we consider a bounded domain Ω in $\mathbb{R}^n$ with Lipschitz boundary. We study the electrostatic equation, which is applicable to model DC currents in semiconductors, *e.g.*, ZnO based varistor ceramics,

$$\begin{cases} -\mathrm{div}\ \sigma_\eta\left(\boldsymbol{x}, \nabla u_\eta(\boldsymbol{x})\right) = f(\boldsymbol{x}), \qquad u_\eta \in W_0^{1,p}(\Omega), \qquad 1 < p < \infty \\ \qquad u_\eta\big|_{\partial\Omega} = 0 \end{cases} \tag{3}$$

where $\sigma_\eta\left(\boldsymbol{x}, \nabla u_\eta(\boldsymbol{x})\right)$ is a nonlinear function of the electric field ∇u_η, and η is a positive parameter that will be passed to zero. It measures the fine scale in the composite. The homogenization analysis will identify a homogenized problem with a solution that, when η is small enough, is a good approximation of the solution to the original equation (3). Further, we assume $f \in W^{-1,q}(\Omega)$, $1/p + 1/q = 1$. We use standard notations for Lebesgue and Sobolev spaces. The Euclidean norm and the scalar product in $\mathbb{R}^n$ are denoted by $|\cdot|$ and $(\cdot,\cdot)$, respectively. The heterogeneous problem is modeled by the use of a unit cell, with periodic boundary conditions, in the higher dimensional space, as in Fig. 1 which is denoted Y^m. The modeling of the quasiperiodic composite is done with the help of a matrix $\mathbf{R}$, which is real valued, with m rows and n columns. The transposed matrix $\mathbf{R}^T$ cuts-and-project the unit cell, as the line does in Fig. 1, and produces a quasiperiodic pattern in $\mathbb{R}^1, \mathbb{R}^2$ or $\mathbb{R}^3$ if $\mathbf{R}$ satisfies criterion (4) given below.

Following de Bruijn (1981) and Duneau and Katz (1985), it is useful to decompose the higher-dimensional periodic space Y^m into the n-dimensional plane $Y_\parallel^m = \{\boldsymbol{y} \in \mathbb{R}^m \mid \left(\mathbf{I}_m - \mathbf{R}\mathbf{R}^T\right) \boldsymbol{y} = \mathbf{0}\}$ and its orthogonal complement $Y_\perp^m = \{\boldsymbol{y} \in \mathbb{R}^m \mid \mathbf{R}\mathbf{R}^T \boldsymbol{y} = \mathbf{0}\}$, which is the essence of the cut-and-projection method. This is illustrated in Fig. 1 where $Y_\parallel^m$ corresponds to the oblique blue line and $Y_\perp^m$ is thus a line perpendicular to it (not shown). This geometric decomposition underpins the decomposition of the functional space in Eq. (37), as well regularity of material constituents, characterized here by the function σ in (3).

The current density is given by a non-linear map, σ, that satisfies assumptions (i)–(vi):

(i) $\sigma(\boldsymbol{x},\cdot,\xi)$ is Y-periodic in $\mathbb{R}^m$, is Lebesgue measurable on $Y_\parallel^m$ and continuous on $Y_\perp^m$, for every $\boldsymbol{x} \in \Omega, \xi \in \mathbb{R}^n$.
(ii) $\sigma(\cdot,\boldsymbol{y},\xi)$ is continuous for almost every $\boldsymbol{y} \in \mathbb{R}^m$ and every $\xi \in \mathbb{R}^n$.
(iii) $\sigma(\boldsymbol{x},\boldsymbol{y},\cdot)$ is continuous for almost every $\boldsymbol{x} \in \Omega$ and $\boldsymbol{y} \in \mathbb{R}^m$.
(iv) $0 \le c|\xi|^p \le (\sigma(\boldsymbol{x},\boldsymbol{y},\xi),\xi)$, $c > 0$, for almost every $\boldsymbol{x} \in \Omega$ and $\boldsymbol{y} \in \mathbb{R}^m$, for any $\xi \in \mathbb{R}^n$.
(v) $\left(\sigma(\boldsymbol{x},\boldsymbol{y},\xi_1) - \sigma(\boldsymbol{x},\boldsymbol{y},\xi_2), \xi_1 - \xi_2\right) \ge c_1 \left|\xi_1 - \xi_2\right|^p$, $c_1 > 0$, for all $\xi_1, \xi_2 \in \mathbb{R}^n$, and almost every $\boldsymbol{x} \in \Omega$ and $\boldsymbol{y} \in \mathbb{R}^m$.
(vi) $|\sigma(\boldsymbol{x},\boldsymbol{y},\xi)| \le c_2\left(1 + |\xi|^{p-1}\right)$, $c_2 > 0, \forall \xi \in \mathbb{R}^n$, almost every $\boldsymbol{x} \in \Omega$ and $\boldsymbol{y} \in \mathbb{R}^m$.

Standard estimates yield solutions that are uniformly bounded in $W_0^{1,p}(\Omega)$ with respect to η.

Remark 1.1. Assumptions (i) and (ii) implies that σ is a Carathéodory function. This is stated and proved in Proposition 1.4 Assumption (v) is needed for a corrector result. It can be replaced by a strict monotone assumption if only the homogenized equation is needed, *e.g.*, see Allaire (1992).

1.2. Two-scale cut-and-projection convergence

In this section, we recall some properties of two-scale convergence in $L^p(\Omega)$, $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$, Allaire (1992), and revisit the extension to the quasiperiodic setting, see Bouchitté et al. (2010) when $p = 2$. More precisely, we consider a real valued matrix $\mathbf{R}$ with m rows and n columns, and we would like to approximate an oscillating sequence $\{u_\eta(\boldsymbol{x})\}_{\eta\in]0,1[}$ by a sequence of two-scale functions $u_0\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)$ where $u_0(\boldsymbol{x},\cdot)$ is Y^m-periodic on $\mathbb{R}^m$, a higher dimensional space paved with periodic cells $Y^m =]0,1[^m$. In what follows, we assume that $\mathbf{R} : \mathbb{R}^n \to \mathbb{R}^m$, $n < m$, fulfills the criterion

$$\mathbf{R}^T \boldsymbol{k} \neq \mathbf{0}\,,\ \forall \boldsymbol{k} \in \mathbb{Z}^m \setminus \{\mathbf{0}\} \tag{4}$$

In fact, when defining a quasiperiodic structure through cut-and-projection, one notes that the matrix $\mathbf{R}$ is not uniquely defined (*e.g.*, an icosahedral phase using a mapping from $\mathbb{R}^6$ or $\mathbb{R}^{12}$ onto $\mathbb{R}^3$, or a Penrose tiling using a mapping from $\mathbb{R}^4$ or $\mathbb{R}^5$ onto $\mathbb{R}^2$, see Duneau and Katz (1985) and Janot (1992)). However, if g is a trigonometric polynomial, then the composite function $f = g\circ\mathbf{R}$ admits the following (uniquely defined) ergodic mean:

$$L(f) = \lim_{T\to+\infty} \frac{1}{(2T)^n} \int_{]-T;T[^n} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{Y^m} g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = [g] \tag{5}$$

where $[g]$ denotes the mean of g over the periodic cell Y^m in $\mathbb{R}^m$. This is the case provided that $\mathbf{R}$ fulfills the criterion (4), see Bouchitté et al. (2010).

This result suggests the following concept of two-scale convergence attached to a matrix $\mathbf{R}$.

Definition 1.1 (*Distributional Two-scale Convergence*)**.** We say that the sequence $\{u_\eta\}$ in $L^p(\Omega)$, $1 < p < \infty$, two-scale converges in the distributional sense towards the function $u_0 \in L^p(\Omega \times Y^m)$ for a matrix $\mathbf{R}$, if for every $\varphi \in \mathcal{D}(\Omega; C^\infty_\sharp(Y^m))$:

$$\lim_{\eta\to0} \int_\Omega u_\eta(\boldsymbol{x})\varphi\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right) \mathrm{d}\boldsymbol{x} = \iint_{\Omega\times Y^m} u_0(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \tag{6}$$

Definition 1.2 (*Weak Two-scale Convergence*)**.** We say that the sequence $\{u_\eta\}$ in $L^p(\Omega)$, two-scale converges weakly towards the function $u_0 \in L^p(\Omega \times Y^m)$ for a matrix $\mathbf{R}$, if for every $\varphi \in L^q(\Omega, C_\sharp(Y^m))$, $1 < p < \infty$, $1/p + 1/q = 1$, (6) holds.

We denote weak two-scale convergence for a matrix $\mathbf{R}$ with $u_\eta \overset{\mathbf{R}}{\rightharpoonup} u_0$. The following result, which is a straightforward extension of a proof in Bouchitté et al. (2010) to L^p case corresponding to Corollary 1.15 in Allaire (1992), ensures the existence of such two-scale limits when the sequence (u_η) is bounded in $L^p(\Omega)$ and $\mathbf{R}$ satisfies (4).

Proposition 1.1. *If $\mathbf{R}$ is a matrix satisfying (4) and $\{u_\eta\}$ is a bounded sequence in $L^p(\Omega)$, $1 < p < \infty$, then there exists a vanishing subsequence η_k and a limit $u_0(\boldsymbol{x},\boldsymbol{y}) \in L^p(\Omega \times Y^m)$ (Y^m-periodic in $\boldsymbol{y}$) such that $u_{\eta_k} \overset{\mathbf{R}}{\rightharpoonup} u_0$, as $\eta_k \to 0$.*

A proof of Proposition 1.1 uses the same arguments as in the periodic case, *e.g.*, see Lukkassen et al. (2002) and can be found in Ferreira et al. (2021).

We will need to pass to the limit in integrals $\int_\Omega u_\eta\, v_\eta\,\mathrm{d}\boldsymbol{x}$ where $u_\eta \overset{\mathbf{R}}{\rightharpoonup} u_0$ and $v_\eta \overset{\mathbf{R}}{\rightharpoonup} v_0$. For this, we introduce the notion of strong two-scale (cut-and-projection) convergence for a matrix $\mathbf{R}$.

Definition 1.3 (*Strong Two-scale Convergence*)**.** A sequence $\{u_\eta\}$ in $L^p(\Omega)$ is said to two-scale converge strongly, for a matrix $\mathbf{R}$, towards a limit u_0 in $L^p(\Omega\times Y^m)$, which we denote $u_\eta \overset{\mathbf{R}}{\to} u_0$, if and only if $u_\eta \overset{\mathbf{R}}{\rightharpoonup} u_0$ and

$$\|u_\eta(\boldsymbol{x})\|_{L^p(\Omega)} \to \|u_0(\boldsymbol{x},\boldsymbol{y})\|_{L^p(\Omega\times Y^m)} \tag{7}$$

This definition expresses that the effective oscillations of the sequence $\{u_\eta\}$ are on the order of η. Moreover, these oscillations are fully identified by u_0.

We note the following two propositions which are useful to establish a link between weak quasiperiodic convergence in Definition 1.2 and strong L^p convergence (see Corollary 1.2).

Proposition 1.2. *Let $\mathbf{R}$ be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfying (4). Let $f(\boldsymbol{x},\boldsymbol{y}) \in L^1(\Omega; C_\sharp(Y^m))$. Then $f\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)$ is a measurable function on Ω such that*

$$\left\| f\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^1(\Omega)} \leq \|f(\boldsymbol{x},\boldsymbol{y})\|_{L^1(\Omega;C_\sharp(Y^m))} := \int_\Omega \sup_{\boldsymbol{y}\in Y^m} |\, f(\boldsymbol{x},\boldsymbol{y})\,|\ d\boldsymbol{x} \tag{8}$$

and

$$\lim_{\eta\to0} \int_\Omega f\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right) \mathrm{d}\boldsymbol{x} = \iint_{\Omega\times Y^m} f(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \tag{9}$$

Proof. Measurability of $f\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)$ follows from Theorem 1 in Lukkassen et al. (2002) (see also Proposition 1.4) that ensures f is of Carathédory type. Inequality (8) is obvious. Finally, (9) follows from Lemma 2.4 in Bouchitté et al. (2010). □

We have the following corollary which is useful to establish a corrector type result (see Proposition 1.3).

Corollary 1.1. *Let $\mathbf{R}$ be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfying (4). Let $\phi(\boldsymbol{x},\boldsymbol{y}) \in L^p(\Omega; C_\sharp(Y^m))$. Then*

$$\lim_{\eta\to0} \int_\Omega \left|\phi\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right|^p \mathrm{d}\boldsymbol{x} = \iint_{\Omega\times Y^m} |\phi(\boldsymbol{x},\boldsymbol{y})|^p\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \tag{10}$$

Proof. We first consider the case when ϕ can be expressed as $\phi(\boldsymbol{x},\boldsymbol{y}) = \tau(\boldsymbol{x})\beta(\boldsymbol{y})$, where $\tau(\boldsymbol{x}) \in L^\infty(\Omega)$ and $\beta(\boldsymbol{y}) \in C_\sharp(Y^m)$. Since $\beta^p \in C_\sharp(Y^m)$, we deduce from Proposition 1.2 that $\beta^p\left(\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)$ converges towards its mean $[\beta^p]$ weakly in $L^1(\Omega)$. As τ^p belongs to $L^\infty(\Omega)$, we obtain

$$\lim_{\eta\to0} \int_\Omega \tau^p(\boldsymbol{x})\beta^p\left(\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right) \mathrm{d}\boldsymbol{x} = \int_\Omega \tau^p(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \int_{Y^m} \beta^p(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$$

From Fubini's theorem, this implies that (10) holds.

This result is extended by linearity to step functions $\phi_k \in S_t(\Omega, C_\sharp(Y^m))$ such that $\phi_k = \sum_{i=1}^k t_i\chi_{A_i}(\boldsymbol{x})\psi_i(\boldsymbol{y})$, where $A_i = \{\boldsymbol{x} \in \Omega,\ \phi_k(\boldsymbol{x},.) = t_i\}$ and $\psi_i(\boldsymbol{y}) \in C_\sharp(Y^m)$. We deduce that (10) holds by density in $L^p(\Omega, C_\sharp(Y^m))$. More precisely, we consider $\phi_k \in S_t(\Omega, C_\sharp(Y^m))$. There exists a sequence of step functions $\phi_k = \sum_{i=1}^k t_i\chi_{A_i}(\boldsymbol{x})\psi_i(\boldsymbol{y})$ such that

$$\lim_{k\to\infty} \int_\Omega \left(\sup_{\boldsymbol{y}\in Y^m} |\,\phi_k(\boldsymbol{x},\boldsymbol{y}) - \phi(\boldsymbol{x},\boldsymbol{y})\,|\right)^p \mathrm{d}\boldsymbol{x} = \lim_{k\to\infty} \|\phi_k - \phi\|^p_{L^p(\Omega,C_\sharp(Y^m))} = 0$$

Moreover, from the triangular inequality and the continuity of the linear map $\mathbf{R}$, we deduce that there exists a constant $C > 0$ such that

$$\begin{aligned}\left\|\phi\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^p(\Omega)} &\leq C\left\|\phi\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right) - \phi_k\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^p(\Omega)} \\ &\quad + \left\|\phi_k\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^p(\Omega)}\end{aligned} \tag{11}$$

Noting that for every $v \in L^p(\Omega, C_\sharp(Y^m))$

$$\int_\Omega \left|v\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right|^p \mathrm{d}\boldsymbol{x} \leq \int_\Omega \left(\sup_{\boldsymbol{y}\in Y^m} |v(\boldsymbol{x},\boldsymbol{y})|\right)^p \mathrm{d}\boldsymbol{x} = \|v\|^p_{L^p(\Omega,C_\sharp(Y^m))}$$

we deduce from (11) that for every integer k

$$\left\|\phi\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^p(\Omega)} \leq C\|\phi - \phi_k\|_{L^p(\Omega,C_\sharp(Y^m))} + \left\|\phi_k\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^p(\Omega)}$$

Since ϕ_k is admissible, we deduce that there exists a constant $C > 0$ such that

$$\begin{aligned}\limsup_{\eta\to 0}\left\|\phi\left(\boldsymbol{x},\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^p(\Omega)} &\leq C\|\phi-\phi_k\|_{L^p(\Omega,C_\sharp(Y^m))}+\|\phi_k\|_{L^p(\Omega\times Y^m)}\\ &\leq C\|\phi-\phi_k\|_{L^p(\Omega,C_\sharp(Y^m))}\\ &+\|\phi_k-\phi\|_{L^p(\Omega\times Y^m)}+\|\phi\|_{L^p(\Omega\times Y^m)}\\ &\leq (C+1)\|\phi-\phi_k\|_{L^p(\Omega,C_\sharp(Y^m))}\\ &+\|\phi\|_{L^p(\Omega\times Y^m)}\end{aligned} \tag{12}$$

Passing to the limit $k\to\infty$ in (12), we obtain

$$\limsup_{\eta\to 0}\left\|\phi\left(\boldsymbol{x},\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^p(\Omega)} \leq \|\phi\|_{L^p(\Omega\times Y^m)} \tag{13}$$

Similar arguments hold for the lower limit. In that case, we obtain that for every integer k

$$\begin{aligned}\liminf_{\eta\to 0}\left\|\phi\left(\boldsymbol{x},\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^p(\Omega)} &\geq \|\phi_k\|_{L^p(\Omega\times Y^m)}-C\|\phi-\phi_k\|_{L^p(\Omega,C_\sharp(Y^m))}\\ &\geq \|\phi\|_{L^p(\Omega\times Y^m)}-\|\phi_k-\phi\|_{L^p(\Omega\times Y^m)}\\ &-C\|\phi-\phi_k\|_{L^p(\Omega,C_\sharp(Y^m))}\\ &\geq \|\phi\|_{L^p(\Omega\times Y^m)}\\ &-(C+1)\|\phi-\phi_k\|_{L^p(\Omega,C_\sharp(Y^m))}\end{aligned} \tag{14}$$

Passing to the limit $k\to\infty$ in (14), we obtain

$$\liminf_{\eta\to 0}\left\|\phi\left(\boldsymbol{x},\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^p(\Omega)} \geq \|\phi\|_{L^p(\Omega\times Y^m)} \tag{15}$$

Equality (10) is established combining inequalities (13) and (15). □

The following proposition provides us with a corrector type result for the sequence $\{u_\eta\}$ when its limit u_0 is smooth enough:

Proposition 1.3. *Let* $\mathbf{R}$ *be a linear map from* $\mathbb{R}^n$ *in* $\mathbb{R}^m$ *satisfying (4). Let* $\{u_\eta\}$ *be a sequence such that* $u_\eta \overset{\mathbf{R}}{\rightharpoonup} u_0(\boldsymbol{x},\boldsymbol{y})$ *(weakly). Then*
(i) u_η *weakly converges in* $L^p(\Omega)$ *towards* $u(\boldsymbol{x})=\int_{Y^m}u_0(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$ *and*

$$\liminf_{\eta\to 0}\|u_\eta\|_{L^p(\Omega)} \geq \|u_0\|_{L^p(\Omega\times Y^m)} \geq \|u\|_{L^p(\Omega)} \tag{16}$$

(ii) Let $\{v_\eta\}$ *be another bounded sequence in* $L^q(\Omega)$, $1/p+1/q=1$, *such that* $v_\eta \overset{\mathbf{R}}{\to} v_0$ *(strongly), then*

$$u_\eta v_\eta \to w(\boldsymbol{x}) \text{ in } \mathcal{D}'(\Omega) \text{ where } w(\boldsymbol{x})=\int_{Y^m}u_0(\boldsymbol{x},\boldsymbol{y})v_0(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \tag{17}$$

(iii) If u_0 *is smooth enough (e.g.,* $u_0\in L^p(\Omega,C_\sharp(Y^m))$*) and*

$$\left\|u_0\left(\boldsymbol{x},\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^p(\Omega)} \to \|u_0(\boldsymbol{x},\boldsymbol{y})\|_{L^p(\Omega\times Y^m)} \tag{18}$$

then

$$\left\|u_\eta - u_0\left(\boldsymbol{x},\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^p(\Omega)} \to 0 \tag{19}$$

Proof. (i) Choosing test functions ϕ in $L^q(\Omega;C_\sharp(Y^m))$ independent of the y variable in Definition 1.2, one has that for every $\phi\in L^q(\Omega)$

$$\begin{aligned}\lim_{\eta\to 0}\int_\Omega u_\eta(\boldsymbol{x})\phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= \iint_{\Omega\times Y^m}u_0(\boldsymbol{x},\boldsymbol{y})\phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\ &= \int_\Omega \phi(\boldsymbol{x})\left(\int_{Y^m}u_0(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\end{aligned}$$

Moreover, $\{u_\eta\}$ is bounded in $L^p(\Omega)$ as a weakly convergent sequence.

Then, let φ_m be a sequence in $L^q(\Omega,C_\sharp(Y^m))$ such that φ_m converges to $|u_0|^{p-2}u_0$ strongly in $L^q(\Omega\times Y^m)$.

We first apply the Young inequality for real numbers a and b, and $1<p<\infty$, $1/p+1/q=1$, which states that $ab\leq |a|^p/p+|b|^q/q$. We consider $a=u_\eta$ and $b=\varphi_m$ to get

$$\int_\Omega |u_\eta(\boldsymbol{x})|^p\,\mathrm{d}\boldsymbol{x} \geq p\int_\Omega u_\eta(\boldsymbol{x})\varphi_m\left(\boldsymbol{x},\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\mathrm{d}\boldsymbol{x}-(p-1)\int_\Omega\left|\varphi_m\left(\boldsymbol{x},\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right|^q\mathrm{d}\boldsymbol{x}$$

We first pass to the limit when η goes to zero:

$$\liminf_{\eta\to 0}\|u_\eta\|^p_{L^p(\Omega)} \geq p\iint_{\Omega\times Y^m}u_0(\boldsymbol{x},\boldsymbol{y})\varphi_m(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}-(p-1)\iint_{\Omega\times Y^m}|\varphi_m(\boldsymbol{x},\boldsymbol{y})|^q\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$$

where we have used that $u_\eta \overset{\mathbf{R}}{\rightharpoonup} u_0(\boldsymbol{x},\boldsymbol{y})$ weakly and $\varphi_m\left(\boldsymbol{x},\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\overset{\mathbf{R}}{\to}\varphi_m(\boldsymbol{x},\boldsymbol{y})$ strongly (making use of Corollary 1.1).

We then pass to the limit when m goes to infinity:

$$\begin{aligned}\liminf_{\eta\to 0}\|u_\eta\|^p_{L^p(\Omega)} &\geq p\iint_{\Omega\times Y^m}|u_0(\boldsymbol{x},\boldsymbol{y})|^p\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}-(p-1)\iint_{\Omega\times Y^m}|u_0(\boldsymbol{x},\boldsymbol{y})|^p\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\ &= \|u_0\|^p_{L^p(\Omega\times Y^m)}\end{aligned} \tag{20}$$

where we have used that $\varphi_m\to|u_0|^{p-2}u_0$ strongly in $L^q(\Omega\times Y^m)$.

Moreover, thanks to Jensen's inequality, we have that

$$\begin{aligned}\|u\|^p_{L^p(\Omega)} &= \int_\Omega\left|\int_{Y^m}u_0(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right|^p\mathrm{d}\boldsymbol{x} \leq \iint_{\Omega\times Y^m}|u_0(\boldsymbol{x},\boldsymbol{y})|^p\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\ &= \|u_0\|^p_{L^p(\Omega\times Y^m)}\end{aligned} \tag{21}$$

We conclude by combining (20) and (21).

(ii) Let ψ_m be a sequence in $L^p(\Omega,C_\sharp(Y^m))$ such that ψ_m converges to u_0 strongly in $L^p(\Omega\times Y^m)$. Let τ be a function in $C_0^\infty(\Omega)$. We note that $v_\eta\tau\overset{\mathbf{R}}{\to}v_0\tau$ (strongly). Thus, passing to the two-scale cut-and-projection limit when η goes to zero in the product of $v_\eta\tau$ and ψ_m, we have that

$$\lim_{\eta\to 0}\int_\Omega\psi_m\left(\boldsymbol{x},\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)v_\eta(\boldsymbol{x})\tau(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \iint_{\Omega\times Y^m}\psi_m(\boldsymbol{x},\boldsymbol{y})\,v_0(\boldsymbol{x},\boldsymbol{y})\tau(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$$

We then pass to the limit when m goes to infinity

$$\begin{aligned}\lim_{m\to\infty}\lim_{\eta\to 0}\int_\Omega\psi_m\left(\boldsymbol{x},\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)v_\eta(\boldsymbol{x})\tau(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\ = \iint_{\Omega\times Y^m}u_0(\boldsymbol{x},\boldsymbol{y})\,v_0(\boldsymbol{x},\boldsymbol{y})\tau(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\end{aligned} \tag{22}$$

where we have used that ψ_m converges to u_0 strongly in $L^p(\Omega\times Y^m)$.

Moreover, from the triangular inequality, we have

$$\begin{aligned}&\left|\int_\Omega u_\eta(\boldsymbol{x})v_\eta(\boldsymbol{x})\tau(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}-\iint_{\Omega\times Y^m}u_0(\boldsymbol{x},\boldsymbol{y})\,v_0(\boldsymbol{x},\boldsymbol{y})\tau(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right|\\ &\leq\left|\int_\Omega\left[u_\eta(\boldsymbol{x})-\psi_m\left(\boldsymbol{x},\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right]v_\eta(\boldsymbol{x})\tau(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right|\\ &+\left|\int_\Omega\psi_m\left(\boldsymbol{x},\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)v_\eta(\boldsymbol{x})\tau(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right.\\ &\left.-\iint_{\Omega\times Y^m}u_0(\boldsymbol{x},\boldsymbol{y})\,v_0(\boldsymbol{x},\boldsymbol{y})\tau(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right|\end{aligned} \tag{23}$$

Combining (23) and (23) we get

$$\begin{aligned}&\limsup_{\eta\to 0}\left|\int_\Omega u_\eta(\boldsymbol{x})v_\eta(\boldsymbol{x})\tau(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}-\iint_{\Omega\times Y^m}u_0(\boldsymbol{x},\boldsymbol{y})\,v_0(\boldsymbol{x},\boldsymbol{y})\tau(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right|\\ &\leq\limsup_{m\to\infty}\limsup_{\eta\to 0}\left|\int_\Omega\left[u_\eta(\boldsymbol{x})-\psi_m\left(\boldsymbol{x},\frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right]v_\eta(\boldsymbol{x})\tau(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right|\end{aligned} \tag{24}$$

It remains to prove that the right-hand side in (24) vanishes. We first recall that $\tau \in C_0^\infty(\Omega)$ and we then invoke Hölder's inequality to deduce that

$$\begin{aligned}
&\left|\int_\Omega \left[u_\eta(\boldsymbol{x}) - \psi_m\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right] v_\eta(\boldsymbol{x})\tau(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right| \\
&\le \max_{\boldsymbol{x}\in\Omega} \mid \tau(\boldsymbol{x}) \mid \left|\int_\Omega \left[u_\eta(\boldsymbol{x}) - \psi_m\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right] v_\eta(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right| \\
&\le \max_{\boldsymbol{x}\in\Omega} \mid \tau(\boldsymbol{x}) \mid \left(\int_\Omega \left|u_\eta(\boldsymbol{x}) - \psi_m\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right|^p \mathrm{d}\boldsymbol{x}\right)^{1/p} \\
&\quad \times\left(\int_\Omega \mid v_\eta(\boldsymbol{x}) \mid^q \mathrm{d}\boldsymbol{x}\right)^{1/q} \\
&\le C(\Omega)\left\|u_\eta - \psi_m\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^p(\Omega)}
\end{aligned} \tag{25}$$

where in the last inequality we have used that v_η is a bounded sequence in $L^q(\Omega)$.

We now invoke the Clarkson inequalities applied to functions u_η and ψ_m in $L^p(\Omega)$:

$$\begin{aligned}
\frac{1}{2^p}\left\|u_\eta - \psi_m\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|^p_{L^p(\Omega)} &\le \frac{1}{2}\left(\|u_\eta\|^p_{L^p(\Omega)} + \left\|\psi_m\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|^p_{L^p(\Omega)}\right) \\
&\quad - \left\|\frac{u_\eta + \psi_m\left(\boldsymbol{x}, \frac{\mathbf{R}x}{\eta}\right)}{2}\right\|^p_{L^p(\Omega)} \quad \text{for } 1 < p < 2
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
\frac{1}{2^q}\left\|u_\eta - \psi_m\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|^q_{L^p(\Omega)} &\le \left(\frac{1}{2}\|u_\eta\|^p_{L^p(\Omega)}\right. \\
&\quad \left.+\frac{1}{2}\left\|\psi_m\left(\boldsymbol{x}, \frac{\mathbf{R}x}{\eta}\right)\right\|^p_{L^p(\Omega)}\right)^{\frac{q}{p}} \\
&\quad - \left\|\frac{u_\eta + \psi_m\left(\boldsymbol{x}, \frac{\mathbf{R}x}{\eta}\right)}{2}\right\|^q_{L^p(\Omega)} \quad \text{for } p \ge 2
\end{aligned} \tag{27}$$

Passing to the 2-scale cut-and-projection limit in (26)

$$\begin{aligned}
\limsup_{\eta\to 0}\left\|u_\eta - \psi_m\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|^p_{L^p(\Omega)} &\le 2^{p-1}\left(\|u_0(\boldsymbol{x},\boldsymbol{y})\|^p_{L^p(\Omega\times Y^m)}\right. \\
&\quad \left.+\|\psi_m(\boldsymbol{x},\boldsymbol{y})\|^p_{L^p(\Omega\times Y^m)}\right) \\
&\quad - 2^p\left\|\frac{u_0(\boldsymbol{x},\boldsymbol{y}) + \psi_m(\boldsymbol{x},\boldsymbol{y})}{2}\right\|^p_{L^p(\Omega\times Y^m)}
\end{aligned} \tag{28}$$

for $1 < p < 2$, and similarly for (27)

$$\begin{aligned}
\limsup_{\eta\to 0}\left\|u_\eta - \psi_m\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|^q_{L^p(\Omega)} &\le 2\left(\|u_0(\boldsymbol{x},\boldsymbol{y})\|^p_{L^p(\Omega\times Y^m)}\right. \\
&\quad \left.+\|\psi_m(\boldsymbol{x},\boldsymbol{y})\|^p_{L^p(\Omega\times Y^m)}\right)^{\frac{1}{p-1}} \\
&\quad - 2^q\left\|\frac{u_0(\boldsymbol{x},\boldsymbol{y}) + \psi_m(\boldsymbol{x},\boldsymbol{y})}{2}\right\|^q_{L^p(\Omega\times Y^m)}
\end{aligned} \tag{29}$$

for $p \ge 2$, where we have used $1/p + 1/q = 1$, thus $q/p = 1/(p-1)$.

We now note that $\left\|\frac{u_0(\boldsymbol{x},\boldsymbol{y})+\psi_m(\boldsymbol{x},\boldsymbol{y})}{2}\right\|_{L^p(\Omega\times Y^m)} \to \|u_0(\boldsymbol{x},\boldsymbol{y})\|_{L^p(\Omega\times Y^m)}$ when m tends to infinity. Thus, taking the lim sup on m in both sides of (28) and (29), we are ensured that

$$\limsup_{m\to\infty}\limsup_{\eta\to 0}\left\|u_\eta - \psi_m\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^p(\Omega)} = 0 \text{ for } 1 < p < \infty \tag{30}$$

We have thus proved that the RHS of (24) vanishes combining (25) and (30).

(iii) We finally want to show (19). Assuming that $u_0 \in L^p(\Omega, C_\sharp(Y^m))$, we can replace ψ_m by u_0 in the Clarkson inequalities (26) and (27), which leads to

$$\limsup_{m\to\infty}\limsup_{\eta\to 0}\left\|u_\eta - u_0\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right\|_{L^p(\Omega)} = 0 \text{ for } 1 < p < \infty$$

which is (19). □

Remark 1.2. Our proof of Proposition 1.3 follows closely that of Theorems 10 and 11 in Lukkassen et al. (2002). We stress that some regularity is needed for u_0 in (iii) of Proposition 1.3. We refer to Theorem 12 in Lukkassen et al. (2002) for a related result on regularity of two-scale limit that remains valid for two-scale cut-and-projection limit (its proof is a straightforward extension of that in Lukkassen et al. (2002), but is rather technical and lengthy).

Remark 1.3. We point out that Proposition 1.3(i) does not hold if weak two-scale convergence $u_{\eta_k} \overset{\mathbf{R}}{\rightharpoonup} u_0$ is replaced by distributional two-scale convergence (see Definition 1.1). Indeed, the choice of space $L^q(\Omega, C_\sharp(Y^m))$ for test functions φ in Definition 1.2 is essential. This is exemplified by the counter-example in Lukkassen et al. (2002) of a sequence $\{u_\eta\}$ in $L^p(0,1)$ defined by $u_\eta(x) = 1/\eta$ if $0 < x < \eta$ and $u_\eta(x) = 0$ if $\eta < x < 1$. One can see that (6) is satisfied for $\varphi \in \mathcal{D}((0,1); C_\sharp^\infty(Y^m))$ and a two-scale limit $u_0(x,y) = 0$. However, considering the test function $g(x) = 1$ which is in $L^q(0,1)$, we get $\lim_{\eta\to 0}\int_0^1 u_\eta(x)g(x)dx = 1$, and so $\{u_\eta\}$ does not converge to $u_0(x,y) = 0$ weakly in $L^p(0,1)$.

We have the following corollary regarding sequences that converge strongly in L^p-spaces

Corollary 1.2. *Let Ω be an open bounded set in $\mathbb{R}^n$ and $Y^m =]0,1[^m$ with $m > n$. Let a sequence $\{u_\eta\}$ converge strongly to u in $L^p(\Omega)$. Then there exist a vanishing subsequence η_k and a limit $u_0(\boldsymbol{x},\boldsymbol{y}) \in L^p(\Omega\times Y^m)$ (Y^m-periodic in $\boldsymbol{y}$) such that $u_{\eta_k} \overset{\mathbf{R}}{\rightharpoonup} u_0 = u$ as $\eta_k \to 0$.*

Proof. By assumption $\{u_\eta\}$ converges strongly. It follows that the sequence is uniformly bounded in $L^p(\Omega)$ and that there is a subsequence that two-scale converges for a matrix $\mathbf{R}$ to $u_0(\boldsymbol{x},\boldsymbol{y}) \in L^p(\Omega\times Y^m)$. The strong convergence implies equalities in (16), *i.e.*,

$$\liminf_{\eta\to 0}\|u_\eta\|_{L^p(\Omega)} = \|u_0\|_{L^p(\Omega\times Y^m)} = \|u\|_{L^p(\Omega)}$$

which gives $u_0(\boldsymbol{x},\boldsymbol{y}) = u(\boldsymbol{x})$, since the weak limit equals the strong limit and the weak limit is given by $u(\boldsymbol{x}) = \int_{Y^m} u_0(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$. □

Classes of functions such that $\left\|u_0\left(\boldsymbol{x}, \frac{\mathbf{R}x}{\eta}\right)\right\|_{L^p(\Omega)} \to \|u_0(\boldsymbol{x},\boldsymbol{y})\|_{L^p(\Omega\times Y^m)}$ are said to be admissible for the two-scale (cut-and-projection) convergence. In particular, classes of functions in $L^p(\Omega, C_\sharp(Y^m))$ (dense subset in $L^p(\Omega\times Y^m)$) are admissible. As mentioned in Allaire (1992) (Section 5), it is not always clear how smooth the test functions have to be to become admissible. One issue to consider is the measurability of the scaled test function $\phi(\boldsymbol{x}, \mathbf{R}\boldsymbol{x}/\eta)$. Continuity in at least one of the variables $\boldsymbol{x}$ or $\boldsymbol{y}$ is usually assumed, but this is not a necessary condition. Although the conductivity does not play the role of a test function we still need to be able to scale the local variable and get a function that is measurable. A sufficient assumption is to make $\sigma(\boldsymbol{x},\boldsymbol{y},\xi)$ continuous in both $\boldsymbol{x}$ and $\boldsymbol{y}$, but that would rule out in principle all realistic composites, *e.g.*, piecewise constant material properties. Assuming continuity with respect to the macroscopic variable, $\boldsymbol{x}$, is less restrictive when modeling realistic composites compared with assuming continuity with respect to the local variable $\boldsymbol{y}$. However, continuity with respect to $\boldsymbol{x}$ is not sufficient. To ensure measurability when scaling $\boldsymbol{y}$, *i.e.*, to make $\sigma(\boldsymbol{x}, \mathbf{R}\boldsymbol{x}/\eta, \xi)$ measurable, we shall assume continuity of $\sigma(\boldsymbol{x},\cdot,\xi)$ in the direction orthogonal to the hyperplane $Y^m_\parallel$, *i.e.*, on $Y^m_\perp$. We have the following proposition on required regularity for the conductivity.

Proposition 1.4. *Let σ satisfy assumptions (i) and (ii), then for any $\eta > 0$, $\sigma(\boldsymbol{x}, \mathbf{R}\boldsymbol{x}/\eta, \xi)$ is measurable on $\Omega \times \mathbb{R}^n$.*

Proof. The image of the mapping $\boldsymbol{x} \to \mathbf{R}\boldsymbol{x}$ is $Y_{\parallel}^m$. It follows that the scaling of the image in $\mathbb{R}^m$, $\mathbf{R}\boldsymbol{x}/\eta$, is a scaling of $Y_{\parallel}^m$, only. Hence, the conductivity function σ satisfies the Carathéodory assumptions (i) and (ii). It follows that $\sigma(\boldsymbol{x}, \mathbf{R}\boldsymbol{x}/\eta, \xi)$ is measurable for each ξ. □

In order to homogenize nonlinear PDEs with a monotone partial differential operator as in (3), we need to identify the differential relationship between χ and u_0, given a bounded sequence $\{u_\eta\}$ in $W^{1,p}(\Omega)$ (such that $u_\eta \overset{\mathbf{R}}{\rightharpoonup} u_0$ and $\nabla u_\eta \overset{\mathbf{R}}{\rightharpoonup} \chi$). This problem was solved by Allaire in the case of periodic functions (Allaire, 1992) and extended by Bouchitté et al. for quasiperiodic functions (Bouchitté et al., 2010) in $W^{1,2}(\Omega)$ and revisited in Wellander et al. (2018, 2019).

2. Function spaces for cut-and-projection partial differential operators

To carry out the homogenization analysis of nonlinear PDEs defined on quasiperiodic domains, we need to pass to the limit when η goes to zero in gradient and divergence operators acting on solutions of PDEs. To do this we introduce some suitable function spaces and for this we will define differential operators acting on the $\mathbb{R}^n$- plane in $\mathbb{R}^m$.

Defined as in Wellander et al. (2018) they are given as

$$\nabla_{\mathbf{R}}\, u(\boldsymbol{y}) = \mathrm{grad}_{\mathbf{R}}\, u(\boldsymbol{y}) = \mathbf{R}^T \mathrm{grad}_{\boldsymbol{y}}\, u(\boldsymbol{y}) = \mathbf{R}^T \nabla_{\boldsymbol{y}}\, u(\boldsymbol{y})$$

$$\mathrm{div}_{\mathbf{R}}\, \boldsymbol{u}(\boldsymbol{y}) = \mathbf{R}^T \nabla_{\boldsymbol{y}} \cdot \boldsymbol{u}(\boldsymbol{y})$$

We define the following functions spaces associated with the differential operators defined above

$$\mathcal{W}_{\sharp}^{p}(\mathrm{grad}_{\mathbf{R}}, Y^m) = \left\{ u \in L_{\sharp}^{p}(Y^m) \;\middle|\; \mathrm{grad}_{\mathbf{R}}\, u \in L_{\sharp}^{p}(Y^m; \mathbb{R}^n) \right\} \tag{31}$$

$$\mathcal{W}_{\sharp}^{p}(\mathrm{div}_{\mathbf{R}}, Y^m) = \left\{ \boldsymbol{u} \in L_{\sharp}^{p}(Y^m; \mathbb{R}^n) \;\middle|\; \mathrm{div}_{\mathbf{R}}\, \boldsymbol{u} \in L_{\sharp}^{p}(Y^m) \right\} \tag{32}$$

$$\mathcal{W}_{\sharp}^{p}(\mathrm{div}_{\mathbf{R}\mathbf{R}^T}, Y^m) = \left\{ \boldsymbol{u} \in L_{\sharp}^{p}(Y^m; \mathbb{R}^m) \;\middle|\; \left(\mathbf{R}\mathbf{R}^T \nabla_y\right) \cdot \boldsymbol{u} \in L_{\sharp}^{p}(Y^m) \right\} \tag{33}$$

and

$$\mathcal{W}^{p}(\mathrm{div}, \Omega) = \left\{ \boldsymbol{u} \in L^{p}(\Omega; \mathbb{R}^n) \;\middle|\; \mathrm{div}\, \boldsymbol{u} \in L^{p}(\Omega) \right\} \tag{34}$$

We have the following integration by parts type generalization to the L^p case of Lemma 6 given in the L^2 setting in Wellander et al. (2019)

Lemma 2.1 (*Green's Identity*). *It holds that*

$$-\int_{Y^m} \left(\mathbf{R}\mathbf{R}^T \nabla_y\right) \cdot \boldsymbol{\phi}(\boldsymbol{y})\, \theta(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} = \int_{Y^m} \boldsymbol{\phi}(\boldsymbol{y}) \cdot \left(\mathbf{R}\mathbf{R}^T \nabla_y\right) \theta(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \tag{35}$$

for $\boldsymbol{\phi} \in \mathcal{W}_{\sharp}^{q}(\mathrm{div}_{\mathbf{R}\mathbf{R}^T}, Y^m)$ and $\theta \in \mathcal{W}_{\sharp}^{p}(\mathrm{grad}_{\mathbf{R}}, Y^m)$, $1/p + 1/q = 1$.

Proof. The proof relies on standard matrix operations and the well known extension from $W^{1,2}$ Sobolev spaces to the setting of $W^{1,p}$ and $W^{1,q}$-duality pairing (Brezis, 2010). The periodic boundary conditions imply

$$-\int_{Y^m} \left[\left(\mathbf{R}\mathbf{R}^T \nabla_y\right) \cdot \boldsymbol{\phi}(\boldsymbol{y})\right] \theta(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} = -\int_{Y^m} \left[\left(\mathbf{R}^T \nabla_y\right) \cdot \mathbf{R}^T \boldsymbol{\phi}(\boldsymbol{y})\right] \theta(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} =$$
$$-\int_{Y^m} \left[\nabla_y \cdot \mathbf{R}\mathbf{R}^T \boldsymbol{\phi}(\boldsymbol{y})\right] \theta(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} = \int_{Y^m} \left[\mathbf{R}\mathbf{R}^T \boldsymbol{\phi}\right] \cdot \nabla_y\, \theta(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}$$
$$= \int_{Y^m} \boldsymbol{\phi}(\boldsymbol{y}) \cdot \mathbf{R}\mathbf{R}^T \nabla_y\, \theta(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}$$

for any pair of functions for $\boldsymbol{\phi} \in \mathcal{W}_{\sharp}^{q}(\mathrm{div}_{\mathbf{R}\mathbf{R}^T}, Y^m)$ and $\theta \in \mathcal{W}_{\sharp}^{p}(\mathrm{grad}_{\mathbf{R}}, Y^m)$, $1/p + 1/q = 1$. □

This motivates us to make the following decomposition as in the L^2-case in Wellander et al. (2019). We decompose $W_{\sharp}^{1,p}(Y^m)$ into two spaces,

$$W_{\sharp}^{1,p}(Y^m) = X_p \oplus X_p^{\perp} \tag{36}$$

where

$$X_p^{\perp} = \left\{ u \in W_{\sharp}^{1,p}(Y^m) | \mathbf{R}\mathbf{R}^T\ \nabla_y u = \mathbf{0} \right\} \tag{37}$$

and

$$X_p = \left\{ u \in W_{\sharp}^{1,p}(Y^m) | \left(\mathbf{I}_m - \mathbf{R}\mathbf{R}^T\right)\ \nabla_y u = \mathbf{0} \right\} \tag{38}$$

Remark 2.1. Note that the projection of a vector $\boldsymbol{v}$ in $\mathbb{R}^m$ on $\mathbb{R}^n$, $\mathbf{R}^T\ \boldsymbol{v} = \mathbf{0}$, where $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$, implies that $\boldsymbol{v}$ is orthogonal to the hyperplane $Y_{\parallel}^m$, *i.e.*, orthogonal to $\mathbb{R}^n$. It follows that $\mathbf{R}\mathbf{R}^T\ \boldsymbol{v} = \mathbf{0}$, where $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$. Vectors $\boldsymbol{w}$ in $\mathbb{R}^m$, orthogonal to $\boldsymbol{v}$, satisfy $\left(\mathbf{I}_m - \mathbf{R}\mathbf{R}^T\right)\ \boldsymbol{w} = \mathbf{0}$. We conclude that X_p contains all functions in $W_{\sharp}^{1,p}(Y^m)$ whose gradients have all their components in the plane, $\mathbb{R}^n$, which means that X_p can be identified with $\mathcal{W}_{\sharp}^{p}(\mathrm{grad}_{\mathbf{R}}, Y^m)$. Moreover, X_p is a subspace of a space much larger than $W_{\sharp}^{1,p}(Y^m)$ which is not differentiable in the direction orthogonal to the plane $Y_{\parallel}^m$, *i.e.*, on $Y_{\perp}^m$.

3. Compactness results

Proposition 3.1. *Let $\{u_\eta\}$ be a uniformly bounded sequence in $W^{1,p}(\Omega)$. Then there exist a subsequence $\{u_{\eta_k}\}$ and functions $u \in W^{1,p}(\Omega)$ and $u_1(\boldsymbol{x}, \boldsymbol{y}) \in L^p(\Omega, X_p)$ such that*

$$u_{\eta_k} \overset{\mathbf{R}}{\to} u(\boldsymbol{x}), \qquad \mathrm{grad}\, u_{\eta_k} \overset{\mathbf{R}}{\rightharpoonup} \mathrm{grad}\, u(\boldsymbol{x}) + \mathrm{grad}_{\mathbf{R}}\, u_1(\boldsymbol{x}, \boldsymbol{y}), \qquad \eta_k \to 0 \tag{39}$$

Remark 3.1. Note that $u_1(\boldsymbol{x}, \boldsymbol{y}) \in L^p(\Omega, X_p)$ implies that $\mathrm{grad}_{\mathbf{R}}\, u_1(\boldsymbol{x}, \boldsymbol{y}) \in L^p(\Omega, L_{\sharp}^p(Y^m; \mathbb{R}^n))$ and that we cannot say if u_1 belongs to $L^p(\Omega, W_{\sharp}^{1,p}(Y^m))$ or to some larger space with lower regularity mentioned in Remark 2.1. Indeed, we cannot say anything about the regularity of u_1 in the direction orthogonal to the hyperplane, $Y_{\parallel}^m$. However, in the decomposition in (36), the gradient of the potential in the "direction" of the hyperplane (or the lower dimensional plane) can be obtained from the gradient of the potential via a rotation of coordinate system. Further note that unlike in Wellander et al. (2018, 2019), in the proof below we use the notion of 2-scale cut-and-projection convergence in distributional sense (Definition 1.1), and not in weak sense (Definition 1.2).

Proof. The first assertion follows by the compact embedding of $L^p(\Omega)$ in $W^{1,p}(\Omega)$, Propositions 1.1, Definition 1.3 and Corollary 1.2. Note that $\mathbf{R}\nabla u_\eta(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega$ is uniformly bounded in $L^p(\Omega; \mathbb{R}^m)$. Let $\boldsymbol{\varphi} \in \mathcal{D}(\Omega; C_{\sharp}^{\infty}(Y^m))^m$. We have the following identities

$$\lim_{\eta\to 0} \int_{\Omega} \mathbf{R}\nabla u_\eta(\boldsymbol{x}) \cdot \boldsymbol{\varphi}\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right) \mathrm{d}\boldsymbol{x} = \lim_{\eta\to 0} \int_{\Omega} \nabla u_\eta(\boldsymbol{x}) \cdot \mathbf{R}^T \boldsymbol{\varphi}\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right) \mathrm{d}\boldsymbol{x} =$$
$$-\lim_{\eta\to 0} \int_{\Omega} u_\eta(\boldsymbol{x}) \left(\nabla_x \cdot \mathbf{R}^T \boldsymbol{\varphi}\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right) + \eta^{-1} \left(\mathbf{R}^T \nabla_y\right) \cdot \mathbf{R}^T \boldsymbol{\varphi}\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right) \mathrm{d}\boldsymbol{x} \tag{40}$$

and

$$\left(\mathbf{R}^T \nabla_y\right) \cdot \mathbf{R}^T \boldsymbol{\varphi}\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right) = \left(\mathbf{R}\mathbf{R}^T \nabla_y\right) \cdot \boldsymbol{\varphi}\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right) \tag{41}$$

Multiplying both sides in (40) with η and Lemma 2.1 gives the limit

$$\begin{aligned} 0 &= \iint_{\Omega\times Y^m} u_0(\boldsymbol{x}, \boldsymbol{y}) \left(\mathbf{R}^T \nabla_y\right) \cdot \mathbf{R}^T \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \\ &= \iint_{\Omega\times Y^m} u_0(\boldsymbol{x}, \boldsymbol{y}) \left(\mathbf{R}\mathbf{R}^T \nabla_y\right) \cdot \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \\ &= \iint_{\Omega\times Y^m} \mathbf{R}\mathbf{R}^T \nabla_y u_0(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \end{aligned} \tag{42}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(\Omega; C_\sharp^\infty(Y^m))^m$. The interpretation of (42) is that $u_0 \in X_p^\perp$, *i.e.*, the gradient $\nabla_y u_0(\boldsymbol{x}, \boldsymbol{y})$ has no component in the hyper plane in $\mathbb{R}^m$ defined by $\mathbf{R} : \mathbb{R}^n \to \mathbb{R}^m$. Indeed, we conclude that the potential $u_0(\boldsymbol{x}, \boldsymbol{y}) = u(\boldsymbol{x})$, is a function of $\boldsymbol{x}$ only due to the compact embedding of $L^p(\Omega)$ in $W^{1,p}(\Omega)$ and that the two-scale limit equals the strong limit, if it exists. Next, let $\boldsymbol{\varphi} \in \mathcal{D}(\Omega; C_\sharp^\infty(Y^m))^m$, and $\boldsymbol{\psi} \in \mathcal{D}(\Omega; C_\sharp^\infty(Y^m))^n$. We have three limits

$$\lim_{\eta\to 0} \int_\Omega \mathbf{R}\nabla u_\eta(\boldsymbol{x}) \cdot \boldsymbol{\varphi}\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right) \mathrm{d}\boldsymbol{x} = \iint_{\Omega\times Y^m} \hat{\chi}_0(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \tag{43}$$

$$\lim_{\eta\to 0} \int_\Omega \mathbf{R}\nabla u_\eta(\boldsymbol{x}) \cdot \boldsymbol{\varphi}\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right) \mathrm{d}\boldsymbol{x} = \lim_{\eta\to 0} \int_\Omega \nabla u_\eta(\boldsymbol{x}) \cdot \mathbf{R}^T \boldsymbol{\varphi}\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right) \mathrm{d}\boldsymbol{x}$$
$$= \iint_{\Omega\times Y^m} \chi_0(\boldsymbol{x}, \boldsymbol{y}) \cdot \mathbf{R}^T \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} = \iint_{\Omega\times Y^m} \mathbf{R}\chi_0(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$$

and

$$\begin{aligned}&\lim_{\eta\to 0} \int_\Omega \nabla u_\eta(\boldsymbol{x}) \cdot \boldsymbol{\psi}\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right) \mathrm{d}\boldsymbol{x} = \lim_{\eta\to 0} \int_\Omega \nabla u_\eta(\boldsymbol{x}) \cdot \boldsymbol{\psi}\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right) \mathrm{d}\boldsymbol{x}\\ &= \iint_{\Omega\times Y^m} \tilde{\chi}_0(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\end{aligned} \tag{44}$$

We find that $\hat{\chi}_0(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{R}\chi_0(\boldsymbol{x}, \boldsymbol{y})$ and $\tilde{\chi}_0(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{R}^T \hat{\chi}_0(\boldsymbol{x}, \boldsymbol{y}) = \chi_0(\boldsymbol{x}, \boldsymbol{y})$. Next, choosing test functions $\boldsymbol{\varphi} \in \mathcal{D}(\Omega; C_\sharp^\infty(Y^m))^m$, such that $\left(\mathbf{R}\mathbf{R}^T \nabla_y\right) \cdot \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y}) = 0$ in (40) with (41) gives

$$\begin{aligned}&\lim_{\eta\to 0} \int_\Omega \mathbf{R}\nabla u_\eta(\boldsymbol{x}) \cdot \boldsymbol{\varphi}\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right) \mathrm{d}\boldsymbol{x}\\ &= -\lim_{\eta\to 0} \int_\Omega u_\eta(\boldsymbol{x}) \left(\nabla_x \cdot \mathbf{R}^T \boldsymbol{\varphi}\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right)\right) \mathrm{d}\boldsymbol{x} =\\ &-\iint_{\Omega\times Y^m} u(\boldsymbol{x}) \left(\nabla_x \cdot \mathbf{R}^T \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y})\right) \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} = \iint_{\Omega\times Y^m} \mathbf{R}\nabla u(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\end{aligned} \tag{45}$$

Hence due to (43) and (45) we have for $\boldsymbol{\varphi} \in \mathcal{D}(\Omega; C_\sharp^\infty(Y^m))^m$, such that $\left(\mathbf{R}\mathbf{R}^T \nabla_y\right) \cdot \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y}) = 0$

$$\iint_{\Omega\times Y^m} \left(\hat{\chi}_0(\boldsymbol{x}, \boldsymbol{y}) - \mathbf{R}\nabla u(\boldsymbol{x})\right) \cdot \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} = 0$$

We deduce, due to orthogonality in the dual pairing sense (35), that there exists $u_1 \in L^p(\Omega; X_p)$ such that

$$\hat{\chi}_0(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{R}\nabla u(\boldsymbol{x}) + \mathbf{R}\mathbf{R}^T \nabla_y u_1(\boldsymbol{x}, \boldsymbol{y})$$

We conclude that the limit of the gradient in (44) becomes

$$\begin{aligned}\tilde{\chi}_0(\boldsymbol{x}, \boldsymbol{y}) &= \mathbf{R}^T \hat{\chi}_0(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{R}^T \left(\mathbf{R}\nabla u(\boldsymbol{x}) + \mathbf{R}\mathbf{R}^T \nabla_y u_1(\boldsymbol{x}, \boldsymbol{y})\right)\\ &= \nabla u(\boldsymbol{x}) + \mathbf{R}^T \nabla_y u_1(\boldsymbol{x}, \boldsymbol{y}) = \operatorname{grad} u(\boldsymbol{x}) + \operatorname{grad}_{\mathbf{R}} u_1(\boldsymbol{x}, \boldsymbol{y})\end{aligned}$$

which completes the proof. □

We define a strictly monotone operator a, which satisfies the following assumptions, (i)–(iv):

(i) $a(\cdot)$ is continuous on $\mathbb{R}^n$
(ii) $0 \le c_1 |\xi|^p \le (a(\xi), \xi)$, $c_1 > 0$, $\forall \xi \in \mathbb{R}^n$
(iii) $\left(a(\xi_1) - a(\xi_2), \xi_1 - \xi_2\right) > 0$ for all $\xi_1, \xi_2 \in \mathbb{R}^n$.
(iv) $|a(\xi)| \le c_2 \left(1 + |\xi|^{p-1}\right)$, $c_2 > 0$, $\forall \xi \in \mathbb{R}^n$

We will use the following Lemma when characterizing the two-scale limit of divergences.

Lemma 3.1. *Let $1 < p < \infty$, $1/p + 1/q = 1$ and assume the operator a satisfies assumptions (i)–(iv) above and that $f(\boldsymbol{x}, \cdot) \in L_\sharp^q(Y^m)$. The equation*

$$-\operatorname{div}_{\mathbf{R}} a\left(\operatorname{grad}_{\mathbf{R}} \theta(\boldsymbol{x}, \cdot)\right) = f(\boldsymbol{x}, \cdot)\,, \qquad a.e.\ \boldsymbol{x} \in \Omega \tag{46}$$

with periodic boundary conditions, has a unique weak solution $\operatorname{grad}_{\mathbf{R}} \theta(\boldsymbol{x}, \cdot)$ *in* $L_\sharp^p(Y^m; \mathbb{R}^n)$,

Proof. The proof follows from Browder (1963) and Minty (1963), *e.g.*, see Lukkassen et al. (2002), page 62. □

Proposition 3.2. *Let $\{\boldsymbol{u}_\eta\}$ be a uniformly bounded sequence in $\mathcal{W}(\operatorname{div}, \Omega)$. Then there exist a subsequence $\{\boldsymbol{u}_{\eta_k}\}$ and functions $\boldsymbol{u}_0 \in \mathcal{W}^p(\operatorname{div}, \Omega; L^p(Y^m))$ and $\boldsymbol{u}_1 \in L^p(\Omega, \mathcal{W}_\sharp^p(\operatorname{div}_{\mathbf{R}}, Y^m))$ such that*

$$\boldsymbol{u}_{\eta_k} \overset{\mathbf{R}}{\rightharpoonup} \boldsymbol{u}_0(\boldsymbol{x}, \boldsymbol{y}), \qquad \operatorname{div} \boldsymbol{u}_{\eta_k} \overset{\mathbf{R}}{\rightharpoonup} \operatorname{div} \boldsymbol{u}(\boldsymbol{x}) + \operatorname{div}_{\mathbf{R}} \boldsymbol{u}_1(\boldsymbol{x}, \boldsymbol{y}), \qquad \eta_k \to 0 \tag{47}$$

with

$$\operatorname{div}_{\mathbf{R}} \boldsymbol{u}_0(\boldsymbol{x}, \boldsymbol{y}) = 0 \tag{48}$$

and

$$\boldsymbol{u}(\boldsymbol{x}) = \int_{Y^m} \boldsymbol{u}_0(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{y}$$

$$\boldsymbol{u} \in \mathcal{W}^p(\operatorname{div}, \Omega).$$

Proof. The proof follows the lines of Lemma 5 and Proposition 6 in Wellander et al. (2019) with appropriately changed function spaces. Let $\phi \in L^q(\Omega)$ and $\psi \in L^q(\Omega, C_\sharp(Y^m))$. We have the weak limit of the divergence,

$$\begin{aligned}&\lim_{\eta\to 0} \int_\Omega \nabla \cdot \boldsymbol{u}_\eta(\boldsymbol{x})\, \phi(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \iint_{\Omega\times Y^m} \nabla \cdot \boldsymbol{u}_0(\boldsymbol{x}, \boldsymbol{y})\, \phi(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\ &= \int_\Omega \nabla \cdot \boldsymbol{u}(\boldsymbol{x})\, \phi(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\,, \ \forall \phi \in L^q(\Omega)\end{aligned}$$

where $\boldsymbol{u}(\boldsymbol{x}) = \int_{Y^m} \boldsymbol{u}_0(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{y}$, where $\boldsymbol{u}_0(\boldsymbol{x}, \boldsymbol{y})$ is the two-scale cut-and-project limit with respect to $\mathbf{R}$. Next, we have the corresponding two-scale cut-and-project limit of the divergence

$$\lim_{\eta\to 0} \int_\Omega \nabla \cdot \boldsymbol{u}_\eta(\boldsymbol{x})\, \psi\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right) \mathrm{d}\boldsymbol{x} = \iint_{\Omega\times Y^m} \chi_0(\boldsymbol{x}, \boldsymbol{y})\, \psi(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\,,$$
$$\forall \psi \in L^q(\Omega, C_\sharp(Y^m))$$

It follows, after an integration by parts (twice) that

$$\begin{aligned}0 &= \lim_{\eta\to 0} \eta \int_\Omega \nabla \cdot \boldsymbol{u}_\eta(\boldsymbol{x})\, \psi\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}\right) \mathrm{d}\boldsymbol{x}\\ &= \iint_{\Omega\times Y^m} \operatorname{div}_{\mathbf{R}} \left(\boldsymbol{u}_0(\boldsymbol{x}, \boldsymbol{y})\right) \psi(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\,, \ \forall \psi \in L^q(\Omega, C_\sharp(Y^m))\end{aligned}$$

which proves (48). Define a function as the difference of the two-scale and the weak limits, *i.e.*, $f(\boldsymbol{x}, \boldsymbol{y}) := \chi_0(\boldsymbol{x}, \boldsymbol{y}) - \nabla \cdot \boldsymbol{u}(\boldsymbol{x})$. We have $f(\boldsymbol{x}, \cdot) \in L_\sharp^p(Y^m)$. Lemma 3.1 implies that there is a unique $\operatorname{grad}_{\mathbf{R}} \theta(\boldsymbol{x}, \cdot)$ in $L_\sharp^q(Y^m; \mathbb{R}^n)$ that solves (46), *i.e.*, $\theta(\boldsymbol{x}, \cdot) \in X_q$, defined in (38). Next, define $\boldsymbol{u}_1(\boldsymbol{x}, \boldsymbol{y}) := -a\left(\operatorname{grad}_{\mathbf{R}} \theta(\boldsymbol{x}, \boldsymbol{y})\right)$ with a as in Lemma 3.1. We get

$$\chi_0(\boldsymbol{x}, \boldsymbol{y}) = \operatorname{div} \boldsymbol{u}(\boldsymbol{x}) + f(\boldsymbol{x}, \boldsymbol{y}) = \operatorname{div} \boldsymbol{u}(\boldsymbol{x}) + \operatorname{div}_{\mathbf{R}} \boldsymbol{u}_1(\boldsymbol{x}, \boldsymbol{y}) \ \in L^p(\Omega \times Y^m)$$

which completes the proof. □

4. Homogenization of a quasiperiodic heterogeneous nonlinear electrostatic problem

Let us now consider the quasiperiodic heterogeneous nonlinear electrostatic problem (3). Standard estimates yield solutions that are uniformly bounded in $W_0^{1,p}(\Omega)$ with respect to η. We can now state the main homogenization result.

Theorem 4.1. *Let $\{u_\eta\}$ be a sequence of solutions to (3). The whole sequence $\{u_\eta\}$ converges weakly in $W_0^{1,p}(\Omega)$ to the solution, u, of the homogenized equation*

$$\begin{cases} -\operatorname{div} \displaystyle\int_{Y^m} \sigma\left(\boldsymbol{x}, \boldsymbol{y}, \nabla u(\boldsymbol{x}) + \mathbf{R}^T \nabla_{\boldsymbol{y}} u_1(\boldsymbol{x}, \boldsymbol{y})\right) \mathrm{d}\boldsymbol{y} = f(\boldsymbol{x})\,, \qquad u \in W_0^{1,p}(\Omega),\\ u|_{\partial\Omega} = 0 \end{cases} \tag{49}$$

$1 < p < \infty$, *where* $\mathbf{R}^T\nabla_y u_1 \in L^p(\Omega; L^p(Y^m_{\parallel}; \mathbb{R}^n))$ *is the unique solution of the local equation*

$$-\operatorname{div}_{\mathbf{R}}\, \sigma\left(\boldsymbol{x}, \boldsymbol{y}, \nabla u(\boldsymbol{x}) + \mathbf{R}^T\nabla_{\boldsymbol{y}} u_1(\boldsymbol{x}, \boldsymbol{y})\right) = 0 \qquad a.e.\ \boldsymbol{y} \in Y^m, \boldsymbol{x} \in \Omega, \tag{50}$$

Proof. From the a priori estimates of sequences u_η and $\sigma_\eta := \sigma\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}, \nabla u_\eta(\boldsymbol{x})\right)$, there is a subsequence such that $u_{\eta_k} \overset{\mathbf{R}}{\to} u(\boldsymbol{x})$, $\operatorname{grad} u_{\eta_k} \overset{\mathbf{R}}{\rightharpoonup} \operatorname{grad} u(\boldsymbol{x}) + \operatorname{grad}_{\mathbf{R}}\, u_1(\boldsymbol{x}, \boldsymbol{y})$ and $\sigma_{\eta_k} \overset{\mathbf{R}}{\rightharpoonup} \sigma_0(\boldsymbol{x}, \boldsymbol{y})$, when $\eta_k \to 0$. Since $f + \operatorname{div}\sigma_\eta = 0$, Eq. (48) yields $\operatorname{div}_{\mathbf{R}}\sigma_0(\boldsymbol{x}, \boldsymbol{y}) = 0$ and $f(\boldsymbol{x}) + \operatorname{div}_{\boldsymbol{x}} \int_{Y^m} \sigma_0(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}\boldsymbol{y} = 0$. We now need to obtain an explicit expression for $\sigma_0(\boldsymbol{x}, \boldsymbol{y})$ in terms of σ, u and u_1. Following, *e.g.*, Allaire (1992), we introduce a test function $\psi_\eta(\boldsymbol{x}) = \nabla\{u(\boldsymbol{x}) + \eta\phi_1(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta})\} + t\phi(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta})$ where $t > 0$, ϕ and ϕ_1 are admissible test functions. This ensures that $\psi_\eta \overset{\mathbf{R}}{\rightharpoonup} \operatorname{grad} u(\boldsymbol{x}) + \operatorname{grad}_{\mathbf{R}}\, \phi_1(\boldsymbol{x}, \boldsymbol{y}) + t\phi(\boldsymbol{x}, \boldsymbol{y})$. Since σ is strictly monotone, we have

$$\int_\Omega \left\{\sigma_\eta - \sigma\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}, \psi_\eta\right)\right\} \cdot (\nabla u_\eta - \psi_\eta)\, \mathrm{d}\boldsymbol{x} \geq 0$$

i.e.,

$$\int_\Omega \left\{-\operatorname{div}\sigma_\eta\, u_\eta - \sigma\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}, \psi_\eta\right) \cdot \nabla u_\eta - \sigma_\eta\psi_\eta + \sigma\left(\boldsymbol{x}, \frac{\mathbf{R}\boldsymbol{x}}{\eta}, \psi_\eta\right)\psi_\eta\right\} \mathrm{d}\boldsymbol{x} \geq 0$$

Using (3), passing to the two-scale limit and using the strong limit to get u yields

$$\int_\Omega \int_{Y^m} \left\{f(\boldsymbol{x})u(\boldsymbol{x}) - \sigma\left(\boldsymbol{x}, \boldsymbol{y}, \psi_0(\boldsymbol{x}, \boldsymbol{y})\right) \cdot (\operatorname{grad} u(\boldsymbol{x}) + \operatorname{grad}_{\mathbf{R}}\, u_1(\boldsymbol{x}, \boldsymbol{y})) - \sigma_0(\boldsymbol{x}, \boldsymbol{y})\psi_0(\boldsymbol{x}, \boldsymbol{y}) + \sigma\left(\boldsymbol{x}, \boldsymbol{y}, \psi_0(\boldsymbol{x}, \boldsymbol{y})\right)\psi_0(\boldsymbol{x}, \boldsymbol{y})\right\} \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \geq 0$$

This equals, after a few integration by parts,

$$\begin{aligned}\int_\Omega \int_{Y^m} &\left\{f(\boldsymbol{x})u(\boldsymbol{x}) + \operatorname{div}_{\boldsymbol{x}}\left(\sigma\left(\boldsymbol{x}, \boldsymbol{y}, \psi_0(\boldsymbol{x}, \boldsymbol{y})\right)\right) u(\boldsymbol{x})\right.\\ &- \sigma\left(\boldsymbol{x}, \boldsymbol{y}, \psi_0(\boldsymbol{x}, \boldsymbol{y})\right) \cdot \operatorname{grad}_{\mathbf{R}}\, u_1(\boldsymbol{x}, \boldsymbol{y})\\ &+ \operatorname{div}_{\boldsymbol{x}}\left(\sigma_0(\boldsymbol{x}, \boldsymbol{y})\right) u(\boldsymbol{x}) + \operatorname{div}_{\mathbf{R}}\left(\sigma_0(\boldsymbol{x}, \boldsymbol{y})\right) \phi_1(\boldsymbol{x}, \boldsymbol{y}) - \sigma_0(\boldsymbol{x}, \boldsymbol{y})t\phi(\boldsymbol{x}, \boldsymbol{y})\\ &- \operatorname{div}_{\boldsymbol{x}}\left(\sigma\left(\boldsymbol{x}, \boldsymbol{y}, \psi_0(\boldsymbol{x}, \boldsymbol{y})\right)\right) u(\boldsymbol{x}) + \sigma\left(\boldsymbol{x}, \boldsymbol{y}, \psi_0(\boldsymbol{x}, \boldsymbol{y})\right) \operatorname{grad}_{\mathbf{R}}\, \phi_1(\boldsymbol{x}, \boldsymbol{y})\\ &\left.+ \sigma\left(\boldsymbol{x}, \boldsymbol{y}, \psi_0(\boldsymbol{x}, \boldsymbol{y})\right) t\phi(\boldsymbol{x}, \boldsymbol{y})\right\} \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \geq 0\end{aligned}$$

The first terms in the first and third rows cancel each other due to the statements above. We also note that the middle term in the third row vanishes due to Eq. (48). The second term in the first row is canceled by the first term in the fourth row. Taking a sequence of functions $\operatorname{grad}_{\mathbf{R}}\, \phi_1$ that converges strongly to $\operatorname{grad}_{\mathbf{R}}\, u_1$ in $L^p(\Omega, L^p_\sharp(Y^m; \mathbb{R}^n))$, yields

$$\begin{aligned}\int_\Omega \int_{Y^m} &\left\{-\sigma(\boldsymbol{x}, \boldsymbol{y}, \operatorname{grad} u(\boldsymbol{x}) + \operatorname{grad}_{\mathbf{R}}\, u_1(\boldsymbol{x}, \boldsymbol{y}) + t\phi(\boldsymbol{x}, \boldsymbol{y})) \cdot \operatorname{grad}_{\mathbf{R}}\, u_1(\boldsymbol{x}, \boldsymbol{y})\right.\\ &- \sigma_0(\boldsymbol{x}, \boldsymbol{y})t\phi(\boldsymbol{x}, \boldsymbol{y})\\ &+ \sigma(\boldsymbol{x}, \boldsymbol{y}, \operatorname{grad} u(\boldsymbol{x}) + \operatorname{grad}_{\mathbf{R}}\, u_1(\boldsymbol{x}, \boldsymbol{y}) + t\phi(\boldsymbol{x}, \boldsymbol{y})) \cdot \operatorname{grad}_{\mathbf{R}}\, u_1(\boldsymbol{x}, \boldsymbol{y})\\ &\left.+ \sigma(\boldsymbol{x}, \boldsymbol{y}, \operatorname{grad} u(\boldsymbol{x}) + \operatorname{grad}_{\mathbf{R}}\, u_1(\boldsymbol{x}, \boldsymbol{y}) + t\phi(\boldsymbol{x}, \boldsymbol{y}))t\phi(\boldsymbol{x}, \boldsymbol{y})\right\} \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \geq 0\end{aligned}$$

The first row cancels the third row. We divide the two terms left by $t > 0$ and send t to zero and obtain

$$\int_\Omega \int_{Y^m} \left[\sigma(\boldsymbol{x}, \boldsymbol{y}, \operatorname{grad} u(\boldsymbol{x}) + \operatorname{grad}_{\mathbf{R}}\, u_1(\boldsymbol{x}, \boldsymbol{y})) - \sigma_0(\boldsymbol{x}, \boldsymbol{y})\right] \phi(\boldsymbol{x}, \boldsymbol{y})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \geq 0$$

for all admissible test functions, *e.g.*, $\phi \in \mathcal{D}(\Omega; C^\infty_\sharp(Y^m))$. It follows that $\sigma_0(\boldsymbol{x}, \boldsymbol{y}) = \sigma(\boldsymbol{x}, \boldsymbol{y}, \operatorname{grad} u(\boldsymbol{x}) + \operatorname{grad}_{\mathbf{R}}\, u_1(\boldsymbol{x}, \boldsymbol{y}))$. Uniqueness of the solution of the limit equation (see e.g. Lions (1969) and Wellander (1998)) implies that the whole sequence converges. ☐

Proposition 4.1 (*Correctors*). *If we assume that* $u_1(\boldsymbol{x}, \boldsymbol{y})$ *is smooth and* σ *is uniformly monotone, then*

$$\lim_{\eta\to 0} \left\|\nabla u_\eta(\boldsymbol{x}) - \nabla\left\{u(\boldsymbol{x}) + \eta u_1\left(\boldsymbol{x}, \frac{R\boldsymbol{x}}{\eta}\right)\right\}\right\|_{L^p(\Omega;\mathbb{R}^n)} = 0$$

Proof. Considering $\psi_\eta(\boldsymbol{x}) = \nabla\left\{u(\boldsymbol{x}) + \eta u_1\left(\boldsymbol{x}, \frac{R\boldsymbol{x}}{\eta}\right)\right\}$ and using that σ is uniformly monotone yields (Allaire, 1992; Wellander, 2002)

$$\int_\Omega \left\{\sigma_\eta - \sigma\left(\boldsymbol{x}, \frac{R\boldsymbol{x}}{\eta}, \psi_\eta\right)\right\} \cdot (\nabla u_\eta - \psi_\eta)\, \mathrm{d}\boldsymbol{x} \geq c \int_\Omega |\nabla u_\eta - \psi_\eta|^p\, \mathrm{d}\boldsymbol{x} \tag{51}$$

where $c > 0$. It follows from the fact that ψ_η are admissible test functions that the left hand side of (51) goes to zero as $\eta \to 0$. The technical details in the proof are similar to the ones in (Allaire, 1992; Wellander, 2002). ☐

5. Concluding remarks

We have applied two-scale cut-and-projection convergence to a canonical nonlinear electrostatic problem for quasiperiodic structures generated by a periodic geometry in a higher dimensional space. Compared with earlier work on homogenization of almost periodic monotone operators (Braides, 1991; Nguetseng and Woukeng, 2007), our annex problem has a simpler, less abstract structure, and should therefore facilitate its numerical implementation in a variety of problems of physical interest, such as in electromagnetism (Wellander, 1998), where intriguing features have been observed, such as transmitted femtosecond pulses developed a trailing diffusive exponential tail that led to some controversy (Ledermann et al., 2009). We further note that our study can be adapted to the nonlinear elasticity case (Ponte Castaneda, 1989), whereby σ would denote a rank-2 stress tensor.

CRediT authorship contribution statement

Niklas Wellander: Concept, Analysis, Writing, Revision of the manuscript. **Sébastien Guenneau:** Concept, Analysis, Writing, Revision of the manuscript. **Elena Cherkaev:** Concept, Analysis, Writing, Revision of the manuscript.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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