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Skein theoretic approach to Yang-Baxter homology

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A B S T R A C T

We introduce skein theoretic techniques to compute the Yang-Baxter (YB) homology and cohomology groups of the R-matrix corresponding to the Jones polynomial. More specifically, we show that the YB operator R for Jones, normalized for homology, admits a skein decomposition $R = I + \beta\alpha$, where $\alpha : V^{\otimes 2} \to k$ is a "cup" pairing map and $\beta : k \to V^{\otimes 2}$ is a "cap" copairing map, and differentials in the chain complex associated to R can be decomposed into horizontal tensor concatenations of cups and caps. We apply our skein theoretic approach to determine the second and third YB homology groups, confirming a conjecture of Przytycki and Wang. Further, we compute the cohomology groups of R, and provide computations in higher dimensions that yield some annihilations of submodules.

1. Introduction

Yang-Baxter (YB) operators, i.e. solutions of the Yang-Baxter equation (YBE), have been first introduced and studied in Statistical Mechanics [5], due to their connection to scattering and integrable systems. They have also played a central role in low-dimensional topology, where they are used to construct link and 3-manifold quantum invariants [9,10], via representations of quantum groups and certain kinds of ribbon categories. Also, the study of set-theoretic YB operators has lead to introducing cocycle invariants of links from algebraic structures such as *quandles* [2].

Subsequently, homology theories for the Yang-Baxter operator have been developed and studied in relation to deformation theories [3], and with applications to knot invariants generalizing the notion of quandle cocycle invariants [1]. In particular, in [1] a (co)homology theory for set-theoretic Yang-Baxter equation was developed. More specifically, a Yang-Baxter set is a pair (X, R), where X is a set and $R : X \times X \to X \times X$ is an invertible map satisfying the equation

$$(R \times 1)(1 \times R)(R \times 1) = (1 \times R)(R \times 1)(1 \times R),$$

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with $1 : X \to X$ denoting the identity map. Given a Yang-Baxter set (X, R), in [1] a (co)chain complex associated to R has been introduced, whose 2-cocycles were used to produce invariants of classical and virtual knots. In [4,6], this homology theory was generalized to the YB operators (R-matrix) on tensor products of vector spaces (or modules): $R : V \otimes V \to V \otimes V$ satisfying

$$(R \otimes \mathbb{1})(\mathbb{1} \otimes R)(R \otimes \mathbb{1}) = (\mathbb{1} \otimes R)(R \otimes \mathbb{1})(\mathbb{1} \otimes R),$$

where $\mathbb{1} : V \to V$ is the identity map. They also provided an alternative diagrammatic of the chain maps that unifies the set-theoretic and tensor YBEs. In [7,11] it was shown that for set-theoretic case, the two homology theories are equivalent.

Also in [7], a family of YB operators corresponding to the Jones and HOMFLYPT polynomials was considered. The original matrices were normalized in order to define chain complexes. Computer based results and a conjecture related to the R-matrix corresponding to Jones polynomial were presented. In [8], the second homology group for the matrices corresponding to the HOMFLYPT polynomial has been computed.

The main purpose of this paper is to develop techniques to compute (co)homology groups of the YB operator corresponding to Jones polynomial. We do so by simplifying the differentials d_n defining its YB homology. More specifically (Theorem 4.1) we decompose d_n in terms of sums of simpler maps g_k, g'_k, h_k and h'_k (see Fig. 11 for a diagrammatic interpretation), by using the skein relation satisfied by the normalized matrix R. As an application we explicitly give the corresponding decompositions of the differentials d_n for $n = 2, 3, 4$ and compute the corresponding matrices and their Smith normal forms, using preliminary results on g_k, g'_k, h_k and h'_k. It might be of interest, as suggested by the referee, to investigate whether the approach of this paper is related to Khovanov homology, which is a categorification of the Jones polynomial.

This article is organized as follows. In Section 2, we recall the definitions of Yang-Baxter differentials and related homology, normalized Kauffman bracket R-matrix and a conjecture of Przytycki and Wang. In Section 3 we show that R satisfies the skein relation $R = \mathbb{1} + \beta\alpha$, where α is a pairing diagrammatically represented by a cup, and β is a copairing represented by a cap. We set up a diagrammatic formalism that will be used in the rest of the paper to simplify proofs and computations. In particular, we apply it to show that the normalized matrix R satisfies the YBE. Section 4 is the central part of the article. Here we show that the differentials corresponding to R, defining YB homology, can be decomposed as tensor products of certain generating maps, that are represented by horizontal concatenations of corresponding diagrams. We therefore proceed, in Section 5, to apply the skein theoretic decomposition of the differentials to compute the homology of R in low dimensions, confirming the case $n = 3$ of Przytycki-Wang conjecture. In Section 6 we dualize our methodology to compute low dimensional cohomology groups of R. Finally, in Section 7 we study the torsion of the homology groups in higher dimensions. Precisely, for $X = (V, R)$ where R is the normalized matrix in [7] on a rank 2 module V, we show that for every odd n, there exists a rank 2 submodule of $H_n(X)$ that is annihilated by multiplication by $y^4 - 1$. For every even n, we show that there exists a rank one submodule K_1 of $Z_n(X)$ that is in the boundary group $B_n(X)$, and a rank one submodule K_2 that is annihilated by multiplication by $y^2 - 1$. Some of the proofs are deferred to the appendices.

2. Preliminary

2.1. Yang-Baxter operators and their normalization

Let V be a k-module over a unital ring k. We say that M is a right (resp. left) V-module if there is a k-morphism (*action*) $\mu_\ell : M \otimes V \to M$ (resp. $\mu_r : V \otimes M \to M$), this unusual choice of conventions comes from the fact that the right action appears in the left differential of YB homology, while the left action appears in the right differential. Below when we focus on the right action, we drop the subscript and use

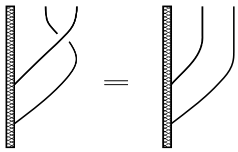

Fig. 1. The left wall condition.

$\mu = \mu_\ell$. In this paper we exclusively consider the ground ring $k = \mathbb{Q}[y, y^{-1}]$ and $M = k$ with *trivial* actions $\mu_\ell(a \otimes x) = a = \mu_r(x \otimes a)$ for all $a \in M$, $x \in V$.

An invertible morphism $R : V \otimes V \to V \otimes V$ is called *Yang-Baxter (YB) operator*, or an *R-matrix*, if it satisfies

$$(R \otimes \mathbb{1})(\mathbb{1} \otimes R)(R \otimes \mathbb{1}) = (\mathbb{1} \otimes R)(R \otimes \mathbb{1})(\mathbb{1} \otimes R).$$

An R-matrix R is said to satisfy the *left wall condition* if it satisfies

$$\mu_\ell(\mu_\ell \otimes \mathbb{1}_V)(\mathbb{1}_M \otimes R) = \mu_\ell(\mu_\ell \otimes \mathbb{1}_V),$$

and the *right wall condition* is defined similarly. An R-matrix satisfies the wall condition if it satisfies both left and right wall conditions. The left wall condition is depicted in Fig. 1. In the figure, the V-module M is represented by the shaded vertical line, and thin lines represent V. The crossing at the left figure represents the map R, and the map $\lambda_\ell : M \otimes V \to M$ is represented by merging two (shaded and thin) lines.

Let R be an R-matrix over $V \otimes V$. It is observed in [7] that for the trivial action $\mu_\ell(1 \otimes e) = 1 = \mu_r(e \otimes 1)$ for every basis vector e to satisfy the wall condition is that the matrix is *column unital*, i.e., the sum of entries of each column is 1. We call the procedure of making a matrix column unital the *normalization*.

2.2. Yang-Baxter differentials

Let the maps

$$d^\ell_{i,n}, d^r_{i,n} \in \mathrm{Hom}(M \otimes V^{\otimes n} \otimes M, M \otimes V^{\otimes (n-1)} \otimes M)$$

be defined by

$$\begin{aligned} d^\ell_{i,n} &= (\mu_\ell \otimes \mathbb{1}^n) \circ (R \otimes \mathbb{1}^{n-2}) \circ \cdots \\ &\qquad \cdots \circ (\mathbb{1}^{i-3} \otimes R \otimes \mathbb{1}^{n-i+1}) \circ (\mathbb{1}^{i-2} \otimes R \otimes \mathbb{1}^{n-i}) \\ d^r_{i,n} &= (\mathbb{1}^n \otimes \mu_r) \circ (\mathbb{1}^{n-2} \otimes R) \circ \cdots \\ &\qquad \cdots \circ (\mathbb{1}^{n-i+1} \otimes R \otimes \mathbb{1}^{i-3}) \circ (\mathbb{1}^{n-i} \otimes R \otimes \mathbb{1}^{i-2}). \end{aligned}$$

We also use the notations $d^s_n = \sum_i d^s_{i,n}$ for $s = l, r$. These maps are diagrammatically represented in Fig. 2.

The differentials of the Yang-Baxter homology are defined by

$$d_n = \sum_{i=1}^{n} (-1)^i [\ d^\ell_{i,n} - d^r_{i,n}\ ].$$

In [4,7], it was proved that $d^2 = 0$, so that d defines a chain complex called Yang-Baxter (YB) homology. The proof of this fact can be observed by diagrammatic means, and is illustrated in Fig. 3.

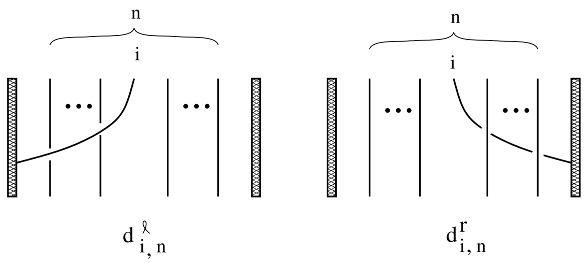

Fig. 2. Left and right curtain maps.

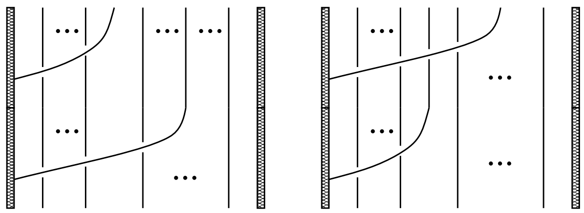

Fig. 3. $d^2 = 0$.

2.3. Kauffman bracket (Jones) R-matrix

For the R-matrix R' that produces the Jones polynomial, the normalization making R' column unital has been performed in [7], where it is shown that the normalized matrix takes the from

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-y^2 & 1 & 0 \\ 0 & y^2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the rest of the paper we focus on this specific R-matrix.

Let e_1, e_2 be the basis elements of the rank 2 free k-module V with respect to which the map R is the above matrix. Specifically, the rows and columns of R are for the basis elements $e_1 \otimes e_1$, $e_1 \otimes e_2$, $e_2 \otimes e_1$, $e_2 \otimes e_2$ in this order. For this specific R-matrix, a conjecture on YB homology groups is stated in [7] as follows, where $X = (V, R)$.

Conjecture 2.1. *$H_n(X) = k^2 \oplus (k/(1-y^2))^{a_n} \oplus (k/(1-y^4))^{s_{n-2}}$ where $s_n = \sum_{i=1}^{n+1} f_i$ is the partial sum of Fibonacci sequence with $f_1 = f_2 = 1$ and a_n is given by $a_1 = 0$ and $2^n = 2 + a_{n-1} + s_{n-3} + a_n + s_{n-2}$.*

We note that s_0 is defined to be 1, though this may not be explicit in [7].

3. Skein for the normalized Kauffman bracket R-matrix

In this section we establish a skein relation for the normalized R-matrix defined above, and define diagrammatic representations. It is not a priori the case that a normalized matrix R of a YB solution R' is a YB solution, but this fact is proved in [7]. We use the skein relation to provide a diagrammatic proof of this fact. We also provide lemmas on maps that appear in the skein that will be used in later sections.

Lemma 3.1. *We have $R = I + J$ where I denotes the identity matrix and $J = \beta\alpha$, where $\alpha : V \otimes V \to k$ and $\beta : k \to V \otimes V$ are defined by*

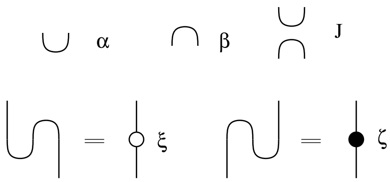

Fig. 4. Cup, cap and zig-zag maps.

$$\begin{aligned} \alpha(e_1 \otimes e_1) = \alpha(e_2 \otimes e_2) &= 0, \\ \alpha(e_1 \otimes e_2) &= -y, \\ \alpha(e_2 \otimes e_1) &= y^{-1}, \\ \beta(1) &= y(e_1 \otimes e_2 - e_2 \otimes e_1). \end{aligned}$$

Proof. Let $J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -y^2 & 1 & 0 \\ 0 & y^2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then we have $R = I + J$. Furthermore J is written as $J = (0, y, -y, 0)^T \cdot (0, -y, y^{-1}, 0)$ where ^{T} denotes the transpose. This means that J is the composition of a pairing $\alpha : V \otimes V \to k$ represented by $(0, -y, y^{-1}, 0)$ and the copairing $\beta : k \to V \otimes V$ represented by $(0, y, -y, 0)^T$, and the result follows. $\square$

Let $\xi, \zeta : V \to V$ be defined by $\xi(e_1) = y^2 e_1$, $\xi(e_2) = e_2$, $\zeta(e_1) = e_1$, $\zeta(e_2) = y^2 e_2$. Then we have the following remark and lemma by straightforward calculations.

Remark 3.2. The maps α, ξ and ζ can be written as follows:

$$\begin{aligned} \alpha(e_i \otimes e_j) &= (-1)^i (1 - \delta_{ij}) y^{j-i}, \\ \zeta(e_i) &= y^{2i-2} e_i, \\ \xi(e_i) &= y^{4-2i} e_i, \end{aligned}$$

where $i = 1, 2$ and δ_{ij} denotes the Kronecker's delta.

Lemma 3.3. *We have*

$$\begin{aligned} (\mathbb{1} \otimes \beta)(\alpha \otimes \mathbb{1}) &= \xi, \\ (\beta \otimes \mathbb{1})(\mathbb{1} \otimes \alpha) &= \zeta. \end{aligned}$$

(See Fig. 4.)

Straightforward computations also show the following lemma, with diagrams found in Fig. 5.

Lemma 3.4. *We have the following:*

(1) $\alpha\beta = -(y^2 + 1)$,
(2) $\xi\zeta = \zeta\xi = y^2 \mathbb{1}$,
(3) $\alpha(\lambda_\ell \otimes \mathbb{1}) = \mu\zeta$.

Theorem 3.5. *The normalized R-matrix R satisfies the YBE.*

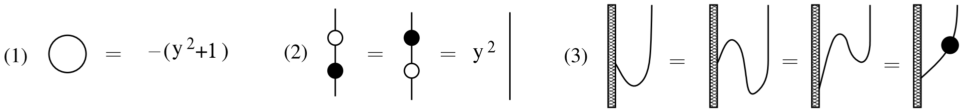

Fig. 5. A few identities.

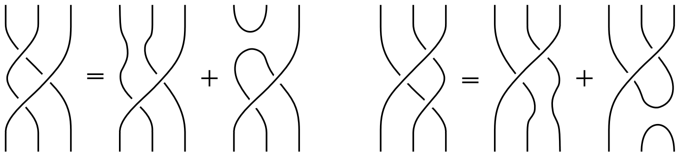

Fig. 6. The normalized R-matrix satisfies the YBE.

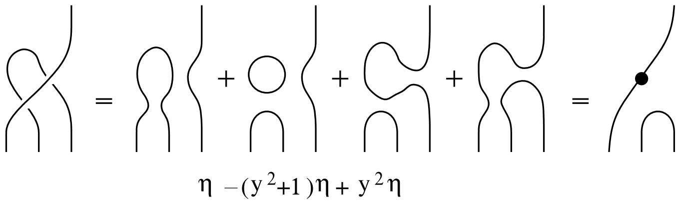

Fig. 7. The normalized R-matrix satisfies the YBE (continued, $\eta = \beta \otimes \mathbb{1}$).

Proof. Applying the skein relation to the Reidemeister move III as in Fig. 6 shows that it is enough to prove that the second summand on the RHS of the first equality is the same as the second summand on the RHS of the second equality. Let us denote these maps $V^{\otimes 3} \longrightarrow V^{\otimes 3}$ by Θ_1 and Θ_2 respectively. We can further simplify Θ_i as the products

$$\Theta_1 = \Phi_1(\alpha \otimes \mathbb{1}),$$
$$\Theta_2 = (\mathbb{1} \otimes \beta)\Phi_2,$$

where $\Phi_1 : V \longrightarrow V^{\otimes 3}$ is depicted in the left side of Fig. 7, and a similar definition is given for $\Phi_2 : V^{\otimes 3} \longrightarrow V$. Applying again the skein relation to the diagrammatic definitions of Φ_i, $i = 1, 2$ and the definitions of cup/cap and zig-zag maps as in Fig. 7 (where Lemma 3.4 (2) was used) we see that

$$\Phi_1 = \zeta \otimes \beta,$$
$$\Phi_2 = \alpha \otimes \zeta,$$

and we obtain

$$\Theta_1 = \Phi_1(\alpha \otimes \mathbb{1}) = (\zeta \otimes \beta)(\alpha \otimes \mathbb{1}) = (\mathbb{1} \otimes \beta)(\alpha \otimes \zeta) = (\mathbb{1} \otimes \beta)\Phi_2 = \Theta_2$$

which concludes the proof. □

Remark 3.6. The modified bracket skein relation $R = I + J$ and Theorem 3.5 imply that this R-matrix defines a braid group representation that factors through a *skew* Temperley-Lieb algebra STL_n defined as follows.

For a positive integer n, STL_n is a k-algebra ($k = \mathbb{Q}[y, y^{-1}]$) generated by h_i, $i = 1, \dots, n-1$ and relations

$$h_i h_i = -(y^2 + 1)h_i \quad i = 1, \dots, n-1,$$

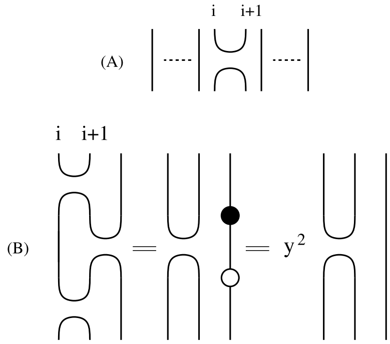

Fig. 8. Skew Temperley-Lieb algebra generator and relation.

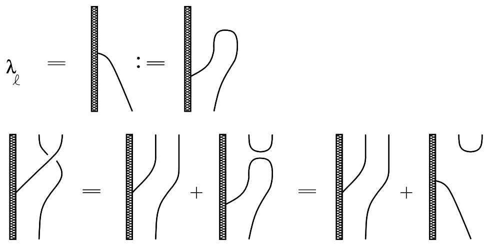

Fig. 9. Definition of λ_ℓ and how it appears in skein.

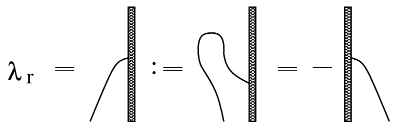

Fig. 10. Definition of λ_r which is the negative of λ_ℓ.

$$
\begin{aligned}
&h_i h_{i+1} h_i = y^2 h_{i+1} \quad i = 1, \dots, n-2, \\
&h_i h_{i-1} h_i = y^2 h_{i-1} \quad i = 2, \dots, n-1, \\
&h_i h_j = h_j h_i \quad |i-j| > 1,\ i, j+1, \dots n-1.
\end{aligned}
$$

Diagrammatic representation of h_i is to place a pair of cup and cap at the i^{th} and $(i+1)^{\text{st}}$ positions as for the Temperley-Lieb algebra (Fig. 8 (A)). The first relation follows from Lemma 3.4 (1). The second relation is depicted in Fig. 8 (B) which follows from Lemma 3.4 (2). These diagrammatic correspondence and computations imply that the assignment $h_i \mapsto \mathbb{1}^{i-1} \otimes (\beta\alpha) \otimes \mathbb{1}^{n-i-1}$ induces a homomorphism $\mathrm{STL}_n \to \mathrm{Aut}(V^{\otimes n})$ where V is a rank 2 k-module. The skein $R = I + J$ where $J = \beta\alpha$ implies that the braid group representation induced from R, $B_n \to \mathrm{Aut}(V^{\otimes n})$ is defined by $\sigma_i \mapsto \mathbb{1}^{i-1} \otimes R \otimes \mathbb{1}^{n-i-1}$.

The left and right curtain maps $\lambda_\ell, \lambda_r : k \to V$ are defined as in Figs. 9 and 10, respectively, and computed as

$$
\lambda_\ell(1) = y(e_2 - e_1) \quad \text{resp.} \quad \lambda_r(1) = y(e_1 - e_2).
$$

In particular, as depicted in Fig. 10, we obtain that $\lambda_r = -\lambda_\ell$.

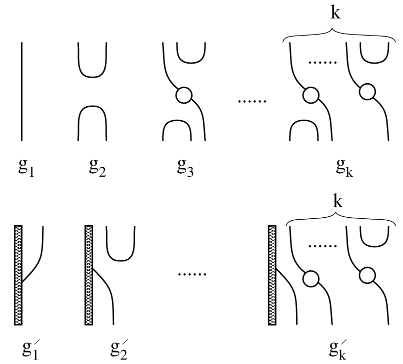

Fig. 11. Left generators.

4. Skein theoretic decomposition of Yang-Baxter differentials

In this section we compute a general decomposition of the differentials d_n, when the Yang-Baxter operator R satisfies the skein relation $R = I + J$.

We introduce the maps $\{g_i\}$, $\{g_j'\}$ corresponding to the diagrams depicted in Fig. 11. Similar definitions and considerations hold for $\{h_k\}$ and $\{h_t'\}$, right differential generators. Our main objective is to show that the Yang-Baxter differentials corresponding to $R = I + J$ can be written as described in Theorem 4.1.

Theorem 4.1. *The left differential is written as*

$$d_n^\ell = \sum_{S(n)} [\ g'_{i_0} g_{i_1}^{k(1)} \cdots g_{i_h}^{k(h)} g_1^{2k}\ ]$$

where

$$S(n) = \{(i_0, \ldots, i_h; k(1), \ldots, k(h)) \mid i_h \neq 1,\ i_0 + i_1^{k(1)} + \cdots + i_h^{k(h)} + 2k = n\}.$$

The n^{th} *Yang-Baxter differential decomposes as*

$$d_n = -\sum [\ g'_{i_0} g_{i_1}^{k(1)} \cdots g_{i_h}^{k(h)} g_1^{2k}\ ] + \sum [\ h_1^{2k} h_{i_k}^{k(h)} \cdots g_{i_1}^{k(1)} h'_{i_0}\ ]$$

if n *is odd, and*

$$d_n = \sum [\ g'_{i_0} g_{i_1}^{k(1)} \cdots g_{i_h}^{k(h)} g_1^{2k}\ ] + \sum [\ h_1^{2k} h_{i_k}^{k(h)} \cdots g_{i_1}^{k(1)} h'_{i_0}\ ]$$

if n *is even, where all sums are over* $S(n)$.

The proof is given in Appendix A. We note that the exponents of the identity maps $g_1(= h_1)$ are even. Therefore it is not the case that all possible horizontal concatenations of generating maps appear in the decompositions.

Specializing to the case in which R is the normalized R-matrix corresponding to Jones polynomial, as in Section 3, we compute the generators g_i and g_i'.

Lemma 4.2. *On basis vectors, the left generators* g_k *and* g_k'*, with* $k \geq 2$*, satisfy*

$$g_k(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \theta(i_1, \ldots, i_k)(e_1 \otimes e_2 - e_2 \otimes e_1) \bigotimes_{t=1}^{k-2} e_{i_t},$$

$$g'_k(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \theta(i_1, \ldots, i_k)(e_2 - e_1) \bigotimes_{t=1}^{k-2} e_{i_t},$$

where $\theta(i_1, \ldots, i_k) := (-1)^{i_{k-1}}(1 - \delta_{i_{k-1} i_k}) y^{i_k - i_{k-1}} y^{4(k-2) - 2\sum_{j=1}^{k-2} i_j} y.$

Proof. The lemma is a direct computation using the definition of g_k and g'_k, along with Remark 3.2. We leave this verification to the reader. □

Similarly, we compute the right generators h_l and h'_l with coefficients $\tau(i_1, \ldots, i_k)$.

Lemma 4.3. *On basis vectors, the right generators h_l and h'_l, with $l \geq 2$, satisfy*

$$h_l(e_{i_1} \otimes \cdots \otimes e_{i_l}) = \tau(i_1, \ldots, i_l) \bigotimes_{j=3}^{l} e_{i_j} \otimes (e_1 \otimes e_2 - e_2 \otimes e_1),$$

$$h'_l(e_{i_1} \otimes \cdots \otimes e_{i_l}) = \tau(i_1, \ldots, i_l) \bigotimes_{j=3}^{l} e_{i_j} \otimes (e_1 - e_2),$$

where $\tau(i_1, \ldots, i_l) := (-1)^{i_1}(1 - \delta_{i_1, i_2}) y^{i_2 - i_1} y^{2\sum_{t=3}^{l} i_t} y^{2(l-2)} y.$

5. Low-dimensional differentials and homology groups

In this section we utilize the skein theoretic procedure described in Section 3, that is similar to the Kauffman bracket, to simplify the differentials and compute Yang-Baxter homology groups in low dimensions, for the normalized Yang-Baxter matrix R. We apply diagrammatic arguments in addition to appealing directly to Theorem 4.1 in order to better illustrate the procedure. Let $X = (V, R)$ be as in Subsection 2.3.

5.1. The first differential

By definition the first differential is $d_1 : V(= k \otimes V = V \otimes k) \to k$, $d_1 = -(\mu_\ell - \mu_r) = 0$. Hence $H_0(X) = 0$ and $Z_1(X) = V$.

5.2. The second differential

Since left and right coactions have opposite signs, it follows that the second differential is identically null, as the following lemma shows.

Lemma 5.1. *We have $d_2 = 0$.*

Proof. Diagrammatic computations are depicted in Fig. 12 where the relation $\lambda_r = -\lambda_\ell$ depicted in Fig. 10 is used at the last step. Alternatively, using Theorem 4.1 and Lemma 4.2 we have that $d_2 = g'_2 + h'_2 = 0$. □

It follows that $H_1(X) = V$ from $Z_1(X) = V$ as noted in the preceding subsection.

5.3. The third differential

We now proceed to computing the third differential. Again, we provide a direct diagrammatic interpretation although the next lemma easily follows from Theorem 4.1.

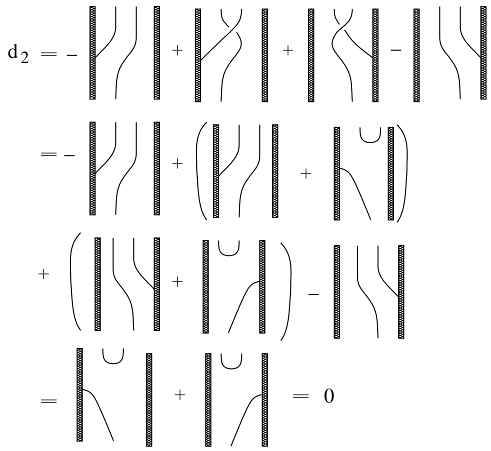

Fig. 12. Second differential.

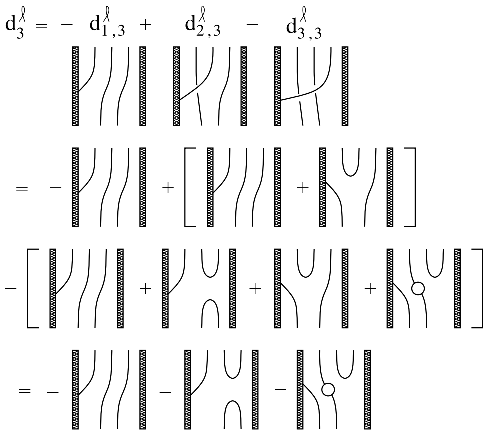

Fig. 13. Third left differential.

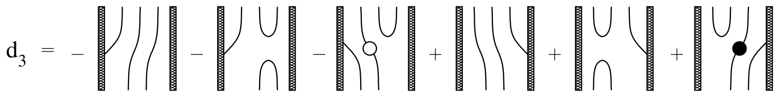

Fig. 14. Third differential.

Lemma 5.2. *The left and right third differentials are given in terms of generators by*

$$d_3^\ell = -g_1' g_1^2 - g_1' g_2 - g_3',$$
$$d_3^r = h_1^2 h_1' + h_2 h_1' + h_3'.$$

Proof. Diagrammatic computations in Fig. 13 give the left differential, where the left walls are abbreviated. The right differential is similar. Together we obtain d_3 as depicted in Fig. 14. $\square$

Lemma 5.3. *The third differential is given on basis vectors by*

$$d_3(e_1 \otimes e_1 \otimes e_2) = (1 - y^4) e_1 \otimes e_1 + (y^2 - 1) e_1 \otimes e_2 + y^2(y^2 - 1) e_2 \otimes e_1,$$
$$d_3(e_1 \otimes e_2 \otimes e_2) = (1 - y^2) e_1 \otimes e_2 + y^2(1 - y^2) e_2 \otimes e_1 + (y^4 - 1) e_2 \otimes e_2,$$

and $d_3(e_i \otimes e_j \otimes e_k) = 0$ *otherwise.*

Proof. On basis vectors $e_i \otimes e_j \otimes e_k$, with $i, j, k = 1, 2$, from Lemma 4.2 and Lemma 5.2 we have

$$\begin{aligned} d_3(e_i \otimes e_j \otimes e_k) = & -e_j \otimes e_k + (-1)^{j+1}(1-\delta_{jk})y^{k-j+1}(e_1 \otimes e_2 - e_2 \otimes e_1) \\ & +(-1)^{j+1}(1-\delta_{jk})y^{k-j-2i+5}(e_2 - e_1) \otimes e_i + e_i \otimes e_j \\ & +(-1)^{i}(1-\delta_{ij})y^{j-i+1}(e_1 \otimes e_2 - e_2 \otimes e_1) \\ & +(-1)^{i}(1-\delta_{ij})y^{j-i+2k-1}e_k \otimes (e_1 - e_2). \end{aligned}$$

When $i = j = k$, since $1 - \delta_{ij} = 1 - \delta_{jk} = 0$, we have

$$d_3(e_i \otimes e_j \otimes e_k) = -e_i \otimes e_i + e_i \otimes e_i = 0.$$

Let us consider the case $i = k \neq j$. Since $i+j = 3$, we have $y^{i-j+1} = y^{2i-2}$, $y^{j-i} = y^{3-2i}$ and $(-1)^i = (-1)^{j+1}$ so that

$$\begin{aligned} d_3(e_i \otimes e_j \otimes e_i) = & -e_j \otimes e_i + (-1)^i y^{2i-2}(e_1 \otimes e_2 - e_2 \otimes e_1) \\ & +(-1)^i y^2(e_2 \otimes e_i - e_1 \otimes e_i) + e_i \otimes e_j \\ & +(-1)^i y^{4-2i}(e_1 \otimes e_2 - e_2 \otimes e_1) \\ & +(-1)^i y^2(e_i \otimes e_1 - e_i \otimes e_2). \end{aligned}$$

Distinguishing the two cases $i = 1$ and $i = 2$ we easily see that either way $d_3(e_i \otimes e_j \otimes e_i) = 0$. Finally, we consider $i = j \neq k$ and $i \neq j = k$. In the first case, since $y^{5-3i+k} = y^{8-4i}$ and $y^{k-i+1} = y^{4-2i}$ we have

$$\begin{aligned} d_3(e_i \otimes e_i \otimes e_k) = & -e_i \otimes e_k + (-1)^{i+1} y^{4-2i}(e_1 \otimes e_2 - e_2 \otimes e_1) \\ & +(-1)^{i+1} y^{8-4i}(e_2 \otimes e_i - e_1 \otimes e_i) + e_i \otimes e_i, \end{aligned}$$

which is readily seen to be zero when $i = 2$ and equal to $(1-y^4)e_1 \otimes e_1 + (y^2-1)e_1 \otimes e_2 + y^2(y^2-1)e_2 \otimes e_1$ when $i = 1$. Similarly, $i \neq j = k$ gives zero when $k = 1$ and $(1-y^2)e_1 \otimes e_2 + y^2(1-y^2)e_2 \otimes e_1 + (y^4-1)e_2 \otimes e_2$ when $k = 2$. This concludes the proof of the lemma. $\square$

We now compute the second homology group.

Theorem 5.4. *We have $H_2(X) = k^2 \oplus k/(y^2-1) \oplus k/(y^4-1)$.*

Proof. From Lemma 5.3, in matrix form with columns for $e_1 \otimes e_1 \otimes e_2$ and $e_1 \otimes e_2 \otimes e_2$, and rows for $e_1 \otimes e_1$, $e_1 \otimes e_2$, $e_2 \otimes e_1$, $e_2 \otimes e_2$ in these orders, the matrix below.

$$\begin{bmatrix} 1-y^4 & 0 \\ y^2-1 & 1-y^2 \\ y^4-y^2 & y^2-y^4 \\ 0 & y^4-1 \end{bmatrix}$$

Then changes of bases are performed as follows.

$$\begin{bmatrix} 1-y^4 & 1-y^4 \\ y^2-1 & 0 \\ y^4-y^2 & 0 \\ 0 & y^4-1 \end{bmatrix} \begin{bmatrix} 1-y^4 & 0 \\ y^2-1 & 0 \\ 0 & 0 \\ 0 & y^4-1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ y^2-1 & 0 \\ 0 & 0 \\ 0 & y^4-1 \end{bmatrix}$$

The right-most matrix represents the group as stated. $\square$

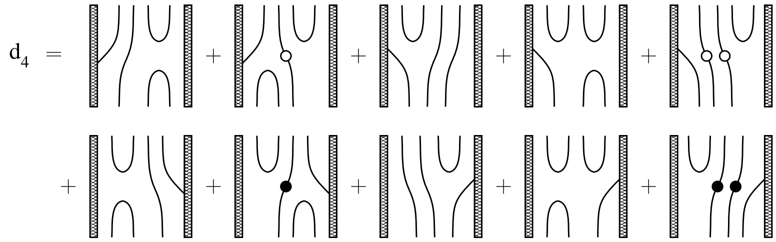

Fig. 15. The fourth differential.

5.4. The fourth differential

Next we compute the fourth differential. As before we have the following.

Lemma 5.5. *The left and right fourth differentials are given in terms of generators by*

$$d_4^\ell = g_1' g_1 g_2 + g_1' g_3 + g_2' g_1^2 + g_2' g_2 + g_3',$$
$$d_4^r = h_2 h_1 h_1' + h_3 h_1' + h_1^2 h_2' + h_2 h_2' + h_3'.$$

The diagrammatic representation of d_4 is found in Fig. 15.

Lemma 5.6. *The matrix form of d_4, with respect to the bases of $V^{\otimes 3}$ and $V^{\otimes 4}$ in lexicographic order with respect to the indices, is given by*

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & & & \\
y^6-y^2 & & y^2-y^4 & & y^4-y^6 & & & \\
1-y^4 & & y^2-1 & & y^4-y^2 & & & \\
 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & & \\
 & & y^4-y^2 & & y^2-y^4 & y^4-y^2 & y^2-y^4 & \\
 & & 1-y^2 & & y^2-y^4 & 0 & y^4-1 & \\
 & & & & & y^2-y^4 & y^4-y^6 & y^6-y^2 \\
 & & & & & 0 & 0 & \\
 & & & & y^4-1 & 1-y^2 & y^2-y^4 & \\
 & & & & & 0 & 0 & \\
 & & & & & y^2-1 & y^4-y^2 & 1-y^4 \\
 & & & & & 0 & & \\
 & & & & & 0 & & \\
 & & & & & 0 & & \\
 & & & & & 0 & &
\end{bmatrix}$$

For exposition the matrix is transposed, so that the eight columns correspond to $e_i \otimes e_j \otimes e_k$ and the rows correspond to $e_i \otimes e_j \otimes e_k \otimes e_\ell$. For example, the second row represents that

$$d_4(e_1 \otimes e_1 \otimes e_1 \otimes e_2)$$
$$= (y^6 - y^2) e_1 \otimes e_1 \otimes e_1 + (y^2 - y^4) e_1 \otimes e_2 \otimes e_1 + (y^4 - y^6) e_2 \otimes e_1 \otimes e_1.$$

Blank entries represent zeros, though some zeros are given to clarify the positions of entries.

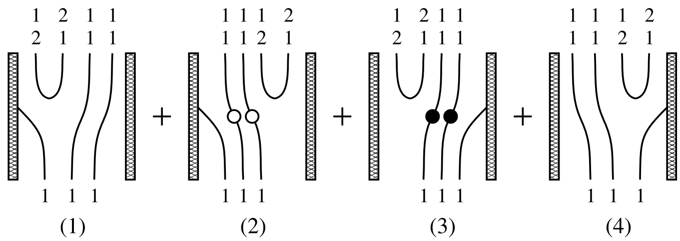

Fig. 16. Sample computations of the fourth differential.

Proof. Either by direct diagrammatic manipulation using the skein relation, or using Theorem 4.1, it follows that d_4 is represented diagrammatically as in Fig. 15. Using Lemma 4.2 we compute d_4 on basis vectors $e_i \otimes e_j \otimes e_k \otimes e_l$ with $i, j, k, l = 1, 2$. We have

$$\begin{aligned} d_4(e_i \otimes e_j \otimes e_k \otimes e_l) = {} & \theta(k,l)e_j \otimes (e_1 \otimes e_2 - e_2 \otimes e_1) + \theta(j,k,l)(e_1 \otimes e_2 - e_2 \otimes e_1) \otimes e_j \\ & + \theta(i,j)(e_2 - e_1) \otimes e_k \otimes e_l \\ & + \theta(i,j)\theta(k,l)(e_2 - e_1) \otimes (e_1 \otimes e_2 - e_2 \otimes e_1) \\ & + \theta(i,j,k,l)(e_2 - e_1) \otimes e_i \otimes e_j \\ & + \tau(i,j)(e_1 \otimes e_2 - e_2 \otimes e_1) \otimes e_k \\ & + \tau(i,j,k)e_k \otimes (e_1 \otimes e_2 - e_2 \otimes e_1) + \tau(k,l)e_i \otimes e_j \otimes (e_1 - e_2) \\ & + \tau(i,j)\tau(k,l)(e_1 \otimes e_2 - e_2 \otimes e_1) \otimes (e_1 - e_2) \\ & + \tau(i,j,k,l)e_k \otimes e_l \otimes (e_1 - e_2). \end{aligned}$$

The differential can be computed directly from this formula.

We illustrate alternative computations aided by diagrams. For computing the coefficient of $e_1 \otimes e_1 \otimes e_1$, we observe that the only contributions are given by four maps whose diagrams are depicted in Fig. 16. For example, the term (2) in Fig. 16 represents the map $\lambda_\ell \cdot \xi \cdot \xi \cdot \alpha$, and it is seen from the diagram that the only terms that give non-zero coefficients for $e_1 \otimes e_1 \otimes e_1$ are $e_1 \otimes e_1 \otimes e_1 \otimes e_2$ and $e_1 \otimes e_1 \otimes e_2 \otimes e_1$. The value for the former is computed as $(\lambda_\ell \cdot \xi \cdot \xi \cdot \alpha)(e_1 \otimes e_1 \otimes e_1 \otimes e_2) = (-y) \cdot y^2 \cdot y^2 \cdot (-y)$. All the other terms are computed similarly using diagrams. □

We can now compute the third homology group of the Yang-Baxter operator R.

Theorem 5.7. *Setting $k := \mathbb{Q}[y, y^{-1}]$ we have*

$$H_3(X) = k^{\oplus 2} \oplus k/(1 - y^2)^{\oplus 2} \oplus k/(1 - y^4)^{\oplus 2}.$$

Proof. Applying a sequence of elementary column and row operations to the matrix d_4 given in Lemma 5.6, we obtain the Smith normal form

$$\begin{bmatrix} 1 - y^2 & & & \\ & 1 - y^2 & & \\ & & 1 - y^4 & \\ & & & 1 - y^4 \end{bmatrix}$$

where, for simplicity, we omit the zero columns on the right of the nontrivial diagonal. Since by Lemma 5.3 the kernel of d_3 is six-dimensional, the assertion follows. □

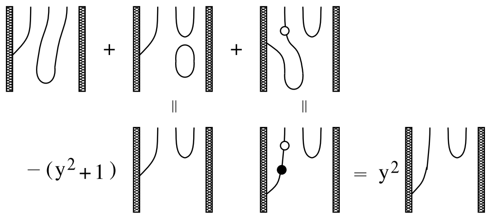

Fig. 17. The pairing α is a 2-cocycle.

6. Yang-Baxter cohomology

Let A be an abelian group. Then by dualizing the chain complex in Subsection 2.2, we obtain a cohomology theory, called Yang-Baxter cohomology, with coefficients in A, and differentials written as $\delta^{n+1} : C^n(X; A) \to C^{n+1}(X; A)$. We denote the cohomology groups by $H^n(X; A)$. We observe that the universal coefficient theorem determines cohomology groups as follows.

Proposition 6.1. *Let $k = \mathbb{Q}[y, y^{-1}]$. Then we have*

$$H^2(X; k) = k^{\oplus 2},$$
$$H^3(X; k) = k^{\oplus 2} \oplus k/(1 - y^2) \oplus k/(1 - y^4).$$

Proof. The universal coefficient theorem reads

$$0 \longrightarrow \mathrm{Ext}^1(\mathrm{H_n(X;A),B}) \longrightarrow \mathrm{H^n(X;B)} \longrightarrow \mathrm{Hom(H_{n-1}(X,A),B)} \longrightarrow 0.$$

We take $A = B = k$. Since $H_1(X; k) = k$, and by Theorem 5.4, we obtain $H^2_R(X; \Bbbk)$ as stated. We have $\mathrm{Ext}^1(k/fk, k) \cong k/fk$ for a Laurent polynomial $f(y)$ in k, hence Theorem 5.4 and Theorem 5.7 determine $H^3_R(X; \Bbbk)$ as stated. $\square$

Remark 6.2. A common argument to show that a n-dimensional cohomology group is nontrivial is to exhibit a non-trivial n-cocycle θ that evaluates an n-cycle x non-trivially, $\theta(x) \neq 0$. We present a diagrammatic method to do this for H^2, even though it is already proved, with a hope that a similar technique might prove productive in higher dimensions.

Specifically, we show that α is a non-trivial 2-cocycle. Fig. 17 shows that the left differential applied to α gives zero. A similar procedure is used for the right differential. By Theorem 5.4 the class represented by $e_1 \otimes e_2$ is non-trivial, and we observe that $\alpha(e_1 \otimes e_2) \neq 0$. Hence α is nontrivial.

Indeed to show that α is a 2-cocycle, one could also explicitly compute

$$\begin{cases} (1 - y^4)\alpha(e_1 \otimes e_1) + (y^2 - 1)\alpha(e_1 \otimes e_2) + y^2(y^2 - 1)\alpha(e_2 \otimes e_1) \\ \quad = 0 + (y^2 - 1)(-y) + y^2(y^2 - 1)y^{-1} = 0, \\ (1 - y^2)\alpha(e_1 \otimes e_2) + y^2(1 - y^2)\alpha(e_2 \otimes e_1) + (y^4 - 1)\alpha(e_2 \otimes e_2) \\ \quad = (1 - y^2)(-y) + y^2(1 - y^2)y^{-1} + 0 = 0 \end{cases}$$

as desired.

7. Further computations in higher dimensions

In this section we exhibit some diagrammatic computations in higher dimensions and observe annihilations by specific polynomials.

For $i = 1, \ldots, n$, let $e^n_{1,i} = e_1 \otimes \cdots \otimes e_1 \otimes e_2 \otimes e_1 \otimes \cdots \otimes e_1$ where there are n factors and e_2 is at the i^{th} position, and similarly $e^n_{2,i} = e_2 \otimes \cdots \otimes e_2 \otimes e_1 \otimes e_2 \otimes \cdots \otimes e_2$. By convention define $e^n_{j,0} = e_j \otimes \cdots \otimes e_j$ for $j = 1, 2$. When understood we suppress the superscript n.

Lemma 7.1. *For all positive integer n, we have $d_n(e^n_{j,0}) = 0$ for $j = 1, 2$.*

Proof. If any of the terms g_k, g'_k, h_k, h'_k, say g_k for some k, that appear in Theorem 4.1 contains α, then $g_k(e_{j,0}) = 0$, since $\alpha(e_j \otimes e_j) = 0$ for $j = 1, 2$. Hence if n is odd, the only non-zero terms in d_n when evaluated by $e_{j,0}$ are

$$(g'_1 g_1^{n-1})(e_{j,0}) = e_{j,0} = (h_1^{n-1} h'_1)(e_{j,0})$$

and they cancel with opposite signs in d_n. If n is even, then there is no term without α, hence the image vanishes. $\square$

Proposition 7.2. *For all positive integers $n > 3$ and $i = 1, \ldots, n$, we have the following. If n is odd, then the coefficient of $e^{n-1}_{1,0}$ and $e^{n-1}_{2,0}$, respectively, is non-zero for the following:*

$$\begin{array}{ll} d_n(e_{1,1}) = (y^2)e_{1,0}, & d_n(e_{1,2i-1}) = (1 - y^{4i-4})e_{1,0}, \\ d_n(e_{1,2i}) = (y^{4i-2} - y^2)e_{1,0}, & d_n(e_{1,n}) = (1 - y^{2(n-1)})e_{1,0}, \\ d_n(e_{2,1}) = (y^{2n-2} - 1)e_{2,0}, & d_n(e_{2,2i+1}) = (y^2 - y^{2n-4i-2})e_{2,0} \\ d_n(e_{2,2i}) = (y^{2n-4i} - 1)e_{2,0}, & \end{array}$$

and zero otherwise. If n is even, then the following terms have non-zero coefficients for $e^{n-1}_{1,0}$ and $e^{n-1}_{2,0}$, and zero otherwise:

$$\begin{array}{ll} d_n(e_{1,2i-1}) = (y^{4i-2} - y^2)e_{1,0}, & d_n(e_{1,2i}) = (1 - y^{4(i-1)})e_{1,0}, \\ d_n(e_{2,2i-1}) = (y^2 - y^{2(n-2i+3)})e_{2,0}, & d_n(e_{2,2i}) = (y^{2(n-2i+2)} - 1)e_{2,0}. \end{array}$$

Proof. Since the image of β has zero coefficients for $e_{j,0}$ for $j = 1, 2$, the only non-zero terms with $e_{j,0}$ in the image are the maps described below. For odd n, the maps are $g'_1 g_1^{n-1}$, $g'_{2i+1} g_1^{n-2i-1}$ for $i = 1, \ldots, (n-1)/2$ (g'_n is the case $i = (n-1)/2$), $h_1^{2i} h'_{n-2i}$ (h'_n is the case $i = 0$), $h_1^{n-1} h'_1$. For even n, the maps are $g'_{2i} g_1^{n-2i}$ for $i = 1, \ldots, n/2$ (g'_n is the case $i = n/2$), $h_1^{2i} h'_{n-2i}$ (h'_n is the case $i = 0$). These maps are represented by diagrams in Fig. 18 (1)–(4) in this order. Note that the requirement in Theorem 4.1 that the exponent of g_1 be even leads to the conditions on the parity.

The actual values can also be computed from diagrams, counting the contributions of ξ and ζ to the powers of y.

For even n and for $e_{1,0}$, and for $i = 1, \ldots, n/2$, we have

$$\begin{aligned} (g'_{2i} g_1^{n-2i})(e_{1,2i-1}) = (-y)(y^{-1})y^{2(2i-2)} e_{1,0} &= -y^{4i-4} e_{1,0} \\ (g'_{2i} g_1^{n-2i})(e_{1,2i}) = (-y)(-y)y^{2(2i-2)} e_{1,0} &= y^{4i-2} e_{1,0} \\ (h_1^{2i} h'_{n-2i})(e_{1,2i+1}) = y(y^{-1}) e_{1,0} &= 1 e_{1,0} \\ (h_1^{2i} h'_{n-2i})(e_{1,2i+2}) = y(-y) e_{1,0} &= -y^2 e_{1,0}. \end{aligned}$$

For example, the first tensor factor of $(g'_{2i} g_1^{n-2i})(e_{1,2i})$ comes from $\lambda_\ell(1) = y(e_2 - e_1)$, and we look at the coefficient of e_1, so that this map contributes $(-y)$. Then $g'_{2i}(e^{2i}_{1,2i})$ contributes $\alpha(e_1 \otimes e_2) = y^{-1}$ and $\xi(e_1)^{2i-2} = y^{2(2i-2)}$. Other terms are computed similarly with the aid of Fig. 18. From these we compute

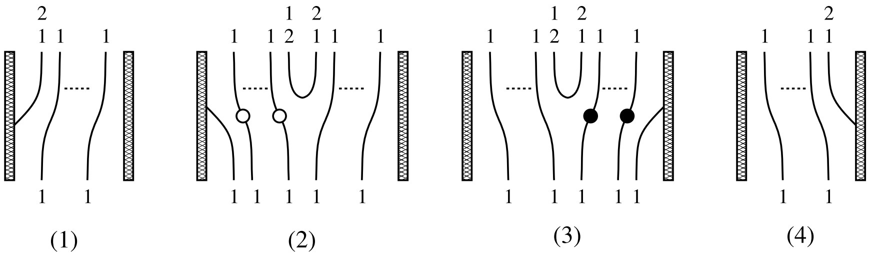

Fig. 18. Maps with image $e_1 \otimes \cdots \otimes e_1$.

$$d_n(e_{1,2i-1}) = (-g'_{2i} g_1^{n-2i} + h_1^{2i-2} h'_{n-2i+2})(e_{1,2i-1}) \quad = \quad (-y^{4i-4}+1)e_{1,0}.$$
$$d_n(e_{1,2i}) = (-g'_{2i} g_1^{n-2i} - h'_{2i-2} h'_{n-2i+2})(e_{1,2i}) \quad = \quad (y^{4i-2}-y^2)e_{1,0}.$$

Other cases are found in Appendix B. $\square$

Corollary 7.3. *For every odd n, there exists a rank 2 submodule of $H_n(X)$ that is annihilated by multiplication by y^4-1.*

For every even n, there exists a rank 1 submodule K_1 of $Z_n(X)$ that is in the boundary group $B_n(X)$, and a rank 1 submodule K_2 that is annihilated by multiplication by y^2-1.

Proof. Let n be odd. Let K be the rank 2 submodule of $C_n(X)$ generated by $e_j \otimes \cdots \otimes e_j$, $j=1,2$. By Lemma 7.1, K is in $Z_n(X)$. Since $n+1$ is even, Proposition 7.2 implies that $\mathrm{Im}(d_{n+1})$ in the submodule generated by $e_{1,0}$ in $Z_n(X)$ is spanned by $\mathrm{GCD}\{(y^{4(i-1)}-1) : i=2,\ldots,(n+1)/2\}e_{1,0}$, and $\mathrm{GCD}\{(y^{4(i-1)}-1) : i=2,\ldots,(n+1)/2\} = y^4-1$. Similarly, $\mathrm{Im}(d_{n+1})$ in the submodule generated by $e_{2,0}$ in $Z_n(X)$ is spanned by $\mathrm{GCD}\{(y^{2(n-2i-2)}-1) : i=2,\ldots,(n+1)/2\}e_{2,0}$, and $\mathrm{GCD}\{(y^{2(n-2i-2)}-1) : i=2,\ldots,(n+1)/2\} = y^4-1$. Hence the rank 2 submodule of $Z_n(X)$ generated by $e_{j,0}$ for $j=1,2$ is annihilated by (y^4-1).

Let n be even. Then $n+1$ is odd. Let K_j be the rank 1 submodule of $Z_n(X)$ generated by $e_{j,0}$ for $j=1,2$, respectively. Since $d_{n+1}(e_{1,1}) = (y^2)e_{1,0}$ from Proposition 7.2 and y^2 is a unit, K_1 is in $\mathrm{Im}(d_{n+1})$. The submodule K_2 is annihilated by the GCD of $y^{2(n+1)-4i}-1$ for $i=1,\ldots,n/2$, which is y^2-1. Thus the statement follows. $\square$

The statement of the preceding corollary supports Przytycki-Wang's conjecture.

Acknowledgement

We are grateful to Jozef Przytycki and Xiao Wang for valuable conversations. Mohamed Elhamdadi was partially supported by Simons Foundation collaboration grant 712462. Masahico Saito was supported in part by NSF DMS 1800443. We thank the referee for raising the question on whether the approach pursued in this article is related to Khovanov homology, as mentioned in the Introduction. Although this is an intriguing question, we are not able to propose a potential answer at this time.

Appendix A. Proof of Theorem 4.1

In this section we use μ and λ instead of μ_ℓ and λ_ℓ, respectively, for brevity. We need a few preliminary maps and results before proving the main theorem. First we introduce a class of operators $V^{\otimes n} \longrightarrow V^{\otimes n+1}$ whose diagrammatic interpretation is similar to the curtain differentials. Namely we set

$$\Psi_n = (\mu \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}) \circ (R \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}) \circ \cdots \circ (\mathbb{1} \otimes \cdots \mathbb{1} \otimes R \otimes \mathbb{1}) \circ (\mathbb{1} \otimes \cdots \mathbb{1} \otimes \beta),$$

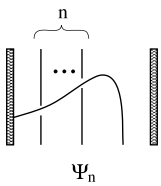

Fig. 19. Diagram representing Ψ_n.

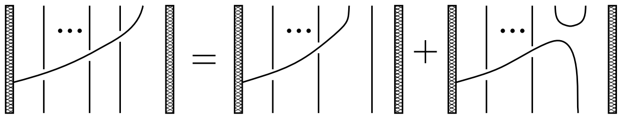

Fig. 20. Skein using Ψ_n.

where μ indicates the action $\mu : k \otimes V \longrightarrow k$. We set $\Psi_0 = \lambda$, where λ is the coaction $k \longrightarrow k \otimes V$. See Fig. 19 for a diagram representing Ψ_n. We similarly define Ψ'_n, by symmetry, where we replace overpassing crossings with underpassing and the left action with the right action.

In the notation below, the dot $\cdot$ represents the horizontal concatenation of diagrams, that represents tensor product of maps. For example, if $f : V \to V$ is a map on V and $\mathbb{1} : V \to V$ denotes the identity map, then $f \cdot \mathbb{1}$ denotes $f \otimes \mathbb{1}$ on $V \otimes V$, and $\mathbb{1}^k$ denotes the identity map on $V^{\otimes k}$. The dot may be abbreviated.

Remark A.1. Lemmas A.2, A.4, and A.5 below are easily adapted, by symmetry, to the case of the right differentials upon exchanging Ψ_n, ξ, μ and λ with Ψ'_n, ζ, μ_r and λ_r, respectively.

Lemma A.2. *The left differentials $d^\ell_{n,n}$ can be decomposed in terms of Ψ_k for all $n \in \mathbb{N}$ as follows:*

$$d^\ell_{n,n} = \sum_{m=2}^{n} \Psi_{n-m} \cdot \alpha \cdot \mathbb{1}^{m-2} + \mu \cdot \mathbb{1}^{n-1}.$$

Proof. The proof is by induction on n, the number of strings in the curtain representing $d^\ell_{n,n}$, i.e. the number of copies of V in the domain of $d^\ell_{n,n}$. The base of the induction is easily verified by direct inspection. For $n = 1$ the statement is in fact vacuously true, while for $n = 2$ it is a consequence of the skein relation. Suppose the equation holds true for all $3 \leq n \leq k$ and set $n = k + 1$. Making use of the skein relation we can write

$$d^\ell_{k+1,k+1} = \Psi_{k-1} \cdot \alpha + d^\ell_{k,k} \cdot \mathbb{1}.$$

See Fig. 20 for diagrams. Applying the inductive hypothesis to $d^\ell_{k,k}$ we obtain

$$d^\ell_{k+1,k+1} = \Psi_{k-1} \cdot \alpha + \Psi_{k-2} \cdot \alpha \cdot \mathbb{1} + \cdots + \Psi_0 \cdot \alpha \mathbb{1}^{k-2} \cdot \mathbb{1} + \mu \cdot \mathbb{1}^{k-1} \cdot \mathbb{1},$$

which concludes the proof. □

We now define the sets $\Gamma^{(n,n+2)}$ and $\Lambda^{(m,m)}$ of diagrams representing maps that will be used to decompose the operators Ψ_n. In general the double superscripts $M^{(m,n)}$ indicate that the set includes maps $V^{\otimes m} \to V^{\otimes n}$. We set (see Fig. 21):

$$\Gamma^{(0,2)} = \{\beta\}, \quad \Gamma^{(1,3)} = \{\mathbb{1}\beta, \beta\xi\}, \quad \Lambda^{(1,1)} = \{\xi\}, \quad \Lambda^{(2,2)} = \{\beta\alpha, \xi^2\}$$

and inductively define

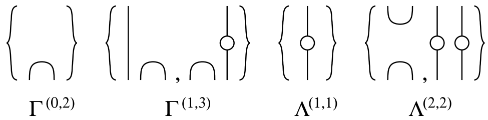

Fig. 21. Γs and Λs.

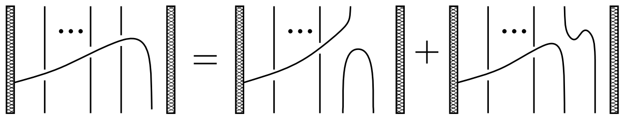

Fig. 22. Skein for Ψ_n.

$$\Gamma^{(n,n+2)} = \Gamma^{(n-1,n+1)} \cdot \xi \cup \bigcup_{m=2}^{n-2} \Gamma^{(n-m,n-m+2)} \cdot \alpha \mathbb{1}^{m-2}\beta \cup \{\mathbb{1}^n \beta\}, \tag{1}$$

$$\Lambda^{(n,n)} = \Lambda^{(n-1,n-1)} \cdot \xi \cup \bigcup_{m=2}^{n-2} \Lambda^{(n-m,n-m)} \cdot \alpha \mathbb{1}^{m-2}\beta \cup \{\alpha \mathbb{1}^{n-2} \beta\}. \tag{2}$$

Remark A.3. It can be seen that the unions defining $\Gamma^{(n,n+2)}$ and $\Lambda^{(n,n)}$ are in fact disjoint.

Lemma A.4. *For all $n \in \mathbb{N}$ the following equation holds:*

$$\Psi_n = \sum_{\psi \in \Gamma^{(n-1,n+1)}} \mu \cdot \psi + \sum_{\phi \in \Lambda^{(n,n)}} \lambda \cdot \phi.$$

Proof. Recall that we use abbreviation $\mu_\ell = \mu$ and $\lambda_\ell = \lambda$. The proof utilizes induction and Lemma A.2. A direct inspection shows that the equation holds for $n = 1, 2$. Indeed by using the skein relation we have

$$\Psi_1 = \mu\beta + \lambda\xi,$$
$$\Psi_2 = \mu\mathbb{1}\beta + \mu\beta\xi + \lambda\alpha\beta + \lambda\xi\xi,$$

and it follows that the statement holds true for $n = 1, 2$. Let us now assume that Ψ_n is of the form given in the statement for all $2 \le n \le k$ and let $n = k + 1$. Applying the skein relation once to Ψ_{k+1} we obtain

$$\Psi_{k+1} = d^\ell_{k+1,k+1} \cdot \beta + \Psi_k \cdot \xi$$

See Fig. 22 for the diagrams.

Using Lemma A.2 we can rewrite the previous equation as

$$\Psi_{k+1} = \Psi_k \cdot \xi + \Psi_{k-1} \cdot \alpha\beta + \Psi_{k-2} \cdot \alpha\mathbb{1}\beta + \cdots + \Psi_0 \cdot \alpha\mathbb{1}^{k-1}\beta + \lambda \cdot \mathbb{1}^k\beta.$$

We now apply the inductive hypothesis to obtain

$$\begin{aligned}\Psi_k = &\sum_{\psi \in \Gamma^{(k-1,k+1)}} \mu \cdot \psi \cdot \xi + \sum_{\phi \in \Lambda^{(k,k)}} \lambda \cdot \phi \cdot \xi \\ &\vdots \\ &+ \sum_{\psi \in \Gamma^{(k-m-1,k-m+1)}} \mu \cdot \psi \cdot \alpha\mathbb{1}^{m-2}\beta + \sum_{\phi \in \Lambda^{(k-m,k-m)}} \lambda \cdot \phi \cdot \alpha\mathbb{1}^{m-2}\beta \\ &\vdots\end{aligned}$$

$$+\lambda \cdot \alpha \mathbb{1}^{k-1}\beta + \mu \cdot \mathbb{1}^k \beta.$$

Using the inductive definition of $\Gamma^{(n,n+2)}$ and $\Lambda^{(m,m)}$ we conclude that

$$\Psi_{k+1} = \sum_{\psi \in \Gamma^{(k,k+2)}} \mu \cdot \psi + \sum_{\phi \in \Lambda^{(k+1,k+1)}} \lambda \cdot \phi,$$

which completes the proof of the lemma. $\square$

Lemma A.5. *The left differential d_n^ℓ can be written in terms of Ψ_i as follows:*

$$d_n^\ell = \begin{cases} \sum_{i=1}^k \Psi_{2(i-1)} \cdot \alpha \cdot \mathbb{1}^{n-2i} & \text{for} \quad n = 2k \\ -\mu \mathbb{1}^{n-1} - \sum_{j=1}^k \Psi_{2j-1} \cdot \alpha \cdot \mathbb{1}^{n-2j+1} & \text{for} \quad n = 2k+1. \end{cases}$$

Proof. Suppose first that n is even and let $n = 2k$ for some k. By definition we have

$$d_n^\ell = \sum_{i=1}^n (-1)^i d_{i,i}^\ell \cdot \mathbb{1}^{n-i}.$$

Since n is even, we can group the terms $d_{i,i}^\ell \cdot \mathbb{1}^{n-i}$ in pairs of consecutive summands $2i$ and $2i+1$ for $i = 0, \ldots, k$. Applying the skein relation to the $(2i)^{\text{th}}$ term, we obtain that, for all i, $d_{2i,2i}^\ell = d_{2i-1,2i-1}^\ell + \Psi_{2i-2} \cdot \alpha$. Here we recall that $\Psi_0 = \lambda$. Putting all terms of the left-hand side of $-d_{2i-1,2i-1}^\ell + d_{2i,2i}^\ell = \Psi_{2i-2} \cdot \alpha$ together and using the fact that consecutive terms appear with opposite signs, we complete the proof for the case n even. If $n = 2k+1$ is odd, we proceed similarly by grouping in pairs the terms $d_{2j,2j}^\ell$ and $d_{2j+1,2j+1}^\ell$ for $j = 1, \ldots, k$. $\square$

Proof of Theorem 4.1. We first consider the case $n = 2s$ for some s. Since in Lemma 5.1 below we show that $d_2^\ell = 0$, we assume that $s \geq 2$. From Lemma A.5 we have

$$d_n^\ell = \sum_{i=1}^s \Psi_{2(i-1)} \cdot \alpha \cdot \mathbb{1}^{n-2i}.$$

Using Lemma A.4 we can rewrite it as

$$d_n^\ell = \sum_{i=2}^s \left(\sum_{\psi \in \Gamma^{(2i-3,2i-1)}} \mu \cdot \psi + \sum_{\phi \in \Lambda^{(2i-2,2i-2)}} \lambda \cdot \phi \right) \cdot \alpha \mathbb{1}^{n-2i}.$$

To complete the proof of the first assertion with even n, it would suffice to show that for each $i = 2, \ldots, s$

$$\sum_{\psi \in \Gamma^{(2i-3,2i-1)}} \mu \cdot \psi \cdot \alpha + \sum_{\phi \in \Lambda^{(2i-2,2i-2)}} \lambda \cdot \phi \cdot \alpha = \sum_{S'(n)} g'_{i_0} g_{i_1}^{k(1)} \cdots g_{i_h}^{k(h)},$$

where, noting that $2k = n = 2i$,

$$S'(n) = \{(i_0, i_1, \ldots, i_h; k(1), \ldots, k(h)) \mid i_h \neq 1,\ i_0 + i_1^{k(1)} + \cdots + i_h^{k(h)} = 2i\}.$$

Since $n - 2i = 2s - 2i$ is even for all $i = 2, \ldots, s$, $g'_1 = \mu$ by definition and g'_{i_0} contains a factor of λ for each $i_0 \geq 2$, the last equality is a consequence of the two set-theoretic equalities:

$$\begin{aligned}\Gamma^{(\ell,\ell+2)}\cdot\alpha &= \{g_{i_1}^{k(1)}\cdots g_{i_h}^{k(h)} \mid i_h\neq 1,\ i_1^{k(1)}+\cdots+i_h^{k(h)}=\ell+2\}\\ &=: S_1'(\ell),\\ \lambda\cdot\Lambda^{(\ell,\ell)}\cdot\alpha &= \{g'_{i_0}\cdot g_{i_1}^{k(1)}\cdots g_{i_h}^{k(h)} \mid i_h\neq 1,\ i_0\neq 1,\ i_0+i_1^{k(1)}+\cdots+i_h^{k(h)}=\ell+2\}\\ &=: S_2'(\ell),\end{aligned}$$

for all n, where $g_{i_1}^{k(1)}\cdots g_{i_h}^{k(h)}$ can be empty in the second last line. We therefore proceed to prove the first equality by induction. First observe that by definition, for $\ell=1$, we have

$$\Gamma^{(1,3)}=\{\mathbb{1}\beta,\beta\xi\},$$

from which

$$\Gamma^{(1,3)}\cdot\alpha=\{\mathbb{1}\beta\alpha,\beta\xi\alpha\}.$$

It is easy to see by direct inspection that

$$\{g_{i_1}^{k(1)}\cdots g_{i_h}^{k(h)} \mid i_h\neq 1,\ i_1^{k(1)}+\cdots+i_h^{k(h)}=3\}=\{g_1\cdot g_2,g_3\}=\{\mathbb{1}\beta\alpha,\beta\xi\alpha\}.$$

So the basis of induction holds true. Let us now assume the equality holds for all ℓ smaller than or equal to r, and suppose $\ell=r+1$. We want to show the inclusion

$$\Gamma^{(r+1,r+3)}\cdot\alpha\subset S_1'(r+1).$$

Let $\psi\in\Gamma^{(r+1,r+3)}$. From the Equality (1), we distinguish three cases

$$\psi\in\begin{cases}\Gamma^{(r,r+2)}\cdot\xi\\ \bigsqcup_{m=2}^{r-1}\Gamma^{(r+1-m,r+3-m)}\cdot\alpha\mathbb{1}^{m-2}\beta\\ \{\mathbb{1}^{r+1}\beta\}.\end{cases}$$

In the last case it is clear that $\psi\alpha=\mathbb{1}^{r+1}\beta\alpha\in S_1'(r+1)$. In the second case, ψ is equal to $\psi'\cdot\alpha\mathbb{1}^{m-2}\beta$, for $\psi'\in\Gamma^{(r+1-m,r+3-m)}$ for some $m=2,\dots,r-1$. Then $\psi\cdot\alpha=\psi'\cdot\alpha\cdot\mathbb{1}^{m-2}\beta\alpha\in S_1'(r+1)$ since $\mathbb{1}^{m-2}\beta\alpha\in S_1'(m)$, $\psi'\cdot\alpha\in S_1'(r-m+1)$ by inductive hypothesis and $S_1'(n)\cdot S_1'(m)\subset S_1'(n+m)$ for all n,m.

Lastly, if $\psi\in\Gamma^{(r,r+2)}\cdot\xi$ we can write $\psi\cdot\alpha=p_1\cdot\xi\cdot\alpha$ for some $p_1\in\Gamma^{(r,r+2)}$. We again distinguish three subcases depending on which of the three cases p_1 belongs to. As before, we see that if p_1 is not in $\Gamma^{(r-1,r+1)}\cdot\xi$ we easily have that $\psi\cdot\alpha\in S_1'(r+1)$. Otherwise we can write $\psi\cdot\alpha=p_2\cdot\xi\xi\cdot\alpha$. So proceeding, at each step we have that either $\psi\cdot\alpha\in S_1'(r+1)$, or we decompose $\psi\cdot\alpha$ as a product of type $p_k\cdot\xi^k\alpha$, with $p_k\in\Gamma^{(1,3)}=\{\mathbb{1}\beta,\beta\xi\}$. Either way $\psi\cdot\alpha\in S_1'(r+1)$ and we have proved that

$$\Gamma^{(r+1,r+3)}\cdot\alpha\subset S_1'(r+1).$$

We now show the opposite inclusion. Let $g=g_{i_1}^{k(1)}\cdots g_{i_h}^{k(h)}\in S_1'(r+1)$ with $i_h\neq 1$. We can therefore write $g=g_{i_1}^{k(1)}\cdots g_{i_{h-1}}^{k(h-1)}\beta\xi^{i_h-2}\alpha\cdots\beta\xi^{i_h-2}\alpha$, where i_h-2 can be possibly zero, and $\beta\xi^{i_h-2}\alpha$ appears $k(h)$ times. We also abbreviate center dots for brevity, such as $\beta\alpha$ for $\beta\cdot\alpha$, with the understanding that these sequences denote the horizontal concatenations instead of compositions of maps. If $i_j=1$ for all $j=1,\dots,h-1$, we have that

$$g = \mathbb{1}^{k(1)+\cdots+k(h-1)} \cdot \beta \cdot \xi^{i_h-2}\beta\alpha\xi^{i_h-2}\beta\alpha\cdots\beta\alpha\xi^{i_h-2} \cdot \alpha \in \Gamma^{(r+1,r+3)} \cdot \alpha,$$

since it is easily seen that

$$\mathbb{1}^{k(1)+\cdots+k(h-1)} \cdot \beta \cdot \xi^{i_h-2}\beta\alpha\xi^{i_h-2}\beta\alpha\cdots\beta\alpha\xi^{i_h-2} \in \Gamma^{(r+1,r+3)}.$$

Otherwise let j be the largest index for which $i_j \neq 1$. Then we have

$$g = g_{i_1}^{k(1)} \cdots g_{i_{j-1}}^{k(j-1)} \cdot \beta\xi^{i_j-2}\alpha\cdots\beta\xi^{i_j-2}\alpha \cdot \mathbb{1}^{i_{j+1}+\cdots+i_{h-1}} \cdot \beta\xi^{i_h-2}\alpha.$$

By the induction hypothesis we can write

$$g = p \cdot \alpha \cdot \beta\xi^{i_j-2}\alpha\mathbb{1}^{i_{j+1}+\cdots+i_{h-1}} \cdot \beta\xi^{i_h-2}\alpha,$$

for $p \in \Gamma^{(m,m+2)}$ for some $m \leq r$. Since

$$p \cdot \alpha \cdot \beta\xi^{i_j-2}\alpha\mathbb{1}^{i_{j+1}+\cdots+i_{h-1}} \cdot \beta\xi^{i_h-2} \in \Gamma^{(r+1,r+3)},$$

we conclude that $g \in \Gamma^{(r+1,r+3)} \cdot \alpha$. Therefore $\Gamma^{(\ell,\ell+2)} \cdot \alpha = S_1'(\ell)$ for all even n. To prove that $\Lambda^{(\ell,\ell)} \cdot \alpha = S_2'(\ell)$, we again proceed by induction. The proof is similar to the case of $\Gamma^{(n,n+2)}$. The base of induction holds true since we have

$$S_2'(1) = \{\lambda\xi\alpha\},$$

and

$$\Lambda^{(1,1)} = \{\xi\}.$$

Let us now suppose that the equality $\Lambda^{(\ell,\ell)} \cdot \alpha = S_2'(\ell)$ holds for all $2 \leq \ell \leq r$. We want to show $\Lambda^{(r+1,r+1)} \cdot \alpha = S_2'(r+1)$. Consider again three different cases

$$\phi \in \begin{cases} \Lambda^{(r,r)} \cdot \xi \\ \bigsqcup_{m=2}^{r} \Lambda^{(r-m,r-m)} \cdot \alpha\mathbb{1}^{m-2}\beta \\ \{\alpha\mathbb{1}^{r-1}\beta\}. \end{cases}$$

In the first case, $\phi = q \cdot \xi$ for some $q \in \Lambda^{r,r}$ and we can proceed backward as for the analogous proof for $\Gamma^{(r+1,r+3)}$ so that at each step we either have $\phi \in S_2'(r+1)$ or we can rewrite $\phi = \tilde{q} \cdot \xi \cdots \xi$, where the product of ξ is r times and $\tilde{q} \in \Lambda^{1,1}$. It follows that in the first case $\lambda\phi\alpha \in S_2'(r+1)$. In the second case we have $\phi = q \cdot \alpha\mathbb{1}^{r-m}\beta$, for some $q \in \Lambda^{(r-m,r-m)}$. Since $q \cdot \alpha \in S_2'(t)$ for some t by induction, and $\mathbb{1}^{r-m}\alpha\beta$ is of type $g_1^d \cdot g_2$, this case follows as well. In the third case, $\lambda\alpha\mathbb{1}^{r-1}\beta \cdot \alpha = \lambda\alpha\mathbb{1}^{r-1}\alpha\beta \in S_2'(r+1)$. It follows that d_n^ℓ decomposes as in the statement of the theorem, when n is even. The case n odd is similar. Let $n = 2s+1$ for some s, then using Lemma A.5, odd case, it holds

$$d_{2s+1}^\ell = -\mu\mathbb{1}^{2s} - \sum_{j=1}^{2} \Psi_{2j-1}\alpha\mathbb{1}^{2s-2j}.$$

Applying Lemma A.4 we obtain

$$d^{\ell}_{2s+1} = -\mu \mathbb{1}^{2s} - \sum_{j=1}^{s} \left(\sum_{\psi \in \Gamma^{(2j-2,2j)}} \mu \cdot \psi + \sum_{\phi \in \Lambda^{(2j-1,2j-1)}} \lambda \cdot \phi \right) \cdot \alpha \mathbb{1}^{2s-2j}.$$

Since $\Gamma^{(2j-2,2j)}$ has even exponents for all j's, using the recursive definition of Γ's it follows that there is no term of type $\mu \cdot \mathbb{1}^d$ in the sum $\sum_{\psi \in \Gamma^{(2j-2,2j)}} \mu \cdot \psi$. So it is enough to show that for each $j = 1, \ldots, s$ we have

$$\left(\sum_{\psi \in \Gamma^{(2j-2,2j)}} \mu \cdot \psi + \sum_{\phi \in \Lambda^{(2j-1,2j-1)}} \lambda \cdot \phi \right) \cdot \alpha = \sum_{S''_2(n)} g'_{i_0} g^{k(1)}_{i_1} \cdots g^{k(h)}_{i_h},$$

where the sum runs over all tuples in

$$S''_2(n) := \{(i_0, i_1, \ldots, i_h; k(1), \ldots, k(h) \mid i_0 + i_1^{k(h)} + \cdots + i_h^{k(h)} = 2j + 1\}.$$

Since $n - 2j = 2s + 1 - 2j$ is odd for all j and g'_{i_0} contains a factor of λ for all $i_0 \geq 2$ it follows that it is enough to prove the set theoretic equalities

$$\Gamma^{(d,d+2)} = S'_1(d)$$
$$\lambda \cdot \Lambda^{(d,d)} \alpha = S'_2(d).$$

These have already been proved above and the proof for $n = 2s + 1$ is complete as well.

It is easy to see that mirroring Lemmas A.2, A.4, A.5 with respect to the y-axis, we obtain a decomposition of d^r_n with right (co)action on k replacing the left (co)action on k, and ζ instead of ξ. So the formula for d^{ℓ}_n just proved can be easily adapted for d^r_n. Putting the two equations together and distinguishing the cases n odd and even, we conclude the proof of the theorem. $\square$

Appendix B. Proof of Proposition 7.2 continued

In this section, we provide proofs of the other cases. For odd n, the following terms result in the non-zero coefficient of $e_{1,0}$ in the image, for $i = 1, \ldots, (n-1)/2$:

$$\begin{aligned}
(g'_1 g_1^{n-1})(e_{1,0}) &= 1e_{1,0} \\
(g'_1 g_1^{n-1})(e_{1,1}) &= 1e_{1,0} \\
(g'_{2i+1} g_1^{n-2i-1})(e_{1,2i}) &= (-y)(y^{-1})y^{2(2i-1)} e_{1,0} = -y^{4i-2} e_{1,0} \\
(g'_{2i+1} g_1^{n-2i-1})(e_{1,2i+1}) &= (-y)(-y)y^{2(2i-1)} e_{1,0} = y^{4i} e_{1,0} \\
(h_1^{2i-2} h'_{n-2i+2})(e_{1,2i-1}) &= (y^{-1})y e_{1,0} = 1e_{1,0} \\
(h_1^{2i-2} h'_{n-2i+2})(e_{1,2i}) &= (-y)y e_{1,0} = -y^2 e_{1,0} \\
(h_1^{n-1} h'_1)(e_{1,0}) &= 1e_{1,0} \\
(h_1^{n-1} h'_1)(e_{1,n}) &= 1e_{1,0}.
\end{aligned}$$

From these we compute

$$\begin{aligned}
d_n(e_{1,0}) &= (-g'_1 g_1^{n-1} + h_1^{n-1} h'_1)(e_{1,0}) = (-1+1)e_{1,0} = 0 \\
d_n(e_{1,1}) &= (-g'_1 g_1^{n-1} - g'_3 g_1^{n-3} + h'_n)(e_{1,1}) = (-1 + y^2 + 1)e_{1,0} = y^2 e_{1,0} \\
d_n(e_{1,2i-1}) &= (-g'_{2i-1} g_1^{n-2i+1} + h_1^{2i-2} h'_{n-2i+2})(e_{1,2i-1}) = (-y^{4i-4} + 1)e_{1,0}
\end{aligned}$$

$$
\begin{aligned}
d_n(e_{1,2i}) = (-g'_{2i+1}g_1^{n-2i-1} + h_1^{2i-2}h'_{n-2i-2})(e_{1,2i}) &= (y^{4i-2} - y^2)e_{1,0} \\
d_n(e_{1,n}) = (-g'_n + h_1^{n-1}h'_1)(e_{1,n}) &= (-y^{2(n-1)} + 1)e_{1,0}.
\end{aligned}
$$

For the coefficient of $e_{2,0}$, we compute

$$
\begin{aligned}
(g'_1g_1^{n-1})(e_{2,0}) &= 1e_{2,0} \\
(g'_1g_1^{n-1})(e_{2,1}) &= 1e_{2,0} \\
(g'_{2i+1}g_1^{n-2i-1})(e_{2,2i}) &= (y)(-y)e_{2,0} = -y^2e_{2,0} \\
(g'_{2i+1}g_1^{n-2i-1})(e_{2,2i+1}) &= (y)(y^{-1})e_{2,0} = 1e_{2,0} \\
(h_1^{2i-2}h'_{n-2i+2})(e_{2,2i-1}) &= (-y)y \cdot y^{2(n-2i)}e_{2,0} = -y^{2n-4i+2}e_{2,0} \\
(h_1^{2i-2}h'_{n-2i+2})(e_{2,2i}) &= (y^{-1})y \cdot y^{2(n-2i)}e_{2,0} = y^{2n-4i}e_{2,0} \\
(h_1^{n-1}h'_1)(e_{2,0}) &= 1e_{2,0} \\
(h_1^{n-1}h'_1)(e_{2,n}) &= 1e_{2,0}.
\end{aligned}
$$

From these we compute

$$
\begin{aligned}
d_n(e_{2,0}) = (-g'_1g_1^{n-1} + h_1^{n-1}h'_1)(e_{2,0}) &= (-1+1)e_{2,0} = 0 \\
d_n(e_{2,1}) = (-g'_1g_1^{n-1} + h'_3h_1^{n-3})(e_{2,1}) &= (-1 + y^{2n-2})e_{2,0} \\
d_n(e_{2,2i+1}) = (-g'_{2i+1}g_1^{n-2i-1} + h_1^{2i}h'_{n-2i-1})(e_{2,2i+1}) &= (y^2 - y^{2n-4i-2})e_{2,0} \\
d_n(e_{2,2i}) = (-g'_{2i+1}g_1^{n-2i-1} + h_1^{2i-2}h'_{n-2i+2})(e_{2,2i}) &= (-1 + y^{2n-4i})e_{2,0} \\
d_n(e_{2,n}) = (-g'_n + h_1^{n-1}h'_1)(e_{2,n}) &= (-1+1)e_{2,0} = 0.
\end{aligned}
$$

For even n and for $e_{2,0}$, we have

$$
\begin{aligned}
(g'_{2i}g_1^{n-2i})(e_{2,2i-1}) &= (-y)(y^{-1})e_{2,0} = -1e_{2,0} \\
(g'_{2i}g_1^{n-2i})(e_{2,2i}) &= (-y)(-y)e_{2,0} = y^2e_{2,0} \\
(h_1^{2i}h'_{n-2i})(e_{1,2i+1}) &= y(y^{-1})y^{2(n-2i)}e_{2,0}. = -y^{2(n-2i+1)}e_{2,0} \\
(h_1^{2i}h'_{n-2i})(e_{1,2i+2}) &= y(-y)y^{2(n-2i+2)}e_{2,0}. = y^{2(n-2i)}e_{2,0}.
\end{aligned}
$$

From these we compute

$$
\begin{aligned}
d_n(e_{2,2i-1}) = (-g'_{2i}g_1^{n-2i} + h_1^{2i-2}h'_{n-2i+2})(e_{1,2i-1}) &= (y^2 - y^{2(n-2i+3)})e_{2,0} \\
d_n(e_{2,2i}) = (-g'_{2i}g_1^{n-2i} - h'_{2i-2}h'_{n-2i+2})(e_{1,2i}) &= (-1 + y^{2(n-2i+2)})e_{2,0}.
\end{aligned}
$$

This completes the proof.

References

[1] J. Scott Carter, Mohamed Elhamdadi, Masahico Saito, Homology theory for the set-theoretic Yang-Baxter equation and knot invariants from generalizations of quandles, Fundam. Math. 184 (2004) 31–54.

[2] J. Scott Carter, Daniel Jelsovsky, Seiichi Kamada, Laurel Langford, Masahico Saito, Quandle cohomology and state-sum invariants of knotted curves and surfaces, Trans. Am. Math. Soc. 355 (10) (2003) 3947–3989, MR 1990571.

[3] Michael Eisermann, Yang–Baxter deformations of quandles and racks, Algebraic Geom. Topol. 5 (2) (2005) 537–562.

[4] Victoria Lebed, Leandro Vendramin, Homology of left non-degenerate set-theoretic solutions to the Yang–Baxter equation, Adv. Math. 304 (2017) 1219–1261.

[5] Michio Jimbo, Introduction to the Yang-Baxter equation, Int. J. Mod. Phys. A 4 (15) (1989) 3759–3777.

[6] Józef H. Przytycki, Knots and Distributive Homology: From Arc Colorings to Yang—Baxter Homology, New Ideas in Low Dimensional Topology, 2015, pp. 413–488.
[7] Józef H. Przytycki, Xiao Wang, Equivalence of two definitions of set-theoretic Yang–Baxter homology and general Yang–Baxter homology, J. Knot Theory Ramif. 27 (7) (2018).
[8] Józef H. Przytycki, Xiao Wang, Second Yang-Baxter homology for the HOMFLYPT polynomial, Preprint, 2020.
[9] Vladimir G. Turaev, The Yang-Baxter equation and invariants of links, Invent. Math. 92 (3) (1988) 527–554.
[10] Vladimir G. Turaev, Quantum Invariants of Knots and 3-Manifolds, vol. 18, Walter de Gruyter GmbH & Co KG, 2016.
[11] Xiao Wang, Knot theory and algebraic structures motivated by and applied to knots, PhD dissertation, The George Washington University, 2018.