Pascal's Triangle Fractal Symmetries

Nayan E. Myerson-Jain,[1] Shang Liu,[2] Wenjie Ji,[1] Cenke Xu,[1] and Sagar Vijay[1]
[1]Department of Physics, University of California, Santa Barbara, California 93106, USA
[2]Kavli Institute for Theoretical Physics, University of California, Santa Barbara, California 93106, USA

(Received 13 October 2021; revised 23 January 2022; accepted 18 February 2022; published 14 March 2022)

We introduce a model of interacting bosons exhibiting an infinite collection of fractal symmetries—termed "Pascal's triangle symmetries"—which provides a natural U(1) generalization of a spin-(1/2) system with Sierpinski triangle fractal symmetries introduced in Newman *et al.*, [Phys. Rev. E **60**, 5068 (1999).]. The Pascal's triangle symmetry gives rise to exact degeneracies, as well as a manifold of low-energy states which are absent in the Sierpinski triangle model. Breaking the U(1) symmetry of this model to Z_p, with prime integer p, yields a lattice model with a unique fractal symmetry which is generated by an operator supported on a fractal subsystem with Hausdorff dimension $d_H = \ln(p(p+1)/2)/\ln p$. The Hausdorff dimension of the fractal can be probed through correlation functions at finite temperature. The phase diagram of these models at zero temperature in the presence of quantum fluctuations, as well as the potential physical construction of the U(1) model, is discussed.

DOI: 10.1103/PhysRevLett.128.115301

Introduction.—In recent years, generalizations of the notion of symmetry have significantly broadened our understanding of states of matter. Highly entangled states of quantum matter, such as Z_2 topological order [1–4] and the $(3+1)D$ algebraic spin liquid with photon excitations [5–7], were previously thought to be beyond the notion of spontaneous symmetry breaking (SSB) of ordinary global symmetries. But it has been recently realized that these phases admit a unified description in terms of the SSB of generalized higher-form symmetries [8–16]. Subsystem symmetries have further enriched our understanding along this line. Long-range-entangled quantum phases with fractionalized excitations that have inherently restricted mobility—termed fracton orders—exhibit emergent subsystem symmetries [17–22]. These symmetries can be further categorized [18]: a type-I subsystem symmetry has generators and conserved charges defined on regular submanifolds such as lines and planes [17–19], while type-II subsystem symmetries have conserved charges defined on a fractal-shaped subsystem [20,23–26], often with noninteger spatial dimensions.

The simplest model with a fractal subsystem symmetry is the Sierpinski triangle model, which was first introduced for the purpose of studying glassy dynamics [27]:

$$H_{\mathrm{ST}} = \sum_{\nabla} - K\sigma_1^z\sigma_2^z\sigma_3^z. \tag{1}$$

Here, σ_i^z is an Ising spin defined on each site of a triangular lattice, and the sum is only over the downward-facing triangular plaquettes of the lattice. This model has the following features: (i) The model has an exotic fractal symmetry, which becomes most explicit when the system is defined on a $L\times L$ lattice with $L = 2^k - 1$: the Hamiltonian is invariant under flipping spins along pairs of extensively large fractal subsystems, each of which forms a Sierpinski triangle, as reviewed in the Supplemental Material [28]. (ii) At finite temperature, the three-point correlation function $\langle\sigma_{0,0}^z\sigma_{r,0}^z\sigma_{0,r}^z\rangle$ of spins arranged on the corners of an equilateral triangle is nonzero only when $r = 2^k$, and it scales as $\sim\exp(-\alpha r^{d_H})$ with $d_H = \ln 3/\ln 2$, which is the Hausdorff dimension of the Sierpinski triangle [23,29,30]. (iii) With the addition of a transverse field $\sum_i -h\sigma_i^x$, there is a quantum phase transition at zero temperature [31,32] when $h = K$, which separates the "fractal-ordered" phase that spontaneously breaks the fractal symmetry ($K > h$) and a disordered phase ($h > K$).

In this Letter, we introduce generalizations of both the classical and quantum Sierpinski triangle models, which lead to the identification of a novel kind of fractal symmetry. These models are obtained from a U(1) parent model with "Pascal's triangle" (also called Yang Hui triangle in China) symmetries, a family of symmetry transformations along a fractal region which are exact in a system with periodic boundary conditions, and for particular system sizes. Even when these symmetries are not exact, the presence of an "approximate" Pascal's triangle symmetry gives rise to low-energy states which are absent in the Sierpinski triangle model [27]. We study the phase diagram of this parent model in the presence of quantum fluctuations, and at nonzero temperature. Descendants of this model are obtained by reducing the U(1) degree of freedom of the parent model to Z_p, with prime integer p. The Z_p models have their own fractal symmetry that are deduced from the Pascal's triangle symmetry, and their degenerate excitations have an emergent fractal structure with Hausdorff dimension $d_H = \ln(p(p+1)/2)/\ln p$.

The U(1) parent model.—The Hamiltonian of the U(1) generalization of the Sierpinski triangle model reads

$$H_{\mathrm{U}(1)} = \sum_{\bigtriangledown} -t\cos(\theta_1 + \theta_2 + \theta_3). \tag{2}$$

It is straightforward to see that Eq. (2) has two conventional U(1) global symmetries:

$$\begin{aligned} \mathrm{U}(1)_1 &: \theta_{j\in A} \to \theta_{j\in A} + \alpha, \qquad \theta_{j\in B} \to \theta_{j\in B} - \alpha, \\ \mathrm{U}(1)_2 &: \theta_{j\in A} \to \theta_{j\in A} + \beta, \qquad \theta_{j\in C} \to \theta_{j\in C} - \beta. \end{aligned} \tag{3}$$

A, B, and C are the three sublattices of the triangular lattice. The ground states of Eq. (2) spontaneously break the two U(1) symmetries. Starting with one of the ground states, say $\theta = 0$ uniformly on the entire lattice, a class of ground states can be generated by rotating θ globally according to Eq. (3). Any ground state obtained this way still has a uniform order of θ on each of the three sublattices, hence the ground states generated through Eq. (3) have a conventional "$\sqrt{3} \times \sqrt{3}$" order, which is the order often observed on the triangular lattice antiferromagnet.

Besides the two ordinary U(1) global symmetries, this model [Eq. (2)] actually contains an infinite series of Z_p distinct fractal symmetries, one for each prime number p, $p \geq 2$. The series of Z_p fractal symmetries exhibited by the U(1) parent model are in the shape of a Pascal's triangle modulo p. For example, when $p = 2$, this Pascal's triangle symmetry reduces down to the familiar Z_2 fractal symmetry of the Sierpinski triangle model; for $p = 3$, the Pascal's triangle modulo 3 reduces to another fractal shape (Fig. 1).

The exact series of fractal transformations of Eq. (2) can be written down as a staggered rotation of the θ_i's in the shape of a Pascal's triangle modulo p, which has a side length of $p^k - 1$, where k is any integer greater than zero. The precise form of the transformation is

$$\theta_i \to \theta_i + \frac{2\pi}{p}(-1)^{i_x+i_y}\binom{i_x + i_y}{i_y} \tag{4}$$

at the points (i_x, i_y) for which $0 \leq i_y \leq i_x$ and $i_x + i_y \in [0, p^k - 1]$. As shown in the Supplemental Material [28], transformations of this form can be used to generate exact symmetries when the system is placed on an $L \times L$ lattice with periodic boundary conditions and with $L = p^k - 1$.

Any Z_p fractal transformation of the U(1) parent model generates fully immobile defects which are analogous to fractons. From the uniform $\theta_i = 0$ ground state, transforming the U(1) degrees of freedom according to Eq. (4) in the shape of a local Pascal's triangle of size $p^k - 1$ creates three defects of energy $t(1 - \cos(2\pi/p))$, one at each downward-facing triangular plaquette located at the corners of the Pascal's triangle, as shown in the Supplemental Material [28], and as indicated schematically in Fig. 1. If we treat these defects as pointlike excitations localized on their downward-facing plaquettes, individual defects cannot be moved by any rotation of θ_i's without creating more excitations and are hence completely immobile.

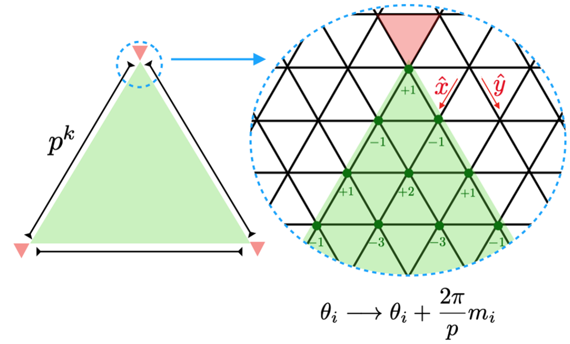

FIG. 1. The U(1) parent model [Eq. (2)] has a family of fractal symmetries, generated by a staggered rotation of the boson phase $\theta_i \to \theta_i + (2\pi/p)m_i$ over a triangular region of side length which is a power of any prime number p. For each p, the fractal symmetry becomes exact for system size L^2 with $L = p^k - 1$. When acting on a classical ground state of the parent model, this transformation generates excitations at the corners of the triangular region. The action of this rotation can be visualized as Pascal's triangle modulo p, which is a fractal with Hausdorff dimension $d_H(p) = \ln(p(p+1)/2)/\ln p$.

At finite temperature, the U(1) parent model is completely disordered, similarly to the Sierpinski triangle model [27]. This can be most easily seen from a duality mapping of the U(1) degrees of freedom on the vertices to new U(1) degrees of freedom on downward-facing plaquettes $(\theta_1 + \theta_2 + \theta_3)_{\bigtriangledown} \to \phi_{\bigtriangledown}$, where $\phi_{\bigtriangledown}$ is defined on the dual site located at the center of each downward-facing triangular plaquette (Fig. 2), and ϕ is still compact (periodically defined). The dual of Eq. (2) is

$$H^d_{\mathrm{U}(1)} = \sum_{\bigtriangledown} -t\cos(\phi_{\bigtriangledown}). \tag{5}$$

Since each ϕ is decoupled from the others, the partition function factorizes into a product of local partition functions for each individual ϕ which does not support any phase transition.

The three-body interactions of the Sierpinski triangle model, as well as those of the U(1) parent model, look artificial. Reference [33] proposed to realize the Sierpinski triangle model with the Rydberg atoms with only two-body van der Waals interactions. In the Supplemental Material [28], we present a more natural construction of the U(1) parent model through a setup with only two-body interactions.

The Z_p models.—From the U(1) parent model in Eq. (2), models with a single fractal symmetry that are natural

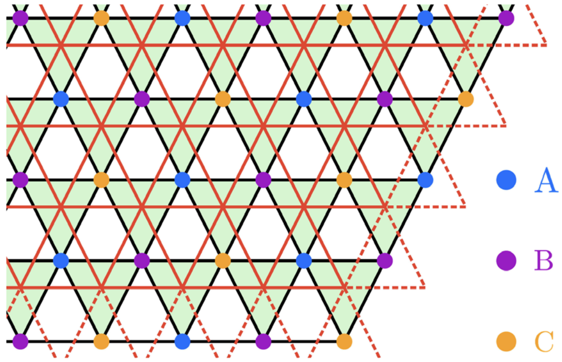

FIG. 2. The models we consider in this Letter involve the sum of all the downward-facing triangles (shaded in green). The dual of the U(1) model [Eqs. (5) and (15)] is defined on the dual triangular lattice, whose sites are the center of each downward-facing triangle of the original lattice.

extensions of the Sierpinski triangle model can be constructed. This is done by breaking the U(1) degrees of freedom down to Z_p clock degrees of freedom, $\sigma_i = e^{i\theta_i}, \theta_i \in (2\pi/p)Z_p$:

$$H_{Z_p} = \sum_{\triangledown} -\frac{t}{2}\sigma_1\sigma_2\sigma_3 + \text{H.c.} \tag{6}$$

The model in Eq. (6) extends many properties of the Sierpinski triangle model to a series of Z_p "Pascal's triangle models," which reduces to Eq. (1) when $p = 2$. We note that these Z_p models for prime integer p were mentioned in Ref. [23], though the symmetries which distinguish them from the Sierpinski case were not explored. Generally, since the Z_p fractal symmetry of the Pascal's triangle models are descended from the U(1) parent model, the fractal symmetry transformation in these models is realized by Eq. (4) with the appropriate choice of p. These models also display the fracton-like defects associated with fractal excitations in the shape of a Pascal's triangle modulo p, as well as spontaneously breaking the Z_p fractal symmetry, yielding a ground-state degeneracy of p^{L-1} when $L = p^k - 1$ (see the Supplemental Material [28] for derivation).

Spontaneous breaking of the Z_p fractal symmetries in the Pascal's triangle models can be diagnosed by a three-point correlation function. Making use of the duality of these models, we can define plaquette degrees of freedom $\tau_\triangledown = (\sigma_1\sigma_2\sigma_3)_\triangledown$ for which the dual Hamiltonian is $H^d_{Z_p} = \sum_\triangledown -(t/2)\tau_\triangledown + \text{H.c.}$ In the thermodynamic limit, each σ variable can be represented as an infinite staggered product of dual τ variables in the shape of a Pascal's triangle modulo p. The three-point function $\mathcal{C}_3(r) = \langle\sigma_{0,0}\sigma_{r,0}\sigma_{0,r}\rangle$, after being rewritten in terms of the dual variables, only has compact support when $r = p^k$, and hence must otherwise vanish. The three-point function factors into a product of single-site expectation values $\langle\tau\rangle, \ldots, \langle\tau^{p-1}\rangle$. From the form of the Hamiltonian, a general expression for the three-point function for arbitrary p, primes can be derived (see the Supplemental Material [28] for details)

$$\mathcal{C}_3(r = p^k) = \prod_{m=1}^{\frac{p-1}{2}} \langle\tau^m\rangle^{N_{m,p-m}(k)}, \tag{7}$$

$N_{m,p-m}(k)$ is the number of times m and $p - m$ appear in a Pascal's triangle modulo p with length $p^k - 1$. Such an expression is complex, but a complete set of recurrence relations is constructed in the Supplemental Material [28] for $N_{m,p-m}(k)$, lending Eq. (7) to efficient numerical evaluation. For small p, this can be done analytically—e.g., for $p = 3$, the three-point function is

$$\mathcal{C}_3(r = 3^k) = \langle\tau\rangle^{r^{d_H}} = \left(\frac{e^{\beta t} - e^{-\beta t/2}}{e^{\beta t} + 2e^{-\beta t/2}}\right)^{r^{d_H}} = e^{-\alpha r^{d_H}}, \tag{8}$$

where $d_H = (\ln 6/\ln 3)$ is the Hausdorff dimension of a Pascal's triangle modulo 3.

As demonstrated by Eq. (7), the decay of the three-point function can be complicated for general p prime. However, a modified version of the Z_p Pascal's triangle models in Eq. (6) can be proposed, for which the three-point function at finite temperature always decays as a simple fractal area law. If we consider an equal-weight summation of plaquette terms,

$$\mathcal{H}_\text{p} = \sum_\triangledown \sum_{m=0}^{\frac{p-1}{2}} -\frac{t}{2}(\sigma_1\sigma_2\sigma_3)^m + \text{H.c.} \tag{9}$$

This model retains the Z_p fractal symmetry of Eq. (6) as it only includes products of the original plaquette terms. As such, the duality $(\sigma_1\sigma_2\sigma_3)_\triangledown \to \tau_\triangledown$ still exists, and the dual of $\mathcal{H}_p$ is

$$\mathcal{H}^d_\text{p} = \sum_\triangledown \sum_{m=0}^{\frac{p-1}{2}} -\frac{t}{2}\tau^m_\triangledown + \text{H.c.} \tag{10}$$

The manner in which the three-point correlation for $\mathcal{H}_\text{p}$ is calculated remains the same as for what it was in Eq. (6), with the exception that $\langle\tau^m\rangle$ no longer depends on power m. As a result, $\mathcal{C}_3(r = p^k)$ decays as a fractal area law no matter what value p takes:

$$\mathcal{C}_3(r = p^k) = \langle\tau\rangle^{r^{d_H}} = e^{-\alpha r^{d_H}}. \tag{11}$$

The Z_p Pascal's triangle model with prime integer p can be further extended to Z_N models with composite integer N. These new composite Z_N models have more than one fractal symmetry. In fact, there is a distinct Z_p fractal symmetry for each unique prime divisor of N—e.g., the $Z_{N=6}$ model has a Z_2 Sierpinski triangle fractal symmetry and a Z_3 Pascal's triangle modulo 3 fractal symmetry. The behavior of the three-point correlations of the Z_N models is further discussed in the Supplemental Material [28].

The quantum phase diagram.—So far, we have only discussed the classical version of the models. To turn quantum fluctuations on in Eq. (2), one can modify the model as

$$H_{Q-\mathrm{U}(1)} = \sum_{\bigtriangledown} -t\cos(\theta_1+\theta_2+\theta_3) + \sum_j \frac{U}{2} n_j^2, \tag{12}$$

where n_j is the boson number operator defined on each site of the triangular lattice, which is conjugate to the boson phase $[n_i, \theta_j] = \mathrm{i}\delta_{ij}$.

As shown previously, a class of ground states of the classical U(1) model in Eq. (2) have the conventional $\sqrt{3}\times\sqrt{3}$ order, which spontaneously breaks the Pascal's triangle symmetry, and the two U(1) symmetries in Eq. (3). We now investigate whether this classical order is stable against quantum fluctuation—i.e., whether it is stable against the U term in Eq. (12). We argue that all symmetries of the Hamiltonian in Eq. (12) are restored by quantum fluctuations.

In an ordered phase that spontaneously breaks the U(1) global symmetry, one can ignore the fact that the phase angle θ is a compact boson (i.e., $\theta \sim \theta + 2\pi$) and hence expand the cosine function of Eq. (12) to the lowest nontrivial order. This procedure leads to an approximate Gaussian Hamiltonian:

$$H^g_{Q-\mathrm{U}(1)} = \sum_{i,j} t\theta_i\theta_j + \sum_j 3t\theta_j^2 + \frac{U}{2}n_j^2. \tag{13}$$

The band structure of θ based on this Gaussian Hamiltonian has minima at $\pm\mathbf{K} = \pm(4\pi/3, 0)$, which is consistent with the $\sqrt{3}\times\sqrt{3}$ order of the classical Hamiltonian. The spectrum of the Gaussian Hamiltonian is gapless.

The Gaussian expansion of the Hamiltonian ignores the compactness of θ. In a quantum model constructed with θ, θ being a compact boson is equivalent to the constraint that its quantum conjugate variable n take discrete values. To check the stability of a semiclassical state of θ under quantum fluctuation, one needs to investigate whether the compactness of θ, or equivalently the discrete nature of n, would destabilize the semiclassical state described by the Gaussian Hamiltonian [Eq. (13)]. For example, the $(2+1)D$ quantum dimer model on the square lattice can be mapped to a compact U(1) gauge theory [34,35]; a Gaussian expansion would lead to gapless photons. But the compactness of the gauge field is always relevant in a semiclassical photon state unless the system is at a fine-tuned multicritical point (the so-called RK point [34]); hence, the Gaussian state is generally unstable against quantum fluctuations. This effect is also referred to as the confinement of a lattice gauge theory. The standard method of the analysis relies on the dual formalism of Eqs. (12) and (13). The dual model is defined on the dual triangular lattice (Fig. 2) by introducing the following variables:

$$\sum_{j\in\bigtriangledown}\theta_j = \phi_{\bar{j}}, \qquad n_j = \sum_{\bar{j}\in\Delta \text{ around } j} -\Psi_{\bar{j}}, \tag{14}$$

where $\bar{j}$ labels the sites of the dual triangular lattice, $\phi_{\bar{j}}$ and $\Psi_{\bar{j}}$ are canonically conjugate variables, $\Psi_{\bar{j}}$ takes discrete values, and $\phi_{\bar{j}}$ is compact. The dual Hamiltonian reads

$$H^d_{Q-\mathrm{U}(1)} = \sum_{\bar{j}} -t\cos(\phi_{\bar{j}}) + \sum_{\bar{\Delta}} \frac{U}{2}(\Psi_{\bar{1}}+\Psi_{\bar{2}}+\Psi_{\bar{3}})^2. \tag{15}$$

Instead of directly dealing with the discrete variable Ψ, we may view $\Psi_{\bar{j}}$ as taking continuous values, and $\phi_{\bar{j}}$ as its noncompact conjugate variable. The discrete nature of Ψ can be enforced through an external potential in the dual Hamiltonian. The dual Hamiltonian becomes

$$H^d_{Q-\mathrm{U}(1)} \sim \sum_{\bar{\Delta}} \frac{U}{2}(\Psi_{\bar{1}}+\Psi_{\bar{2}}+\Psi_{\bar{3}})^2 - \sum_{\bar{j}} t\cos(\phi_{\bar{j}}) - \alpha\cos(2\pi\Psi_{\bar{j}}). \tag{16}$$

The next step is to temporarily ignore the α terms, and expand $-t\cos(\phi_j)$ to the lowest nontrivial order. After this procedure, the dual Hamiltonian takes a Gaussian form, and it is the dual of the Gaussian Hamiltonian in Eq. (13). The goal of this analysis is to check the role of the α terms at this Gaussian state. This dual Gaussian Hamiltonian can be solved, leading to a band structure of Ψ. The minima of the band structure of Ψ are located at the two corners of the Brillouin zone, $\pm\mathbf{K} = (\pm 4\pi/3, 0)$. We then expand $\Psi_{\mathbf{r}}$ at $\pm\mathbf{K}$:

$$\Psi(\mathbf{r}) \sim e^{\mathrm{i}\mathbf{K}\cdot\mathbf{r}}\psi(\mathbf{r}) + e^{-\mathrm{i}\mathbf{K}\cdot\mathbf{r}}\psi^*(\mathbf{r}). \tag{17}$$

The Lagrangian of the dual theory expanded at $\pm\mathbf{K}$ becomes

$$\mathcal{L}^d_{Q-\mathrm{U}(1)} = (\partial_\tau\vec{\psi})^2 + \rho_2(\nabla\vec{\psi})^2 - \sum_a \alpha\cos(\vec{e}_a\cdot\vec{\psi}), \tag{18}$$

where $\vec{\psi} = (\mathrm{Re}(\psi), \mathrm{Im}(\psi))$; $a = A$, B, and C label the three sublattices of the dual triangular lattice; and $e_A = 2\pi(1,0)$, $e_B = 2\pi(-1/2, \sqrt{3}/2)$, and $e_C = 2\pi(-1/2, -\sqrt{3}/2)$ [36].

The last three terms in Eq. (18) arise from rewriting the last term of Eq. (16) by expanding Ψ at $\pm\mathbf{K}$. After this expansion, the last term of Eq. (16) becomes $-\alpha\cos(\vec{e}_a\cdot\vec{\psi}(\mathbf{r}))$ for $\mathbf{r}$ belonging to sublattice a ($a = A$, B, and C) of the dual triangular lattice. Hence, at long scales, a nonvanishing term would survive. The α term in Eq. (18) will be relevant for the Gaussian theory with nonzero ρ_2, which implies that the compactness of θ, or the discrete nature of n in Eq. (12) destabilizes the semiclassical Gaussian state, and the spectrum of Eq. (12) should be gapped even with small U.

The dual description of the U(1) model studied here captures the spectrum of the gapless modes arising from spontaneously breaking the global U(1) symmetries, though it does not reproduce the spectrum at $U = 0$ that arise due to the Pascal triangle symmetries. Nevertheless, the nature of the ground state of the system when the pinning potential flows to strong coupling can still be inferred. A strong α would pin Ψ to integer values, which implies that a relevant α would drive the system into an eigenstate of n in Eq. (12), and the U term will lead to a unique and gapped ground state without any spontaneous symmetry breaking. Hence, we postulate that quantum fluctuations of Eq. (12) restore all the symmetries of the model in Eq. (2), and continuously connect to the large-U limit of Eq. (12). The analysis here would be more involved if n took half-integer values in Eq. (12).

One possible quantum generalization of Eq. (6) is

$$H_{Q-Z_p} = \sum_{\bigtriangledown} - t\sigma_1^z\sigma_2^z\sigma_3^z - \sum_j h\sigma_j^x + \text{H.c.}, \qquad (19)$$

for which the clock operators σ^z and σ^x obey $(\sigma^z)^p = (\sigma^x)^p = 1$ and $\sigma^z\sigma^x = e^{2\pi i/p}\sigma^x\sigma^z$ [one can also take $\sigma^z = \exp(i\theta)$ and $\sigma^x = \exp(i2\pi n/p)$, and restrict θ to take values in $(2\pi/p)Z_p$]. Unlike the quantum U(1) model, Eq. (19) is exactly self-dual with the introduction of the dual plaquette variables

$$\tau_j^x = \sigma_1^z\sigma_2^z\sigma_3^z, \qquad \tau_1^z\tau_2^z\tau_3^z = \sigma_j^x, \qquad (20)$$

for which the dual Hamiltonian takes the same form as Eq. (19) with t and h switched. Since the spectrum of the Z_p models at $h = 0$ is gapped, and it takes an infinite order of perturbations of h to mix two different ground states in the thermodynamics limit, the classical fractal order of the Z_p Pascal's triangle models is not destroyed upon the introduction of quantum fluctuations. Furthermore, the exact self-duality implies that there should be one or more quantum phase transitions that separate the fractal ordered phase ($t \gg h$) and the disordered phase ($h \gg t$).

Discussion.—Although we demonstrated that the semiclassical order of Eq. (12) is unstable against quantum fluctuation, some deformation of Eq. (12) can support a stable semiclassical order. In the Supplemental Material [28], we will show that if we sum over three-boson interactions for both upward-facing and downward-facing triangles, the semiclassical $\sqrt{3} \times \sqrt{3}$ order becomes stable against quantum fluctuations. Also, the system may be tuned to a multicritical point where ρ_2 in Eq. (18) vanishes, and the low-energy dynamics is controlled by $\rho_4(\nabla^2\vec{\psi})^2$. The system can remain gapless for a finite range of ρ_4, though it takes tuning multiple parameters to reach this state [37–39].

The nature of the quantum phase transition(s) in the quantum Z_p model is a challenging subject. So far, there is no well-established paradigm for understanding quantum phase transitions involving spontaneous breaking of a fractal symmetry. Any approach to studying the quantum phase transition of the Z_p models (such as the quantum Z_2 Sierpinski triangle model) through the U(1) generalization would need to address the enlarged Pascal's triangle symmetry pointed out in the current work.

C. X. is supported by NSF Grant No. DMR-1920434 and the Simons Investigator program. S. L. is supported by the Gordon and Betty Moore Foundation under Grant No. GBMF8690 and the National Science Foundation under Grant No. NSF PHY-1748958. S. V. acknowledges that part of this work was performed at the Aspen Center for Physics, which is supported by National Science Foundation Grant No. PHY-1607611.

[1] N. Read and S. Sachdev, Phys. Rev. Lett. **66**, 1773 (1991).
[2] X. G. Wen, Phys. Rev. B **44**, 2664 (1991).
[3] R. Moessner and S. L. Sondhi, Phys. Rev. Lett. **86**, 1881 (2001).
[4] A. Kitaev, Ann. Phys. (Amsterdam) **303**, 2 (2003).
[5] X.-G. Wen, Phys. Rev. B **68**, 115413 (2003).
[6] R. Moessner and S. L. Sondhi, Phys. Rev. B **68**, 184512 (2003).
[7] M. Hermele, M. P. A. Fisher, and L. Balents, Phys. Rev. B **69**, 064404 (2004).
[8] Z. Nussinov and G. Ortiz, Ann. Phys. (Amsterdam) **324**, 977 (2009).
[9] O. Aharony, N. Seiberg, and Y. Tachikawa, J. High Energy Phys. 08 (2013) 115.
[10] S. Gukov and A. Kapustin, arXiv:1307.4793.
[11] A. Kapustin and R. Thorngren, arXiv:1308.2926.
[12] A. Kapustin and R. Thorngren, arXiv:1309.4721.
[13] A. Kapustin and N. Seiberg, J. High Energy Phys. 04 (2014) 001.
[14] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, J. High Energy Phys. 02 (2015) 172.
[15] P.-S. Hsin, H. T. Lam, and N. Seiberg, SciPost Phys. **6**, 039 (2019).
[16] C. Córdova, T. T. Dumitrescu, and K. Intriligator, J. High Energy Phys. 02 (2019) 184.
[17] C. Chamon, Phys. Rev. Lett. **94**, 040402 (2005).
[18] S. Vijay, J. Haah, and L. Fu, Phys. Rev. B **94**, 235157 (2016).
[19] S. Vijay, J. Haah, and L. Fu, Phys. Rev. B **92**, 235136 (2015).
[20] J. Haah, Phys. Rev. A **83**, 042330 (2011).
[21] R. M. Nandkishore and M. Hermele, Annu. Rev. Condens. Matter Phys. **10**, 295 (2019).

[22] M. Pretko, X. Chen, and Y. You, Int. J. Mod. Phys. A **35**, 2030003 (2020).
[23] B. Yoshida, Phys. Rev. B **88**, 125122 (2013).
[24] C. Castelnovo and C. Chamon, Philos. Mag. **92**, 304 (2012).
[25] I. H. Kim, arXiv:1202.0052.
[26] D. J. Williamson, Phys. Rev. B **94**, 155128 (2016).
[27] M. E. J. Newman and C. Moore, Phys. Rev. E **60**, 5068 (1999).
[28] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.128.115301 for a physical realization of the U(1) parent model, a derivation and discussion of the Pascal's triangle symmetries, a computation of the 3-point correlations in models with the fractal symmetries that we consider, and a related model of bosons in which the ordered phase is stable against the inclusion of quantum fluctuations, but which lacks the Pascal's triangle fractal symmetries.
[29] J. P. Garrahan and M. E. J. Newman, Phys. Rev. E **62**, 7670 (2000).
[30] J. P. Garrahan, J. Phys. Condens. Matter **14**, 1571 (2002).
[31] L. M. Vasiloiu, T. H. E. Oakes, F. Carollo, and J. P. Garrahan, Phys. Rev. E **101**, 042115 (2020).
[32] Z. Zhou, X.-F. Zhang, F. Pollmann, and Y. You, arXiv:2105.05851.
[33] N. E. Myerson-Jain, S. Yan, D. Weld, and C. Xu, Phys. Rev. Lett. **128**, 017601 (2022).
[34] D. S. Rokhsar and S. A. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988).
[35] E. Fradkin and S. Kivelson, Mod. Phys. Lett. B **04**, 225 (1990).
[36] C. Xu and A. Vishwanath (to be published).
[37] E. Fradkin, D. A. Huse, R. Moessner, V. Oganesyan, and S. L. Sondhi, Phys. Rev. B **69**, 224415 (2004).
[38] E. Ardonne, P. Fendley, and E. Fradkin, Ann. Phys. (Amsterdam) **310**, 493 (2004).
[39] A. Vishwanath, L. Balents, and T. Senthil, Phys. Rev. B **69**, 224416 (2004).