Uniqueness of Power Flow Solutions Using Graph-Theoretic Notions

Haixiang Zhang, SangWoo Park, Javad Lavaei, and Ross Baldick

***Abstract*—This article extends the uniqueness theory in (Park *et al.*, 2021) and establishes general necessary and sufficient conditions for the uniqueness of P–Θ power flow solutions in an AC power system using some properties of the monotone regime and the power network topology. We show that the necessary and sufficient conditions can lead to tighter sufficient conditions for the uniqueness in several special cases. Our results are based on the existing notion of maximal girth and our new notion of maximal eye. Moreover, we develop a series–parallel reduction method and search-based algorithms for computing the maximal eye and the maximal girth, which are necessary for the uniqueness analysis. Reduction to a single line using the proposed reduction method is guaranteed for 2-vertex-connected series–parallel graphs. The relations between the parameters of the network before and after reduction are obtained. It is verified on real-world networks that the computation of the maximal eye can be reduced to the analysis of a much smaller power network, while the maximal girth is computed during the reduction process.**

***Index Terms*—Graph theory, monotone operators, power systems, power flow analysis.**

I. Introduction

THE AC power flow problem plays a crucial role in various aspects of power systems, e.g., the daily operations in contingency analysis and security-constrained dispatch of electricity markets. In essence, the goal of the AC power flow problem is to solve for the complex voltage of each bus that determines the power system set point. However, the nonlinear nature of the AC power flow equations makes it difficult to analytically solve the equations, if not impossible. Moreover, the uniqueness of the AC power flow problem is not guaranteed, even when either voltage magnitudes or phase angle differences are limited to the "physically realizable" regime [1]–[4]. Hence, unexpected operating points may appear for some system conditions and can jeopardize the normal operations of power systems. Conditions that ensure the existence of a unique "physically realizable" power flow solution are important but not fully understood.

For a special case of the AC power flow problem, the uniqueness property of the real power–phase (P–Θ) power flow problem [5] has been studied in [1]. In the P–Θ power flow problem, the magnitude of the complex voltage at each node is given and the objective is to find a set of voltage phases such that the power flow equations are satisfied. The "physically realizable" constraint requires that the angular difference across every line lies within the stability limit of $\pi/2$ for lossless networks. Sufficient conditions (on the angular differences) that depend on the topological properties of the power network are established in [1]. Specifically, the authors proposed the notion of monotone regime and an upper bound on the angular differences based on the power network topology, which together can ensure the uniqueness of solutions. However, due to the nonlinear property of sinusoidal functions and the low-rank structure of angular differences, it is unclear to what extent the sufficient conditions given in [1] are necessary.

The goal of this article is to provide more general necessary and sufficient conditions for the uniqueness, using the notion of maximal eye defined in Section III and the notion of maximal girth introduced in [1]. This article also designs algorithms to compute these graph-theoretic parameters.

A. Main Results

In this article, we extend the uniqueness theory of the P–Θ power flow problem proposed in [1]. We focus on the uniqueness of the power flow problem in a stronger sense and derive general necessary and sufficient conditions that *depend only on the choice of the monotone regime and the network topology*. Under certain circumstances, the general conditions can be simplified to obtain tighter sufficient conditions. In addition, some algorithms for computing the maximal eye and the maximal girth of undirected graphs are proposed. A reduction method is designed to reduce the size of graphs and accelerate the computation process. More specifically, the contributions of this article are threefold.

1) We extend the uniqueness theory of the P–Θ problem to a stronger sense. The new uniqueness property is named strong uniqueness, and a constant called the maximal eye

Manuscript received July 9, 2021; revised November 8, 2021; accepted December 29, 2021. Date of publication January 25, 2022; date of current version May 26, 2022. The work of Haixiang Zhang, SangWoo Park, and Javad Lavaei was supported in part by the Army Research Office, in part by the Office of Naval Research, in part by the Air Force Office of Scientific Research, and in part by the National Science Foundation. The work of Ross Baldick was supported in part by the University of Texas Faculty Development Program. Recommended by Associate Editor Prabir Barooah. *(Corresponding author: Javad Lavaei.)*

Haixiang Zhang is with the Department of Mathematics, University of California, Berkeley, CA 94704 USA (e-mail: haixiang_zhang@berkeley.edu).

SangWoo Park and Javad Lavaei are with the Department of Industrial Engineering and Operations Research, University of California, Berkeley, CA 94704 USA (e-mail: spark111@berkeley.edu; lavaei@berkeley.edu).

Ross Baldick is with the Department of Electrical and Computer Engineering, University of Texas at Austin, Austin, TX 78712 USA (e-mail: baldick@ece.utexas.edu).

Digital Object Identifier 10.1109/TCNS.2022.3145749

is developed to classify all network topologies that ensure the strong uniqueness. Numerical results show that the maximal eye gives more reasonable conditions compared to its counterpart for the weak uniqueness defined in [1] and is known as the maximal girth.

2) We propose general necessary and sufficient conditions for both the strong and the weak uniqueness. The conditions are derived by Farkas' lemma, which are associated with the dual to the negation of the uniqueness problem. Sufficient conditions for the strong and the weak uniqueness are derived directly from the general conditions. In the special case when the power network is a single cycle or is lossless, necessary and sufficient conditions that do not contain sinusoidal functions are derived.
3) Finally, we develop a reduction method, named iterative series–parallel reduction (ISPR) method, that can accelerate the computation of the maximal eye and the maximal girth. The ISPR method is proved to reduce 2-vertex-connected series–parallel (SP) graphs to a single line, independent of the choice of the slack bus. The relationship between the maximal eye (girth) of graphs before and after the reduction is unveiled. When applying the ISPR method to real-world examples, the maximal eye is usually not changed over the reduction process, while the maximal girth is computed during the reduction process. We also design search-based algorithms for computing the maximal eye and the maximal girth, which are able to compute the exact value for graphs with up to 100 nodes before reduction in a reasonable amount of time.

In summary, this article constitutes a substantial generalization of the uniqueness theory in [1]. A stronger notion of uniqueness is proposed, and general necessary and sufficient conditions are proposed. These two in combination provide a tool for analyzing large-scale power networks and enable a deeper understanding of the uniqueness of the P–Θ power flow problem.

B. Related Work

The study of solutions to the power flow problem has a long history dating back to [6], which gave an example showing the general nonuniqueness of solutions for the power flow problem. Then, the number of solutions of the power flow problem was estimated in [7], which also characterized the stability region for the power flow problem. However, these early works only considered lossless transmission networks consisting of photovoltaic (PV) buses.

The fully coupled AC power flow equations are extremely difficult to analyze, and the theoretical results that can be obtained are often highly conservative or complicated to interpret. One approach to overcoming this difficulty is to study two decoupled power flow problems (the P–Θ problem and the reactive power–voltage (Q–V) problem) as in [5]. The intuition comes from the fact that the sensitivity of real power with respect to the change in angle differences outweighs the sensitivity with respect to the change in voltage magnitudes when angle differences are small and voltage magnitudes are close to 1 per unit (the opposite relationship holds for reactive power). This simplification should be differentiated from the DC approximations, which greatly simplifies the AC power flow equations by linearizing the equations and discarding all of the nonlinearities in the problem. Note that the P–Θ problem is still highly nonlinear. Under the assumption that resistive losses are negligible, conditions for the existence and uniqueness of both the real P–Θ problem and the reactive Q–V problem were derived in [5] and [8].

In another line of work, the topology structure of the power network was also considered to derive stronger conditions for the uniqueness. The number of solutions was estimated for radial networks in [2] and [9] and later for general networks. Moreover, a more recent work [10] gave several algorithms to compute the unique high-voltage solution. Delabays *et al.*[11] established upper bounds on the number of linearly stable fixed-point solutions for locally coupled Kuramoto models, which can be applied toward a lossless power flow problem. In this article, we consider the P–Θ problem [5] for general lossy power networks and utilize the topology information. We refer to [1] for a more detailed review of the existing literature.

The fixed-point technique is often used for proving the existence and uniqueness of equations. For the power flow problem, the fixed-point technique was first utilized in [12] and was further developed by several works [13]–[18]. Another more recently applied approach is to treat the P–Θ power flow problem as a rank-1 matrix sensing problem and solve its convex relaxation counterpart [19], [20]. Reference [21] also considered the domain of voltages over which the power flow operator is monotone. However, the relation between the rank-1-constrained problem and its convexification is not clear for general power networks.

Reference [22] presented a unifying framework for network problems on the n-torus. The framework applies to the AC power flow problem when the power networks are lossless. The idea of considering the regime when the power flow on each line is monotone was extended to lossy power networks in [1]. The regime where the power flow on a line increases monotonically with the angle difference across the line—called the monotone regime in this article—was proposed. In [1], it was also shown that the solution of the P–Θ problem is unique under the assumption that angle differences across the lines are bounded by some limit related to the maximal girth of the network, which is defined in [23]. We refer the reader to the survey paper [24] for an overview.

The existing algorithms in the literature cannot be directly used to compute the maximal eye (introduced in Section III) or the maximal girth. A related problem is computing the maximal chordless cycle as an upper bound to these parameters. The computation of maximal chordless cycles was proved to be $\mathcal{NP}$-complete in [25]. Efficient algorithms for enumerating chordless cycles were proposed in [26] and [27], and both take linear time to enumerate a single chordless cycle. The algorithms for enumerating maximal chordless cycles can be easily modified to compute the minimal chordless cycle containing a given edge. A series–parallel reduction (SPR) method was introduced as an alternative definition of generalized series–parallel (GSP) graphs in [28]. Under the assumption that the slack bus is the last bus to be reduced, all GSP graphs can be reduced to a single line [1]. However, whether the SPR method can still reduce GSP graphs without the assumption on the slack bus is not known. In this article, we show that 2-vertex-connected[1] SP graphs can be reduced to a single line without the assumption.

C. Notations

We start with some mathematical notations. We use $\mathbb{N}, \mathbb{Z}, \mathbb{R},$ and $\mathbb{C}$ to denote the set of all natural numbers, integers, real numbers, and complex numbers, respectively. We denote $[n] := \{1, \ldots, n\}$ for any $n \in \mathbb{N}$. The symbol j denotes the unit imaginary number. The notations $(\cdot)^T$ and $(\cdot)^H$ denote the transpose and Hermitian transpose of a matrix, respectively. For a complex number z, $|z|$ denotes its magnitude, and for a set X, the symbol $|X|$ denotes its cardinality. $\Re(\cdot)$ denotes the real part of a given scalar or matrix.

For an undirected graph, the set of vertices and the set of edges are denoted as $\mathbb{V}$ and $\mathbb{E}$, respectively. Suppose that the edges of an undirected graph are weighted with the weights captured by a matrix $W \in \mathbb{R}^{|\mathbb{V}|\times|\mathbb{V}|}$, where W_{ij} is the weight of edge $\{i, j\}$. Then, the graph is represented as $(\mathbb{V}, \mathbb{E}, W)$. For a directed graph $(\mathbb{V}, \mathbb{E}, A)$, the matrix $A \in \mathbb{R}^{|\mathbb{V}|\times|\mathbb{V}|}$ gives the orientation of each line, where $A_{ij} = 1$ (respectively, $A_{ij} = -1$) represents the direction $i \to j$ (respectively, $j \to i$). The undirected edge connecting two vertices k and ℓ is denoted by a set notation $\{k, \ell\}$, whereas (k, ℓ) denotes a directed edge coming out of vertex k and going into ℓ. For parallel edges, we use $\{k, \ell, i\}$ to represent different edges connecting k and ℓ, where $i \in \mathbb{Z}_+$ is the index of each parallel edge.

A power network $\mathbb{G} = (\mathbb{V}, \mathbb{E}, Y)$ consists of two parts: the underlying undirected graph $(\mathbb{V}, \mathbb{E})$ and the complex admittance matrix $Y \in \mathbb{C}^{n\times n}$, where n is the number of vertices in the underlying graph. The underlying graph is assumed to be a simple and connected graph. The set of vertices $\mathbb{V}$ and the set of edges $\mathbb{E}$ correspond to the set of buses and the set of lines of the power network. The series element of the equivalent Π-model of each line $\{k, \ell\}$ is modeled by admittance $Y_{k\ell} = G_{k\ell} - \mathrm{j}B_{k\ell}$, where $G_{k\ell}, B_{k\ell} \geq 0$.

We denote $\mathrm{v} \in \mathbb{C}^n$ as the vector of complex bus voltages. The complex voltage at bus k can be written in the polar form using its magnitude and phase angle $v_k = |v_k|e^{\mathrm{j}\Theta_k}$ for all $k \in [n]$, where $|v_k| \in \mathbb{R}$ and $\Theta_k \in \mathbb{R}$ denote the voltage magnitude and the phase angle, respectively. We denote $\Theta_{k\ell} := \Theta_k - \Theta_\ell \in [-\pi, \pi)$ as the phase difference modulus by 2π for all $\{k, \ell\} \in \mathbb{E}$. In the rest of this article, we use the corresponding values in $[-\pi, \pi)$ for phase differences.

D. Article Organization

The rest of this article is organized as follows. Section II gives the necessary background knowledge about the P–Θ power flow problem and the existing uniqueness theory for the P–Θ problem. The notions of strong uniqueness and weak uniqueness are also introduced. In Section III, we propose the general analysis framework of the uniqueness theory that only depends on the monotone regime and the topological structure. We show that necessary and sufficient conditions can be fully characterized by a feasibility problem, which has fewer variables than that of the P–Θ problem. Sufficient conditions for the uniqueness are derived, and it is shown that the uniqueness conditions in [1] follow as a natural corollary. Then, we consider three special cases in Section IV by assuming specific topological structures for the underlying graph or a specific monotone regime. In these special cases, the necessary and sufficient conditions are simplified and the intricate sinusoidal functions are avoided in the verification of those conditions. Furthermore, the sufficient conditions proposed in Section III are proved to be tight when no information beyond the monotone regime and the topological structure is available. Finally, a reduction method and search-based algorithms for computing the maximal girth and the maximal eye are given in Section V. We provide numerical illustrations in Section VI. Proofs are delineated in the technical report [29]. Finally, Section VII concludes this article.

[1]A graph is called 2-vertex-connected if it is connected after the deletion of any single vertex.

II. Preliminaries

A. P–Θ Problem Formulation

As mentioned in the introduction, we focus our attention to the P–Θ problem, which describes the relationship between the voltage phasor angles and the real power injections. We first make the following assumptions.

Assumption 1: The slack bus and the reference bus are bus 1. All other buses except the slack bus are PV buses.

Recall that the following injection operator describes the P–Θ problem, where the shunt elements are assumed to be purely reactive.

Definition 1: Given $\mathbb{G} = (\mathbb{V}, \mathbb{E}, Y)$, define $\hat{P}_k : \{0\} \times \mathbb{R}^{n-1} \to \mathbb{R}$ as the map from the vector of phasor angles to the real power injection at bus k:

$$\hat{P}_k(\Theta) := \Re\{(Yv)_k^H v_k\} \quad \forall\Theta \in \{0\} \times \mathbb{R}^{n-1}.$$

Moreover, define the injection operator $\hat{P} : \{0\} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ as

$$\hat{P}(\Theta) := [\hat{P}_2(\Theta), \ldots, \hat{P}_n(\Theta)].$$

The goal of the P–Θ problem is, given $P \in \mathbb{R}^{n-1}$, to find the voltage phasor angles $\Theta \in \{0\} \times \mathbb{R}^{n-1}$ such that

$$\hat{P}(\Theta) = P. \tag{1}$$

B. Monotone Regime and Allowable Sets

We are interested in the uniqueness property of the solution to problem (1). In general, the number of solutions to problem (1) is hard to estimate because of the periodic behavior of sinusoidal functions, especially when there is no symmetrical structure in the power network. Thus, we limit the phase angle vectors to the monotone regime, within which the real power flow from bus k to bus ℓ increases monotonically with respect to the phase difference $\Theta_{k\ell}$ for each line $\{k, \ell\} \in \mathbb{E}$. The monotone regime is defined in [1] as follows.

Definition 2: The *monotone regime* of a power network $(\mathbb{V}, \mathbb{E}, Y)$ is the set

$$\{\Theta \in \mathbb{R}^n \mid \Theta_1 = 0, \Theta_{k\ell} \in [-\gamma_{k\ell}, \gamma_{k\ell}] \, \forall \{k, \ell\} \in \mathbb{E}\},$$

where $\gamma_{k\ell} := \tan^{-1}(B_{k\ell}/G_{k\ell}) \in [0, \pi/2]$ for all $\{k, \ell\} \in \mathbb{E}$.

Due to the periodicity of sinusoidal functions, the solution to the P–Θ problem is trivially nonunique if there is no constraint on the phase angles. In this article, we consider the case when the voltage phase angles are within the monotone regime. It is noted in [30] that $\gamma_{k\ell}$ is generally larger than $2\pi/5$, while $\Theta_{k\ell}$ is rarely larger than $\pi/6$ due to stability and thermal limits. The constraint that the angular difference across every line lies within the stability limit of $[-\gamma_{k\ell}, \gamma_{k\ell}]$ is equivalent to the steady-state stability limit if each line is considered individually. As shown in [1], the phase angle vectors of leaf buses except the slack bus are uniquely determined by the phase angle vectors of nonleaf buses in the monotone regime. Hence, we assume that all vertices in the underlying graph except vertex 1 have degree at least 2.

Assumption 2: The graph $(\mathbb{V}, \mathbb{E})$ is connected. All vertices except vertex 1 in the graph $(\mathbb{V}, \mathbb{E})$ have degree at least 2.

We focus on finding a neighborhood of a solution, in which there is no other solution to the P–Θ problem. The neighborhood is defined as follows.

Definition 3: The set of *allowable perturbations* is defined as

$$\mathcal{W} := \{\omega_{k\ell} \geq 0 \mid \forall \{k, \ell\} \in \mathbb{E}\}.$$

Suppose that Θ is a solution to the P–Θ problem in the monotone regime. Then, the set of *neighboring phases* is defined as

$$\mathcal{N}(\mathbb{G}, \Theta, \mathcal{W}) := \{\tilde{\Theta} \in \mathbb{R}^n \mid \tilde{\Theta}_1 = 0,$$
$$\tilde{\Theta}_{k\ell} \in [-\gamma_{k\ell}, \gamma_{k\ell}] \cap [\Theta_{k\ell} - \omega_{k\ell}, \Theta_{k\ell} + \omega_{k\ell}] \, \forall \{k, \ell\} \in \mathbb{E}\}.$$

We note that $\tilde{\Theta}_{k\ell}$ refers to the value of $\tilde{\Theta}_k - \tilde{\Theta}_\ell$ modulo 2π.

Without loss of generality, we assume that $\omega_{k\ell} \leq 2\gamma_{k\ell}$ for all $\{k, \ell\} \in \mathbb{E}$, since the width of the monotone regime is $2\gamma_{k\ell}$, setting $\omega_{k\ell} > 2\gamma_{k\ell}$ will not enlarge the set of neighboring phases compared to setting $\omega_{k\ell} = 2\gamma_{k\ell}$.

Assumption 3: The perturbation width satisfies $\omega_{k\ell} \leq 2\gamma_{k\ell}$ for all $\{k, \ell\} \in \mathbb{E}$.

It is desirable to analyze the uniqueness of the solution in the neighborhood $\mathcal{N}(\mathbb{G}, \Theta, \mathcal{W})$. Park *et al.* [1] considered the *set of allowable angles*, which is defined as

$$\{\tilde{\Theta} \in \mathbb{R}^n | \tilde{\Theta}_1 = 0, \tilde{\Theta}_{k\ell} \in [-\omega_{k\ell}/2, \omega_{k\ell}/2] \, \forall \{k, \ell\} \in \mathbb{E}\}.$$

Note that the set of allowable angles is a special case of the set of allowable perturbations, since any two phase vectors in the set of allowable angles are in the corresponding sets of neighboring phases of each other. In this article, we use the *set of allowable perturbations*, but the sufficient conditions we derive can be naturally applied to using the *set of allowable angles*.

C. Notions of Weak and Strong Uniqueness

Informally, we say that the P–Θ problem (1) has a unique solution Θ under the allowable perturbation set $\mathcal{W}$, if there exists at most one solution in the set $\mathcal{N}(\mathbb{G}, \Theta, \mathcal{W})$. We give two different definitions of the uniqueness. First, we introduce the uniqueness in the weak sense.

Definition 4: We say that a solution Θ to the P–Θ problem (1) is weakly unique with the given set of allowable perturbations $\mathcal{W}$ if, for any solution $\tilde{\Theta} \in \mathcal{N}(\mathbb{G}, \Theta, \mathcal{W})$, there exists a line $\{k, \ell\} \in \mathbb{E}$ such that $\Theta_{k\ell} = \tilde{\Theta}_{k\ell}$.

In other words, two solutions are different according to Definition 4 if and only if they have different phase differences for every line. Next, we extend the definition of the weak uniqueness to a stronger sense that is also more useful and usual.

Definition 5: We say that a solution Θ to the P–Θ problem (1) is strongly unique with the given set of allowable perturbations $\mathcal{W}$ if, for any solution $\tilde{\Theta} \in \mathcal{N}(\mathbb{G}, \Theta, \mathcal{W})$ and any $\{k, \ell\} \in \mathbb{E}$, we have $\Theta_{k\ell} = \tilde{\Theta}_{k\ell}$.

In other words, two solutions are different according to Definition 5 if and only if the phase differences are different on at least one line.

III. Uniqueness Theory for General Graphs

In this section, we derive necessary and sufficient conditions on the set of allowable perturbations $\mathcal{W}$ such that the solution to problem (1) becomes strongly or weakly unique. In particular, we aim to analyze *the impact of the power system topology and the size of the monotone regime* on the uniqueness property. Namely, given the topological structure and the monotone regime, we aim to find conditions on $\mathcal{W}$ such that the uniqueness of solutions holds. To achieve this, we need to derive conditions under which all power networks with the same topological structure and monotone regime have unique solutions. To formalize the problem, we fix the underlying graph $(\mathbb{V}, \mathbb{E})$ and the angles specifying the monotone regime $\Gamma := \{\gamma_{k\ell} \in (0, \pi/2] \mid \{k, \ell\} \in \mathbb{E}\}$. We define the set of possible admittances with the same monotone regime as

$$\mathcal{S}(\gamma) := \{(C\cos(\gamma), C\sin(\gamma)) \mid C > 0\} \quad \forall \gamma \in [0, \pi/2].$$

The set of complex admittance matrices with the same monotone regime is defined as

$$\mathcal{Y}(\mathbb{V}, \mathbb{E}, \Gamma) := \{Y \text{ is an admittance matrix} \mid$$
$$Y_{k\ell} = G_{k\ell} - \mathbf{j}B_{k\ell}, (G_{k\ell}, B_{k\ell}) \in \mathcal{S}(\gamma_{k\ell}), \{k, \ell\} \in \mathbb{E}\}.$$

Then, we define the set of power networks with the same topological structure and the same monotone regime as

$$\mathcal{G}(\mathbb{V}, \mathbb{E}, \Gamma) := \{\mathbb{G} = (\mathbb{V}, \mathbb{E}, Y) \mid Y \in \mathcal{Y}(\mathbb{V}, \mathbb{E}, \Gamma)\}$$

or simply $\mathcal{G}$ if there is no confusion about $\mathbb{V}$, $\mathbb{E}$, and Γ. Hence, the problem under study in this article can be stated as follows.

1) What are the necessary and sufficient conditions on the allowable perturbations $\mathcal{W}$ such that the solution to problem (1) is unique within the set of allowable perturbations for any power network $\mathbb{G} \in \mathcal{G}$?

The necessary and sufficient conditions provide two sides on the uniqueness theory. The sufficient conditions give a guarantee for the uniqueness of solutions for any single power network with the given topological structure and monotone regime, while the necessary conditions bound the optimal conditions we can derive only using the knowledge of the topological structure and the monotone regime. We first give an equivalent characterization of strong and weak uniqueness.

Lemma 1 (Necessary and sufficient conditions for uniqueness): Given the set of power networks $\mathcal{G}(\mathbb{V},\mathbb{E},\Gamma)$ and the set of allowable perturbations $\mathcal{W}$, the following two statements are equivalent.

1) For any power network $\mathbb{G} \in \mathcal{G}(\mathbb{V},\mathbb{E},\Gamma)$ and any power injection $P \in \mathbb{R}^{|\mathbb{V}|-1}$ such that problem (1) is feasible in the monotone regime, the solution to problem (1) in the monotone regime is strongly unique in $\mathcal{N}(\mathbb{G},\Theta,\mathcal{W})$.
2) For any power network $\mathbb{G} \in \mathcal{G}(\mathbb{V},\mathbb{E},\Gamma)$ and any two phase angle vectors Θ^1, Θ^2 in the monotone regime with the property $\Theta^2 \in \mathcal{N}(\mathbb{G},\Theta^1,\mathcal{W})$, there exists a vector $y \in \mathbb{R}^{|\mathbb{V}|}$ such that $y_1 = 0$ and

$$\sin(\gamma_{k\ell} + \Theta^1_{k\ell}/2 + \Theta^2_{k\ell}/2) \cdot y_k \geq \sin(\gamma_{k\ell} - \Theta^1_{k\ell}/2 - \Theta^2_{k\ell}/2) \cdot y_\ell \quad \forall \{k,\ell\} \in \mathbb{E} \ \text{ s.t. } \Theta^1_{k\ell} - \Theta^2_{k\ell} > 0, \tag{2}$$

where at least one of the inequalities above is strict.

The equivalence between statements 1 and 2 still holds true even after replacing the strong uniqueness with the weak uniqueness in statement 1, provided that the phase angle vector Θ^2 in statement 2 is required to satisfy $\Theta^1_{k\ell} \neq \Theta^2_{k\ell}$ for all $\{k,\ell\} \in \mathbb{E}$.

Intuitively, the above lemma studies the uniqueness of solutions through its dual form. The existence of multiple solutions can be formulated as a linear feasibility problem. Then, the strong duality of linear programming allows us to equivalently consider the dual of the feasibility problem. The dual form is preferred since the dual problem has fewer variables and its solution is easier to construct. We, then, derive several sufficient conditions using Lemma 1. We first show that we only need to verify statement 2 in Lemma 1 for two phase angle vectors Θ^1 and Θ^2 that induce a (weakly) feasible orientation, which we will define below. We define the orientation induced by two phase angle vectors.

Definition 6: Suppose that Θ^1 and Θ^2 are two phase angle vectors of the graph. Then, we define the induced orientation of $\Delta := \Theta^1 - \Theta^2$ as $A_{k\ell} := \mathrm{sign}(\Delta_{k\ell})$ for all $\{k,\ell\} \in \mathbb{E}$, where the sign function $\mathrm{sign}(\cdot)$ is defined as

$$\mathrm{sign}(x) := \begin{cases} +1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}.$$

In the definition of induced orientations, we assign one of the three directions $+1, -1$, and 0 to each edge. The first two directions are "normal" directions for directed graphs. An edge with direction $+1$ or -1 is called a normal edge. Edges with direction 0 are viewed as an undirected edge and reachable in both directions. In addition, edges with direction 0 are not considered when computing the in-degree and the out-degree. We only need to consider orientations induced by two different phase angle vectors $\Theta^1 - \Theta^2$ such that $\hat{P}(\Theta^1) = \hat{P}(\Theta^2)$. However, a precise characterization of those orientations is difficult, and we consider a larger set that contains those orientations.

Definition 7: An orientation assigned to an undirected graph is called a *feasible orientation* if all edges are normal and each vertex except vertex 1 has nonzero in-degree and out-degree.

According to the analysis in [1], the induced orientation of two solutions Θ^1 and Θ^2 in the monotone regime that are different according to Definition 4 must be a feasible orientation. Then, we give the definition of weakly feasible orientations as the counterpart for the strong uniqueness.

Definition 8: An orientation assigned to an undirected graph is called a *weakly feasible orientation* if two properties are satisfied: 1) there exists at least one normal edge and 2) the in-degree and the out-degree of any vertex except vertex 1 are both zero or both nonzero.

Edges with direction 0 are lines with the same angular difference for the two phase angle vectors Θ^1 and Θ^2. By the same discussion as in Section II, we can view a weakly feasible orientation as a feasible orientation for the subgraph that only has normal edges. The next lemma shows that we only need to consider weakly feasible orientations or feasible orientations when checking the conditions in statement 2 of Lemma 1.

Lemma 2: If two different phase angle vectors Θ^1 and Θ^2 in the monotone regime satisfy $\Theta^2 \in \mathcal{N}(\mathbb{G},\Theta^1,\mathcal{W})$ and the induced orientation of $\Theta^1 - \Theta^2$ is not weakly feasible, then there exists a vector $y \in \mathbb{R}^{|\mathbb{V}|}$ such that statement 2 of Lemma 1 holds. The result holds true for the weak uniqueness property as well, provided that the induced orientation of $\Theta^1 - \Theta^2$ is not a feasible orientation.

Combining Lemmas 1 and 2, we obtain sufficient conditions for the strong uniqueness and the weak uniqueness.

Theorem 3 (Sufficient conditions for uniqueness): Given the set of allowable perturbations $\mathcal{W}$, suppose that for any two different phase angle vectors $\Theta^1 - \Theta^2$ in the monotone regime satisfying $\Theta_2 \in \mathcal{N}(\mathbb{G},\Theta^1,\mathcal{W})$, the induced orientation of $\Theta^1 - \Theta^2$ is not a weakly feasible orientation. Then, the solution to problem (1) is strongly unique for all power networks in $\mathcal{G}$. The result holds true for the weak uniqueness as well, provided that the induced orientation of $\Theta^1 - \Theta^2$ is not a feasible orientation.

The sufficient condition given above is a generalization of [1, Th. 4], which ensures the weak uniqueness of solutions in the set of allowable phases. Using Theorem 3, we can derive a corollary similar to [1, Th. 4].

Corollary 4: Consider an arbitrary set of allowable perturbations $\mathcal{W}$. The solution to problem (1) in the monotone regime is strongly unique for any power network $\mathbb{G} \in \mathcal{G}$ if, for any weakly feasible orientation of the underlying graph $(\mathbb{V},\mathbb{E})$, there exists a directed cycle $(k_1,\ldots,k_t)$ containing at least one normal edge such that the allowable perturbations satisfy the inequality

$$\sum_{\{k_i,k_{i+1}\} \text{ is normal}} \omega_{k_i k_{i+1}} < 2\pi,$$

where $k_{t+1} := k_1$. The same result holds true for the weak uniqueness if we substitute weakly feasible orientations with feasible orientations.

Now, we consider a special case where all constants $\omega_{k\ell}$ in the set of allowable perturbations are equal, i.e., there exists a constant $\omega \geq 0$ such that the set of allowable perturbation is

$$\mathcal{W}_\omega := \{\omega_{k\ell} = \omega \, \forall \{k,\ell\} \in \mathbb{E}\}.$$

The problem we consider in this case is as follows.

2) What is the sufficient condition on ω such that the solution to problem (1) is unique with the allowable perturbation set $\mathcal{W}_\omega$?

We derive an upper bound on the constant ω to guarantee the uniqueness. We first define the maximal eye and the maximal girth of an undirected graph.

Definition 9: Consider an undirected graph $(\mathbb{V}, \mathbb{E})$. For any weakly feasible orientation assigned to the graph $(\mathbb{V}, \mathbb{E})$, we define the minimal length of directed cycles that contain at least one normal edge as the *size of eye* of this orientation, where edges with direction 0 are considered as bidirectional edges. We define the *maximal eye* of the graph $(\mathbb{V}, \mathbb{E})$ as the maximum of the size of eye over all possible weakly feasible orientations. We denote the maximal eyes of the graph $(\mathbb{V}, \mathbb{E})$, a power network $\mathbb{G}$, and a group of power networks $\mathcal{G}$ as $\mathrm{e}(\mathbb{V}, \mathbb{E})$, $\mathrm{e}(\mathbb{G})$, and $\mathrm{e}(\mathcal{G})$, respectively.

Remark 1: There always exists a directed cycle containing normal edges when the underlying graph is under a weakly feasible orientation. To understand this, we first choose an arbitrary normal edge $(k_1, k_2) \in \mathbb{E}$. Since the vertex k_2 has nonzero in-degree, it also has nonzero out-degree. Hence, there exists another vertex k_3 such that $(k_2, k_3) \in \mathbb{E}$. Continuing this procedure will result in the existence of a vertex k_t such that $v_t = k_s$ for some $s < t$. This generates a directed cycle $(k_s, k_{s+1}, \ldots, k_{t-1})$ containing only normal edges. Hence, the size of eye is well defined.

The counterpart of the maximal eye, known as the maximal girth, is defined in [1], and we restate the definition as follows.

Definition 10: Consider an undirected graph $(\mathbb{V}, \mathbb{E})$. For any feasible orientation assigned to the underlying graph $(\mathbb{V}, \mathbb{E})$, we define the minimal size of directed cycles as the *girth* of this feasible orientation. We define the *maximal girth* of the graph $(\mathbb{V}, \mathbb{E})$ as the maximum of the girth over all feasible orientations. We denote the maximal girths of the graph $(\mathbb{V}, \mathbb{E})$, a power network $\mathbb{G}$, and a group of power networks $\mathcal{G}$ as $\mathrm{g}(\mathbb{V}, \mathbb{E})$, $\mathrm{g}(\mathbb{G})$, and $\mathrm{g}(\mathcal{G})$, respectively.

Remark 2: Similar to the discussion in Remark 1, there exists at least one directed cycle when the graph is under a feasible orientation. The maximal eye can be equivalently defined as the maximum of the maximal girth over all subgraphs that do not have degree-1 vertices.

We provide an upper bound for ω using the maximal eye and the maximal girth, which follows from Corollary 4.

Corollary 5: If the inequality

$$\omega_{k\ell} < \frac{2\pi}{\mathrm{e}(\mathcal{G})} \quad \forall \{k,\ell\} \in \mathbb{E} \tag{3}$$

is satisfied, then the solution to problem (1) in the monotone regime is strongly unique for any power network $\mathbb{G} \in \mathcal{G}$. The same result holds true for the weak uniqueness, provided that $\mathrm{e}(\mathcal{G})$ in (3) is substituted by $\mathrm{g}(\mathcal{G})$.

In Section V, we design search-based algorithms to calculate the maximal eye and the maximal girth. However, computing the maximal eye or the maximal girth is challenging for graphs with more than 100 nodes. Hence, we seek upper bounds and lower bounds for the maximal eye and the maximal girth. In this article, we obtain a simple upper bound for both the maximal girth and the maximal eye. We define $\kappa(\mathbb{G})$ and $\kappa(\mathcal{G})$ as the sizes of the longest chordless cycles of the underlying graph of the power network $\mathbb{G}$ and any power network in the power network class $\mathcal{G}$, respectively. The upper bound on the maximal girth and the maximal eye will be provided in the following.

Theorem 6: For any power network $\mathbb{G}$, it holds that

$$\mathrm{g}(\mathbb{G}) \leq \mathrm{e}(\mathbb{G}) \leq \kappa(\mathbb{G}) \tag{4}$$

and that $\mathrm{g}(\mathcal{G}) \leq \mathrm{e}(\mathcal{G}) \leq \kappa(\mathcal{G})$.

We note that although computing the longest chordless cycle is $\mathcal{NP}$-complete [25], the computation of the longest chordless cycle is faster than the computation of the maximal eye and the maximal girth in practice.

IV. Uniqueness Theory for Three Special Cases

In this section, we consider three special cases. For each case, the power network has either a special topological structure or a special monotone regime. In the first two cases, the underlying graph of the power network is a single cycle or a 2-vertex-connected SP graph. When the underlying graph is a single cycle, the sufficient conditions in Corollary 4 are also necessary. If the underlying graph is a 2-vertex-connected SP graph, we prove that the sufficient conditions for the weak uniqueness in Corollary 5 also ensure the strong uniqueness. In the last case, the power network is assumed to be lossless. In this case, the monotone regime of each line reaches the maximum possible size $[-\pi/2, \pi/2]$. Sinusoidal functions can, then, be avoided in statement 2 of Lemma 1, and therefore, the verification of conditions is easier.

A. Single Cycles

We first consider the case when the underlying graph $(\mathbb{V}, \mathbb{E})$ is a single cycle. We first show that the weak uniqueness is equivalent to the strong uniqueness in this case.

Lemma 7: Suppose that the underlying graph is a single cycle with the edges $(1,2), (2,3), \ldots, (n,1)$. Then, given the set of allowable perturbations $\mathcal{W}$, the solution to problem (1) in the monotone regime is weakly unique if and only if it is strongly unique.

Next, we prove that the sufficient conditions derived in Corollary 4 are also necessary for a single cycle with the nontrivial monotone regime.

Theorem 8: Suppose that the underlying graph is a single cycle with the edges $(1,2), (2,3), \ldots, (n,1)$, and that the set of allowable perturbations satisfies $0 < \omega_{i,i+1} \leq \gamma_{i,i+1}$ for all $i \in [n]$, where $\gamma_{n,n+1} := \gamma_{n,1}$ and $\omega_{n,n+1} := \omega_{n,1}$. The solution to problem (1) in the monotone regime is strongly unique for any power network $\mathbb{G} \in \mathcal{G}(\mathbb{V}, \mathbb{E}, \Gamma)$ and any power injection $P \in \mathbb{R}^{n-1}$ that makes problem (1) feasible *if and only if* the set of allowable perturbations $\mathcal{W}$ satisfies

$$\sum_{i=1}^{n} \omega_{i,i+1} < 2\pi$$

where $\omega_{n,n+1} := \omega_{n,1}$.

In contrast to requiring $\omega_{i,i+1} > 0$ in the above theorem, the condition that $\omega_{i,i+1} = 0$ for some i is sufficient but not necessary for the uniqueness of solutions. Under this condition, two solutions Θ^1 and Θ^2 in the monotone regime such that $\Theta^2 \in \mathcal{N}(\mathbb{G}, \Theta^1, \mathcal{W})$ must satisfy $\Theta^1_{i,i+1} = \Theta^2_{i,i+1}$. Hence, any solution is strongly unique with this set of allowable perturbations. However, by Theorem 8, this condition is not necessary for the uniqueness of solutions.

B. SP Graphs

In this subsection, we consider another special class of graphs, namely, the 2-vertex-connected SP graphs. The objective is to find an upper bound on the constant ω to guarantee that the solution to problem (1) is unique. Corollary 5 shows that the solution is strongly unique if $\omega < 2\pi/\mathrm{e}(\mathbb{G})$ and is weakly unique if $\omega < 2\pi/\mathrm{g}(\mathbb{G})$. However, for a 2-vertex-connected SP graph, we can prove a stronger theorem. We first prove that the maximal eye is equal to the maximal girth for a 2-vertex-connected SP graph. The main tool is the ear decomposition of an undirected graph [31].

Definition 11: An *ear* of an undirected graph $(\mathbb{V}, \mathbb{E})$ is a simple path or a single cycle. An *ear decomposition* of an undirected graph $(\mathbb{V}, \mathbb{E})$, denoted as $\mathcal{D} := (L_0, \ldots, L_{r-1})$, is a partition of $\mathbb{E}$ into an ordered sequence of ears such that one or two endpoints of each ear L_k are contained in an earlier ear, i.e., an ear L_ℓ with $\ell < k$, and the internal vertices of each ear do not belong to any earlier ear. We call $\mathcal{D}$ a *proper ear decomposition* if each ear L_k is a simple path for all $k = 1, \ldots, r-1$. A *tree ear decomposition* is a proper ear decomposition in which the first ear is a single edge, and for each subsequent ear L_k, there is a single ear L_ℓ with $\ell < k$, such that both endpoints of L_k lie on L_ℓ. A *nested ear decomposition* is a tree ear decomposition such that, within each ear L_ℓ, the set of pairs of endpoints of other ears L_k that lie within L_ℓ forms a set of nested intervals.

The following theorem in [32] provides an equivalent characterization of 2-vertex-connected SP graphs through the ear decomposition.

Theorem 9: A 2-vertex-connected graph is SP if and only if it has a nested ear decomposition.

With the help of the nested ear decomposition, we will prove that the maximal girth is equal to the maximal eye for 2-vertex-connected SP graphs. The intuition behind the proof is that we first choose two vertices as the "source" and the "sink" for the power flow network. For each edge with direction 0, we first consider the directed path that contains this edge and goes from the "source" to the "sink" and, then, assign a normal direction (± 1) to this edge according to the directed path. This step ensures that the first inequality in (4) holds as equality.

Lemma 10: Suppose that $(\mathbb{V}, \mathbb{E})$ is a 2-vertex-connected SP graph. Then, the following equality holds true:

$$\mathrm{g}(\mathbb{V}, \mathbb{E}) = \mathrm{e}(\mathbb{V}, \mathbb{E}).$$

Therefore, combining the above lemma with Corollary 5, we obtain a stronger sufficient condition for 2-vertex-connected SP graphs. This result implies that the sufficient conditions for the weak uniqueness in Corollary 5 also guarantee the strong uniqueness.

Theorem 11: Suppose that the underlying graph $(\mathbb{V}, \mathbb{E})$ is a 2-vertex-connected SP graph. The solution to problem (1) is strongly unique for any power network $\mathbb{G} \in \mathcal{G}$ in the monotone regime if

$$\omega < \frac{2\pi}{\mathrm{g}(\mathcal{G})}.$$

C. Lossless Networks

Finally, we consider the case when the power network is lossless, namely, when $\gamma_{k\ell} = \pi/2$ for all $\{k, \ell\} \in \mathbb{E}$. In this case, we prove that the strong uniqueness holds if and only if there does not exist another solution in the set of neighboring phases such that the induced orientation has strictly more strongly connected components than weakly connected components. This result makes it possible to avoid nonlinear sinusoidal functions in statement 2 of Lemma 1, and therefore, the uniqueness of solutions becomes easier to verify. We first define the subgraph induced by two phase angle vectors.

Definition 12: Suppose that Θ^1 and Θ^2 are two different phase angle vectors, and that the orientation A is the induced orientation of $\Theta^1 - \Theta^2$. Then, the *induced subgraph* of $\Theta^1 - \Theta^2$ is constructed as a directed subgraph of $(\mathbb{V}, \mathbb{E}, A)$ by first deleting all edges with direction 0 and then deleting all degree-1 vertices.

In what follows, we establish a necessary and sufficient condition for the uniqueness of the solution that does not contain sinusoidal functions.

Theorem 12: Consider the set of allowable perturbations $\mathcal{W}$. If the monotone regime satisfies $\gamma_{k\ell} = \pi/2$ for all $\{k, \ell\} \in \mathbb{E}$, then the following two statements are equivalent.

1) For any power network $\mathbb{G} \in \mathcal{G}(\mathbb{V}, \mathbb{E}, \Gamma)$ and any power injection $P \in \mathbb{R}^{|\mathbb{V}|-1}$ such that problem (1) is feasible, the solution to problem (1) in the monotone regime is strongly unique in $\mathcal{N}(\mathbb{G}, \Theta, \mathcal{W})$.
2) For any power network $\mathbb{G} \in \mathcal{G}(\mathbb{V}, \mathbb{E}, \Gamma)$ and any two phase angle vectors Θ^1 and Θ^2 in the monotone regime with the property $\Theta^2 \in \mathcal{N}(\mathbb{G}, \Theta^1, \mathcal{W})$, the induced subgraph of $\Theta^1 - \Theta^2$ has strictly more strongly connected components than weakly connected components.

The equivalence between statements 1 and 2 still holds true even after replacing the strong uniqueness with the weak uniqueness in statement 1, provided that the phase angle vectors Θ^2 in statement 2 is required to satisfy $\Theta^1_{k\ell} \neq \Theta^2_{k\ell}$ for all $\{k, \ell\} \in \mathbb{E}$.

The result of the above theorem is stronger than the sufficient conditions in Theorem 3. This is because any (weakly) *infeasible* orientation has strictly more strongly connected components than weakly connected components. Hence, the sufficient conditions in Theorem 3 ensure that all induced orientations are (weakly) *infeasible*. Then, statement 2 of this theorem holds true, and the solution becomes strongly (weakly) unique.

V. Iterative Series–Parallel Reduction

In the preceding sections, we have shown that the maximal eye and the maximal girth play important roles in the uniqueness theory. However, computing the maximal eye or the maximal girth is cumbersome for large graphs. Hence, we develop an iterative reduction method to design a reduced graph and, then, prove the relationship between the maximal eye or the maximal girth of the original graph and those of the reduced graph. Next, we test the performance of those algorithms on real-world problems. Search-based algorithms for computing the maximal eye and the maximal girth are given in [29].

A. ISPR Method

In this subsection, we propose an iterative reduction method, named as the ISPR method, that can reduce the size of the underlying graph for computing the maximal eye and the maximal girth. The ISPR method is different from the SPR method introduced in [1] in two aspects. First, the purpose of the ISPR method is to accelerate the computation of the maximal eye and the maximal girth, while the focus of the SPR method is to facilitate the verification of uniqueness conditions. Second, we prove that all 2-vertex-connected SP graphs can be reduced to a single edge (K_2) without the assumption in [1] that the slack bus is the last to be reduced.

Before introducing the ISPR method, we extend the definition of the maximal eye and the maximal girth to weighted graphs with "multiple slack buses." This generalized class of graphs appear during the reduction process. By defining the length of a cycle as the sum of the weights of the edges on the cycle, the maximal eye and the maximal girth can be generalized to weighted graphs. Next, we define (weakly) feasible orientations for graphs with "multiple slack buses," namely, the slack nodes.

Definition 13: For a weighted undirected graph $(\mathbb{V}, \mathbb{E}, W)$, a subset of vertices $\mathbb{V}_s \subseteq \mathbb{V}$ is called the set of *slack nodes*. An orientation A assigned to the graph is called a *weakly feasible orientation* if each edge has one of the directions $\{+1, -1, 0\}$ and each vertex not in $\mathbb{V}_s$ either has nonzero in-degree and nonzero out-degree or has zero in-degree and zero out-degree. An orientation A assigned to the graph is called a *feasible orientation* if each edge has one of the directions $\{+1, -1\}$ and each vertex not in $\mathbb{V}_s$ has nonzero in-degree and nonzero out-degree.

Now, we can define the maximal eye for graphs with slack nodes by taking the maximum of the size of eye over weakly feasible orientations. The maximal girth can be defined in a similar way. For power networks, the only slack node is the slack bus of the power network. Hence, the extended definitions of the maximal eye and the maximal girth are consistent with their original definitions. The ISPR method is based on three types of operation.

1) *Type I Operation:* Replacement of a set of parallel edges with a single edge that connects their common endpoints. The weight of the new single edge is the minimum over the weights of the deleted parallel edges.
2) *Type II Operation:* Replacement of the two edges incident to a degree-2 vertex with a single edge, if the vertex has exactly two neighboring vertices and is not a slack node. The weight of the new edge is the sum of the weights of the two deleted edges.
3) *Type III Operation:* Deletion of a vertex that has only a single neighboring vertex. If the deleted vertex is a slack node, or if the deleted vertex has degree at least 2 for the problem of computing the maximal girth, then we define its neighboring vertex as a slack node.

Algorithm 1: ISPR Method.

Input: Undirected unweighted graph $(\mathbb{V}, \mathbb{E})$, slack bus k
Output: Reduced undirected weighted graph $(\mathbb{V}_R, \mathbb{E}_R, W_R)$, two constants α_1, α_2 defined in Theorems 14 and 16, set of slack nodes $\mathbb{V}_s$
Set the initial weight for each edge to be 1.
Set the initial set of slack nodes as $\mathbb{V}_s \leftarrow \{k\}$.
while at least one operation is implementable **do**
  if Type I Operations are implementable **then**
    Implement Type I Operation.
    Update values α_1 and α_2 according to their definitions in Theorems 14 and 16.
    continue
  end if
  if Type II Operations are implementable **then**
    Implement Type II Operation.
    continue
  end if
  if Type III Operations are implementable **then**
    Implement Type III Operation.
    Update values α_1 and α_2 according to their definitions in Theorems 14 and 16.
    Update the set of slack nodes $\mathbb{V}_s$.
    continue
  end if
end while
Return reduced graph $(\mathbb{V}_R, \mathbb{E}_R, W_R)$, set of slack nodes $\mathbb{V}_s$, and values α_1 and α_2.

The update scheme of weights and slack nodes is designed to control the change of the maximal eye or the maximal girth. The ISPR method successively reduces the size of the graph by applying Type I–III Operations; the pseudocode of the ISPR method is given in Algorithm 1.

We note that after the reduction process, there is no parallel edge or pendant (degree-1) vertex in the reduced graph. Ignoring the weights of the edges and the set of the slack nodes, the operations in the ISPR method can cover the operations in the classical SPR [28], which are as follows.

1) *Type I′ operation:* Replacement of parallel edges with a single edge that connects their common endpoints.
2) *Type II′ operation:* Replacement of the two edges incident to a degree-2 vertex with a single edge.

3) *Type III′ operation:* Deletion of a pendant vertex.

Hence, the ISPR method can be viewed as a generalization of the classical SPR. We first consider the change of the maximal eye after each operation.

Lemma 13: Given a weighted undirected graph $(\mathbb{V}, \mathbb{E}, W)$, let e denote its maximal eye. Assume that one of Type I–III Operations is implemented on the graph. By denoting the new graph and its maximal eye as $(\tilde{\mathbb{V}}, \tilde{\mathbb{E}}, \tilde{W})$ and $\tilde{\mathrm{e}}$, the following statements hold.

1) If Type I Operation is implemented, then

$$\tilde{\mathrm{e}} \le \mathrm{e} \le \max\{\tilde{\mathrm{e}}, W_{\max} + W_{\min}\},$$

where $W_{\max}$ and $W_{\min}$ are the maximal and minimal weights of the deleted parallel edges, respectively.

2) If Type II Operation is implemented, then $\mathrm{e} = \tilde{\mathrm{e}}$.

3) If Type III Operation is implemented and the deleted vertex has degree 1, then $\mathrm{e} = \tilde{\mathrm{e}}$.

4) If Type III Operation is implemented and the deleted vertex has degree larger than 1, then

$$\mathrm{e} = \max\{\tilde{\mathrm{e}}, W_{\max} + W_{\min}\},$$

where $W_{\max}$ and $W_{\min}$ are the maximal and minimal weights of the deleted parallel edges, respectively.

Using the above lemma, we have the following theorem.

Theorem 14: Given a power network with the underlying graph $(\mathbb{V}, \mathbb{E})$, let e denote the maximal eye of the graph. Denote the graph after reduction and its maximal eye as $(\mathbb{V}_R, \mathbb{E}_R, W_R)$ and e_R, respectively. Then, we have

$$\max\{\mathrm{e}_R, \alpha_2\} \le \mathrm{e} \le \max\{\mathrm{e}_R, \alpha_1, \alpha_2\},$$

where α_1 and α_2 are the maximum of $W_{\max} + W_{\min}$ over Type I and III Operations, respectively. Here, $W_{\max}$ and $W_{\min}$ are defined in Lemma 13. If Type I or III Operations is never implemented, then we set α_1 or α_2 to 0.

Similarly, we can prove the relation between the maximal girth of the original graph and that of the reduced graph. We first show the change of the maximal girth after each operation.

Lemma 15: Given a weighted undirected graph $(\mathbb{V}, \mathbb{E}, W)$, let g denote its maximal girth. Assume that one of Type I–III Operations is implemented on the graph. By denoting the new graph and its maximal girth of new graph as $(\tilde{\mathbb{V}}, \tilde{\mathbb{E}}, \tilde{W})$ and $\tilde{\mathrm{g}}$, the following statements hold.

1) If Type I Operation is implemented, then

$$\tilde{\mathrm{g}} \le \mathrm{g} \le \max\{\tilde{\mathrm{g}}, W_{\max} + W_{\min}\},$$

where $W_{\max}$ and $W_{\min}$ are the maximal and minimal weights of the deleted parallel edges, respectively.

2) If Type II Operation is implemented, then $\mathrm{g} = \tilde{\mathrm{g}}$.

3) If Type III Operation is implemented and the deleted vertex has degree 1, then $\mathrm{g} = \tilde{\mathrm{g}}$.

4) If Type III Operation is implemented, the deleted vertex is a slack node and has degree larger than 1, then

$$\tilde{\mathrm{g}} \le \mathrm{g} \le \max\{\tilde{\mathrm{g}}, W_{\max} + W_{\min}\},$$

where $W_{\max}$ and $W_{\min}$ are the maximal and minimal weights of the deleted parallel edges, respectively.

TABLE I

NUMBER OF VERTICES AND EDGES BEFORE AND AFTER THE ISPR METHOD FOR MAXIMAL EYE ALONG WITH VALUES COMPUTED DURING THE REDUCTION PROCESS

Power Network	Original Size	Reduced Size	α_1	α_2	e_R
Case 14	(14,20)	(2,1)	6	3	0
Case 30	(30,41)	(8,13)	4	3	8
Case 39	(39,46)	(8,12)	4	5	8
Case 57	(57,78)	(22,39)	4	-	23
Case 118	(118,179)	(44,83)	5	-	13
Case 300	(300,409)	(109,196)	8	4	≥10
Case 1354	(1354,1710)	(263,500)	9	8	TLE
Case 2383	(2383,2886)	(499,949)	11	5	TLE

5) If Type III Operation is implemented, the deleted vertex is not a slack node and has degree larger than 1, then

$$\mathrm{g} = \min\{\tilde{\mathrm{g}}, W_{\max} + W_{\min}\},$$

where $W_{\max}$ and $W_{\min}$ are the maximal and minimal weights of the deleted parallel edges, respectively.

By the above lemma, the relationship between the maximal girth of the original graph and that of the reduced graph will be discovered below.

Theorem 16: Given a power network with the underlying graph $(\mathbb{V}, \mathbb{E})$, let g denote its maximal girth. By denoting the graph after reduction and its maximal girth as $(\mathbb{V}_R, \mathbb{E}_R, W_R)$ and g_R, we have

$$\min\{\mathrm{g}_R, \alpha_2\} \le \mathrm{g} \le \min\{\max\{\mathrm{g}_R, \alpha_1\}, \alpha_2\},$$

where α_1 is the maximum of $W_{\max} + W_{\min}$ over Type I Operations and the second case of Type III Operations, and α_2 is the minimum of $W_{\max} + W_{\min}$ over the third case of Type III Operations. Here, $W_{\max}$ and $W_{\min}$ are defined in Lemma 13. If operations for computing α_1 or α_2 are never implemented, then we set α_1 to 0 or α_2 to $+\infty$.

Based on the numerical results in Table I and [29, Table II] for large power networks, the values of α_1 and α_2 in Theorems 14 and 16 are usually smaller than e_R and g_R. Hence, we have the approximation

$$\mathrm{e} \approx \mathrm{e}_R, \quad \mathrm{g} \approx \alpha_2. \tag{5}$$

The above relations imply that for large power networks, computing the maximal eye is equivalent to computing the maximal eye of a reduced graph, while the maximal girth is already computed during the reduction process. Finally, we prove that 2-vertex-connected SP graphs can be reduced to a single edge by the ISPR method.

Theorem 17: If the underlying graph $(\mathbb{V}, \mathbb{E})$ of a power network is a 2-vertex-connected SP graph, then the ISPR method reduces the underlying graph to a single edge.

For an undirected graph without slack nodes, the classical SPR (Type I′–III′ Operations) can reduce the graph to a single edge if and only if the graph is a GSP graph [28]. We note that 2-vertex-connected SP graphs are a special class of GSP graphs, and it is unclear whether the reduction guarantee for the ISPR method can be extended to any GSP graphs in the presence of slack nodes.

VI. Numerical Results

In this section, we verify the theoretical results of this article and test the performance of the proposed algorithms. First, we show that, using the ISPR method, the computation of the maximal eye can be reduced to a smaller graph, while the computation of the maximal girth is finished during the process of reduction. Then, we show that Corollary 5 gives a valid sufficient condition for the strong uniqueness. We use IEEE power networks in MATPOWER [33] to perform experiments. Finally, the proximity between the P–Θ problem and the AC power flow problem is numerically illustrated.

A. Computation of the Maximal Eye and the Maximal Girth

We first consider the computation of the maximal eye. The results are listed in Table I. Here, we use "–" to denote the case when this value does not exist and use "TLE" (time limit exceeded) to denote the case when the algorithm does not find any leaf node in two days. The lower bounds for the maximal eye are derived by stopping the algorithm before it terminates. It can be observed that the ISPR method can largely reduce the size of the graph and, therefore, can accelerate the computing process. Moreover, the values of α_1 and α_2 are small compared to the maximal eye of the reduced graph. Hence, the approximation in (5) holds and the maximal eye of the original graph is equal to the maximal eye of the reduced graph. Although the algorithm achieves acceleration compared to the brute-force search method, we are only able to compute the maximal eye for graphs with up to 118 vertices. Note that since graph problems have exponential complexities, solving them for graphs having as low as 200 nodes is still beyond the current computational capabilities. However, this does not undermine the usefulness of the introduced graph parameters, since it is shown in this article that those parameters accurately decide whether the power flow problem has a unique solution.

Next, we consider the computation of the maximal girth. We use the same algorithms and the results are listed in the technical report [29]. In this case, it can be observed that α_2 is equal to 3 for large power networks. This is because the underlying graphs of large power networks considered in Table I have "pendant triangles." Pendant triangles are triangles that have only one vertex connected to the rest of the graph. Furthermore, the approximation in Theorem 16 holds and the maximal girth of the original graph is equal to $\alpha_2 = 3$. Hence, the maximal girth can be computed during the reduction process. This shows that the conditions for the weak uniqueness is significantly loose and requiring $\omega_{k\ell}$ to be at most $2\pi/3$ for all edges $\{k, \ell\}$ is enough. However, for 2-vertex-connected SP graphs, we have shown that the maximal girth is equal to the maximal eye, and the requirement for the weak uniqueness is the same as that for the strong uniqueness.

B. Verification of Corollary 5

In this subsection, we validate the results in Corollary 5, i.e., showing that there does not exist a different solution in the monotone regime with the set of allowable perturbations being $\mathcal{W}_{2\pi/\mathrm{e}(\mathcal{G})}$.

A random power flow set point is generated by first choosing a random vector of voltages. The voltage magnitudes and angles are randomly sampled from a uniform distribution ranging from user-set min/max values

$$|v_i^0| \sim \mathcal{U}(V_{\min}, V_{\max}) \text{ for all } i \in \mathbb{V},$$

$$|\Theta_i^0| \sim \mathcal{U}(\Theta_{\min}, \Theta_{\max}) \text{ for all } i \in \mathbb{V},$$

where $\mathcal{U}(a, b)$ is the uniform distribution on $[a, b]$. The voltage angles are rejected and discarded if they do not belong to the monotone regime. A new random sample is chosen until the angles belong to the monotone regime. Finally, once we have a voltage profile belonging to the monotone regime, we use the information to calculate the real power injections, namely, P^0. The values of $|v^0|$ and P^0 are provided as an input to the power flow algorithm. Note that Θ^0 is always a solution to the P–Θ problem $\hat{P}(\Theta) = P^0$. In this sense, we refer to Θ^0 the ground truth solution. There are usually other solutions, and the goal of this experiment is to analyze where those other solutions are situated with respect to the ground truth solution.

In order to explore different parts of the solution space, we randomly sample an initial point around the ground truth Θ^0 and feed it into MATPOWER. The current setting is to consider a normal distribution around the ground truth, with some specified standard deviation. Intuitively, if the random initial point is close enough to the ground truth solution, then the algorithm will converge to the ground truth solution. However, if we start the algorithm with a suitably far initial point, then the power flow algorithms may converge to a different solution. Note that initializing too far away can lead to convergence issues of the algorithm.

Next, we define a metric that can capture the distance between two solutions to the P–Θ problem. Consider a solution of the P–Θ problem, Θ^i, where i corresponds to the random initialization number ($i \in \mathcal{R} := \{1, \ldots, 10000\}$). Let Θ_k^i denote the voltage angle at bus k for the ith experiment. We define $\mathrm{dist}(\Theta^i)$ to be the distance between the particular solution Θ^i and the ground truth solution, characterized in terms of their angle differences

$$\mathrm{dist}(\Theta^i) := \max_{\{k,\ell\} \in \mathbb{E}} |\Theta_{k\ell}^i - \Theta_{k\ell}^0|.$$

Now, define $\mathrm{dist}^m(\mathbb{G})$ to be the smallest nonzero distance among all solutions in the monotone region for a given power system $\mathbb{G}$. More concretely, we let the symbol $\mathcal{M}$ represent the set of indices i such that the solution Θ^i belongs to the monotone region defined in this article and define

$$\mathrm{dist}^m(\mathbb{G}) := \min_{i \in \mathcal{M} \cap \mathcal{R}} \mathrm{dist}(\Theta^i) \quad \text{s.t. } \mathrm{dist}(\Theta^i) \neq 0.$$

As a specific scenario, we consider the case when all the line properties are the same and the voltage magnitudes are fixed to be one. In other words, $V_{\max} = V_{\min} = 1$. Furthermore, the lines are close to being lossless. We note that when we experimented with significantly lossy lines, different solutions were not found within the monotone region. This is because the monotone regime is small when the lines are very lossy.

TABLE II
DISTANCE MEASURE FOR DIFFERENT TEST CASES

Power Networks	dist^m	$2\pi/e$
Case 14	∞	∞
Case 30	71.8	45
Case 39	53.8	45
Case 57	37.8	15.7
Case 118	66.1	27.7

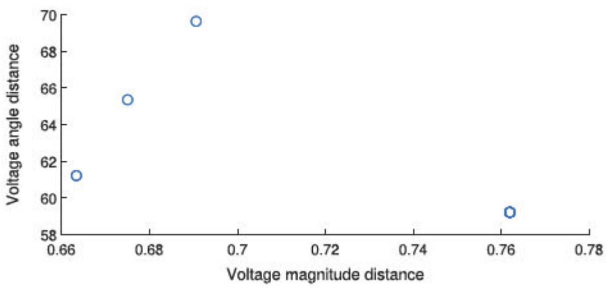

Fig. 1. Distance between AC power flow solutions for IEEE-39.

The values of $\mathrm{dist}(\Theta^i)$ and dist^m are calculated for different networks and are summarized in Table II. The results in the table are twofolds. First, the distance dist^m provides an upper bound on the allowable perturbations such that the solution is strongly unique. On the other hand, the results in the last column are the theoretical lower bound on the allowable perturbations to guarantee the strong uniqueness. We can see that the numerical results verify our theoretical findings, although there exists a gap between the maximal possible allowable perturbations that ensure the uniqueness and the bounds obtained from our theoretical results.

C. Implication for AC Power Flow Problem

The P–Θ problem discussed in this article makes the assumption that all buses are PV buses. In order to show the connection between the P–Θ problem and the full AC power flow problem, we numerically demonstrate the proximity of the power flow solutions under the two problem settings. A random set point is generated, as we did in the previous subsection, by producing a random voltage profile $(|v^0|, \Theta^0)$ and computing the corresponding real/reactive powers. Then, this set point is utilized as input parameters to solve for the AC power flow problem (without the assumption that all buses are PV buses) with random initial points around $|v^0|, \Theta^0$. Note that $|v^0|, \Theta^0$ (call it the reference solution) obviously comprises one solution to the AC power flow problem, but there are potentially other solutions that satisfy the AC power flow equations. Let us define the distance between two solutions as we did in the previous subsection. Fig. 1 shows that none of the other solutions are within the allowable perturbation bound obtained in Corollary 5 when compared to the reference solution. Furthermore, the voltage magnitude distance shows that these are unrealistic solutions, since voltage magnitudes are usually maintained to be within 5% of the nominal value. Similar experiments conducted for various set points and all the power networks mentioned in Table II lead to the same results.

VII. CONCLUSION

In this article, we extended the uniqueness theory of P–Θ power flow solutions developed in [1] for an AC power system. The notion of strong uniqueness was introduced to characterize the uniqueness in the common sense. We proposed a general necessary and sufficient condition for the uniqueness of the solution, which depends only on the monotone regime and the network topology. These conditions can be greatly simplified in certain scenarios. When the underlying graph of the power network is a single cycle, sufficient conditions in [1] are proved to be necessary. For 2-vertex-connected SP graphs, we showed that the maximal eye is equal to the maximal girth, which means that the sufficient condition for the weak uniqueness also implies the strong uniqueness. When the power network is lossless, we derived a necessary and sufficient condition that does not contain sinusoidal functions, and its sufficient part is stronger than the general sufficient conditions. A reduction method, named the ISPR method, was proposed to reduce the size of the power network and accelerate the computation of the maximal eye and the maximal girth. The ISPR method was proved to reduce a 2-vertex-connected SP graph to a single edge, and the relation between the graphs before and after the reduction was analyzed. Some algorithms based on the depth-first search (DFS) method with pruning were designed to compute the maximal eye and the maximal girth.

APPENDIX

Algorithms for Computing the Maximal Eye and the Maximal Girth

In the appendix, we propose search-based algorithms for computing the maximal eye and the maximal girth. Our approach is based on the DFS method and utilizes the pruning technique to accelerate the computing process. We first describe a common subprocedure that will be used in both algorithms. The subprocedure computes the minimal directed chordless cycle containing a given edge. Given a truncation length $T \geq 1$, the subprocedure returns the truncation length if there does not exist a directed chordless cycle that contains the given edge and has length at most T. The subprocedure is also based on the DFS method with pruning and borrows the idea of *blocking* from [34] to accelerate the searching process. The pseudocode of the subprocedure is listed in the online technical report [29].

Next, we propose the algorithms for computing the maximal eye and the maximal girth. Since the algorithm of the maximal girth is similar to the algorithm for the maximal eye, we only present the algorithm for computing the maximal eye and offer the other one in [29]. The algorithm is also based on the DFS method with pruning, and the pseudocode is provided in the online technical report [29]. We first order all edges and gradually assign one of the directions $\{0, -1, +1\}$ to each edge following the ordering of the edges. The search space consists of the orientations for the first several edges (intermediate states) and the orientations for the entire graph (final states). One can verify that all intermediate states and final states form a trinomial[2] tree, since each orientation for the first $k < |\mathbb{E}|$ edges leads to three different orientations for the first $k+1$ edges. Then, the algorithm for computing the maximal eye searches in the same way as the classical DFS method on a directed tree. For each node, we consider the subgraph consisting of those edges that have been assigned a direction. We compute the length of the minimal directed chordless cycle in the subgraph, which contains the last edge in the subgraph, using the subprocedure. The truncation length can be decided as follows. Since a DFS method is implemented on a trinomial tree, there exists a directed path from the root node of the trinomial tree to the current node. The truncation length can be chosen as the minimal length computed on the preceding nodes of the path. When the search reaches a leaf node, we obtain an orientation for the entire graph, and the size of the eye becomes the minimal length on the path to the root node. By searching over all leaf nodes, we find the maximal eye. Similarly, one can use the pruning technique to reduce the search space. The current node is pruned if it cannot be extended to a weakly feasible orientation for the entire graph, or the size of the eye of the subgraph is smaller than the known maximal size of the eye.

References

[1] S. Park, R. Y. Zhang, J. Lavaei, and R. Baldick, "Uniqueness of power flow solutions using monotonicity and network topology," *IEEE Trans. Control Netw. Syst.*, vol. 8, no. 1, pp. 319–330, Mar. 2021.

[2] H.-D. Chiang and M. E. Baran, "On the existence and uniqueness of load flow solution for radial distribution power networks," *IEEE Trans. Circuits Syst.*, vol. 37, no. 3, pp. 410–416, Mar. 1990.

[3] I. A. Hiskens and R. J. Davy, "Exploring the power flow solution space boundary," *IEEE Trans. Power Syst.*, vol. 16, no. 3, pp. 389–395, Aug. 2001.

[4] H. D. Nguyen and K. S. Turitsyn, "Appearance of multiple stable load flow solutions under power flow reversal conditions," in *Proc. IEEE Power Energy Soc. Gen. Meeting*, National Harbor, MD, USA, Jul. 2014, pp. 1–5.

[5] M. Ilic, "Network theoretic conditions for existence and uniqueness of steady state solutions to electric power circuits," in *Proc. IEEE Int. Symp. Circuits Syst.*, 1992, pp. 2821–2828.

[6] A. J. Korsak, "On the question of uniqueness of stable load-flow solutions," *IEEE Trans. Power App. Syst.*, vol. PAS-91, no. 3, pp. 1093–1100, May 1972.

[7] A. Araposthatis, S. Sastry, and P. Varaiya, "Analysis of power-flow equation," *Int. J. Elect. Power Energy Syst.*, vol. 3, no. 3, pp. 115–126, Jul. 1981.

[8] J. Thorp, D. Schulz, and M. Ilic-Spong, "Reactive power-voltage problem: Conditions for the existence of solution and localized disturbance propagation," *Int. J. Elect. Power Energy Syst.*, vol. 8, pp. 66–74, 1986.

[9] K. Miu and H.-D. Chiang, "Existence, uniqueness, and monotonic properties of the feasible power flow solution for radial three-phase distribution networks," *IEEE Trans. Circuits Syst.*, vol. 47, no. 10, pp. 1502–1514, Oct. 2000.

[10] K. Dvijotham, E. Mallada, and J. Simpson-Porco, "High-voltage solution in radial power networks: Existence, properties, and equivalent algorithms," *IEEE Control Syst. Lett.*, vol. 1, no. 2, pp. 322–327, Oct. 2017.

[11] R. Delabays, T. Coletta, and P. Jacquod, "Multistability of phase-locking in equal-frequency Kuramoto models on planar graphs," *J. Math. Phys.*, vol. 58, no. 3, 2017, Art. no. 032703.

[12] F. F. Wu, "Theoretical study of the convergence of the fast decoupled load flow," *IEEE Trans. Power App. Syst.*, vol. PAS-96, no. 1, pp. 268–275, Jan. 1977.

[2]A directed tree is called a trinomial tree if there is a root node and each nonleaf node has exactly three descendant nodes.

[13] S. Z. S. Bolognani, "On the existence and linear approximation of the power flow solution in power distribution networks," *IEEE Trans. Power Syst.*, vol. 31, no. 1, pp. 163–172, Jan. 2016.

[14] C. Wang, A. Bernstein, J.-Y. Le Boudec, and M. Paolone, "Explicit conditions on existence and uniqueness of load-flow solutions in distribution networks," *IEEE Trans. Smart Grid*, vol. 9, no. 2, pp. 953–962, Mar. 2018.

[15] J. W. Simpson-Porco, "A theory of solvability for lossless power flow equations—Part I: Fixed-point power flow," *IEEE Trans. Control Netw. Syst.*, vol. 5, no. 3, pp. 1361–1372, Sep. 2018.

[16] J. W. Simpson-Porco, "A theory of solvability for lossless power flow equations—Part II: Conditions for radial networks," *IEEE Trans. Control Netw. Syst.*, vol. 5, no. 3, pp. 1373–1385, Sep. 2018.

[17] A. Bernstein, C. Wang, E. Dall'Anese, J.-Y. L. Boudec, and C. Zhao, "Load-flow in multiphase distribution networks: Existence, uniqueness, non-singularity and linear models," *IEEE Trans. Power Syst.*, vol. 33, no. 6, pp. 5832–5843, Nov. 2018.

[18] B. Cui and X. A. Sun, "Solvability of power flow equations through existence and uniqueness of complex fixed point," 2019, *arXiv:1904.08855*.

[19] R. Madani, J. Lavaei, and R. Baldick, "Convexification of power flow problem over arbitrary networks," in *Proc. IEEE 54th Conf. Decis. Control*, 2015, pp. 1–8.

[20] Y. Zhang, R. Madani, and J. Lavaei, "Conic relaxations for power system state estimation with line measurements," *IEEE Trans. Control Netw. Syst.*, vol. 5, no. 3, pp. 1193–1205, Sep. 2018.

[21] K. Dvijotham, M. Chertkov, and S. Low, "Monotone operator approach to power flow solutions," in *Proc. Amer. Control Conf.*, 2016, pp. 1769–1769.

[22] S. Jafarpour, E. Y. Huang, K. D. Smith, and F. Bullo, "Flow and elastic networks on the n-torus: Geometry, analysis, and computation," *SIAM Rev.*, vol. 64, no. 1, pp. 59–104, 2022.

[23] S. Pemmaraju and S. Skiena, *Computational Discrete Mathematics: Combinatorics and Graph Theory With Mathematica*. Cambridge, U.K.: Cambridge Univ. Press, 2003.

[24] F. Dörfler, J. W. Simpson-Porco, and F. Bullo, "Electrical networks and algebraic graph theory: Models, properties, and applications," *Proc. IEEE*, vol. 106, no. 5, pp. 977–1005, May 2018.

[25] M. R. Garey and D. S. Johnson, *Computers and Intractability*, vol. 174. San Francisco, CA, USA: Freeman, 1979.

[26] E. S. Dias, D. Castonguay, H. Longo, and W. A. R. Jradi, "Efficient enumeration of chordless cycles," 2013, *arXiv:1309.1051*.

[27] T. Uno and H. Satoh, "An efficient algorithm for enumerating chordless cycles and chordless paths," in *Proc. Int. Conf. Discov. Sci.*, 2014, pp. 313–324.

[28] N. Korneyenko, "Combinatorial algorithms on a class of graphs," *Discrete Appl. Math.*, vol. 54, pp. 215–217, 1994.

[29] H. Zhang, S. Park, J. Lavaei, and R. Baldick, "Uniqueness of power flow solutions using graph-theoretic notions," 2020. [Online]. Available: https://lavaei.ieor.berkeley.edu/Eye_PF_2020_1.pdf

[30] S. Sojoudi, S. Fattahi, and J. Lavaei, "Convexification of generalized network flow problem," *Math. Program.*, vol. 173, no. 1, pp. 353–391, 2019.

[31] D. Eppstein, "Parallel recognition of series-parallel graphs," *Inf. Comput.*, vol. 98, no. 1, pp. 41–55, 1992.

[32] S. Khuller, "Ear decompositions," *SIGACT News*, vol. 20, no. 1, p. 128, 1989.

[33] R. Zimmerman, C. Murillo-Sanchez, and R. Thomas, "MATPOWER: Steady-state operations, planning and analysis tools for power systems research and education," *IEEE Trans. Power Syst.*, vol. 26, no. 1, pp. 12–19, Feb. 2011.

[34] J. C. Tiernan, "An efficient search algorithm to find the elementary circuits of a graph," *Commun. ACM*, vol. 13, no. 12, pp. 722–726, 1970.

Haixiang Zhang received the B.S. degree in applied mathematics and computer science from Peking University, Beijing, China, in 2019. He is currently working toward the Ph.D. degree in applied mathematics with the Department of Mathematics, University of California, Berkeley, CA, USA.

SangWoo Park received the B.S. degree in environmental engineering from Johns Hopkins University, Baltimore, MD, USA, in 2016, and the M.S. degree in industrial engineering and operations research in 2017 from the University of California, Berkeley, CA, USA, where he is currently working toward the Ph.D. degree in industrial engineering and operations research with the Department of Industrial Engineering and Operations Research.

Javad Lavaei received the Ph.D. degree in control and dynamical systems from the California Institute of Technology, Pasadena, CA, USA, in 2011.

He is currently an Associate Professor with the Department of Industrial Engineering and Operations Research, University of California, Berkeley, CA, USA.

Dr. Lavaei is the recipient of multiple awards, including the National Science Foundation CAREER Award, the Office of Naval Research Young Investigator Award, and the Donald P. Eckman Award.

Ross Baldick received the B.Sc. degree in mathematics and physics, and the B.E. degree in electrical engineering from the University of Sydney, Sydney, NSW, Australia, in 1983 and 1985, respectively. He received the M.S. and Ph.D. degrees in electrical engineering and computer sciences from the University of California, Berkeley, CA, USA, in 1988 and 1990, respectively.

He is currently an Emeritus Professor with the Department of Electrical and Computer Engineering, University of Texas at Austin, Austin, TX, USA.