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Braided Frobenius algebras from certain Hopf algebras

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A braided Frobenius algebra is a Frobenius algebra with a Yang–Baxter operator that commutes with the operations, that are related to diagrams of compact surfaces with boundary expressed as ribbon graphs. A heap is a ternary operation exemplified by a group with the operation $(x, y, z) \mapsto xy^{-1}z$, that is ternary self-distributive. Hopf algebras can be endowed with the algebra version of the heap operation. Using this, we construct braided Frobenius algebras from a class of certain Hopf algebras that admit integrals and cointegrals. For these Hopf algebras we show that the heap operation induces a Yang–Baxter operator on the tensor product, which satisfies the required compatibility conditions. Diagrammatic methods are employed for proving commutativity between Yang–Baxter operators and Frobenius operations.

Keywords: Self-distributivity; heap operation; Yang–Baxter operators; compact orientable surfaces with boundary.

Mathematics Subject Classification 2020: 17B38, 16T05, 16T25, 57K12

1. Introduction

Frobenius algebras have been studied in recent decades in relation to 2-dimensional topological quantum field theories (TQFTs) [15], and to Khovanov homology [14] in knot theory, that is a categorification of the Jones polynomial [11]. Braid groups have been extensively used in relation to generalizations of the Jones polynomial, and braided monoidal categories have been developed to further extend knot invariants to ribbon graphs [20], that consist of disk vertices and ribbon edges. Spatial graphs with a move that corresponds to handle slides have been studied for handlebody-links [9]. Corresponding algebraic structures that have multiplication and braiding at the same time, with compatibility conditions, have also been studied [2, 17]. Compact surfaces with boundary can be represented by ribbon graphs, and their moves [18] and their invariants [10] have been studied. For algebraic objects having both Frobenius and braiding structures, Frobenius objects in braided monoidal categories were proposed in [4], and relations to a certain tangle category were discussed. Hopf-Frobenius algebras were considered e.g. in [3], and invariants that use heaps for framed links were considered in [21, 22].

Motivated from these developments, in this paper, we present a construction of braided Frobenius algebras from certain Hopf algebras. A *braided Frobenius algebra* is a Frobenius algebra $X = (V, \mu, \eta, \Delta, \epsilon)$ (multiplication, unit, comultiplication, counit) over a unital ring $\Bbbk$, along with a Yang–Baxter (YB) operator β, such that operations of X commute with the operator β, as explicitly formulated in Definition 5.1. This commutation is represented by diagrams depicted in Fig. 1, where the multiplication and the YB operator are represented by trivalent vertices and crossings, respectively, and these are part of moves for spatial graph diagrams.

The idea of the construction is based on *heaps*. A heap is an abstraction of a group endowed with the ternary operation $a \times b \times c \mapsto T(a, b, c) = ab^{-1}c$. It is computed that this operation on a group satisfies the *ternary self-distributive law* (TSD) $T((x, y, z), u, v) = T(T(x, u, v), T(y, u, v), T(z, u, v))$ for all x, y, z, u, v. Binary self-distributive operations have been studied in relation to the Yang–Baxter operators through tensor categories (e.g. [1]). In [6] a diagrammatic interpretation of TSD was given in terms of framed links, providing set-theoretic Yang–Baxter operators. The assignment of heap elements on arcs and the heap operations to crossings are depicted in Fig. 2, together with the TSD property corresponding to a braid relation (the type III Reidemeister move in knot theory). In [5], the constructions of TSD operations from heaps were generalized to monoidal categories. Those in the category of finite dimensional Hopf algebras over a field are called *quantum heaps*. We use quantum heaps X to construct a Frobenius algebra structure on $V = X \otimes X$ that commute with the Yang–Baxter operator induced by the TSD operations. A key method of proofs is an extensive use of diagrams.

The paper is organized as follows. In Sec. 2 we review basic definitions and facts regarding heap structures, Hopf algebras, Frobenius algebras and Yang–Baxter operators. In Sec. 3 we deal with ternary self-distributive (TSD) structures in coalgebras, and construct a Yang–Baxter operator associated to a TSD structure arising from quantum heaps in Hopf algebras. In Sec. 4 (co)pairings are constructed that

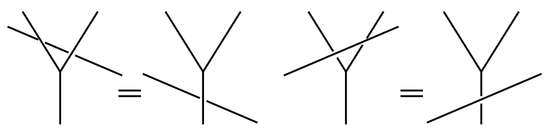

Fig. 1. Axioms of a braided Frobenius algebra.

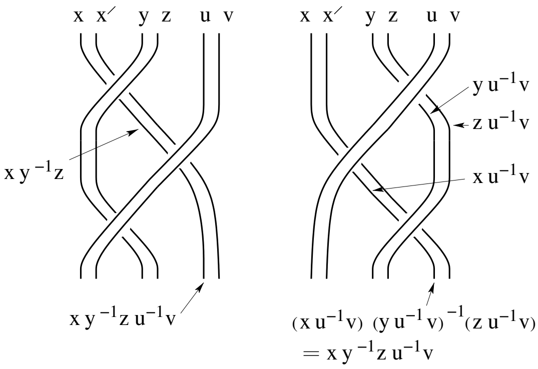

Fig. 2. Heap operation and braid relation.

commute with Yang–Baxter operators. These (co)pairing are used for (co)units for Frobenius structures. In Sec. 5 we introduce the notion of braided Frobenius algebra and show that there is a class of these structures arising from quantum heaps where a Frobenius algebra is defined via Hopf algebra (co)integrals. Section 6 discusses relations to compact surfaces with boundary embedded in 3-space, and issues of twists in braided Frobenius algebras.

2. Preliminaries

In this section, we review materials used in this paper.

2.1. *Heaps*

We recall the definition and basic properties of heaps. Given a set X with a ternary operation $[-]$, the set of equalities

$$[[x_1, x_2, x_3], x_4, x_5] = [x_1, [x_4, x_3, x_2], x_5] = [x_1, x_2, [x_3, x_4, x_5]]$$

is called para-associativity. The equations $[x, x, y] = y$ and $[x, y, y] = x$ are called the degeneracy conditions. A *heap* is a non-empty set with a ternary operation satisfying the para-associativity and the degeneracy conditions [5]. A typical example of a heap is a group G where the ternary operation is given by $[x, y, z] = xy^{-1}z$, which we call a *group heap*.

Let X be a set with a ternary operation $(x, y, z) \mapsto T(x, y, z)$. The condition $T((x, y, z), u, v) = T(T(x, u, v), T(y, u, v), T(z, u, v))$ for all $x, y, z, u, v \in X$, is called *ternary self-distributivity*, TSD for short. It is known and easily checked that the heap operation $(x, y, z) \mapsto [x, y, z] = T(x, y, z)$ is ternary self-distributive. We focus on the TSD property of heaps.

2.2. *Hopf algebras*

A *Hopf algebra* $(X, \mu, \eta, \Delta, \epsilon, S)$ (a module over a unital ring $\Bbbk$, multiplication, unit, comultiplication, counit, antipode, respectively), is defined as follows. First, a bialgebra X has a multiplication $\mu : X \otimes X \to X$ with unit η and a comultiplication $\Delta : X \to X \otimes X$ with counit ϵ such that the compatibility condition $\Delta \circ \mu = (\mu \otimes \mu)\tau \circ (\Delta \otimes \Delta)$ holds. Then a Hopf algebra is a bialgebra endowed with a map $S : X \to X$, called *antipode*, satisfying the equations $\mu \circ (\mathbb{1} \otimes S) \circ \Delta = \eta \circ \epsilon = \mu \circ (S \otimes \mathbb{1}) \circ \Delta$, called the *antipode condition*.

The diagrammatic representation of the algebraic operations appearing in a Hopf algebra is given in Fig. 3. Diagrams are read from top to bottom. For example, the top two arcs of the trivalent vertex for μ (the leftmost diagram) represent $X \otimes X$, the vertex represents μ, and the bottom arc represents X. In Fig. 4 some of the defining axioms of a Hopf algebra are translated into diagrammatic equalities. Specifically, diagrams represent (A) associativity of μ, (B) unit condition, (C), compatibility between μ and Δ, (D) the antipode condition. The coassociativity and counit conditions are represented by diagrams that are vertical mirrors of (A) and (B), respectively.

Any Hopf algebra satisfies the equality $S\mu\tau = \mu(S \otimes S)$, where τ denotes the transposition $\tau(x \otimes y) = y \otimes x$ for simple tensors. This equality is depicted in Fig. 5. A Hopf algebra is called *involutory* if $S^2 = \mathbb{1}$, the identity. It is known, [13, Theorem III.3.4], that if a Hopf algebra is commutative or cocommutative it follows that it is also involutory. In what follows, we will not mention that our Hopf algebras are involutory when they are (co)commutative, and freely apply the fact that $S^2 = \mathbb{1}$ without further mention.

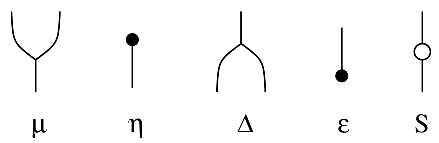

Fig. 3. Operations of Hopf algebras.

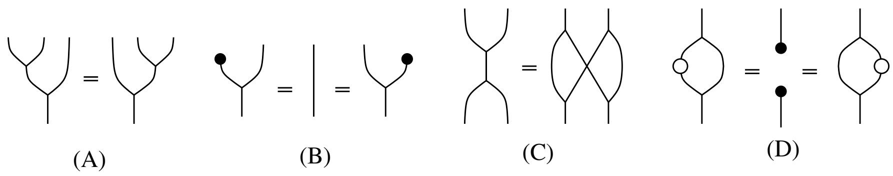

Fig. 4. Axioms of Hopf algebras.

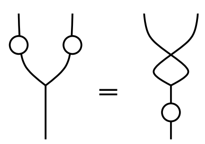

Fig. 5. Twisting μ with antipodes.

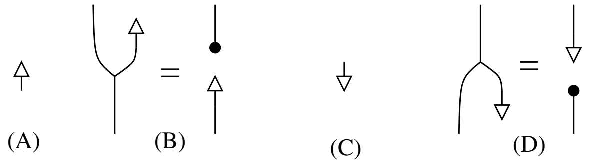

Fig. 6. Left (co)integral of Hopf algebras.

For the comultiplication, we use Sweedler's notation $\Delta(x) = x^{(1)} \otimes x^{(2)}$ supressing the summation. Further, we use $(\Delta \otimes \mathbb{1})\Delta(x) = (x^{(11)} \otimes x^{(12)}) \otimes x^{(2)}$ and $(\mathbb{1} \otimes \Delta)\Delta(x) = x^{(1)} \otimes (x^{(21)} \otimes x^{(22)})$, both of which are also written as $x^{(1)} \otimes x^{(2)} \otimes x^{(3)}$ from the coassociativity.

A left integral of X is an element $\lambda \in X$ such that $x\lambda = \epsilon(x)\lambda$ for all $x \in X$. A right integral, a (two-sided) integral, cointegrals are defined similarly. Diagrams for integral conditions are depicted in Fig. 6. The diagram (A) represents an integral, (B) represents the defining equation of a left integral, and similar for cointegrals in (C) and (D). The existence of integrals is a fundamental tool to endow a Hopf algebra with a Frobenius structure (defined in what follows). It is known that the space of integrals of a finite dimensional free Hopf algebra over a PID is 1-dimensional, see [16]. More generally, a finitely generated projective Hopf algebra over a ring admits a left integral space of rank one [19]. Observe that when a Hopf algebra is (co)commutative, it follows that a left (co)integral is also a right (co)integral.

2.3. *Frobenius algebras*

We use the following definition: A *Frobenius algebra* $(V, \mu, \eta, \Delta, \epsilon)$ is an associative algebra (V, μ, η) with multiplication μ and unit $\eta : \Bbbk \to V$, and a coassociative coalgebra (V, Δ, ϵ) with comultiplication Δ and counit $\epsilon : V \to \Bbbk$, over a unital ring $\Bbbk$, such that μ and Δ satisfy the Frobenius compatibility condition: $(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \Delta) = \Delta\mu = (\mathbb{1} \otimes \mu)(\Delta \otimes \mathbb{1})$. This condition is depicted in Fig. 7.

2.4. *The Yang–Baxter operator*

Let X be a module over a ring and let $R : X \otimes X \to X \otimes X$ be an operator (i.e. a linear map). The Yang–Baxter equation, YBE for short, for R is the functional

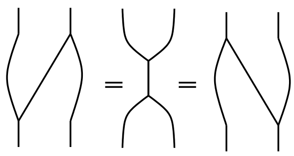

Fig. 7. Frobenius compatibility condition.

equation

$$(R \otimes \mathbb{1}) \circ (\mathbb{1} \otimes R) \circ (R \otimes \mathbb{1}) = (\mathbb{1} \otimes R) \circ (R \otimes \mathbb{1}) \circ (\mathbb{1} \otimes R),$$

where left-hand side and right-hand side are both endomorpshism of $X \otimes X \otimes X$. The YBE is well known to be represented by the type III Reidemeister move in knot theory, and has been widely studied in low-dimensional topology because it produces invariants of knots. If the operator R satisfies the YBE, then it is said to be a *pre Yang–Baxter operator*. If, in addition, R is invertible then we say that R is a *Yang–Baxter operator*, YB operator for short.

3. Ternary Self-Distributive Operations in Coalgebras and YB Operators

In this section, we provide a method of producing YB operators from ternary self-distributive (TSD) operations.

Definition 3.1 ([6]). A coalgebra morphism $T : V^{\otimes 3} \to V$ for a coalgebra V over a unital ring $\mathbb{k}$ is called *ternary self-distributive* (TSD for short) if it satisfies, when expressed in simple tensors,

$$T(T(x \otimes y \otimes z) \otimes u \otimes v) = T(T(x \otimes u^{(1)} \otimes v^{(1)}) \otimes T(y \otimes u^{(2)} \otimes v^{(2)}) \otimes T(z \otimes u^{(3)} \otimes v^{(3)})),$$

where the coalgebra structure on $V^{\otimes 3}$ is induced by the comultiplication of V according to

$$\Delta_{\otimes 3}(x \otimes y \otimes z) = x^{(1)} \otimes y^{(1)} \otimes z^{(1)} \otimes x^{(2)} \otimes y^{(2)} \otimes z^{(2)} \otimes x^{(3)} \otimes y^{(3)} \otimes z^{(3)}.$$

A diagram representing the TSD condition is depicted in Fig. 8.

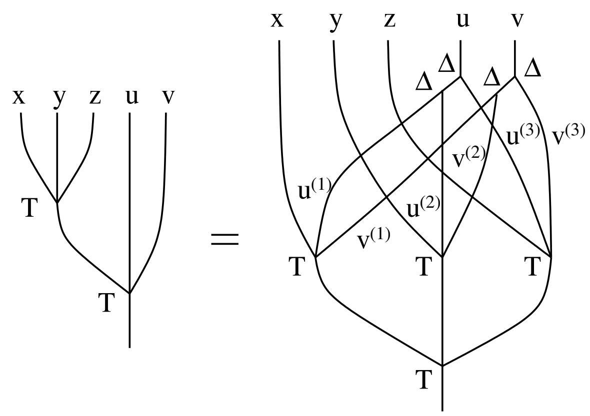

Fig. 8. TSD condition for coalgebras.

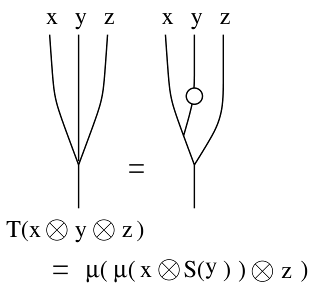

Fig. 9. Quantum heap operation as a TSD.

Lemma 3.2 ([6]). *Let* $(X, \mu, \Delta, \iota, \epsilon, S)$ *be an involutory Hopf algebra. Let* $T(x \otimes y \otimes z) = xS(y)z = \mu(\mu(x \otimes S(y)) \otimes z)$ *expressed in simple tensors, where the concatenation denotes the multiplication. Then* T *is TSD.*

This construction is represented by the diagrams in Fig. 9.

Definition 3.3. A TSD morphism $T : V^{\otimes 3} \to V$ for a module V over a unital ring $\Bbbk$ is called *reversible* if it satisfies

$$T(T(x \otimes y^{(2)} \otimes z^{(2)}) \otimes z^{(1)} \otimes y^{(1)}) = \epsilon(y)\epsilon(z) \cdot x$$

for all $x, y, z \in V$.

Lemma 3.4. *Let* $(X, \mu, \Delta, \iota, \epsilon, S)$ *be a cocommutative (therefore involutory) Hopf algebra, and let* $T(x \otimes y \otimes z) = xS(y)z$ *be as defined in Lemma* 3.2. *Then* T *is reversible.*

Proof. One computes

$$\begin{aligned} T(T(x \otimes y^{(2)} \otimes z^{(2)}) \otimes z^{(1)} \otimes y^{(1)}) &= xS(y^{(2)})z^{(2)}S(z^{(1)})y^{(1)} \\ &= xS(y^{(2)})S(z^{(1)})z^{(2)}y^{(1)} \\ &= \epsilon(z) \cdot xS(y^{(2)})y^{(1)} \\ &= \epsilon(y)\epsilon(z) \cdot x \end{aligned}$$

as desired. ◻

Lemma 3.5. *Let* (X, Δ) *be a cocommutative coalgebra over a unital ring* $\Bbbk$. *Let* $T : X^{\otimes 3} \to X$ *be a reversible TSD morphism. Then the map* $\gamma : X^{\otimes 3} \to X^{\otimes 3}$ *defined for simple tensors by* $\gamma(x \otimes y \otimes z) = y^{(1)} \otimes z^{(1)} \otimes T(x \otimes y^{(2)} \otimes z^{(2)})$ *is invertible with inverse* $\gamma^{-1}(y \otimes z \otimes x) = T(x \otimes z^{(2)} \otimes y^{(2)}) \otimes y^{(1)} \otimes z^{(1)}$, *so that* $\gamma\gamma^{-1} = \mathbb{1}$ *and* $\gamma^{-1}\gamma = \mathbb{1}$.

Proof. The proof is an application of the reversibility condition of T. On simple tensors we have

$$\begin{aligned}\gamma^{-1}\gamma(x\otimes y\otimes z) &= \gamma^{-1}(y^{(1)}\otimes z^{(1)}\otimes T(x\otimes y^{(2)}\otimes z^{(2)}))\\ &= T(T(x\otimes y^{(2)}\otimes z^{(2)})\otimes z^{(12)}\otimes y^{(12)})\otimes y^{(11)}\otimes z^{(11)}\\ &= T(T(x\otimes y^{(22)}\otimes z^{(22)})\otimes z^{(21)}\otimes y^{(21)})\otimes y^{(1)}\otimes z^{(1)}\\ &= \epsilon(y^{(2)})\epsilon(z^{(2)})x\otimes y^{(1)}\otimes z^{(1)}\\ &= x\otimes y\otimes z,\end{aligned}$$

which shows that $\gamma^{-1}\gamma = \mathbb{1}$. Similar considerations imply that $\gamma\gamma^{-1} = \mathbb{1}$ as well. In this proof we used the notation of the form $y^{(11)}$ and $y^{(12)}$ to indicate that these terms are obtained from the term $y^{(1)}$ by applying comultiplication. $\square$

Diagrammatic representations of morphisms γ and γ^{-1} in Lemma 3.5 are depicted in the left and right of Fig. 10, respectively. The first equality $\gamma^{-1}\gamma = \mathbb{1}$ in the lemma is represented by Fig. 11.

Lemma 3.6. *Let (X,Δ) be a cocommutative coalgebra over a unital ring $\Bbbk$. Let $T : X^{\otimes 3}\to X$ be a reversible TSD coalgebra morphism. Let $V = X\otimes X$ be endowed with the tensor coalgebra structure induced by (X,Δ). Then the map $\beta : V^{\otimes 2}\to V^{\otimes 2}$ defined for simple tensors by*

$$\beta((x\otimes y)\otimes(z\otimes w)) := (z^{(1)}\otimes w^{(1)})\otimes T(x\otimes z^{(2)}\otimes w^{(2)})\otimes T(y\otimes z^{(3)}\otimes w^{(3)})$$

satisfies the YBE. Furthermore, there is an inverse

$$\beta^{-1}((z\otimes w)\otimes(x\otimes y)) = (T(x\otimes w^{(2)}\otimes z^{(2)})\otimes T(y\otimes w^{(3)}\otimes z^{(3)}))\otimes z^{(1)}\otimes w^{(1)}.$$

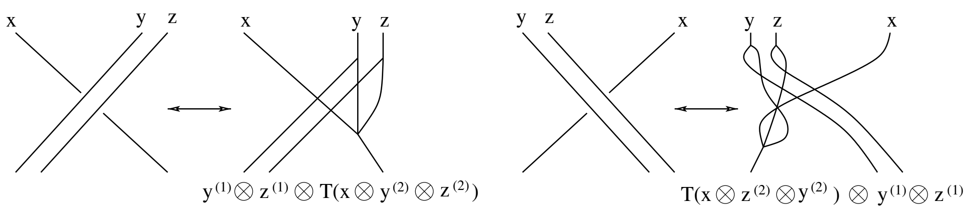

Fig. 10. Hopf algebra maps corresponding to crossings.

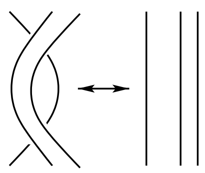

Fig. 11. The type II Reidemeister move with a single under-arc and double over-arcs.

Proof. We show that the YBE holds on simple tensors. Let $x, y, z, w, u, v \in X$, then the left-hand side of the YBE is computed as

$$\begin{aligned}
&(\beta \otimes \mathbb{1})(\mathbb{1} \otimes \beta)(\beta \otimes \mathbb{1})(x \otimes y \otimes z \otimes w \otimes u \otimes v) \\
&= u^{(1)} \otimes v^{(1)} \otimes T(z^{(1)} \otimes u^{(2)} \otimes v^{(2)}) \otimes T(w^{(1)} \otimes u^{(3)} \otimes v^{(3)}) \\
&\quad \otimes T(T(x \otimes z^{(2)} \otimes w^{(2)}) \otimes u^{(4)} \otimes v^{(4)}) \\
&\quad \otimes T(T(y \otimes z^{(3)} \otimes w^{(3)}) \otimes u^{(5)} \otimes v^{(5)}).
\end{aligned}$$

The right-hand side computed on $x \otimes y \otimes z \otimes w \otimes u \otimes v$ gives

$$\begin{aligned}
&(\mathbb{1} \otimes \beta) \circ (\beta \otimes \mathbb{1}) \circ (\mathbb{1} \otimes \beta)(x \otimes y \otimes z \otimes w \otimes u \otimes v) \\
&= u^{(1)} \otimes v^{(1)} \otimes T(z^{(1)} \otimes u^{(4)} \otimes v^{(4)}) \otimes T(w^{(1)} \otimes u^{(7)} \otimes v^{(7)}) \\
&\quad \otimes T(T(x \otimes u^{(2)} \otimes v^{(2)}) \otimes T(z^{(2)} \otimes u^{(5)} \otimes v^{(5)}) \otimes T(w^{(2)} \otimes u^{(8)} \otimes v^{(8)})) \\
&\quad \otimes T(T(y \otimes u^{(3)} \otimes v^{(3)}) \otimes T(z^{(3)} \otimes u^{(6)} \otimes v^{(6)}) \otimes T(w^{(3)} \otimes u^{(9)} \otimes v^{(9)})).
\end{aligned}$$

To apply the TSD property of T using the fact that T is a coalgebra morphism, one needs to have a proper consecutive order of terms that appear under comultiplication. To rearrange the terms in the last line above, the cocommutativity of Δ is applied. Then we see that both sides coincide. To show that β is reversible observe that, since Δ is cocommutative, one has $\beta = (\gamma \otimes \mathbb{1}) \circ (\mathbb{1} \otimes \gamma)$. Since γ is invertible by Lemma 3.5, it follows that β is invertible. □

Figure 10 shows the diagrammatic interpretation of the braiding and its inverse in Lemma 3.6 on a single edge of a ribbon. The full braiding, as well as its inverse, is obtained by repeating the procedure on both edges that delimit a ribbon.

Lemma 3.7. *Let $(X, \mu, \eta, \Delta, \epsilon, S)$ be a cocommutative Hopf algebra. Then the map $\beta : X^{\otimes 2} \to X^{\otimes 2}$ defined on simple tensors as*

$$x \otimes y \otimes z \otimes w \mapsto z^{(1)} \otimes w^{(1)} \otimes xS(z^{(2)})w^{(2)} \otimes yS(z^{(3)})w^{(3)}$$

is a Yang–Baxter operator.

Proof. The statement follows directly by applying Lemma 3.6 to the quantum heap construction of Lemma 3.2. The invertibility follows from Lemma 3.4. □

4. YB Operators and Pairings in Quantum Heaps

In this section, we introduce pairings and copairings that commute with the YB operators constructed in the preceding section. We obtain such (co)pairing using integrals of Hopf algebras.

Definition 4.1. A pairing $\cup : V \otimes V \to \Bbbk$ and a copairing $\cap : \Bbbk \to V \otimes V$ in a module V over a unital ring $\Bbbk$ are said to have (or satisfy) the *switchback property*

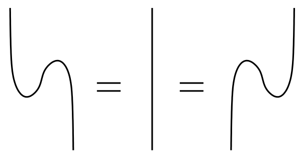

Fig. 12. The switchback property.

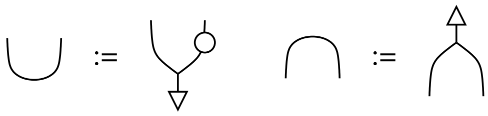

Fig. 13. Defining cup and cap by left (co)integrals.

if they satisfy the equalities

$$(\cup \otimes \mathbb{1})(\mathbb{1} \otimes \cap) = \mathbb{1} = (\mathbb{1} \otimes \cup)(\cap \otimes \mathbb{1}).$$

The conditions imply that $\cup$ is non-singular.

These conditions are depicted diagrammatically in Fig. 13.

Definition 4.2. Let $(X, \mu, \eta, \Delta, \epsilon, S)$ be a finitely generated projective Hopf algebra over a (unital) ring $\Bbbk$. Then X has an integral and a cointegral [19]. Let us indicate them by λ and γ, respectively. We define a cup on X by $\cup := \lambda\mu(\mathbb{1} \otimes S)$ and $\cap := \Delta\gamma$, as depicted in Fig. 13.

For a Hopf algebra $(X, \mu, \eta, \Delta, \epsilon, S)$, the following module $P(H^*)$ was considered in [19]. Let $\chi : X^* \to X^* \otimes X$ be a right X-comodule structure on X^* defined by the left H^*-module structure. Then $P(H^*)$ was defined by $P(H^*) = \{x^* \in X^* \mid \chi(x^*) = x^* \otimes 1\}$.

Lemma 4.3. *Let $(X, \mu, \eta, \Delta, \epsilon, S)$ be a finitely generated projective Hopf algebra over a ring $\Bbbk$, such that $P(X^*) \cong \Bbbk$, and $\cup, \cap$ be as in Definition 4.2. Then $\cup$ and $\cap$ satisfy the switchback property.*

Proof. In [19], it is proved, under the assumptions, that there exists an integral λ and cointegral γ such that $\cup' = \lambda\mu$ and $\cap' = (S \otimes \mathbb{1})\Delta\gamma$ satisfy the switchback property. It then follows that so do $\cup$ and $\cap$ in Definition 4.2 as well. $\square$

Since we use this lemma extensively from here forward, we will assume that every Hopf algebra satisfies the assumption of this lemma. As pointed out in [19], the condition that $P(H^*) \cong \Bbbk$ is automatically satisfied when $\mathrm{pic}(\Bbbk) = 0$. This is the case for instance when $\Bbbk$ is a PID or a local ring. In particular, one obtains the result of Larson and Sweedler in [16], where the ground ring is taken to be a PID.

Definition 4.4. Let V be a coalgebra over a unital ring $\Bbbk$, with ternary morphism (of coalgebras) $T : V^{\otimes 3} \to V$. A pairing $\cup : V \otimes V \to \Bbbk$ is said to have (or satisfy) the *passcup property with respect to T* if it satisfies

$$(\mathbb{1}^{\otimes 2} \otimes \cup)(u^{(1)} \otimes v^{(1)} \otimes T(x \otimes u^{(2)} \otimes v^{(2)}) \otimes y)$$
$$= (\cup \otimes \mathbb{1}^{\otimes 2})(x \otimes T(y \otimes v^{(2)} \otimes u^{(2)}) \otimes u^{(1)} \otimes v^{(1)})$$

for all $x, y, u, v \in V$. In other words, $\cup$ satisfies the passcup property with respect to T, if the map γ induced by T, in Lemma 3.5, satisfies

$$(\mathbb{1}^{\otimes 2} \otimes \cup)(\gamma \otimes \mathbb{1}) = (\cup \otimes \mathbb{1}^{\otimes 2})(\mathbb{1} \otimes \gamma^{-1}).$$

The passcup property is depicted in Fig. 14.

Lemma 4.5. *Let $(X, \mu, \eta, \Delta, \epsilon, S)$ be a cocommutative Hopf algebra. Then the pairing $\cup$ defined in Definition* 4.2 *satisfies the passcup property with respect to the TSD defined in Lemma* 3.2.

Proof. In order to prove the passcup property, we proceed as in Fig. 15. The first equality corresponds to rewriting one negative crossing using the definition of inverse of quantum heap operation T, equality (1) utilizes naturality of the switching map $X \otimes X \to X \otimes X$, equality (2) corresponds to the compatibility relation between the antipode S and the comultiplication Δ of X, equality (3) is given by redrawing the diagram using naturality of switching map, equality (4) corresponds to the fact that λ is both a right and left integral. Involutority is used at step (3). This completes the proof of the passcup property. $\square$

Lemma 4.6. *Let $(X, \mu, \eta, \Delta, \epsilon, S)$ be a commutative and cocommutative Hopf algebra and set $V = X \otimes X$. Then the cup and cap defined in Definition* 13 *commute*

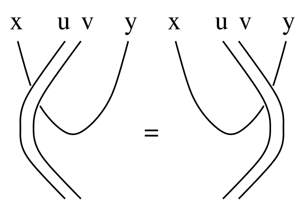

Fig. 14. The passcup property.

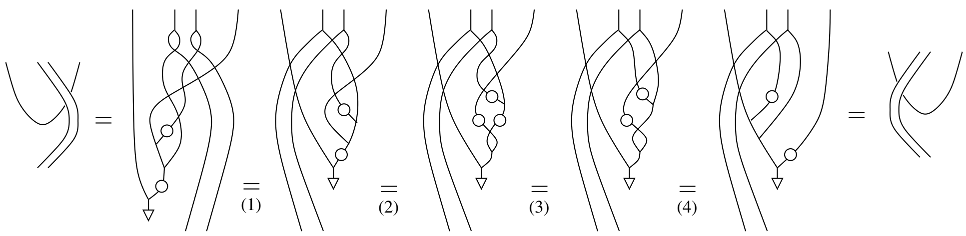

Fig. 15. Proof of the passcup property.

with the YB operator β defined in Lemma 3.7. *Specifically, it holds that* $(\mathbb{1} \otimes \cup)\beta = \cup \otimes \mathbb{1}$ *and* $(\cup \otimes \mathbb{1})\beta = \mathbb{1} \otimes \cup$ *as morphisms* $V \otimes V \to V$, $(\mathbb{1} \otimes \cap)\beta = \cap \otimes \mathbb{1}$ *and* $\beta(\cap \otimes \mathbb{1}) = \mathbb{1} \otimes \cap$ *as morphisms* $V \to V \otimes V$.

Proof. Diagrammatic sketch proofs are found in Figs. 16–19. In Fig. 16, $(\mathbb{1} \otimes \cup)\beta = \cup \otimes \mathbb{1}$ is proved by Lemmas 4.5 and 3.5 successively. Other equalities are proved as depicted, using Hopf algebra axioms and the definition of integrals. In Fig. 17, other than axioms, commutativity is used in the 4th equality, and cocommutativity is used in the 5th equality. While the diagrams in Figs. 17 and 18 treat the single-stranded

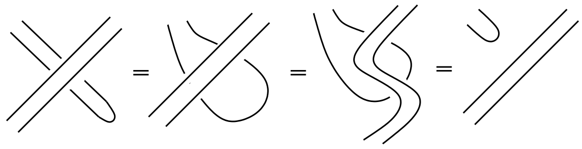

Fig. 16. Proof of $(\mathbb{1} \otimes \cup)\beta = \cup \otimes \mathbb{1}$.

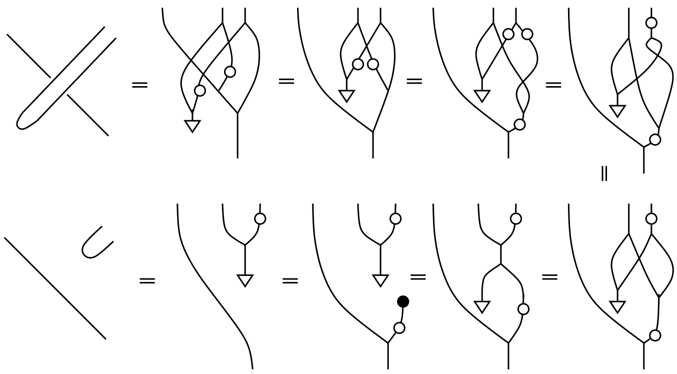

Fig. 17. Proof of $(\cup \otimes \mathbb{1})\beta = \mathbb{1} \otimes \cup$.

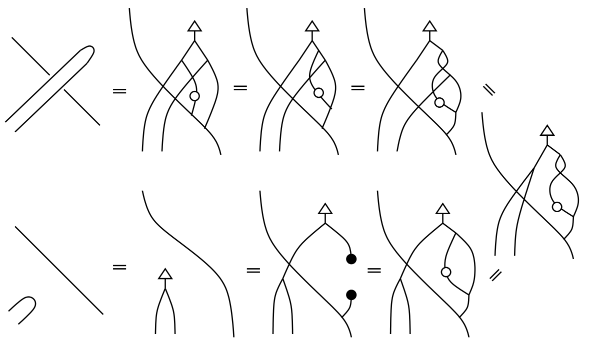

Fig. 18. Proof of $(\mathbb{1} \otimes \cap)\beta = \cap \otimes \mathbb{1}$.

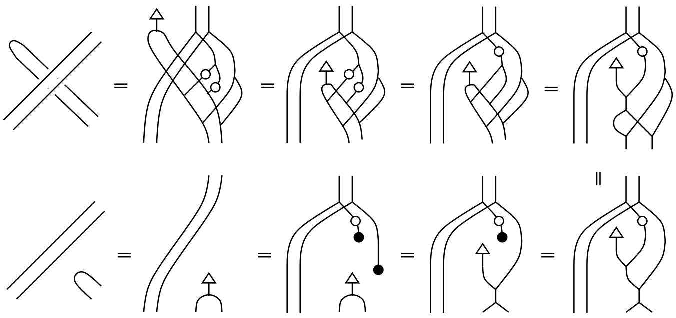

Fig. 19. Proof of $\beta(\cap \otimes \mathbb{1}) = \mathbb{1} \otimes \cap$.

case, an iteration of the diagrammatic proof implies the case with two edges sliding. Cocommutativity is used in the 3rd and 5th equalities in Fig. 18 and 3rd equality in Fig. 19. Observe that the equalities $S\eta = \eta$ and $\epsilon S = \epsilon$, which are consequences of S being an anti-homomorphism, have been used. $\square$

5. Construction of Braided Frobenius Algebras

In [4], a *braided Frobenius object* is defined to be a Frobenius object in a braided monoidal category. It is natural to define a braided Frobenius algebra to be, more generally, a Frobenius algebra endowed with a YB operator β, that satisfies the coherence conditions of a Frobenius object in a braided monoidal category with β playing the role of the braiding.

Definition 5.1 (cf. [4]). A *braided Frobenius algebra* is a Frobenius algebra $X = (V, \mu, \eta, \Delta, \epsilon)$ (multiplication, unit, comultiplication, counit) over unital ring $\Bbbk$, endowed with a YB operator $\beta : V \otimes V \to V \otimes V$, such that the Frobenius operations commute with β as follows:

$$\begin{aligned}
&(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \beta)(\beta \otimes \mathbb{1}) = \beta \otimes (\mathbb{1} \otimes \mu), && (\mathbb{1} \otimes \mu)(\beta \otimes \mathbb{1})(\mathbb{1} \otimes \beta) = \beta \otimes (\mu \otimes \mathbb{1}),\\
&(\Delta \otimes \mathbb{1})\beta = (\beta \otimes \mathbb{1})(\beta \otimes \mathbb{1})(\mathbb{1} \otimes \Delta), && (\mathbb{1} \otimes \Delta)\beta = (\beta \otimes \mathbb{1})(\beta \otimes \mathbb{1})(\Delta \otimes \mathbb{1}),\\
&(\mathbb{1} \otimes \eta)\beta = \eta \otimes \mathbb{1}, && \beta(\eta \otimes \mathbb{1}) = \mathbb{1} \otimes \eta,\\
&(\mathbb{1} \otimes \epsilon)\beta = \epsilon \otimes \mathbb{1}, && (\epsilon \otimes \mathbb{1})\beta = \mathbb{1} \otimes \epsilon.
\end{aligned}$$

The commuting conditions for a braided Frobenius algebra for multiplication are depicted in Fig. 1. Those for comultiplication are represented by the upside down diagrams. The commuting conditions for the (co)unit are depicted in Fig. 20.

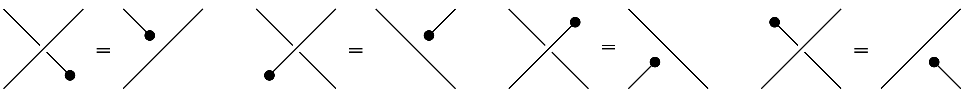

Fig. 20. Commutation of (co)unit and the YB operator.

It is expected that our results can be applied to the case of Frobenius objects in monoidal categories, since our main methods are diagrammatic. Although the main exception is the proof of (co)associativity in Theorem 5.2, one can replace the proof with diagrammatic arguments as well. Since the main scope of this paper is to obtain the algebraic background to have invariants of compact surfaces with boundary based on Hopf algebras, we restrict ourselves to working in the setting of modules over unital rings, as in Definition 5.1.

We now proceed to construct a family of braided Frobenius algebras from a class of Hopf algebras. We mention that the monoid structure in the next theorem also appears in [8, Secs. 4 and 5], under the name of *pair of pants monoid*, for dagger pivotal categories. In our construction, the fact that Frobenius monoids (e.g. algebras) are self-dual allows us to discard the duality in $X^* \otimes X$.

Theorem 5.2. *Let* $(X, \mu, \eta, \Delta, \epsilon, S)$ *be a commutative and cocommutative Hopf algebra. Then* $V = X \otimes X$ *has a braided Frobenius algebra structure.*

Proof. A product $\mu_{\otimes 2} : X^{\otimes 2} \otimes X^{\otimes 2} \to X^{\otimes 2}$ is defined by means of $\cup$ as

$$\mu_{\otimes 2} := \mathbb{1} \otimes \cup \otimes \mathbb{1}.$$

The coproduct $\Delta_{\otimes 2} : X^{\otimes 2} \to X^{\otimes 2} \otimes X^{\otimes 2}$ is obtained from $\cap$ by the definition

$$\Delta_{\otimes 2} := \mathbb{1} \otimes \cap \otimes \mathbb{1}.$$

The unit $\eta_{\otimes 2} : \Bbbk \to X^{\otimes 2} \otimes X^{\otimes 2}$ is defined by $\cap$. The unit condition follows from the switchback condition, as depicted in Fig. 21. The counit $\epsilon_{\otimes 2}$ is defined by $\epsilon_{\otimes 2} = \cup$ and the counit condition follows similarly. It is checked that these operations define a Frobenius structure in a manner similar to arguments found in [8].

Since X is an involutory Hopf algebra, applying Lemma 3.7 it follows that $X \otimes X$ has a YB operator β that is induced by the quantum heap structure of X. Hence, $X \otimes X$ is endowed with a Frobenius structure and a YB operator induced by the quantum heap operation.

To complete the proof, we need to show that YB operator and Frobenius morphisms commute in the sense of Definition 5.1. The commutations between (co)units and YB operator follow from Lemma 4.6 (see Figs. 16–19). For doubled strands, the commutations between multiplication and YB operator are depicted in Fig. 22. These follow from commutations between counits and YB operator. The commutations between comultiplication and YB operator are represented by the upside

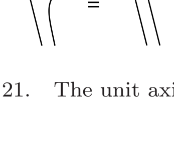

Fig. 21. The unit axiom.

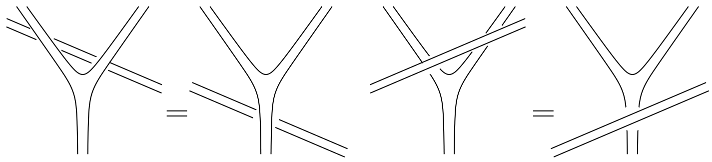

Fig. 22. Braided Frobenius conditions for a doubled Hopf algebra.

down (the vertical mirror) figures of Fig. 22, and follow from commutations between units and YB operator. □

Example 5.3. Let $X = \Bbbk[G]$ be a group ring of a group heap G with the TSD operation defined by linearlization of the group heap operation $T(x \otimes y \otimes z) := xy^{-1}z$ for $x, y, z \in G$. Endow X with the Hopf algebra structure, where μ is defined by the linearlized group multiplication, group $\Delta(x) = x \otimes x$ for $x \in G$, unit defined by $\eta(1) = e \in G$ (the identity element), and counit defined by $\epsilon(x) = 1$ for $x \in G$. The integral is defined by $\sum_{x\in G} x$ and cointegral by $e \mapsto 1$, $e \neq g \mapsto 0$. All conditions in Theorem 5.2 are checked. The YB operator is defined from T, and for group elements $\beta((x \otimes y) \otimes (u \otimes v)) = (u \otimes v) \otimes (xu^{-1}v \otimes yu^{-1}v)$. Thus, the YB operator is the linearlization of group heap YB operator as depicted in Fig. 2. If the group G is abelian, X satisfies the assumption of Theorem 5.2. Moreover, so does the dual Hopf algebra $\Bbbk[G]^*$.

Example 5.4. Let $\Bbbk$ be a PID or a local ring of characteristic p. Then the truncated polynomial algebra $H = \Bbbk[X]/(X^{p^k})$ is a finitely generated free (hence projective) Hopf algebra for any $k \geq 1$. As previously pointed out, H satisfies $P(X^*) \cong \Bbbk$ since $\Bbbk$ is either a PID or a local ring. We can therefore apply Lemma 4.3 and Theorem 5.2, since H is commutative and cocommutative. Explicitly, the algebra structure of H is determined by multiplication of polynomials, the comultiplication is obtained extending $\Delta(X) = 1 \otimes X + X \otimes 1$ to be an algebra homomorphism (note that it is here crucial that H is truncated at a power of the characteristic of the ground ring), the counit is defined by $\epsilon(1) = 1$, $\epsilon(X) = 0$ and the antipode is given by $S(X) = -X$. This construction can be generalized to truncated polynomial algebras with more than one indeterminate.

We note that considering local rings gives a wider class of objects with respect to that of PID's in [16]. For instance, the ring $\mathbb{Z}_p[Y_1, Y_2]/(Y_1, Y_2)^2$ is a local ring that is not a PID to which the previous construction can be applied.

6. Twists in Braided Frobenius Algebras

In this section, we introduce twists in braided Frobenius algebras, and discuss relations to tortile category structure and surfaces with boundary embedded in 3-space.

Definition 6.1. Let (V, Δ, ϵ) be a finite dimensional coalgebra over a field $\Bbbk$ with a TSD morphism $T : V^{\otimes 3} \to V$ (Definition 3.1). Then the morphism $\theta : V \otimes V \to V \otimes V$ defined by

$$\theta(x \otimes y) = T(x^{(1)} \otimes x^{(2)} \otimes y^{(2)}) \otimes T(y^{(1)} \otimes x^{(3)} \otimes y^{(3)})$$

is called a *twist* by T.

Remark 6.2. The twisting introduced in Definition 6.1 is motivated by a "quantum" version of the core quandle [7] operation $(x, y) \mapsto yx^{-1}y$ defined on groups. In fact we have

$$\begin{aligned} x \otimes y &\mapsto x^{(1)} S(x^{(2)}) y^{(2)} \otimes y^{(1)} S(x^{(3)}) y^{(3)} \\ &= \epsilon(x^{(1)}) y^{(2)} \otimes y^{(1)} S(x^{(2)}) y^{(3)} \\ &= y^{(2)} \otimes y^{(1)} S(x) y^{(3)}, \end{aligned}$$

where the second term in the tensor product can be identified with the core quandle operation between $y^{(1)}$ and x.

The operation θ is written by maps as follows. Fix a basis $\{e_i : i = 1, \ldots, n\}$ for V, and define the pairing $\vee : V \otimes V^* \to \Bbbk$ for the dual space V^* by $\vee(x_i \otimes x_j^*) = \delta_{i,j}$ with the Kronecker's delta, and copairing $\wedge : \Bbbk \to V \otimes V^*$ by $\wedge(1) = \sum_{i=1}^n x_i \otimes x_i^*$. Then θ is written as

$$\theta = (\mathbb{1}^{\otimes 2} \otimes \vee)(\mathbb{1}^{\otimes 3} \otimes \vee \otimes \mathbb{1})(\beta \otimes \mathbb{1}^{\otimes 2})(\mathbb{1}^{\otimes 3} \otimes \wedge \otimes \mathbb{1})(\mathbb{1}^{\otimes 2} \otimes \wedge)$$

with the YB operator β induced by T (Lemma 3.6). Diagrammatically, θ is represented by Fig. 23, and corresponds to a full twist as in the right of the figure. In the figure, the maxima and minima correspond to $\wedge$ and $\vee$, respectively, and they are indicated by such notations to distinguish them from $\cap$ and $\cup$.

Proposition 6.3. *Let (V, Δ, ϵ) be a cocommutative coalgebra over a unital ring $\Bbbk$ with a reversible TSD morphism $T : V^{\otimes 3} \to V$ (Definition 3.1). Let θ be the twist in Definition 6.1. Then θ commutes with the YB operator β induced by T. Specifically, we have $\beta(\theta \otimes \mathbb{1}) = (\mathbb{1} \otimes \theta)\beta$ and $\beta(\mathbb{1} \otimes \theta) = (\theta \otimes \mathbb{1})\beta$.*

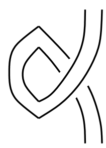

Fig. 23. Twisting a ribbon.

Proof. On simple tensors we have

$$\begin{aligned}\beta(\theta\otimes\mathbb{1})(x\otimes y\otimes z\otimes w) &= \beta(T(x^{(1)}\otimes x^{(2)}\otimes y^{(2)})\otimes T(y^{(1)}\otimes x^{(3)}\otimes y^{(3)})\otimes z\otimes w)\\ &= z^{(1)}\otimes w^{(1)}\otimes T(T(x^{(1)}\otimes x^{(2)}\otimes y^{(2)})\otimes z^{(2)}\otimes w^{(2)})\\ &\quad\otimes T(T(y^{(1)}\otimes x^{(3)}\otimes y^{(3)})\otimes z^{(3)}\otimes w^{(3)})\end{aligned}$$

and also

$$\begin{aligned}&(\mathbb{1}\otimes\theta)\beta(x\otimes y\otimes z\otimes w)\\ &\quad= (\mathbb{1}\otimes\theta)(z^{(1)}\otimes w^{(1)}\otimes T(x\otimes z^{(2)}\otimes w^{(2)})\otimes T(y\otimes z^{(3)}\otimes w^{(3)}))\\ &\quad= z^{(1)}\otimes w^{(1)}\otimes T(T(x\otimes z^{(2)}\otimes w^{(2)})^{(1)}\otimes T(x\otimes z^{(2)}\otimes w^{(2)})^{(2)}\\ &\qquad\otimes T(y\otimes z^{(3)}\otimes w^{(3)})^{(2)})\otimes T(T(y\otimes z^{(3)}\otimes w^{(3)})^{(1)}\\ &\qquad\otimes T(x\otimes z^{(2)}\otimes w^{(2)})^{(3)}\otimes T(y\otimes z^{(3)}\otimes w^{(3)})^{(3)})\\ &\quad= z^{(1)}\otimes w^{(1)}\otimes T(T(x^{(1)}\otimes z^{(21)}\otimes w^{(21)})\otimes T(x^{(2)}\otimes z^{(22)}\otimes w^{(22)})\\ &\qquad\otimes T(y^{(2)}\otimes z^{(32)}\otimes w^{(32)}))\otimes T(T(y^{(1)}\otimes z^{(31)}\otimes w^{(31)})\\ &\qquad\otimes T(x^{(3)}\otimes z^{(23)}\otimes w^{(23)})\otimes T(y^{(3)}\otimes z^{(33)}\otimes w^{(33)})),\end{aligned}$$

where the fact that, by definition, T is a coalgebra morphism has been applied in the second equality of the second set of equations. Applying cocommutativity (and coassociativity) of Δ we can rearrange the z and w terms to have the equality

$$\begin{aligned}&(\mathbb{1}\otimes\theta)\beta(x\otimes y\otimes z\otimes w)\\ &\quad= z^{(1)}\otimes w^{(1)}\otimes T(T(x^{(1)}\otimes z^{(2)}\otimes w^{(2)})\otimes T(x^{(2)}\otimes z^{(3)}\otimes w^{(3)})\\ &\qquad\otimes T(y^{(2)}\otimes z^{(4)}\otimes w^{(4)}))\otimes T(T(y^{(1)}\otimes z^{(5)}\otimes w^{(5)})\\ &\qquad\otimes T(x^{(3)}\otimes z^{(6)}\otimes w^{(6)})\otimes T(y^{(3)}\otimes z^{(7)}\otimes w^{(7)})),\end{aligned}$$

which shows that $\beta(\theta\otimes\mathbb{1})(x\otimes y\otimes z\otimes w)$ and $(\mathbb{1}\otimes\theta)\beta(x\otimes y\otimes z\otimes w)$ differ by an application of the TSD condition of T. This shows the equality $\beta(\theta\otimes\mathbb{1}) = (\mathbb{1}\otimes\theta)\beta$.

Let us now consider the equation $\beta(\mathbb{1}\otimes\theta) = (\theta\otimes\mathbb{1})\beta$. For the left-hand side we have

$$\begin{aligned}&\beta(\mathbb{1}\otimes\theta)(x\otimes y\otimes z\otimes w)\\ &\quad= T(z^{(11)}\otimes z^{(21)}\otimes w^{(21)})\otimes T(w^{(11)}\otimes z^{(31)}\otimes w^{(31)})\\ &\qquad\otimes T(x\otimes T(z^{(12)}\otimes z^{(22)}\otimes w^{(22)})\otimes T(w^{(12)}\otimes z^{(32)}\otimes w^{(32)})\\ &\qquad\otimes T(y\otimes T(z^{(13)}\otimes z^{(23)}\otimes w^{(23)})\otimes T(w^{(13)}\otimes z^{(33)}\otimes w^{(33)}),\end{aligned}$$

while for the right-hand side we have

$$\begin{aligned}(\theta\otimes\mathbb{1})\beta(x\otimes y\otimes z\otimes w) &= T(z^{(11)}\otimes z^{(12)}\otimes w^{(12)})\otimes T(w^{(11)}\otimes z^{(13)}\otimes w^{(13)})\\ &\quad\otimes T(x\otimes z^{(2)}\otimes w^{(2)})\otimes T(y\otimes z^{(3)}\otimes w^{(3)}).\end{aligned}$$

To complete the proof we see that it is enough to show the equality

$$T(x \otimes T(z^{(1)} \otimes z^{(2)} \otimes w^{(2)}) \otimes T(w^{(1)} \otimes z^{(3)} \otimes w^{(3)})) = T(x \otimes z \otimes w). \tag{1}$$

We have

$$\begin{aligned} &T(x \otimes T(z^{(1)} \otimes z^{(2)} \otimes w^{(2)}) \otimes T(w^{(1)} \otimes z^{(3)} \otimes w^{(3)})) \\ &\quad = \epsilon(z^{(2)})\epsilon(w^{(2)}) \cdot T(x \otimes T(z^{(1)} \otimes z^{(3)} \otimes w^{(3)}) \otimes T(w^{(1)} \otimes z^{(4)} \otimes w^{(4)})) \\ &\quad = T(T(T(x \otimes w^{(3)} \otimes z^{(3)}) \otimes z^{(2)} \otimes w^{(2)}) \otimes T(z^{(1)} \otimes z^{(4)} \otimes w^{(4)}) \\ &\qquad \otimes T(w^{(1)} \otimes z^{(5)} \otimes w^{(5)})), \end{aligned}$$

where the first equality uses the definition of counit ϵ, and the second equality makes use of the invertibility condition of T. Let us now apply the TSD property of T to the terms $T(x \otimes w^{(2)} \otimes z^{(2)})$, $z^{(1)}$, $w^{(1)}$, $z^{(3)}$ and $w^{(3)}$, where we set $T(x \otimes w^{(3)} \otimes z^{(3)}) = q$ for convenience. We get

$$\begin{aligned} &T(T(q \otimes z^{(2)} \otimes w^{(2)}) \otimes T(z^{(1)} \otimes z^{(4)} \otimes w^{(4)}) \otimes T(w^{(1)} \otimes z^{(5)} \otimes w^{(5)})) \\ &\quad = T(T(q \otimes z^{(2)} \otimes w^{(2)}) \otimes z^{(1)} \otimes w^{(1)}) = \epsilon(z^{(1)})\epsilon(w^{(1)}) \cdot T(x \otimes z^{(2)} \\ &\qquad \otimes w^{(2)}) = T(x \otimes z \otimes w), \end{aligned}$$

where invertibility of T, as well as cocommutativity of Δ, has been used in the second equality. This shows that Eq. (1) holds. □

Remark 6.4. Here we discuss relations to the notion of *tortile category*. A braided monoidal category with duals is called *tortile* [12] (or *ribbon* [8]) if it is endowed with natural isomorphisms $\theta_X : X \to X$, called *twists*, such that $\theta_{X \otimes Y} = \beta_{Y,X}\beta_{X,Y}(\theta_X \otimes \theta_Y)$ for all objects X, Y, where β denotes the braiding, $\theta_I = \mathbb{1}$, where I denotes the unit object, and $(\theta_X)^* = \theta_{X^*}$ for all X.

Let (V, Δ, ϵ) be a finite dimensional coalgebra over a field $\Bbbk$ with a TSD operation $T : V^{\otimes 3} \to V$ (Definition 3.1). Then Proposition 6.3 implies that a twist on $V^{\otimes 2k}$ can be defined from the twist in Definition 6.1 (and the pairing and copairing defined in the paragraph preceding Proposition 6.3). More specifically, the twist θ_k on $V^{\otimes 2k}$ is defined by parallel loops, that are defined by taking k-fold parallel ribbons. The case $k = 2$ is depicted in Fig. 24 left. The equality $\theta_2 = \beta_{V,V}\beta_{V,V}(\theta \otimes \theta)$ is indicated

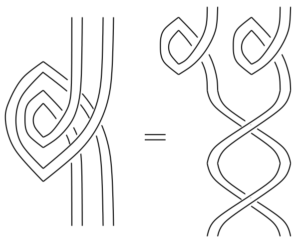

Fig. 24. Twisting a doubled ribbon.

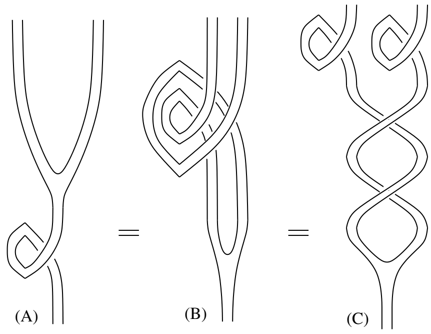

Fig. 25. Commutation between a twist and multiplication.

in the figure. More general statements on parallel string constructions and tortile categories are found in [12].

Proposition 6.5. *Let X be as in Theorem* 5.2, *and let $V = X \otimes X$ denote the associated braided Frobenius structure on the doubled vector space. Let θ be the twist in Definition* 6.1. *Then the twist θ commutes with the multiplication and comultiplication. This means, with notations as in Remark* 6.4, *that $\theta_V\, \mu_{\otimes 2} = \mu_{\otimes 2}\, \theta_{V,V}$ and $\Delta_{\otimes 2}\theta_V = \theta_{V,V}\Delta_{\otimes 2}$ hold.*

The commutation between the twist and multiplication is depicted in the left equality $(A) = (B)$ of Fig. 25. The right equality $(B) = (C)$ is a consequence of Fig. 24. We note that the resulting equality $(A) = (C)$ corresponds diagrammatically to twisting the trivalent vertex by one full twist.

Proof. We verify equality $\theta_V\, \mu_{\otimes 2} = \mu_{\otimes 2}\, \theta_{V,V}$ on simple tensors $x \otimes y \otimes z \otimes w$. For the left-hand side we have

$$\theta_V\, \mu_{\otimes 2}(x \otimes y \otimes z \otimes w) = \gamma(yS(z)) \cdot w^{(2)} \otimes w^{(1)} S(x) w^{(3)}.$$

The right-hand side is given as

$$\begin{aligned}
\mu_{\otimes 2}\, \theta_{V,V}(x \otimes y \otimes z \otimes w) &= \gamma(y^{(1)} S(x^{(3)}) y^{(3)} S(z^{(3)}) w^{(3)} S(w^{(4)}) z^{(4)} S(y^{(4)}) \\
&\quad \times x^{(4)} S(z^{(1)})) \\
&\quad \cdot x^{(1)} S(x^{(2)}) y^{(2)} S(z^{(2)}) w^{(2)} \otimes w^{(1)} S(x^{(5)}) y^{(5)} S(z^{(5)}) w^{(5)} \\
&= \gamma(y^{(1)} S(z^{(1)})) \cdot y^{(2)} S(z^{(2)}) w^{(2)} \\
&\quad \otimes w^{(1)} S(x) y^{(3)} S(z^{(3)}) w^{(3)} \\
&= \gamma(yS(z)) \cdot w^{(2)} \otimes w^{(1)} S(x) w^{(3)},
\end{aligned}$$

where the first equality is obtained by unraveling the definitions, the second equality is a multiple application of the counit axiom, and the third equality follows by

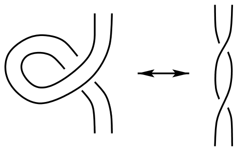

Fig. 26. Twisting a ribbon by a loop.

applying the definition of cointegral γ twice. Equality $\Delta_{\otimes 2}\theta_V = \theta_{V,V}\Delta_{\otimes 2}$ is proven on simple tensors in a similar fashion. □

Remark 6.6. For the braided Frobenius algebra V constructed in Theorem 5.2, a twist Θ can be defined using $\cap$ and $\cup$ instead of $\wedge$ and $\vee$ as depicted in Fig. 26 left. Specifically,

$$\Theta = (\mathbb{1}^{\otimes 2} \otimes \cup)(\mathbb{1}^{\otimes 3} \otimes \cup \otimes \mathbb{1})(\beta \otimes \mathbb{1}^{\otimes 2})(\mathbb{1}^{\otimes 3} \otimes \cap \otimes \mathbb{1})(\mathbb{1}^{\otimes 2} \otimes \cap)$$

with the YB operator β induced by T (Lemma 3.6). Since all maps that appear in this formula commute with the YB operator β from earlier lemmas, Θ commute with β. By the same argument as Remark 6.4, we obtain a twist Θ_k on $V^{\otimes 2k}$. Similarly, Θ commutes with μ and Δ. A sketch proof of the commutation between μ and Θ is depicted in Fig. 27.

We close the paper with remarks on invariants of embedded surfaces with boundary. It is of interest to find invariants, using braided Frobenius algebras, of compact orientable surfaces with boundary represented by ribbon graph diagrams, as considered in [18], in a way analogous to quantum knot invariants. In this approach, a height function is fixed on the plane, and building blocks of diagrams consist of cups and caps in addition to crossings and trivalent vertices. Although a complete set of moves for ribbon graph diagrams for certain embedded surfaces was given in [18], height functions were not considered. It is desirable to have a list of additional moves. For example, the passcup move and passcap move (the upside

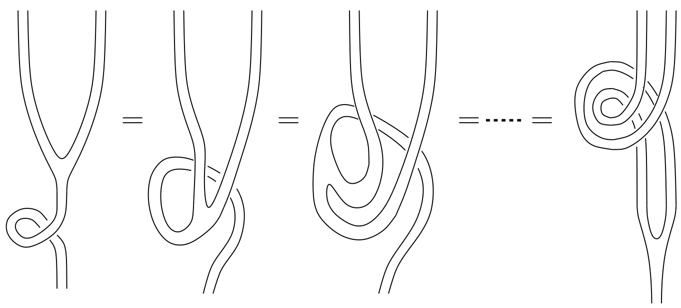

Fig. 27. Sketch picture proof of commutation.

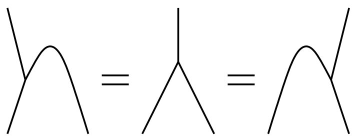

Fig. 28. Conversion of μ to Δ through $\cap$.

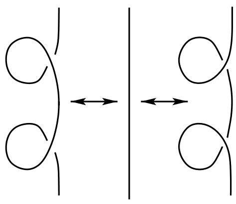

Fig. 29. Canceling a pair of loops.

down of passcup) are such moves, and they are satisfied by braided Frobenius algebras constructed in this paper. Another move depicted in Fig. 28 is also satisfied from Frobenius algebra axioms. Although most moves in [18] for orientable surfaces (without half twists), with appropriate choices of height functions, are satisfied by our resulting braided Frobenius algebras, it is not clear at this time whether the equation corresponding to the move depicted in Fig. 29 is satisfied, in general, under our construction. However, it may be satisfied by some specific examples, and may provide invariants for such surfaces.

For non-orientable surfaces, ribbon graph diagrams [18] contain half-twists, and there is a move of twisting a vertex as indicated in Fig. 30, that involve half twists of ribbons merging at a vertex. From the topological correspondence of the twist θ to a full twists as in Fig. 26, such a hypothetical half twist, which we denote by $\sqrt{\theta}$, would be required to satisfy $\sqrt{\theta} \circ \sqrt{\theta} = \theta$ (thus the notation). We have not found such a morphism in braided Frobenius algebras constructed in Theorem 5.2, and raise a question: For the twists (θ and Θ) defined in this section for the braided Frobenius algebras constructed in Theorem 5.2, are there half twists $\sqrt{\theta}$ and $\sqrt{\Theta}$? We point out a curious fact that the composition of a half-twist of a vertex in Fig. 30 twice is a full twist of a vertex represented by Fig. 25, which is satisfied by the braided Frobenius algebras constructed in this paper.

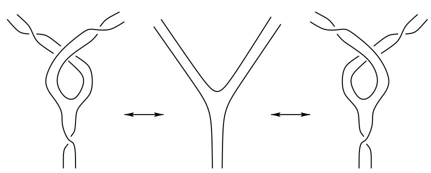

Fig. 30. Twisting a vertex of a ribbon.

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