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Strong L^2 convergence of time Euler schemes for stochastic 3D Brinkman–Forchheimer–Navier–Stokes equations

Hakima Bessaih[1] · **Annie Millet**[2,3]

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Abstract
We prove that some time Euler schemes for the 3D Navier–Stokes equations modified by adding a Brinkman–Forchheimer term and a random perturbation converge in $L^2(\Omega)$. This extends previous results concerning the strong rate of convergence of some time discretization schemes for the 2D Navier Stokes equations. Unlike the 2D case, our proposed 3D model with the Brinkman–Forchheimer term allows for a strong rate of convergence of order almost 1/2, that is independent of the viscosity parameter.

Keywords Stochastic Navier–Sokes equations · Time Euler schemes · Strong convergence · Implicit time discretization · Brinkman–Forchheimer

Mathematics Subject Classification 60H15 · 60H35 · 76D06 · 76M35

1 Introduction

An incompressible fluid flow dynamic can be described by the so-called incompressible Navier–Stokes equations (NSEs). The fluid flow is defined by a velocity field

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✉ Annie Millet
annie.millet@univ-paris1.fr

Hakima Bessaih
hbessaih@fiu.edu

1 Mathematics and Statistics Department, Florida International University, 11200 SW 8th Street, Miami, FL 33199, USA

2 SAMM, EA 4543, Université Paris 1 Panthéon Sorbonne, 90 Rue de Tolbiac, 75634 Paris Cedex, France

3 LPSM, UMR 8001, Universités Paris 6-Paris 7, Paris Cedex, France

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u and a pressure term π that evolve in a very particular way. These equations are parametrized by the viscosity coefficient $\nu > 0$. Many questions are open in the 3D setting. In this paper, we will focus on the 3D incompressible Navier–Stokes equations with a smoothing term of Brinkman–Forchheimer type, in a bounded domain $D = [0, L]^3$ of $\mathbb{R}^3$, and subject to an external forcing defined as:

$$\begin{aligned} \partial_t u - \nu \Delta u + (u \cdot \nabla) u + a|u|^{2\alpha} u + \nabla \pi = G(u) dW \quad \text{in} \quad (0, T) \times D, \\ \text{div} u = 0 \quad \text{in} \quad (0, T) \times D, \end{aligned} \tag{1.1}$$

for $a > 0, \alpha \in [1, +\infty)$ and some terminal time $T > 0$. The process $u : \Omega \times [0, T] \times D \to \mathbb{R}^3$ is the velocity field with initial condition u_0 in D, and periodic boundary conditions $u(t, x + Lv_i) = u(t, x)$ on $(0, T) \times \partial D$, where v_i, $i = 1, 2, 3$ denotes the canonical basis of $\mathbb{R}^3$, and $\pi : \Omega \times [0, T] \times D \to \mathbb{R}$ is the pressure. Note that similar computations using the restriction to a bounded domain as a technical step would enable to deal with $D = \mathbb{R}^3$ (with no boundary condition). In order to focus on the main issue, this will not be treated here.

Here G is a diffusion coefficient with global Lipschitz conditions and linear growth and the driving noise W is a Wiener process defined of a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. In 2D, there is an extensive literature concerning the deterministic NSEs and we refer to the books of Temam; see [27, 28] for known results. The stochastic setting has also been widely investigated in dimension 2, see [19] for some very general results and the references therein. Unique global weak and strong solutions (in the PDE sense) are constructed for both additive and multiplicative noise, and without being exhaustive, we refer to [11, 15].

Global well posedness in the 3D case is a famous open problem, and can be proved with some additional smoothing term such as either a Brinkman–Forchheimer nonlinearity to model porous media, or some rotating fluid term. Let us mention that these models can be used with some anisotropic viscosity, that is no viscosity in one direction (see e.g. [10, 14]). The stochastic case has been investigated as well by several authors among which Flandoli et al.; see for example [18] for an account of remaining open problems. The anisotropic 3D case with a stochastic perturbation has been studied in [20] for rotating fluids, and in [6] for a Brinkman–Forchheimer modification.

Numerical schemes and algorithms were introduced to best approximate and construct solutions for PDEs. A similar approach has started to emerge for stochastic models, in particular SPDEs, and has known a strong interest by the probability community. Many algorithms based on either finite difference, finite elements or spectral Galerkin methods (for the space discretization), and on either Euler, Crank–Nicolson or splitting schemes (for the temporal discretization) have been introduced for both the linear and nonlinear cases. Their rates of convergence have been widely investigated. The literature on numerical analysis for SPDEs is now very extensive. Models having either linear, global Lipschitz properties or more generally some monotonicity properties are well developed in an extensive literature, see [3, 4]. In this case the convergence is proven to be in mean square. When nonlinearities are involved that are not of Lipschitz or monotone type, a rate of convergence in mean square is more difficult to obtain. Indeed, because of the stochastic perturbation, there is no way of using the Gronwall lemma after taking the expectation of the error bound because it involves a nonlinear term that is usually in a quadratic form. One way of getting around it is to localize the nonlinear term in order to get a linear inequality, and then use the Gronwall lemma. This gives rise to a rate of convergence in probability, that was first introduced by Printems [26].

Discretizations of the 2D stochastic Navier–Stokes equations with a multiplicative noise were investigated in several papers. The following ones provide a rate of convergence in probability of time implicit Euler or splitting schemes [5, 12, 13, 17]. The Euler scheme is coupled with a finite element space discretization. Note that [17] tackles the problem of weak convergence, that is convergence in distribution, while in case of an additive noise [11] proves almost sure and mean square convergence without giving an explicit rate.

Strong (i.e. $L^2(\Omega)$) convergence for a time splitting scheme, for an implicit time Euler scheme—coupled with a finite elements approximation—of the stochastic 2D Navier–Stokes equations were proven in [7, 8] for either a multiplicative noise or an “additive” noise. In the latter case a polynomial (suboptimal) speed of convergence is proven.

In [9], strong convergence of a space-time discretization (implicit Euler scheme in time and finite elements approximation in space) for stochastic 2D Navier–Stokes equations on the torus with an additive noise is studied. The rate of convergence is “optimal”, namely almost 1/2 in time and 1 in space. However, since exponential moments of the H^1-norm of the solution is used, some constraints on the strength of the noise have to be imposed. In the additive case, no localization is needed and the argument is based on a direct use of the discrete Gronwall lemma.

In this paper, we study a time implicit Euler scheme (5.1) for a stochastic 3D Navier–Stokes equation with a modification, by adding a smoothing term of Brinkman–Forchheimer type. Unlike the 2D case—and thanks to this extra term—neither localization nor exponential moments are needed, and we obtain the “optimal” convergence rate with no constraint on the noise and the viscosity. For technical reasons, we only have to assume that the exponent α of the Brinkman–Forchheimer term $|u|^{2\alpha}u$ in (1.1) belongs to the interval $[1, \frac{3}{2}]$. The proof is based on a careful study of the time regularity of the solution in both the L^2 and H^1 norms, and the discrete Gronwall lemma.

The paper is organized as follows. Section 2 describes the functional setting of the model. In Sect. 3 we describe the stochastic perturbation, state the global well posedness of the solution to (1.1) and its moment estimates in various norms. If the exponent $\alpha = 1$ we have to impose that the coefficient a is “large”. The way the Brinkman–Forchheimer term helps to obtain estimates for the bilinear part is described in Sect. 7.1 of the Appendix. The proof of the existence and uniqueness relies on a Galerkin approximation. It is quite classical, similar to the anisotropic case described in [6]. The proof is sketched in Sects. 7.2 and 7.3 of the Appendix for the sake of completeness. Section 4 is devoted to the moment time increments of the solution to (1.1) in L^2 and H^1; the results are crucial to obtain the optimal strong convergence rate. In Sect. 5 we describe the fully implicit time Euler scheme, prove its existence and some moment estimates. Finally, in Sect. 6 we prove the strong (that is $L^2(\Omega)$) convergence rate of this scheme.

As usual, except if specified otherwise, C denotes a positive constant that may change throughout the paper, and $C(a)$ denotes a positive constant depending on some parameter a.

2 Notations and preliminary results

Let $D=[0,L]^3$ with periodic boundary conditions, $\mathbb{L}^p:=L^p(D)^3$ (resp. $\mathbb{W}^{k,p}:=W^{k,p}(D)^3$) be the usual Lebesgue and Sobolev spaces of vector-valued functions endowed with the norms $\|\cdot\|_{\mathbb{L}^p}$ (resp. $\|\cdot\|_{\mathbb{W}^{k,p}}$). If $p=2$, set $\mathbb{H}^k:=\mathbb{W}^{k,2}$ and we denote by $\|\cdot\|_k$ the $\mathbb{H}^k$ norm, $k=0,1,\ldots$; note that $\|\,.\,\|_0=\|\cdot\|_{\mathbb{L}^2}$. In what follows, we will consider velocity fields that have zero divergence on D. Let H (resp. V) be the subspace of $\mathbb{L}^2$ (resp. $\mathbb{H}^1$) defined by

$$
\begin{aligned}
H &:= \{u\in\mathbb{L}^2\,:\,\operatorname{div} u=0 \text{ weakly in } D \text{ with periodic boundary conditions}\},\\
V &:= H\cap\mathbb{W}^{1,2}.
\end{aligned}
$$

H and V are separable Hilbert spaces. The space H inherits its inner product denoted by $(\cdot,\cdot)$ and its norm $\|\cdot\|_H$ from $\mathbb{L}^2$. The norm in V, inherited from $\mathbb{W}^{1,2}$, is denoted by $\|\cdot\|_V$; we let $(\cdot,\cdot)_V$ denote the associated inner product. Moreover, let V' be the dual space of V with respect to the pivot space H, and $\langle\cdot,\cdot\rangle$ denotes the duality between V' and V.

Let $\Pi:\mathbb{L}^2\to H$ denote the Leray projection, and set $A=-\Pi\Delta$ with its domain $\mathrm{Dom}(A)=\mathbb{W}^{2,2}\cap H$.

Let $b:V^3\to\mathbb{R}$ denote the trilinear map defined by

$$
b(u_1,u_2,u_3):=\int_D\big(\big[u_1(x)\cdot\nabla\big]u_2(x)\big)\cdot u_3(x)\,dx,
$$

which by the incompressibility condition satisfies $b(u_1,u_2,u_3)=-b(u_1,u_3,u_2)$ for $u_i\in V$, $i=1,2,3$. There exists a continuous bilinear map $B:V\times V\mapsto V'$ such that

$$
\langle B(u_1,u_2),u_3\rangle=b(u_1,u_2,u_3),\quad \text{for all } u_i\in V,\ i=1,2,3.
$$

The map B satisfies the following antisymmetry relations:

$$
\langle B(u_1,u_2),u_3\rangle=-\langle B(u_1,u_3),u_2\rangle,\quad \langle B(u_1,u_2),u_2\rangle=0,\quad \forall u_i\in V. \tag{2.1}
$$

For $u,v\in V$, set $B(u,v):=\Pi\big(\big[u\cdot\nabla\big]v\big)$.

In dimension 3, the Gagliardo–Nirenberg inequality implies that for $p\in[2,6]$, $\mathbb{H}^1\subset\mathbb{L}^p$; more precisely

$$
\|u\|_{\mathbb{L}^4}\le\bar{C}_4\,\|u\|_{\mathbb{L}^2}^{\frac14}\,\|\nabla u\|_{\mathbb{L}^2}^{\frac34}\quad\text{and}\quad\|u\|_{\mathbb{L}^3}\le\bar{C}_3\,\|u\|_{\mathbb{L}^2}^{\frac12}\,\|\nabla u\|_{\mathbb{L}^2}^{\frac12},\ \forall u\in\mathbb{H}^1, \tag{2.2}
$$

for some positive constants $\bar{C}_3$ and $\bar{C}_4$.

Furthermore, the Gagliardo–Nirenberg inequality implies that $\mathbb{H}^2 \subset \mathbb{L}^p$ for any $p \in [2, \infty)$, and for $u \in \mathbb{H}^2$

$$\|u\|_{\mathbb{L}^p} \le C(p)\, \|Au\|_{\mathbb{L}^2}^{\beta(p)}\, \|u\|_{\mathbb{L}^2}^{1-\beta(p)} \quad \text{for} \quad \beta(p) = \frac{3}{2}\Big(\frac{1}{2} - \frac{1}{p}\Big). \tag{2.3}$$

Note that for $p = 6$ we have $\beta(6) = \frac{1}{2}$. Furthermore, $\|u\|_{\mathbb{L}^\infty} \le C\|u\|_{\mathbb{H}^2}$ for $u \in \mathbb{H}^2$.

Let $\alpha \in (1, +\infty)$ and let $f, g, h : D \to \mathbb{R}$ be regular functions. Given any positive constants ε_0 and ε_1 and some constant C_α depending on α, the following upper estimates are straightforward consequences of the Hölder and Young inequalities

$$\int_D \big|f(x)g(x)h(x)\big|dx \le \big\| |f||g|^{\frac{1}{\alpha}} \big\|_{L^{2\alpha}} \big\| |g|^{1-\frac{1}{\alpha}} \big\|_{L^{\frac{2\alpha}{\alpha-1}}} \|h\|_{L^2}. \tag{2.4}$$

$$\le \epsilon_0 \|h\|_{L^2}^2 + \frac{\varepsilon_1}{4\varepsilon_0} \big\| |f|^\alpha g \big\|_{L^2}^2 + \frac{C_\alpha}{\varepsilon_0 \varepsilon_1^{\frac{1}{\alpha-1}}} \|g\|_{L^2}^2. \tag{2.5}$$

Let $\Omega_T = \Omega \times [0, T]$ be endowed with the product measure $d\mathbb{P} \otimes ds$ on $\mathcal{F} \otimes \mathcal{B}(0, T)$. The following functional notations will be used throughout the paper. Set

$$X_0 = L^\infty(0, T; H) \cap L^2(0, T; V) \cap L^{2\alpha+2}([0, T] \times D; \mathbb{R}^3), \tag{2.6}$$

$$\mathcal{X}_0 = L^4\big(\Omega; L^\infty(0, T; H)\big) \cap L^2\big(\Omega; L^2(0, T; V)\big) \cap L^{2\alpha+2}(\Omega_T \times D; \mathbb{R}^3), \tag{2.7}$$

$$X_1 = L^\infty(0, T; V) \cap L^2(0, T; \mathrm{Dom}A) \cap \Big\{u : [0, T] \times D \to \mathbb{R}^3 :$$
$$\int_0^T \big[\|u(t)\|_{\mathbb{L}^{2\alpha+2}}^{2\alpha+2} + \big\| |u(t)|^\alpha \nabla u(t) \big\|_{\mathbb{L}^2}^2\big]dt < \infty\Big\}, \tag{2.8}$$

$$\mathcal{X}_1 = L^4\big(\Omega; L^\infty(0, T; V)\big) \cap L^2\big(\Omega; L^2(0, T; \ \mathrm{Dom}\ A)\big) \cap \Big\{u : \Omega_T \times D \to \mathbb{R}^3 :$$
$$\mathbb{E}\int_0^T \big[\|u(t)\|_{\mathbb{L}^{2\alpha+2}}^{2\alpha+2} + \big\| |u(t)|^\alpha \nabla u(t) \big\|_{\mathbb{L}^2}^2\big]dt < \infty\Big\}. \tag{2.9}$$

3 Global well posedness and first moment estimates

For technical reasons, we assume that the initial condition u_0 belongs to $L^p(\Omega; V)$ for some $p \in [2, \infty]$, and only consider *strong solutions* in the PDE sense. We prove that the stochastic 3D Navier–Stokes equation with Brinkman–Forchheimer smoothing (1.1) has a unique solution on any time interval $[0, T]$ and prove moment estimates of this solution. This requires some hypotheses on the driving noise W and the diffusion coefficient G.

3.1 The driving noise and the diffusion coefficient

Let $(e_k, k \geq 1)$ be an orthonormal basis of H whose elements belong to $\mathbb{H}^2 := W^{2,2}(D;\mathbb{R}^3)$ and are orthogonal in V. Let $\mathcal{H}_n = \text{span}(e_1, \ldots, e_n)$ and let P_n (resp. $\tilde{P}_n$) denote the orthogonal projection from H (resp. V) onto $\mathcal{H}_n$. Furthermore, given $i \neq j$ we have

$$(Ae_i\ ,\ e_j) = (\nabla e_i\ ,\ \nabla e_j) = 0$$

since the basis $\{e_n\}_n$ is orthogonal in V. Hence $Au \in \mathcal{H}_n$ for every $u \in \mathcal{H}_n$.

We deduce that for $u \in V$ we have $P_n u = \tilde{P}_n u$. Indeed, for $v \in \mathcal{H}_n$ and $u \in V$:

$$(P_n u, v) = (u, v), \text{ and } (\nabla P_n u, \nabla v) = -(P_n u, Av) = -(u, Av) = (\nabla u, \nabla v). \quad (3.1)$$

Hence given $u \in V$, we have $(P_n u, v)_V = (u, v)_V$ for any $v \in \mathcal{H}_n$.

Let K be a separable Hilbert space and Q be a symmetric, positive trace-classe operator on K. Let $(W(t), t \in [0, T])$ be a K-valued Wiener process with covariance operator Q, defined on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Let $\{\zeta_j\}_{j\geq 1}$ denote an orthonormal basis of K made of eigenfunctions of Q, with eigenvalues $\{q_j\}_{j\geq 1}$ and $\text{Tr}Q = \sum_{j\geq 1} q_j < \infty$. Then

$$W(t) = \sum_{j=1}^{\infty} \sqrt{q_j}\, \beta^j(t)\, \zeta_j, \qquad \forall t \in [0, T],$$

where $\{\beta_j\}_{j\geq 1}$ are independent one-dimensional Brownian motions defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. For details concerning this Wiener process we refer to [16].

Let $\mathcal{L} \equiv \mathcal{L}(K; H)$ (resp. $\tilde{\mathcal{L}} \equiv \mathcal{L}(K; V)$) be the space of continuous linear operators from K to H (resp. V) with norm $\|\,.\,\|_{\mathcal{L}}$ (resp. $\|\,.\,\|_{\tilde{\mathcal{L}}}$).

The noise intensity of the stochastic perturbation $G : V \to \tilde{\mathcal{L}}$ which we put in (1.1) satisfies the following classical growth and Lipschitz conditions (i) and (ii). Note that due to the 3D framework, we have to impose growth conditions both on the $\|\cdot\|_{\mathcal{L}}$ and $\|\cdot\|_{\tilde{\mathcal{L}}}$ norms.

The diffusion coefficient G satisfies the following assumption:

Condition (G) Assume that $G : V \to \tilde{\mathcal{L}}$ satisfies the following conditions:

(i) **Growth condition** There exist positive constants $K_i, \tilde{K}_i, i = 0, 1$, such that

$$\|G(u)\|_{\mathcal{L}}^2 \leq K_0 + K_1\|u\|_H^2, \quad \forall u \in H, \quad (3.2)$$
$$\|G(u)\|_{\tilde{\mathcal{L}}}^2 \leq \tilde{K}_0 + \tilde{K}_1\|u\|_V^2, \quad \forall u \in V. \quad (3.3)$$

(ii) **Lipschitz condition** There exists a positive constant L such that

$$\|G(u) - G(v)\|_{\mathcal{L}}^2 \leq L\|u - v\|_H^2, \quad \forall u, v \in H. \quad (3.4)$$

We define a weak pathwise solution (that is strong probabilistic solution in the weak deterministic sense) of (1.1) as follows:

Definition 1 We say that Eq. (1.1) has a strong solution if:

- u is an adapted V-valued process which belongs a.s. to X_1,
- $\mathbb{P}$ a.s. we have $u \in C([0,T];V)$, and

$$\begin{aligned}
\big(u(t),\phi\big) + \nu \int_0^t \big(\nabla u(s), \nabla\phi\big)ds + \int_0^t \big\langle [u(s)\cdot\nabla]u(s), \phi\big\rangle ds \\
+ a\int_0^t \int_D |u(s,x)|^{2\alpha} u(s,x)\phi(x)dxds \\
= \big(u_0,\phi\big) + \int_0^t \big(\phi, G(u(s))dW(s)\big)
\end{aligned}$$

for every $t \in [0,T]$ and every $\phi \in V$.

3.2 Global well-posedness and moment estimates of the solution

We next prove that if $\mathbb{E}(\|u_0\|_V^4) < \infty$, then (1.1) has a unique solution u in $\mathcal{X}_1$.

Theorem 2 *Let $\alpha \in [1,+\infty)$, and for $\alpha = 1$ suppose that $4\nu a > 1$. Let $u_0 \in L^{2p}(\Omega;V)$, for some $p \in [1,\infty)$, be independent of W, and G satisfy the growth and Lipschitz conditions* **(G)**. *Then Eq. (1.1) has a unique solution in $\mathcal{X}_1$ such that a.s. $u \in C([0,T];V)$. Furthermore,*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|u(t)\|_V^{2p} + \int_0^T \|Au(t)\|_{\mathbb{L}^2}^2 dt + \int_0^T \|u(t)\|_{\mathbb{L}^{2\alpha+2}}^{2\alpha+2} dt\Big) \le C\big[1 + \mathbb{E}\big(\|u_0\|_V^{2p}\big)\big]. \tag{3.5}$$

The proof, which is quite classical, is sketched in Sect. 7.3 of the Appendix.

4 Moment estimates of time increments of the solution

In this section we prove moment estimates for various norms of time increments of the solution to (1.1). This will be crucial to deduce the speed of convergence of numerical schemes. Let $u_0 \in L^{2p}(\Omega;V)$ for some $p \in [2,\infty)$ and u be the solution to (1.1), that is

$$\begin{aligned}
u(t) = S(t)u_0 - \int_0^t S(t-s)B(u(s),u(s))ds - a\int_0^t S(t-s)\Pi|u(s)|^{2\alpha}u(s)ds \\
+ \int_0^t S(t-s)G(u(s))dW(s), \quad \forall t \in [0,T], \quad \mathbb{P} \text{ a.s.}
\end{aligned} \tag{4.1}$$

where $S(t) = e^{-\nu t A}$ is the analytic semi group generated by the Stokes operator A multiplied by the viscosity ν. Then (see e.g. [13, Lemma 2.2] and [26, Lemma 2.1]), for $b > 0$ and $t \in [0, T]$,

$$\big\| A^b e^{-\nu t A} \big\|_{\mathcal{L}(\mathbb{L}^2;\mathbb{L}^2)} \le C(b,\nu)\, t^{-b}, \tag{4.2}$$

$$\big\| A^{-b}\big(\mathrm{Id} - e^{-\nu t A}\big) \big\|_{\mathcal{L}(\mathbb{L}^2;\mathbb{L}^2)} \le \widetilde{C(b,\nu)}\, t^{b}, \tag{4.3}$$

for some positive constants $C(b, \nu)$ and $\widetilde{C(b,\nu)}$.

The following regularity result for the bilinear term will be crucial in the proof of time regularity.

Lemma 1 *(i) There exists a positive constant M such that*

$$\|A^{-\frac14} B(u,u)\|_{\mathbb{L}^2} \le M \|A^{\frac12} u\|^2_{\mathbb{L}^2} \le M \|u\|_V^2, \quad \forall u \in V. \tag{4.4}$$

(ii) For $\delta \in (0, \frac34)$,

$$\|A^{-\delta} B(u,u)\|_{\mathbb{L}^2} \le C \|Au\|_{\mathbb{L}^2}^{\frac34-\delta} \|u\|_{\mathbb{H}^1}^{\frac54+\delta}, \quad \forall u \in \mathrm{Dom}(A). \tag{4.5}$$

Proof (i) Using [21, Lemma 2.2] we deduce that given positive constants δ, θ, ρ such that $0 \le \delta < \frac12 + \frac34$, $\theta > 0$, $\rho > 0$ such that $\rho + \delta > \frac12$ and $\delta + \theta + \rho \ge \frac54$, there exists a constant $M := M(\delta, \theta, \rho)$ such that for u, v regular enough

$$\|A^{-\delta} B(u,v)\|_{\mathbb{L}^2} \le M \|A^{\theta} u\|_{\mathbb{L}^2} \|A^{\rho} v\|_{\mathbb{L}^2}.$$

Choosing $\delta = \frac14$, $\theta = \rho = \frac12$, we deduce (4.4).

(ii) For $u \in \mathbb{H}^2$, we have

$$\|A^{-\delta} B(u,u)\|_{\mathbb{L}^2} = \sup \Big\{ \int_D |\nabla u|\, |u|\, |\phi|\, dx;:\ \|\phi\|_{\mathbb{H}^{2\delta}} \le 1 \Big\}.$$

In dimension 3, the Sobolev embedding theorem (see e.g. [1, Theorem 7.57, p. 217]) implies $\mathbb{W}^{\beta,p}(D) \subset \mathbb{L}^q(D)$ if $3 > \beta p$, $\beta > 0$, $1 < p < 3$ and $p \le q \le \frac{3p}{3-\beta p}$. Hence for $\delta \in (0, \frac34)$, choosing $\beta = 2\delta$, $p = 2$ and $q = \frac{6}{3-4\delta}$, we obtain $\mathbb{W}^{2\delta,2}(D) = \mathbb{H}^{2\delta}(D) \subset \mathbb{L}^q(D)$. Let $\bar p = \frac{3}{2\delta}$; then $\frac{1}{\bar p} + \frac12 + \frac1q = 1$, and the Hölder inequality yields

$$\|A^{-\delta} B(u,u)\|_{\mathbb{L}^2} \le C \|\nabla u\|_{\mathbb{L}^2} \|u\|_{\mathbb{L}^{\bar p}}.$$

The Gagliardo–Nirenberg inequality (2.3) implies $\|u\|_{\mathbb{L}^{\bar p}} \le C \|Au\|_{\mathbb{L}^2}^{\frac34-\delta} \|u\|_{\mathbb{L}^2}^{\frac14+\delta}$. This concludes the proof of (4.5). □

The following result proves regularity of the Brinkman–Forchheimer term. To have a regularity similar to that of the bilinear term, we have to impose some restriction on the exponent α.

Lemma 2 *Let $\alpha \in [1, \frac32]$.*

(i) there exists a positive constant C such that

$$\big\| A^{-\frac14}\big(|u|^{2\alpha}u\big)\big\|_{\mathbb{L}^2} \le C\|u\|_V^{2\alpha+1}, \quad \forall u\in V. \tag{4.6}$$

(ii) Furthermore, for any $\delta\in(0,\frac34)$ there exists $C>0$ such that

$$\big\| A^{-\delta}\big(|u|^{2\alpha}u\big)\big\|_{\mathbb{L}^2} \le C\|Au\|_{\mathbb{L}^2}^{\frac34-\delta}\|u\|_V^{2\alpha+\frac14+\delta}, \quad \forall u\in \mathrm{Dom}(A). \tag{4.7}$$

Proof We use once more the Sobolev embedding theorem $\mathbb{W}^{\beta,p}(D)\subset\mathbb{L}^r(D)$ if $3>\beta p$, $\beta>0$, $1<p<3$ and $p\le r\le\frac{3p}{3-\beta p}$.

(i) Choosing $\beta=\frac12$, $p=2$ and $r=3$, we obtain $\mathbb{W}^{\frac12,2}(D)=H^{\frac12}(D)\subset\mathbb{L}^3(D)$, while $\beta=1$, $p=2$ and $r\in[2,6]$ yields $H^1(D)\subset\mathbb{L}^r(D)$. Given $u\in\mathbb{H}^1$, we have

$$\big\| A^{-\frac14}\big(|u|^{2\alpha}u\big)\big\|_{\mathbb{L}^2} = \sup\Big\{\int_D |u(x)|^{2\alpha}u(x)\phi(x)dx : \|\phi\|_{\mathbb{H}^{\frac12}}\le 1\Big\}.$$

Using Hölder's inequality with exponents 2,6 and 3, we obtain for $\delta\in[\frac14,\frac34)$

$$\big\| A^{-\frac14}\big(|u|^{2\alpha}u\big)\big\|_{\mathbb{L}^2} \le \sup\{\|u\|_{\mathbb{L}^{4\alpha}}^{2\alpha}\|u\|_{\mathbb{L}^6}\|\phi\|_{\mathbb{L}^3} : \|\phi\|_{\mathbb{H}^{\frac12}}\le 1\}\le C\|u\|_V^{2\alpha+1},$$

where the last upper estimate is a consequence of the inequality $4\alpha\in[4,6]$. This completes the proof of (4.6).

(ii) As in the proof of Lemma 4.1 (ii) we choose $q=\frac{6}{3-4\delta}$ to ensure $\mathbb{H}^{2\delta}\subset\mathbb{L}^q$ and $p=\frac{3}{2\delta}$. The Hölder and Gagliardo–Nirenberg inequalities imply

$$\begin{aligned}\|A^{-\delta}(|u|^{2\alpha}u)\|_{\mathbb{L}^2} = &\ \sup\Big\{\int |u|^{2\alpha}|u|\phi : \|\phi\|_{\mathbb{H}^{2\delta}}\le 1\Big\}\le \|u\|_{\mathbb{L}^{4\alpha}}^{2\alpha}\|u\|_{\mathbb{L}^p}\|\phi\|_{\mathbb{L}^q}\\ \le &\ C\|u\|_{\mathbb{L}^{4\alpha}}^{2\alpha}\|Au\|_{\mathbb{L}^2}^{\frac34-\delta}\|u\|_{\mathbb{L}^2}^{\frac14+\delta}.\end{aligned}$$

Since $\alpha\in[1,\frac32]$, the Sobolev embedding $\mathbb{H}^1\subset\mathbb{L}^\gamma$ for $\gamma\in[4,6]$ concludes the proof. □

The following proposition gives upper estimates for moments of time increments of the solution to the stochastic 3D modified Navier Stokes equation u defined in equation (4.1).

Proposition 3 *Let u_0 be $\mathcal{F}_0$-measurable and let $\alpha\in[1,\frac32]$ with $4\nu a>1$ if $\alpha=1$. Suppose that the diffusion coefficient G satisfies Condition* **(G)** *and let u be the solution to (1.1). Then for $\lambda\in(0,\frac12)$ we have*

(i) Suppose $u_0\in L^{(2\alpha+1)p}(\Omega;V)$ for some $p\in[2,\infty)$. There exists a positive constant $C:=C(T,a,p,\mathrm{Tr}Q)$ such that for $0\le t_1<t_2\le T$,

$$\mathbb{E}\big(\|u(t_2)-u(t_1)\|_H^p\big)\le C\,|t_2-t_1|^{\lambda p}\big[1+\mathbb{E}\big(\|u_0\|_V^{(2\alpha+1)p}\big)\big]. \tag{4.8}$$

(ii) Let $N \geq 1$ be an integer and for $k = 0, \dots, N$ set $t_k = \frac{kT}{N}$. Then there exists $C := C(T, a, \mathrm{Tr}\,Q, \lambda) > 0$ (independent of N) such that for $p(\lambda) = \frac{2+8\alpha-2\lambda}{1-\lambda}$ and $u_0 \in \mathbb{L}^{p(\lambda)}(\Omega; V)$

$$\mathbb{E}\Big(\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \big[\|\nabla(u(s)-u(t_j))\|_{\mathbb{L}^2}^2 + \|\nabla(u(s)-u(t_{j-1}))\|_{\mathbb{L}^2}^2\big]ds\Big) \\ \leq C\Big(\frac{T}{N}\Big)^{2\lambda}\Big[1+\mathbb{E}\Big(\|u_0\|_V^{p(\lambda)}\Big)\Big]. \tag{4.9}$$

Proof The proof relies on a semi-group argument.
(i) Let $t_1 < t_2$ belong to the time interval $[0, T]$. Then $u(t_2)-u(t_1) = \sum_{i=1}^4 T_i$, where

$$T_1 = S(t_2)u_0 - S(t_1)u_0,$$
$$T_2 = -\int_0^{t_2} S(t_2-s)B(u(s),u(s))ds + \int_0^{t_1} S(t_1-s)B(u(s),u(s))ds,$$
$$T_3 = -a\int_0^{t_2} S(t_2-s)\big(|u(s)|^{2\alpha}u(s)\big)ds + a\int_0^{t_1} S(t_1-s)\big(|u(s)|^{2\alpha}u(s)\big)ds,$$
$$T_4 = \int_0^{t_2} S(t_2-s)G(u(s))dW(s) - \int_0^{t_1} S(t_1-s)G(u(s))dW(s).$$

Then using (4.3) and the upper estimate $\sup_{t\in[0,T]}\|S(t)\|_{\mathcal{L}(\mathbb{L}^2;\mathbb{L}^2)} < \infty$ we deduce

$$\|T_1\|_{\mathbb{L}^2} = \big\|S(t_1)A^{-\frac12}\big[S(t_2-t_1)-\mathrm{Id}\big]A^{\frac12}u_0\big\|_{\mathbb{L}^2} \\ \leq C\|S(t_1)\|_{\mathcal{L}(\mathbb{L}^2;\mathbb{L}^2)}\,|t_2-t_1|^{\frac12}\,\|A^{\frac12}u_0\|_{\mathbb{L}^2} \leq C\,|t_2-t_1|^{\frac12}\,\|u_0\|_V.$$

Hence taking expected values, we deduce for every $p \in [2, \infty)$

$$\mathbb{E}\big(\|T_1\|_{\mathbb{L}^2}^p\big) \leq C^p\,|t_2-t_1|^{\frac{p}{2}}\,\mathbb{E}(\|u_0\|_V^p). \tag{4.10}$$

Furthermore, $T_2 = -T_{2,1} - T_{2,2}$, where

$$T_{2,1} = \int_0^{t_1} S(t_1-s)\big[S(t_2-t_1)-\mathrm{Id}\big]B(u(s),u(s))ds,$$
$$T_{2,2} = \int_{t_1}^{t_2} S(t_2-s)B(u(s),u(s))ds.$$

Using the Minkowski inequality, (4.2), (4.3) and (4.4), we deduce that for $\varepsilon \in \big(0, \frac14\big)$,

$$\|T_{2,1}\|_{\mathbb{L}^2} \leq \int_0^{t_1} \|A^{1-\epsilon}S(t_1-s)\;A^{-(\frac34-\varepsilon)}\big[S(t_2-t_1)-\mathrm{Id}\big] \\ \times A^{-\frac14}B(u(s),u(s))\|_{\mathbb{L}^2}ds$$

$$\leq \; C\,|t_2-t_1|^{\frac34-\varepsilon} \sup_{s\in[0,t_1]} \|u(s)\|_V^2 \int_0^{t_1} (t_1-s)^{-1+\varepsilon} ds.$$

Hence (3.5) implies that if $\mathbb{E}(\|u_0\|_V^{2p}) < \infty$ for some $p \in [1,\infty)$, we have

$$\mathbb{E}\big(\|T_{2,1}\|_{\mathbb{L}^2}^p\big) \leq C(T)\,|t_2-t_1|^{(\frac34-\varepsilon)p}\big[1+\mathbb{E}\|u_0\|_V^{2p}\big]. \tag{4.11}$$

The Minkowski inequality, (4.2) and (4.4) imply

$$\begin{aligned}\|T_{2,2}\|_{\mathbb{L}^2} &\leq \int_{t_1}^{t_2} \|A^{\frac14} S(t_2-s) A^{-\frac14} B(u(s),u(s))\|_{\mathbb{L}^2} ds\\ &\leq \; C \sup_{s\in[t_1,t_2]} \|u(s)\|_V^2 \int_{t_1}^{t_2} (t_2-s)^{-\frac14} ds.\end{aligned}$$

Using once more (3.5) we deduce that if $\mathbb{E}(\|u_0\|_V^{2p}) < \infty$ for some $p \in [1,\infty)$,

$$\mathbb{E}\big(\|T_{2,2}\|_{\mathbb{L}^2}^p\big) \leq C\,|t_2-t_1|^{\frac34 p}\big[1+\mathbb{E}\|u_0\|_V^{2p}\big]. \tag{4.12}$$

A similar decomposition yields $T_3 = -a\big(T_{3,1}+T_{3,2}\big)$, where

$$\begin{aligned}T_{3,1} &= \int_0^{t_1} S(t_1-s)\big[S(t_2-t_1)-\mathrm{Id}\big]|u(s)|^{2\alpha} u(s) ds,\\ T_{3,2} &= \int_{t_1}^{t_2} S(t_2-s)|u(s)|^{2\alpha} u(s) ds.\end{aligned}$$

The Minkowski inequality and the upper estimates (4.2), (4.3) and (4.6) imply that for $\varepsilon \in \big(0,\frac14\big)$,

$$\begin{aligned}\|T_{3,1}\|_{\mathbb{L}^2} \leq & \int_0^{t_1} \|A^{1-\varepsilon} S(t_1-s)\, A^{-(\frac34-\varepsilon)}\big[S(t_2-t_1)-\mathrm{Id}\big]\\ &\times\; A^{-\frac14}\big(|u(s)|^{2\alpha} u(s)\big)\|_{\mathbb{L}^2} ds\\ \leq & \; C\,|t_2-t_1|^{\frac34-\varepsilon} \sup_{s\in[0,t_1]} \|u(s)\|_V^{2\alpha+1} \int_0^{t_1} (t_1-s)^{-(1-\varepsilon)} ds,\end{aligned}$$

and the upper estimate (3.5) implies that for $p \in [1,\infty)$,

$$\mathbb{E}\big(\|T_{3,1}\|_{\mathbb{L}^2}^p\big) \leq C\;|t_2-t_1|^{(\frac34-\varepsilon)p}\;\big[1+\mathbb{E}\|u_0\|_V^{(2\alpha+1)p}\big]. \tag{4.13}$$

The Minkowski inequality and the upper estimates (4.2) and (4.6) imply

$$\|T_{3,2}\|_{\mathbb{L}^2} \leq \int_{t_1}^{t_2} \|A^{\frac14} S(t_2-s) A^{-\frac14}\big(|u(s)|^{2\alpha} u(s)\big)\|_{\mathbb{L}^2} ds$$

$$\leq \ C(T)\int_{t_1}^{t_2}(t_2-s)^{-\frac{1}{4}}\|u(s)\|_V^{2\alpha+1}ds \leq C(T)|t_2-t_1|^{\frac{3}{4}}\sup_{s\in[t_1,t_2]}\|u(s)\|_V^{2\alpha+1}.$$

Then using once more (3.5) we obtain for $p \in [1,\infty)$,

$$\mathbb{E}(\|T_{3,2}\|_{\mathbb{L}^2}^p) \leq C(T,p)|t_2-t_1|^{\frac{3}{4}p}\big[1+\mathbb{E}\|u_0\|_V^{(2\alpha+1)p}\big]. \tag{4.14}$$

A similar decomposition of the stochastic integral yields $T_4 = T_{4,1}+T_{4,2}$, where

$$T_{4,1} = \int_0^{t_1} S(t_1-s)\big[S(t_2-t_1)-\mathrm{Id}\big]G(u(s))dW(s),$$
$$T_{4,2} = \int_{t_1}^{t_2} S(t_2-s)G(u(s))dW(s).$$

The Burkholder–Davis–Gundy inequality, the growth condition (3.2), (4.2) and (4.3) imply for $\epsilon \in \big(0,\frac{1}{2}\big)$ and $p\in[1,\infty)$,

$$\begin{aligned}
\mathbb{E}(\|T_{4,1}\|_{\mathbb{L}^2}^{2p}) \leq&\ C_p\,\mathbb{E}\Big(\Big|\int_0^{t_1}\|S(t_1-s)\big[S(t_2-t_1)-\mathrm{Id}\big]G(u(s))\|_{\mathcal{L}}^2\,\mathrm{Tr}Q\,ds\Big|^p\Big)\\
\leq&\ C_p\,\mathbb{E}\Big(\Big|\int_0^{t_1}\|A^{\frac{1}{2}-\varepsilon}S(t_1-s)\|_{\mathcal{L}(\mathbb{L}^2;\mathbb{L}^2)}^2\\
&\times\|A^{-(\frac{1}{2}-\varepsilon)}\big[S(t_2-t_1)-\mathrm{Id}\big]\|_{\mathcal{L}(\mathbb{L}^2;\mathbb{L}^2)}^2\|G(u(s))\|_{\mathcal{L}}^2\,\mathrm{Tr}Q\,ds\Big|^p\Big)\\
\leq&\ C_p\,(\mathrm{Tr}Q)^p\ |t_2-t_1|^{(1-2\varepsilon)p}\Big[K_0^p+K_1^p\,\mathbb{E}\Big(\sup_{s\in[0,t_1]}\|u(s)\|_H^{2p}\Big)\Big]\\
&\times\Big(\int_0^{t_1}(t_1-s)^{-1+2\varepsilon}ds\Big)^p\\
\leq&\ C(T,p,\mathrm{Tr}Q)\,|t_2-t_1|^{(1-2\varepsilon)p}\big[1+\mathbb{E}\big(\|u_0\|_V^{2p}\big)\big],
\end{aligned} \tag{4.15}$$

where the last upper estimate is deduced from (3.5).

Finally, using once more the Burkholder–Davis–Gundy inequality, $\sup_{t\in[0,T]}\|S(t)\|_{\mathcal{L}(\mathbb{L}^2;\mathbb{L}^2)}<\infty$, the growth condition (3.2) and (3.5), we obtain for $p\in[1,\infty)$

$$\begin{aligned}
\mathbb{E}(\|T_{4,2}\|_{\mathbb{L}^2}^{2p}) \leq&\ C_p\,\mathbb{E}\Big(\Big|\int_{t_1}^{t_2}\|S(t_2-s)G(u(s))\|_{\mathcal{L}}^2\,\mathrm{Tr}Q\,ds\Big|^p\Big)\\
\leq&\ C_p\,(\mathrm{Tr}Q)^p\ \mathbb{E}\Big(\Big|\int_{t_1}^{t_2}\|S(t_2-s)\|_{\mathcal{L}(\mathbb{L}^2;\mathbb{L}^2)}^2\big[K_0+K_1\|u(s)\|_H^2\big]ds\Big|^p\Big)\\
\leq&\ C_p\,(\mathrm{Tr}Q)^p\ |t_2-t_1|^p\big[1+\mathbb{E}\big(\|u_0\|_V^{2p}\big)\big].
\end{aligned} \tag{4.16}$$

The upper estiimates (4.10)–(4.16) conclude the proof of (4.8).

(ii) For $1\leq j\leq N$, $s\in[t_{j-1},t_j)$ we have $\nabla u(t_j)-\nabla u(s)=\sum_{i=1}^4 T_i(s,j)$, where

$$T_1(s,j) = \ \nabla S(t_j)u_0-\nabla S(s)u_0,$$

$$T_2(s,j) = -\int_0^{t_j} \nabla S(t_j - r) B(u(r), u(r)) dr + \int_0^s \nabla S(s-r) B(u(r), u(r)) dr,$$
$$T_3(s,j) = -a\int_0^{t_j} \nabla S(t_j - r)\big(|u(r)|^{2\alpha} u(r)\big) dr + a\int_0^s \nabla S(s-r)\big(|u(r)|^{2\alpha} u(r)\big) dr,$$
$$T_4(s,j) = \int_0^{t_j} S(t_j - r)\nabla G(u(r)) dW(r) - \int_0^s S(s-r)\nabla G(u(r)) dW(r).$$

Using the upper estimates (4.2) and (4.3) we obtain

$$\|T_1(s,j)\|_{\mathbb{L}^2} = \|A^\delta S(s)\ A^{-\delta}\big[S(t_j - s) - \mathrm{Id}\big] A^{\frac12} u_0\|_{\mathbb{L}^2} \le C s^{-\delta} |t_j - s|^\delta \|u_0\|_V$$

for any $\delta \in (0, 1]$. Therefore, given any $\delta \in (0, \frac12)$, we deduce

$$\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|T_1(s,j)\|_{\mathbb{L}^2}^2 ds \le C\Big(\frac{T}{N}\Big)^{2\delta} \|u_0\|_V^2 \int_0^T s^{-2\delta} ds = C(T,\lambda)\Big(\frac{T}{N}\Big)^{2\delta} \|u_0\|_V^2. \tag{4.17}$$

As in the proof of (i), let $T_2(s,j) = -\big(T_{2,1}(s,j) + T_{2,2}(s,j)\big)$, where

$$T_{2,1}(s,j) = \int_0^s \nabla S(s-r)\big[S(t_j - s) - \mathrm{Id}\big] B(u(r), u(r)) dr,$$
$$T_{2,2}(s,j) = \int_s^{t_j} \nabla S(t_j - r) B(u(r), u(r)) dr.$$

The Minkowski inequality and the upper estimates (4.2), (4.3) and (4.5) imply for $\delta \in \big(0, \frac12)$ and $\gamma \in (0, \frac12 - \delta)$

$$\begin{aligned}
&\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|T_{2,1}(s,j)\|_{\mathbb{L}^2}^2 ds \\
&\quad \le \sum_{j=1}^N \int_{t_{j-1}}^{t_j} ds \Big\{ \int_0^s \|A^{\frac12+\delta+\gamma} S(s-r) A^{-\gamma}\big[S(t_j - s) - \ \mathrm{Id}\ \big] \\
&\qquad \times A^{-\delta} B(u(r), u(r)) dr\|_{\mathbb{L}^2} dr \Big\}^2 \\
&\le C\Big(\frac{T}{N}\Big)^{2\gamma} \sum_{j=1}^N \int_{t_{j-1}}^{t_j} ds \Big\{ \int_0^s (s-r)^{-(\frac12+\delta+\gamma)} \|Au(r)\|_{\mathbb{L}^2}^{\frac34-\delta} \|u(r)\|_V^{\frac54+\delta} dr\Big\}^2 \\
&\le C\Big(\frac{T}{N}\Big)^{2\gamma} \int_0^T ds \Big(\int_0^s (s-r)^{-(\frac12+\delta+\gamma)} dr\Big) \\
&\qquad \times \Big(\int_0^s (s-r)^{-(\frac12+\delta+\gamma)} \|Au(r)\|_{\mathbb{L}^2}^{2(\frac34-\delta)} dr\Big) \sup_{r\in[0,T]} \|u(r)\|_V^{2(\frac54+\delta)},
\end{aligned} \tag{4.18}$$

where in the last upper estimate, we have used the Cauchy–Schwarz inequality with respect to the measure $(s-r)^{-\frac{1}{2}-\delta-\gamma}1_{(0,s)}(r)dr$.

Since $\int_r^T (s-r)^{-(\frac{1}{2}+\delta+\gamma)}ds \le \int_0^T s^{-(\frac{1}{2}+\delta+\gamma)}ds = C(T,\delta,\gamma)$ for any $r\in[0,T)$, and $\int_0^s (s-r)^{-(\frac{1}{2}+\delta+\gamma)}dr \le C(T,\delta,\gamma)$ for any $s\in[0,T)$, using the Fubini theorem, Hölder's and Jensen's inequalities with respect to dP with conjugate exponents $\frac{1}{\frac{1}{4}+\delta}$ and $\frac{1}{\frac{3}{4}-\delta}$, we deduce

$$\begin{aligned}
\mathbb{E}\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|T_{2,1}(s,j)\|_{\mathbb{L}^2}^2 ds &\le C\Big(\frac{T}{N}\Big)^{2\gamma} C(T,\delta,\gamma)\\
&\quad\times \mathbb{E}\Big(\sup_{r\in[0,T]} \|u(r)\|_V^{2(\frac{5}{4}+\delta)} \int_0^T dr \|Au(r)\|_{\mathbb{L}^2}^{2(\frac{3}{4}-\delta)} \int_r^T (s-r)^{-(\frac{1}{2}+\delta+\gamma)}ds\Big)\\
&\le C\Big(\frac{T}{N}\Big)^{2\gamma} C(T,\delta,\gamma)^2 \Big\{\mathbb{E}\Big(\sup_{r\in[0,T]} \|u(r)\|_V^{\frac{2(5+4\delta)}{1+4\delta}}\Big)\Big\}^{\frac{1}{4}+\delta}\\
&\quad\times \Big\{\mathbb{E}\Big(\int_0^T \|Au(r)\|_{\mathbb{L}^2}^2 dr\Big)\Big\}^{\frac{3}{4}-\delta}.
\end{aligned}$$

Let $\lambda\in(0,\frac{1}{2})$, $\delta=\frac{1-2\lambda}{4}\in(0,\frac{1}{4})$ and $\gamma\in(0,\frac{1}{2}-2\delta)$. Using (3.5) we infer

$$\mathbb{E}\Big(\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|T_{2,1}(s,j)\|_{\mathbb{L}^2}^2 ds\Big) \le C(T,\delta)\Big(\frac{T}{N}\Big)^{2\lambda}\Big[1+\mathbb{E}\Big(\|u_0\|_V^{\frac{6-2\lambda}{1-\lambda}}\Big)\Big]. \tag{4.19}$$

Using the Minkowski inequality, (4.2), (4.5) and Hölder's inequality for the measure $1_{[t_{j-1},t_j]}(s)ds$ with conjugate exponents $p_1=\frac{2}{\frac{3}{4}-\delta}$ and $p_2=\frac{2}{\frac{5}{4}+\delta}$ we have $p_2(\frac{1}{2}+\delta)<1$ for $\delta\in(0,\frac{1}{4})$, and deduce

$$\begin{aligned}
&\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|T_{2,2}(s,j)\|_{\mathbb{L}^2}^2 ds\\
&\le C\sum_{j=1}^N \int_{t_{j-1}}^{t_j} ds\Big\{\int_s^{t_j} \big\|A^{\frac{1}{2}+\delta}S(t_j-r)\, A^{-\delta}B(u(r),u(r))\big\|_{\mathbb{L}^2} dr\Big\}^2\\
&\le C\sum_{j=1}^N \int_{t_{j-1}}^{t_j} ds\Big\{\int_s^{t_j} (t_j-r)^{-(\frac{1}{2}+\delta)}\|Au(r)\|_{\mathbb{L}^2}^{\frac{3}{4}-\delta}\|u(r)\|_V^{\frac{5}{4}+\delta} dr\Big\}^2\\
&\le C\sup_{r\in[0,T]}\|u(r)\|_V^{\frac{5}{2}+2\delta}\sum_{j=1}^N \int_{t_{j-1}}^{t_j}\Big(\int_s^{t_j}(t_j-r)^{-p_2(\frac{1}{2}+\delta)}dr\Big)^{\frac{2}{p_2}}\\
&\quad\times\Big(\int_s^{t_j}\|Au(r)\|_{\mathbb{L}^2}^2 dr\Big)^{\frac{2}{p_1}} ds
\end{aligned}$$

$$\leq C \sup_{r\in[0,T]} \|u(r)\|_V^{\frac52+2\delta} \Big(\frac{T}{N}\Big)^{\frac14-\delta} \sum_{j=1}^N \int_{t_{j-1}}^{t_j} ds \Big(\int_{t_{j-1}}^{t_j} \|Au(r)\|_{\mathbb{L}^2}^2 dr\Big)^{\frac{2}{p_1}}.$$

The Hölder inequality for the counting measure on $\{1,\ldots,N\}$ with conjugate exponents $\frac{p_1}{2} = \frac{1}{\frac34-\delta}$ and $\frac{1}{\frac14+\delta}$ yields

$$\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|T_{2,2}(s,j)\|_{\mathbb{L}^2}^2 ds \leq C(T,\delta)\Big(\frac{T}{N}\Big)^{\frac54-\delta} \sup_{r\in[0,T]} \|u(r)\|_V^{\frac52+2\delta}$$
$$\times \Big\{ \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|Au(r)\|_{\mathbb{L}_2}^2 dr\Big\}^{\frac34-\delta} N^{\frac14+\delta}$$
$$\leq C(T,\delta)\Big(\frac{T}{N}\Big)^{1-2\delta} \sup_{r\in[0,T]} \|u(r)\|_V^{\frac52+2\delta} \Big\{ \int_0^T \|Au(r)\|_{\mathbb{L}_2}^2 dr\Big\}^{\frac34-\delta}.$$

Hölder's inequality with respect to dP with conjugate exponents $\frac{1}{\frac34-\delta}$ and $\frac{1}{\frac14+\delta}$ implies

$$\mathbb{E}\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|T_{2,2}(s,j)\|_{\mathbb{L}^2}^2 ds \leq C(T,\delta)\Big(\frac{T}{N}\Big)^{1-2\delta} \Big\{\mathbb{E}\Big(\sup_{r\in[0,T]} \|u(r)\|_V^{\frac{10+8\delta}{1+4\delta}}\Big)\Big\}^{\frac14+\delta}$$
$$\times \Big\{\mathbb{E}\Big(\int_0^T \|Au(r)\|_{\mathbb{L}^2}^2 dr\Big)\Big\}^{\frac34-\delta}. \tag{4.20}$$

Let $\lambda\in(0,\frac12)$ and $\delta=\frac{1-2\lambda}{4}\in(0,\frac14)$. The inequalities (4.19), (4.20) and (3.5) imply

$$\mathbb{E}\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|T_2(s,j)\|_{\mathbb{L}^2}^2 ds \leq C(T,\lambda)\Big(\frac{T}{N}\Big)^{2\lambda}\Big[1+\mathbb{E}\Big(\|u_0\|_V^{\frac{6-2\lambda}{1-\lambda}}\Big)\Big]. \tag{4.21}$$

A similar decomposition yields $T_3(s,j) = -a\big(T_{3,1}(s,j)+T_{3,2}(s,j)\big)$, where

$$T_{3,1}(s,j) = \int_0^s \nabla S(s-r)\big[S(t_j-s)-\mathrm{Id}\big]\big(|u(r)|^{2\alpha}u(r)\big)dr,$$
$$T_{3,2}(s,j) = \int_s^{t_j} \nabla S(t_j-r)\big(|u(r)|^{2\alpha}u(r)\big)dr.$$

The Minkowski inequality and the upper estimates (4.2), (4.3), (4.7) imply for $\delta\in(0,\frac12)$ and $\gamma\in(0,\frac12-\delta)$,

$$\|T_{3,1}(s,j)\|_{\mathbb{L}^2} \leq \int_0^s \Big\| A^{\frac12+\delta+\gamma} S(s-r)\; A^{-\gamma}\big[S(t_j-s)-\mathrm{Id}\big]$$
$$\times A^{-\delta}\big(|u(r)|^{2\alpha}u(r)\big)\Big\|_{\mathbb{L}^2} dr$$

$$\leq C(t_j-s)^{\gamma}\int_0^s (s-r)^{-(\frac12+\delta+\gamma)}\|Au(r)\|_{\mathbb{L}^2}^{\frac34-\delta}\|u(r)\|_V^{2\alpha+\frac14+\delta}dr.$$

Therefore, given $\delta\in\big(0,\frac12\big)$ and $\gamma\in(0,\frac12-\delta)$

$$\sum_{j=1}^N\int_{t_{j-1}}^{t_j}\|T_{3,1}(s,j)\|_{\mathbb{L}^2}^2ds\leq C\Big(\frac{T}{N}\Big)^{2\gamma}\int_0^T\Big\{\int_0^s(s-r)^{-(\frac12+\delta+\gamma)}\|Au(r)\|_{\mathbb{L}^2}^{\frac34-\delta}\times\|u(r)\|_V^{2\alpha+\frac14+\delta}dr\Big\}^2ds,$$

which is similar to (4.18) replacing the exponent $\frac54+\delta$ of $\|u(r)\|_V$ by $2\alpha+\frac14+\delta$. Therefore, we deduce for $\delta\in(0,\frac14)$

$$\mathbb{E}\sum_{j=1}^N\int_{t_{j-1}}^{t_j}\|T_{3,1}(s,j)\|_{\mathbb{L}^2}^2ds\leq C(T,\delta)\Big(\frac{T}{N}\Big)^{1-4\delta}\Big[1+\mathbb{E}\Big(\|u_0\|_V^{\frac{16\alpha+2+8\delta}{1+4\delta}}\Big)\Big].\quad(4.22)$$

The Minkowski inequality, (4.2) and (4.7) imply for $\delta\in(0,\frac14)$

$$\begin{aligned}\sum_{j=1}^N\int_{t_{j-1}}^{t_j}\|T_{3,2}(s,j)\|_{\mathbb{L}^2}^2\,ds&\leq\sum_{j=1}^N\int_{t_{j-1}}^{t_j}ds\Big\{\int_s^{t_j}\Big\|A^{\frac12+\delta}S(t_j-r)\\&\quad\times A^{-\delta}\big(|u(r)|^{2\alpha}u(r)\big)\Big\|_{\mathbb{L}^2}dr\Big\}^2\\&\leq C\sum_{j=1}^N\int_{t_{j-1}}^{t_j}ds\Big\{\int_s^{t_j}(t_j-r)^{-(\frac12+\delta)}\|Au(r)\|_{\mathbb{L}^2}^{\frac34-\delta}\|u(r)\|_V^{2\alpha+\frac14+\delta}dr\Big\}^2.\end{aligned}$$

The arguments for proving (4.20) imply

$$\begin{aligned}\mathbb{E}\sum_{j=1}^N\int_{t_{j-1}}^{t_j}\|T_{3,2}(s,j)\|_{\mathbb{L}^2}^2ds&\leq C(T,\delta)\Big(\frac{T}{N}\Big)^{1-2\delta}\Big\{\mathbb{E}\Big(\sup_{r\in[0,T]}\|u(r)\|_V^{\frac{16\alpha+2+8\delta}{1+4\delta}}\Big)\Big\}^{\frac14+\delta}\\&\quad\times\Big\{\mathbb{E}\Big(\int_0^T\|Au(r)\|_{\mathbb{L}^2}^2dr\Big)\Big\}^{\frac34-\delta}.\end{aligned}\quad(4.23)$$

The inequalities (4.22), (4.23) and (3.5) imply that for $\lambda\in(0,\frac12)$ and $\delta=\frac{1-2\lambda}{4}\in\big(0,\frac14\big)$,

$$\mathbb{E}\sum_{j=1}^N\int_{t_{j-1}}^{t_j}\|T_3(s,j)\|_{\mathbb{L}^2}^2ds\leq C(T,a,\lambda)\Big(\frac{T}{N}\Big)^{2\lambda}\Big[1+\mathbb{E}\Big(\|u_0\|_V^{p(\lambda)}\Big)\Big].\quad(4.24)$$

Finally, the stochastic integral can be decomposed as follows: $T_4(s,j) = T_{4,1}(s,j) + T_{4,2}(s,j)$, where

$$T_{4,1}(s,j) = \int_0^s S(s-r)\big[S(t_j-s) - \mathrm{Id}\big]\nabla G(u(r))dW(r),$$
$$T_{4,2}(s,j) = \int_s^{t_j} S(t_j-r)\nabla G(u(r))dW(r).$$

The $L^2(\Omega)$-isometry, (4.2), (4.3) and the growth condition (3.3) imply for $\delta \in \big(0, \frac{1}{2}\big)$

$$\begin{aligned}
&\mathbb{E}\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|T_{4,1}(s,j)\|_{\mathbb{L}^2}^2 ds \\
&\quad\leq \mathbb{E}\sum_{j=1}^N \int_{t_{j-1}}^{t_j}\int_0^s \big\| S(s-r)\big[S(t_j-s)-\mathrm{Id}\big]A^{\frac{1}{2}}G(u(r))\big\|_{\mathcal{L}}^2 \,\mathrm{Tr}Q\, dr ds \\
&\quad\leq \mathbb{E}\sum_{j=1}^N \int_{t_{j-1}}^{t_j} ds \int_0^s \big\|A^{\frac{1}{2}-\delta}S(s-r)\big\|_{\mathcal{L}(\mathbb{L}^2;\mathbb{L}^2)}^2 \\
&\qquad\times \big\|A^{-(\frac{1}{2}-\delta)}\big[S(t_j-s)-\mathrm{Id}\big]\big\|_{\mathcal{L}(\mathbb{L}^2;\mathbb{L}^2)}^2 \|G(u(r))\|_{\tilde{\mathcal{L}}}^2 \,\mathrm{Tr}Q\, dr \\
&\quad\leq \mathrm{Tr}Q\, \mathbb{E}\int_0^T ds \int_0^s (s-r)^{-1+2\delta}(t_j-s)^{1-2\delta}\big[\tilde{K}_0 + \tilde{K}_1\|u(r)\|_V^2\big]dr \\
&\quad\leq \mathrm{Tr}Q\,\Big[\tilde{K}_0 + \tilde{K}_1\,\mathbb{E}\Big(\sup_{r\in[0,T]}\|u(r)\|_V^2\Big)\Big]\Big(\frac{T}{N}\Big)^{1-2\delta}\int_0^T s^{2\delta}ds \\
&\quad\leq C(T,\mathrm{Tr}Q,\delta)\Big(\frac{T}{N}\Big)^{1-2\delta}\big[1+\mathbb{E}(\|u_0\|_V^2)\big],
\end{aligned} \tag{4.25}$$

Finally, the $L^2(\Omega)$-isometry, $\sup_r \|S(r)\|_{\mathcal{L}(\mathbb{L}^2;\mathbb{L}^2)}$, the growth condition (3.3) and (3.5) imply

$$\begin{aligned}
&\mathbb{E}\Big(\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|T_{4,2}(s,j)\|_{\mathbb{L}^2}^2 ds\Big) \\
&\quad\leq \mathbb{E}\sum_{j=1}^N \int_{t_{j-1}}^{t_j}\int_s^{t_j} \|S(t_j-r)\|_{\mathcal{L}(\mathbb{L}^2;\mathbb{L}^2)}^2 \|G(u(r))\|_{\tilde{\mathcal{L}}}^2 \,\mathrm{Tr}Q\, dr ds \\
&\quad\leq \mathrm{Tr}Q\,\mathbb{E}\sum_{j=1}^N \int_{t_{j-1}}^{t_j} ds \int_s^{t_j} \big[\tilde{K}_0 + \tilde{K}_1\|u(r)\|_V^2\big]dr \\
&\quad\leq C(T,\mathrm{Tr}Q)\,\frac{T}{N}\,\big[1+\mathbb{E}(\|u_0\|_V^2)\big].
\end{aligned} \tag{4.26}$$

For $\alpha \in [1, \frac{3}{2}]$ and $\lambda \in (0, \frac{1}{2})$, $2 < \frac{6-2\lambda}{1-\lambda} < p(\lambda) := \frac{2+8\alpha-2\lambda}{1-\lambda}$. Therefore, the upper estimates (4.17), (4.21), (4.24)–(4.26) imply for $\lambda \in (0, \frac{1}{2})$

$$\mathbb{E}\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\|\nabla(u(s)-u(t_j))\|_{\mathbb{L}^2}^2 ds \le C(T, a, \mathrm{Tr}Q, \lambda)\Big(\frac{T}{N}\Big)^{2\lambda}\Big[1+\mathbb{E}\Big(\|u_0\|_V^{p(\lambda)}\Big)\Big].$$

Small changes in the proof of this upper estimate prove that under similar assumptions

$$\mathbb{E}\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\|\nabla(u(s)-u(t_{j-1}))\|_{\mathbb{L}^2}^2 ds \le C(T, \mathrm{Tr}Q, \lambda)\Big(\frac{T}{N}\Big)^{2\lambda}\Big[1+\mathbb{E}\Big(\|u_0\|_V^{p(\lambda)}\Big)\Big].$$

This completes the proof of (4.9). □

Remark 1 Note that the above proof shows that when time increments of the gradient of the solution are dealt with, due to the term containing the initial condition, one cannot obtain moments of $\mathbb{E}(\|u(t)-u(s)\|_V^2)$ uniformly in s, t with $0 \le s < t \le T$. Furthermore, in order to obtain the "optimal" time regularity, that is almost $\frac{1}{2}$, we also need a time integral.

5 Well-posedness and moment estimates of the implicit time Euler scheme

We first prove the existence of the fully time implicit time Euler scheme. Fix $N \in \{1, 2, \ldots\}$, let $h := \frac{T}{N}$ denote the time mesh, and for $j = 0, 1, \ldots, N$ set $t_j := j\frac{T}{N}$.

The fully implicit time Euler scheme $\{u^k; k = 0, 1, \ldots, N\}$ is defined by $u^0 = u_0$ and for $\varphi \in V$

$$\begin{aligned}\Big(u^k - u^{k-1} + h\nu A u^k + hB\big(u^k, u^k\big) + h\,a\,|u^k|^{2\alpha}u^k, \varphi\Big)\\ = \big(G(u^{k-1})[W(t_k)-W(t_{k-1})]\,,\,\varphi\big), \qquad k = 1, 2, \ldots, N.\end{aligned} \tag{5.1}$$

Set $\Delta_j W := W(t_j) - W(t_{j-1})$, $j = 1, \ldots, N$.

The following proposition states the existence and uniqueness of the sequence $\{u^k\}_{k=0,\ldots,N}$ and provides moment estimates which do not depend on N.

Proposition 4 *Let $\alpha \in [1, \frac{3}{2}]$ and Condition* **(G)** *be satisfied. The time fully implicit scheme (5.1) has a solution $\{u^k\}_{k=1,\ldots,N} \in V \cap \mathbb{H}^2$ Furthermore,*

$$\begin{aligned}\sup_{N\ge 1}\mathbb{E}\Big(\max_{k=0,\ldots,N}\|u^k\|_V^2 + \frac{T}{N}\sum_{k=1}^{N}\|Au^k\|_{\mathbb{L}^2}^2\\ + \frac{T}{N}\sum_{k=1}^{N}\big[\|u^k\|_{\mathbb{L}^{2\alpha+2}}^{2\alpha+2} + \||u^k|^{\alpha}\nabla u^k\|_{\mathbb{L}^2}^2\big]\Big) < \infty.\end{aligned} \tag{5.2}$$

Proof The proof is divided in two steps.
Step 1: Existence of the scheme We first prove that for fixed $N \geq 1$ (5.1) has a solution in $V \cap \mathbb{L}^{2\alpha+2}$. For technical reasons we consider a Galerkin approximation. As in Sect. 3 let $\{e_l\}_l$ denote an orthonormal basis of H made of elements of $\mathbb{H}^2$ which are orthogonal in V. Since $\alpha \in [1, \frac{3}{2}]$, the Gagliardo–Nirenberg inequality implies that $\mathbb{H}^1 \subset \mathbb{L}^{2\alpha+2}$.
For $m = 1, 2, \dots$ let $V_m = \text{ span } (e_1, \dots, e_m) \subset \mathbb{H}^2$ and let $P_m : V \to V_m$ denote the projection from V to V_m. In order to find a solution to (5.1) we project this equation on V_m, that is we define by induction a sequence $\{u^k(m)\}_{k=0,\dots,N} \in V_m$ such that $u^0(m) = P_m(u_0)$, and for $k = 1, \dots, N$ and $\varphi \in V_m$

$$\begin{aligned}\big(u^k(m) - u^{k-1}(m), \varphi\big) + h\Big[\nu\big(\nabla u^k(m), \nabla\varphi\big) + \big\langle B\big(u^k(m), u^k(m)\big), \varphi\big\rangle \\ + a\,\big(|u^k(m)|^{2\alpha} u^k(m), \varphi\big)\Big] = \big(G(u^{k-1}(m))\Delta_k W\,,\ \varphi\big).\end{aligned} \tag{5.3}$$

For almost every ω set $R(0, \omega) := \|u_0(\omega)\|_{\mathbb{L}^2}$. Fix $k = 1, \dots, N$ and suppose that for $j = 0, \dots, k-1$ the $\mathcal{F}_{t_j}$- measurable random variables $u^j(m)$ have been defined, and that

$$R(j, \omega) := \sup_{m\geq 1} \|u^j(m, \omega)\|_{\mathbb{L}^2} < \infty \quad \text{for almost every } \omega.$$

We prove that $u^k(m)$ exists and satisfies a.s. $\sup_{m\geq 1} \|u^k(m, \omega)\|_{\mathbb{L}^2} < \infty$. The argument is based on the following result [22, Cor 1.1, p. 279], which can be deduced from Brouwer's theorem.

Proposition 5 *Let H be a Hilbert space of finite dimension, $(.,.)_H$ denote its inner product, and $\Phi : H \to H$ be continuous such that for some $\mu > 0$,*

$$\big(\Phi(f), f\big)_H \geq 0, \quad \text{for all } f \in H \text{ with} \|f\|_H = \mu.$$

Then there exists $f \in H$ such that $\Phi(f) = 0$ and $\|f\|_H \leq \mu$.

For $\omega \in \Omega$ let $\Phi^k_{m,\omega} : V_m \to V_m$ be defined for $f \in V_m$ as the solution of

$$\begin{aligned}\big(\Phi^k_{m,\omega}(f), \varphi\big) = &\ \big(f - u^{k-1}(m, \omega), \varphi\big) \\ &+ h\Big[\nu\big(\nabla f, \nabla\varphi\big) + \big\langle P_m B(f, f), \varphi\big\rangle + a\big(P_m(|f|^{2\alpha} f), \varphi\big)\Big] \\ &- \big(P_m G(u^{k-1}(m, \omega)\Delta_k W(\omega), \varphi\big), \qquad \forall \varphi \in V_m.\end{aligned}$$

Then

$$\begin{aligned}\big(\Phi^k_{m,\omega}(f), f\big) = &\ \|f\|^2_{\mathbb{L}^2} - \big(u^{k-1}(m, \omega), f\big) + h\nu\|\nabla f\|^2_{\mathbb{L}^2} + h\,a\|f\|^{2\alpha+2}_{\mathbb{L}^{2\alpha+2}} \\ &- \big(G(u^{k-1}(m, \omega)\big)\Delta_k W(\omega), f\big).\end{aligned}$$

The Young inequality implies $\big|\big(u^{k-1}(m,\omega), f\big) \le \frac{1}{2}\|f\|_{\mathbb{L}^2}^2 + \frac{1}{2}\|u^{k-1}(m,\omega)\|_{\mathbb{L}^2}^2$ and the growth condition (3.2) implies

$$\begin{aligned}\big|\big(G(u^{k-1}(m,\omega)\Delta_k W(\omega), f\big)\big| &\le \big\|G\big(u^{k-1}(m,\omega)\big)\|_{\mathcal{L}}\|\Delta_k W(\omega)\|_K\,\|f\|_{\mathbb{L}^2}\\ &\le \frac{1}{4}\|f\|_{\mathbb{L}^2}^2 + \big[K_0 + K_1\|u^{k-1}(m,\omega)\|_{\mathbb{L}^2}^2\big]\,\|\Delta_k W(\omega)\|_K^2.\end{aligned}$$

Hence

$$\begin{aligned}\big(\Phi_{m,\omega}^k(f), f\big) \ge\; &\frac{1}{4}\|f\|_{\mathbb{L}^2}^2 - \frac{1}{2}\|u^{k-1}(m,\omega)\|_{\mathbb{L}^2}^2\\ &- \big[K_0 + K_1\|u^{k-1}(m,\omega)\|_{\mathbb{L}^2}^2\big]\|\Delta_k W(\omega)\|_K^2 \ge 0\end{aligned}$$

if

$$\|f\|_{\mathbb{L}^2}^2 = R^2(k,\omega) := 4\Big[K_0\|\Delta_k W(\omega)\|_K^2 + R^2(k-1,\omega)\Big(\frac{1}{2} + K_1\|\Delta_k W(\omega)\|_K^2\Big)\Big].$$

Proposition 5 implies the existence of $u^k(m,\omega) \in V_m$ such that $\Phi_{m,\omega}^k\big(u^k(m,\omega)\big) = 0$, and $\|u^k(m,\omega)\|_{\mathbb{L}^2}^2 \le R^2(k,\omega)$; note that this element $u^k(m,\omega)$ need not be unique. Furthermore, the random variable $u^k(m)$ is $\mathcal{F}_{t_k}$-measurable.

The definition of $u^k(m)$ implies that it is a solution to (5.3). Taking $\varphi = u^k(m)$ in (5.3) and using the Young inequality, we obtain

$$\begin{aligned}&\|u^k(m)\|_{\mathbb{L}^2}^2 + h\,\nu\|\nabla u^k(m)\|_{\mathbb{L}^2}^2 + h\,a\|u^k(m)\|_{\mathbb{L}^{2\alpha+2}}^{2\alpha+2}\\ &\quad= \big(u^{k-1}(m), u^k(m)\big) + \big(G(u^{k-1}(m)\Delta_k W, u^k(m)\big)\\ &\quad\le \frac{1}{4}\|u^k(m)\|_{\mathbb{L}^2}^2 + \|u^{k-1}(m)\|_{\mathbb{L}^2}^2 + \frac{1}{4}\|u^k(m)\|_{\mathbb{L}^2}^2\\ &\qquad+ \big[K_0 + K_1\|u^{k-1}(m)\|_{\mathbb{L}^2}^2\big]\|\Delta_k W\|_K^2.\end{aligned}$$

Hence a.s.

$$\begin{aligned}&\sup_{m\ge 1}\Big[\frac{1}{2}\|u^k(m,\omega)\|_{\mathbb{L}^2}^2 + h\,\nu\|\nabla u^k(m,\omega)\|_{\mathbb{L}^2}^2 + h\,a\|u^k(m,\omega)\|_{\mathbb{L}^{2\alpha+2}}^{2\alpha+2}\Big]\\ &\quad\le R^2(k-1,\omega)\big[1 + K_1\|\Delta_k W(\omega)\|_K^2\big] + K_0\|\Delta_k W(\omega)\|_K^2,\end{aligned}$$

Therefore, for k and almost every ω, the sequence $\{u^k(m,\omega)\}_m$ is bounded in $V \cap \mathbb{L}^{2\alpha+2}$; it has a subsequence (still denoted $\{u^k(m,\omega)\}_m$) which converges weakly in $V \cap \mathbb{L}^{2\alpha+2}$ to $\phi_k(\omega)$. The random variable ϕ_k is $\mathcal{F}_{t_k}$-measurable.

Since D is bounded, the embedding of V in H is compact; hence the subsequence $\{u^k(m,\omega)\}_m$ converges strongly to $\phi_k(\omega)$ in $\mathbb{L}^2$.

Then by definition $u^0(m)$ converges strongly to u_0. We next prove by induction on k that ϕ^k solves (5.1). Fix a positive integer m_0 and consider the equation (5.3) for

$k = 1, \dots, N$, $\varphi \in V_{m_0}$, and $m \ge m_0$. As $m \to \infty$ we have a.s.

$$\big(u^k(m) - u^{k-1}(m), \varphi\big) \to \big(\phi^k - \phi^{k-1}, \varphi\big).$$

Furthermore, the antisymmetry of B (2.1) and the Gagliardo–Nirenberg inequality $\|g\|_{\mathbb{L}^4} \le C\|\nabla g\|_{\mathbb{L}^2}^{\frac{3}{4}} \|g\|_{\mathbb{L}^2}^{\frac{1}{4}}$ yield a.s.

$$\begin{aligned}
&\big|\big\langle B\big(u^k(m), u^k(m)\big) - B(\phi^k, \phi^k), \varphi\big\rangle\big| \\
&\quad \le \big|\big\langle B\big(u^k(m) - \phi^k, \varphi\big), u^k(m)\big\rangle\big| + \big|\big\langle B\big(\phi^k, \varphi\big), u^k(m) - \phi^k\big\rangle\big| \\
&\quad \le \|\nabla\varphi\|_{\mathbb{L}^2} \|u^k(m) - \phi^k\|_{\mathbb{L}^4} \big[\|u^k(m)\|_{\mathbb{L}^4} + \|\phi^k\|_{\mathbb{L}^4}\big] \\
&\quad \le C\, \|\varphi\|_{\mathbb{L}^2} \big[\max_m \|u^k(m)\|_V^{\frac{7}{4}} + \|\phi^k\|_V^{\frac{7}{4}}\big] \|u^k(m) - \phi^k\|_{\mathbb{L}^2}^{\frac{1}{4}} \to 0
\end{aligned}$$

as $m \to \infty$. The inequality (7.8) implies

$$\begin{aligned}
&\big|\big(|u^k(m)|^{2\alpha} u^k(m) - |\phi^k|^{2\alpha}\phi^k, \varphi\big)\big| \\
&\quad \le C \int |u^k(m) - \phi^k| \big(|u^k(m)|^{2\alpha} + |\phi^k|^{2\alpha}\big)\, |\varphi|\, dx \\
&\quad \le C\, \|\varphi\|_{\mathbb{L}^\infty} \|u^k(m) - \phi^k\|_{\mathbb{L}^2} \big(\|u^k(m)\|_{L^{4\alpha}}^{4\alpha} + \|\phi^k\|_{\mathbb{L}^{4\alpha}}^{4\alpha}\big) \\
&\quad \le C\, \|\varphi\|_{\mathbb{H}^2} \big(\max_m \|u^k(m)\|_V^{4\alpha} + \|\phi^k\|_V^{4\alpha}\big) \|u^k(m) - \phi^k\|_{\mathbb{L}^2} \to 0
\end{aligned}$$

as $m \to \infty$. Note that the last upper estimate follows from the inclusion $\mathbb{H}^1 \subset \mathbb{L}^p$ for $p \in [2, 6]$, and $\alpha \in [1, \frac{3}{2}]$. Finally, the Cauchy–Schwarz inequality and the Lipschitz condition (3.4) imply

$$\begin{aligned}
&\big|\big(G\big(u^{k-1}(m)\big)\Delta_k W, \varphi\big) - \big(G\big(\phi^{k-1}\big)\Delta_k W, \varphi\big)\big| \\
&\quad \le \|\varphi\|_{\mathbb{L}^2} \|G(u^{k-1}(m) - G(\phi^{k-1})\|_{\mathcal{L}} \|\Delta_k W\|_K \\
&\quad \le \sqrt{L}\, \|\varphi\|_{\mathbb{L}^2}\, \|u^{k-1}(m) - \phi^{k-1}\|_{\mathbb{L}^2}\, \|\Delta_k W\|_K \to 0
\end{aligned}$$

as $m \to \infty$. Therefore, letting $m \to \infty$ in (5.3), we deduce

$$\Big(\phi^k - \phi^{k-1} + h\nu A\phi^k + hB\big(\phi^k, \phi^k\big) + h\, a\, |\phi^k|^{2\alpha}\phi^k, \varphi\Big) = \big(G(\phi^{k-1})\Delta_k W\,,\, \varphi\big)$$

for every $\varphi \in V_{m_0}$. Since $\cup_{m_0} V_{m_0}$ is dense in V, we deduce that ϕ^k is a solution to (5.1).

Step 2: Moment estimates We next prove (5.2) for any $\{u^k\}_{k=0,\dots,N}$ solution to (5.1). We first study the $\mathbb{L}^2$-norm of the sequence. Write (5.1) with $\varphi = u^k$ and use the identity $(f, f-g) = \frac{1}{2}\big[\|f\|_{\mathbb{L}^2} - \|g\|_{\mathbb{L}^2}^2 + \|f-g\|_{\mathbb{L}^2}^2\big]$. Using the Cauchy–Schwarz and Young inequalities, and the growth condition (3.2), this yields for $k = 1, \dots, N$

$$\frac{1}{2}\|u^k\|_{\mathbb{L}^2}^2 - \frac{1}{2}\|u^{k-1}\|_{\mathbb{L}^2}^2 + \frac{1}{2}\|u^k - u^{k-1}\|_{\mathbb{L}^2}^2 + h\nu\|\nabla u^k\|_{\mathbb{L}^2}^2 + ha\|u^k\|_{\mathbb{L}^{2\alpha+2}}^{2\alpha+2}$$

$$
\begin{aligned}
&= \big(G(u^{k-1})\Delta_k W\,,\, u^k - u^{k-1}\big) + \big(G(u^{k-1})\Delta_k W\,,\, u^{k-1}\big)\\
&\le \frac{1}{2}\|u^k - u^{k-1}\|^2_{\mathbb{L}^2} + \frac{1}{2}\big[K_0 + K_1\|u^{k-1}\|^2_{\mathbb{L}^2}\big]\|\Delta_k W\|^2_K\\
&\quad + \big(G(u^{k-1})\Delta_k W\,,\, u^{k-1}\big).
\end{aligned}
$$

For any $K = 1, \ldots, N$, adding the above inequalities for $k = 1, ..., K$ we deduce

$$
\begin{aligned}
&\|u^K\|^2_{\mathbb{L}^2} + 2h\nu \sum_{k=1}^{K} \|\nabla u^k\|^2_{\mathbb{L}^2} + 2ha \sum_{k=1}^{K} \|u^k\|^{2\alpha+2}_{\mathbb{L}^{2\alpha+2}} \le \|u_0\|^2_{\mathbb{L}^2}\\
&\quad + \sum_{k=1}^{K} \big[K_0 + K_1\|u^{k-1}\|^2_{\mathbb{L}^2}\big]\|\Delta_k W\|^2_K + 2\sum_{k=1}^{K} \big(G(u^{k-1})\Delta_k W\,,\; u^{k-1}\big). \qquad (5.4)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\Big(\max_{1\le K\le N} \|u^K\|^2_{\mathbb{L}^2}\Big) + 2h\mathbb{E}\Big(\sum_{k=1}^{N} \big[\nu\|\nabla u^k\|^2_{\mathbb{L}^2} + a\|u^k\|^{2\alpha+2}_{\mathbb{L}^{2\alpha+2}}\big]\Big)\\
&\quad \le\; 2\mathbb{E}\Big(\max_{1\le K\le N} \Big[\|u^K\|^2_{\mathbb{L}^2} + 2h\sum_{k=1}^{K} \big(\nu\|\nabla u^k\|^2_{\mathbb{L}^2} + a\|u^k\|^{2\alpha+2}_{\mathbb{L}^{2\alpha+2}}\big)\Big]\Big)\\
&\quad \le\; 2\mathbb{E}(\|u_0\|^2_{\mathbb{L}^2}) + 2h\mathrm{Tr}(Q)\sum_{k=0}^{N-1} \big[K_0 + K_1\mathbb{E}(\|u^k\|^2_{\mathbb{L}^2})\big]\\
&\qquad + 4\mathbb{E}\Big(\max_{1\le K\le N} \sum_{k=1}^{K} \big(G(u^{k-1})\Delta_k W\,,\, u^{k-1}\big)\Big).
\end{aligned}
$$

The Davis and then Young inequalities imply

$$
\begin{aligned}
&\mathbb{E}\Big(\max_{1\le K\le N} \sum_{k=1}^{K} \big(G(u^{k-1})\Delta_k W\,,\, u^{k-1}\big)\Big)\\
&\quad \le 3\mathbb{E}\Big(\Big\{\sum_{k=0}^{N-1} \|u^k\|^2_{\mathbb{L}^2}\big[K_0 + K_1\|u^k\|^2_{\mathbb{L}^2}\big]h\mathrm{Tr}\,Q\Big\}^{\frac{1}{2}}\Big)\\
&\quad \le \frac{1}{4}\mathbb{E}\Big(\max_{0\le k\le N-1} \|u^k\|^2_{\mathbb{L}^2}\Big) + 9\mathbb{E}\Big(h\mathrm{Tr}\,Q\sum_{k=0}^{N-1} \big[K_0 + K_1\|u^k\|^2_{\mathbb{L}^2}\big]\Big).
\end{aligned}
$$

Hence we deduce

$$
\frac{1}{2}\mathbb{E}\Big(\max_{1\le K\le N} \|u^K\|^2_{\mathbb{L}^2}\Big) + 2h\mathbb{E}\Big(\sum_{k=1}^{N} \big[\nu\|\nabla u^k\|^2_{\mathbb{L}^2} + a\|u^k\|^{2\alpha+2}_{\mathbb{L}^{2\alpha+2}}\big]\Big)
$$

$$\leq\ 2\mathbb{E}(\|u_0\|_{\mathbb{L}^2}^2) + 74TK_0\mathrm{Tr}Q + 74K_1\mathrm{Tr}Q\sum_{k=0}^{N-1} h\mathbb{E}(\|u^k\|_{\mathbb{L}^2}^2). \tag{5.5}$$

Neglecting the sum in the left hand side and using the discrete Gronwall lemma, we obtain

$$\sup_{N\geq 1}\mathbb{E}\Big(\max_{1\leq K\leq N}\|u^K\|_{\mathbb{L}^2}^2\Big) \leq C(T, \mathrm{Tr}Q, \|u_0\|_{\mathbb{L}^2}^2, K_0, K_1).$$

Plugging this upper estimate in (5.5), we obtain

$$\sup_{N\geq 1}\mathbb{E}\Big(\max_{k=0,\ldots,N}\|u^k\|_{\mathbb{L}^2}^2 + \frac{T}{N}\sum_{k=1}^{N}\big[\nu\|\nabla u^k\|_{\mathbb{L}^2}^2 + a\|u^k\|_{\mathbb{L}^{2\alpha+2}}^{2\alpha+2}\big]\Big) < \infty.$$

A similar argument with $\varphi = Au^k$, integrating by parts, and using Lemma 4 and inequality (7.14) yields

$$\sup_{N\geq 1}\mathbb{E}\Big(\max_{1\leq K\leq N}\|\nabla u^K\|_{\mathbb{L}^2}^2 + \frac{T}{N}\sum_{k=1}^{N}\big[\|Au^k\|_{\mathbb{L}^2}^2 + \||u^k|^\alpha\nabla u^k\|_{\mathbb{L}^2}^2\big]\Big) = C_2(\alpha) < \infty.$$

This completes the proof of the proposition. □

6 Strong convergence of the implicit time Euler scheme

Let u be the solution to (1.1) and $\{u^j\}_{j=0,\ldots,N}$ solve the fully implicit time Euler scheme defined in (5.1). Let $e_j := u(t_j) - u^j$. Using (1.1) and (5.1), we deduce $e_0 = 0$ and for $j = 1,\ldots,N$ and $\varphi \in V$

$$\begin{aligned}
&\big(e_j - e_{j-1}\,,\,\varphi\big) + \nu\int_{t_{j-1}}^{t_j}\big(\nabla u(s) - \nabla u^j\,,\,\nabla\varphi\big)ds\\
&\quad + \int_{t_{j-1}}^{t_j}\big\langle B(u(s),u(s)) - B(u^j,u^j)\,,\,\varphi\big\rangle ds\\
&\quad + a\int_{t_{j-1}}^{t_j}\big(|u(s)|^{2\alpha}u(s) - |u^j|^{2\alpha}u^j\,,\,\varphi\big)ds\\
&= \int_{t_{j-1}}^{t_j}\big([G(u(s)) - G(u^{j-1})]dW(s)\,,\,\varphi\big).
\end{aligned} \tag{6.1}$$

Note that since $\alpha \in [1,\frac{3}{2}]$ and $\mathbb{H}^1 \subset \mathbb{L}^p$ for $p \in [2,6]$, Hölder's inequality with exponents 2, 3 and 6 implies that the space integral defining the inner product $\big(|u(s)|^{2\alpha}u(s) - |u^j|^{2\alpha}u^j\,,\,\varphi\big)$ is converging for $u(s), u^j, \varphi \in V$. The following convergence theorem is one of the main results of this paper.

Theorem 6 *Suppose that condition* **(G)** *holds. Let* $\alpha \in [1, \frac{3}{2}]$*; when* $\alpha = 1$*, suppose that* $4\nu a(1 \wedge \kappa) > 1$*, where* $\kappa > 0$ *is the constant defined in inequality* (7.9)*.*
Fix $\lambda \in (0, \frac{1}{2})$ *and set* $p(\lambda) = \frac{2+8\alpha-2\lambda}{1-\lambda}$*. Let* $u_0 \in L^{p(\lambda)}(\Omega; V)$*,* u *be the solution to* (1.1) *and* $\{u^j\}_{j=0,\ldots,N}$ *solve the fully implicit scheme* (5.1)*. Then there exists a positive constant* $C := C(\nu, \alpha, a, \kappa, \mathrm{Tr} Q)$ *independent of* N *such that for* N *large enough*

$$\mathbb{E}\Big(\max_{1\leq j\leq N} \|u(t_j) - u^j\|_{\mathbb{L}^2}^2 + \frac{T}{N} \sum_{j=1}^N \|\nabla[u(t_j) - u^j]\|_{\mathbb{L}^2}^2 \Big) \leq C \Big(\frac{T}{N}\Big)^{2\lambda} \Big[1 + \mathbb{E}\Big(\|u_0\|_V^{p(\lambda)}\Big)\Big]. \tag{6.2}$$

Remark 2 Note that the various parameters of the model $\nu, \alpha, a, \mathrm{Tr} Q$ only appear in the multiplicative constant C in the right hand side of (6.2), but not in the exponent λ which can be chosen arbitrarily close to $\frac{1}{2}$ if $u_0 \in V$ is deterministic, or if u_0 is a V-valued Gaussian random variable independent of W.

Proof of Theorem 6 (i) We first suppose that $\alpha \in (1, \frac{3}{2}]$.

Using the identity (6.1) with $\varphi = e_j$, the equality $(f, f-g) = \frac{1}{2}\big[\|f\|_{\mathbb{L}^2}^2 - \|g\|_{\mathbb{L}^2}^2 + \|f-g\|_{\mathbb{L}^2}^2\big]$ and the estimate (7.18), we deduce that for some $\kappa > 0$ we have for $j = 1, \ldots, N$

$$\frac{1}{2}\big(\|e_j\|_{\mathbb{L}^2}^2 - \|e_{j-1}\|_{\mathbb{L}^2}^2\big) + \frac{1}{2}\|e_j - e_{j-1}\|_{\mathbb{L}^2}^2 + \nu h \|\nabla e_j\|_{\mathbb{L}^2}^2 + a\kappa h \||u(t_j)|^\alpha e_j\|_{\mathbb{L}^2}^2 + a\kappa h \||u^j|^\alpha e_j\|_{\mathbb{L}^2}^2 \leq \sum_{l=1}^7 T_{j,l}, \tag{6.3}$$

where by the antisymmetry property (2.1) we have

$$T_{j,1} = -\int_{t_{j-1}}^{t_j} \big\langle B\big(u(s) - u(t(j)), u(s)\big)\,,\ e_j \big\rangle ds,$$

$$T_{j,2} = -\int_{t_{j-1}}^{t_j} \big\langle B\big(e_j, u(s)\big)\,,\ e_j \big\rangle ds,$$

$$T_{j,3} = -\int_{t_{j-1}}^{t_j} \big\langle B\big(u^j, u(s) - u^j\big)\,, e_j \big\rangle ds = -\int_{t_{j-1}}^{t_j} \big\langle B\big(u^j, u(s) - u(t_j)\big)\,,\ e_j \big\rangle ds,$$

$$T_{j,4} = -\nu \int_{t_{j-1}}^{t_j} \big(\nabla(u(s) - u(t_j)), \nabla e_j\big) ds,$$

$$T_{j,5} = -a \int_{t_{j-1}}^{t_j} \big(|u(s)|^{2\alpha} u(s) - |u(t_j)|^{2\alpha} u(t_j), e_j\big) ds,$$

$$T_{j,6} = \int_{t_{j-1}}^{t_j} \big([G(u(s)) - G(u^{j-1}) dW(s), e_j - e_{j-1}\big),$$

$$T_{j,7} = \int_{t_{j-1}}^{t_j} \big([G(u(s)) - G(u^{j-1})] dW(s), e_{j-1} \big).$$

We next prove upper estimates of the terms $T_{j,l}$ for $l = 1, ..., 5$, and of the expected value of $T_{j,6}$ and $T_{j,7}$.

Using the Hölder inequality with exponents 2, 3, 6, the Sobolev embedding $\mathbb{H}^1 \subset \mathbb{L}^6$ and the Gagliardo–Nirenberg inequality (2.2), we deduce for $\epsilon_1 > 0$

$$\begin{aligned}
|T_{j,1}| &\leq \int_{t_{j-1}}^{t_j} \|u(s) - u(t_j)\|_{\mathbb{L}^3} \|\nabla u(s)\|_{\mathbb{L}^2} \|e_j\|_{\mathbb{L}^6} ds \\
&\leq C_6 C_3 \|e_j\|_{\mathbb{H}^1} \int_{t_{j-1}}^{t_j} \|u(s) - u(t_j)\|_{\mathbb{L}^2}^{\frac{1}{2}} \|\nabla[u(s) - u(t_j)]\|_{\mathbb{L}^2}^{\frac{1}{2}} \|\nabla u(s)\|_{\mathbb{L}^2} ds \\
&\leq \epsilon_1 \nu h \|e_j\|_{\mathbb{H}^1}^2 + \frac{(C_6 C_3)^2}{4\epsilon_1 \nu} \sup_{s\in[0,T]} \|u(s)\|_V^2 \Big(\int_{t_{j-1}}^{t_j} \|u(s) - u(t_j)\|_{\mathbb{L}^2}^2 ds \Big)^{\frac{1}{2}} \\
&\quad \times \Big(\int_{t_{j-1}}^{t_j} \|\nabla[u(s) - u(t_j)]\|_{\mathbb{L}^2}^2 ds \Big)^{\frac{1}{2}} \\
&\leq \epsilon_1 \nu h \big[\|e_j\|_{\mathbb{L}^2}^2 + \|\nabla e_j\|_{\mathbb{L}^2}^2 \big] + \int_{t_{j-1}}^{t_j} \|\nabla[u(s) - u(t_j)]\|_{\mathbb{L}^2}^2 ds \\
&\quad + \frac{(C_6 C_3)^4}{64 \epsilon_1^2 \nu^2} \sup_{s\in[0,T]} \|u(s)\|_V^4 \int_{t_{j-1}}^{t_j} \|u(s) - u(t_j)\|_{\mathbb{L}^2}^2 ds, \qquad (6.4)
\end{aligned}$$

where the last inequalities are deduced from the Cauchy–Schwarz and Young inequalities.

Let $T_{j,2} = -T_{j,2,1} - T_{j,2,2} + T_{j,2,3}$, where

$$T_{j,2,1} = \int_{t_{j-1}}^{t_j} \big\langle B\big(e_j, u(t_j)\big), e_j \big\rangle ds, \quad T_{j,2,2} = \int_{t_{j-1}}^{t_j} \big\langle B\big(e_j, u(s) - u(t_j)\big), u(t_j) \big\rangle ds,$$
$$T_{j,2,3} = \int_{t_{j-1}}^{t_j} \big\langle B\big(e_j, u(s) - u(t_j)\big), u^j \big\rangle ds.$$

The antisymmetry (2.1) implies

$$\big\langle B\big(e_j, u(t_j)\big), e_j \big\rangle = -\big\langle B\big(e_j, e_j\big), u(t_j) \big\rangle = -\sum_{k,l=1}^{3} \int_D (e_j)_k \partial_k (e_j)_l u(t_j)_l dx.$$

Hence the upper estimate (2.5) with $f = u(t_j)_l$, $g = (e_j)_k$ and $h = \partial_k (e_j)_l$ yields for $\epsilon_2, \bar{\epsilon}_2 > 0$

$$\big|\big\langle B\big(e_j, u(t_j)\big), e_j\big\rangle\big| \le \epsilon_2\nu \sum_{k,l} \|\partial_k (e_j)_l\|_{\mathbb{L}^2}^2 + \sum_{k,l} \frac{\bar{\epsilon}_2 a\kappa}{4\epsilon_2\nu} \| |u(t_j)_l|^\alpha (e_j)_k\|_{\mathbb{L}^2}^2 + \frac{C_\alpha}{\epsilon_2\nu(\bar{\epsilon}_2 a\kappa)^{\frac{1}{\alpha-1}}} \|(e_j)_k\|_{\mathbb{L}^2}^2,$$

which implies

$$|T_{j,2,1}| \le \epsilon_2\nu\, h\, \|\nabla e_j\|_{\mathbb{L}^2}^2 + \frac{\bar{\epsilon}_2 a\kappa}{4\epsilon_2\nu}\, h\, \| |u(t_j)|^\alpha e_j\|_{\mathbb{L}^2}^2 + \frac{C(\alpha,\nu,a,\kappa)}{\epsilon_2(\bar{\epsilon}_2)^{\frac{1}{\alpha-1}}}\, h\, \|e_j\|_{\mathbb{L}^2}^2.$$

Using a similar computation based on (2.5) with $f = u(t_j)_l$, $g = (e_j)_k$ and $h = \partial_k[u(s) - u(t_j)]_l$ for $k, l = 1, 2, 3$, summing on k, l and integrating on the time interval $[t_{j-1}, t_j]$, we obtain for $\tilde{\epsilon}_2 > 0$

$$|T_{j,2,2}| \le \int_{t_{j-1}}^{t_j} \|\nabla[u(s) - u(t_j)]\|_{\mathbb{L}^2}^2 ds + \frac{\tilde{\epsilon}_2 a\kappa}{4}\, h\, \| |u(t_j)|^\alpha e_j\|_{\mathbb{L}^2}^2 + \frac{\bar{C}(\alpha, a, \kappa)}{(\tilde{\epsilon}_2)^{\frac{1}{\alpha-1}}}\, h\, \|e_j\|_{\mathbb{L}^2}^2.$$

Replacing $f = u(t_j)$ by $f = u^j$ in the above estimate, we obtain

$$|T_{j,2,3}| \le \int_{t_{j-1}}^{t_j} \|\nabla[u(s) - u(t_j)]\|_{\mathbb{L}^2}^2 ds + \frac{\tilde{\epsilon}_2 a\kappa}{4}\, h\, \| |u^j|^\alpha e_j\|_{\mathbb{L}^2}^2 + \frac{\bar{C}(\alpha, a, \kappa)}{(\tilde{\epsilon}_2)^{\frac{1}{\alpha-1}}} h\, \|e_j\|_{\mathbb{L}^2}^2.$$

The three previous inequalities imply for $\epsilon_2, \bar{\epsilon}_2, \tilde{\epsilon}_2 > 0$,

$$\begin{aligned} |T_{j,2}| \le\; & \Big[\frac{C(\alpha,\nu,a,\kappa)}{\epsilon_2(\bar{\epsilon}_2)^{\frac{1}{\alpha-1}}} + \frac{2\bar{C}(\alpha,a,\kappa)}{(\tilde{\epsilon}_2)^{\frac{1}{\alpha-1}}}\Big] h\, \|e_j\|_{\mathbb{L}^2}^2 + \epsilon_2\nu\, h\, \|\nabla e_j\|_{\mathbb{L}^2}^2 \\ & + \frac{\tilde{\epsilon}_2}{4} a\kappa\, h\, \| |u^j|^\alpha e_j\|_{\mathbb{L}^2}^2 + \Big[\frac{\bar{\epsilon}_2}{4\epsilon_2\nu} + \frac{\tilde{\epsilon}_2}{4}\Big] a\kappa\, h\, \| |u(t_j)|^\alpha e_j\|_{\mathbb{L}^2}^2 \\ & + 2\int_{t_{j-1}}^{t_j} \|\nabla[u(s) - u(t_j)]\|_{\mathbb{L}^2}^2 ds. \end{aligned} \tag{6.5}$$

Using once more (2.5) with $f = (u^j)_k$, $g = (e_j)_l$ and $h = \partial_k\big([u(s) - u(t_j)]_l\big)$ for $k, l = 1, 2, 3$, and summing on k, l, we obtain for $\epsilon_3 > 0$,

$$\big|\big\langle B\big(u^j, u(s) - u(t_j)\big), e_j\big\rangle\big| \le \|\nabla[u(s) - u(t_j)]\|_{\mathbb{L}^2}^2 + \frac{\epsilon_3 a\kappa}{4} \| |u^j|^\alpha e_j\|_{\mathbb{L}^2}^2 + \frac{C_\alpha}{(\epsilon_3 a\kappa)^{\frac{1}{\alpha-1}}} \|e_j\|_{\mathbb{L}^2}^2.$$

Integrating on $[t_{j-1}, t_j]$ we deduce for $\epsilon_3 > 0$

$$|T_{j,3}| \le \frac{C_\alpha h}{(\epsilon_3 a\kappa)^{\frac{1}{\alpha-1}}} \|e_j\|_{\mathbb{L}^2}^2 + \frac{\epsilon_3 a\kappa h}{4} \||u^j|^\alpha e_j\|_{\mathbb{L}^2}^2 + \int_{t_{j-1}}^{t_j} \|\nabla[u(s) - u(t_j)|\|_{\mathbb{L}^2}^2 ds. \quad (6.6)$$

The Cauchy–Schwarz and Young inequalities imply that for $\epsilon_4 > 0$,

$$|T_{j,4}| \le \epsilon_4 \nu\, h\, \|\nabla e_j\|_{\mathbb{L}^2}^2 + \frac{\nu}{4\epsilon_4} \int_{t_{j-1}}^{t_j} \|\nabla[u(s) - u(t_j)]\|_{\mathbb{L}^2}^2 ds. \quad (6.7)$$

Since $\big||f|^{2\alpha} f - |g|^{2\alpha} g\big| \le C(\alpha)|f-g|\big(|f|^{2\alpha} + |g|^{2\alpha}\big)$, the Hölder inequality with exponents 2, 3 and 6 implies

$$\begin{aligned}
&\big|\big(|u(s)|^{2\alpha} u(s) - |u(t_j)|^{2\alpha} u(t_j), e_j\big)\big| \\
&\quad\le\; C(\alpha) \int_{\mathbb{R}^3} \big[|u(s)|^{2\alpha} + |u(t_j)|^{2\alpha}\big] |u(s) - u(t_j)||e_j| dx \\
&\quad\le\; C(\alpha)\big[\|u(s)\|_{\mathbb{L}^{4\alpha}}^{2\alpha} + \|u(t_j)\|_{\mathbb{L}^{4\alpha}}^{2\alpha}\big] \|u(s) - u(t_j)\|_{\mathbb{L}^3} \|e_j\|_{\mathbb{L}^6}.
\end{aligned}$$

The Sobolev embedding $\mathbb{H}^1 \subset \mathbb{L}^6$ and the Gagliardo–Nirenberg inequality (2.2) yield for $\epsilon_5 > 0$

$$\begin{aligned}
|T_{j,5}| \le\; & C(\alpha) \sup_{s\in[0,T]} \|u(s)\|_V^{2\alpha} \int_{t_{j-1}}^{t_j} \|e_j\|_{\mathbb{H}^1} \|u(s) - u(t_j)\|_{\mathbb{L}^2}^{\frac12} \|\nabla[u(s) - u(t_j)]\|_{\mathbb{L}^2}^{\frac12} ds \\
\le\; & \epsilon_5 \nu\, h \big[\|e_j\|_{\mathbb{L}^2}^2 + \|\nabla e_j\|_{\mathbb{L}^2}^2\big] + \frac{C(\alpha)^2}{8\epsilon_5 \nu} \int_{t_{j-1}}^{t_j} \|\nabla[u(s) - u(t_j)]\|_{\mathbb{L}^2}^2 ds \\
& + \frac{C(\alpha)^2}{8\epsilon_5\nu} \sup_{s\in[0,T]} \|u(s)\|_V^{8\alpha} \int_{t_{j-1}}^{t_j} \|u(s) - u(t_j)\|_{\mathbb{L}^2}^2 ds, \quad (6.8)
\end{aligned}$$

where the last upper estimate is deduced from the Hölder inequality with exponents 2,4 and 4 and the Young inequality.

Fix $J \in \{1, 2, \ldots, N\}$; adding the inequalities (6.3) for $j = 1, \ldots, J$, using the identity $e_0 = 0$ and the upper estimates (6.4)–(6.8) we deduce that for any positive numbers ϵ_j, $j = 1, \ldots, 5$, $\bar\epsilon_2$ and $\tilde\epsilon_2$, we have

$$\begin{aligned}
& \frac12 \|e_J\|_{\mathbb{L}^2}^2 + \frac12 \sum_{j=1}^J \|e_j - e_{j-1}\|_{\mathbb{L}^2}^2 + \nu\, h \sum_{j=1}^J \|\nabla e_j\|_{\mathbb{L}^2}^2 \\
& + a\kappa\, h \sum_{j=1}^J \Big[\||u(t_j)|^\alpha e_j\|_{\mathbb{L}^2}^2 + \||u^j|^\alpha e_j\|_{\mathbb{L}^2}^2\Big] \le \sum_{j=1}^J \sum_{l=6}^7 T_{j,l} \\
& + \Big[\epsilon_1 \nu + \frac{2\,\bar C(\alpha, a, \kappa)}{(\tilde\epsilon_2)^{\frac{1}{\alpha-1}}} + \frac{C(\alpha, \nu, a, \kappa)}{\epsilon_2 \nu (\bar\epsilon_2)^{\frac{1}{\alpha-1}}} + \frac{C_\alpha}{(\epsilon_3 a\kappa)^{\frac{1}{\alpha-1}}} + \epsilon_5 \nu\Big] h \sum_{j=1}^J \|e_j\|_{\mathbb{L}^2}^2
\end{aligned}$$

$$
\begin{aligned}
&+ \Big(\epsilon_1 + \epsilon_2 + \epsilon_4 + \epsilon_5\Big) \nu\, h \sum_{j=1}^J \|\nabla e_j\|_{\mathbb{L}^2}^2 \\
&+ \Big(\frac{\bar{\epsilon}_2}{4\epsilon_2 \nu} + \frac{\tilde{\epsilon}_2}{4}\Big) a\kappa\, h \sum_{j=1}^J \| |u(t_j)|^\alpha e_j\|_{\mathbb{L}^2}^2 + \frac{\tilde{\epsilon}_2 + \epsilon_3}{4} a\kappa\, h \sum_{j=1}^J \| |u^j|^\alpha e_j\|_{\mathbb{L}^2}^2 \\
&+ \frac{(C_6 C_3)^4}{64 \epsilon_1^2 \nu^2} \sup_{s\in[0,T]} \|u(s)\|_V^4 \sum_{j=1}^J \int_{t_{j-1}}^{t_j} \|u(s) - u(t_j)\|_{\mathbb{L}^2}^2 ds \\
&+ \frac{C(\alpha)^2}{8\epsilon_5 \nu} \sup_{s\in[0,T]} \|u(s)\|_V^{8\alpha} \sum_{j=1}^J \int_{t_{j-1}}^{t_j} \|u(s) - u(t_j)\|_{\mathbb{L}^2}^2 ds \\
&+ \Big[4 + \frac{\nu}{4\epsilon_4} + \frac{C(\alpha)^2}{8\epsilon_5 \nu}\Big] \sum_{j=1}^J \int_{t_{j-1}}^{t_j} \|\nabla[u(s) - u(t_j)]\|_{\mathbb{L}^2}^2 ds. \qquad (6.9)
\end{aligned}
$$

Choose positive $\epsilon_1, \epsilon_2, \epsilon_4$ and ϵ_5 such that $\epsilon_1 + \epsilon_2 + \epsilon_4 + \epsilon_5 \le \frac{1}{2}$; then choose positive $\bar{\epsilon}_2$, $\tilde{\epsilon}_2$ and ϵ_3 such that $\frac{\bar{\epsilon}_2}{4\epsilon_2\nu} + \frac{\tilde{\epsilon}_2}{4} \le 1$ and $\frac{\tilde{\epsilon}_2 + \epsilon_3}{4} \le 1$. We deduce the existence of positive constants C_i, $i = 1, 2, 3$ depending on $\nu, a, \kappa, \epsilon_j$ for $j = 1, ..., 5$, $\bar{\epsilon}_2$ and $\tilde{\epsilon}_2$, such that

$$
\begin{aligned}
&\frac{1}{2}\|e_J\|_{\mathbb{L}^2}^2 + \frac{1}{2}\sum_{j=1}^J \|e_j - e_{j-1}\|_{\mathbb{L}^2}^2 + \frac{\nu}{2} h \sum_{j=1}^J \|\nabla e_j\|_{\mathbb{L}^2}^2 \le C_1\, h \sum_{j=1}^J \|e_j\|_{\mathbb{L}^2}^2 \\
&\quad + C_2 \Big[1 + \sup_{s\in[0,T]} \|u(s)\|_V^{8\alpha}\Big] \sum_{j=1}^J \int_{t_{j-1}}^{t_j} \|u(s) - u(t_j)\|_{\mathbb{L}^2}^2 ds \\
&\quad + C_3 \sum_{j=1}^J \int_{t_{j-1}}^{t_j} \|\nabla[u(s) - u(t_j)\|_{\mathbb{L}^2}^2 ds + \sum_{j=1}^J \sum_{l=6}^7 T_{j,l}.
\end{aligned}
$$

Let N be large enough to ensure $C_1 \frac{T}{N} < \frac{1}{4}$. Note that for non negative numbers $\{x(J), y(J); J = 1, \ldots, N\}$ we have $\frac{1}{2}\big[\sup_{J\le N} a(J) + \sup_{J\le N} b(J)\big] \le \sup_{J\le N}[a(J) + b(J)]$. Therefore, using this upper estimate and then taking expected values in the above inequality, using the Cauchy–Schwarz and Hölder inequalities with conjugate exponents $p, q \in (1, \infty)$, we deduce

$$
\begin{aligned}
&\frac{1}{8}\mathbb{E}\Big(\max_{J\le N} \|e_J\|_{\mathbb{L}^2}^2\Big) + \frac{1}{4}\sum_{j=1}^N \mathbb{E}(\|e_j - e_{j-1}\|_{\mathbb{L}^2}^2) + \frac{\nu}{4} h \sum_{j=1}^N \mathbb{E}(\|\nabla e_j\|_{\mathbb{L}^2}^2) \\
&\le C_1\, h \sum_{j=0}^{N-1} \mathbb{E}(\|e_j\|_{\mathbb{L}^2}^2) + \mathbb{E}\Big(\sum_{k=1}^N |T_{j,6}|\Big) + \mathbb{E}\Big(\max_{K\le N} \sum_{j=1}^K T_{j,7}\Big)
\end{aligned}
$$

$$
+ C_2\Big\{1+\mathbb{E}\Big(\sup_{s\in[0,T]}\|u(s)\|_V^{16\alpha}\Big)\Big\}^{\frac12}\Big\{N\,h\sum_{j=1}^N \mathbb{E}\int_{t_{j-1}}^{t_j}\|u(s)-u(t_j)\|_{\mathbb{L}^2}^4 ds\Big\}^{\frac12}
$$

$$
+ C_3\mathbb{E}\Big(\sum_{j=1}^N\int_{t_{j-1}}^{t_j}\|\nabla[u(s)-u(t_j)\|_{\mathbb{L}^2}^2 ds\Big). \tag{6.10}
$$

We next find upper estimates of the expected value of the sum of the stochastic terms $T_{j,l}, l=6,7$.

For $j\in\{1,\ldots,N\}$, the Cauchy–Schwarz and Young inequalities, the Lipschitz condition (3.4), the Cauchy–Schwarz and Young inequalities imply for $\epsilon_6>0$

$$
\begin{aligned}
\mathbb{E}\big|T_{j,6}\big| &\le \mathbb{E}\Big(\Big\|\int_{t_{j-1}}^{t_j}\big[G(u(s))-G(u^{j-1})\big]dW(s)\Big\|_{\mathbb{L}^2}\|e_j-e_{j-1}\|_{\mathbb{L}^2}\Big)\\
&\le \epsilon_6\,\mathbb{E}\big(\|e_j-e_{j-1}\|_{\mathbb{L}^2}^2\big)\\
&\quad+\frac{2}{4\epsilon_6}\mathbb{E}\int_{t_{j-1}}^{t_j}\big[L\|u(s)-u(t_{j-1})\|_{\mathbb{L}^2}^2+L\|e_{j-1}\|_{\mathbb{L}^2}^2\big]\,\mathrm{Tr}Q\,ds\\
&\le \epsilon_6\,\mathbb{E}\big(\|e_j-e_{j-1}\|_{\mathbb{L}^2}^2\big)+h\,\frac{L\,\mathrm{Tr}Q}{2\epsilon_6}\,\mathbb{E}(\|e_{j-1}\|_{\mathbb{L}^2}^2)\\
&\quad+\frac{L\,\mathrm{Tr}Q}{2\epsilon_6}\,\mathbb{E}\int_{t_{j-1}}^{t_j}\|u(s)-u(t_{j-1})\|_{\mathbb{L}^2}^2\,ds.
\end{aligned}\tag{6.11}
$$

Using the Davis inequality and the Lipschitz condition (3.4), we deduce that for $\epsilon_7>0$

$$
\begin{aligned}
&\mathbb{E}\Big(\max_{K\le N}\sum_{j=1}^N T_{j,7}\Big)\\
&\le\ 3\mathbb{E}\Big(\Big\{\sum_{j=1}^J\int_{t_{j-1}}^{t_j}\|G\big(u(s)\big)-G\big(u^{j-1}\big)\|_{\mathcal{L}}^2\,\|e_{j-1}\|_{\mathbb{L}^2}^2\,\mathrm{Tr}Q\,ds\Big\}^{\frac12}\Big)\\
&\le\ 3\,\mathbb{E}\Big(\max_{0\le j\le N-1}\|e_j\|_{\mathbb{L}^2}\Big\{\sum_{j=1}^N\int_{t_{j-1}}^{t_j}\|G\big(u(s)\big)-G\big(u^{j-1}\big))\|_{\mathcal{L}}^2\,\mathrm{Tr}Q\,ds\Big\}^{\frac12}\Big)\\
&\le\ \epsilon_7\,\mathbb{E}\Big(\max_{1\le j\le N}\|e_j\|_{\mathbb{L}^2}^2\Big)\\
&\quad+\frac{18\,L\,\mathrm{Tr}Q}{4\epsilon_7}\mathbb{E}\Big(\sum_{j=1}^N\int_{t_{j-1}}^{t_j}\big[\|u(s)-u(t_{j-1})\|_{\mathbb{L}^2}^2+\|e_{j-1}\|_{\mathbb{L}^2}^2\big]\,ds\Big),
\end{aligned}\tag{6.12}
$$

where in the last inequality we have used $e_0=0$ and Young's inequality.

Choose $\epsilon_6 = \frac{1}{4}$ and $\epsilon_7 = \frac{1}{16}$; the upper estimates (6.10)–(6.12) imply

$$\begin{aligned}
&\frac{1}{16}\mathbb{E}\Big(\max_{J\le N}\|e_J\|_{\mathbb{L}^2}^2\Big) + \frac{\nu}{4}\,h\sum_{j=1}^N \mathbb{E}(\|\nabla e_j\|_{\mathbb{L}^2}^2)\\
&\quad\le \big(C_1 + 74\,L\,\mathrm{Tr}Q\big)\,h\sum_{j=0}^{N-1}\mathbb{E}(\|e_j\|_{\mathbb{L}^2}^2)\\
&\qquad + C_2 T\Big\{1+\mathbb{E}\Big(\sup_{s\in[0,T]}\|u(s)\|_V^{16\alpha}\Big)\Big\}^{\frac{1}{2}}\Big\{\sum_{j=1}^N\mathbb{E}\int_{t_{j-1}}^{t_j}\|u(s)-u(t_j)\|_{\mathbb{L}^2}^4 ds\Big\}^{\frac{1}{2}}\\
&\qquad + C(T,L,\mathrm{Tr}Q)\sum_{j=1}^N\int_{t_{j-1}}^{t_j}\mathbb{E}\big(\|u(s)-u(t_{j-1})\|_{\mathbb{L}^2}^2\big)ds\\
&\qquad + C_3\mathbb{E}\Big(\sum_{j=1}^N\int_{t_{j-1}}^{t_j}\|\nabla[u(s)-u(t_j)]\|_{\mathbb{L}^2}^2 ds\Big).
\end{aligned}$$

Let $\lambda\in(0,1)$ and set $\delta=\frac{1}{4}(1-\lambda)$. The moment estimates (4.8) and (4.9) imply

$$\begin{aligned}
&\frac{1}{16}\mathbb{E}\Big(\max_{j\le N}\|e_j\|_{\mathbb{L}^2}^2\Big) + \frac{\nu}{4}\,h\sum_{j=1}^N\mathbb{E}(\|\nabla e_j\|_{\mathbb{L}^2}^2) \le \big(C_1+74\,L\,\mathrm{Tr}Q\big)\,h\sum_{j=0}^{N-1}\mathbb{E}(\|e_j\|_{\mathbb{L}^2}^2)\\
&\quad + C(T)\big\{1+\mathbb{E}(\|u_0\|_V^{16\alpha})\big\}^{\frac{1}{2}}\,h^\lambda + C\Big[1+\mathbb{E}\Big(\|u_0\|_V^{\frac{16\alpha+2+8\delta}{1+4\delta}}\Big)\Big]h^\lambda \qquad (6.13)
\end{aligned}$$

for some constant $C := C(T,\nu,\alpha,a,p,\mathrm{Tr}Q)$. Note that for $\delta\in\big(0,\frac{1}{32\alpha-4}\big)$ we have $\frac{16\alpha+2+8\delta}{1+4\delta}\ge 16\alpha$. Neglecting the second term in the left hand side of (6.13) and using the discrete Gronwall lemma, we deduce that, for some positive constants C (resp. C_1) depending on $T,\nu,\alpha,a,\mathrm{Tr}Q$ and $\mathbb{E}\Big(\|u_0\|_V^{\frac{16\alpha+2+8\delta}{1+4\delta}}\Big)$ (resp. depending on ν,α,a,κ) such that

$$\mathbb{E}\Big(\max_{j\le N}\|e_j\|_{\mathbb{L}^2}^2\Big)\le C\,h^\lambda\,e^{16(C_1+74\,L\,\mathrm{Tr}Q)T}.$$

Plugging this inequality in (6.13) we deduce (6.2); this completes the proof when $\alpha\in(1,\frac{3}{2}]$.

(ii) We next let $\alpha=1$ and assume $4\nu a>1$ and $4\nu a\kappa>1$; we only point out the differences in the proof.

We have to use a different argument to obtain upper estimates of the terms $\{T_{j,2,i}, i=1,2,3\}$ and $T_{j,3}$. The Cauchy–Schwarz and Young inequalities prove that for $\epsilon_2,\bar\epsilon_2,\tilde\epsilon_2>0$,

$$|T_{j,2,1}|\le\epsilon_2\nu\,h\,\|\nabla e_j\|_{\mathbb{L}^2}^2+\frac{1}{4\epsilon_2\nu}\,h\,\||u(t_j)|e_j\|_{\mathbb{L}^2}^2,$$

$$|T_{j,2,2}| \le \bar{\epsilon}_2\, h\, \||u(t_j)|e_j\|_{\mathbb{L}^2}^2 + \frac{1}{4\bar{\epsilon}_2}\int_{t_{j-1}}^{t_j} \|\nabla[u(s)-u(t_j)]\|_{\mathbb{L}^2}^2 ds,$$

$$|T_{j,2,3}| \le \tilde{\epsilon}_2\, h\, \||u^j|e_j\|_{\mathbb{L}^2}^2 + \frac{1}{4\tilde{\epsilon}_2}\int_{t_{j-1}}^{t_j} \|\nabla[u(s)-u(t_j)]\|_{\mathbb{L}^2}^2 ds.$$

This implies

$$|T_{j,2}| \le \ \epsilon_2 \nu\, h\, \|\nabla e_j\|_{\mathbb{L}^2}^2 + \Big(\frac{1}{4\epsilon_2\nu} + \bar{\epsilon}_2\Big) h\, \||u(t_j)|e_j\|_{\mathbb{L}^2}^2 + \tilde{\epsilon}_2\, h\, \||u^j|e_j\|_{\mathbb{L}^2}^2 + \Big(\frac{1}{4\bar{\epsilon}_2} + \frac{1}{4\tilde{\epsilon}_2}\Big)\int_{t_{j-1}}^{t_j} \|\nabla[u(s)-u(t_j)]\|_{\mathbb{L}^2}^2 ds. \tag{6.14}$$

Using once more the Cauchy–Schwarz and Young inequalities, we obtain for $\epsilon_3 > 0$

$$|T_{j,3}| \le \epsilon_3\, h\, \||u^j|e_j\|_{\mathbb{L}^2}^2 + \frac{1}{4\epsilon_3}\int_{t_{j-1}}^{t_j} \|\nabla[u(s)-u(t_j)]\|_{\mathbb{L}^2}^2 ds. \tag{6.15}$$

The upper estimates (6.4), (6.14), (6.15), (6.7) and (6.8) imply for any positive numbers ϵ_j, $j=1,\ldots,5$, $\bar{\epsilon}_2$ and $\tilde{\epsilon}_2$

$$\begin{aligned}
&\frac12 \|e_J\|_{\mathbb{L}^2}^2 + \frac12 \sum_{j=1}^J \|e_j - e_{j-1}\|_{\mathbb{L}^2}^2 + \nu\, h \sum_{j=1}^J \|\nabla e_j\|_{\mathbb{L}^2}^2 \\
&\quad + a\kappa\, h \sum_{j=1}^J \Big[\||u(t_j)|e_j\|_{\mathbb{L}^2}^2 + \||u^j|e_j\|_{\mathbb{L}^2}^2 \Big] \le \sum_{j=1}^J \sum_{l=6}^7 T_{j,l} \\
&\quad + \Big[\epsilon_1\nu + \epsilon_5\nu\Big] h \sum_{j=1}^J \|e_j\|_{\mathbb{L}^2}^2 + \Big(\epsilon_1+\epsilon_2+\epsilon_4+\epsilon_5\Big)\nu\, h \sum_{j=1}^J \|\nabla e_j\|_{\mathbb{L}^2}^2 \\
&\quad + \Big(\frac{1}{4\epsilon_2\nu} + \bar{\epsilon}_2\Big) h \sum_{j=1}^J \||u(t_j)|e_j\|_{\mathbb{L}^2}^2 + \big(\tilde{\epsilon}_2 + \epsilon_3\big)\, h \sum_{j=1}^J \||u^j|e_j\|_{\mathbb{L}^2}^2 \\
&\quad + \frac{(C_6C_3)^4}{64\epsilon_1^2\nu^2} \sup_{s\in[0,T]} \|u(s)\|_V^8 \sum_{j=1}^J \int_{t_{j-1}}^{t_j} \|u(s)-u(t_j)\|_{\mathbb{L}^2}^2 ds \\
&\quad + \Big[1 + \frac{1}{4\bar{\epsilon}_2} + \frac{1}{4\tilde{\epsilon}_2} + \frac{1}{4\epsilon_3} + \frac{\nu}{4\epsilon_4} + \frac{C(\alpha)^2}{8\epsilon_5\nu}\Big] \sum_{j=1}^J \int_{t_{j-1}}^{t_j} \|\nabla[u(s)-u(t_j)]\|_{\mathbb{L}^2}^2 ds.
\end{aligned} \tag{6.16}$$

Fix $\epsilon \in (0,\frac12)$ such that $(1-2\epsilon)^2 4\nu a\kappa > 1$, let $\epsilon_2 = 1-2\epsilon$, and then choose positive numbers ϵ_1, ϵ_4 and ϵ_5 such that $\epsilon_1+\epsilon_2+\epsilon_4+\epsilon_5 = 1-\epsilon$. Choose $\bar{\epsilon}_2 \in (0, \epsilon a\kappa)$,

$\tilde{\epsilon}_2 + \epsilon_3 \leq a\kappa$. The choice of ϵ_2 and $\bar{\epsilon}_2$ implies $\frac{1}{4\epsilon_2 \nu} + \bar{\epsilon}_2 < a\kappa$. Therefore,

$$\begin{aligned}
&\frac{1}{2}\|e_J\|^2_{\mathbb{L}^2} + \frac{1}{2}\sum_{j=1}^J \|e_j - e_{j-1}\|^2_{\mathbb{L}^2} + \epsilon\nu\, h \sum_{j=1}^J \|\nabla e_j\|^2_{\mathbb{L}^2} \leq C_1\, h \sum_{j=1}^J \|e_j\|^2_{\mathbb{L}^2} \\
&\quad + \sum_{j=1}^J \sum_{l=6}^7 T_{j,l} + C_2 \sup_{s\in[0,T]} \|u(s)\|^8_V \sum_{j=1}^J \int_{t_{j-1}}^{t_j} \|u(s) - u(t_j)\|^2_{\mathbb{L}^2} ds \\
&\quad + C_3 \sum_{j=1}^J \int_{t_{j-1}}^{t_j} \|\nabla[u(s) - u(t_j)\|^2_{\mathbb{L}^2} ds.
\end{aligned}$$

As in the case $\alpha \in (1, \frac{3}{2}]$, using (6.11) and (6.12) with $\epsilon_6 = \frac{1}{4}$ and $\epsilon_7 = \frac{1}{16}$, we deduce

$$\begin{aligned}
&\frac{1}{16}\mathbb{E}\Big(\sup_{J\leq N} \|e_J\|^2_{\mathbb{L}^2}\Big) + \frac{\epsilon\nu}{4}\, h \sum_{j=1}^N \mathbb{E}(\|\nabla e_j\|^2_{\mathbb{L}^2}) \\
&\leq \big(C_1 + 74\, L\, \mathrm{Tr} Q\big)\, h \sum_{j=0}^{N-1} \mathbb{E}(\|e_j\|^2_{\mathbb{L}^2}) \\
&\quad + C_2 T \Big\{1 + \mathbb{E}\Big(\sup_{t\in[0,T]} \|u(s)\|^{16}_V\Big)\Big\}^{\frac{1}{2}} \Big\{\sum_{j=1}^N \mathbb{E}\int_{t_{j-1}}^{t_j} \|u(s) - u(t_j)\|^4_{\mathbb{L}^2} ds\Big\}^{\frac{1}{2}} \\
&\quad + C_3 \mathbb{E}\Big(\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|\nabla[u(s) - u(t_j)]\|^2_{\mathbb{L}^2} ds\Big).
\end{aligned}$$

We conclude the proof as in the case $\alpha \in (1, \frac{3}{2}]$. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

7 Appendix

In this section, we provide the proof of the well-posedness result stated in Sect. 3.

7.1 Proofs of preliminary estimates

The following results gather some estimates of the bilinear term, and more generally of the non linear part in (1.1). They are deduced from the Brinkman–Forchheimer smoothing term. The proofs are somewhat similar to the corresponding ones in [6] in a different functional setting.

The next lemma gathers further properties of B.

Lemma 3 *Suppose that $\alpha \in [1, +\infty)$.*

(i) Let $u \in L^\infty(0,T;H) \cap L^{2\alpha+2}([0,T]\times D;\mathbb{R}^3)$, $v \in X_0$. Then

$$\int_0^T \big|\langle B(u(t),u(t)), v(t)\rangle\big| dt \\ \leq \|\nabla v\|_{L^2(0,T;\mathbb{L}^2)} \operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|_{\mathbb{L}^2}^{\frac{\alpha-1}{\alpha}} \; \|u\|_{L^{2\alpha+2}([0,T]\times D;\mathbb{R}^3)}^{\frac{\alpha+1}{\alpha}} \; T^{\frac{\alpha-1}{2\alpha}}. \tag{7.1}$$

$$\int_0^T \big|\langle B(u(t),u(t)) - B(v(t),v(t)), u(t)-v(t)\rangle\big| dt \leq \|\nabla v\|_{L^2(0,T;H)} \\ \times \operatorname*{ess\,sup}_{t\in[0,T]} \|(u-v)(t)\|_H^{\frac{\alpha-1}{\alpha}} \; \|u-v\|_{L^{2\alpha+2}([0,T]\times D;\mathbb{R}^3)}^{\frac{\alpha+1}{\alpha}} \; T^{\frac{\alpha-1}{2\alpha}}. \tag{7.2}$$

(ii) Let $u \in L^4(\Omega; L^\infty(0,T;H)) \cap L^{2\alpha+2}(\Omega_T \times D;\mathbb{R}^3)$ and $v \in \mathcal{X}_0$. Then

$$\mathbb{E}\int_0^T \big|\langle B(u(t),u(t)), v(t)\rangle\big| dt \leq \Big\{\mathbb{E}\Big|\int_0^T \|\nabla v(t)\|_{\mathbb{L}^2}^2 dt\Big|^2\Big\}^{\frac14} \\ \times \Big\{\mathbb{E}\Big(\operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|_H^4\Big)\Big\}^{\frac{\alpha-1}{4\alpha}} \Big\{\mathbb{E}\int_0^T dt \int_D |u(t,x)|^{2\alpha+2} dx\Big\}^{\frac{1}{2\alpha}} \; T^{\frac{\alpha-1}{2\alpha}}, \tag{7.3}$$

$$\mathbb{E}\int_0^T \big|\langle B(u(t),u(t)) - B(v(t),v(t)), u(t)-v(t)\rangle\big| dt \\ \leq T^{\frac{\alpha-1}{2\alpha}} \Big\{\mathbb{E}\Big|\int_0^T \|\nabla v(t)\|_{\mathbb{L}^2}^2 dt\Big|^2\Big\}^{\frac14} \Big\{\mathbb{E}\Big(\operatorname*{ess\,sup}_{t\in[0,T]} \|(u-v)(t)\|_H^4\Big)\Big\}^{\frac{\alpha-1}{4\alpha}} \\ \times \Big\{\mathbb{E}\int_0^T dt \int_D |(u-v)(t,x)|^{2\alpha+2} dx\Big\}^{\frac{1}{2\alpha}}. \tag{7.4}$$

Proof (i) Suppose $\alpha > 1$. Using (2.4) with $h = \partial_i v_j$, $f = u_i$ and $g = u_j$, we deduce

$$|\langle B(u,u), v\rangle| = |-\langle B(u,v), u\rangle| \leq \sum_{i,j=1}^3 \int_D |u_i(x)\partial_i v_j(x) u_j(x)| dx \\ \leq \big\| |u|\, |u|^{\frac{1}{\alpha}} \big\|_{\mathbb{L}^{2\alpha}} \; \big\| |u|^{1-\frac{1}{\alpha}} \big\|_{\mathbb{L}^{\frac{2\alpha}{\alpha-1}}} \; \|\nabla v\|_{\mathbb{L}^2}.$$

Integrating on the time interval $[0,T]$ and using the Cauchy–Schwarz inequality, we obtain

$$\int_0^T \big|\langle B(u(t),u(t)),v(t)\rangle\big|dt \le \operatorname*{ess\,sup}_{t\in[0,T]}\|u(t)\|_H^{\frac{\alpha-1}{\alpha}} \Big(\int_0^T \|u(t)\|_{\mathbb{L}^{2\alpha+2}}^{\frac{2\alpha+2}{\alpha}}dt\Big)^{\frac12} \times \Big(\int_0^T \|\nabla v(t)\|_{\mathbb{L}^2}^2 dt\Big)^{\frac12}.$$

Hölder's inequality implies

$$\int_0^T \|u(t)\|_{\mathbb{L}^{2\alpha+2}}^{\frac{2\alpha+2}{\alpha}}dt \le \|u\|_{L^{2\alpha+2}([0,T]\times D;\mathbb{R}^3)}^{\frac{2\alpha+2}{\alpha}}\, T^{\frac{\alpha-1}{\alpha}}.$$

This completes the proof of (7.1) for $\alpha>1$.

If $\alpha=1$, since $|\langle B(u,u),v\rangle|\le \big\|u\big\|_{\mathbb{L}^4}^2\,\|\nabla v\|_{\mathbb{L}^2}$, a straightforward computation implies (7.1).

Since $\langle B(u,u)-B(v,v)\,,\,u-v\rangle=\langle B(u-v,v)\,,\,u-v\rangle$, using the antisymmetry (2.1) it is easy to see that the upper estimate (7.1) implies (7.2).

(ii) For $\alpha>1>\frac23$, we have $\frac{4\alpha}{3\alpha-2}>1$. Using Hölder's inequality for the expected value with exponents 4, $\frac{4\alpha}{3\alpha-2}$ and 2α in (7.1), we deduce

$$\mathbb{E}\int_0^T \big|\langle B(u(t),u(t)),v(t)\rangle\big|dt \le \Big\{\mathbb{E}\Big(\|\nabla v\|_{L^2(0,T;\mathbb{L}^2)}^4\Big)\Big\}^{\frac14} \times \Big\{\mathbb{E}\Big(\operatorname*{esssup}_{t\in[0,T]}\|u(t)\|_{\mathbb{L}^2}^{\frac{4(\alpha-1)}{3\alpha-2}}\Big)\Big\}^{\frac{3\alpha-2}{4\alpha}} \times \Big\{E\int_0^T dt\int_D |u(t,x)|^{2\alpha+2}dx\Big\}^{\frac{1}{2\alpha}}\, T^{\frac{\alpha-1}{2\alpha}}.$$

Since $\alpha>\frac12$ we have $\frac{4(\alpha-1)}{3\alpha-2}<4$; this completes the proof of (7.3) for $\alpha>1$.
For $\alpha=1$, using the antisymmetry (2.1), and twice the Cauchy–Schwarz inequality, we deduce

$$\mathbb{E}\int_0^T \big|\langle B(u(t),u(t)),v(t)\rangle\big|dt \le \Big\{\mathbb{E}\int_0^T \|\nabla v(t)\|_{\mathbb{L}^2}^2dt\Big\}^{\frac12}\Big\{\mathbb{E}\int_0^T \|u(t)\|_{\mathbb{L}^4}^4dt\Big\}^{\frac12}$$
$$\le \Big\{\mathbb{E}\Big|\int_0^T \|\nabla v(t)\|_{\mathbb{L}^2}^2dt\Big|^2\Big\}^{\frac14}\Big\{\mathbb{E}\int_0^T \|u(t)\|_{\mathbb{L}^4}^4dt\Big\}^{\frac12}.$$

This completes the proof of (7.3).
A similar argument based on the identity $\langle B(u,u)-B(v,v)\,,\,u-v\rangle=\langle B(u-v,v)\,,\,u-v\rangle$ shows (7.4). □

We next prove upper estimates for the gradient of the bilinear term.

Lemma 4 *(i) There exists a positive constant C such that for $\alpha \in (1,\infty)$, some constant $C_\alpha > 0$, any constants $\varepsilon_0, \varepsilon_1 > 0$ we have for $u \in X_1$,*

$$|\langle A^{1/2}B(u,u)\,,\ A^{1/2}u\rangle| \le C\Big[\varepsilon_0\|Au\|^2_{\mathbb{L}^2} + \frac{\varepsilon_1}{4\varepsilon_0}\big\||u|^\alpha\nabla u\big\|^2_{\mathbb{L}^2} + \frac{C_\alpha}{\varepsilon_0\varepsilon_1^{\frac{1}{\alpha-1}}}\|\nabla u\|^2_{\mathbb{L}^2}\Big]. \tag{7.5}$$

(ii) Let $\alpha = 1$; for every $\epsilon > 0$, we have for some constant $C > 0$ and any $u \in X_1$

$$|\langle A^{1/2}B(u,u)\,,\ A^{1/2}u\rangle| \le \varepsilon\|Au\|^2_{\mathbb{L}^2} + \frac{1}{4\varepsilon}\big\||u|\nabla u\big\|^2_{\mathbb{L}^2}. \tag{7.6}$$

Proof (i) Let $\alpha > 1$ and $u \in X_1$. Then

$$\langle A^{1/2}B(u,u)\,,\ A^{1/2}u\rangle = \sum_{i,j,k=1}^{3}\int_D \partial_k\big[u_i\,\partial_i u_j\big]\,\partial_k u_j dx = T_1 + T_2,$$

where, using the antisymmetry property (2.1), we get

$$T_1 = \sum_{i,j,k=1}^{3}\int_D \partial_k u_i\ \partial_i u_j\ \partial_k u_j dx,$$

$$T_2 = \sum_{i,j,k=1}^{3}\int_D u_i\ \partial_k\partial_i u_j\ \partial_k u_j dx = \sum_{k=1}^{3}\langle B(u,\partial_k u)\,,\ \partial_k u\rangle = 0.$$

Using integration by parts, we deduce $T_1 = T_{1,1} + T_{1,2}$, where since $\operatorname{div} u = 0$

$$T_{1,1} = -\sum_{j,k=1}^{3}\int_D \partial_k\Big(\sum_{i=1}^{3}\partial_i u_i\Big)\,u_j\ \partial_k u_j dx = 0,$$

$$T_{1,2} = -\sum_{i,j,k=1}^{3}\int_D \partial_k u_i\ u_j\ \partial_i\partial_k u_j dx.$$

The inequality (2.5) applied with $f = u_j$, $g = \partial_k u_i$ and $h = \partial_i\partial_k u_j$ implies

$$|T_{1,2}| \le \sum_{i,j,k=1}^{3}\varepsilon_0\|\partial_i\partial_k u_j\|^2_{L^2} + \sum_{i,j,k=1}^{3}\frac{\varepsilon_1}{4\varepsilon_0}\big\||u_j|^\alpha\partial_k u_i\|^2_{L^2} + \sum_{i,j,k=1}^{3}\frac{C_\alpha}{\varepsilon_0\varepsilon_1^{\frac{1}{\alpha-1}}}\|\partial_k u_i\|^2_{L^2}.$$

This completes the proof of (7.5).

(ii) Let $\alpha = 1$ and $u \in X_1$. Then an integration by parts implies

$$\begin{aligned}\langle A^{1/2}B(u,u)\,,\ A^{1/2}u\rangle &= \sum_{i,j,k=1}^{3}\int_D \partial_k\big[u_i\,\partial_i u_j\big]\,\partial_k u_j dx\\ &= -\sum_{i,j=1}^{3}\int_D u_i\,\partial_i u_j\ \Delta u_j\,dx.\end{aligned}$$

The Cauchy–Schwarz and Young inequalities imply (7.6).

□

For $\varphi \in X_0$, set

$$F(\varphi) = -\nu A\varphi - B(\varphi,\varphi) - a\Pi|\varphi|^{2\alpha}\varphi. \tag{7.7}$$

Lemma 2.2 page 415 in [2] provides upper and lower bounds of the non linear Brinkman–Forchheimer term. Let $\alpha \in [1,\infty)$; there exist positive constants C and κ such that for $u, v \in \mathbb{R}^3$

$$\big||u|^{2\alpha}u - |v|^{2\alpha}v\big| \le C|u-v|\,\big(|u|^{2\alpha} + |v|^{2\alpha}\big), \tag{7.8}$$

$$\big(|u|^{2\alpha}u - |v|^{2\alpha}v\big)\cdot(u-v) \ge \kappa|u-v|^2\big(|u|+|v|\big)^{2\alpha}. \tag{7.9}$$

The following lemma gives upper bounds of F for any $\alpha \in [1,\infty)$.

Lemma 5 *Let* $\alpha \in [1,+\infty)$.

(i) Let $u \in X_0$, $v \in L^2(0,T;V) \cap L^{2\alpha+2}([0,T]\times D;\mathbb{R}^3)$. *Then*

$$\begin{aligned}\int_0^T |\langle F(u(t)), v(t)\rangle| dt \le C\Big[&\|v\|_{L^2(0,T;V)}\|u\|_{L^2(0,T;V)}\\ &+ \|v\|_{L^{2\alpha+2}([0,T]\times D;\mathbb{R}^3)}\|u\|^{2\alpha+1}_{L^{2\alpha+2}([0,T]\times D;\mathbb{R}^3)}\\ &+ \|v\|_{L^2(0,T;V)}\ \operatorname*{esssup}_{t\in[0,T]}\|u(t)\|_H^{\frac{\alpha-1}{\alpha}}\ \|u\|^{\frac{\alpha+1}{\alpha}}_{L^{2\alpha+2}([0,T]\times D;\mathbb{R}^3)}T^{\frac{\alpha-1}{2\alpha}}\Big]\end{aligned} \tag{7.10}$$

for some positive constant C.

(ii) Let $u \in \mathcal{X}_0$, $v \in L^4(\Omega; L^2(0,T;V)) \cap L^{2\alpha+2}(\Omega_T\times D;\mathbb{R}^3)$. *Then*

$$\begin{aligned}\mathbb{E}\int_0^T |\langle F(u(t)), v(t)\rangle| dt \le C\Big[&\|v\|_{L^2(\Omega_T;V)}\|u\|_{L^2(\Omega_T;V)}\\ &+ \|v\|_{L^{2\alpha+2}(\Omega_T\times D;\mathbb{R}^3)}\|u\|^{2\alpha+1}_{L^{2\alpha+2}(\Omega_T\times D;\mathbb{R}^3)}\\ &+ \|v\|_{L^4(\Omega;L^2(0,T;V))}\Big\{\mathbb{E}\Big(\operatorname*{ess\,sup}_{t\in[0,T]}\|u(t)\|_H^4\Big)\Big\}^{\frac{\alpha-1}{\alpha}}\ \|u\|^{\frac{\alpha+1}{\alpha}}_{L^{2\alpha+2}(\Omega_T\times D;\mathbb{R}^3)}T^{\frac{\alpha-1}{2\alpha}}\Big]\end{aligned} \tag{7.11}$$

for some positive constant C.

Proof Integration by parts and the Cauchy–Schwarz inequality imply

$$\nu \int_0^T |\langle Au(t)\,,\ v(t)\rangle| dt = \int_0^T \Big| -\nu \int_D A^{\frac12} u(t,x) A^{\frac12} v((t,x) dx \Big| dt \\ \le \nu \|u\|_{L^2(0,T;V)} \|v\|_{L^2(0,T;V)}.$$

Furthermore, Hölder's inequality with conjugate exponents $2\alpha+2$ and $\frac{2\alpha+2}{2\alpha+1}$ yields

$$\int_0^T \Big| \int_D |u(t,x)|^{2\alpha} u(t,x) v(t,x) dx \Big| dt \le \big\| |u|^{2\alpha} u \big\|_{L^{\frac{2\alpha+2}{2\alpha+1}}([0,T]\times D;\mathbb{R}^3)} \\ \times \|v\|_{L^{2\alpha+2}([0,T]\times D;\mathbb{R}^3)}.$$

Using the above upper estimates with the inequality (7.1) concludes the proof of (7.10).

(ii) The upper estimate (7.11) is a straightforward consequence of the upper estimates (7.3), (7.10), the Cauchy–Schwarz and Hölder inequalities. □

The next lemma provides estimates of the gradient of $F(u)$ for $\alpha \in [1, +\infty)$. Note that when $\alpha = 1$, this requires that the coefficient a in front of the Brinkman–Forchheimer smoothing term is "not too smal" compared to the viscosity ν.

Lemma 6 *(i) Let $\alpha > 1$. For $\eta \in (0, \nu)$, $\tilde a \in (0, a)$, there exists a positive constant $C := C(\alpha, \eta, \tilde a)$ such that for $u \in X_1$ and $t \in [0, T]$,*

$$\int_0^t \langle A^{1/2} F(u(s)), A^{1/2} u(s)\rangle ds \\ \le -\eta \int_0^t \|Au(s)\|_{\mathbb{L}^2}^2 ds - \tilde a \int_0^t \big\| |u(s)|^\alpha \nabla u(s) \big\|_{\mathbb{L}^2}^2 ds + C \int_0^t \|\nabla u(s)\|_{\mathbb{L}^2}^2 ds. \tag{7.12}$$

(ii) Let $\alpha = 1$ and suppose $4\nu a > 1$. Then for $\eta \in \big(0, \nu - \frac{1}{4a}\big)$ and $\tilde a = a - \frac{1}{4(\nu-\eta)}$ we have

$$\int_0^t \langle A^{1/2} F(u(s)), A^{1/2} u(s)\rangle ds \le -\eta \int_0^t \|Au(s)\|_{\mathbb{L}^2}^2 ds \\ -\tilde a \int_0^t \big\| |u(s)|^\alpha \nabla u(s) \big\|_{\mathbb{L}^2}^2 ds. \tag{7.13}$$

Proof (i) Let $\alpha \in (1, \infty)$. For $u \in X_1$, integration by parts implies for a.e. $s \in [0, t]$,

$$\nu \langle A^{\frac12} \Delta u(s), A^{\frac12} u(s)\rangle = -\nu \|Au(s)\|_{\mathbb{L}^2}^2.$$

Furthermore,

$$\int_D \nabla\big(|u(s)|^{2\alpha} u(s)\big) \cdot \nabla u(s) dx$$

$$= \int_D \big[|u(s)|^{2\alpha}\nabla u(s)\cdot\nabla u(s) + 2\alpha|u(s)|^{2(\alpha-1)}\big(u(s)\cdot\nabla u(s)\big)^2\big]dx$$
$$\geq \int_D |u(s)|^{2\alpha}\nabla u(s)\cdot\nabla u(s)dx = \big\||u(s)|^\alpha\nabla u(s)\big\|_{\mathbb{L}^2}^2. \tag{7.14}$$

Hence, using (7.5) with $C\,\varepsilon_0 \in (0, \nu-\eta)$, then ε_1 such that $C\,\frac{\varepsilon_1}{4\varepsilon_0} \in (0, a-\tilde{a})$, we deduce that for a.e. $s \in [0,T]$,

$$\langle A^{1/2}F(u(s)), A^{1/2}u(s)\rangle \leq\ -\eta\|Au(s)\|_{\mathbb{L}^2}^2 - \tilde{a}\big\||u(s)|^\alpha\nabla u(t)\|_{\mathbb{L}^2}^2$$
$$+ C(\alpha,\eta,\tilde{a})\|\nabla u(s)\|_{\mathbb{L}^2}^2. \tag{7.15}$$

Integrating this inequality on the time interval $[0,t]$ concludes the proof of (7.12).

(ii) Let $\alpha = 1$. Then using (7.6) and (7.14), we deduce for $\epsilon > 0$ and $s \in [0,T]$

$$\langle A^{1/2}F(u(s)), A^{1/2}u(s)\rangle \leq\ -(\nu-\epsilon)\|Au(s)\|_{\mathbb{L}^2}^2 + \frac{1}{4\epsilon}\||u(s)|\nabla u(s)\|_{\mathbb{L}^2}^2$$
$$- a\||u(s)|\nabla u(s)\|_{\mathbb{L}^2}^2.$$

Since $4a\nu > 1$, for $\eta \in \big(0, \nu - \frac{1}{4a}\big)$, $\epsilon = \nu-\eta$ and $\tilde{a} = a - \frac{1}{4(\nu-\eta)}$ we deduce

$$\langle A^{1/2}F(u(s)), A^{1/2}u(s)\rangle \leq -\eta\|Au(s)\|_{\mathbb{L}^2}^2 - \tilde{a}\big\||u(s)|^\alpha\nabla u(s)\|_{\mathbb{L}^2}^2. \tag{7.16}$$

Integrating on the time interval $[0,t]$, we deduce (7.13).

□

We finally prove upper estimates of increments $F(u) - F(v)$ for $\alpha \in [1,\infty)$.

Lemma 7 *There exists a positive constant κ depending on $\alpha \in [1,+\infty)$, and for $\eta \in (0,\nu)$ a positive constant $\bar{C}(\eta)$, such that for $u, v \in V \cap L^{2\alpha+2}(D;\mathbb{R}^3)$,*

$$\langle F(u) - F(v), u-v\rangle \leq\ -\eta\|\nabla(u-v)\|_{\mathbb{L}^2}^2 - a\kappa\big\|(|u|+|v|)^\alpha(u-v)\big\|_{\mathbb{L}^2}^2$$
$$+ \bar{C}(\eta)\|\nabla v\|_{\mathbb{L}^2}^4\|u-v\|_{\mathbb{L}^2}^2. \tag{7.17}$$

Proof Using integration by parts, we obtain

$$\nu\langle \Delta(u-v), u-v\rangle = -\nu\|\nabla(u-v)\|_{\mathbb{L}^2}^2.$$

The monotonicity property (7.9) implies

$$a\int_D \big(|u(x)|^{2\alpha}u(x) - |v(x)|^{2\alpha}v(x)\big)\cdot\big(u(x)-v(x)\big)dx$$
$$\geq a\kappa\big\|(|u|+|v|)^\alpha(u-v)\big\|_{\mathbb{L}^2}^2. \tag{7.18}$$

Finally, Hölder's inequality and the Gagliardo–Nirenberg inequality (2.2) for the $\mathbb{L}^4$ norm imply

$$\begin{aligned}|\langle B(u,u)-B(v,v),u-v\rangle| &= |\langle B(u-v,v),u-v\rangle|\\ &\leq \|u-v\|_{\mathbb{L}^4}^2\|\nabla v\|_{\mathbb{L}^2} \leq \bar{C}_4^2\|u-v\|_{\mathbb{L}^2}^{\frac{1}{2}}\|\nabla(u-v)\|_{\mathbb{L}^2}^{\frac{3}{2}}\|\nabla v\|_{\mathbb{L}^2}\\ &\leq \frac{3}{4}\varepsilon^{\frac{4}{3}}\|\nabla(u-v)\|_{\mathbb{L}^2}^2+\frac{1}{4}\frac{1}{\varepsilon^4}\bar{C}_4^8\|\nabla v\|_{\mathbb{L}^2}^4\|u-v\|_{\mathbb{L}^2}^2,\end{aligned}$$

where the last inequality holds for any $\varepsilon>0$ by Young's inequality. Choosing $\frac{3}{4}\varepsilon^{\frac{4}{3}}\in(0,\nu-\eta)$, we conclude the proof of (7.17). □

We next prove that (1.1) has a unique strong solution in $\mathcal{X}_1$. The outline is quite classical, based on some Galerkin approximation and a priori estimates.

7.2 Galerkin approximation and a priori estimates

Recall that D is periodic domain of $\mathbb{R}^3$. Let $(e_n, n\geq 1)$ be the orthonormal basis of H defined in Sect. 3.1 (that is made of functions in H which are also orthogonal in V). For every integer $n\geq 1$ we set $\mathcal{K}_n:=\text{span}(\zeta_1,\ldots,\zeta_n)$ where $\{\zeta_j\}_{j\geq 1}$ is an ONB of K mode of eigenfunctions of Q. Let Π_n denote the projection from K onto $Q^{1/2}(\mathcal{K}_n)$, and let $W_n(t)=\sum_{j=1}^n\sqrt{q_j}\zeta_j\beta_j(t)=\Pi_n W(t)$.

Recall that if $\mathcal{H}_n=\text{span}(e_1,\ldots,e_n)$, the orthogonal projection P_n of H onto $\mathcal{H}_n$ restricted to V coincides with the orthogonal projection of V onto $\mathcal{H}_n$.

Fix $n\geq 1$ and consider the following stochastic ordinary differential equation on the n-dimensional space $\mathcal{H}_n$ defined by $u_n(0)=P_n u_0$, and for $t\in[0,T]$ and $v\in\mathcal{H}_n$:

$$d(u_n(t),v)=\big\langle P_nF(u_n(t)),v\big\rangle dt+(P_n\,G(u_n(t))\,\Pi_n\,dW(t),v),\quad \mathbb{P}\text{ a.s.},\qquad(7.19)$$

where F is defined in (7.7). Then for $k=1,\ldots,n$ we have for $t\in[0,T]$:

$$d(u_n(t),e_k)=\big\langle P_nF(u_n(t)),e_k\big\rangle dt+\sum_{j=1}^n q_j^{\frac{1}{2}}\big(P_n\,G(u_n(t))\zeta_j\,,\,e_k\big)\,d\beta_j(t),\quad \mathbb{P}\text{ a.s.}$$

Note that for $v\in\mathcal{H}_n$ the map $u\in\mathcal{H}_n\mapsto\langle F(u)\,,\,v\rangle$ is locally Lipschitz. Indeed, $\mathbb{H}^2\subset\mathbb{L}^{2\alpha+2}$ and there exists some constant $C(n)$ such that $\|v\|_{\mathbb{H}^2}\leq C(n)\|v\|_{\mathbb{L}^2}$ for $v\in\mathcal{H}_n$. Let $\varphi,\psi,v\in\mathcal{H}_n$; integration by parts implies that

$$|\langle\Delta\varphi-\Delta\psi,v\rangle|\leq\|\varphi-\psi\|_V\,\|v\|_V\leq C(n)^2\|\varphi-\psi\|_{\mathbb{L}^2}\,\|v\|_{\mathbb{L}^2}.$$

In the polynomial nonlinear term, the upper estimate (7.8), the Hölder inequality with exponents $\frac{\alpha+1}{\alpha}$, $2\alpha+2$, and $2\alpha+2$, and the Sobolev embedding $\mathbb{H}^2\subset\mathbb{L}^{2\alpha+2}$ imply

$$\begin{aligned}&\Big|\int_D\big(|\varphi(x)|^{2\alpha}\varphi(x)-|\psi(x)|^{2\alpha}\psi(x)\big)v(x)dx\Big|\\ &\quad\leq C\,\big(\|\varphi\|_{\mathbb{L}^{2\alpha+2}}^{2\alpha}+\|\psi\|_{\mathbb{L}^{2\alpha+2}}^{2\alpha}\big)\,\|\varphi-\psi\|_{\mathbb{L}^{2\alpha+2}}\,\|v\|_{\mathbb{L}^{2\alpha+2}}\end{aligned}$$

$$\leq C\,C(n)^{2(\alpha+1)}\big(\|\varphi\|_{\mathbb{L}^2}^{2\alpha} + \|\psi\|_{\mathbb{L}^2}^{2\alpha}\big)\,\|\varphi-\psi\|_{\mathbb{L}^2}\,\|v\|_{\mathbb{L}^2}.$$

Finally, using integration by parts, the Hölder and Gagliardo–Nirenberg inequalities, we deduce:

$$\begin{aligned}|\langle B(\varphi,\varphi)-B(\psi,\psi),v\rangle| &= \big|-\langle B(\varphi-\psi,v)\,,\,\varphi\rangle - \langle B(\psi,v)\,,\,\varphi-\psi\rangle\big|\\ &\leq C\,\|\varphi-\psi\|_{\mathbb{L}^4}\big(\|\varphi\|_{\mathbb{L}^4}+\|\psi\|_{\mathbb{L}^4}\big)\|\nabla v\|_{\mathbb{L}^2}\\ &\leq CC(n)^3\|\varphi-\psi\|_{\mathbb{L}^2}\big(\|\varphi\|_{\mathbb{L}^2}+\|\psi\|_{\mathbb{L}^2}\big)\|v\|_{\mathbb{L}^2}.\end{aligned}$$

Condition (**G**) implies that the map $u\in\mathcal{H}_n\mapsto\big(\sqrt{q_j}\,\big(G(u)\zeta_j\,,\,e_k\big):1\leq j,k\leq n\big)$ satisfies the classical global linear growth and Lipschitz conditions from $\mathcal{H}_n$ to $n\times n$ matrices uniformly in $t\in[0,T]$. Hence by a well-known result about existence and uniqueness of solutions to stochastic differential equations (see e.g. [24]), there exists a maximal solution $u_n=\sum_{k=1}^n(u_n\,,\,e_k)\,e_k\in\mathcal{H}_n$ to (7.19), i.e., a stopping time $\tau_n^*\leq T$ such that (7.19) holds for $t<\tau_n^*$ and if $\tau_n^*<T$, $\|u_n(t)\|_{L^2}\to\infty$ as $t\uparrow\tau_n^*$.

The following proposition shows that $\tau_n^*=T$ a.s., and provides a priori estimates on norms of u_n, which do not depend on n.

Proposition 7 *Let $\alpha\in[1,\infty)$, and if $\alpha=1$, suppose that $4\nu a>1$.*

(i) Let u_0 be $\mathcal{F}_0$-measurable such that $\mathbb{E}\big(\|u_0\|_H^2\big)<\infty$, $T>0$ and G satisfy (3.2) and (3.4). Then the evolution equation (7.19) with initial condition P_nu_0 has a unique global solution on $[0,T]$ (i.e., $\tau_n^*=T$ a.s.) with a modification $u_n\in C([0,T];\mathcal{H}_n)$. Furthermore, if $\mathbb{E}\big(\|u_0\|_H^{2p}\big)<\infty$ for some $p\in[1,\infty)$, we have $u_n\in\mathcal{X}_0$ and

$$\sup_n\mathbb{E}\Big(\sup_{t\in[0,T]}\|u_n(t)\|_H^{2p}+\int_0^T\big[\|u_n(t)\|_V^2+\|u_n(t)\|_{\mathbb{L}^{2\alpha+2}}^{2\alpha+2}\big]\|u_n(t)\|_H^{2p-2}\,dt\Big)\\ \leq C\big[1+\mathbb{E}(\|u_0\|_H^{2p})\big]. \tag{7.20}$$

(ii) If $\mathbb{E}(\|u_0\|_V^{2p})<\infty$ for some $p\in[1,\infty)$ and G satisfies also (3.3), we have furthermore

$$\sup_n\mathbb{E}\Big(\sup_{t\in[0,T]}\|u_n(t)\|_V^{2p}\\ +\int_0^T\big[\|Au_n(t)\|_{\mathbb{L}^2}^2+\||u_n(t)|^\alpha\nabla u_n(t)\|_{\mathbb{L}^2}^2\big]\|u_n(t)\|_V^{2p-2}dt\Big)\\ \leq C\big[1+\mathbb{E}(\|u_0\|_V^{2p})\big]. \tag{7.21}$$

Proof (i) For fixed $N>0$ set $\tau_N:=\inf\{t\geq0:\|u_n(t)\|_H\geq N\}\wedge\tau_n^*$. Itô's formula and the antisymmetry property of B imply

$$\|u_n(t\wedge\tau_N)\|_H^2 = \|P_n u_0\|_H^2 - 2\int_0^{t\wedge\tau_N}\big[\nu\|\nabla u_n(s)\|_{\mathbb{L}^2}^2 + a\|u_n(s)\|_{L^{2\alpha+2}}^{2\alpha+2}\big]ds + \sum_{i=1}^2 T_i(t), \tag{7.22}$$

where

$$T_1(t) = 2\int_0^{t\wedge\tau_n}\big(G(u_n(s))\,dW_n(s)\,,\,u_n(s)\big),$$
$$T_2(t) = \int_0^{t\wedge\tau_n}\|P_nG(u_n(s))\Pi_n\|_{\mathcal{L}}^2\,ds.$$

Apply once more the Itô formula to $z\mapsto z^p$ and $z=\|u_n(t\wedge\tau_N)\|_H^2$ for $p\in[2,\infty)$. We obtain

$$\|u_n(t\wedge\tau_N)\|_H^{2p} = \|P_nu_0\|_H^{2p} + \sum_{i=1}^3\bar{T}_i(t) - 2p\int_0^{t\wedge\tau_N}\big[\nu\|\nabla u_n(s)\|_{\mathbb{L}^2}^2 + a\|u_n(s)\|_{L^{2\alpha+2}}^{2\alpha+2}\big]\|u_n(s)\|_H^{2p-2}ds, \tag{7.23}$$

where

$$\bar{T}_1(t) = 2p\int_0^{t\wedge\tau_N}\big(P_nG(u_n(s))\,dW_n(s)\,,\,u_n(s)\big)\,\|u_n(s)\|_H^{2p-2},$$
$$\bar{T}_2(t) = p\int_0^{t\wedge\tau_N}\|P_nG(u_n(s))\Pi_n\|_{\mathcal{L}}^2\,\|u_n(s)\|_H^{2p-2}\,ds,$$
$$\bar{T}_3(t) = 2p(p-1)\int_0^{t\wedge\tau_N}\big\|\big(G(u_n(s))\,\Pi_n\big)^*u_n(s)\big\|_K^2\,\|u_n(s)\|_H^{2p-4}\,ds.$$

The growth condition (3.2) implies

$$\bar{T}_2(t)+\bar{T}_3(t) \le p(2p-1)\int_0^t[K_0+K_1\|u_n(s\wedge\tau_N)\|_H^2]\,\|u_n(s\wedge\tau_N)\|_H^{2p-2}\,\mathrm{Tr}Q\,ds.$$

Using the Davis inequality, the growth condition (3.2) and Young's inequality, we deduce for $\beta\in(0,1)$,

$$\begin{aligned}\mathbb{E}\Big(\sup_{s\le t\wedge\tau_n}\bar{T}_1(s)\Big) &\le 6p\,\mathbb{E}\Big(\Big\{\int_0^{t\wedge\tau_N}\|G(u_n(s))\|_{\mathcal{L}}^2\,\|u_n(s)\|_H^{4p-2}\,\mathrm{Tr}Q\,ds\Big\}^{\frac12}\Big)\\ &\le \beta\,\mathbb{E}\Big(\sup_{s\le t}\|u_n(s\wedge\tau_N)\|_H^{2p}\Big)\\ &\quad+\frac{9p^2}{\beta}\mathbb{E}\int_0^t\big[K_0+K_1\|u_n(s\wedge\tau_N)\|_H^2\big]\|u_n(s\wedge\tau_N)\|_H^{2p-2}\,\mathrm{Tr}Q\,ds.\end{aligned}$$

Neglecting the first integral in the right hand side of (7.23), using the above upper estimates of $\bar{T}_i$ and the Gronwall lemma, we deduce that for $\beta \in (0,1)$,

$$\sup_{n\geq 1} \mathbb{E}\Big(\sup_{s\leq T} \|u_n(s\wedge \tau_N)\|_H^{2p}\Big) \leq C(\beta, p, K_0, K_1, \mathrm{Tr}Q)\big[1+\mathbb{E}(\|u_0\|_H^{2p}\big]. \tag{7.24}$$

As $N\to\infty$, the sequence of stopping times τ_N increases to τ_n^* and on the set $\{\tau_n^* < T\}$, we have $\sup_{s\in[0,\tau_N]} \|u_n(s)\|_H \to \infty$. Hence (7.24) implies $P(\tau_n^* < T)=0$ and for almost every ω, for $N(\omega)$ large enough we have $\tau_{N(\omega)}(\omega)=T$. Plugging the upper estimate (7.24) in (7.23), we conclude the proof of (7.20).

Note that the above argument based on (7.22) instead of (7.23) proves that if $\mathbb{E}(\|u_0\|_H^2) < \infty$ we have once more $\tau_{N(\omega)}(\omega) = T$ for $N(\omega)$ large enough and a.e. ω, and that (7.20) holds for $p=1$.

We next prove that $u_n\in\mathcal{X}_0$. Plugging the above upper estimate for $p=1$ in (7.22), taking expected values and using Condition (3.2), we obtain

$$\mathbb{E}\int_0^T \big[\|u_n(s)\|_V^2 + \|u_n(s)\|_{L^{2\alpha+2}}^{2\alpha+2}\big]ds < \infty.$$

A similar argument using (7.24) in (7.23) completes the proof of (7.20) when the H-norm of the initial condition has $2p$ moments.

(ii) Taking the gradient of both hand sides of (7.19), using the Itô formula and (3.1), we deduce for $\tilde{\tau}_N := \inf\{s\geq 0 \,:\, \|u_n(s)\|_V \geq N\}\wedge T$,

$$\begin{aligned}
\|A^{\frac12} u_n(t\wedge\tilde{\tau}_N)\|_{\mathbb{L}^2}^2 &= \|A^{\frac12}P_n u_0\|_{\mathbb{L}^2}^2 + 2\int_0^{t\wedge\tilde{\tau}_N} \langle A^{\frac12}P_nF(u_n(s)), A^{\frac12}u_n(s)\rangle\, ds\\
&\quad + 2\int_0^{t\wedge\tilde{\tau}_N}\big(A^{\frac12}P_nG(u_n(s))dW_n(s), A^{\frac12}u_n(s)\big)\\
&\quad + \int_0^{t\wedge\tilde{\tau}_N} \|A^{\frac12}P_nG(u_n(s))\Pi_n\|_{\mathcal{L}}^2\, ds\\
&= \|A^{\frac12}P_n u_0\|_{\mathbb{L}^2}^2 + 2\int_0^{t\wedge\tilde{\tau}_N} \langle A^{\frac12}F(u_n(s)), A^{\frac12}u_n(s)\rangle\, ds\\
&\quad + 2\int_0^{t\wedge\tilde{\tau}_N}\big(A^{\frac12}G(u_n(s))\Pi_n dW(s), A^{\frac12}u_n(s)\big)\\
&\quad + \int_0^{t\wedge\tilde{\tau}_N} \|A^{\frac12}P_nG(u_n(s))\Pi_n\|_{\mathcal{L}}^2\, ds.
\end{aligned}$$

Indeed, since $u_n(s)\in V$ for $s\leq t\wedge\tilde{\tau}_N$, we deduce $A^{\frac12}u_n(s)\in H$ and $A^{\frac12}G(u_n(s))\in \mathcal{L}(K,H)$.

Using once more the Itô formula for the function $z\mapsto z^p$ for $p\in[2,\infty)$, we obtain

$$\|A^{\frac12}u_n(t\wedge\tilde{\tau}_N)\|_{\mathbb{L}^2}^{2p} \leq \|A^{\frac12}u_0\|_{\mathbb{L}^2}^{2p} + \sum_{i=1}^3 \tilde{T}_i(t)$$

$$+ 2\int_0^{t\wedge\tilde\tau_N} \langle A^{\frac12}\nabla F(u_n(s)),\, A^{\frac12}u_n(s)\rangle\, \|A^{\frac12}u_n(s)\|_{\mathbb{L}^2}^{2(p-1)} ds, \tag{7.25}$$

where

$$\begin{aligned}
\tilde T_1(t) =&\ 2p\int_0^{t\wedge\tilde\tau_N} \big(A^{\frac12}G(u_n(s))dW_n(s)\,,\ A^{\frac12}u_n(s)\big)\, \|A^{\frac12}u_n(s)\|_{\mathbb{L}^2}^{2(p-1)} ds,\\
\tilde T_2(t) =&\ p\int_0^{t\wedge\tilde\tau_N} \|G(u_n(s))\varPi_n\|^2_{\tilde{\mathcal{L}}}\ \|A^{\frac12}u_n(s)\|_{\mathbb{L}^2}^{2(p-1)} ds,\\
\tilde T_3(t) =&\ 2p(p-1)\int_0^{t\wedge\tilde\tau_N} \|\big(A^{\frac12}G(u_n(s))\,\varPi_n\big)^* A^{\frac12}u_n(s)\|_K^2\ \|A^{\frac12}u_n(s)\|_{\mathbb{L}^2}^{2(p-2)} ds.
\end{aligned}$$

Since

$$\|\big(A^{\frac12}G(u_n(s))\,\varPi_n\big)^*\|_{\mathcal{L}(H;K)} \le \|A^{\frac12}G(u_n(s))\|_{\mathcal{L}(K;H)} \le \|G(u_n(s))\|_{\mathcal{L}(K;V)},$$

the growth condition (3.3) and Young's inequality imply

$$\tilde T_2(t)+\tilde T_3(t) \le C(p,T,\mathrm{Tr}Q,\tilde K_0,\tilde K_1)\Big[1+\int_0^{t\wedge\tilde\tau_N}\big(\|u_n(s)\|_H^{2p}+\|\nabla u_n(s)\|_{\mathbb{L}^2}^{2p}\big)ds\Big].$$

The growth condition (3.3), the Gundy and Young inequalities imply that for $\tilde\beta \in (0,1)$,

$$\begin{aligned}
\mathbb{E}\Big(\sup_{s\le t}\tilde T_1(s)\Big) \le&\ C(p)\mathbb{E}\Big(\Big\{\int_0^{t\wedge\tilde\tau_N}\big[\tilde K_0+\tilde K_1\|u_n(s)\|_V^2\big]\|\nabla u_n(s)\|_{\mathbb{L}^2}^{4p-2}\,\mathrm{Tr}Q ds\Big\}^{\frac12}\Big)\\
\le&\ \tilde\beta\mathbb{E}\Big(\sup_{s\le t}\|u_n(s\wedge\tilde\tau_N)\|_{\mathbb{L}^2}^{2p}\Big)+\tilde\beta\mathbb{E}\Big(\sup_{s\le t}\|\nabla u_n(s\wedge\tilde\tau_N)\|_{\mathbb{L}^2}^{2p}\Big)\\
&+C(\tilde\beta,\mathrm{Tr}Q,\tilde K_0,\tilde K_1)\Big[1+\mathbb{E}\Big(\int_0^t \|\nabla u_n(s\wedge\tilde\tau_N)\|_{\mathbb{L}^2}^{2p}ds\Big)\Big].
\end{aligned}$$

Let $\rho\in(0,\nu)$ and $\tilde a\in(0,a)$. Using (7.12) for $\alpha>1$ and (7.13) for $\alpha=1$, (7.20) and the Gronwall lemma, we deduce

$$\mathbb{E}\Big(\sup_{s\le\tilde\tau_N}\|u_n(s)\|_V^{2p}\Big)\le C\big[1+\mathbb{E}(\|u_0\|_V^{2p})\big]$$

for some positive constant C which does not depend on N and n. For fixed n, letting $N\to\infty$ and using the monotone convergence theorem we deduce $u_n\in L^{2p}(\varOmega;L^\infty(0,T;V))$. Plugging this in (7.25) and taking expected values, we conclude the proof of (7.21). □

7.3 Proof of global well-posedness of the solution

The proof of Theorem 2 is classical and uses the upper estimates (7.2) and (7.4) for the uniqueness; see e.g. [6] for details.

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