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Spatiotemporal dynamics of a diffusive predator–prey model with fear effect*

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Abstract. This paper concerned with a diffusive predator–prey model with fear effect. First, some basic dynamics of system is analyzed. Then based on stability analysis, we derive some conditions for stability and bifurcation of constant steady state. Furthermore, we derive some results on the existence and nonexistence of nonconstant steady states of this model by considering the effect of diffusion. Finally, we present some numerical simulations to verify our theoretical results. By mathematical and numerical analyses, we find that the fear can prevent the occurrence of limit cycle oscillation and increase the stability of the system, and the diffusion can also induce the chaos in the system.

Keywords: diffusion, stability, fear effect, predator–prey model.

1 Introduction

Since Lotka [11] and Volterra [17] proposed famous Lotka–Volterra equations, the construction and study of models for the population dynamics of predator–prey interactions has been an important topic in theoretical ecology. According to different background, researchers have proposed many types of predator–prey models, and rich dynamics of these systems have been investigated extensively [6, 8, 18, 21]. In the wild, it is easy to observe that the reduction of prey is due to the direct killing of predators, which is reflected by functional responses in the predator–prey model such as Holling type and Beddington–DeAngelis [1, 7, 9, 10, 16, 24].

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However, a new study suggested that the behavior of the prey can be changed by the predator, and it could have a greater impact than direct killing. In fact, Zanette et al. [22] found that the offspring production of the song sparrows reduced by 40% because of the fear from predator. To model the fear effect in predator–prey interactions, Wang et al. [19] proposed a predator–prey model as follows:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = ur_0 f(k, v) - du - au^2 - g(u)v,$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = v\bigl(-m + cg(u)\bigr),$$

where r_0 is the birth rate of the prey, d is the natural death rate of the prey, a represents the death rate due to intraspecies competition. The parameter k refers to the level of fear, which reflects the reduction of prey growth rate due to the antipredator behavior. With the increase of k, the growth rate of prey decreases. In [19], the authors consider that the functional response $g(u)$ is the linear ($g(u) = pu$) or the Holling type II ($g(u) = p/(1 + qu)$). Their theoretical results show that fear effect could improve the stability of the predator–prey system.

It is considered that the trait effect has reduced the growth rate of the prey due to fear of predators, and the prey has been subjected to a strong Allee effect caused by mating during reproduction. Inspired by this idea, [14] considered a predator–prey model with the trait effect that reduced the growth rate of the prey due to fear of predators, and the prey has been subjected to a strong Allee effect caused by mating during reproduction. Their results showed that the fear effect does not affect the stability of the equilibria, but with the increasing of the cost of fear, the equilibrium density of predator decreases. Sasmal and Takeuchi [15] studied a predator–prey model that incorporates fear effect due to the presence of predator and group defense. Wang et al. [23] investigated a predator–prey model incorporating the fear of predators and a prey refuge, and they found that the fear effect can not only reduce the population density of predator, but also stabilize the system by excluding the existence of periodic solutions. Here, we remark that all models in these papers did not consider the factor of diffusion.

It should be pointed out that in real life, species are heterogeneous in space, so individuals tend to migrate to areas with low population density, which will increase the possibility of survival. Hence, some researchers considered reaction–diffusion predator–prey model by incorporating the fear effect into prey population. Niu et al. [4] investigated a diffusive predator–prey model with the fear effect. Taking the mature delay as bifurcation parameter, they found that the delay can induce Hopf and Hopf–Hopf bifurcations. Wang and Zou [20] proposed and analyzed a reaction–diffusion–advection predator–prey model. [3] investigated a diffusive predator–prey model with fear effect. Their results show that for the Holling type II predator functional response case, there exist no nonconstant positive steady states for large conversion rate.

Motivated by these pioneer work and note that none of the above mentioned models considered the Holling III functional response, we are led to consider a diffusive predator–

prey model as follows:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = d_1\Delta u + \frac{ru}{1+kv} - \gamma_1 u - au^2 - \frac{m_1u^2v}{b+u^2}, \quad (x,t)\in \Omega\times(0,+\infty),\\
&\frac{\partial v}{\partial t} = d_2\Delta v + v\left(-\gamma_2 + \frac{m_2u^2}{b+u^2}\right), \quad (x,t)\in \Omega\times(0,+\infty),\\
&u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad x\in\overline{\Omega},\\
&\frac{\partial u(t,x)}{\partial n} = \frac{\partial v(t,x)}{\partial n} = 0, \quad t>0,\ x\in\partial\Omega,
\end{aligned} \tag{1}$$

where $u(x,t)$, $v(x,t)$ denote the density of the prey and the predator at location x and time t, respectively. r is the birth rate of prey, γ_1 is the natural death rate of prey, a represents the death rate due to intraspecies competition. The parameter k reflects the level of fear, which drives antipredator behaviours of the prey. $m_1u^2/(b+u^2)$ is Holling type-III function (see [5]). The parameter γ_2 is the death rate of predator. $\Omega\subseteq\mathbb{R}^N$ is a bounded region with smooth boundary $\partial\Omega$, and n denotes the outward normal vector to the boundary $\partial\Omega$. The homogeneous Neumann boundary condition indicates that there is no population flow across the boundary.

We also assume that $u_0(x), v_0(x)\in C([-\tau,0],X)$, and X is defined by

$$X=\left\{u,v\in W^{2,2}(\Omega)\colon \frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0,\ x\in\Omega\right\}.$$

In this paper, our goal is to investigate the dynamical properties of (1) such as global existence of the solutions, stability and bifurcation of the constant steady state. In addition, we will use energy estimates to obtain of the dynamic and steady state solutions and so to discuss the nonexistence and existence of spatial patterns.

Our paper is organized as follows. In Section 2, we study some basic dynamics of the system. In Section 3, we obtain the stability and bifurcation of the equilibria. In Section 4, we investigate the nonexistence and existence of the nonconstant steady state. In Section 5, numerical results are presented to verify the theoretical results.

2 Basic dynamics

In this section, we discuss some basic dynamics of system (1) including the existence of solution and the priori bound of the solution.

First, we let $|\Omega|$ be the Lebesgue measure of Ω and denote

$$\left\|\varphi(\cdot,t)\right\|_{L^1(\Omega)} = \int_\Omega \left|\varphi(x,t)\right|\,\mathrm{d}x, \qquad \left\|\varphi(\cdot,t)\right\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega}\left|\varphi(x,t)\right|,$$

$$\left\|\varphi(\cdot,t)\right\|_{C(\overline{\Omega})} = \max_{x\in\overline{\Omega}}\left|\varphi(x,t)\right|.$$

Theorem 1. *For system* (1)*, the following conclusions are true*:

(i) *If* $u_0(x) \geqslant 0$*,* $v_0(x) \geqslant 0$*, then system* (1) *admits a unique solution* $(u(t,x), v(t,x)) > (0,0)$ *for* $t \in (0,+\infty)$ *and* $x \in \bar{\Omega}$;

(ii) *If* $r < \gamma_1$*, then*

$$\lim_{t\to\infty,\, x\in\overline{\Omega}} \max u(x,t) = 0, \qquad \lim_{t\to\infty,\, x\in\overline{\Omega}} \max v(x,t) = 0;$$

(iii) *If* $r > \gamma_1$*, then any solution* $(u(x,t), v(x,t))$ *of* (1) *satisfies*

$$\limsup_{t\to+\infty} u(t,x) \leqslant \frac{r-\gamma_1}{a}.$$

In addition,

$$\|u(\cdot,t)\|_{C(\bar{\Omega})} \leqslant K_1, \qquad \|v(\cdot,t)\|_{C(\bar{\Omega})} \leqslant C^*,$$

where $K_1 = \max\{K, \max_{\bar{\Omega}} u_0(x)\}$*, and* C^* *depends on* r*,* γ_1*,* γ_2*,* m_1*,* m_2*,* $u_0(x)$*,* $v_0(x)$ *and* Ω;

(iv) *If* $r < \gamma_1 + a\sqrt{\gamma_2 b/(m_2-\gamma_2)}$*, then*

$$\lim_{t\to\infty,\, x\in\overline{\Omega}} \max v(x,t) = 0.$$

Proof. (i) Define

$$f(u,v) = \frac{ru}{1+kv} - \gamma_1 u - au^2 - \frac{m_1u^2v}{b+u^2}, \qquad g(u,v) = v\left(-\gamma_2 + \frac{m_2u^2}{b+u^2}\right),$$

then $f_v = -kru/(1+kv)^2 - m_1u^2/(b+u^2) \leqslant 0$ and $g_u = 2m_2buv/(b+u^2)^2 \geqslant 0$ in $\overline{\mathbb{R}^2_+} = \{u \geqslant 0,\ v \geqslant 0\}$. Consequently, system (1) is a mixed quasimonotone system. Consider the following ordinary differential equation model:

$$\begin{aligned} &\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{ru}{1+kv} - \gamma_1 u - au^2, \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = v\left(-\gamma_2 + \frac{m_2u^2}{b+u^2}\right), \\ &u(0) = \bar{u}_0, \qquad v(0) = \bar{v}_0. \end{aligned} \tag{2}$$

where $\bar{u}_0 = \sup_{\overline{\Omega}} u_0(x)$, $\bar{v}_0 = \sup_{\overline{\Omega}} v_0(x)$. Let $(\tilde{u}(t), \tilde{v}(t))$ be the unique solution of system (2). Then $(0,0)$ and $(\tilde{u}(t), \tilde{v}(t))$ are the lower solution and upper solution of system (1). Thus, according to the [13, Thm. 8.3.3], system (1) has a unique globally defined solution $(u(x,t), v(x,t))$, which satisfies

$$0 \leqslant u(x,t) \leqslant \tilde{u}(t), \quad 0 \leqslant v(x,t) \leqslant \tilde{v}(t).$$

The strong maximum principle ensures that $u(x,t), v(x,t) > 0$ $(x \in \bar{\Omega})$.

(ii) The first equation of system (2) implies that

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{ru}{1+kv} - \gamma_1 u - au^2 \leqslant u(r-\gamma_1).$$

Obviously, $r < \gamma_1$ leads $\tilde{u} \to 0$ as $t \to \infty$. Consequently, $v(x,t) \to 0$.

(iii) It is noted that

$$\frac{\partial u}{\partial t} - d_1\Delta u = \frac{ru}{1+kv} - \gamma_1 u - au^2 - \frac{m_1u^2v}{b+u^2} \leqslant ru - \gamma_1 u - au^2.$$

Thus, by the comparison principle, one have

$$\limsup_{t\to+\infty} \max_{\bar{\Omega}} u(x,t) \leqslant \frac{r-\gamma_1}{a}.$$

The maximum principle ensures that $\|u(\cdot,t)\|_{C(\bar{\Omega})} \leqslant K_1$ for all $t \geqslant 0$.

Let $U(t) = \int_\Omega u(x,t)\,\mathrm{d}x$, $V(t) = \int_\Omega v(x,t)\,\mathrm{d}x$, then

$$\begin{aligned}\frac{\mathrm{d}U}{\mathrm{d}t} &= \int_\Omega u_t\,\mathrm{d}x = d_1\int_\Omega \Delta u\,\mathrm{d}x + \int_\Omega \left[\frac{ru}{1+kv} - \gamma_1 u - au^2 - \frac{m_1u^2v}{b+u^2}\right]\mathrm{d}x \\ &= \int_\Omega \left[\frac{ru}{1+kv} - \gamma_1 u - au^2 - \frac{m_1u^2v}{b+u^2}\right]\mathrm{d}x, \qquad (3)\\ \frac{\mathrm{d}V}{\mathrm{d}t} &= \int_\Omega v_t\,\mathrm{d}x = d_2\int_\Omega \Delta v\,\mathrm{d}x + \int_\Omega \left(v\left(-\gamma_2 + \frac{m_2u^2}{b+u^2}\right)\right)\mathrm{d}x \\ &= \int_\Omega \left(v\left(-\gamma_2 + \frac{m_2u^2}{b+u^2}\right)\right)\mathrm{d}x. \qquad (4)\end{aligned}$$

Multiplying both sides of Eq. (3) by m_2/m_1, then combining with Eq. (4), we obtain

$$\begin{aligned}\left(\frac{m_2}{m_1}U + V\right)_t &= -\gamma_2 V + \frac{m_2}{m_1}\int_\Omega \left(\frac{ru}{1+kv} - \gamma_1 u - au^2\right)\mathrm{d}x \\ &\leqslant -\gamma_2\left(\frac{m_2}{m_1}U + V\right) + \left(\frac{m_2}{m_1}(r-\gamma_1) + \gamma_2\right)U.\end{aligned}$$

Noting that $\|u(\cdot,t)\|_{C(\bar{\Omega})} \leqslant K_1$ proved above, we have $U(t) \leqslant K_1|\Omega|$. Thus

$$\left(\frac{m_2}{m_1}U + V\right)_t \leqslant -\gamma_2\left(\frac{m_2}{m_1}U + V\right) + M_2, \qquad (5)$$

where $M_2 = ((m_2/m_1)(r-\gamma_1) + \gamma_2)K_1|\Omega|$.

The integration of inequity (5) results in

$$\begin{aligned}\int_\Omega v(x,t)\,\mathrm{d}x = V(t) &< \frac{m_2}{m_1}U(t) + V(t) \\ &\leqslant \left(\frac{m_2}{m_1}U(0) + V(0)\right)\mathrm{e}^{-\gamma_2 t} + \frac{M_2}{\gamma_2}\left(1 - \mathrm{e}^{-\gamma_2 t}\right),\end{aligned}$$

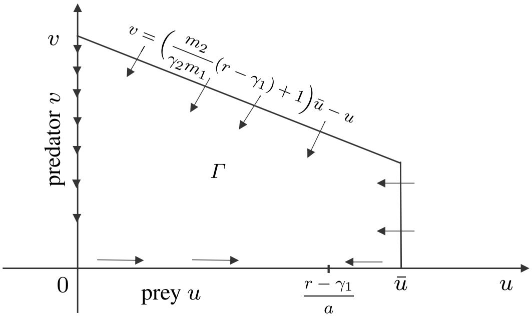

Figure 1. Invariant region $\mathcal{R}_\alpha$ for system (1).

which means that

$$\left\|v(\cdot,t)\right\|_{L^1(\Omega)} \leqslant \frac{m_2}{m_1}\left\|u_0(\cdot)\right\|_{L^1(\Omega)} + \left\|v_0(\cdot)\right\|_{L^1(\Omega)} + \frac{M_2}{\gamma_2} := K_2.$$

From [2, Thm. 3.1] we have $\|v(\cdot,t)\|_{L^\infty(\Omega)} \leqslant K_3$, where K_3 depends on K_2 and $\|v_0(x)\|_{L^\infty(\Omega)}$. As a result, there is a C^* such that $\|v(\cdot,t)\|_{C(\overline{\Omega})} \leqslant C^*$.

(iv) Obviously, $\limsup_{t\to+\infty} \max_{\bar{\Omega}} u(x,t) \leqslant (r-\gamma_1)/a$ proved above that if $r < \gamma_1 + a\sqrt{\gamma_2 b/(m_2-\gamma_2)}$, then $v(x,t) \to 0$ uniformly on $\overline{\Omega}$ as $t \to \infty$. □

Theorem 2. *The trapezoidal region*

$$\mathcal{R}_\alpha = \left\{(u,v) \;\middle|\; 0 \leqslant u \leqslant \bar{u},\ 0 \leqslant v \leqslant \left(\frac{m_2}{\gamma_2 m_1}(r-\gamma_1)+1\right)\bar{u} - u\right\},$$

$\bar{u} \geqslant (r-\gamma_1)/a$, *is a positively invariant region for system* (1) (*see Fig*. 1).

Proof. The reaction kinetics do not point out of $\mathcal{R}_\alpha$ along $u = 0$, $v = 0$, and $u = \bar{u}$. Setting

$$W(u,v) = v - \left(\frac{m_2(r-\gamma_1)}{\gamma_2 m_1} + 1\right)\bar{u} + u$$

and denoting the outward normal to $\mathcal{R}_\alpha$ along the line $W(u,v)$ by $\partial W := (\partial W/\partial u, \partial W/\partial v) = (1,1)$, then denoting $\mathbf{f} = (f,g)^{\mathrm{T}}$, one obtain

$$\begin{aligned}
&\partial W \cdot \mathbf{f}|_{v=(m_2(r-\gamma_1)/(\gamma_2 m_1)+1)\bar{u}-u} \\
&\quad = \frac{ru}{1+kv} - \gamma_1 u - au^2 - \frac{\gamma_2 m_1}{m_2}v \leqslant (r-\gamma_1)u - \frac{\gamma_2 m_1}{m_2}v \\
&\quad = \left(r - \gamma_1 + \frac{\gamma_2 m_1}{m_2}\right)(u-\bar{u}) \leqslant 0
\end{aligned}$$

as $0 \leqslant u \leqslant \bar{u}$. Consequently, $\mathcal{R}_\alpha$ is an invariant region. □

3 Constant steady state solutions, stability and bifurcation

3.1 Constant steady state solutions

Theorem 3. *For system* (1)*, the following conclusions hold*:

(i) *If* $r \leqslant \gamma_1$*, then system* (1) *has only the trivial constant solution* $E_0 = (0, 0)$;

(ii) *If* $\gamma_1 < r \leqslant \gamma_1 + a\sqrt{\gamma_2 b/(m_2 - \gamma_2)}$*, then system* (1) *has a predator-free constant steady state solution* $E_1 = ((r - \gamma_1)/a, 0)$ *denoting the extinction of predator, while system has no positive constant steady state solution*;

(iii) *If* $r > \gamma_1 + a\sqrt{\gamma_2 b/(m_2 - \gamma_2)}$*, then system* (1) *has a unique positive constant steady state solution.*

Proof. Obviously, (i) and (ii) hold. The positive constant steady state solution $E^* = (u^*, v^*)$ satisfies

$$\frac{r}{1+kv} - \gamma_1 - au - \frac{m_1 uv}{b + u^2} = 0, \qquad -\gamma_2 + \frac{m_2 u^2}{b + u^2} = 0. \tag{6}$$

From the second equation of (6) we have $u^* = \sqrt{\gamma_2 b/(m_2 - \gamma_2)}$. Then according to the first equation of (6), we obtain

$$B_1 v^2 + B_2 v + B_3 = 0, \tag{7}$$

where

$$\begin{aligned} B_1 &= k m_1 u^*, \\ B_2 &= m_1 u^* + \gamma_1 k u^{*2} + b\gamma_1 k + a k u^{*3} + abku^*, \\ B_3 &= (\gamma_1 - r)\big(b + u^{*2}\big) + a u^{*3} + abu^*. \end{aligned}$$

Clearly, $B_1, B_2 > 0$. Therefore, Eq. (7) has at most a positive root as that $B_3 < 0$, which means that $r > \gamma_1 + a\sqrt{\gamma_2 b/(m_2 - \gamma_2)}$. Therefore, we have the conclusion. □

3.2 Stability

Recall that $0 = \mu_0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots \to \infty$ are the eigenvalues of the Laplace operator $-\Delta$ on Ω under homogeneous Neumann boundary condition, and $S(\mu_n)$ is the space of eigenfunctions corresponding to μ_i in $W^{1,2}(\Omega)$. Let $\mathbf{X} = [W^{1,2}(\Omega)]^2$ and $\{\phi_{ij}\colon j = 1, \ldots, \dim[S(\mu_n)]\}$ be an orthonormal basis of $S(\mu_n)$, and $\mathbf{X}_{i,j} = \{\mathbf{c}\phi_{i,j}\colon \mathbf{c} \in \mathbf{R}^2\}$. Then

$$\mathbf{X} = \bigoplus_{i=1}^{\infty} \mathbf{X}_i \quad \text{and} \quad \mathbf{X}_i = \bigoplus_{j=1}^{\dim[S(\mu_j)]} \mathbf{X}_{ij}.$$

Assume that (u, v) is a constant solution of system (1), then we have

$$\begin{pmatrix} \phi_t \\ \psi_t \end{pmatrix} = L \begin{pmatrix} \phi \\ \psi \end{pmatrix} = D \begin{pmatrix} \Delta\phi \\ \Delta\psi \end{pmatrix} + J_{(u,v)} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

with domain $X = \{(\phi,\psi) \in H^2(\Omega) \times H^2(\Omega)\colon \partial\phi/\partial = 0\}$, where

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \qquad J(u,v) = \begin{pmatrix} A_1(u,v) & A_2(u,v) \\ A_3(u,v) & A_4(u,v) \end{pmatrix}$$

and

$$A_1(u,v) = \frac{r}{1+kv} - \gamma_1 - 2au - \frac{2m_1buv}{(b+u^2)^2}, \qquad A_2(u,v) = -\frac{rkuv}{(1+kv)^2} - \frac{m_1u^2}{b+u^2},$$
$$A_3(u,v) = \frac{2m_2buv}{(b+u^2)^2}, \qquad A_4(u,v) = -\gamma_2 + \frac{m_2u^2}{b+u^2}.$$

For each $i = 0,1,\dots$, $\mathbf{X}_i$ is invariant under the operator L, and λ is an eigenvalue of L on $\mathbf{X}_i$ if and only if λ is an eigenvalue of $-\mu_n D + J(u,v)$ for all $n \in \{0,1,2,\dots\} := N_0$.

The direct calculation shows

$$\lambda^2 - T_n\lambda + D_n = 0, \tag{8}$$

where

$$T_n = -(d_1+d_2)\mu_n + A_1(u,v) + A_4(u,v),$$
$$D_n = d_1d_2\mu_n^2 - \big(A_1(u,v)d_2 + A_4(u,v)d_1\big)\mu_n + A_1(u,v)A_4(u,v) - A_2(u,v)A_3(u,v).$$

Theorem 4.

(i) *If $r < \gamma_1$, then $E_0 = (0,0)$ is globally asymptotically stable.*

(ii) *If $\gamma_1 < r \leqslant \gamma_1 + a\sqrt{\gamma_2 b/(m_2-\gamma_2)}$, then $E_1 = ((r-\gamma_1)/a, 0)$ is globally asymptotically stable.*

(iii) *If $m_1v^*(u^{*2} - b)/(b+u^{*2})^2 - a < 0$, then E^* is locally asymptotically stable.*

Proof. (i) For $E_0 = (0,0)$, the corresponding characteristic equation is

$$(\lambda + d_1\mu_n - r + \gamma_1)(\lambda + d_2\mu_n + \gamma_2) = 0.$$

Clearly, we obtain

$$\lambda_1 = r - \gamma_1 - d_1\mu_n, \qquad \lambda_2 = -\gamma_2 - d_2\mu_n.$$

Hence, if $r < \gamma_1$, then E_0 is locally asymptotically stable. Note that there is no other constant steady states in this case. This means that E_0 is indeed globally asymptotically stable.

(ii) For $E_1 = ((r-\gamma_1)/a, 0)$, the corresponding characteristic equation is

$$(\lambda + d_1\mu_n + r - \gamma_1)\left(\lambda + d_2\mu_n + \gamma_2 - \frac{m(r-\gamma_1)^2}{a^2b + (r-\gamma_1)^2}\right) = 0.$$

Obviously,

$$\lambda_1 = \gamma_1 - r - d_1\mu_n, \qquad \lambda_2 = -\gamma_2 + \frac{m(r-\gamma_1)^2}{a^2b + (r-\gamma_1)^2} - d_2\mu_n.$$

Consequently, if $\gamma_1 < r \leqslant \gamma_1 + a\sqrt{\gamma_2 b/(m_2 - \gamma_2)}$, then $E_1 = ((r - \gamma_1)/a, 0)$ is locally asymptotically stable. In fact, E_1 is globally asymptotically stable.

It follows from Theorem 1 that $\limsup_{t\to+\infty} \max_{\bar{\Omega}} u(\cdot, t) \leqslant (r-\gamma_1)/a$, so for $\epsilon > 0$,

$$u(\cdot, t) \leqslant \frac{r - \gamma_1}{a} + \epsilon, \quad t \geqslant t_1.$$

By the second equation of (1) we have

$$v_t - d_2\Delta v \leqslant v\left(\frac{m_2(\frac{r-\gamma_1}{a} + \epsilon)^2}{b + (\frac{r-\gamma_1}{a} + \epsilon)^2} - \gamma_2\right), \quad t \geqslant t_1.$$

Therefore, $\limsup_{t\to+\infty} \max_{\bar{\Omega}} v(\cdot, t) \leqslant 0$, and there exists $t_2 > t_1$ such that $v(\cdot, t) \leqslant \epsilon$, $t \geqslant t_2$. Then by first equation of (1), one have

$$u_t - d_1\Delta u \geqslant \frac{ru}{1 + k\epsilon} - \gamma_1 u - au^2 - \frac{m_1 u^2 \epsilon}{b + u^2}, \quad t > t_2,\ x \in \Omega.$$

Then we obtain that $\liminf_{t\to+\infty} \min_{\bar{\Omega}} u(\cdot, t) \geqslant (r - \gamma_1)/a$. Combining with $\limsup_{t\to+\infty} \max_{\bar{\Omega}} u(\cdot, t) \leqslant (r - \gamma_1)/a$ allows us to derive

$$\lim_{t\to+\infty,\,\bar{\Omega}} \max \left| u(\cdot, t) - \frac{r - \gamma_1}{a} \right| = 0.$$

Hence, E_1 is globally asymptotically stable when $\gamma_1 < r \leqslant \gamma_1 + a\sqrt{\gamma_2 b/(m_2 - \gamma_2)}$.

(iii) For the positive steady state $E^* = (u^*, v^*)$, $A_4(u^*, v^*) = 0$ and $A_1(u^*, v^*) = u^*(m_1 v^*(u^{*2} - b)/(b + u^{*2})^2 - a)$. Hence, the corresponding characteristic equation is

$$\begin{aligned}\lambda^2 - \bigl(A_1(u^*, v^*) - (d_1 + d_2)\mu_n\bigr)\lambda + d_1 d_2 \mu_n^2 \\ - A_1(u^*, v^*) d_2 \mu_n - A_2(u^*, v^*) A_3(u^*, v^*) = 0.\end{aligned}$$

Obviously,

$$\begin{aligned}\lambda_1 + \lambda_2 &= -\mu_n(d_1 + d_2) + A_1(u^*, v^*), \\ \lambda_1\lambda_2 &= d_1 d_2 \mu_n^2 - A_1(u^*, v^*) d_2 \mu_n - A_2(u^*, v^*) A_3(u^*, v^*).\end{aligned} \tag{9}$$

All roots of (9) have negative real parts if

$$\frac{m_1 v^*(u^{*2} - b)}{(b + u^{*2})^2} - a < 0. \tag{10}$$

Therefore, the positive constant steady state $E^* = (u^*, v^*)$ is locally asymptotically stable when condition (10) holds. □

Remark 1. Theorems 3 and 4 show that when $r \in (0, \gamma_1]$, system has only trivial constant solution $E_0 = (0, 0)$, and it is globally asymptotically stable; when r increases and enter the interval $(\gamma_1, \gamma_1 + a\sqrt{\gamma_2 b/(m_2 - \gamma_2)})$, E_0 loses its stability to a predator-free constant steady state E_1; and when r further passes $\gamma_1 + a\sqrt{\gamma_2 b/(m_2 - \gamma_2)}$, E_1 loses its stability to a positive steady state E^*. We can conclude that as the parameter r increases, the model experiences two bifurcations of constant steady state.

Remark 2. Obviously, the conditions of Theorem 4 are independent of the diffusion. Consequently, the conclusions of Theorem 4 are still valid for the corresponding ODE model. In addition, we can also conclude that the diffusion cannot destabilize the positive steady state E^*. Therefore, the PDE system (1) cannot occur Turing instability/bifurcation.

3.3 Hopf bifurcation

In this subsection, we will discuss the bifurcation of system (1). Let the parameters r, k, a, b, γ_1, γ_2, m_1, m_2 and d_1 be fixed, and take $d_2 > 0$ as a bifurcation parameter.

Theorem 5.

(i) *If* $m_1 v^*(u^{*2} - b)/(b + {u^*}^2)^2 = a$ *holds, then spatially homogeneous Hopf bifurcation occurs.*

(ii) *If* $d_1\mu_1 < A_1(u^*, v^*)$, *let* n_0 *be the largest positive integer such that* $A_1(u^*, v^*) - d_1\mu_{n_0} > 0$. *In addition, we assume that* $d_{2n_1} \neq d_{2n_2}$ *whenever* $n_1 \neq n_2$, $1 \leqslant n_1, n_2 < n_0$, *and*

$$\frac{A_1}{2d_1} < \mu_1 < \min\left\{-A_1 + \sqrt{-A_2A_3},\ \frac{A_1}{d_1}\right\}. \tag{11}$$

Then system (1) *undergoes spatially inhomogeneous Hopf bifurcation at* (d_{2n}^H, E^*) *for* $1 \leqslant n \leqslant n_0$, *where* $d_{2n}^H = (A_1(u^*, v^*) - d_1\mu_n)/\mu_n$.

Proof. (i) If $m_1 v^*(a^2 - b)/(b + {u^*}^2)^2 = a$ holds, then $T_0 = 0$, $T_n \neq 0$ $(n \geqslant 1)$, $D_0 > 0$, and $D_n > 0$ $(n \geqslant 1)$. Therefore, Eq. (8) has a pair of pure imaginary roots $\lambda = \{\pm\sqrt{D_0}\imath\}/$, which means that spatially homogeneous Hopf bifurcation occurs.

(ii) From the assumption it follows that $T_1(d_{21}^H) = 0$, $T_n(d_{21}^H) \neq 0$ for $n \geqslant 0$ and $T_0(d_{20}) = B_1(u^*, v^*) > 0$. In addition, $D_0(d_{20}) = -A_2(u^*, v^*)A_3(u^*, v^*) > 0$ for any $d_2 > 0$. Clearly,

$$D_1(d_{21}^H) = -d_1^2\mu_1^2 + 2d_1A_1\mu_1 - A_1^2 - A_2A_3.$$

Obviously, if condition (11) holds, then $D_1(d_{21}^H) > 0$. Moreover, if $\mu_1 > B_1/(2d_1)$, then

$$\frac{\mathrm{d}D_n}{\mathrm{d}\mu_n} = 2d_1d_2\mu_n - A_1d_2 \geqslant 2d_1d_2\mu_1 - A_1d_2 > 0.$$

Therefore, $D_n(d_{21}^H)$ is nondecreasing with respect to n. Hence, when $n \geqslant 2$, $D_k(d_{21}^H) \geqslant D_1(d_{21}^H) > 0$. Therefore, when d_2 is near d_{21}^H, Eq. (8) has a pair of conjugate eigenvalues

$$\lambda = \frac{1}{2}\left\{-T_1(d_2) \pm \sqrt{T_1^2(d_2) - 4D_1^2(d_2)}\right\}.$$

Clearly, $\mathrm{Re}'(\lambda) = -\mu_1/2 \neq 0$.

As a result, Hopf bifurcation occurs at (d_{21}^H, E^*), which also means that system (1) has a family of inhomogeneous periodic solutions near E^*. □

4 Nonconstant steady states

In this section, we will discuss nonexistence and existence of nonconstant steady state of system (1). To this end, we consider the following elliptic system:

$$\begin{aligned} -d_1\Delta u &= \frac{ru}{1+kv} - \gamma_1 u - au^2 - \frac{m_1 u^2 v}{b+u^2}, \quad x \in \Omega, \\ -d_2\Delta v &= v\left(\gamma_2 - \frac{m_1 u^2}{b+u^2}\right), \quad x \in \Omega. \end{aligned} \tag{12}$$

4.1 A priori estimates

To derive some priori estimates for nonnegative solutions of system (12), we need the following technical lemma [12].

Lemma 1 [Maximum principle]. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ and $g \in C(\bar{\Omega}) \times \mathbb{R}$. If $z \in H^1(\Omega)$ is a weak solution of the inequalities*

$$\Delta z + g\bigl(x, z(x)\bigr) \geqslant 0 \quad \text{in } \Omega, \qquad \frac{\partial z(x)}{\partial n} \leqslant 0 \quad \text{on } \partial\Omega$$

and if there is a constant K such that $g(x,z) < 0$ for $z > K$, then $z \leqslant K$ a.e. in Ω.

Lemma 2 [Harnack inequality]. *Suppose that $c(x) \in C(\Omega)$ and $\omega \in C^2(\Omega) \cap C^1(c\bar{\Omega})$ is a positive classical solution to $\Delta\omega(x) + c(x)\omega(x) = 0$ in Ω subject to the homogenuous Neumann boundary condition. Then there exists a positive constant $C_* = C_*(\|c(x)\|_\alpha, \Omega)$ such that*

$$\max_{\bar{\Omega}} \omega(x) \leqslant C_* \min_{\bar{\Omega}} \omega(x).$$

For the sake of discussion, we shall write $\wedge = \wedge(d_1, d_2, r, k, \gamma_1, \gamma_2, m_1, m_2, a, b)$.

Theorem 6 [Upper bounds]. *Suppose that $(u(x), v(x))$ is a nonnegative solution of (12), then either $(u(x), v(x))$ is one of constant solutions $(0,0)$ and $((r-\gamma_1)/a, 0)$ or for $x \in \overline{\Omega}$, $(u(x), v(x))$ satisfies*

$$0 < u(x) < M_1, \quad 0 < v(x) < M_2, \tag{13}$$

where

$$M_1 = \frac{r-\gamma_1}{a}, \qquad M_2 = \frac{m_1(r - \gamma_1 + \frac{d_1\gamma_2}{d_2})^2}{4am_2\gamma_2}.$$

Proof. If there exists $x_0 \in \overline{\Omega}$ satisfying $v(x_0) = 0$, then by the strong maximum principle, $v(x) \equiv 0$ and

$$\begin{aligned} -d_1\Delta u &= (r-\gamma_1)u - au^2, \quad x \in \overline{\Omega}, \\ \frac{\partial u}{\partial n} &= 0, \quad x \in \partial\Omega. \end{aligned}$$

Thus, $u \equiv 0$ or $u \equiv (r-\gamma_1)/a$. Otherwise, $u(x) > 0$ and $v(x) > 0$ for $x \in \overline{\Omega}$.

Further, from Lemma 1 we obtain that $u(x) \leqslant (r-\gamma_1)/a := M_1$, and by the strong maximum principle, we have $u(x) < M_1$ for all $x \in \overline{\Omega}$. Then

$$\begin{aligned}
-\left(d_1\Delta u + \frac{m_2}{m_1}d_2\Delta v\right) &= \frac{ru}{1+kv} - \gamma_1 u - au^2 - \frac{m_1\gamma_2 v}{m_2} \\
&\leqslant \left(\left(r-\gamma_1+\frac{d_1\gamma_2}{d_2}\right)u - au^2\right) - \frac{\gamma_2}{d_2}\left(d_1 u + \frac{m_2}{m_1}d_2 v\right) \\
&\leqslant \frac{(r-\gamma_1+\frac{d_1\gamma_2}{d_2})^2}{4a} - \frac{\gamma_2}{d_2}\left(d_1 u + \frac{m_2}{m_1}d_2 v\right).
\end{aligned}$$

It can be obtained from the maximum principle that

$$d_1 u + \frac{m_2}{m_1}d_2 v < \frac{d_2(r-\gamma_1+\frac{d_1\gamma_2}{d_2})^2}{4a\gamma_2}.$$

Therefore,

$$v \leqslant \frac{m_1}{d_2 m_2}\left(d_1 u + \frac{m_2}{m_1}d_2 v\right) < \frac{m_1(r-\gamma_1+\frac{d_1\gamma_2}{d_2})^2}{4am_2\gamma_2} := M_2. \qquad \square$$

Theorem 7. *Let d^* be a fixed positive constant. Then for $d_1, d_2 \geqslant d^*$, there exists two positive constants $\underline{C}$ and $\overline{C}$ with $\underline{C} < \overline{C}$ depending possibly on $\wedge$ such that any solutions $(u(x), v(x))$ of system* (12) *satisfies*

$$\underline{C} \leqslant u(x), v(x) \leqslant \overline{C}.$$

Proof. We choose $\overline{C} = \max\{M_1, M_2\}$, so $u(x), v(x) \leqslant \overline{C}$ for any $x \in \overline{\Omega}$.

Next, we shall prove $u(x), v(x) \geqslant \underline{C}$. Let

$$\begin{aligned}
c_1(x) &= d_1^{-1}\left(\frac{r}{1+kv(x)} - \gamma_1 - au(x) - \frac{m_1 u(x)v(x)}{b+u^2(x)}\right), \\
c_2(x) &= d_2^{-1}\left(-\gamma_2 + \frac{m_1 u^2(x)}{b+u^2(x)}\right).
\end{aligned}$$

Thus,

$$\left|c_1(x)\right| \leqslant d_1^{-1}(r-\gamma_1+a\overline{C}), \qquad \left|c_2(x)\right| \leqslant d_2^{-1}\left(\gamma_2 + \frac{m_1\overline{C}^2}{b+\overline{C}^2}\right).$$

Lemma 2 shows that there exists a positive constant C_2 such that

$$\max_{\bar{\Omega}} u(x) \leqslant C_2 \min_{\bar{\Omega}} u(x), \qquad \max_{\bar{\Omega}} v(x) \leqslant C_2 \min_{\bar{\Omega}} v(x).$$

Hence, now it remains to prove that there exists $C_3 > 0$ such that

$$\max_{\bar{\Omega}} u(x) \geqslant C_3, \qquad \max_{\bar{\Omega}} v(x) \geqslant C_3. \tag{14}$$

Contrariwise, let us assume that (14) does not hold. Then there exists a sequence $(u_n(x), v_n(x))$ such that

$$\max_{\bar{\Omega}} u_n \to 0 \quad \text{or} \quad \max_{\bar{\Omega}} v_n \to 0 \quad \text{as } n \to +\infty. \tag{15}$$

By the regularity theory for elliptic equations, there exists a subsequence of $\{(u_n, v_n)\}$, which will be denoted again by $\{(u_n, v_n)\}$ such that $\{(u_n, v_n)\} \to (u_0, v_0)$ in $C^2(\bar{\Omega})$ as $n \to +\infty$. Note that $u_0 \leqslant (r - \gamma_1)/a$ and from (15) either $u_0 \equiv 0$ or $v_0 \equiv 0$. Therefore, we have that

(i) $u_0 \equiv 0$, $v_0 \not\equiv 0$; or $u_0 \equiv 0$, $v_0 \equiv 0$;
(ii) $u_0 \not\equiv 0$, $v_0 \equiv 0$.

Also, $\{(u_n, v_n)\}$ satisfy (13), so do u and v. Letting $n \to \infty$, we get that $\{(u_n, v_n)\}$ is a positive solution of (12). Therefore, by integrating Eq. (12) for u_n and v_n over Ω, we have

$$\int_{\Omega} \left(\frac{r u_n}{1 + k v_n} - \gamma_1 u_n - a u_n^2 - \frac{m_1 u_n^2 v_n}{b + u_n^2} \right) \mathrm{d}x = 0,$$

$$\int_{\Omega} v_n \left(-\gamma_2 + \frac{m_1 u_n^2}{b + u_n^2} \right) \mathrm{d}x = 0.$$

(i) In this case, $u_0 \equiv 0$, then

$$-\gamma_2 + \frac{m_1 u_n^2}{b + u_n^2} \to -\gamma_2 < 0$$

and $v_n > 0$, then

$$\int_{\Omega} v_n \left(-\gamma_2 + \frac{m_1 u_n^2}{b + u_n^2} \right) < 0$$

for sufficiently large n. So, we obtain a contradiction.

(ii) If $u_0 \not\equiv 0$, $v_0 \equiv 0$, using the first equation of (12). So, $u_0 \equiv (r - \gamma_1)/a$ for large n. Thus

$$-\gamma_2 + \frac{m_1 u_n^2}{b + u_n^2} \to \frac{m_1 \frac{(r-\gamma_1)^2}{a^2}}{b + \frac{(r-\gamma_1)^2}{a^2}} - \gamma_2 \neq 0.$$

So, we have

$$\int_{\Omega} v_n \left(-\gamma_2 + \frac{m_1 u_n^2}{b + u_n^2} \right) \neq 0$$

for a sufficiently large n, which is a contradiction. This completes the proof. □

4.2 Nonexistence of nonconstant positive steady states

In this subsection, we show the nonexistence of positive steady state solutions when the diffusion coefficients d_1 and d_2 are large.

Theorem 8. *For any fixed r, k, γ_1, γ_2, m_1, m_2, b and a, there exists a positive constant d^* such that if $\min\{d_1, d_2\} > d^*$, then (12) has no nonconstant solutions.*

Proof. Assume that $(u(x), v(x))$ is nonnegative solution of (12). Denote

$$\bar{u} = \frac{\int_\Omega u(x)\,\mathrm{d}x}{|\Omega|} \quad \text{and} \quad \bar{v} = \frac{\int_\Omega v(x)\,\mathrm{d}x}{|\Omega|}.$$

Obviously, $\int_\Omega (u - \bar{u})\,\mathrm{d}x = 0$ and $\int_\Omega (v - \bar{v})\,\mathrm{d}x = 0$. For the purpose of discussions, let $H(u, v) = u^2 v/(b + u^2)$. By the mean value theorem of bivariate functions, we have

$$H(u, v) - H(\bar{u}, \bar{v}) = H'_u(\xi, \eta)(u - \bar{u}) + H'_v(\xi, \eta)(v - \bar{v}).$$

Obviously, $H_u = 2buv/(b+u^2)^2 \leqslant K_1$, $H_v = u^2/(b+u^2) \leqslant 1$, where $K_1 = 2M_1M_2/b$.

Multiplying both sides of the first equation of (12) by $u - \bar{u}$ and using Theorem 6, we get

$$\begin{aligned} d_1 \int_\Omega \left|\nabla(u-\bar{u})\right|^2 \mathrm{d}x &= \delta \int_\Omega (u-\bar{u})\left(\frac{ru}{1+kv} - \gamma_1 u - au^2 - H(u,v) - \frac{r\bar{u}}{1+k\bar{v}} + \gamma_1\bar{u} - a\bar{u}^2 - H(\bar{u},\bar{v})\right) \mathrm{d}x \\ &\leqslant (r + rkM_2 + \gamma_1 + 2M_1 a) \int_\Omega (u-\bar{u})^2\,\mathrm{d}x + (rkM_1 + 1) \int_\Omega \|u-\bar{u}\|\|v-\bar{v}\|\,\mathrm{d}x \\ &\leqslant \left(r + rkM_2 + \gamma_1 + 2M_1 a + \frac{rkM_1 + 1}{2}\right) \int_\Omega (u-\bar{u})^2\,\mathrm{d}x \\ &\quad + \frac{rkM_1 + 1}{2} \int_\Omega (v-\bar{v})^2\,\mathrm{d}x. \end{aligned} \tag{16}$$

Applying Theorem 6 and by multiplying $v - \bar{v}$ to the second equation in (12) and then integrating on Ω, we have

$$\begin{aligned} d_2 \int_\Omega \left|\nabla(v - \bar{v})\right|^2 \mathrm{d}x &= \int_\Omega (v - \bar{v})\big(\gamma_2(v - \bar{v}) + H(u, v) - H(\bar{u}, \bar{v})\big)\,\mathrm{d}x \\ &\leqslant (\gamma_2 + 1) \int_\Omega (v - \bar{v})^2\,\mathrm{d}x + K_1 \int_\Omega \|u - \bar{u}\|\|v - \bar{v}\|\,\mathrm{d}x, \\ &\leqslant \left(\gamma_2 + 1 + \frac{K_1}{2}\right) \int_\Omega (v - \bar{v})^2\,\mathrm{d}x + \frac{K_1}{2} \int_\Omega (u - \bar{u})^2\,\mathrm{d}x. \end{aligned} \tag{17}$$

Using the Poincaré inequality,

$$\mu_1 \int_\Omega (u-\bar{u})^2 \,\mathrm{d}x \leqslant \int_\Omega \left|\nabla(u-\bar{u})\right|^2 \mathrm{d}x, \qquad \mu_1 \int_\Omega (v-\bar{v})^2 \,\mathrm{d}x \leqslant \int_\Omega \left|\nabla(v-\bar{v})\right|^2 \mathrm{d}x,$$

where μ_1 is the second eigenvalue of the Laplace operator $-\Delta$ on Ω under homogeneous Neumann boundary condition.

Combining (16) and (17) leads to

$$\begin{aligned} d_1\mu_1 \int_\Omega (u-\bar{u})^2 \,\mathrm{d}x + d_2\mu_1 \int_\Omega (v-\bar{v})^2 \,\mathrm{d}x \\ \leqslant A \int_\Omega (u-\bar{u})^2 \,\mathrm{d}x + B \int_\Omega (v-\bar{v})^2 \,\mathrm{d}x, \end{aligned}$$

where

$$A = r + rkM_2 + \gamma_1 + 2M_1 a + \frac{rkM_1 + 1}{2} + \frac{K_1}{2},$$
$$B = \frac{rkM_1}{2} + \frac{3}{2} + \gamma_2 + \frac{K_1}{2}.$$

This implies that

$$\min\{d_1, d_2\} > d^* = \frac{1}{\mu_1} \max\{A, B\},$$

then we can conclude that

$$\nabla(u-\bar{u}) = \nabla(v-\bar{v}) = 0.$$ □

4.3 Existence of nonconstant positive steady states

To study the existence of nonconstant positive solutions, we use Leray–Schauder degree theory. Let $\mathbf{w} = (u, v)$ and

$$\mathbf{F}(\mathbf{w}) = \left(\frac{ru}{1+kv} - \gamma_1 u - au^2 - \frac{m_1 u^2 v}{b+u^2},\ v\left(-\gamma_2 + \frac{m_2 u^2}{b+u^2}\right)\right)^{\mathrm{T}}.$$

Thus, (12) can be rewritten as

$$-D\Delta\mathbf{w} = \mathbf{F}(\mathbf{w}) \quad \text{in} \ \frac{\partial \mathbf{w}}{\partial n} = 0 \ \text{on} \ \Omega,$$

or equivalently,

$$\mathcal{F}(\mathbf{w}) := \mathbf{w} - (I-\Delta)^{-1}\left(D^{-1}\mathbf{F}(\mathbf{w}) + \mathbf{w}\right) \quad \text{on} \ X, \tag{18}$$

where $(I-\Delta)^{-1}$ represents the inverse of $I-\Delta$ with the homogeneous Neumann boundary condition. From (18), by a direct computation, we have

$$\mathcal{F}_{\mathbf{w}}(\mathbf{w}^*) = I - (I-\Delta)^{-1}\left(D^{-1}\mathbf{F}_{\mathbf{w}}(\mathbf{w}^*) + I\right), \quad \mathbf{w}^* = (u^*, v^*).$$

Clearly,

$$H(d_1, d_2; \mu) = \det\bigl[\mu I - D^{-1}\mathbf{F}_\mathbf{w}(\mathbf{w}^*)\bigr] = \frac{1}{d_1 d_2}\det\bigl[\mu D - \mathbf{F}_\mathbf{w}(\mathbf{w}^*)\bigr],$$
$$= \mu^2 - \frac{A_1(u^*, v^*)}{d_1}\mu_1 - \frac{A_2(u^*, v^*)A_3(u^*, v^*)}{d_1 d_2},$$

where

$$A_1(u^*, v^*) = u^*\left(\frac{m_1 v^*(a^2 - b)}{(b + u^*)^2} - a\right),$$
$$A_2(u^*, v^*) = -\left(\frac{rku^*v^*}{(1 + kv^*)^2} + \frac{m_1 v^*}{b + u^{*2}}\right) < 0,$$
$$A_3(u^*, v^*) = \frac{2m_2 bu^*v^*}{(b + u^{*2})^2} > 0.$$

Obviously, if $A_1(u^*, v^*) < 0$, then $H(d_1, d_2; \mu) > 0$ for all $\mu > 0$. If

$$d_2 A_1(u^*, v^*) > \sqrt{-4d_1 d_2 A_2(u^*, v^*)A_3(u^*, v^*)},$$

then $H(d_1, d_2; \mu) = 0$ has two positive roots as follows:

$$\mu^- = \frac{d_2 A_1(u^*, v^*) - \sqrt{d_2^2 A_1^2(u^*, v^*) + 4d_1 d_2 A_2(u^*, v^*)A_3(u^*, v^*))}}{2d_1 d_2},$$
$$\mu^+ = \frac{d_2 A_1(u^*, v^*) + \sqrt{d_2^2 A_1^2(u^*, v^*) + 4d_1 d_2 A_2(u^*, v^*)A_3(u^*, v^*))}}{2d_1 d_2}.$$

Set $\varGamma = \{\mu_0, \mu_1, \mu_2, \dots\}$ and $\varLambda = \{\mu \geqslant 0\colon\ \mu^- < \mu < \mu^+\}$. Obviously,

$$\lim_{d_2 \to \infty} \mu^-(d_1, d_2) = 0, \qquad \lim_{d_2 \to \infty} \mu^+(d_1, d_2) = \frac{B_1(u^*, v^*)}{d_1}.$$

Theorem 9. *Assume that*

$$\frac{d_2}{d_1} > \frac{-4A_2(u^*, v^*)A_3(u^*, v^*)}{A_1^2(u^*, v^*)}$$

and there exist $i, j \in N$ *such that* $0 \leqslant \mu_j < \mu^- < \mu_{j+1} \leqslant \mu_i < \mu^+ < \mu_{i+1}$, *and* $\sum_{k=j+1}^{i} m(\mu_k)$ *is odd. Then there exists at least one nonconstant solution of* (12).

Proof. Let d^* be defined in Theorem 8, and for $t \in [0, 1]$,

$$\mathcal{A}_t(\mathbf{w}) \triangleq (-\Delta + I)^{-1}\begin{pmatrix} u + (\frac{1-t}{d^*} + \frac{t}{d_1})f(u, v) \\ v + (\frac{1-t}{d^*} + \frac{t}{d_2})g(u, v) \end{pmatrix}.$$

Consider the following problem:

$$\mathcal{A}_t(\mathbf{w}) = \mathbf{w} \quad \text{in } \varOmega, \qquad \frac{\partial \mathbf{w}}{\partial n} = 0 \quad \text{on } \partial\varOmega. \tag{19}$$

It is easy to see that solving (12) is equivalent to find a fixed point of $A_t(w)$ with $t = 1$. $\mathbf{w}^*$ is the unique constant solution of (19) for any $t \in [0, 1]$. By the definition of d^* in Theorem 8, one have that E^* is the only fixed point of $\mathcal{A}_0$.

$$\deg(I - \mathcal{A}_0, \Lambda, 0) = \operatorname{index}(I - \mathcal{A}_0, \Lambda, E^*) = 1.$$

Since $\mathcal{F} = I - H(\cdot, 1)$ and if (12) has no nonconstant positive solution, then we have

$$\deg\bigl(I - \mathcal{A}_1, \Lambda, (0,0)\bigr) = \operatorname{index}(\mathcal{F}, \mathbf{w}^*) = (-1)^{\sum_{k=j+1}^{i} m(\mu_k)} = -1.$$

In addition, by the homotopy invariance of the topological degree,

$$\deg(I - \mathcal{A}_0, \Lambda, 0) = \deg(I - \mathcal{A}_1, \Lambda, 0),$$

which is a contradiction. □

5 Numerical results and discussions

In this section, we take some numerical simulations to discuss the effect of diffusion and the cost of fear.

5.1 The effect of diffusion

In order to discuss the effect of diffusion, in $\Omega = (0, 20)$, we assume the parameters values: $r = 0.8$, $k = 30$, $\gamma_1 = 0.2$, $a = 0.02$, $b = 0.02$, $m_1 = 0.6$, $m_2 = 0.3$, $\gamma_2 = 0.2$, $d_1 = 0.001$ and $d_2 = 0.5$. A direct calculation shows that system (1) has a positive steady state $E^* = (0.2, 0.0528)$. According to Theorem 4, the positive steady state is unstable. Figure 2 shows that system (1) has a stable limit cycle around the positive steady state E^* with the initial conditions $u_0(x) = 0.2 + 4 \cdot 10^{-4} \cos(2x)$, $v_0(x) = 0.0528 + 5 \cdot 10^{-4} \cos(2x)$.

However, if we change the diffusion rate of d_2 to be $d_2 = 1$, we find that the stable limit cycle is broken with the occurrence of spatial pattern (see Fig. 3), where the periodic pattern disappears and a strip pattern appears.

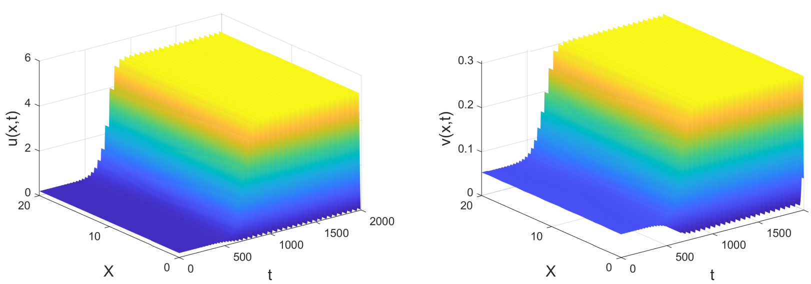

Figure 2. The positive steady state $E^* = (0.2, 0.0528)$ is unstable, and there exists a stable limit cycle with the initial values $u_0(x) = 0.2 + 4 \cdot 10^{-4} \cos(2x)$, $v_0(x) = 0.0528 + 5 \cdot 10^{-4} \cos(2x)$ and the diffusion rate $d_2 = 0.5$.

Furthermore, if we vary the diffusion coefficient d_2 from $d_2 = 1$ to $d_2 = 0.002$, then we find that system (1) is chaotic (see Fig. 4).

We further find that different initial conditions with the same diffusion rate $d_2 = 1.2$ can lead to different spatial patterns that can be stationary or periodic (Fig. 5).

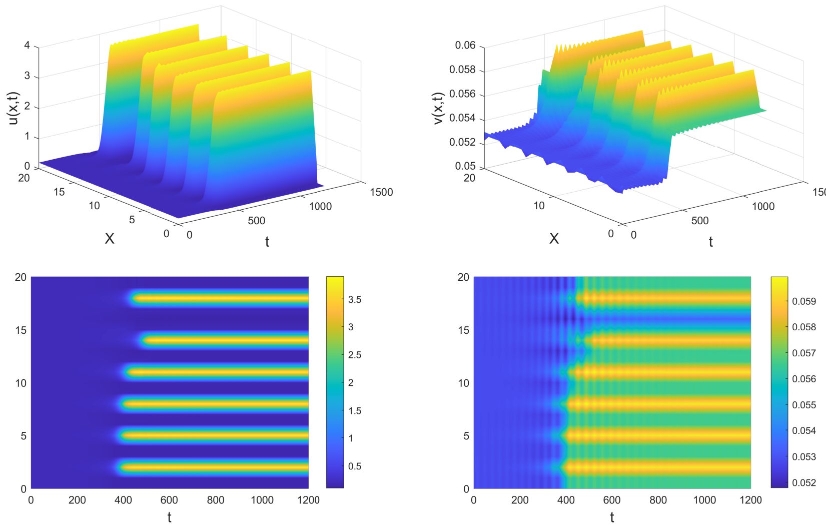

Figure 3. The emergent stationary spatial pattern with the initial values $u_0(x) = 0.2 + 4 \cdot 10^{-4}\cos(2x)$, $v_0(x) = 0.0528 + 5 \cdot 10^{-4}\cos(2x)$ and the diffusion rate $d_2 = 1$.

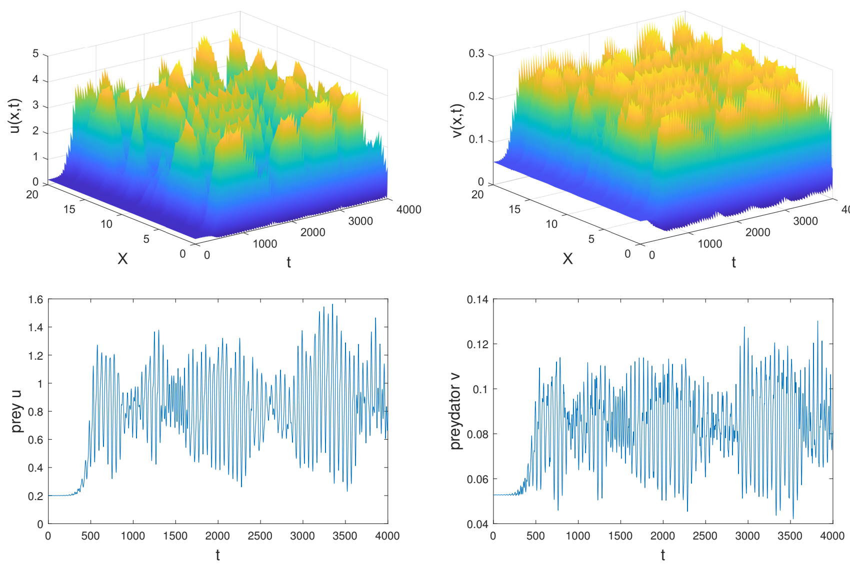

Figure 4. System (1) is chaotic with the initial values $u_0(x) = 0.2 + 4 \cdot 10^{-4}\cos(2x)$, $v_0(x) = 0.0528 + 5 \cdot 10^{-4}\cos(2x)$ and the diffusion rate $d_2 = 0.002$.

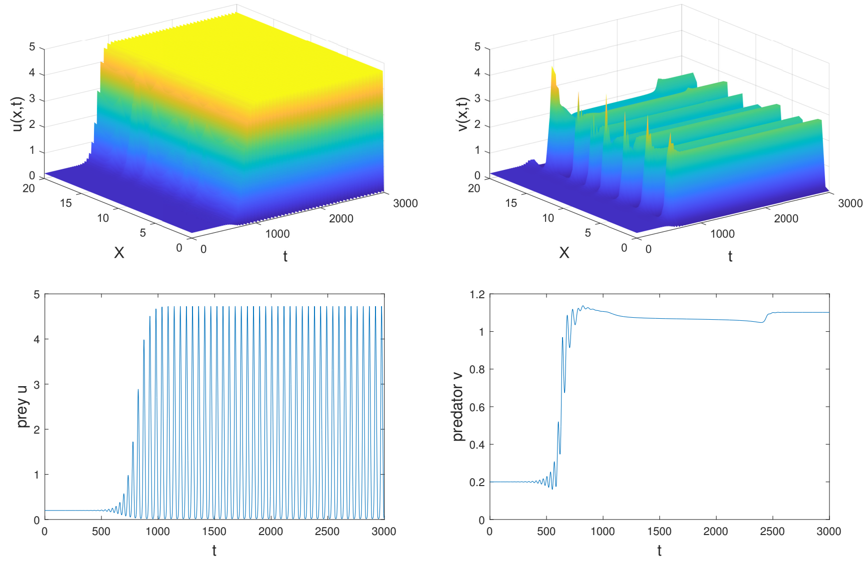

Figure 5. Spatial patterns and spatially averaged population dynamics for different random perturbed initial conditions with the same diffusion rate $d_2 = 1.2$.

5.2 Effect of the cost of fear

Choose $r = 0.8$, $k = 50$, $\gamma_1 = 0.2$, $a = 0.2$, $b = 0.02$, $m_1 = 0.6$, $m_2 = 0.3$, $\gamma_2 = 0.2$, $d_1 = 0.001$, $d_2 = 1$ and $\Omega = (0, \pi)$. Calculations show that system (1) has a unique positive steady state $E^* = (0.0325, 0.2000)$. According to Theorem 4, we observe that the positive steady state E^* of system (1) is locally asymptotically stable, and the dynamic behaviors of system (1) is illustrated graphically in Fig. 6.

From above discussions we can obtain that fear can affect the stability of the positive steady state, and it can induce the Hopf bifurcation, which is different from the results found in [14, 19] with linear functional response (see Fig. 9). Figure 9 shows that there exists a threshold value k_0 such that when $k \in (0, k_0]$, system (1) has a periodic solution. But when k passes the threshold value, then system becomes stable.

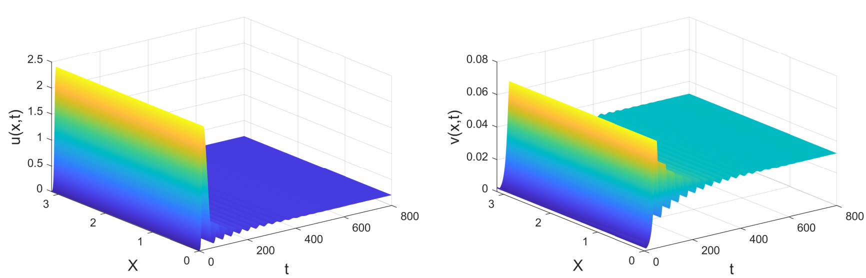

Figure 6. The positive steady state $E^* = (0.0325, 0.2000)$ is locally asymptotically stable.

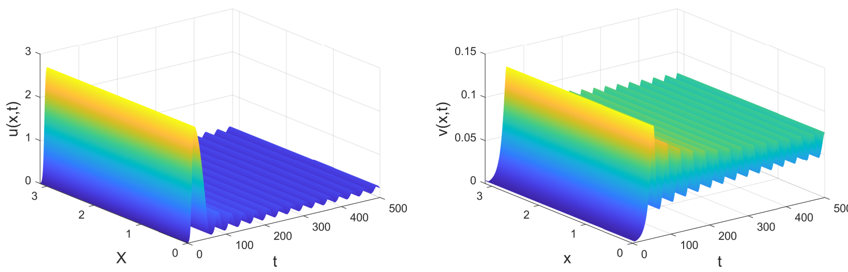

Figure 7. Hopf bifurcation of system (1) with $k = 20.37$.

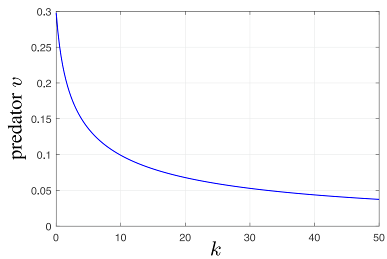

Figure 8. The positive steady state v^* with varying the cost of fear k.

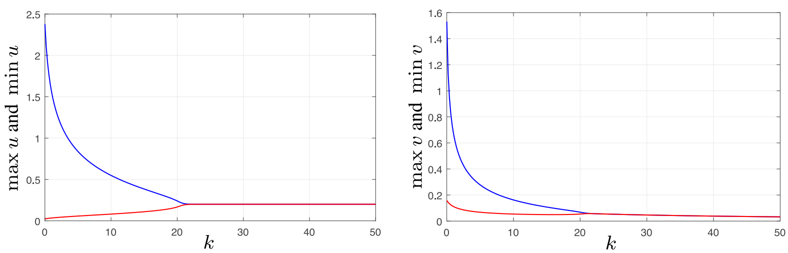

Figure 9. The maximum and minimum of prey u and predator v with the cost of fear k varying in $[0, 50]$.

If we choose $k = 20.37$, while other parameters do not change, according to Theorem 5, system (1) undergoes spatial homogeneous Hopf bifurcation (see Fig. 7). It is shown that system (1) has spatially homogeneous periodic solutions emerged from the positive steady state E^*.

In addition, we find that the positive steady state can be changed by the different value of the cost of fear. Figure 8 shows that the positive steady state v^* decreases with increasing of the cost of fear.

6 Conclusion

A diffusive predator–prey model with the fear effect is studied in our paper. We derive some basic dynamics of the system and give condition for the existence of the positive steady state. According to eigenvalue analysis method, we investigate the stability and bifurcation of the positive constant steady state. We also give some conditions for the nonexistence and existence of nonconstant solutions of the system.

Theorems 3 and 4 show that the birth rate of prey r can not only induce the static bifurcation, but also can induce saddle-node bifurcation.

Theorem 4 indicates that the diffusion can not induce the Turing instability/bifurcation. However, Theorem 5 provides that the diffusion can induce the inhomogeneous Hopf bifurcation, which can lead to the formation of spatial patterns. Furthermore, Theorem 9 shows that system (12) has at least one nonconstant positive solution under the effect of the diffusion. From Section 5.1 we can obtain that the different diffusion rate d_2 can lead to different spatial patterns, which can be periodic (Fig. 2), stationary (Fig. 3) and chaotic (Fig. 4). In addition, we also find that system has different spatial patterns with the different initial conditions (Fig. 5).

We further obtain that the fear effect can reduce the density of predator: with increasing the cost of fear, the density of predator population decreases at the positive steady state (see Fig. 8). From Section 5.2 it is obtained that the fear can prevent the occurrence of limit cycle oscillation and increase the stability of the system (see Fig. 9).

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