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Stochastic elliptic–parabolic system arising in porous media

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We prove the existence of a pathwise weak solution to the single-phase, miscible displacement of one incompressible fluid by another in a porous medium with random forcing. Our system is described by a parabolic concentration equation driven by an additive noise coupled with an elliptic pressure equation. We use a pathwise argument combined with Schauder's fixed point theorem.

Keywords: Porous medium; flow and transport; elliptic–parabolic system; noise, stochastic processes.

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1. Introduction and Main Result

1.1. *Description of the model*

The oil recovery process in petroleum engineering uses the technique of injecting a fluid or a solvent into special reservoir wells in order to reduce the resident oil viscosity and thereby enhance its recovery at the production wells. When a miscible fluid is pumped into injection wells, the evolution of the mixture during this

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process is physically described as a miscible displacement flow model in porous media, see [20]. Under appropriate physical assumptions, such model was mathematically introduced and studied by Peaceman and Rachford in [18], see also [17] for more description of the model. We are interested in the single-phase, miscible displacement flow model in a porous medium, of one incompressible fluid by another. This model is described by a nonlinear coupled elliptic–parabolic system.

The invading fluid is injected and produced into the reservoir wells. These wells were first modeled as modified Dirac masses. However, the length of the wells is represented by a wellbore diameter, so that wells could be later modeled by measures and by distribution functions supported over the wellbore diameter. Hence, the choice of these particular external forces is dictated by physical and mathematical motivations. In this paper, we consider a stochastic perturbation into the miscible displacement flow model in a porous medium. In particular, we represent the wells source and sink terms with random distribution. We first present the deterministic models.

Let $\mathcal{U}$ be a bounded open domain in $\mathbb{R}^2$ with a Lipschitz continuous boundary $\partial\mathcal{U}$, that represent the porous medium, and $[0,T]$ be the time interval of the displacement of the fluid. The model is described by the following coupled elliptic–parabolic system (1.1)–(1.3):

$$\nabla \cdot v(x,t) = q^I(x,t) - q^P(x,t) \quad \text{on } \mathcal{U} \times [0,T], \tag{1.1}$$

where v is the Darcy velocity of the fluid, defined by

$$v(x,t) = -\frac{k(x)}{\mu(c(x,t))}(\nabla p(x,t) - \rho(c(x,t))g(x,t)), \tag{1.2}$$

and μ and ρ are, respectively, the viscosity and the density of the fluid mixture, k is the absolute permeability of the porous medium, and g represents the gravitational vector.

$$\begin{cases} \phi(x)\partial_t c(x,t) - \nabla \cdot (D\,\nabla c - c\,v)(x,t) + (q^P c)(x,t) = (q^I \hat{c})(x,t) & \text{on } \mathcal{U} \times [0,T], \\ c(x,0) = c_0(x) & \text{on } \mathcal{U}, \end{cases} \tag{1.3}$$

where ϕ is the porosity of the medium, $\hat{c}$ the concentration of the injected fluid in the medium, D is the diffusion–dispersion tensor or coefficient and $q^I, q^P \geq 0$ represent the sum of injection well source terms and production well sink terms, respectively. The coupled elliptic parabolic system (1.1)–(1.3) describes the behavior of the total fluid pressure of the mixture p, the Darcy velocity v of the fluid mixture, computed with respect to the concentration c of one of the components in the mixture with initial condition $c_0(x)$,

$$(p, v, c) : \mathcal{U} \times [0,T] \to \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R},$$
$$(x,t) \mapsto (p(x,t), v(x,t), c(x,t)).$$

There is an extensive literature devoted to the study of this model both theoretically and numerically. The miscible displacement system of fluids in porous media was first studied by Peaceman and Rachford in [16, 18], where a numerical investigation for approximating solutions to the system in dimension $d = 2$, was obtained and compared to the laboratory data for the displacements of oil with less-viscous solvents, see also [17]. After that, various numerical techniques and methods to approximate the discrete solution of the above system were obtained and reported by various authors, see, for instance, [3, 4, 6, 7, 9, 23] and references cited therein. Moreover, the displacement flow model in porous medium was theoretically studied and analyzed by Sammon [19], where the viscosity μ was assumed independent of the concentration c, and that $D > 0$ the diffusion coefficient is independent of the velocity v. Furthermore, the author considered that the injection q^I and production q^P well source/sink terms are supported near its well location (i.e. as a modified Dirac delta function, which is nonzero only near its well location) which leads to a singular behavior of the solution near wells. In [15], the author established well-posedness results for the stationary displacement problem depending on the viscosity (i.e. for Mobility ratio $M = \frac{\mu(0)}{\mu(1)}$ sufficiently close to one), assuming a concentration-dependent viscosity $\mu(c)$, a regularized velocity-dependent diffusion–dispersion tensor $D(v)$, and under the assumptions that the injection and production wells are non-negative elements of $L^r(\mathcal{U}), r \in (d, \infty)$, d being the spatial dimension.

In [11], Feng extended Sammon's [19] result but for concentration-dependent viscosity $\mu(c)$, a velocity-dependent dispersion–diffusion tensor $D(v)$, and under the assumption that the source function q^I and q^P are not supported or concentrated near the wells, but they are smoothly distributed over the reservoir (i.e. square integrable), see also [5] for specific boundary condition, and [1] for asymptotic behavior of the solutions when the molecular diffusion effects are neglected.

In [10], Fabrie and Gallouët have studied the displacement of miscible and immiscible flow with wells action modeled by measures, i.e. the source terms q^I, q^P can be written in terms of a spatial measure. The author obtained existence result under assumption that a concentration-dependent viscosity $\mu(c)$, and that diffusion–dispersion tensor $D(v)$ is bounded. This result was later extended to a generalized diffusion–dispersion tensor in [8].

In this paper, we propose the induced sources terms to be stochastic, in particular, we introduce an additive noise in the parabolic concentration equation. The stochastic perturbation is described by an additive noise $\dot{W}(t)$, where $W(t)$ is an $H^1(\mathcal{U})$-valued Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$ with expectation $\mathbb{E}$, and covariance $\mathcal{Q}$, bounded linear operator on $L^2(\mathcal{U})$ of trace class:

$$\begin{cases} W(x,t,\omega) \in L^2(\Omega, C([0,T]; H^1(\mathcal{U}))), \\ D(x)\nabla W(x,t,\omega) \cdot \vec{n} = 0 \quad \text{for } x \in \partial\mathcal{U} \text{ and for a.e. } (t,\omega) \in [0,T] \times \Omega. \end{cases} \tag{1.4}$$

We prove the existence of weak solution to the stochastic miscible displacement-type model in a porous medium, using a pathwise argument similar to [2, 12]. First, for a fixed $\omega \in \Omega$, we solve for $(p(t), v(t), c(t))$, the system (2.2)–(2.4), then we prove the measurability of the process. Thus, we obtain the existence of the process $(p(t), v(t), c(t))$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Let us mention that our work is the first step on studying the following stochastic model. From now on, we denote by

$$\begin{aligned}
\kappa(c(x,t,\omega)) &:= \frac{k(x)}{\mu(c(x,t,\omega))}, \quad v(x,t,\omega) = -\kappa(c(x,t,\omega))\nabla p(x,t,\omega),\\
f(x,t,\omega) &:= (q^I \hat{c})(x,t,\omega), \quad h(x,t,\omega) := q^I(x,t,\omega) - q^P(x,t,\omega),\\
q(x,t,\omega) &:= q^P(x,t,\omega),
\end{aligned}$$

where $f, h, q : \mathcal{U} \times [0,T] \times \Omega \to \mathbb{R}$. Neglecting gravity g in the Darcy velocity and assuming $\phi \equiv 1$, the model reads

$$\begin{cases}
v(t) = -\kappa(c(t))\nabla p(t),\\
\nabla \cdot v(t) = h(t),\\
d\,c(t) - \nabla \cdot (D(x)\nabla c(t) - c(t)\,v(t))dt + q(t)\,c(t)dt = f(t)dt + dW(t)
\end{cases} \tag{1.5}$$

with the initial condition $c(x,0) = c_0(x), (\omega, x) \in \mathcal{U} \times \Omega$. The main variable is the stochastic process $(p(t), v(t), c(t))$

$$\begin{aligned}
(p, v, c) : \mathcal{U} \times [0,T] \times \Omega &\to \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R},\\
(x,t,\omega) &\mapsto (p(x,t,\omega), v(x,t,\omega), c(x,t,\omega)),
\end{aligned}$$

that satisfies the no-flow boundary conditions, which results from vanishing permeability at the reservoir boundary $\partial\mathcal{U}$,

$$D\nabla c(x,t) \cdot \vec{n} = 0, \quad (x,t) \in \partial\mathcal{U} \times [0,T], \tag{1.6}$$

$$v(x,t) \cdot \vec{n} = 0, \quad (x,t) \in \partial\mathcal{U} \times [0,T], \tag{1.7}$$

where $\vec{n}$ is the outward normal vector to $\partial\mathcal{U}$. According to boundary condition (1.7), we assume the following compatibility condition:

$$\int_{\mathcal{U}} h(t,x)dx := \int_{\mathcal{U}} q^I(x,t) - q^P(x,t)dx = 0 \quad \forall\, t \in [0,T].$$

Moreover, we normalize the pressure p by an average condition, to eliminate any arbitrary constants in the solution p of the elliptic equation,

$$\int_{\mathcal{U}} p(x,t)dx = 0, \quad t \in [0,T]. \tag{1.8}$$

Note that throughout, we assume that and sink source $q^I, q^P \geq 0$.

1.2. *Functional setting and notations*

The usual Sobolev spaces $H^{1,p}(\mathcal{U})$, where $p \in [1, \infty]$, with norm $\| \cdot \|_{H^{1,k}} := \| \cdot \|_{1,k}$ will be used. When $p = 2$, we simply write $H^1(\mathcal{U})$, and $\| \cdot \|_1$ the H^1-norm. The $H^{-1}(\mathcal{U})$ is defined as the dual space of H^1, and we denote by $\langle \cdot, \cdot \rangle_{(H^{-1},H^1)}$ the dual pair. Since $\mathcal{U}$ is a bounded domain in $\mathbb{R}^2$, then the H^1-norm and the semi-norm are equivalent: There exists $b_0 > 0$, such that $b_0\|u\|_1 \leq \|\nabla u\|_{L^2} \leq \|u\|_1$. We also use the Sobolev embedding $H^1(\mathcal{U}) \subset L^r(\mathcal{U})$, for $r \geq 2$.

The L^2 inner product is simply denoted by $\langle \cdot, \cdot \rangle$, i.e. for $f, g \in L^2(\mathcal{U})$, $\langle f, g \rangle := \int_{\mathcal{U}} f(x)g(x)dx$, unless otherwise state for the dual. We denote by $\| \cdot \|_0$ the L^2-norm. Throughout this paper, C denotes a positive constant, that may change from line to line, which depends on the domain $\mathcal{U}$, dependence on other parameters will be specified.

Let $\{e_k\}_{k=1}^{\infty} \subset H^1(\mathcal{U})$ be an orthonormal basis of $L^2(\mathcal{U})$, which also an orthogonal basis in $H^1(\mathcal{U})$. The construction of such a basis can be obtained from normalizing eigenpairs of the Laplace differential operator $\mathcal{U}$. Throughout this paper, we use this basis to construct Galerkin approximations of solution to the concentration evolution (parabolic equation). To this end, for a fixed $m \in \mathbb{N}$, we denote by $\mathcal{H}_m := \mathrm{span}\{e_k\}_{k=1}^{k=m}$ and we define Π_m to be the orthogonal projection of H^1 onto $\mathcal{H}_m$.

For a Banach space X, we define $L^p(0, T; X)$ the space of measurable functions $u : [0, T] \to X$, such that for $0 < p < \infty$,

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|_X^p dt \right)^{\frac{1}{p}} < \infty \quad \text{and}$$

$$\|u\|_{L^\infty(0,T;X)} := \operatorname{ess} \sup_{0 \leq t \leq T} |u(t)\|_X.$$

We denote by $C^{0,\sigma}(0, T; X)$ the space of Hölder continuous functions $u : [0, T] \to X$, equipped with the norm

$$\|u\|_{C^{0,\sigma}(0,T;X)} := \sup_{s \in [0,T]} \|u(s)\|_X + \sup_{\substack{s,t \in [0,T] \\ s \neq t}} \frac{\|u(s) - u(t)\|_X}{(s - t)^\sigma}.$$

We also recall the following compactness embedding result, which is an application of Yu A. Dubinsky's Theorem, as $H^1(\mathcal{U}) \subset L^2(\mathcal{U}) \subset H^{-1}(\mathcal{U})$ and we refer the reader to [22, Theorem 4.1] for more details and general setting.

Proposition 1.1. (Dubinsky's Theorem) *Let $1 < q < \infty$, and M be a bounded set in $L^q(0, T; H^1(\mathcal{U}))$ consisting of function equicontinuous in $C([0, T]; H^{-1}(\mathcal{U}))$. Then, M is relatively compact in $L^q(0, T; L^2(\mathcal{U}))$ and $C(0, T; H^{-1}(\mathcal{U}))$.*

1.3. *Assumptions and main results*

We establish the existence of a stochastic process to the nonlinear coupled elliptic–parabolic stochastic system, under some hypotheses on the data.

Assume the following assumptions hold, for $\mathbb{P}$-a.e. $\omega \in \Omega$:

(A.1) $c_0 \in L^2(\mathcal{U})$.
(A.2) $h, q \in L^\infty([0,T]; L^2(\mathcal{U}))$.
(A.3) $f \in L^2(0,T; L^2(\mathcal{U}))$.
(A.4) $D \in L^\infty(\mathcal{U})$ and there exist $D^*, D_* > 0$ such that $D_* < D(x) \leq D^*$ for a.e. $x \in \mathcal{U}$.
(A.5) $\kappa \in \mathcal{C}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, with $0 < \kappa_* \leq \kappa(\xi) \leq \kappa^*$ for a.e. $\xi \in \mathbb{R}$.
(A.6.1) $W \in L^2(\Omega, L^2(0,T; H^1(\mathcal{U})))$.
(A.6.2) $D(x)\nabla W(x,t,\omega) \cdot \vec{n} = 0$, for $x \in \partial\mathcal{U}$ and for a.e. $(t,\omega) \in [0,T] \times \Omega$.

Definition 1.1. A stochastic process $(p(t), v(t), c(t))$ is called a weak solution to the system (1.5) on $[0,T]$, with initial condition $c_0 \in L^2(\mathcal{U})$, if the triple $(p(t), v(t), c(t))$ satisfies for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$p(\cdot,\omega) \in L^\infty(0,T; H^{1,r}(\mathcal{U})), \quad v(\cdot,\omega) \in L^\infty(0,T; L^r(\mathcal{U})^2), \tag{1.9}$$

$$c(\cdot,\omega) \in C([0,T]; L^2(\mathcal{U})) \cap L^2(0,T; H^1(\mathcal{U})), \tag{1.10}$$

where $2 \leq r < \infty$, and for a.e. $0 \leq t \leq T$,

$$v(x,t) = -\kappa(c(x,t))\nabla p(x,t), \qquad \int_{\mathcal{U}} p(t,x)dx = 0, \tag{1.11}$$

$$\Gamma(v(t),\phi) = \langle h(t),\phi\rangle \quad \forall\, \phi \in H^1(\mathcal{U}), \tag{1.12}$$

$$\langle c(t),\psi\rangle + \int_0^t \Lambda(c(s),v(s),\psi)ds = \langle c_0,\psi\rangle + \int_0^t \langle f(s),\psi\rangle ds + \langle W(t),\psi\rangle \quad \forall\, \psi \in H^1(\mathcal{U}), \tag{1.13}$$

where

$$\Lambda(c(t),v(t),\psi) := \langle D\nabla c(t),\nabla\psi\rangle - \langle c(t)\, v(t),\nabla\psi\rangle + \langle q(t)c(t),\psi\rangle, \tag{1.14}$$

$$\Gamma(v,\phi) := -\langle v(t),\nabla\phi\rangle. \tag{1.15}$$

Theorem 1.1. *Under the assumptions*: (A.1)–(A.7), *there exists a stochastic process* (p,v,c) *solution to the coupled elliptic–parabolic system* (1.5) *in the sense of the Definition* 1.1, *and for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$.

$$\|p\|_{L^\infty(0,T;H^1(\mathcal{U}))} \leq (\kappa_* b_0^2)^{-1}\|h\|_{L^\infty(0,T;L^2(\mathcal{U}))}, \tag{1.16}$$

$$\|c\|^2_{L^\infty_T(0,T;L^2(\mathcal{U}))} \leq 2\beta e^T + 2\|W\|^2_{L^\infty(0,T;L^2(\mathcal{U}))}, \tag{1.17}$$

$$\|c\|^2_{L^2(0,T;H^1(\mathcal{U}))} \leq 2(D_* b_0^2)^{-1}(\beta+1)e^T + 2\|W\|^2_{L^2(0,T;H^1(\mathcal{U}))}, \tag{1.18}$$

where

$$\beta := \|c_0\|_0^2 + \frac{3}{D_*}\|W\|^2_{L^2(0,T;H^1(\mathcal{U}))}\left(\frac{1}{{b_0}^2}\|q\|^2_{L^\infty(0,T;L^2(\mathcal{U}))} + (D^*)^2 + (\kappa^*)^2\|h\|^2_{L^\infty(0,T;L^2(\mathcal{U}))}\right) + \|f\|^2_{L^2(0,T;L^2(\mathcal{U}))}. \tag{1.19}$$

Moreover,

$$\sup_{0\le t\le T}\|p(t)\|_{1,r}\le C(r,\kappa_*,\kappa^*)\|h\|_{L^\infty(0,T;L^2(\mathcal{U}))}. \tag{1.20}$$

The following section is devoted to the proof of Theorem 1.1. We use path-wise argument combined with Schauder's fixed point theorem and a measurability argument.

The proof is organized as follows; in Sec. 2.1.1, for a given concentration $c(t) := \tilde\alpha(t)+W(t)$ we prove the existence of the pressure $p_{\tilde\alpha}(t)$ and the velocity $v_{\tilde\alpha}(t)$, using the Lax–Milgram theorem for the elliptic equation. In Sec. 2.1.2, for a fixed $(p_{\tilde\alpha}, v_{\tilde\alpha})$, we construct $\tilde\alpha(t)$, respectively, the concentration $c_{\tilde\alpha}(t)$, solution to the parabolic equation through a Galerkin approximation. In Sec. 2.1.3, we obtain the existence of a pathwise solution $(p(t), v(t), c(t))$ using Schauder's fixed point theorem. We prove that there exists a fixed point $\tilde\alpha(t) = \alpha(t)$, respectively, $c_{\tilde\alpha}(t) = c(t)$, such that $(p(t), v(t), c(t))$ is a solution to (1.5) in sense of Definition 1.1, for $\mathbb{P}$-a.e., $\omega\in\Omega$. Finally, in Sec. 2.2 we conclude the proof of Theorem 1.1, by checking the measurability of the process $(p(t), v(t), c(t))$.

2. Proofs of Main Results

The existence a weak solution to the system (1.5) is achieved via pathwise and measurability argument. In order to construct a weak solution to the system (1.5) for a fixed $\omega\in\Omega$, we define

$$\begin{cases}\alpha(x,t) := c(x,t) - W(x,t), \text{ where } W(t) \\ \qquad\qquad \text{is the Brownian motion defined in (1.4)}, \\ \alpha_0(x) := c(x,0) + W(x,0) = c_0(x).\end{cases}$$

Since by assumption $D(x)\nabla c(x,t)\cdot\vec n = D(x)\nabla W(x,t)\cdot\vec n = 0$ on $\partial\mathcal{U}$, then $\alpha(t)$ satisfies the boundary condition

$$D(x)\nabla\alpha(x,t)\cdot\vec n = 0 \quad \text{for } x\in\partial\mathcal{U} \text{ and for a.e. } t\in[0,T]. \tag{2.1}$$

Hence, we rewrite the elliptic–parabolic system (1.11), (1.12) and (1.13) for a.e. $0\le t\le T$,

$$v(t) := -\kappa(\alpha(t)+W(t))\nabla p(t), \tag{2.2}$$

$$\Gamma(v(t),\phi) := -\langle v(t),\nabla\phi\rangle = \langle h(t),\phi\rangle \quad \forall\,\phi\in H^1(\mathcal{U}) \tag{2.3}$$

and

$$\begin{aligned}\langle\partial_t\alpha(t),\psi\rangle + \Lambda(\alpha(t),v(t),\psi) &= -\Lambda(W(t),v(t),\psi) \\ &\quad + \langle f(t),\psi\rangle \quad \forall\,\psi\in H^1(\mathcal{U}),\end{aligned} \tag{2.4}$$

where

$$\Lambda(\alpha(t),v(t),\psi) = \langle D\nabla\alpha(t),\nabla\psi\rangle - \langle\alpha(t)\,v(t),\nabla\psi\rangle + \langle q(t)\alpha(t),\psi\rangle,$$

$$\Lambda(W(t),v(t),\psi) = \langle D\nabla W(t),\nabla\psi\rangle - \langle W(t)\,v(t),\nabla\psi\rangle + \langle q(t)W(t),\psi\rangle.$$

2.1. *Existence of pathwise weak solution*

2.1.1. *Existence and uniqueness of solution to the elliptic equation*

In this section, we prove existence and uniqueness of a solution $(p(t), v(t))$ to Eqs. (2.2) and (2.3), respectively, to (1.11) and (1.12), for a given concentration $c_{\tilde{\alpha}}(t)$. Indeed, for $\tilde{\alpha}(t) \in L^2(0,T;L^2(\mathcal{U}))$ (i.e. for a given concentration $c_{\tilde{\alpha}}(t) := \tilde{\alpha}(t) + W(t)$), we prove that there exist a unique solution $p(t) := p_{\tilde{\alpha}}(t)$ for the elliptic equation (2.3), using Lax–Milgram theorem and the fact that p satisfies (1.8). Consequently, we obtain the existence and uniqueness of $v(t) := v_{\tilde{\alpha}}(t)$, which satisfies (2.2).

Lemma 2.1. *Given $\omega \in \Omega, \tilde{\alpha} \in L^2(0,T;L^2(\mathcal{U}))$ and $h \in L^\infty(0,T;L^2(\mathcal{U}))$ then there exist $(p_{\tilde{\alpha}}(t), v_{\tilde{\alpha}}(t))$ satisfying (2.3) and (2.2) for a.e. $t \in [0,T]$, such that*

$$p_{\tilde{\alpha}} \in L^\infty(0,T;H^1(\mathcal{U})), \quad v_{\tilde{\alpha}} \in L^\infty(0,T;L^2(\mathcal{U})^2)$$

and

$$\|p_{\tilde{\alpha}}(t)\|_1 \leq (\kappa_* b_0^2)^{-1} \|h(t)\|_0. \tag{2.5}$$

Moreover, there exists $2 < r_0 \leq \infty$ such that for $r \in [2, r_0)$, $p_{\tilde{\alpha}} \in L^\infty(0,T;H^{1,r}(\mathcal{U}))$ and

$$\|p_{\tilde{\alpha}}(t)\|_{1,r} \leq C(r, \kappa_*, \kappa^*) \|h(t)\|_0, \tag{2.6}$$

where r_0 depends only on $\kappa_*, \kappa^*, \mathcal{U}$.

Proof of Lemma 2.1. Let $\omega \in \Omega, t \in [0,T], \tilde{\alpha} \in L^2(0,T;L^2(\mathcal{U})), h \in L^\infty(0,T;L^2(\mathcal{U}))$ fixed. Recall that, $\Gamma(v(t),\phi) = \langle \kappa(\tilde{\alpha}(t) + W)\nabla p(t), \nabla\phi\rangle$, then one can see that Γ is a bilinear mapping in $H^1(\mathcal{U})$ with respect to p and ϕ. Moreover, Γ is bounded and coercive in H^1,

$$|\Gamma(v(t),\phi)| = |\langle \kappa(\tilde{\alpha}(t) + W(t))\nabla p(t), \nabla\phi\rangle| \leq \kappa^* \|\nabla p(t)\|_0 \|\nabla\phi\|_0$$

and

$$\begin{aligned} |\Gamma(v(t),p(t))| &= |\langle \kappa(\tilde{\alpha}(t) + W(t))\nabla p(t), \nabla p(t)\rangle| \\ &\geq \kappa_* \|\nabla p(t)\|_0^2 \geq b_0^2 \kappa_* \|p(t)\|_1^2. \end{aligned} \tag{2.7}$$

Then, by the Lax–Milgram theorem there is a unique solution $p := p_{\tilde{\alpha}}(\cdot,t) \in H^1(\mathcal{U})$ to (2.3) for all $\phi \in H^1(\mathcal{U})$, and a unique $v_{\tilde{\alpha}} := \kappa(\tilde{\alpha} + W)\nabla p_{\tilde{\alpha}} \in L^2(\mathcal{U})^2$, for a.e. $t \in [0,T]$. Thus, there exists a unique $(p_{\tilde{\alpha}}(t), v_{\tilde{\alpha}}(t))$ solution to Eqs. (2.2) and (2.3), for a.e $t \in [0,T]$. Using the fact that $\Gamma(v_{\tilde{\alpha}}(t), p_{\tilde{\alpha}}(t)) = \langle h(t), p_{\tilde{\alpha}}(t)\rangle$ in (2.7), we obtain (2.5) and that $p_{\tilde{\alpha}} \in L^\infty(0,T;H^1(\mathcal{U})), v_{\tilde{\alpha}} \in L^\infty(0,T;L^2(\mathcal{U})^2)$.

By Meyer's type estimate (see, for examples, [13, Theorem 2; [14]), there exist $r_0 > 2$ such that if $p \in H^1(\mathcal{U})$ is a solution of $\langle \kappa(\tilde{\alpha} + W)\nabla p, \nabla\phi\rangle = F(\phi), \forall\, \phi \in$

$H^1(\Omega)$ and $F \in H^{-1,r}(\mathcal{U})$ for $r \in [2, r_0)$, then $p \in H^{1,r}(\mathcal{U})$ and there exists constant $\tilde{C}(r)$ that depends only on $\kappa_*, \kappa^*, \mathcal{U}$ and r such that

$$\|p\|_{1,r} \leq \tilde{C}(r)\|F\|_{-1,r} \quad \forall\, r \in [2, r_0). \tag{2.8}$$

Moreover, r_0 depends on $\kappa_*, \kappa^*, \mathcal{U}$. In our case, $p = p_{\tilde{\alpha}}$ and $F(\phi) = \langle h, \phi\rangle$ where $h \in L^2(\mathcal{U})$, then

$$|F(\phi)| \leq \|h\|_0\|\phi\|_1 \leq C\|h\|_0\|\phi\|_{1,r},$$

which implies that $\|F\|_{-1,r} \leq C(r)\|h\|_0$, where $C(r)$ depends only on $\kappa_*, \kappa^*, \mathcal{U}$ and r. This concludes the proof of Lemma 2.1. □

2.1.2. *Existence of concentration, for fixed velocity*

We construct a weak solution to the parabolic equation (2.4), respectively, to (1.13), using a Galerkin approximation in finite-dimensional $\mathcal{H}_m$ subspace of H^1, for fixed $\omega \in \Omega$.

Recall that, we assumed that $c(x, t) := \alpha(x, t) + W(x, t)$, with initial condition $c_0(x) = \alpha(x, 0)$. We fix $(v_{\tilde{\alpha}}, p_{\tilde{\alpha}})$ the unique solution to Eqs. (2.2) and (2.3) given by Lemma 2.1.

Let $\{e_k\}_{k=1}^{\infty} \subset H^1(\mathcal{U})$ be an orthonormal basis of $L^2(\mathcal{U})$, which also an orthogonal basis in $H^1(\mathcal{U})$. For a fixed $m \in \mathbb{N}$, we denote by $\mathcal{H}_m := \text{span}\{e_k\}_{k=1}^{k=m}$ and we consider Π_m to be the orthogonal projection of H^1 onto $\mathcal{H}_m$. Next, we prove the existence and uniqueness of solution the projected system of (2.4).

We denote by c_0^m the projection of c_0 onto $\mathcal{H}_m$, thus $c_0^m = \alpha_m(0) = \sum_{k=0}^{m}\langle c_0, e_k\rangle e_k$. Applying the projection Π_m into the solution $\alpha(t)$ of Eq. (2.4), we have $\Pi_m(\alpha(t, x)) := \alpha_m(t, x) : [0, T] \to H^1(\mathcal{U})$, with the form

$$\alpha_m(t, x) := \sum_{j=1}^{m} \mathcal{X}_{j,m}(t)e_j(x), \tag{2.9}$$

where, $\mathcal{X}_{j,m}(t)$ satisfies for $t = 0, \mathcal{X}_{j,m}(0) := \langle c_0, e_j\rangle$.

Lemma 2.2. *Fix $m \in \mathbb{N}, \tilde{\alpha} \in L^2(0, T; L^2(\mathcal{U}))$ and $v_{\tilde{\alpha}} = \kappa(\tilde{\alpha} + W)\nabla p_{\tilde{\alpha}}$, where $p_{\tilde{\alpha}}$ is a unique solution of (2.3). Then there exists a unique solution $\alpha_m(t)$ of*

$$\langle \partial_t\alpha_m, \psi\rangle + \Lambda(\alpha_m, v_{\tilde{\alpha}}, \psi) = -\Lambda(W, v_{\tilde{\alpha}}, \psi) + \langle f, \psi\rangle \quad \forall\, \psi \in \mathcal{H}_m. \tag{2.10}$$

Moreover,

$$\alpha_m \in L^\infty(0, T; L^2(\mathcal{U})) \cap L^2(0, T; H^1(\mathcal{U})), \quad \partial_t\alpha_m \in L^2(0, T; H^{-1}(\mathcal{U})), \tag{2.11}$$

$$\alpha_m \in C^{0,\sigma}(0, T; H^{-1}(\mathcal{U})), \quad \textit{for } 0 \leq \sigma \leq \frac{1}{2}, \quad \textit{uniformly w.r.t. to } m. \tag{2.12}$$

Proof.

- **Step 1.** Existence and uniqueness of α_m: Let $\mathcal{X}_{j,m} = [0,T] \to \mathbb{R}$, $j = 1, \ldots, m.$ and set

$$\alpha_m(t,x) := \sum_{j=1}^{m} \mathcal{X}_{j,m}(t) e_j(x), \quad \text{with } \mathcal{X}_{j,m}(0) := \langle c_0, e_j \rangle.$$

We take $\psi = e_k$ in (2.10), we have

$$\langle \partial_t \alpha_m, e_k \rangle + \Lambda(\alpha_m, p_{\tilde{\alpha}}, e_k) = -\Lambda(W, p_{\tilde{\alpha}}, e_k) + \langle f, e_k \rangle.$$

Denote by $\lambda_k(t) := -\Lambda(W, p_{\tilde{\alpha}}, e_k) + \langle f, e_k \rangle$ and using the orthogonality condition, we have

$$\langle \partial_t \alpha_m, e_k \rangle = \sum_{j=1}^{m} \mathcal{X}'_{j,m}(t) \langle e_j, e_k \rangle = \mathcal{X}'_{k,m}(t),$$

$$\Lambda(\alpha_m, p_{\tilde{\alpha}}, e_k) := \sum_{j=1}^{m} \mathcal{X}_{j,m}(t) \Lambda_m^{j,k}(t), \quad \text{where}$$

$$\Lambda_m^{j,k}(t) := \langle D \nabla e_j, \nabla e_k \rangle - \langle e_j \, v_{\tilde{\alpha}}(t), \nabla e_k \rangle + \langle q(t) e_j, e_k \rangle.$$

This yields to an equivalent ordinary differential equation (ODE) system:

$$\begin{cases} \mathcal{X}'_{k,m}(t) + \sum_{j=1}^{m} \mathcal{X}_{k,m}(t) \Lambda_m^{j,k}(t) = \lambda_k(t), \quad k = 1, \ldots, m, \\ \mathcal{X}_{k,m}(0) := \langle c_0, e_k \rangle. \end{cases} \tag{2.13}$$

Letting

$$\mathcal{X}'_m(t) := (\mathcal{X}'_{1,m}, \ldots, \mathcal{X}'_{m,m})^T, \quad \mathcal{X}_m(t) := (\mathcal{X}_{1,m}, \ldots, \mathcal{X}_{m,m})^T,$$

$$(M_m(t))_{j,k} := (\{\Lambda_1^{j,k}(t), \ldots, \Lambda_m^{j,k}(t))^T, \quad \lambda(t) := (\lambda_1(t), \ldots, \lambda_m(t))^T.$$

We rewrite (2.13) as

$$\mathcal{X}'_m(t) + M_m(t) \mathcal{X}_m(t) = \lambda(t),$$

then for each fixed $m \in \mathbb{N}$, we obtain a linear ODE. Using the standard existence theory for the first-order initial value problems (IVPs), there exists a unique and absolute continuous $\mathcal{X}_m(t)$ solution to (2.13), thus there exists a unique solution $\alpha_m(t)$ for Eq. (2.10) for a.e. $t \in [0,T]$, with e_k as a test function. Finally, there exists a unique $\alpha_m(t)$ solution of (2.10) for a.e. $t \in [0,T]$ and for any $\psi \in \mathcal{H}_m$, i.e. for $\psi = \sum_{k=1}^{m} \langle \psi, e_k \rangle e_k$.

- **Step 2.** Regularity of $\alpha_m(t)$. First, we prove that $\alpha_m \in L^\infty(0,T;L^2(\mathcal{U}))$. Using $\alpha_m(t)$ as a test function in (2.10), with the fact that $D_* < D(x)$ and that

$$-\langle \alpha_m(t)v_{\tilde{\alpha}}(t), \nabla\alpha_m\rangle + \langle q(t)\alpha_m(t), \alpha_m\rangle = \frac{1}{2}\langle h(t) + 2q(t), \alpha_m(t)^2\rangle,$$

we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\alpha_m(t)\|_0^2 + D_*\|\nabla\alpha_m(t)\|_0^2 + \frac{1}{2}\langle h(t) + 2q(t), \alpha_m^2(t)\rangle \\ \le J_1(t) + J_2(t) + J_3(t) + J_4(t),\end{aligned}$$

where

$$J_1(t) := |\langle D\nabla W(t), \nabla\alpha_m(t)\rangle|, \quad J_2(t) := |\langle W(t)v_{\tilde{\alpha}}(t), \nabla\alpha_m(t)\rangle|,$$
$$J_3(t) := |\langle q(t)W(t), \alpha_m(t)\rangle|, \quad J_4(t) := |\langle f(t), \alpha_m(t)\rangle|.$$

We estimate each term separately, using Hölder estimate, (2.5) and (2.6),

$$\begin{aligned}J_1(t) &\le D^*\|W(t)\|_1\|\nabla\alpha_m(t)\|_0,\\ J_2(t) &\le \|W(t)\|_{L^s}\|v_{\tilde{\alpha}}(t)\|_{L^r}\|\nabla\alpha_m(t)\|_0\\ &\le \kappa^*\|W(t)\|_1\|p_{\tilde{\alpha}}\|_{1,r}\|\nabla\alpha_m(t)\|_0\\ &\le \kappa^*\|W(t)\|_1\|h(t)\|_0\|\nabla\alpha_m(t)\|_0,\end{aligned}$$

where $\frac{1}{r} + \frac{1}{s} = \frac{1}{2}$. In the last estimate, we used the fact that $H^1(\mathcal{U}) \subset L^r(\mathcal{U})$, for $r \in [2,\infty)$. Next, we have

$$\begin{aligned}J_3(t) &\le \frac{1}{b_0}\|q(t)\|_0\|W(t)\|_1\|\nabla\alpha_m(t)\|_0,\\ J_4(t) &\le \frac{1}{2}\|f(t)\|_0^2 + \frac{1}{2}\|\alpha_m(t)\|_0^2.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\alpha_m(t)\|_0^2 + D_*\|\nabla\alpha_m(t)\|_0^2 \le \frac{1}{2}\|f(t)\|_0^2 + \frac{1}{2}\|\alpha_m(t)\|_0^2 + \frac{D_*}{2}\|\nabla\alpha_m(t)\|_0^2\\ + \frac{3}{2D_*}\|W(t)\|_1^2\left(\frac{1}{{b_0}^2}\|q(t)\|_0^2 + (\kappa^*)^2\|h(t)\|_0^2 + (D^*)^2\right).\end{aligned}$$

Thus,

$$\frac{d}{dt}\|\alpha_m(t)\|_0^2 + D_*\|\nabla\alpha_m(t)\|_0^2 \le \xi(t) + \|\alpha_m(t)\|_0^2, \tag{2.14}$$

where $\xi(t) := \frac{3}{D_*}\|W(t)\|_1^2(\frac{1}{{b_0}^2}\|q(t)\|_0^2 + (\kappa^*)^2\|h(t)\|_0^2 + (D^*)^2) + \|f(t)\|_0^2$.

Integrating in time on $[0,t]$, for $0 < t < T$, we obtain

$$\|\alpha_m(t)\|_0^2 \leq \|\alpha_m(0)\|_0^2 + \int_0^t \xi(s)ds + \int_0^t \|\alpha_m(s)\|_0^2 ds. \tag{2.15}$$

Using Grönwall's inequality, we get

$$\|\alpha_m(t)\|_0^2 \leq e^t[\|\alpha_m(0)\|_0^2 + \int_0^t \xi(s)ds] \leq e^t[\|\alpha_m(0)\|_0^2 + C],$$

where C does not depend on m, as $W \in L^2(0,T;H^1(\mathcal{U}))$, $f \in L^2(0,T;L^2(\mathcal{U}))$ and $h, q \in L^\infty(0,T;L^2(\mathcal{U}))$. Using the fact that by construction $\|\alpha_m(0)\|_0 \leq \|c_0\|_0$, we obtain

$$\|\alpha_m(t)\|^2_{L^\infty(0,T;L^2(\mathcal{U}))} \leq \beta e^T, \tag{2.16}$$

where β is defined as (1.19),

$$\beta := \frac{3}{D_*}\|W(t)\|^2_{L^2(0,T;H^1(\mathcal{U}))}\bigg(\frac{1}{{b_0}^2}\|q(t)\|^2_{L^\infty(0,T;L^2(\mathcal{U}))} + (\kappa^*)^2\|h(t)\|^2_{L^\infty(0,T;L^2(\mathcal{U}))} + (D^*)^2\bigg) + \|f(t)\|^2_{L^2(0,T;L^2(\mathcal{U}))} + \|c_0\|_0^2,$$

which yield $\alpha_m \in L^\infty(0,T;L^2(\mathcal{U}))$ and $\alpha_m \in L^2(0,T;L^2(\mathcal{U}))$.

Next, we prove that $\alpha_m \in L^2(0,T;H^1(\mathcal{U}))$ and $\partial_t\alpha_m \in L^2(0,T;H^{-1}(\mathcal{U}))$. Integrating (2.14) on $[0,T]$, we get

$$\|\alpha_m(T)\|_0^2 + D_*\|\nabla\alpha_m(t)\|^2_{L^2(0,T;L^2(\mathcal{U}))} \leq \|\alpha_m(0)\|_0^2 + \int_0^T \xi(s)ds + \|\alpha_m\|^2_{L^2(0,T;L^2(\mathcal{U}))}.$$

Hence,

$$b_0^2 D_*\|\alpha_m(t)\|^2_{L^2(0,T;H^1(\mathcal{U}))} \leq \beta + \|\alpha_m\|^2_{L^2(0,T;L^2(\mathcal{U}))},$$

which yields

$$\|\alpha_m(t)\|^2_{L^2(0,T;H^1(\mathcal{U}))} \leq (b_0^2 D_*)^{-1}(\beta+1)e^T.$$

Thus, $\alpha_m \in L^2(0,T;H^1(\mathcal{U}))$.

Next, we prove that $\partial_t\alpha_m \in L^2(0,T;H^{-1}(\mathcal{U}))$. As the test function $\psi \in H_m$ in (2.10), one may consider any function $\varphi \in H^1(\mathcal{U})$, such that $\varphi = \psi + \tilde{\psi}$, where $\psi \in H_m$ and $\tilde{\psi}$ in $(H_m)^\perp$ the orthogonal subspace to H_m. Using the form (2.9) of $\alpha_m(t)$ and the orthogonality of the basis $\{e_k\}_{k=1}^\infty$, we have

$$\langle\partial_t\alpha_m(t), \varphi\rangle_{(H^{-1},H^1)} = \langle\partial_t\alpha_m(t), \psi + \tilde{\psi}\rangle = \langle\partial_t\alpha_m(t), \psi\rangle.$$

Using, $\varphi = \psi + \tilde{\psi} \in H^1(\mathcal{U})$, as a text function in (2.10), taking the supremum over $\varphi \in H^1(\mathcal{U})$, such that $\|\varphi\|_1 \leq 1$, with a similar estimate as for $J_k(t)$, $k = 1, 2, 3, 4$, and integrating in time from 0 to T, we obtain

$$\|\partial_t \alpha_m\|^2_{L^2(0,T;H^{-1}(\mathcal{U}))} \leq [(\|\alpha_m(t)\|_{L^2(0,T;H^1(\mathcal{U}))} + \|W(t)\|_{L^2(0,T;H^1(\mathcal{U}))}) \times (D^* + \kappa^* \|h(t)\|_{L^\infty(0,T;L^2(\mathcal{U}))} + \|q(t)\|_{L^\infty(0,T;L^2(\mathcal{U}))}) + \|f(t)\|_{L^2(0,T;L^2(\mathcal{U}))}]^2.$$

Since $\alpha_m, W \in L^2(0,T;H^1(\mathcal{U})), f \in L^2(0,T;L^2(\mathcal{U}))$ and $h, q \in L^\infty(0,T;L^2(\mathcal{U}))$ then $\partial_t \alpha_m \in L^2(0,T;H^{-1}(\mathcal{U}))$. This concludes the proof of (2.11).

Finally, we prove $\alpha_m \in C^{0,\sigma}(0,T;H^{-1})$. Integrating (2.10) over (s,t), such that $0 \leq s < t \leq T$, for $\psi \in H^1(\mathcal{U})$, with $\|\psi\|_1 \leq 1$. We have

$$|\langle \alpha_m(t) - \alpha_m(s), \psi \rangle| \leq I_1 + I_2,$$

where $I_1 := \int_s^t |\Lambda(\alpha_m(\tau), v_{\tilde{\alpha}}, \psi)| d\tau$ and $I_2 := \int_s^t |\Lambda(W(\tau), v_{\tilde{\alpha}}, \psi)| + |\langle f, \psi \rangle| d\tau$.

Using Cauchy–Schwarz and Hölder's inequalities with Sobolev embedding $H^1(\mathcal{U}) \subset L^s(\mathcal{U})$, for $s \in [2, \infty)$, we obtain

$$I_1 + I_2 \leq \int_s^t \|\psi\|_1 (\|\alpha_m(\tau)\|_1 + \|W(\tau)\|_1) \left(D^* + \frac{\kappa^*}{b_0^2 \kappa_*} \|h(\tau)\|_0 + \|q(\tau)\|_0 \right) + \|f(\tau)\|_0 \|\psi\|_0 d\tau.$$

By (2.11) and Hölder's inequalities, we obtain

$$I_1 + I_2 \leq \|\psi\|_1 (t-s)^{\frac{1}{2}} (\|\alpha_m\|_{L^2(0,T,H^1(\mathcal{U}))} + \|W\|_{L^2(0,T,H^1(\mathcal{U}))}) \times \left(D^* + \frac{\kappa^*}{b_0^2 \kappa_*} \|h\|_{L^\infty(0,T,L^2(\mathcal{U}))} + \|q\|_{L^\infty(0,T,L^2(\mathcal{U}))} \right) + \|f\|_{L^2(0,T,L^2(\mathcal{U}))} \|\psi\|_0 (t-s)^{\frac{1}{2}}.$$

By taking the supremum over all $s, t \in [0,T]$, we obtain $\alpha_m \in C^{0,\frac{1}{2}}(0,T;H^{-1})$, thus $\alpha_m \in C^{0,\sigma}(0,T;H^{-1})$, for $0 \leq \sigma \leq \frac{1}{2}$. This concludes the proof of Lemma 2.2. □

Lemma 2.3. *Given $c_0 \in L^2(\mathcal{U}), \tilde{\alpha} \in L^2(0,T;L^2(\mathcal{U})), p_{\tilde{\alpha}} \in L^2(0,T;H^1(\mathcal{U}))$ and $v_{\tilde{\alpha}} \in L^2(0,T;L^2(\mathcal{U})^2)$ solution of (2.2) and (2.3), then there exists a unique solution $\alpha(t)$ of (2.4), such that*

$$\alpha \in C(0,T;L^2(\mathcal{U})) \cap L^2(0,T;H^1(\mathcal{U})), \quad \alpha \in C^{0,\sigma}(0,T;H^{-1}(\mathcal{U})),$$
$$\textit{for } 0 \leq \sigma \leq \frac{1}{2} \quad \textit{and} \quad \partial_t \alpha \in L^2(0,T;H^{-1}(\mathcal{U})).$$

Proof.

- **Step 1.** Existence of $\alpha(t)$.
  Due the regularity of $\alpha_m(t)$, given by (2.11) and (2.12), the sequence of the functions $\{\alpha_m\}_{m \in \mathbb{N}}$ is bounded in $L^2(0,T;H^1(\mathcal{U}))$ and in $C^{0,\sigma}(0,T;H^{-1}(\mathcal{U}))$, for

$0 \leq \sigma \leq \frac{1}{2}$. Then, by Proposition 1.1, there exists a subsequence $\{\alpha_{m_j}(t)\}_{j\in\mathbb{N}}$, and $\alpha(t)$ such that

$$\alpha_{m_j} \to \alpha \text{ strongly in } L^2(0,T;L^2(\mathcal{U})),$$
$$\alpha_{m_j} \to \alpha \text{ strongly in } C(0,T;H^{-1}(\mathcal{U})).$$

Moreover, $\{\alpha_m\}_{m\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2(\mathcal{U})) \cap L^2(0,T;H^1(\mathcal{U}))$ then one can obtain

$$\alpha_{m_j} \rightharpoonup \alpha \text{ weak-star in } L^\infty(0,T;L^2(\mathcal{U})),$$
$$\alpha_{m_j} \rightharpoonup \alpha \text{ weakly in } L^2(0,T;H^1(\mathcal{U})).$$

Applying (2.10) for any $\psi \in H^1(\mathcal{U})$, as a test function and integrating in time over $(0,t)$, we obtain

$$\langle \alpha_m(t), \psi\rangle + \int_0^t \Lambda(\alpha_m(s), v_{\tilde{\alpha}}(s), \psi)ds = \langle c_0^m, \psi\rangle$$
$$- \int_0^t \Lambda(W(s), v_{\tilde{\alpha}}(s), \psi) + \langle f(s), \psi\rangle ds.$$

Passing to the limit to each term and using the fact by construction c_0^m converge to c_0, we get

$$\langle \alpha(t), \psi\rangle + \int_0^t \Lambda(\alpha(s), v_{\tilde{\alpha}}(s), \psi)ds = \langle c_0, \psi\rangle$$
$$- \int_0^t \Lambda(W(s), v_{\tilde{\alpha}}(s), \psi) + \langle f(s), \psi\rangle ds \quad \forall\, \psi \in H^1(\mathcal{U}). \tag{2.17}$$

Using standard argument, one can check that $\alpha(0) = c_0$. Now, we rewrite (2.17) as

$$\langle \partial_t\alpha(t), \psi\rangle + \Lambda(\alpha(t), v_{\tilde{\alpha}}(t), \psi) = -\Lambda(W(t), v_{\tilde{\alpha}}(t), \psi)$$
$$+ \langle f(t), \psi\rangle \quad \forall\, \psi \in H^1(\mathcal{U}).$$

This equality and the fact that $\alpha \in L^2(0,T;H^1(\mathcal{U})) \cap L^\infty(0,T;L^2(\mathcal{U}))$ implies that $\partial_t\alpha \in L^2(0,T;H^{-1}(\mathcal{U}))$. As a consequence of [21, Lemma 1.2, Chap. 3], we have $\alpha \in C(0,T;L^2(\mathcal{U}))$. Furthermore, using the same argument as in proof of Lemma 2.2 (Step 2), we have $\alpha \in C^{0,\sigma}(0,T;H^{-1}(\mathcal{U}))$, for $0 \leq \sigma \leq \frac{1}{2}$.

- **Step 2.** Uniqueness of $\alpha(t)$. Let $\alpha_1(t)$ and $\alpha_2(t)$ two weak solution of (2.4), with $\alpha_1(0) = \alpha_2(0)$. We have

$$\langle \partial_t(\alpha_1(t) - \alpha_2(t)), \psi\rangle + \Lambda((\alpha_1 - \alpha_2), v_{\tilde{\alpha}}, \psi) = 0, \quad \psi \in H^1(\mathcal{U}).$$

Recall that

$$\Lambda((\alpha_1 - \alpha_2), v_{\tilde{\alpha}}, \psi) = \langle D\nabla(\alpha_1 - \alpha_2), \nabla\psi\rangle + \langle(\alpha_1 - \alpha_2)v_{\tilde{\alpha}}, \nabla\psi\rangle$$
$$+ \langle q(\alpha_1 - \alpha_2), \psi\rangle, \tag{2.18}$$

and due to the fact that $q^I, q^P \geq 0$, we have

$$h(t,x) + 2q(t,x) \geq 0, \quad \text{for a.e, } x \in \mathcal{U} \quad \forall t \in [0,T]. \tag{2.19}$$

Take $\psi = \alpha_1 - \alpha_2$, and using (2.19), we obtain

$$\frac{1}{2}\frac{d}{dt}\|(\alpha_1 - \alpha_2)(t)\|_0^2 = -\frac{1}{2}\|D\nabla(\alpha_1 - \alpha_2)(t)\|_0^2$$
$$- \frac{1}{2}\langle h(t) + 2q(t), ((\alpha_1 - \alpha_2)(t))^2\rangle \leq 0.$$

Integrating in time over $(0,t)$, and as $\alpha_1(0) = \alpha_2(0)$, we get $\|\alpha_1(t) - \alpha_2(t)\|_0 = 0$, thus the solution $\alpha(t)$ of (2.4) is unique and this concludes the proof of Lemma 2.3. □

2.1.3. *Proof of Theorem* 1.2 (*fixed point argument*)

In this section, we prove the existence of a weak solution $(p(t), v(t), c(t))$ to the system (1.5), for fixed $\omega \in \Omega$. Using Schauder's fixed point theorem, we prove that there exists an $\tilde{\alpha}(t) = \alpha(t)$ such that $(p_\alpha(t), v_\alpha(t), \alpha(t))$ is a solution of (2.2)–(2.4). Since, $c(t) = \alpha(t) + W(t)$, then we obtain the existence $(p(t), v(t), c(t))$ to the system (1.5).

Lemma 2.4. *Given* $c_0 \in L^2(\mathcal{U})$, *then there exists a stochastic process* $(p(t), v(t), c(t))$, *weak solution to the system* (1.5) *on* $[0,T]$, *with initial condition* $c(0) = c_0$ *in* $L^2(\mathcal{U})$, *such that* $(p(t), v(t), c(t))$ *satisfies assumptions of Definition* 1.1.

Proof. Define

$$\Phi : L^2(0,T; L^2(\mathcal{U})) \to L^2(0,T; L^2(\mathcal{U}))$$
$$\tilde{\alpha} \mapsto \Phi(\tilde{\alpha}) := \alpha(x,t),$$

where $\alpha(t)$ is the unique solution of (2.4) with $\alpha(0) = c_0$, given by Lemma 2.3.

- **Step 1.** Continuity of the mapping $\Phi : L^2(0,T; L^2(\mathcal{U})) \to L^2(0,T; L^2(\mathcal{U}))$.

Let $\tilde{\alpha}_n \subset L^2(0,T; L^2(\mathcal{U}))$ such that $\tilde{\alpha}_n \to \tilde{\alpha}$, in $L^2(0,T; L^2(\mathcal{U}))$. Denote by $\kappa(\tilde{\alpha} + W) := \kappa_{\tilde{\alpha}}, \kappa(\tilde{\alpha}_n + W) := \kappa_{\tilde{\alpha}_n}$ and let $(p_{\tilde{\alpha}_n}, v_{\tilde{\alpha}_n})$ and $(p_{\tilde{\alpha}}, v_{\tilde{\alpha}})$ be the corresponding solution to $\tilde{\alpha}_n$, respectively, to $\tilde{\alpha}$ of the elliptic equation (2.3) and (2.2). Then, we claim the following.

Claim 2.1.

$$\lim_{n\to+\infty} \|v_{\tilde{\alpha}_n} - v_{\tilde{\alpha}}\|_{L^2(0,T;L^2(\mathcal{U}))} = 0. \tag{2.20}$$

Proof. Since $\tilde{\alpha}_n \to \tilde{\alpha}$, in $L^2(0,T; L^2(\mathcal{U}))$, then there exists a subsequence, which we will denote it by $\{\tilde{\alpha}_n\}$, such that $\tilde{\alpha}_n(t,x) \to \tilde{\alpha}(t,x)$ for a.e., $(t,x) \in [0,T] \times \mathcal{U}$.

Moreover, $\kappa \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, thus by Lebesgue dominated convergence theorem we have

$$\|(\kappa_{\tilde{\alpha}} - \kappa_{\tilde{\alpha}n})\nabla p_{\tilde{\alpha}}\|_{L^2(0,T:L^2(\mathcal{U}))} \to 0, \quad \text{as } n \to \infty. \tag{2.21}$$

From (2.2) and (2.3), we have

$$\langle \kappa_{\tilde{\alpha}n}\nabla p_{\tilde{\alpha}n} - \kappa_{\tilde{\alpha}}\nabla p_{\tilde{\alpha}}, \nabla\phi\rangle = 0 \quad \forall\, \phi \in H^1(\mathcal{U}). \tag{2.22}$$

By adding and subtracting the mixed term $\kappa_{\tilde{\alpha}n}\nabla p_{\tilde{\alpha}}$ and taking $\phi = p_{\tilde{\alpha}n} - p_{\tilde{\alpha}}$ in (2.22), we have

$$\kappa_*\|\nabla(p_{\tilde{\alpha}n} - p_{\tilde{\alpha}})\|_0^2 \le |\langle(\kappa_{\tilde{\alpha}n} - \kappa_{\tilde{\alpha}})\nabla p_{\tilde{\alpha}}, \nabla(p_{\tilde{\alpha}n} - p_{\tilde{\alpha}})\rangle|. \tag{2.23}$$

Integrating in time $[0,T]$ and using Young's and Cauchy–Schwarz inequalities, we obtain

$$\begin{aligned}\kappa_*\|\nabla(p_{\tilde{\alpha}n} - p_{\tilde{\alpha}})\|^2_{L^2(0,T;L^2(\mathcal{U}))} \le{}& \frac{1}{2\kappa_*}\|(\kappa_{\tilde{\alpha}n} - \kappa_{\tilde{\alpha}})\nabla p_{\tilde{\alpha}}\|^2_{L^2(0,T;L^2(\mathcal{U}))} \\ &+ \frac{\kappa_*}{2}\|\nabla(p_{\tilde{\alpha}n} - p_{\tilde{\alpha}})\|^2_{L^2(0,T;L^2(\mathcal{U}))}.\end{aligned}$$

Hence,

$$\|\nabla(p_{\tilde{\alpha}n} - p_{\tilde{\alpha}})\|^2_{L^2(0,T;L^2(\mathcal{U}))} \le \frac{1}{\kappa_*^2}\|(\kappa_{\tilde{\alpha}n} - \kappa_{\tilde{\alpha}})\nabla p_{\tilde{\alpha}}\|^2_{L^2(0,T;L^2(\mathcal{U}))}. \tag{2.24}$$

Adding and subtracting the mixed term $\kappa_{\tilde{\alpha}n}\nabla p_{\tilde{\alpha}}$ and using (2.23) and (2.24), we get

$$\begin{aligned}&\|\kappa_{\tilde{\alpha}n}\nabla p_{\tilde{\alpha}n} - \kappa_{\tilde{\alpha}}\nabla p_{\tilde{\alpha}}\|^2_{L^2(0,T;L^2(\mathcal{U}))} \\ &\quad\le \|\kappa_{\tilde{\alpha}n}\nabla(p_{\tilde{\alpha}n} - p_{\tilde{\alpha}})\|^2_{L^2(0,T;L^2(\mathcal{U}))} + \|(\kappa_{\tilde{\alpha}n} - \kappa_{\tilde{\alpha}})\nabla p_{\tilde{\alpha}}\|^2_{L^2(0,T;L^2(\mathcal{U}))} \\ &\quad\le \left(\frac{(\kappa^*)^2}{(\kappa_*)^2} + 1\right)\|(\kappa_{\tilde{\alpha}n} - \kappa_{\tilde{\alpha}})\nabla p_{\tilde{\alpha}}\|^2_{L^2(0,T;L^2(\mathcal{U}))}.\end{aligned}$$

Passing to the limit, as n goes to ∞ and using (2.21), we obtain (2.20) and this concludes the proof of Claim 2.1. □

Since $\Phi(\tilde{\alpha}_n) = \alpha_n(t,x)$, where $\alpha_n(t)$ is a sequence of solution to Eq. (2.4):

$$\begin{cases}\langle \partial_t\alpha_n, \psi\rangle + \Lambda(\alpha_n, v_{\tilde{\alpha}n}, \psi) = -\Lambda(W, v_{\tilde{\alpha}n}, \psi) + \langle f, \psi\rangle \quad \forall\, \psi \in H^1(\mathcal{U}), \\ \alpha_n(0) = c_0.\end{cases} \tag{2.25}$$

From Lemma 2.3, we have

$$\Phi(\tilde{\alpha}_n) \in C(0,T;L^2(\mathcal{U})) \cap L^2(0,T;H^1(\mathcal{U})), \tag{2.26}$$

$$\Phi(\tilde{\alpha}_n) \in C^{0,\sigma}(0,T;H^{-1}(\mathcal{U})), \quad \text{for } 0 \le \sigma \le \frac{1}{2}, \quad \partial_t\Phi(\tilde{\alpha}_n) \in L^2(0,T;H^{-1}(\mathcal{U})), \tag{2.27}$$

which yields,

$$\Phi(\tilde{\alpha}_n) = \alpha_n \rightharpoonup Y \text{ weakly in } L^2(0,T;H^1(\mathcal{U})) \cap L^\infty(0,T;L^2(\mathcal{U})), \tag{2.28}$$

and from Proposition 1.1,

$$\Phi(\tilde{\alpha}_n) = \alpha_n \to Y \text{ strongly in } L^2(0,T;L^2(\mathcal{U})) \cap C(0,T;H^{-1}(\mathcal{U})). \tag{2.29}$$

Integrating (2.25) in time over $(0,t)$, we obtain

$$\begin{aligned} &\langle \alpha_n(t), \psi\rangle + \int_0^t \Lambda(\alpha_n(s), v_{\tilde{\alpha}_n}(s), \psi) ds \\ &\quad = \langle c_0, \psi\rangle + \int_0^t \langle f(s), \psi\rangle ds - \int_0^t \Lambda(W(s), v_{\tilde{\alpha}_n}(s), \psi) ds \quad \forall\, \psi \in H^1(\mathcal{U}). \end{aligned} \tag{2.30}$$

Passing to the limit in (2.30) and using Claim 2.1, (2.29) and (2.28), we obtain

$$\begin{aligned} &\langle Y(t), \psi\rangle + \int_0^t \Lambda(Y, v_{\tilde{\alpha}}(s), \psi) ds \\ &\quad = \langle c_0, \psi\rangle - \int_0^t \Lambda(W(s), v_{\tilde{\alpha}}(s), \psi) ds + \int_0^t \langle f(s), \psi\rangle ds, \quad \forall\, \psi \in H^1(\mathcal{U}). \end{aligned} \tag{2.31}$$

Note that, the convergence of the advection term $\langle \alpha_n v_{\tilde{\alpha}_n}, \nabla\psi\rangle$ in (2.30) follows from the fact that the sequence α_n converges to Y strongly in $L^2(0,T;L^2(\mathcal{U}))$, and Claim 2.1 (one may use standard density argument, if necessary).

Using a standard argument (see [21, Sec. 1.3, Chap. 3]), one can check that $Y(0) = c_0$. Thus,

$$\langle \partial_t Y, \psi\rangle + \Lambda(Y, v_{\tilde{\alpha}}, \psi) = -\Lambda(W, v_{\tilde{\alpha}}, \psi) + \langle f, \psi\rangle \quad \forall\, \psi \in H^1(\mathcal{U}).$$

From Lemma 2.3, the solution of Eq. (2.4) is unique, we conclude that $Y = \alpha := \Phi(\tilde{\alpha})$. By (2.29), we have $\Phi(\tilde{\alpha}_n) = \alpha_n \to \Phi(\tilde{\alpha}) = \alpha$ in $L^2(0,T;L^2(\mathcal{U}))$. Then, the mapping Φ is continuous.

- **Step 2.** Existence of a weak solution c.

From Lemma 2.3, we have $\Phi(\tilde{\alpha}) = \alpha(t) \in L^2(0,T;H^1(\mathcal{U})) \cap C^{0,\sigma}(0,T;H^{-1})$, for $0 \le \sigma \le \frac{1}{2}$. By Proposition 1.1, the range of Φ is relatively compact in $L^2(0,T;L^2(\mathcal{U}))$. Since Φ is a continuous mapping with a relatively compact range then by Schauder's fixed point theorem there exists a fixed point $\tilde{\alpha}$ such that $\Phi(\tilde{\alpha}(t)) = \tilde{\alpha}(t)$, where $\tilde{\alpha}(t)$ is a solution of

$$\langle \partial_t \tilde{\alpha}, \psi\rangle + \Lambda(\tilde{\alpha}, v_{\tilde{\alpha}}, \psi) = -\Lambda(W, v_{\tilde{\alpha}}, \psi) + \langle f, \psi\rangle \quad \forall\, \psi \in H^1(\mathcal{U}).$$

Since $c(t) = \tilde{\alpha}(t) + W(t)$, then $c(t)$ is a solution of

$$\langle c(t), \psi\rangle + \int_0^t \Lambda(c(s), v_c(s), \psi) ds$$
$$= \langle c_0, \psi\rangle + \int_0^t \langle f(s), \psi\rangle ds + \langle W(t), \psi\rangle \quad \forall \psi \in H^1(\mathcal{U}). \tag{2.32}$$

Then, there exists a weak solution $c(t)$ to the parabolic equation in (1.5), which satisfies (1.13).

- **Step 3.** Conclusion.

Finally, from Step 2 and Lemma 2.1 there exist $(p(t), v(t), c(t))$ weak solution to the system (1.5) on $[0, T]$ for fixed $\omega \in \Omega$, with initial condition $c_0 \in L^2(\mathcal{U})$. Moreover, the triple $(p(t), v(t), c(t))$ satisfies assumptions of Definition 1.1. This concludes the proof of Lemma 2.4. □

2.2. *Measurability*

The measurability of the process $(p(t), v(t), c(t))$, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_t, \mathbb{P})$ is obtained by using that the mapping

$$F : (\Omega, \mathcal{F}) \to C([0, T]; H^1(\mathcal{U})), \quad \omega \mapsto W(\cdot)(w)$$

is measurable, and for every $z \in L^2(\mathcal{U}), t \in [0, T]$, by checking that the mapping

$$G : C([0, t); H^1(\mathcal{U})) \to \mathbb{R} \times \mathbb{R},$$
$$W \mapsto (\langle c(t), z\rangle, \langle p(t), z\rangle)$$

is continuous. The continuity is proved using the same strategy for proving the convergence in Step 1 of the proof of Lemma 2.4. Hence composing these two mapping, the mapping $G \circ F(\omega) = (\langle c(t)(\omega), z\rangle, \langle p(t)(\omega), z\rangle) \in \mathbb{R}^2$ is measurable, which means that $\omega \mapsto c(t)(\omega)$ and $\omega \mapsto p(t)(\omega)$ are $\mathcal{F}$-measurable. The measurability of the process $v(t)$ is a straightforward consequence from the measurability of the process $p(t)$.

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