

TRACY–WIDOM AT EACH EDGE OF REAL COVARIANCE AND MANOVA ESTIMATORS

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We study the sample covariance matrix for real-valued data with general population covariance, as well as MANOVA-type covariance estimators in variance components models under null hypotheses of global sphericity. In the limit as matrix dimensions increase proportionally, the asymptotic spectra of such estimators may have multiple disjoint intervals of support, possibly intersecting the negative half line. We show that the distribution of the extremal eigenvalue at each regular edge of the support has a GOE Tracy–Widom limit. Our proof extends a comparison argument of Ji Oon Lee and Kevin Schnelli, replacing a continuous Green function flow by a discrete Lindeberg swapping scheme.

1. Introduction. Consider a matrix $\widehat{\Sigma} = X'TX$, where $X \in \mathbb{R}^{M \times N}$ has random independent entries, and $T \in \mathbb{R}^{M \times M}$ is deterministic. We study eigenvalue fluctuations at the edges of the spectrum of $\widehat{\Sigma}$, when $M \asymp N$ are both large.

At the largest edge and for $T \succ 0$, a substantial literature, reviewed below, shows that the fluctuations of the largest eigenvalue of $\widehat{\Sigma}$ follow the Tracy–Widom distribution. In this paper, we extend the validity of this Tracy–Widom limit to matrices T with both positive and negative eigenvalues, and to all “regular” edges of the spectrum of $\widehat{\Sigma}$. Our main result is stated informally as follows:

THEOREM (Informal). *Let $\widehat{\Sigma} = X'TX$, where $\sqrt{N}X \in \mathbb{R}^{M \times N}$ has independent entries with mean 0, variance 1, and bounded higher moments, and $T \in \mathbb{R}^{M \times M}$ is diagonal with bounded entries. Let μ_0 be the deterministic approximation for the spectrum of $\widehat{\Sigma}$ and let E_* be any regular edge of the support of μ_0 . Then for $\lambda(\widehat{\Sigma})$ the extremal eigenvalue of $\widehat{\Sigma}$ near E_* , and for a scale constant $\gamma > 0$,*

$$\pm(\gamma N)^{2/3}(\lambda(\widehat{\Sigma}) - E_*) \xrightarrow{L} \mu_{TW}.$$

Here, $\xrightarrow{L} \mu_{TW}$ denotes weak convergence to the GOE Tracy–Widom law [30], and the sign $\pm$ is chosen according to whether E_* is a left or right edge. A formal statement is provided in Theorem 2.9, and we comment on the assumption of diagonal T in Remark 1.1 below.

Our study of this model is motivated by two applications in statistics and genetics. In the first well-studied setting, $\mathbf{y}_1, \dots, \mathbf{y}_n \in \mathbb{R}^p$ are observations of p variables, or “traits”, in n independent samples. When the traits are distributed with mean 0 and covariance $\Sigma \in \mathbb{R}^{p \times p}$, the sample covariance matrix $\tilde{\Sigma} = n^{-1}Y'Y$ provides an unbiased estimate of Σ , where $Y \in \mathbb{R}^{n \times p}$ is a row-wise stacking of $\mathbf{y}_1, \dots, \mathbf{y}_n$. Assuming a representation $Y = n^{1/2}X'\Sigma^{1/2}$, this takes the form

$$(1) \quad \tilde{\Sigma} = \Sigma^{1/2}X'X'\Sigma^{1/2}.$$

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The nonzero eigenvalues of $\tilde{\Sigma}$ are the same as those of its “companion” matrix $\hat{\Sigma} = X' \Sigma X$. Here $T = \Sigma$ is positive definite, and since $\mathbf{y}_1, \dots, \mathbf{y}_n$ are independent and identically distributed, there is a single level of variation.

In the second setting, we consider models with multiple levels of variation which induce *dependence* among the observations. For example, suppose the samples are divided into I groups of size $J = n/I$, and modeled by a random effects linear model where the traits for sample j of group i are given by

$$\mathbf{y}_{i,j} = \boldsymbol{\alpha}_i + \boldsymbol{\varepsilon}_{i,j} \in \mathbb{R}^p.$$

Here, $\boldsymbol{\alpha}_i, \boldsymbol{\varepsilon}_{i,j}$ are independent vectors capturing variation at the group and individual levels, with mean 0 and respective covariances $\Sigma_1, \Sigma_2 \in \mathbb{R}^{p \times p}$. The traditional (MANOVA) estimate of the variance component Σ_1 is

$$(2) \quad \hat{\Sigma} = Y' B Y,$$

where again $Y \in \mathbb{R}^{n \times p}$ is a row-wise stacking of the observations $\mathbf{y}_{i,j}$. The matrix B is *not* positive definite, having $n - I$ negative eigenvalues: Loosely speaking, one subtracts a scaled estimate of the second-level noise Σ_2 to estimate Σ_1 . Under a null hypothesis of “global sphericity” where $\Sigma_1, \Sigma_2 \propto \text{Id}$, and introducing a representation $Y = UX$ detailed in Section 2.4, we obtain $\hat{\Sigma} = X' T X$ with $T = U' B U$ having positive and negative eigenvalues in nonvanishing proportions. [6], Boxes 1 and 2, has an example from quantitative genetics, and our main result resolves an open question stated there about Tracy–Widom limits and scaling constants in this model.

Returning to the general discussion, when $M, N \rightarrow \infty$ proportionally, the empirical spectrum of $\hat{\Sigma}$ is well approximated by a deterministic law μ_0 [21, 26, 27, 31]. Under a “sphericity” null hypothesis that $T = \text{Id}$, the law μ_0 is the Marcenko–Pastur distribution, and the largest and smallest eigenvalues of $\hat{\Sigma}$ converge to the edges of the support of μ_0 [3, 13, 32] and have asymptotic GUE/GOE Tracy–Widom fluctuations [12, 15, 16, 24, 25, 29]. In statistics and genetics, these results have enabled the application of Roy’s largest root test in high-dimensional principal components analysis [16, 23].

In this paper, we study $\hat{\Sigma}$ in the setting $T \neq \text{Id}$. For $T \geq 0$, [1] showed that all eigenvalues of $\hat{\Sigma}$ converge to the support of μ_0 , and [2, 17] proved exact separation of eigenvalues and eigenvalue rigidity. For complex Gaussian X and $T \succ 0$, [7, 22] established GUE Tracy–Widom fluctuations of the largest eigenvalue, under an edge regularity condition introduced in [7]. For complex Gaussian X , this was extended to each regular edge of the support in [14]. For real X and diagonal $T \succ 0$, [19] established GOE Tracy–Widom fluctuations of the largest eigenvalue, using different techniques based on earlier work for the deformed Wigner model in [18]. Universality results of [5, 17] lift these assumptions that X is Gaussian and/or T is diagonal.

We build on the proof in [19] to extend the above picture in two directions: First, we establish a GOE Tracy–Widom limit at each regular edge of the support for real X , including the interior edges. This extension is new even in the Gaussian setting. Second, we extend the notion of edge regularity and associated analysis to T having both positive and negative eigenvalues. This is important for our study of random effects models with multiple levels of variation, whose edge behavior is obtained here for the first time.

REMARK 1.1. We restrict attention as in [19] to diagonal T . By rotational invariance, this encompasses the case of nondiagonal T and real Gaussian X . Existing universality results of [5, 17] imply that our conclusions hold also for nondiagonal $T \geq 0$. We believe that, with minor modifications to the proof, the results of [17] may be further extended to T having negative eigenvalues, but we will not pursue this extension here.

1.1. *Strategy of proof.* Our proof generalizes the resolvent comparison argument of [19] for the largest eigenvalue. Let E_* denote an edge of the deterministic spectral support of $\widehat{\Sigma}$. (We define this formally in Section 2.) We will consider

$$\widehat{\Sigma}^{(L)} = X' T^{(L)} X$$

for a different matrix $T^{(L)}$, and compare the eigenvalue behavior of $\widehat{\Sigma}$ near E_* with that of $\widehat{\Sigma}^{(L)}$ near an edge $E_*^{(L)}$.

In [19], E_* is the rightmost edge of support. The comparison between T and $T^{(L)}$ is achieved by a continuous interpolation over $l \in [0, L]$, where $T^{(0)} = T$ and each $T^{(l)}$ has diagonal entries $\{t_\alpha^{(l)} : \alpha = 1, \dots, M\}$ given by

$$(3) \quad (t_\alpha^{(l)})^{-1} = e^{-l} (t_\alpha^{(0)})^{-1} + (1 - e^{-l}).$$

(See [19], equation (6.1).) Taking $L = \infty$, $T^{(\infty)}$ is a multiple of the identity, and Tracy–Widom fluctuations are known for $\widehat{\Sigma}^{(\infty)}$. Along this interpolation, the edge $E_*^{(l)}$ evolves continuously. Defining a smooth resolvent approximation

$$(4) \quad \mathbb{P}[\widehat{\Sigma}^{(l)} \text{ has no eigenvalues in } E_*^{(l)} + [s_1, s_2]] \approx \mathbb{E}[K(\mathfrak{X}^{(l)}(s_1, s_2))],$$

[19] establishes the bound

$$(5) \quad \left| \frac{d}{dl} \mathbb{E}[K(\mathfrak{X}^{(l)}(s_1, s_2))] \right| \leq N^{-1/3+\varepsilon}$$

for a small constant $\varepsilon > 0$ and s_1, s_2 on the $N^{-2/3}$ scale. This is applied to compare the probability in (4) for $l = 0$ and $l = \infty$.

We extend this argument by showing that the continuous interpolation in (3) may be replaced by a discrete Lindeberg sequence $T^{(0)}, T^{(1)}, \dots, T^{(L)}$ for an integer $L \leq O(N)$, swapping one diagonal entry of T at a time. Letting E_* be any regular edge of $\widehat{\Sigma}$, each matrix $\widehat{\Sigma}^{(l)} \equiv X' T^{(l)} X$ will have a corresponding edge $E_*^{(l)}$ such that

$$(6) \quad |E_*^{(l+1)} - E_*^{(l)}| \leq O(1/N).$$

Each of these L discrete steps may be thought of as corresponding to a time interval $\Delta l = O(N^{-1})$ in the continuous interpolation (3). We show that the above conditions are sufficient to establish a discrete analogue of (5),

$$(7) \quad |\mathbb{E}[K(\mathfrak{X}^{(l+1)}(s_1, s_2))] - \mathbb{E}[K(\mathfrak{X}^{(l)}(s_1, s_2))]| \leq N^{-4/3+\varepsilon}.$$

As $L \leq O(N)$, summing over $l = 0, \dots, L - 1$ establishes the desired comparison between $T^{(0)}$ and $T^{(L)}$.

In contrast to the continuous flow (3), our swapping sequence is well-defined even for negative $t_\alpha^{(0)}$. Furthermore, by swapping the diagonal entries of T from one support interval to another without continuously evolving them between such intervals, we may preserve an interior edge E_* even as the other intervals of support disappear.

Section 3 reviews prerequisite proof ingredients. Section 4 constructs the interpolating sequence. Finally, Section 5 establishes (7). The main step of Section 5 is to generalize the “decoupling lemma” of [19], Lemma 6.2, to a setting involving two different resolvents G and $\check{G}$ corresponding to $T \equiv T^{(l)}$ and $\check{T} \equiv T^{(l+1)}$.

2. Model and results.

2.1. Deterministic spectral law. Let $T = \text{diag}(t_1, \dots, t_M) \in \mathbb{R}^{M \times M}$ be a deterministic diagonal matrix, whose diagonal values $t_1, \dots, t_M$ may be positive, negative, or zero. Let $X \in \mathbb{R}^{M \times N}$ be a random matrix with independent entries of mean 0 and variance $1/N$. We study the matrix

$$\widehat{\Sigma} = X' T X$$

in the limit as $N, M \rightarrow \infty$ proportionally. In this limit, the empirical spectrum of $\widehat{\Sigma}$ is well-approximated by a deterministic law μ_0 .¹ We review in this section the definition of μ_0 and its relevant properties.

When $T = \text{Id}$, μ_0 is the Marcenko–Pastur law [21]. More generally, the law μ_0 may be defined by a fixed-point equation in its Stieltjes transform: For each $z \in \mathbb{C}^+$, there is a unique value $m_0(z) \in \mathbb{C}^+$ which satisfies

$$(8) \quad z = -\frac{1}{m_0(z)} + \frac{1}{N} \sum_{\alpha=1}^M \frac{t_\alpha}{1 + t_\alpha m_0(z)}.$$

This is oftentimes called the Marcenko–Pastur equation, and it defines implicitly the Stieltjes transform $m_0 : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ of a law μ_0 on $\mathbb{R}$ [21, 26, 27]. This law μ_0 admits a continuous density f_0 at each $x \in \mathbb{R}_*$, given by

$$(9) \quad f_0(x) = \lim_{z \in \mathbb{C}^+ \rightarrow x} \frac{1}{\pi} \text{Im } m_0(z),$$

where

$$(10) \quad \mathbb{R}_* = \begin{cases} \mathbb{R} & \text{if } \text{rank}(T) > N, \\ \mathbb{R} \setminus \{0\} & \text{if } \text{rank}(T) \leq N. \end{cases}$$

For $x \neq 0$, this is shown in [28]; we extend this to $x = 0$ when $\text{rank}(T) > N$ in Appendix A [10].

This law μ_0 may have multiple disjoint intervals of support, and two such cases are depicted in Figures 1 and 2 of Appendix A. We denote the support of μ_0 by $\text{supp}(\mu_0)$, and we call $E_* \in \mathbb{R}$ a right (or left) edge if it is a right (or left) endpoint of one of the disjoint intervals constituting $\text{supp}(\mu_0)$. When 0 is a point mass of μ_0 , we do not consider it an edge.

The support intervals and edge locations of μ_0 are described in a simple way by (8), as explained in [17, 28]: Define $P = \{0\} \cup \{-t_\alpha^{-1} : t_\alpha \neq 0\}$, and consider $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Consider the formal inverse of $m_0(z)$,

$$(11) \quad z_0(m) = -\frac{1}{m} + \frac{1}{N} \sum_{\alpha=1}^M \frac{t_\alpha}{1 + t_\alpha m},$$

as a real-valued function on $\bar{\mathbb{R}} \setminus P$ with the convention $z_0(\infty) = 0$. Two examples are also plotted in Figures 1 and 2 of Appendix A. Then the local extrema of z_0 are in 1-to-1 correspondence with edges of μ_0 , with the scale of square-root decay at each edge inversely related to the curvature of z_0 .

¹We define μ_0 as an N -dependent law depending directly on M/N and T , rather than assuming that M/N and the spectrum of T converge to certain limiting quantities.

PROPOSITION 2.1. Let $m_1, \dots, m_n \in \bar{\mathbb{R}} \setminus P$ denote the local minima and local maxima² of z_0 , ordered such that $0 > m_1 > \dots > m_k > -\infty$ and $\infty \geq m_{k+1} > \dots > m_n > 0$. Let $E_j = z_0(m_j)$ for each $j = 1, \dots, n$. Then:

- (a) μ_0 has exactly $n/2$ support intervals and n edges, which are given by $E_1, \dots, E_n$.
- (b) E_j is a right edge if m_j is a local minimum, and a left edge if m_j is a local maximum.
- (c) The edges are ordered as $E_1 > \dots > E_k > E_{k+1} > \dots > E_n$.
- (d) For each E_j where $m_j \neq \infty$, we have $E_j \in \mathbb{R}_*$ and $z_0''(m_j) \neq 0$. Defining $\gamma_j = \sqrt{2/|z_0''(m_j)|}$, the density of μ_0 satisfies $f_0(x) \sim (\gamma_j/\pi)\sqrt{|E_j - x|}$ as $x \rightarrow E_j$ with $x \in \text{supp}(\mu_0)$.

DEFINITION 2.2. For an edge E_* of μ_0 , the local minimum/maximum m_* of z_0 such that $z_0(m_*) = E_*$ is its m -value. The edge is *soft* if $m_* \neq \infty$ and *hard* if $m_* = \infty$. For a soft edge, $\gamma = \sqrt{2/|z_0''(m_*)|}$ is its associated *scale*.

The statements of Proposition 2.1 are known for $T \geq 0$, and we describe the extension to general T in Appendix A. When $T \geq 0$, an edge at 0 is usually called hard and all other edges soft. Definition 2.2 extends this to general T : A hard edge is always 0 and can occur when $\text{rank}(T) = N$. If T has negative eigenvalues, then a soft edge may also be 0 when $\text{rank}(T) > N$. We thus distinguish hard edges by the m -value rather than the edge location.

2.2. *Edge regularity and extremal eigenvalues.* We state our assumptions on T and X . We also introduce the notion of a regular edge, which is similar to the definitions of [7, 14, 17] for $T \geq 0$.

ASSUMPTION 2.3. $T = \text{diag}(t_1, \dots, t_M) \in \mathbb{R}^{M \times M}$, where $|t_\alpha| < C$ for some constant $C > 0$ and each $\alpha = 1, \dots, M$.

ASSUMPTION 2.4. $X \in \mathbb{R}^{M \times N}$ is random with independent entries. For all indices (α, i) , all $\ell \geq 1$, and some constants $C, C_1, C_2, \dots > 0$,

$$C^{-1} < M/N < C, \quad \mathbb{E}[X_{\alpha i}] = 0, \quad \mathbb{E}[X_{\alpha i}^2] = 1/N, \quad \mathbb{E}[|\sqrt{N}X_{\alpha i}|^\ell] \leq C_\ell.$$

DEFINITION 2.5. Let $E_* \in \mathbb{R}$ be a soft edge of μ_0 with m -value m_* and scale γ . Then E_* is *regular* if there is a constant $\tau > 0$ such that $|m_*| < \tau^{-1}$, $\gamma < \tau^{-1}$, and $|m_* + t_\alpha^{-1}| > \tau$ for all $\alpha \in \{1, \dots, M\}$ such that $t_\alpha \neq 0$.

A smaller constant τ indicates a weaker assumption. We will say E_* is τ -regular if we wish to emphasize the role of τ . All subsequent constants may implicitly depend on τ .

The existence of any regular edge will imply that the average value of $|t_\alpha|$ is of constant order; see Proposition 3.1. An interpretation of regularity is the following, whose proof we defer to Appendix B.

PROPOSITION 2.6. Suppose Assumption 2.3 holds and the edge E_* is regular. Then there exist constants $C, c, \delta > 0$ (independent of N) such that:

- (a) (*Separation*) The interval $(E_* - \delta, E_* + \delta)$ belongs to $\mathbb{R}_*$ and contains no edge other than E_* .

² $m_* \in \bar{\mathbb{R}} \setminus P$ is a local minimum of z_0 if $z_0(m) \geq z_0(m_*)$ for all m in a sufficiently small neighborhood of m_* , with the convention that $m_* = \infty$ is a local minimum if z_0 is positive over $(C, \infty) \cup (-\infty, -C)$ for some $C > 0$. Local maxima are defined similarly.

(b) (*Square-root decay*) For all $x \in \text{supp}(\mu_0) \cap (E_* - \delta, E_* + \delta)$, the density f_0 of μ_0 satisfies $c\sqrt{|E_* - x|} \leq f_0(x) \leq C\sqrt{|E_* - x|}$.

We will study the extremal eigenvalue of $\widehat{\Sigma}$ at each regular edge. This is well-defined by the following results establishing closeness of eigenvalues of $\widehat{\Sigma}$ to the support of μ_0 . Such results were shown in [1, 17] for $T > 0$, and we discuss the extension to general T in Appendix C.

THEOREM 2.7 (No eigenvalues outside support). *Suppose Assumptions 2.3 and 2.4 hold. Fix any constants $\delta, D > 0$. There exists a constant $N_0 \equiv N_0(\delta, D)$ such that for all $N \geq N_0$, with probability at least $1 - N^{-D}$, all eigenvalues of $\widehat{\Sigma}$ are within distance δ of $\text{supp}(\mu_0)$.*

THEOREM 2.8 ($N^{-2/3}$ concentration). *Suppose Assumptions 2.3 and 2.4 hold, and E_* is a regular right edge. Then there exists a constant $\delta > 0$ such that for any $\varepsilon, D > 0$, some $N_0 \equiv N_0(\varepsilon, D)$, and all $N \geq N_0$,*

$$\mathbb{P}[\text{no eigenvalue of } \widehat{\Sigma} \text{ belongs to } [E_* + N^{-2/3+\varepsilon}, E_* + \delta]] > 1 - N^{-D}.$$

The analogous statement holds if E_ is a regular left edge, with no eigenvalue of $\widehat{\Sigma}$ belonging to $[E_* - \delta, E_* - N^{-2/3+\varepsilon}]$.*

2.3. *Tracy–Widom fluctuations.* The following is our main result.

THEOREM 2.9. *Let $\widehat{\Sigma} = X'TX$. Suppose that Assumptions 2.3 and 2.4 hold for T and X , and that E_* is a τ -regular edge of the law μ_0 . Let E_* have scale γ as defined in Definition 2.2. Then there exists a τ -dependent constant $\delta > 0$ such that as $N, M \rightarrow \infty$:*

(a) *For E_* a right edge and $\lambda_{\max}$ the largest eigenvalue of $\widehat{\Sigma}$ in $E_* + [-\delta, \delta]$,*

$$(\gamma N)^{2/3}(\lambda_{\max} - E_*) \xrightarrow{L} \mu_{TW}.$$

(b) *For E_* a left edge and $\lambda_{\min}$ the smallest eigenvalue of $\widehat{\Sigma}$ in $E_* + [-\delta, \delta]$,*

$$(\gamma N)^{2/3}(E_* - \lambda_{\min}) \xrightarrow{L} \mu_{TW}.$$

Here, μ_{TW} is the GOE Tracy–Widom law. The notation $\xrightarrow{L}$ indicates convergence in law. As E_* is N -dependent, let us clarify that this means

$$|\mathbb{P}[(\gamma N)^{2/3}(\lambda_{\max} - E_*) \leq x] - \mu_{TW}((-\infty, x])| \leq o(1)$$

for any fixed $x \in \mathbb{R}$, where E_* is any (deterministic) choice of τ -regular edge, and $o(1)$ denotes a term vanishing as $N, M \rightarrow \infty$ and depending only on x, τ , and the constants in Assumptions 2.3 and 2.4. Note that Assumption 2.4 requires M/N to be bounded, and one may accordingly consider $M \equiv M(N)$ where N is the fundamental large parameter.

When $T \geq 0$, the above result holds also for the sample covariance matrix with the same values of E_* and γ , since this has the same eigenvalues as $\widehat{\Sigma}$ except for a set of $|N - M|$ zeros.

COROLLARY 2.10. *Under the conditions of Theorem 2.9, suppose $T \geq 0$, and let $\tilde{\Sigma} = T^{1/2}XX'T^{1/2}$. Then Theorem 2.9 holds also for $\tilde{\Sigma}$.*

When $T = \text{Id}$, the equation $0 = z'_0(m_*)$ may be solved explicitly to yield

$$m_* = -\sqrt{N}/(\sqrt{N} \pm \sqrt{M}), \quad E_* = (\sqrt{N} \pm \sqrt{M})^2/N,$$

$$(\gamma N)^{-2/3} = \frac{|\sqrt{N} \pm \sqrt{M}|}{N} \left| \frac{1}{\sqrt{M}} \pm \frac{1}{\sqrt{N}} \right|^{1/3}$$

for the upper and lower edges. These centering and scaling constants are the same as those of [12, 24, 29] and differ from those of [16, 20] in small $O(1)$ adjustments to N and M . These adjustments do not affect the validity of Theorem 2.9, although the proper adjustments are shown in [20] to lead to an improved second-order rate of convergence.

2.4. Application to linear mixed models. Consider $Y \in \mathbb{R}^{n \times p}$ representing p traits in n samples, modeled by a Gaussian random effects linear model

$$(12) \quad Y = U_1 \alpha_1 + \cdots + U_k \alpha_k.$$

Each random effect matrix $\alpha_r \in \mathbb{R}^{m_r \times p}$ has independent rows with distribution $\mathcal{N}(0, \Sigma_r)$. The deterministic incidence matrix $U_r \in \mathbb{R}^{n \times m_r}$ determines how the random effect contributes to the observations Y . For simplicity, we omit here possible additional fixed effects, and we present an example with a fixed mean effect in Example E.3 of Appendix E.

In many examples, a canonical unbiased MANOVA estimator exists for each covariance Σ_r and takes the form (2), where $B \equiv B_r \in \mathbb{R}^{n \times n}$ is a symmetric matrix that is constructed based on $U_1, \dots, U_k$. Spectral properties of MANOVA estimators in the regime $n, p, m_1, \dots, m_k \rightarrow \infty$ were studied in [9, 11], which contain additional discussion and examples.

Theorem 2.9 provides the basis for an asymptotic test of the global sphericity null hypothesis

$$(13) \quad H_0 : \Sigma_r = \sigma_r^2 \text{Id} \quad \text{for every } r = 1, \dots, k$$

in this model, based on outlier eigenvalues of $\hat{\Sigma}$. While this test may be performed using any matrix B in (2), to yield power against nonisotropic alternatives for a particular covariance Σ_r , we suggest choosing $B \equiv B_r$ such that $\hat{\Sigma}$ is the MANOVA estimator for Σ_r . Under H_0 , let us set $N = p$ and write $\alpha_r = \sqrt{N} \sigma_r X_r$ where $X_r \in \mathbb{R}^{m_r \times N}$ has independent $\mathcal{N}(0, 1/N)$ entries. Defining $M = m_1 + \cdots + m_k$, $F_{rs} = N \sigma_r \sigma_s U_r' B U_s \in \mathbb{R}^{m_r \times m_s}$, and

$$(14) \quad X = \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix} \in \mathbb{R}^{M \times N}, \quad F = \begin{pmatrix} F_{11} & \cdots & F_{1k} \\ \vdots & \ddots & \vdots \\ F_{k1} & \cdots & F_{kk} \end{pmatrix} \in \mathbb{R}^{M \times M},$$

the MANOVA estimator (2) takes the form

$$\hat{\Sigma} = Y' B Y = \sum_{r,s=1}^k \alpha_r' U_r' B U_s \alpha_s = X' F X.$$

Rotational invariance of X implies $\hat{\Sigma} \stackrel{L}{=} X' T X$ where $T = \text{diag}(t_1, \dots, t_M)$ is the diagonal matrix of eigenvalues of F . Under mild conditions for the model, as discussed in [9, 11], Assumptions 2.3 and 2.4 hold for $\hat{\Sigma}$.

In general, depending on the model design, the bulk eigenvalue distribution of $\hat{\Sigma}$ may have multiple disjoint intervals of support. [11] studies the spectral behavior in spiked alternatives to the null hypothesis (13), showing that outlier eigenvalues may appear in any interval in the complement of this support. By Theorem 2.9, the deviation of such an outlier to the closest

bulk edge may be compared to the Tracy–Widom null distribution as an assessment of its statistical significance against the null hypothesis (13).

We focus on the case of the largest eigenvalue of $\widehat{\Sigma}$, that is, testing the significance of an outlier above the largest bulk edge. In detail, such a test may be performed as follows:

1. Construct the above matrix F . Let $t_1, \dots, t_M$ be its eigenvalues.
2. Plot the function $z_0(m)$ from (11) over $m \in \mathbb{R}$, and locate the value m_* closest to 0 such that $z'_0(m_*) = 0$ and $m_* < 0$.
3. Compute the center and scale $E_* = z_0(m_*)$ and $\gamma = \sqrt{2/z''_0(m_*)}$.
4. Compare $(\gamma N)^{2/3}(\lambda_{\max} - E_*)$ to the GOE Tracy–Widom law μ_{TW} .

Asymptotic validity of this test requires regularity of the rightmost edge of μ_0 . We provide a sufficient condition for this in Proposition E.1, which encompasses many balanced classification designs. More generally, edge regularity is quantified by the separation between m_* and the poles of $z_0(m)$, and by the curvature of $z_0(m)$ at m_* . One may visually inspect the plot of $z_0(m)$ for a qualitative diagnostic check of this assumption.

Constructing F and computing $z_0(m)$ requires knowledge of $\sigma_1^2, \dots, \sigma_k^2$. If any σ_r^2 is unknown, it may be replaced by the $1/n$ -consistent estimate

$$\hat{\sigma}_r^2 = p^{-1} \text{Tr } \widehat{\Sigma}_r,$$

where $\widehat{\Sigma}_r$ is an unbiased MANOVA estimator for Σ_r . We verify this in Appendix E, where we also discuss the concrete example of the balanced one-way design, and provide numerical simulation results to assess approximation accuracy in finite samples.

3. Preliminaries and tools. The remainder of this paper is devoted to the proof of Theorem 2.9. The proof can be separated conceptually into a “deterministic component”, which constructs the interpolation from $T^{(0)} = T$ to $T^{(L)}$ to satisfy certain deterministic properties, and a “stochastic component”, which then uses resolvent-based techniques to obtain the desired estimates for this interpolation. The former component is more model-specific, but the latter can potentially be applied to other models where interior edges arise.

We collect in this section some tools for the proof. The deterministic interpolation arguments are then presented in Section 4, and the stochastic estimates in Section 5.

3.1. Notation. We denote $\mathcal{I}_M = \{1, \dots, M\}$, $\mathcal{I}_N = \{1, \dots, N\}$, and $\mathcal{I} \equiv \mathcal{I}_N \sqcup \mathcal{I}_M$ considering $\mathcal{I}_N$ and $\mathcal{I}_M$ as disjoint. We index rows and columns of $\mathbb{C}^{(N+M) \times (N+M)}$ by $\mathcal{I}$ and consistently use lower-case Roman letters i, j , etc. for indices in $\mathcal{I}_N$, Greek letters α, β , etc. for indices in $\mathcal{I}_M$, and upper-case Roman letters A, B , etc. for general indices in $\mathcal{I}$.

We typically write $z = E + i\eta$ where $E = \text{Re } z$ and $\eta = \text{Im } z$. $\mathbb{C}^+$ and $\overline{\mathbb{C}^+}$ denote the open and closed upper-half complex planes. X' denotes the transpose of a matrix X . $\|\mathbf{v}\|$ denotes the Euclidean norm for vectors, and $\|X\| = \sup_{\mathbf{v}: \|\mathbf{v}\|=1} \|X\mathbf{v}\|$ the operator norm for matrices. $C, c > 0$ denote constants changing from instance to instance and may depend on τ in the context of a regular edge. $a_N \asymp b_N$ means $cb_N \leq a_N \leq Ca_N$.

3.2. Edge regularity. The following are consequences of edge regularity. Similar properties were established for $T \geq 0$ in [4, 17], and we defer proofs for general T to Appendix B.

PROPOSITION 3.1. *Suppose Assumption 2.3 holds, and E_* is a regular edge with m -value m_* and scale γ . Then there exist constants $C, c > 0$ such that for all $\alpha = 1, \dots, M$,*

$$c < |m_*| < C, \quad c < \gamma < C, \quad |E_*| < C, \quad |1 + t_\alpha m_*| > c.$$

Furthermore, if any regular edge E_* exists, then T satisfies

$$(15) \quad |\{\alpha \in \{1, \dots, M\} : |t_\alpha| > c\}| > cM$$

for a constant $c > 0$, and if $T \geq 0$, then also $E_* > c > 0$.

PROPOSITION 3.2. *Suppose Assumption 2.3 holds and E_* is a regular edge with m -value m_* . Then there exist constants $c, \delta > 0$ such that for all $m \in (m_* - \delta, m_* + \delta)$, if E_* is a right edge then $z_0''(m) > c$, and if E_* is a left edge then $z_0''(m) < -c$.*

PROPOSITION 3.3. *Suppose Assumption 2.3 holds and E_* is a regular edge. Then there exist constants $C, c, \delta > 0$ such that the following hold: Define*

$$\mathbf{D}_0 = \{z \in \mathbb{C}^+ : \operatorname{Re} z \in (E_* - \delta, E_* + \delta), \operatorname{Im} z \in (0, 1]\}.$$

Then for all $z \in \mathbf{D}_0$ and $\alpha \in \{1, \dots, M\}$,

$$c < |m_0(z)| < C, \quad c < |1 + t_\alpha m_0(z)| < C.$$

Furthermore, for all $z \in \mathbf{D}_0$, denoting $z = E + i\eta$ and $\kappa = |E - E_*|$,

$$c\sqrt{\kappa + \eta} \leq |m_0(z) - m_*| \leq C\sqrt{\kappa + \eta}, \quad cf(z) \leq \operatorname{Im} m_0(z) \leq Cf(z)$$

where

$$f(z) = \begin{cases} \sqrt{\kappa + \eta} & \text{if } E \in \operatorname{supp}(\mu_0), \\ \eta & \text{if } E \notin \operatorname{supp}(\mu_0). \end{cases}$$

3.3. Resolvent bounds and identities. For $z \in \mathbb{C}^+$, denote the resolvent and Stieltjes transform of $\widehat{\Sigma}$ by

$$(16) \quad G_N(z) = (\widehat{\Sigma} - z \operatorname{Id})^{-1} \in \mathbb{C}^{N \times N}, \quad m_N(z) = N^{-1} \operatorname{Tr} G_N(z).$$

These satisfy the basic properties

$$(17) \quad |m_N(z)| \leq 1/\eta, \quad |G_{ij}(z)| \leq 1/\eta,$$

$$(18) \quad |m_N(z) - m_N(z')| \leq |z - z'|/\eta^2, \quad |G_{ij}(z) - G_{ij}(z')| \leq |z - z'|/\eta^2.$$

As in [17, 19], define the linearized resolvent $G(z)$ by

$$H(z) = \begin{pmatrix} -z \operatorname{Id} & X' \\ X & -T^{-1} \end{pmatrix} \in \mathbb{C}^{(N+M) \times (N+M)}, \quad G(z) = H(z)^{-1}.$$

The Schur-complement formula yields the alternative form

$$(19) \quad G(z) = \begin{pmatrix} G_N(z) & G_N(z)X'T \\ TXG_N(z) & TXG_N(z)X'T - T \end{pmatrix},$$

which is understood as the definition of $G(z)$ when T is not invertible. We will omit the argument z in m_0, m_N, G_N, G when the meaning is clear.

For any $A \in \mathcal{I}$, define $H^{(A)}$ as the submatrix of H with row and column A removed, and define $G^{(A)} = (H^{(A)})^{-1}$. When T is not invertible, $G^{(A)}$ is defined by the alternative form analogous to (19). We index $G^{(A)}$ by $\mathcal{I} \setminus \{A\}$.

Note that G and $G^{(A)}$ are symmetric, in the sense $G' = G$ and $(G^{(A)})' = G^{(A)}$ without complex conjugation. The entries of G and $G^{(A)}$ are related by the following Schur-complement identities from [17], Lemma 4.4.

LEMMA 3.4 (Resolvent identities). *Fix $z \in \mathbb{C}^+$.*

(a) *For any $i \in \mathcal{I}_N$ and $\alpha \in \mathcal{I}_M$,*

$$G_{ii} = -\frac{1}{z + \sum_{\alpha, \beta \in \mathcal{I}_M} G_{\alpha\beta}^{(i)} X_{\alpha i} X_{\beta i}}, \quad G_{\alpha\alpha} = -\frac{t_\alpha}{1 + t_\alpha \sum_{i, j \in \mathcal{I}_N} G_{ij}^{(\alpha)} X_{\alpha i} X_{\alpha j}}.$$

(b) *For any $i \neq j \in \mathcal{I}_N$ and $\alpha \neq \beta \in \mathcal{I}_M$,*

$$G_{ij} = -G_{ii} \sum_{\beta \in \mathcal{I}_M} G_{\beta j}^{(i)} X_{\beta i}, \quad G_{\alpha\beta} = -G_{\alpha\alpha} \sum_{j \in \mathcal{I}_N} G_{j\beta}^{(\alpha)} X_{\alpha j}.$$

For any $\alpha \in \mathcal{I}_M$ and $i \in \mathcal{I}_N$,

$$G_{i\alpha} = -G_{ii} \sum_{\beta \in \mathcal{I}_M} G_{\beta\alpha}^{(i)} X_{\beta i} = -G_{\alpha\alpha} \sum_{j \in \mathcal{I}_N} G_{ij}^{(\alpha)} X_{\alpha j}.$$

(c) *For any $A, B, C \in \mathcal{I}$ with $A \neq C$ and $B \neq C$,*

$$G_{AB}^{(C)} = G_{AB} - \frac{G_{AC} G_{CB}}{G_{CC}}.$$

3.4. *Stochastic domination.* For a nonnegative scalar Ψ (either random or deterministic), we write

$$\xi \prec \Psi \quad \text{and} \quad \xi = O_{\prec}(\Psi)$$

if, for any constants $\varepsilon, D > 0$ and all $N \geq N_0(\varepsilon, D)$,

$$(20) \quad \mathbb{P}[|\xi| > N^\varepsilon \Psi] < N^{-D}.$$

Here, $N_0(\varepsilon, D)$ may depend on ε, D , and quantities which are explicitly constant in the context of the statement.

Several known elementary properties of stochastic domination pertaining to union bounds and expectations are reviewed in Appendix D.

3.5. *Local law.* We will require a local law for entries of $G(z)$, when $z \in \mathbb{C}^+$ close to a regular edge E_* . This was established in [17] for $T \geq 0$, and we discuss the extension to general T in Appendix C.

THEOREM 3.5 (Entrywise local law at regular edges). *Suppose Assumptions 2.3 and 2.4 hold, and E_* is a τ -regular edge. Then for a τ -dependent constant $\delta > 0$, the following holds: Fix any constant $a > 0$ and define*

$$(21) \quad \mathbf{D} = \{z \in \mathbb{C}^+ : \operatorname{Re} z \in (E_* - \delta, E_* + \delta), \operatorname{Im} z \in [N^{-1+a}, 1]\}.$$

For $A \in \mathcal{I}$, denote $t_A = 1$ if $A \in \mathcal{I}_N$ and $t_A = t_\alpha$ if $A = \alpha \in \mathcal{I}_M$. Set

$$(22) \quad \Pi(z) = \begin{pmatrix} m_0(z) \operatorname{Id} & 0 \\ 0 & -T(\operatorname{Id} + m_0(z)T)^{-1} \end{pmatrix} \in \mathbb{C}^{(N+M) \times (N+M)}.$$

Then for all $z \equiv E + i\eta \in \mathbf{D}$ and $A, B \in \mathcal{I}$,

$$(23) \quad (G_{AB}(z) - \Pi_{AB}(z)) / (t_A t_B) \prec \sqrt{(\operatorname{Im} m_0(z)) / (N\eta)} + 1 / (N\eta),$$

and also

$$m_N(z) - m_0(z) \prec 1 / (N\eta).$$

COROLLARY 3.6. *Under the assumptions of Theorem 3.5, for any $\varepsilon, D > 0$ and all $N \geq N_0(\varepsilon, D)$, with probability at least $1 - N^{-D}$,*

$$|G_{AB}(z) - \Pi_{AB}(z)|/|t_A t_B| \leq N^\varepsilon (\sqrt{\operatorname{Im} m_0(z)/(N\eta)} + 1/(N\eta))$$

holds simultaneously for every $z \in \mathbf{D}$ and $A, B \in \mathcal{I}$.

Here, $N_0(\varepsilon, D)$ may depend on the constant a defining $\mathbf{D}$. It is verified from (19) that the quantity on the left of (23) is alternatively written as

$$(24) \quad \frac{G_{AB} - \Pi_{AB}}{t_A t_B} = \left(\begin{array}{cc} G_N - m_0 \operatorname{Id} & G_N X' \\ X G_N & X G_N X' - m_0 (\operatorname{Id} + m_0 T)^{-1} \end{array} \right)_{AB}.$$

This is understood as its definition when either t_A and/or t_B is 0.

3.6. *Resolvent approximation.* Fix a regular edge E_* . For $s_1, s_2 \in \mathbb{R}$ and $\eta > 0$, define

$$(25) \quad \mathfrak{X}(s_1, s_2, \eta) = N \int_{E_* + s_1}^{E_* + s_2} \operatorname{Im} m_N(y + i\eta) dy.$$

For η much smaller than $N^{-2/3}$ and s_1, s_2 on the $N^{-2/3}$ scale, we expect

$$\#(E_* + s_1, E_* + s_2) \approx \pi^{-1} \mathfrak{X}(s_1, s_2, \eta),$$

where the left side denotes the number of eigenvalues of $\widehat{\Sigma}$ in this interval. The following is a version of this approximation, similar to [8], Corollary 6.2. We provide a self-contained proof in Appendix D.

LEMMA 3.7. *Suppose Assumptions 2.3 and 2.4 hold, and E_* is a regular right edge. Let $K : \mathbb{R} \rightarrow [0, 1]$ be such that $K(x) = 1$ for all $x \leq 1/3$ and $K(x) = 0$ for all $x \geq 2/3$. Then for sufficiently small constants $\delta, \varepsilon > 0$:*

Let $\lambda_{\max}$ be the maximum eigenvalue of $\widehat{\Sigma}$ in $(E_ - \delta, E_* + \delta)$. Set $s_+ = N^{-2/3+\varepsilon}$, $l = N^{-2/3-\varepsilon}$, and $\eta = N^{-2/3-9\varepsilon}$. For any $D > 0$, all $N \geq N_0(\varepsilon, D)$, and all $s \in [-s_+, s_+]$,*

$$\begin{aligned} \mathbb{E}[K(\pi^{-1} \mathfrak{X}(s - l, s_+, \eta))] - N^{-D} &\leq \mathbb{P}[\lambda_{\max} \leq E_* + s] \\ &\leq \mathbb{E}[K(\pi^{-1} \mathfrak{X}(s + l, s_+, \eta))] + N^{-D}. \end{aligned}$$

4. The interpolating sequence. In this section, we construct the interpolating sequence $T^{(0)}, \dots, T^{(L)}$ described in the Introduction. We consider only the case of a right edge; this is without loss of generality, as the edge can have arbitrary sign and we may take the reflection $T \mapsto -T$. For each pair $T \equiv T^{(l)}$ and $\check{T} \equiv T^{(l+1)}$, the following definition captures the relevant property that will be needed in the subsequent computation.

DEFINITION 4.1. Let $T, \check{T} \in \mathbb{R}^{M \times M}$ be two diagonal matrices satisfying Assumption 2.3. Let E_* be a right edge of the law μ_0 defined by T , and let $\check{E}_*$ be a right edge of $\check{\mu}_0$ defined by $\check{T}$. (T, E_*) and $(\check{T}, \check{E}_*)$ are *swappable* if, for a constant $\phi > 0$, both of the following hold.

- Letting $t_\alpha, \check{t}_\alpha$ be the diagonal entries of $T, \check{T}$, we have $\sum_\alpha |t_\alpha - \check{t}_\alpha| < \phi$.
- The m -values $m, \check{m}_*$ of $E_*, \check{E}_*$ satisfy $|m_* - \check{m}_*| < \phi/N$.

We say that (T, E_*) and $(\check{T}, \check{E}_*)$ are ϕ -swappable if we wish to emphasize the role of ϕ . All subsequent constants may implicitly depend on ϕ .

One method to construct a swappable pair $T, \check{T}$ is to ensure $|t_\alpha - \check{t}_\alpha| \leq \phi/M$ for every $\alpha = 1, \dots, M$, and such a condition would hold for each pair $T^{(l)}, T^{(l+1)}$ of a suitable discretization of the continuous flow in [19]. However, to study interior edges of the spectrum, we will instead consider swappable pairs of a ‘‘Lindeberg’’ form where there is an $O(1)$ difference between t_α and $\check{t}_\alpha$ for a single index α .

We first establish some basic deterministic properties of a swappable pair, including closeness of the edges $E_*, \check{E}_*$ as claimed in (6).

LEMMA 4.2. *Suppose $T, \check{T}$ are diagonal matrices satisfying Assumption 2.3, $E_*, \check{E}_*$ are regular right edges, and (T, E_*) and $(\check{T}, \check{E}_*)$ are swappable. Let m_*, γ and $\check{m}_*, \check{\gamma}$ be the m -values and scales of $E_*, \check{E}_*$. Denote $s_\alpha = (1 + t_\alpha m_*)^{-1}$ and $\check{s}_\alpha = (1 + \check{t}_\alpha \check{m}_*)^{-1}$. Then there exists a constant $C > 0$ such that all of the following hold:*

(a) *For all integers $i, j \geq 0$ satisfying $i + j \leq 4$,*

$$\left| \frac{1}{N} \sum_{\alpha=1}^M t_\alpha^i s_\alpha^i \check{t}_\alpha^j \check{s}_\alpha^j - \frac{1}{N} \sum_{\alpha=1}^M t_\alpha^{i+j} s_\alpha^{i+j} \right| \leq C/N.$$

(b) *(Closeness of edge location) $|E_* - \check{E}_*| \leq C/N$ and*

$$(26) \quad \left| (E_* - \check{E}_*) - \frac{1}{N} \sum_{\alpha=1}^M (t_\alpha - \check{t}_\alpha) s_\alpha \check{s}_\alpha \right| \leq C/N^2.$$

(c) *(Closeness of scale) $|\gamma - \check{\gamma}| \leq C/N$.*

PROOF. By Proposition 3.1, $|t_\alpha|, |s_\alpha|, \gamma < C, c < |m_*| < C$ and similarly for $\check{t}_\alpha, \check{s}_\alpha, \check{m}_*, \check{\gamma}$. From the definitions of s_α and $\check{s}_\alpha$, we verify

$$(27) \quad t_\alpha s_\alpha - \check{t}_\alpha \check{s}_\alpha = (t_\alpha - \check{t}_\alpha) s_\alpha \check{s}_\alpha + (\check{m}_* - m_*) t_\alpha s_\alpha \check{t}_\alpha \check{s}_\alpha.$$

Then, denoting $A_{i,j} = N^{-1} \sum_{\alpha} t_\alpha^i s_\alpha^i \check{t}_\alpha^j \check{s}_\alpha^j$, swappability implies

$$|A_{i,j} - A_{i+1,j-1}| \leq \frac{1}{N} \sum_{\alpha=1}^M |t_\alpha^i s_\alpha^i \check{t}_\alpha^{j-1} \check{s}_\alpha^{j-1}| |t_\alpha \check{s}_\alpha - t_\alpha s_\alpha| \leq C/N.$$

Iteratively applying this yields (a). For (b), note by (27) that

$$\begin{aligned} E_* - \check{E}_* &= -\frac{1}{m_*} + \frac{1}{\check{m}_*} + \frac{1}{N} \sum_{\alpha=1}^M (t_\alpha s_\alpha - \check{t}_\alpha \check{s}_\alpha) \\ &= (m_* - \check{m}_*) \left(\frac{1}{m_* \check{m}_*} - A_{1,1} \right) + \frac{1}{N} \sum_{\alpha=1}^M (t_\alpha - \check{t}_\alpha) s_\alpha \check{s}_\alpha. \end{aligned}$$

Recall $0 = z'_0(m_*) = m_*^{-2} - A_{2,0}$. Then part (b) follows from the definition of swappability, together with $|A_{1,1} - m_*^{-2}| = |A_{1,1} - A_{2,0}| \leq C/N$ and $|m_*^{-2} - m_*^{-1} \check{m}_*^{-1}| \leq C/N$. For (c), we have $\gamma^{-2} = z''_0(m_*)/2 = -m_*^{-3} + A_{3,0}$. Then (c) follows from $|\gamma^{-2} - \check{\gamma}^{-2}| \leq |m_*^{-3} - \check{m}_*^{-3}| + |A_{3,0} - A_{0,3}| \leq C/N$. $\square$

We now prove the existence of an interpolating sequence. Note that to ensure the final edge $E_*^{(L)}$ is not a hard edge at 0, we allow the final matrix $T^{(L)}$ to have two distinct values $\{0, t\}$.

LEMMA 4.3. Suppose T is diagonal and satisfies Assumption 2.3, and E_* is a τ -regular right edge with scale $\gamma = 1$. Then there exist τ -dependent constants $C', \tau', \phi > 0$, a sequence of diagonal matrices $T^{(0)}, T^{(1)}, \dots, T^{(L)}$ in $\mathbb{R}^{M \times M}$ for $L \leq 2M$, and a sequence of right edges $E_*^{(0)}, E_*^{(1)}, \dots, E_*^{(L)}$ of the corresponding laws $\mu_0^{(l)}$ defined by $T^{(l)}$, such that:

1. $T^{(0)} = T$ and $E_*^{(0)} = E_*$.
2. $T^{(L)}$ has at most two distinct diagonal entries 0 and t , for some $t \in \mathbb{R}$.
3. Each $T^{(l)}$ satisfies Assumption 2.3 with constant C' .
4. Each $E_*^{(l)}$ is τ' -regular.
5. $(T^{(l)}, E_*^{(l)})$ and $(T^{(l+1)}, E_*^{(l+1)})$ are ϕ -swappable for each $l = 0, \dots, L-1$.
6. (Scaling) Each $E_*^{(l)}$ has associated scale $\gamma^{(l)} = 1$.

We first ignore the scaling property 6, and construct $T^{(0)}, \dots, T^{(L)}$ and $E_*^{(0)}, \dots, E_*^{(L)}$ satisfying properties 1–5. We will use a Lindeberg swapping construction, where each $T^{(l+1)}$ differs from $T^{(l)}$ in only one diagonal entry. It is useful to write z'_0 and z''_0 as

$$z'_0(m) = \frac{1}{m^2} - \frac{1}{N} \sum_{\alpha: t_\alpha \neq 0} \frac{1}{(m + t_\alpha^{-1})^2}, \quad z''_0(m) = -\frac{2}{m^3} + \frac{2}{N} \sum_{\alpha: t_\alpha \neq 0} \frac{1}{(m + t_\alpha^{-1})^3},$$

and to think about swapping entries of T as swapping or removing poles of z'_0 and z''_0 . In particular, for each fixed $m \in \mathbb{R}$, we can easily deduce from the above whether a given swap increases or decreases $z'_0(m)$ and $z''_0(m)$.

Upon defining a swap $T \rightarrow \check{T}$, the identification of the new right edge $\check{E}_*$ for $\check{T}$ uses the following continuity lemma.

LEMMA 4.4. Suppose T is a diagonal matrix satisfying Assumption 2.3, and E_* is a τ -regular right edge with m -value m_* . Let $\check{T}$ be a matrix that replaces a single diagonal entry t_α of T by a value $\check{t}_\alpha$, such that $|\check{t}_\alpha| \leq \|T\|$ and either $\check{t}_\alpha = 0$ or $|m_* + \check{t}_\alpha^{-1}| > \tau$. Let $z_0, \check{z}_0$ denote the function (11) defined by $T, \check{T}$. Then there exist τ -dependent constants $N_0, \phi > 0$ such that whenever $N \geq N_0$:

- $\check{T}$ has a right edge $\check{E}_*$ with m -value $\check{m}_*$ satisfying $|m_* - \check{m}_*| < \phi/N$.
- The interval between m_* and $\check{m}_*$ does not contain any pole of z_0 or $\check{z}_0$.
- $\text{sign}(m_* - \check{m}_*) = \text{sign}(\check{z}'_0(m_*))$.

(We define $\text{sign}(x) = 1$ if $x > 0$, -1 if $x < 0$, and 0 if $x = 0$.)

PROOF. By Proposition 3.1, $|m_*| > \nu$ for a constant ν . Take $\delta < \min(\tau/2, \nu/2)$. Then the given conditions for $\check{t}_\alpha$ imply that $(m_* - \delta, m_* + \delta)$ does not contain any pole of z_0 or $\check{z}_0$, and

$$|z'_0(m) - \check{z}'_0(m)| < C/N$$

for some $C > 0$ and all $m \in (m_* - \delta, m_* + \delta)$. For sufficiently small δ , Proposition 3.2 also ensures $z''_0(m) > c$ for all $m \in (m_* - \delta, m_* + \delta)$. If $\check{z}'_0(m_*) < 0 = z'_0(m_*)$, this implies $\check{z}_0$ must have a local minimum in $(m_*, m_* + C/N)$, for a constant $C > 0$ and all $N \geq N_0$. Similarly, if $\check{z}'_0(m_*) > 0$, then $\check{z}_0$ has a local minimum in $(m_* - C/N, m_*)$, and if $\check{z}'_0(m_*) = 0$, then $\check{z}_0$ has a local minimum at m_* . The result follows from Proposition 2.1 upon setting $\check{E}_* = \check{z}_0(\check{m}_*)$. $\square$

The basic idea for proving Lemma 4.3 is to take a Lindeberg sequence $T^{(0)}, \dots, T^{(L)}$ and apply the above lemma for each swap. We cannot do this naively for any Lindeberg sequence,

because in general if $E_*^{(l)}$ is τ_l -regular, then the above lemma only guarantees that $E_*^{(l+1)}$ is τ_{l+1} -regular for $\tau_{l+1} = \tau_l - C/N$ and a τ_l -dependent constant $C > 0$. Thus edge regularity, as well as the edge itself, may vanish after $O(N)$ swaps.

To circumvent this, we consider a specific construction of the Lindeberg sequence, apply Lemma 4.4 along this sequence to identify an edge $\check{E}_*$ for each successive $\check{T}$, and use a separate argument to show that $\check{E}_*$ must be τ' -regular for a fixed constant $\tau' > 0$. Hence we may continue to apply Lemma 4.4 along the whole sequence.

We consider separately the cases $m_* < 0$ and $m_* > 0$.

LEMMA 4.5. *Suppose (the right edge) E_* has m -value $m_* < 0$. Then for some τ -dependent constant N_0 , whenever $N \geq N_0$, Lemma 4.3 holds without the scaling condition, property 6.*

PROOF. We construct a Lindeberg sequence that first reflects about m_* each pole of z_0 to the right of m_* , and then replaces each pole by the one closest to m_* .

Suppose, first, that there are K_1 nonzero diagonal entries t_α of T (positive or negative) where $-t_\alpha^{-1} > m_*$. Consider a sequence of matrices $T^{(0)}, T^{(1)}, \dots, T^{(K_1)}$ where $T^{(0)} = T$, and each $T^{(k+1)}$ replaces one such diagonal entry t_α of $T^{(k)}$ by the value $\check{t}_\alpha$ such that $-\check{t}_\alpha^{-1} < m_*$ and $|m_* + \check{t}_\alpha^{-1}| = |m_* + t_\alpha^{-1}|$. For each such swap $T \rightarrow \check{T}$, we verify $|\check{t}_\alpha| \leq |t_\alpha| \leq \|T\|$, $\check{z}'_0(m_*) = z'_0(m_*) = 0$, and $\check{z}''_0(m_*) > z''_0(m_*) > 0$. Thus we may take $\check{m}_* = m_*$ in Lemma 4.4, and the new edge $\check{E}_* = \check{z}_0(m_*)$ remains τ -regular for the same constant τ .

All diagonal entries of $T^{(K_1)}$ are now nonnegative. Let $t = \|T^{(K_1)}\|$ be the maximal such entry. By the above construction, $-t^{-1} < m_* < 0$. Since $E_*^{(K_1)}$ is τ -regular, (15) implies $t > c$ for a constant $c > 0$. Let K_2 be the number of positive diagonal entries of $T^{(K_1)}$ strictly less than t , and consider a sequence $T^{(K_1+1)}, \dots, T^{(K_1+K_2)}$ where each $T^{(k+1)}$ replaces one such diagonal entry in $T^{(k)}$ by t . Applying Lemma 4.4 to each such swap $T \rightarrow \check{T}$, we verify $\check{z}'_0(m_*) < z'_0(m_*) = 0$, so $m_* < \check{m}_* < 0$. Then $|\check{m}_*| < |m_*|$ and $\min_\alpha |\check{m}_* + \check{t}_\alpha^{-1}| > \min_\alpha |m_* + t_\alpha^{-1}|$. Also $\check{m}_* + \check{t}_\alpha^{-1} > 0$ for all $\check{t}_\alpha \neq 0$, so $\check{z}''_0(\check{m}_*) > -2/\check{m}_*^3 > 2t^3$. This verifies $\check{E}_* = \check{z}_0(\check{m}_*)$ is τ' -regular for a fixed constant $\tau' > 0$. (We may take any $\tau' < \min(\tau, t^{3/2})$.)

The total number of swaps $L = K_1 + K_2$ is at most $2M$, and all diagonal entries of $T^{(L)}$ belong to $\{0, t\}$. This concludes the proof, with property 5 verified by Lemma 4.4. $\square$

LEMMA 4.6. *Lemma 4.5 holds also when E_* has m -value $m_* > 0$.*

PROOF. Proposition 2.1 implies m_* is a local minimum of z_0 . The interval $(0, m_*)$ must contain a pole of z_0 —otherwise, by the boundary condition of z_0 at 0, there would exist a local maximum m of z_0 in $(0, m_*)$ satisfying $z_0(m) > z_0(m_*)$, which would contradict the edge ordering in Proposition 2.1(c). Let $-t^{-1}$ be the pole in $(0, m_*)$ closest to m_* . Note that $t < 0$ and $|t| > |m_*|^{-1} > \tau$. We construct a Lindeberg sequence that first replaces a small but constant fraction of entries of T by t , then replaces all nonzero $t_\alpha > t$ by 0, and finally replaces all $t_\alpha < t$ by 0.

First, fix a small constant $c_0 > 0$, let $K_1 = \lfloor c_0 M \rfloor$, and consider a sequence of matrices $T^{(0)}, T^{(1)}, \dots, T^{(K_1)}$ where $T^{(0)} = T$ and each $T^{(k+1)}$ replaces a different (arbitrary) diagonal entry of $T^{(k)}$ by t . For c_0 sufficiently small, we claim that we may apply Lemma 4.4 to identify an edge $E_*^{(k)}$ for each $k = 1, \dots, K_1$, such that each $E_*^{(k)}$ is $\tau/2$ -regular. Indeed, let $k \in \{0, \dots, K_1 - 1\}$ and suppose inductively that we have identified this edge $E_*^{(j)}$ for $j = 0, \dots, k$. Let m_j, γ_j be the m -value and scale for $E_*^{(j)}$. Then Lemmas 4.4 and 4.2 ensure $|m_j - m_{j-1}| < C/N$ and $|\gamma_j - \gamma_{j-1}| < C/N$ for a τ -dependent constant $C > 0$. This yields $|m_k - m_0| < c_0(M/N)C$ and $|\gamma_k - \gamma_0| < c_0(M/N)C$. As the original edge $E_*^{(0)}$ is τ -regular,

for sufficiently small c_0 this implies $E_*^{(k)}$ is in fact $3\tau/4$ -regular. Applying Lemma 4.4, we may identify an edge $E_*^{(k+1)}$ for $T^{(k+1)}$ with m -value m_{k+1} satisfying $|m_{k+1} - m_k| < \phi/N$ when $N \geq N_0$. Thus $E_*^{(k+1)}$ is $\tau/2$ -regular, completing the induction.

$T^{(K_1)}$ now has at least $c_0 M$ diagonal entries equal to t . By the condition in Lemma 4.4 that the swap $m_* \rightarrow \check{m}_*$ does not cross any pole of z_0 or $\check{z}_0$, we have that $-t^{-1}$ is still the pole in $(0, m_*^{(K_1)})$ closest to $m_*^{(K_1)}$. Let K_2 be the number of nonzero diagonal entries t_α of $T^{(K_1)}$ (positive or negative) such that $t_\alpha > t$. Consider a sequence $T^{(K_1+1)}, \dots, T^{(K_1+K_2)}$ where each $T^{(k+1)}$ replaces one such entry in $T^{(k)}$ by 0. Note that each swap $T \rightarrow \check{T}$ of this sequence satisfies $\check{z}'_0(m) > z'_0(m)$ at every value m . Then in particular, $\check{z}'_0(m_*) > z'_0(m_*) = 0$, so Lemma 4.4 yields a new edge $\check{E}_*$ for which $-t^{-1} < \check{m}_* < m_*$. For every α such that $-\check{t}_\alpha^{-1} > -t^{-1}$, we have $-\check{t}_\alpha^{-1} > m_*$ because $-t^{-1}$ is the closest pole to the left of m_* . Then, since $\check{m}_* < m_*$, this shows $\min_{\alpha: -\check{t}_\alpha^{-1} > -t^{-1}} |\check{m}_* + \check{t}_\alpha^{-1}| > \min_{\alpha: -t_\alpha^{-1} > -t^{-1}} |m_* + t_\alpha^{-1}| > \tau/2$. The conditions $\check{m}_* > |t|^{-1} > c$ and

$$0 = \check{z}'_0(\check{m}_*) \leq \frac{1}{\check{m}_*^2} - \frac{c_0 M}{N} \frac{1}{(\check{m}_* + t^{-1})^2}$$

ensure that $\check{m}_* + t^{-1} > \nu$ for a constant $\nu > 0$, and hence $\min_\alpha |\check{m}_* + \check{t}_\alpha^{-1}| > \min(\nu, \tau/2)$ for the minimum over all α . To bound $\check{z}''_0(\check{m}_*)$, let us introduce the function

$$f(m) = -\frac{2}{N} \sum_{\alpha=1}^M \frac{t_\alpha^2 m^3}{(1 + t_\alpha m)^3}$$

and define analogously $\check{f}(m)$ for $\check{T}$. We have $f'(m) < 0$ for all m , so $f(\check{m}_*) > f(m_*)$. Furthermore, if t_α was the value which was replaced by 0, then $1 + t_\alpha \check{m}_* > 0$. (This is obvious for positive t_α ; for negative t_α , it follows from $-t^{-1} < \check{m}_* < m_* < -t_\alpha^{-1}$, as $-t^{-1}$ is the closest pole to the left of $\check{m}_*$.) Then $\check{f}(\check{m}_*) > f(\check{m}_*) > f(m_*)$. Applying the condition $0 = z'_0(m_*)$, we verify $f(m_*) = m_*^4 z''_0(m_*)$. Then

$$\check{z}''_0(\check{m}_*) > \frac{m_*^4}{\check{m}_*^4} z''_0(m_*) > z''_0(m_*).$$

This shows that $\check{E}_* = \check{z}_0(\check{m}_*)$ is τ' -regular for a fixed constant $\tau' > 0$. (We may take $\tau' = \min(\nu, \tau/2)$ as above.)

Finally, $T^{(K_1+K_2)}$ now has at least $c_0 M$ diagonal entries equal to t , and all nonzero diagonal entries t_α satisfy $t_\alpha < t < 0$. Let K_3 be the number of such entries and consider a sequence $T^{(K_1+K_2+1)}, \dots, T^{(K_1+K_2+K_3)}$ where each $T^{(k+1)}$ replaces one such entry of $T^{(k)}$ by 0. Again, each such swap satisfies $\check{z}'_0(m_*) > z'_0(m_*) = 0$, so by Lemma 4.4, $-t^{-1} < \check{m}_* < m_*$. As in the K_2 swaps above, this implies $\min_\alpha |\check{m}_* + \check{t}_\alpha^{-1}| > c$ for a constant $c > 0$. The condition $\check{t}_\alpha < t$ for all nonzero $\check{t}_\alpha$ implies that $1 + \check{t}_\alpha \check{m}_* < 0$ for all nonzero $\check{t}_\alpha$, so we have

$$\check{f}(\check{m}_*) \geq -\frac{2c_0 M}{N} \frac{t^2 \check{m}_*^3}{(1 + t \check{m}_*)^3} > c$$

for a constant $c > 0$, by Proposition 3.1. Applying again $\check{f}(\check{m}_*) = \check{m}_*^4 \check{z}''_0(\check{m}_*)$, this yields $\check{z}''_0(\check{m}_*) > c' > 0$, so $\check{E}_*$ is τ' -regular for a constant $\tau' > 0$.

The total number of swaps $L = K_1 + K_2 + K_3$ is at most $2M$. All diagonal entries of $T^{(L)}$ belong to $\{0, t\}$, so this concludes the proof. $\square$

We now establish Lemma 4.3 for all properties 1–6 by rescaling.

PROOF OF LEMMA 4.3. By Lemmas 4.5 and 4.6, there exist sequences $T^{(0)}, \dots, T^{(L)}$ and $E_*^{(0)}, \dots, E_*^{(L)}$ satisfying conditions 1–5. By Lemma 4.2, the associated scales $\gamma_0, \dots, \gamma_L$ satisfy $|\gamma_{l+1} - \gamma_l| \leq C/N$ for a ϕ, τ' -dependent constant $C > 0$ and each $l = 0, \dots, L-1$.

We verify from the definitions of E_*, m_*, γ that under the rescaling $T \mapsto cT$ for any $c > 0$, we have

$$E_* \mapsto cE_*, \quad m_* \mapsto c^{-1}m_*, \quad \gamma \mapsto c^{-3/2}\gamma.$$

Consider then the matrices $\tilde{T}^{(l)} = \gamma_l^{2/3}T^{(l)}$ and edges $\tilde{E}_*^{(l)} = \gamma_l^{2/3}E_*^{(l)}$. We check properties 1–6 for $\tilde{T}^{(l)}$ and $\tilde{E}_*^{(l)}$: Properties 1, 2, and 6 are obvious. Since $T^{(0)}, \dots, T^{(L)}$ are all τ' -regular, Proposition 3.1 implies $c < \gamma_l < C$ for constants $C, c > 0$ and every l . Then it is easy to check that properties 3, 4, and 5 also hold with adjusted constants. $\square$

Finally, we record here a deterministic estimate for any swappable pair (T, E_*) and $(\check{T}, \check{E}_*)$ that satisfies also the scaling condition $\gamma = \check{\gamma} = 1$. In the proof of [19] for a continuous interpolation $T^{(l)}$, denoting $\dot{i}_\alpha$ and $\dot{m}_*$ the derivatives with respect to l , the differential analogue of the following lemma is the pair of identities

$$\sum_{\alpha} \dot{i}_{\alpha} t_{\alpha} s_{\alpha}^3 = N \dot{m}_*, \quad \sum_{\alpha} \dot{i}_{\alpha} t_{\alpha}^2 s_{\alpha}^4 = N \dot{m}_* (A_4 - m_*^{-4})$$

where $A_4 \equiv A_{4,0} = N^{-1} \sum_{\alpha} t_{\alpha}^4 s_{\alpha}^4$.

LEMMA 4.7. Suppose $T, \check{T}$ satisfy Assumption 2.3, $E_*, \check{E}_*$ are associated regular right edges with scales $\gamma = \check{\gamma} = 1$, and (T, E_*) and $(\check{T}, \check{E}_*)$ are swappable. Define $s_{\alpha} = (1 + t_{\alpha} m_*)^{-1}$, $\check{s}_{\alpha} = (1 + \check{t}_{\alpha} \check{m}_*)^{-1}$, $A_4 = N^{-1} \sum_{\alpha} t_{\alpha}^4 s_{\alpha}^4$,

$$(28) \quad \mathcal{P}_{\alpha} = s_{\alpha} \check{s}_{\alpha} (t_{\alpha} s_{\alpha} + \check{t}_{\alpha} \check{s}_{\alpha}), \quad \mathcal{Q}_{\alpha} = s_{\alpha} \check{s}_{\alpha} (t_{\alpha}^2 s_{\alpha}^2 + t_{\alpha} s_{\alpha} \check{t}_{\alpha} \check{s}_{\alpha} + \check{t}_{\alpha}^2 \check{s}_{\alpha}^2).$$

Then for some constant $C > 0$, both of the following hold:

$$(29) \quad \left| 2N(m_* - \check{m}_*) - \sum_{\alpha=1}^M (t_{\alpha} - \check{t}_{\alpha}) \mathcal{P}_{\alpha} \right| \leq C/N,$$

$$(30) \quad \left| 3N(m_* - \check{m}_*)(A_4 - m_*^{-4}) - \sum_{\alpha=1}^M (t_{\alpha} - \check{t}_{\alpha}) \mathcal{Q}_{\alpha} \right| \leq C/N.$$

PROOF. For (29), we have from $0 = z'_0(m_*)$ applied to T and $\check{T}$

$$(31) \quad m_*^{-2} - \check{m}_*^{-2} = \frac{1}{N} \sum_{\alpha} t_{\alpha}^2 s_{\alpha}^2 - \check{t}_{\alpha}^2 \check{s}_{\alpha}^2.$$

The left side may be written as

$$(32) \quad m_*^{-2} - \check{m}_*^{-2} = (\check{m}_* - m_*)(\check{m}_* + m_*)m_*^{-2}\check{m}_*^{-2} = 2(\check{m}_* - m_*)m_*^{-3} + O(N^{-2}),$$

where the second equality applies $|m_*|, |\check{m}_*| \asymp 1$ and $|\check{m}_* - m_*| \leq C/N$. The right side may be written as

$$\frac{1}{N} \sum_{\alpha} t_{\alpha}^2 s_{\alpha}^2 - \check{t}_{\alpha}^2 \check{s}_{\alpha}^2 = \frac{1}{N} \sum_{\alpha} (t_{\alpha} - \check{t}_{\alpha}) t_{\alpha} s_{\alpha}^2 + (s_{\alpha}^2 - \check{s}_{\alpha}^2) t_{\alpha} \check{t}_{\alpha} + (t_{\alpha} - \check{t}_{\alpha}) \check{t}_{\alpha} \check{s}_{\alpha}^2.$$

Including the identities $(1 + t_\alpha m_*)s_\alpha = 1$ and $(1 + \check{t}_\alpha \check{m}_*)\check{s}_\alpha = 1$,

$$\begin{aligned}
 & \frac{1}{N} \sum_{\alpha} t_{\alpha}^2 s_{\alpha}^2 - \check{t}_{\alpha}^2 \check{s}_{\alpha}^2 \\
 &= \frac{1}{N} \sum_{\alpha} (t_{\alpha} - \check{t}_{\alpha}) (t_{\alpha} s_{\alpha}^2 (1 + \check{t}_{\alpha} \check{m}_*) \check{s}_{\alpha} + \check{t}_{\alpha} \check{s}_{\alpha}^2 (1 + t_{\alpha} m_*) s_{\alpha}) + (s_{\alpha}^2 - \check{s}_{\alpha}^2) t_{\alpha} \check{t}_{\alpha} \\
 &= \frac{1}{N} \sum_{\alpha} (t_{\alpha} - \check{t}_{\alpha}) s_{\alpha} \check{s}_{\alpha} (t_{\alpha} s_{\alpha} + \check{t}_{\alpha} \check{s}_{\alpha} + t_{\alpha} s_{\alpha} \check{t}_{\alpha} \check{m}_* + \check{t}_{\alpha} \check{s}_{\alpha} t_{\alpha} m_*) + (s_{\alpha}^2 - \check{s}_{\alpha}^2) t_{\alpha} \check{t}_{\alpha} \\
 &\equiv \frac{1}{N} \sum_{\alpha} (t_{\alpha} - \check{t}_{\alpha}) s_{\alpha} \check{s}_{\alpha} (t_{\alpha} s_{\alpha} + \check{t}_{\alpha} \check{s}_{\alpha}) + R_{\alpha},
 \end{aligned} \tag{33}$$

where we define R_{α} as the remainder term. Noting that

$$s_{\alpha}^2 - \check{s}_{\alpha}^2 = (s_{\alpha} - \check{s}_{\alpha})(s_{\alpha} + \check{s}_{\alpha}) = (\check{t}_{\alpha} \check{m}_* - t_{\alpha} m_*) s_{\alpha} \check{s}_{\alpha} (s_{\alpha} + \check{s}_{\alpha}),$$

we have

$$\begin{aligned}
 R_{\alpha} &= t_{\alpha} \check{t}_{\alpha} s_{\alpha} \check{s}_{\alpha} (t_{\alpha} s_{\alpha} \check{m}_* + t_{\alpha} \check{s}_{\alpha} m_* - \check{t}_{\alpha} s_{\alpha} \check{m}_* - \check{t}_{\alpha} \check{s}_{\alpha} m_*) \\
 &\quad + \check{t}_{\alpha} s_{\alpha} \check{m}_* + \check{t}_{\alpha} \check{s}_{\alpha} \check{m}_* - t_{\alpha} s_{\alpha} m_* - t_{\alpha} \check{s}_{\alpha} m_*) \\
 &= t_{\alpha} s_{\alpha} \check{t}_{\alpha} \check{s}_{\alpha} (\check{m}_* - m_*) (t_{\alpha} s_{\alpha} + \check{t}_{\alpha} \check{s}_{\alpha}).
 \end{aligned}$$

Then, denoting $A_{i,j} = N^{-1} \sum_{\alpha} t_{\alpha}^i s_{\alpha}^i \check{t}_{\alpha}^j \check{s}_{\alpha}^j$ and applying Lemma 4.2(a),

$$\frac{1}{N} \sum_{\alpha} R_{\alpha} = (\check{m}_* - m_*) (A_{2,1} + A_{1,2}) = 2(\check{m}_* - m_*) A_{3,0} + O(N^{-2}).$$

By the scaling $\gamma = 1$, we have $A_{3,0} = 1 + m_*^{-3}$. Combining this with (31), (32), and (33) and multiplying by N yields (29).

The identity (30) follows similarly: The condition $\gamma = \check{\gamma}$ implies

$$m_*^{-3} - \check{m}_*^{-3} = \frac{1}{N} \sum_{\alpha} t_{\alpha}^3 s_{\alpha}^3 - \check{t}_{\alpha}^3 \check{s}_{\alpha}^3.$$

The left side is

$$(\check{m}_* - m_*) (m_*^2 + m_* \check{m}_* + \check{m}_*^2) m_*^{-3} \check{m}_*^{-3} = 3(\check{m}_* - m_*) m_*^{-4} + O(N^{-2}).$$

Applying $(1 + t_{\alpha} m_*)s_{\alpha} = 1$ and $(1 + \check{t}_{\alpha} \check{m}_*)\check{s}_{\alpha} = 1$, the right side is

$$\begin{aligned}
 & \frac{1}{N} \sum_{\alpha} t_{\alpha}^3 s_{\alpha}^3 - \check{t}_{\alpha}^3 \check{s}_{\alpha}^3 = \frac{1}{N} \sum_{\alpha} (t_{\alpha} s_{\alpha} - \check{t}_{\alpha} \check{s}_{\alpha}) (t_{\alpha}^2 s_{\alpha}^2 + t_{\alpha} s_{\alpha} \check{t}_{\alpha} \check{s}_{\alpha} + \check{t}_{\alpha}^2 \check{s}_{\alpha}^2) \\
 &= \frac{1}{N} \sum_{\alpha} (t_{\alpha} - \check{t}_{\alpha}) s_{\alpha} \check{s}_{\alpha} (t_{\alpha}^2 s_{\alpha}^2 + t_{\alpha} s_{\alpha} \check{t}_{\alpha} \check{s}_{\alpha} + \check{t}_{\alpha}^2 \check{s}_{\alpha}^2) \\
 &\quad + t_{\alpha} s_{\alpha} \check{t}_{\alpha} \check{s}_{\alpha} (\check{m}_* - m_*) (t_{\alpha}^2 s_{\alpha}^2 + t_{\alpha} s_{\alpha} \check{t}_{\alpha} \check{s}_{\alpha} + \check{t}_{\alpha}^2 \check{s}_{\alpha}^2) \\
 &= \left(\frac{1}{N} \sum_{\alpha} (t_{\alpha} - \check{t}_{\alpha}) \mathcal{Q}_{\alpha} \right) + 3(\check{m}_* - m_*) A_4 + O(N^{-2}).
 \end{aligned}$$

Combining the above and multiplying by N yields (30). $\square$

5. Resolvent comparison and proof of Theorem 2.9. We will conclude the proof of Theorem 2.9 by establishing the following estimate.

THEOREM 5.1 (Resolvent comparison). *Fix $\varepsilon > 0$ a sufficiently small constant, and let $s_1, s_2, \eta \in \mathbb{R}$ be such that $|s_1|, |s_2| < N^{-2/3+\varepsilon}$ and $\eta \in [N^{-2/3-\varepsilon}, N^{-2/3}]$. Let $T, \check{T} \in \mathbb{R}^{M \times M}$ be two diagonal matrices and $E_*, \check{E}_*$ two corresponding regular right edges, such that (T, E_*) and $(\check{T}, \check{E}_*)$ are swappable and their scales satisfy $\gamma = \check{\gamma} = 1$. Suppose Assumptions 2.3 and 2.4 hold.*

Let $m_N, \check{m}_N$ be the Stieltjes transforms as in (16) corresponding to $T, \check{T}$, and define

$$\mathfrak{X} = N \int_{E_*+s_1}^{E_*+s_2} \operatorname{Im} m_N(y+i\eta) dy, \quad \check{\mathfrak{X}} = N \int_{\check{E}_*+s_1}^{\check{E}_*+s_2} \operatorname{Im} \check{m}_N(y+i\eta) dy.$$

Let $K : \mathbb{R} \rightarrow \mathbb{R}$ be any function such that K and its first four derivatives are uniformly bounded by a constant. Then

$$(34) \quad \mathbb{E}[K(\mathfrak{X}) - K(\check{\mathfrak{X}})] \prec N^{-4/3+16\varepsilon}.$$

PROOF OF THEOREM 2.9. By symmetry under $T \mapsto -T$, it suffices to consider a right edge. By rescaling $T \mapsto \gamma^{2/3}T$, it suffices to consider $\gamma = 1$.

Let $T^{(0)}, \dots, T^{(L)}, E_*^{(0)}, \dots, E_*^{(L)}$ satisfy Lemma 4.3. Define $\mathfrak{X}^{(k)}(s_1, s_2, \eta)$ as in (25) for each $(T^{(k)}, E_*^{(k)})$. For a small constant $\varepsilon > 0$, let η, s_+, l and $K : [0, \infty) \rightarrow [0, 1]$ be as in Lemma 3.7, where K has bounded derivatives of all orders. Fix $x \in \mathbb{R}$ and let $s = xN^{-2/3}$. Applying Lemma 3.7,

$$\mathbb{P}[\lambda_{\max}(\widehat{\Sigma}) \leq E_* + s] \leq \mathbb{E}[K(\pi^{-1}\mathfrak{X}^{(0)}(s+l, s_+, \eta))] + N^{-1}.$$

Setting $\varepsilon' = 9\varepsilon$ and applying Theorem 5.1,

$$\mathbb{E}[K(\pi^{-1}\mathfrak{X}^{(k)}(s+l, s_+, \eta))] \leq \mathbb{E}[K(\pi^{-1}\mathfrak{X}^{(k+1)}(s+l, s_+, \eta))] + N^{-4/3+17\varepsilon'}$$

for each $k = 0, \dots, L-1$. Finally, defining $\widehat{\Sigma}^{(L)} = X'T^{(L)}X$ and $\lambda_{\max}(\widehat{\Sigma}^{(L)})$ as its largest eigenvalue in $(E_*^{(L)} - \delta', E_*^{(L)} + \delta')$ for some $\delta' > 0$, applying Lemma 3.7 again yields

$$\mathbb{E}[K(\pi^{-1}\mathfrak{X}^{(L)}(s+l, s_+, \eta))] \leq \mathbb{P}[\lambda_{\max}(\widehat{\Sigma}^{(L)}) \leq E_*^{(L)} + s + 2l] + N^{-1}.$$

Recalling $L \leq 2M$ and combining the above bounds,

$$\mathbb{P}[N^{2/3}(\lambda_{\max}(\widehat{\Sigma}) - E_*) \leq x] \leq \mathbb{P}[N^{2/3}(\lambda_{\max}(\widehat{\Sigma}^{(L)}) - E_*^{(L)}) \leq x + 2N^{-\varepsilon}] + o(1).$$

The matrix $T^{(L)}$ has all diagonal entries 0 or t , so $\widehat{\Sigma}^{(L)} = t\tilde{X}'\tilde{X}$ for $\tilde{X} \in \mathbb{R}^{\tilde{M} \times N}$ having independent entries satisfying the moment conditions of Assumption 2.4. The corresponding law $\mu_0^{(L)}$ has a single support interval and a unique right edge, so $E_*^{(L)}$ must be this edge. Regularity of $E_*^{(L)}$ and (15) imply $|t| \asymp 1$ and $\tilde{M}/N \asymp 1$. If $E_*^{(L)} > 0$, then $t > 0$. If $E_*^{(L)} < 0$, then $t < 0$, and edge regularity implies $\tilde{M}/N$ is bounded away from 1. Then we obtain

$$(35) \quad \mathbb{P}[N^{2/3}(\lambda_{\max}(\widehat{\Sigma}^{(L)}) - E_*^{(L)}) \leq x + 2N^{-\varepsilon}] = F_1(x) + o(1),$$

where F_1 is the distribution function of μ_{TW} , by applying the results of [12, 17] to either the largest eigenvalue of $\widehat{\Sigma}^{(L)}$ or the smallest positive eigenvalue of $-\widehat{\Sigma}^{(L)}$. Combining the above, we obtain

$$\mathbb{P}[N^{2/3}(\lambda_{\max}(\widehat{\Sigma}) - E_*) \leq x] \leq F_1(x) + o(1).$$

The reverse bound is analogous, concluding the proof. $\square$

In the remainder of this section, we prove Theorem 5.1.

5.1. *Individual resolvent bounds.* For diagonal T and for $z = y + i\eta$ as appearing in Theorem 5.1, we record here simple resolvent bounds that follow from the local law. Similar bounds were used in [8, 19]. We also introduce the shorthand notation that will be used in the computation.

Let E_* be a regular right edge. Fix a small constant $\varepsilon > 0$, and fix s_1, s_2, η such that $|s_1|, |s_2| \leq N^{-2/3+\varepsilon}$ and $\eta \in [N^{-2/3-\varepsilon}, N^{-2/3}]$. Changing variables, we write

$$\mathfrak{X} \equiv \mathfrak{X}(s_1, s_2, \eta) = N \int_{s_1}^{s_2} \operatorname{Im} m_N(y + E_* + i\eta) dy.$$

For $y \in [s_1, s_2]$, we write as shorthand

$$\begin{aligned} z &\equiv z(y) = y + E_* + i\eta, & G &\equiv G(z(y)), \\ m_N &\equiv m_N(z(y)), & G^{(\alpha)} &\equiv G^{(\alpha)}(z(y)), \\ m_N^{(\alpha)} &\equiv \frac{1}{N} \sum_{i \in \mathcal{I}_N} G_{ii}^{(\alpha)}(z(y)), & \mathfrak{X}^{(\alpha)} &\equiv N \int_{s_1}^{s_2} \operatorname{Im} m_N^{(\alpha)}(\tilde{y} + E_* + i\eta) d\tilde{y}. \end{aligned}$$

We use the simplified summation notation

$$\sum_{i,j} \equiv \sum_{i,j \in \mathcal{I}_N}, \quad \sum_{\alpha,\beta} \equiv \sum_{\alpha,\beta \in \mathcal{I}_M}$$

where sums over lower-case Roman indices are over $\mathcal{I}_N$ and sums over Greek indices are over $\mathcal{I}_M$. We use also the simplified integral notation

$$\int \tilde{G}_{AB} \equiv \int_{s_1}^{s_2} G(z(\tilde{y}))_{AB} d\tilde{y}, \quad \int \tilde{m}_N \equiv \int_{s_1}^{s_2} m_N(z(\tilde{y})) d\tilde{y},$$

so that integrals are implicitly over $[s_1, s_2]$, and we denote by $\tilde{F}$ the function F evaluated at $F(z(\tilde{y}))$ for $\tilde{y}$ the variable of integration. In this notation, $\mathfrak{X}$ and $\mathfrak{X}^{(\alpha)}$ are simply

$$\mathfrak{X} = \sum_i \operatorname{Im} \int \tilde{G}_{ii}, \quad \mathfrak{X}^{(\alpha)} = \sum_i \operatorname{Im} \int \tilde{G}_{ii}^{(\alpha)}.$$

We introduce the fundamental small parameter

$$(36) \quad \Psi = N^{-1/3+3\varepsilon}.$$

We will eventually bound all quantities in the computation by powers of Ψ . In fact, as shown in Lemmas 5.2 and 5.3 below, nonintegrated resolvent entries are controlled by powers of the smaller quantity $N^{-1/3+\varepsilon}$. However, integrated quantities will require the additional slack of $N^{2\varepsilon}$. We will pass to using Ψ for all bounds after this distinction is no longer needed.

We have the following corollaries of Proposition 3.3 and Theorem 3.5:

LEMMA 5.2. *Under the assumptions of Theorem 5.1, for all $y \in [s_1, s_2]$, $i \neq j \in \mathcal{I}_N$, and $\alpha \neq \beta \in \mathcal{I}_M$,*

$$\begin{aligned} G_{ii} &< 1, & \frac{1}{G_{ii}} &< 1, & \frac{G_{\alpha\alpha}}{t_\alpha} &< 1, & \frac{t_\alpha}{G_{\alpha\alpha}} &< 1, & G_{ij} &< N^{-1/3+\varepsilon}, \\ \frac{G_{i\alpha}}{t_\alpha} &< N^{-1/3+\varepsilon}, & \frac{G_{\alpha\beta}}{t_\alpha t_\beta} &< N^{-1/3+\varepsilon}, & m_N - m_* &< N^{-1/3+\varepsilon}. \end{aligned}$$

If T is singular, these are defined by continuity and the form (19) for G .

PROOF. Proposition 3.3 implies $\operatorname{Im} m_0(z(y)) \leq C\sqrt{\kappa + \eta} \leq CN^{-1/3+\varepsilon/2}$, while $\eta \geq N^{-2/3-\varepsilon}$ by assumption. Then Theorem 3.5 yields $(t_A t_B)^{-1}(G - \Pi)_{AB} \prec N^{-1/3+\varepsilon}$ for all $A, B \in \mathcal{I}$. Proposition 3.3 also implies $|m_0(z)| \asymp 1$ and $|1 + t_\alpha m_0(z)| \asymp 1$, from which all of the entrywise bounds on G follow. The bound on m_N follows from $|m_0 - m_*| \leq C\sqrt{\kappa + \eta} \leq CN^{-1/3+\varepsilon/2}$ and $|m_N - m_0| \prec N^{-1/3+\varepsilon}$. $\square$

LEMMA 5.3. *Under the assumptions of Theorem 5.1, for all $i \in \mathcal{I}_N$ and $\alpha \in \mathcal{I}_M$,*

$$\sum_k G_{ik}^{(\alpha)} X_{\alpha k} \prec N^{-1/3+\varepsilon}, \quad \sum_{p,q} G_{pq}^{(\alpha)} X_{\alpha p} X_{\alpha q} - m_* \prec N^{-1/3+\varepsilon}.$$

PROOF. Applying Lemmas 3.4(b) and 5.2,

$$\sum_k G_{ik}^{(\alpha)} X_{\alpha k} = -G_{i\alpha}/G_{\alpha\alpha} \prec N^{-1/3+\varepsilon}.$$

Similarly, applying Lemma 3.4(a) and Theorem 3.5,

$$\sum_{p,q} G_{pq}^{(\alpha)} X_{\alpha p} X_{\alpha q} - m_* = -\frac{1}{G_{\alpha\alpha}} - \frac{1}{t_\alpha} - m_* = \frac{1}{\Pi_{\alpha\alpha}} - \frac{1}{G_{\alpha\alpha}} + (m_0 - m_*) \prec N^{-1/3+\varepsilon}. \quad \square$$

REMARK 5.4. All probabilistic bounds such as the above are derived from Theorem 3.5. Thus they in fact hold in the uniform sense of Corollary 3.6. We continue to use the notation $\prec$ for convenience, with the understanding that we may take union bounds and integrals over $y \in [s_1, s_2]$.

We record one trivial bound for an integral that will be repeatedly used, and which explains the appearance of Ψ .

LEMMA 5.5. *Suppose the assumptions of Theorem 5.1 hold, $F(z(y)) \prec N^{a(-1/3+\varepsilon)}$ for some $a \geq 2$, and we may take a union bound of this statement over $y \in [s_1, s_2]$ (in the sense of Lemma D.3). Then, with $\Psi = N^{-1/3+3\varepsilon}$,*

$$N \int \tilde{F} \prec \Psi^{a-1}.$$

PROOF. We have $N(s_2 - s_1)N^{a(-1/3+\varepsilon)} \leq 2N^{1/3+\varepsilon}N^{a(-1/3+\varepsilon)} \leq 2\Psi^{a-1}$. $\square$

The next lemma allows us to “remove the superscript” in the computation.

LEMMA 5.6. *Under the assumptions of Theorem 5.1, for any $y \in [s_1, s_2]$, $i, j \in \mathcal{I}_N$ (possibly equal), and $\alpha \in \mathcal{I}_M$,*

$$G_{ij} - G_{ij}^{(\alpha)} \prec N^{2(-1/3+\varepsilon)}, \quad m_N - m_N^{(\alpha)} \prec N^{2(-1/3+\varepsilon)}, \quad \mathfrak{X} - \mathfrak{X}^{(\alpha)} \prec \Psi.$$

PROOF. Applying the last resolvent identity from Lemma 3.4,

$$G_{ij} - G_{ij}^{(\alpha)} = \frac{G_{i\alpha} G_{j\alpha}}{G_{\alpha\alpha}} = G_{i\alpha} \frac{G_{j\alpha}}{t_\alpha} \frac{t_\alpha}{G_{\alpha\alpha}},$$

so the first statement follows from Lemma 5.2. Taking $i = j$ and averaging over $\mathcal{I}_N$ yields the second statement. The third statement follows from Lemma 5.5 and $\mathfrak{X} - \mathfrak{X}^{(\alpha)} = \operatorname{Im} N \int (\tilde{m}_N - \tilde{m}_N^{(\alpha)})$. $\square$

5.2. *Resolvent bounds for a swappable pair.* We now record bounds for a swappable pair (T, E_*) and $(\check{T}, \check{E}_*)$, where $E_*, \check{E}_*$ are both regular. We denote by $\check{m}_N, \check{G}, \check{\mathfrak{X}}$ the analogues of $m_N, G, \mathfrak{X}$ for $\check{T}$. For $\varepsilon, s_1, s_2, \eta$ and $y \in [s_1, s_2]$ as in Section 5.1, we write as shorthand

$$\check{z} \equiv \check{z}(y) = y + \check{E}_* + i\eta, \quad \check{G} \equiv \check{G}(\check{z}(y)), \quad \check{m}_N \equiv \check{m}_N(\check{z}(y)).$$

The results of the preceding section hold equally for $\check{G}, \check{m}_N$, and $\check{\mathfrak{X}}$.

The desired bound (34) arises from the following identity: Suppose first that T and $\check{T}$ are invertible. Applying $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$,

$$G - \check{G} = G \begin{pmatrix} (-\check{z} + z) \text{Id} & 0 \\ 0 & -\check{T}^{-1} + T^{-1} \end{pmatrix} \check{G}.$$

Hence, as $z - \check{z} = E_* - \check{E}_*$,

$$(37) \quad G_{ij} - \check{G}_{ij} = \sum_k G_{ik} \check{G}_{jk}(E_* - \check{E}_*) - \sum_\alpha \frac{G_{i\alpha}}{t_\alpha} \frac{\check{G}_{j\alpha}}{\check{t}_\alpha} (t_\alpha - \check{t}_\alpha).$$

This holds by continuity when T is singular, using the form (19).

The following lemma allows us to “remove the check” in the computation.

LEMMA 5.7. *Suppose the assumptions of Theorem 5.1 hold. Let $\Psi = N^{-1/3+3\varepsilon}$. Then for any $y \in [s_1, s_2]$ and $i, j \in \mathcal{I}_N$ (possibly equal),*

$$G_{ij} - \check{G}_{ij} \prec N^{2(-1/3+\varepsilon)}, \quad m_N - \check{m}_N \prec N^{2(-1/3+\varepsilon)}, \quad \mathfrak{X} - \check{\mathfrak{X}} \prec \Psi.$$

PROOF. Applying Lemma 5.2 for both G and $\check{G}$, and also the definition of swappability and Lemma 4.2, we have from (37)

$$G_{ij} - \check{G}_{ij} \prec |E_* - \check{E}_*| \cdot N \cdot N^{2(-1/3+\varepsilon)} + \sum_\alpha |t_\alpha - \check{t}_\alpha| N^{2(-1/3+\varepsilon)} \prec N^{2(-1/3+\varepsilon)}.$$

(The contribution from $k = i$ or $k = j$ in the first sum of (37) is of lower order.) Taking $i = j$ and averaging over $\mathcal{I}_N$ yields the second statement, and integrating over $y \in [s_1, s_2]$ and applying Lemma 5.5 yields the third. $\square$

In many cases, we may strengthen the above lemma by an additional factor of Ψ if we take an expectation. (This may be seen by taking $Y = Y^{(\alpha)} = 1$ and $a = 0$ in Lemma 5.9 below.) To take expectations of remainder terms, we will invoke Lemma D.2 combined with the following basic bound:

LEMMA 5.8. *Under the assumptions of Theorem 5.1, let $P \equiv P(z(y))$ be any polynomial in the entries of X and G with bounded degree, bounded (possibly random) coefficients, and at most N^C terms for a constant $C > 0$. Then for a constant $C' > 0$ and all $y \in [s_1, s_2]$, we have $\mathbb{E}[|P|] \leq N^{C'}$.*

PROOF. By the triangle inequality and Holder’s inequality, it suffices to consider a bounded power of a single entry of G or X . Then the result follows from (17) and the form (19) for G . $\square$

LEMMA 5.9. *Under the assumptions of Theorem 5.1, let Y be any quantity such that $Y \prec \Psi^a$ for some constant $a \geq 0$. Suppose that for each $\alpha \in \mathcal{I}_M$, there exists a quantity $Y^{(\alpha)}$ such that $Y - Y^{(\alpha)} \prec \Psi^{a+1}$, and $Y^{(\alpha)}$ is independent of row α of X . Suppose furthermore that $\mathbb{E}[|Y|^\ell] \leq N^{C_\ell}$ for each integer $\ell > 0$ and some constants $C_1, C_2, \dots > 0$.*

Then, for all $i, j \in \mathcal{I}_N$ (possibly equal) and $y \in [s_1, s_2]$,

$$\mathbb{E}[(G_{ij} - \check{G}_{ij})Y] \prec N^{2(-1/3+\varepsilon)}\Psi^{a+1} \prec \Psi^{a+3},$$

$$\mathbb{E}[(m_N - \check{m}_N)Y] \prec N^{2(-1/3+\varepsilon)}\Psi^{a+1} \prec \Psi^{a+3}.$$

$$\mathbb{E}[(\mathfrak{X} - \check{\mathfrak{X}})Y] \prec \Psi^{a+2}.$$

PROOF. Applying (26), the bound $N^{-1} \prec \Psi^3$, and Lemma 5.2 to (37),

$$\begin{aligned} (G_{ij} - \check{G}_{ij})Y &= \sum_k G_{ik} \check{G}_{jk} (E_* - \check{E}_*)Y - \sum_\alpha \frac{G_{i\alpha}}{t_\alpha} \frac{\check{G}_{j\alpha}}{\check{t}_\alpha} (t_\alpha - \check{t}_\alpha)Y \\ &= \sum_\alpha (t_\alpha - \check{t}_\alpha) \left(s_\alpha \check{s}_\alpha \frac{1}{N} \sum_k G_{ik} \check{G}_{jk} - \frac{G_{i\alpha}}{t_\alpha} \frac{\check{G}_{j\alpha}}{\check{t}_\alpha} \right) Y + O_{\prec}(\Psi^{a+5}). \end{aligned}$$

By swappability and Lemma 5.2, the explicit term on the right is of size $O_{\prec}(N^{2(-1/3+\varepsilon)}\Psi^a)$. (The contributions from $k = i$ and $k = j$ in the summation are of lower order.) Applying the assumption $Y - Y^{(\alpha)} \prec \Psi^{a+1}$ as well as Lemma 5.6, we may replace Y with $Y^{(\alpha)}$, G_{ik} with $G_{ik}^{(\alpha)}$, and $\check{G}_{jk}$ with $\check{G}_{jk}^{(\alpha)}$ above while introducing an $O_{\prec}(N^{2(-1/3+\varepsilon)}\Psi^{a+1})$ error. Hence,

$$\begin{aligned} (38) \quad (G_{ij} - \check{G}_{ij})Y &= \sum_\alpha (t_\alpha - \check{t}_\alpha) \left(s_\alpha \check{s}_\alpha \frac{1}{N} \sum_k G_{ik}^{(\alpha)} \check{G}_{jk}^{(\alpha)} - \frac{G_{i\alpha}}{t_\alpha} \frac{\check{G}_{j\alpha}}{\check{t}_\alpha} \right) Y^{(\alpha)} \\ &\quad + O_{\prec}(N^{2(-1/3+\varepsilon)}\Psi^{a+1}). \end{aligned}$$

Applying the resolvent identities from Lemma 3.4,

$$\frac{G_{i\alpha}}{t_\alpha} = \frac{G_{\alpha\alpha}}{t_\alpha} \sum_k G_{ik}^{(\alpha)} X_{\alpha k} = - \frac{1}{1 + t_\alpha \sum_{p,q} G_{pq}^{(\alpha)} X_{\alpha p} X_{\alpha q}} \sum_k G_{ik}^{(\alpha)} X_{\alpha k}.$$

Recalling $s_\alpha = (1 + t_\alpha m_*)^{-1}$, and applying Lemma 5.3 and a Taylor expansion of $(1 + t_\alpha x)^{-1}$ around $x = m_*$,

$$\frac{G_{i\alpha}}{t_\alpha} = -s_\alpha \sum_k G_{ik}^{(\alpha)} X_{\alpha k} + O_{\prec}(N^{2(-1/3+\varepsilon)}),$$

where the explicit term on the right is of size $O_{\prec}(N^{-1/3+\varepsilon}) \prec \Psi$. A similar expansion holds for $\check{G}_{j\alpha}/\check{t}_\alpha$. Substituting into (38),

$$\begin{aligned} (G_{ij} - \check{G}_{ij})Y &= \sum_\alpha (t_\alpha - \check{t}_\alpha) s_\alpha \check{s}_\alpha \left(\frac{1}{N} \sum_k G_{ik}^{(\alpha)} \check{G}_{jk}^{(\alpha)} - \sum_{k,l} G_{ik}^{(\alpha)} X_{\alpha k} \check{G}_{jl}^{(\alpha)} X_{\alpha l} \right) Y^{(\alpha)} \\ &\quad + O_{\prec}(N^{2(-1/3+\varepsilon)}\Psi^{a+1}). \end{aligned}$$

Denoting by $\mathbb{E}_\alpha$ the partial expectation over only row α of X (i.e., conditional on $X_{\beta j}$ for all $\beta \neq \alpha$), we have

$$\mathbb{E}_\alpha \left[\frac{1}{N} \sum_k G_{ik}^{(\alpha)} \check{G}_{jk}^{(\alpha)} - \sum_{k,l} G_{ik}^{(\alpha)} X_{\alpha k} \check{G}_{jl}^{(\alpha)} X_{\alpha l} \right] = 0,$$

while the remainder term remains $O_{\prec}(N^{2(-1/3+\varepsilon)}\Psi^{a+1})$ by Lemma D.2, where the moment condition of Lemma D.2 is verified by Lemma 5.8, the moment assumption on Y , and Cauchy–Schwarz. Then the first statement follows. The second statement follows from applying this with $i = j$ and averaging over $i \in \mathcal{I}_N$. The third statement follows from integrating over $y \in [s_1, s_2]$ and noting $N^{1/3+\varepsilon} N^{2(-1/3+\varepsilon)} = \Psi$ as in Lemma 5.5. (If Y also depends on the spectral parameter $z(y)$, we evaluate m_N and $\check{m}_N$ at a different parameter $\tilde{y}$ and integrate over $\tilde{y}$.) $\square$

5.3. *Proof of resolvent comparison.* We use the notation of Sections 5.1 and 5.2.

The proof of Theorem 5.1 is a lengthy computation using the preceding lemmas. To help organize the various terms which appear in this computation, we denote them as $\mathfrak{X}_{k,*}$ for $k = 3, 4$ and $*$ a label describing the form of this term. The meaning of the index $k \in \{3, 4\}$ is to denote that the typical size of this term $\mathfrak{X}_{k,*}$ is at most $O_{\prec}(\Psi^k)$ —this is verified from Lemmas 5.2 and 5.5. We choose the label $*$ to indicate the form of this term: Roughly speaking, 1 indicates a term $m_N - m_*$, 2, 3, or 4 indicate a product of 2, 3, or 4 resolvent entries G_{ij} , the mark $'$ indicates that a resolvent entry is squared, and the superscript $\sim$ denotes that this quantity is contained inside $\text{Im } f$. (A small exception is made for the notation $\mathfrak{X}_{4,22'}$, which has the term $(m_N - m_*)^2$.) All of these terms depend implicitly on a fixed index $i \in \mathcal{I}_N$ and $y \in [s_1, s_2]$, which we omit for notational brevity.

$$\begin{aligned}
 \mathfrak{X}_{3,12'} &= K'(\mathfrak{X})(m_N - m_*) \frac{1}{N} \sum_k G_{ik}^2, \\
 \mathfrak{X}_{3,3} &= K'(\mathfrak{X}) \frac{1}{N^2} \sum_{k,l} G_{ik} G_{kl} G_{il}, \\
 \mathfrak{X}_{3,2\tilde{2}} &= K''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,k,l} G_{ik} G_{il} \text{Im} \int \tilde{G}_{jk} \tilde{G}_{jl}, \\
 \mathfrak{X}_{3,2'\tilde{2}'} &= K''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,k,l} G_{ik}^2 \text{Im} \int \tilde{G}_{jl}^2, \\
 \mathfrak{X}_{4,22'} &= K'(\mathfrak{X})(m_N - m_*)^2 \frac{1}{N} \sum_k G_{ik}^2, \\
 \mathfrak{X}_{4,13} &= K'(\mathfrak{X})(m_N - m_*) \frac{1}{N^2} \sum_{k,l} G_{ik} G_{kl} G_{il}, \\
 \mathfrak{X}_{4,4} &= K'(\mathfrak{X}) \frac{1}{N^3} \sum_{j,k,l} G_{ij} G_{jk} G_{kl} G_{il}, \\
 \mathfrak{X}_{4,4'} &= K'(\mathfrak{X}) \frac{1}{N^3} \sum_{j,k,l} G_{ik}^2 G_{jl}^2, \\
 \mathfrak{X}_{4,12\tilde{2}} &= K''(\mathfrak{X})(m_N - m_*) \frac{1}{N^2} \sum_{j,k,l} G_{ik} G_{il} \text{Im} \int \tilde{G}_{jk} \tilde{G}_{jl}, \\
 \mathfrak{X}_{4,12'\tilde{2}'} &= K''(\mathfrak{X})(m_N - m_*) \frac{1}{N^2} \sum_{j,k,l} G_{ik}^2 \text{Im} \int \tilde{G}_{jl}^2, \\
 \mathfrak{X}_{4,3\tilde{2}} &= K''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,p,q,r} G_{ip} G_{iq} G_{pr} \text{Im} \int \tilde{G}_{jq} \tilde{G}_{jr}, \\
 \mathfrak{X}_{4,3'\tilde{2}} &= K''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,p,q,r} G_{ir}^2 G_{pq} \text{Im} \int \tilde{G}_{jp} \tilde{G}_{jq}, \\
 \mathfrak{X}_{4,3\tilde{2}'} &= K''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,p,q,r} G_{iq} G_{ir} G_{qr} \text{Im} \int \tilde{G}_{jp}^2,
 \end{aligned}$$

$$\begin{aligned}
\mathfrak{X}_{4,2\tilde{1}\tilde{2}} &= K''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,k,l} G_{ik} G_{il} \operatorname{Im} \int (\tilde{m}_N - m_*) \tilde{G}_{jk} \tilde{G}_{jl}, \\
\mathfrak{X}_{4,2\tilde{1}\tilde{2}'} &= K''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,k,l} G_{il}^2 \operatorname{Im} \int (\tilde{m}_N - m_*) \tilde{G}_{jk}^2, \\
\mathfrak{X}_{4,2\tilde{3}} &= K''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,p,q,r} G_{ip} G_{iq} \operatorname{Im} \int \tilde{G}_{jp} \tilde{G}_{jr} \tilde{G}_{qr}, \\
\mathfrak{X}_{4,2\tilde{3}'} &= K''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,p,q,r} G_{ip} G_{iq} \operatorname{Im} \int \tilde{G}_{jr}^2 \tilde{G}_{pq}, \\
\mathfrak{X}_{4,2\tilde{3}} &= K''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,p,q,r} G_{ip}^2 \operatorname{Im} \int \tilde{G}_{jq} \tilde{G}_{jr} \tilde{G}_{qr}, \\
\mathfrak{X}_{4,2\tilde{2}\tilde{2}} &= K'''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,k,p,q,r} G_{ip} G_{iq} \left(\operatorname{Im} \int \tilde{G}_{jp} \tilde{G}_{jr} \right) \left(\operatorname{Im} \int \tilde{G}_{kq} \tilde{G}_{kr} \right), \\
\mathfrak{X}_{4,2\tilde{2}\tilde{2}'} &= K'''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,k,p,q,r} G_{ip}^2 \left(\operatorname{Im} \int \tilde{G}_{jq} \tilde{G}_{jr} \right) \left(\operatorname{Im} \int \tilde{G}_{kq} \tilde{G}_{kr} \right), \\
\mathfrak{X}_{4,2\tilde{2}\tilde{2}'} &= K'''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,k,p,q,r} G_{ip} G_{iq} \left(\operatorname{Im} \int \tilde{G}_{jp} \tilde{G}_{jq} \right) \left(\operatorname{Im} \int \tilde{G}_{kr}^2 \right), \\
\mathfrak{X}_{4,2\tilde{2}\tilde{2}'} &= K'''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,k,p,q,r} G_{ip}^2 \left(\operatorname{Im} \int \tilde{G}_{jq}^2 \right) \left(\operatorname{Im} \int \tilde{G}_{kr}^2 \right).
\end{aligned}$$

We define the aggregate quantities

$$\begin{aligned}
\mathfrak{X}_3 &= \mathfrak{X}_{3,12'} + \mathfrak{X}_{3,3} + \mathfrak{X}_{3,2\tilde{2}}, \\
\mathfrak{X}_4 &= 3\mathfrak{X}_{4,22'} + 6\mathfrak{X}_{4,13} + 12\mathfrak{X}_{4,4} + 3\mathfrak{X}_{4,4'} + 4\mathfrak{X}_{4,12\tilde{2}} + 8\mathfrak{X}_{4,3\tilde{2}} + 4\mathfrak{X}_{4,3'\tilde{2}} \\
&\quad + 2\mathfrak{X}_{4,2\tilde{1}\tilde{2}} + 2\mathfrak{X}_{4,2\tilde{3}'} + 4\mathfrak{X}_{4,2\tilde{3}} + 4\mathfrak{X}_{4,2\tilde{2}\tilde{2}}, \\
\mathfrak{X}_4^- &= \mathfrak{X}_{4,2\tilde{1}\tilde{2}} + \mathfrak{X}_{4,2\tilde{3}'} + 2\mathfrak{X}_{4,2\tilde{3}} - \mathfrak{X}_{4,12\tilde{2}} - \mathfrak{X}_{4,3'\tilde{2}} - 2\mathfrak{X}_{4,3\tilde{2}}.
\end{aligned}$$

Theorem 5.1 is a consequence of the following two technical lemmas. (There are several terms $\mathfrak{X}_{3,*}$ and $\mathfrak{X}_{4,*}$ above which do not appear in the aggregate quantities $\mathfrak{X}_3$, $\mathfrak{X}_4$, $\mathfrak{X}_4^-$ or in the statements of these lemmas. They appear in the intermediate calculations in the proofs, and for convenience we have collected all required definitions above.)

LEMMA 5.10 (Decoupling). *Under the assumptions of Theorem 5.1, denote $\mathfrak{X}_\lambda = \lambda\mathfrak{X} + (1-\lambda)\check{\mathfrak{X}}$ for $\lambda \in [0, 1]$. For fixed $i \in \mathcal{I}_N$ and $y \in [s_1, s_2]$, define $\mathfrak{X}_3$, $\mathfrak{X}_4$, and $\mathfrak{X}_4^-$ as above. For fixed $\alpha \in \mathcal{I}_M$, let $s_\alpha = (1 + t_\alpha m_*)^{-1}$ and $\check{s}_\alpha = (1 + \check{t}_\alpha \check{m}_*)^{-1}$, define $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ as in (28), and*

$$\mathcal{R}_\alpha = s_\alpha \check{s}_\alpha (t_\alpha s_\alpha - \check{t}_\alpha \check{s}_\alpha)^2.$$

Then

$$\begin{aligned}
&\int_0^1 \mathbb{E} \left[K'(\mathfrak{X}_\lambda) \frac{G_{i\alpha}}{t_\alpha} \frac{\check{G}_{i\alpha}}{\check{t}_\alpha} \right] d\lambda \\
&= s_\alpha \check{s}_\alpha \int_0^1 \mathbb{E} \left[K'(\mathfrak{X}_\lambda) \frac{1}{N} \sum_k G_{ik} \check{G}_{ik} \right] d\lambda
\end{aligned}$$

$$- \mathcal{P}_\alpha \mathbb{E}[\mathfrak{X}_3] + \frac{1}{3} \mathcal{Q}_\alpha \mathbb{E}[\mathfrak{X}_4] + \frac{1}{3} \mathcal{R}_\alpha \mathbb{E}[\mathfrak{X}_4^-] + O_{\prec}(\Psi^5).$$

LEMMA 5.11 (Optical theorems). *Under the assumptions of Theorem 5.1, for fixed $i \in \mathcal{I}_N$ and $y \in [s_1, s_2]$, define $\mathfrak{X}_3$ and $\mathfrak{X}_4$ as above. Let $A_4 = N^{-1} \sum_{\alpha} t_{\alpha}^4 s_{\alpha}^4$. Then*

$$2 \operatorname{Im} \mathbb{E}[\mathfrak{X}_3] = (A_4 - m_*^{-4}) \operatorname{Im} \mathbb{E}[\mathfrak{X}_4] + O_{\prec}(\Psi^5).$$

Lemma 5.10 generalizes [19], Lemma 6.2, to a swappable pair. We will present its proof in Section 5.4. We introduce the interpolation $\mathfrak{X}_{\lambda} = \lambda \mathfrak{X} + (1 - \lambda) \check{\mathfrak{X}}$ as a device to bound $K(\mathfrak{X}) - K(\check{\mathfrak{X}})$. (This is different from a continuous interpolation between the entries of T and $\check{T}$.) Let us make several additional remarks:

1. The proof in [19] requires this lemma in “differential form”, where $T = \check{T}$. In this case, we have $G = \check{G}$, $\mathfrak{X}_{\lambda} = \mathfrak{X}$ for every $\lambda \in [0, 1]$, $s_{\alpha} = \check{s}_{\alpha}$, and $t_{\alpha} = \check{t}_{\alpha}$. Then the integral over λ is irrelevant, and Lemma 5.10 reduces to the full version of [19], Lemma 6.2.

2. The term $\mathfrak{X}_4^-$ does not appear in [19] and is not canceled by the optical theorems of Lemma 5.11. (When $T = \check{T}$, we have $\mathcal{R}_{\alpha} = 0$ so this term is not present.) The cancellation instead occurs by symmetry of its definition, upon integrating over y : Momentarily writing $\mathfrak{X}_{k,*}$ as $\mathfrak{X}_{k,*}(y)$, and noting that $K(\mathfrak{X})$ is real-valued, we obtain

$$(39) \quad \operatorname{Im} \int \mathfrak{X}_{4,2\bar{1}\bar{2}}(\tilde{y}) d\tilde{y} = \operatorname{Im} \int \mathfrak{X}_{4,1\bar{2}\bar{2}}(\tilde{y}) d\tilde{y}$$

from the symmetric definition of these two terms. A similar cancellation occurs for the pairs $(\mathfrak{X}_{4,2\bar{3}}, \mathfrak{X}_{4,3\bar{2}})$ and $(\mathfrak{X}_{4,2\bar{3}}, \mathfrak{X}_{4,3\bar{2}}^-)$ which comprise $\mathfrak{X}_4^-$.

3. An important simplification in the proof is that we may use Lemmas 5.7 and 5.9 to convert $O_{\prec}(\Psi^3)$ and $O_{\prec}(\Psi^4)$ terms to involve only G and not $\check{G}$ —hence $\mathfrak{X}_3$, $\mathfrak{X}_4$, $\mathfrak{X}_4^-$ are defined only by T and not $\check{T}$.

The other technical ingredient, Lemma 5.11, is identical to the full version of [19], Lemma B.1, as the terms $\mathfrak{X}_3$ and $\mathfrak{X}_4$ depend only on the single matrix T . We briefly discuss the breakdown of its proof in Section 5.5.

In [19], for expositional clarity, these lemmas were stated and proven only in the special case $K' \equiv 1$. Full proofs were presented for an analogous deformed Wigner model in [18]. Although more cumbersome, we will demonstrate the full proof of Lemma 5.10 for general K in Section 5.4, as much of the additional complexity in our calculation due to two resolvents G and $\check{G}$ arises from the interpolation $\mathfrak{X}_{\lambda}$ and the Taylor expansion of K' .

We establish Theorem 5.1 using the above two results:

PROOF OF THEOREM 5.1. We write

$$(40) \quad K(\mathfrak{X}) - K(\check{\mathfrak{X}}) = \int_0^1 \frac{d}{d\lambda} K(\mathfrak{X}_{\lambda}) d\lambda = \int_0^1 K'(\mathfrak{X}_{\lambda})(\mathfrak{X} - \check{\mathfrak{X}}) d\lambda.$$

Recalling $\mathfrak{X} = \sum_i \operatorname{Im} \int \tilde{G}_{ii}$ and applying (37),

$$\mathfrak{X} - \check{\mathfrak{X}} = \sum_i \operatorname{Im} \int \left(\sum_k \tilde{G}_{ik} \check{\tilde{G}}_{ik}(E_* - \check{E}_*) - \sum_{\alpha} \frac{\tilde{G}_{i\alpha}}{t_{\alpha}} \frac{\check{\tilde{G}}_{i\alpha}}{\check{t}_{\alpha}} (t_{\alpha} - \check{t}_{\alpha}) \right).$$

($\tilde{G}$ and $\check{\tilde{G}}$ denote G and $\check{G}$ evaluated at the variable of integration $\tilde{y}$.) Further applying (26), Lemma 5.2, and the trivial bound $N^{-2/3+\varepsilon} \prec \Psi^2$,

$$\mathfrak{X} - \check{\mathfrak{X}} = \sum_i \operatorname{Im} \int \sum_{\alpha} (t_{\alpha} - \check{t}_{\alpha}) \left(s_{\alpha} \check{s}_{\alpha} \frac{1}{N} \sum_k \tilde{G}_{ik} \check{\tilde{G}}_{ik} - \frac{\tilde{G}_{i\alpha}}{t_{\alpha}} \frac{\check{\tilde{G}}_{i\alpha}}{\check{t}_{\alpha}} \right) + O_{\prec}(\Psi^4).$$

Applying this to (40), taking the expectation, exchanging orders of summation and integration, and noting that $K'(\mathfrak{X}_\lambda)$ is real,

$$\begin{aligned} & \mathbb{E}[K(\mathfrak{X}) - K(\check{\mathfrak{X}})] \\ &= \sum_i \sum_\alpha (t_\alpha - \check{t}_\alpha) \operatorname{Im} \int \int_0^1 \mathbb{E} \left[K'(\mathfrak{X}_\lambda) \left(s_\alpha \check{s}_\alpha \frac{1}{N} \sum_k \tilde{G}_{ik} \tilde{G}_{ik} - \frac{\tilde{G}_{i\alpha}}{t_\alpha} \frac{\tilde{G}_{i\alpha}}{\check{t}_\alpha} \right) \right] d\lambda d\tilde{y} \\ & \quad + O_{\prec}(\Psi^4), \end{aligned}$$

where the expectation of the remainder term is still $O_{\prec}(\Psi^4)$ by Lemmas D.2 and 5.8. Denoting by $\tilde{\mathfrak{X}}_3(i)$, $\tilde{\mathfrak{X}}_4(i)$, and $\tilde{\mathfrak{X}}_4^-(i)$ the quantities $\mathfrak{X}_3$, $\mathfrak{X}_4$, and $\mathfrak{X}_4^-$ defined by $\tilde{y}$ and the outer index of summation i , Lemma 5.10 implies

$$\begin{aligned} & \mathbb{E}[K(\mathfrak{X}) - K(\check{\mathfrak{X}})] \\ &= \sum_i \sum_\alpha (t_\alpha - \check{t}_\alpha) \operatorname{Im} \int \left(\mathcal{P}_\alpha \mathbb{E}[\tilde{\mathfrak{X}}_3(i)] - \frac{1}{3} \mathcal{Q}_\alpha \mathbb{E}[\tilde{\mathfrak{X}}_4(i)] - \frac{1}{3} \mathcal{R}_\alpha \mathbb{E}[\tilde{\mathfrak{X}}_4^-(i)] \right) d\tilde{y} \\ & \quad + O_{\prec}(N^{1/3+\varepsilon} \Psi^5), \end{aligned}$$

where the error is $N^{1/3+\varepsilon} \Psi^5$ because $\sum_\alpha |t_\alpha - \check{t}_\alpha| \leq C$ and the range of integration is contained in $[-N^{-2/3+\varepsilon}, N^{-2/3+\varepsilon}]$. We note, from the identity (39) and the analogous cancellation for the other two pairs of terms, that $\operatorname{Im} \int \tilde{\mathfrak{X}}_4^-(i) d\tilde{y} = 0$, so this term vanishes. Then, applying Lemma 5.11,

$$\begin{aligned} & \mathbb{E}[K(\mathfrak{X}) - K(\check{\mathfrak{X}})] \\ (41) \quad &= \sum_i \sum_\alpha (t_\alpha - \check{t}_\alpha) \left(\mathcal{P}_\alpha \frac{A_4 - m_*^{-4}}{2} - \frac{\mathcal{Q}_\alpha}{3} \right) \operatorname{Im} \int \mathbb{E}[\tilde{\mathfrak{X}}_4(i)] d\tilde{y} + O_{\prec}(N^{1/3+\varepsilon} \Psi^5). \end{aligned}$$

Finally, applying Lemma 4.7, we have

$$(42) \quad \sum_\alpha (t_\alpha - \check{t}_\alpha) \left(\mathcal{P}_\alpha \frac{A_4 - m_*^{-4}}{2} - \frac{\mathcal{Q}_\alpha}{3} \right) \leq C/N.$$

Thus the first term of (41) is of size $O_{\prec}(N \cdot 1/N \cdot N^{-2/3+\varepsilon} \cdot \Psi^4)$, which is of smaller order than the remainder $N^{1/3+\varepsilon} \Psi^5$. (In [19] for the differential version of Lemma 5.10, this first term is zero due to the exact cancellation of the analogue of (42).) Hence $\mathbb{E}[K(\mathfrak{X}) - K(\check{\mathfrak{X}})] \prec N^{1/3+\varepsilon} \Psi^5 = N^{-4/3+16\varepsilon}$. $\square$

5.4. Proof of decoupling lemma. In this section, we prove Lemma 5.10. We will implicitly use the resolvent bounds of Lemma 5.2 throughout.

Step 1: Consider first a fixed value $\lambda \in [0, 1]$. Let $\mathbb{E}_\alpha$ denote the partial expectation over row α of X (i.e., conditional on all $X_{\beta j}$ for $\beta \neq \alpha$). In anticipation of computing $\mathbb{E}_\alpha$ for the quantity on the left, we expand

$$K'(\mathfrak{X}_\lambda) \frac{G_{i\alpha}}{t_\alpha} \frac{\check{G}_{i\alpha}}{\check{t}_\alpha}$$

as a polynomial of entries of row α of X , with coefficients independent of all entries in this row.

Applying the resolvent identities,

$$\frac{G_{i\alpha}}{t_\alpha} = \frac{G_{\alpha\alpha}}{t_\alpha} \sum_k G_{ik}^{(\alpha)} X_{\alpha k} = - \frac{1}{1 + t_\alpha \sum_{p,q} G_{pq}^{(\alpha)} X_{\alpha p} X_{\alpha q}} \sum_k G_{ik}^{(\alpha)} X_{\alpha k}.$$

Applying Lemma 5.3 and a Taylor expansion of the function $(1 + t_\alpha x)^{-1}$ around $x = m_*$,

$$(43) \quad \begin{aligned} \frac{G_{i\alpha}}{t_\alpha} &= -s_\alpha \sum_k G_{ik}^{(\alpha)} X_{\alpha k} + t_\alpha s_\alpha^2 \left(\sum_{p,q} G_{pq}^{(\alpha)} X_{\alpha p} X_{\alpha q} - m_* \right) \sum_k G_{ik}^{(\alpha)} X_{\alpha k} \\ &\quad - t_\alpha^2 s_\alpha^3 \left(\sum_{p,q} G_{pq}^{(\alpha)} X_{\alpha p} X_{\alpha q} - m_* \right)^2 \sum_k G_{ik}^{(\alpha)} X_{\alpha k} + O_{\prec}(\Psi^4) \\ &\equiv U_1 + U_2 + U_3 + O_{\prec}(\Psi^4), \end{aligned}$$

where we defined the three explicit terms of sizes $O_{\prec}(\Psi)$, $O_{\prec}(\Psi^2)$, $O_{\prec}(\Psi^3)$ as U_1, U_2, U_3 . Similarly

$$(44) \quad \frac{\check{G}_{i\alpha}}{\check{t}_\alpha} = \check{U}_1 + \check{U}_2 + \check{U}_3 + O_{\prec}(\Psi^4),$$

where $\check{U}_i$ are defined analogously with $\check{s}_\alpha, \check{t}_\alpha, \check{m}_*, \check{G}$ in place of $s_\alpha, t_\alpha, m_*, G$.

For $K'(\mathfrak{X}_\lambda)$, define $\mathfrak{X}_\lambda^{(\alpha)} = \lambda \mathfrak{X}^{(\alpha)} + (1 - \lambda) \check{\mathfrak{X}}^{(\alpha)}$ and note from Lemma 5.6 that $\mathfrak{X}_\lambda - \mathfrak{X}_\lambda^{(\alpha)} \prec \Psi$. Taylor expanding $K'(x)$ around $x = \mathfrak{X}_\lambda^{(\alpha)}$,

$$(45) \quad K'(\mathfrak{X}_\lambda) = K'(\mathfrak{X}_\lambda^{(\alpha)}) + K''(\mathfrak{X}_\lambda^{(\alpha)})(\mathfrak{X}_\lambda - \mathfrak{X}_\lambda^{(\alpha)}) + \frac{K'''(\mathfrak{X}_\lambda^{(\alpha)})}{2}(\mathfrak{X}_\lambda - \mathfrak{X}_\lambda^{(\alpha)})^2 + O_{\prec}(\Psi^3).$$

Applying the definition of $\mathfrak{X}, \mathfrak{X}^{(\alpha)}$ and the resolvent identities,

$$\mathfrak{X} - \mathfrak{X}^{(\alpha)} = \text{Im} \int \sum_j (\tilde{G}_{jj} - \tilde{G}_{jj}^{(\alpha)}) = \text{Im} \int \sum_j \frac{\tilde{G}_{j\alpha}^2}{\tilde{G}_{\alpha\alpha}} = \text{Im} \int \tilde{G}_{\alpha\alpha} \sum_{j,p,q} \tilde{G}_{jp}^{(\alpha)} X_{\alpha p} \tilde{G}_{jq}^{(\alpha)} X_{\alpha q}.$$

Further applying the resolvent identity for $\tilde{G}_{\alpha\alpha}$, a Taylor expansion as above, and Lemma 5.5,

$$(46) \quad \begin{aligned} \mathfrak{X} - \mathfrak{X}^{(\alpha)} &= -t_\alpha s_\alpha \text{Im} \int \sum_{j,p,q} \tilde{G}_{jp}^{(\alpha)} X_{\alpha p} \tilde{G}_{jq}^{(\alpha)} X_{\alpha q} \\ &\quad + t_\alpha^2 s_\alpha^2 \text{Im} \int \sum_{r,s} (\tilde{G}_{rs}^{(\alpha)} X_{\alpha r} X_{\alpha s} - m_*) \sum_{j,p,q} \tilde{G}_{jp}^{(\alpha)} X_{\alpha p} \tilde{G}_{jq}^{(\alpha)} X_{\alpha q} + O_{\prec}(\Psi^3) \\ &\equiv V_1 + V_2 + O_{\prec}(\Psi^3), \end{aligned}$$

where $V_1 \prec \Psi$ and $V_2 \prec \Psi^2$. Analogously we may write

$$(47) \quad \check{\mathfrak{X}} - \check{\mathfrak{X}}^{(\alpha)} = \check{V}_1 + \check{V}_2 + O_{\prec}(\Psi^3),$$

where $\check{V}_1, \check{V}_2$ are defined with $\check{s}_\alpha, \check{t}_\alpha, \check{m}_*, \check{G}$ in place of $s_\alpha, t_\alpha, m_*, G$. Substituting (46) and (47) into (45), and combining with (43) and (44), we obtain

$$(48) \quad K'(\mathfrak{X}_\lambda) \frac{G_{i\alpha}}{t_\alpha} \frac{\check{G}_{i\alpha}}{\check{t}_\alpha} = W_2 + W_3 + W_4 + O_{\prec}(\Psi^5),$$

where the $O_{\prec}(\Psi^2), O_{\prec}(\Psi^3), O_{\prec}(\Psi^4)$ terms are respectively

$$\begin{aligned} W_2 &= K'(\mathfrak{X}_\lambda^{(\alpha)}) U_1 \check{U}_1, \\ W_3 &= K'(\mathfrak{X}_\lambda^{(\alpha)}) (U_2 \check{U}_1 + U_1 \check{U}_2) + K''(\mathfrak{X}_\lambda^{(\alpha)}) (\lambda V_1 + (1 - \lambda) \check{V}_1) U_1 \check{U}_1, \\ W_4 &= K'(\mathfrak{X}_\lambda^{(\alpha)}) (U_3 \check{U}_1 + U_2 \check{U}_2 + U_1 \check{U}_3) + K''(\mathfrak{X}_\lambda^{(\alpha)}) (\lambda V_1 + (1 - \lambda) \check{V}_1) (U_2 \check{U}_1 + U_1 \check{U}_2) \\ &\quad + \left[K''(\mathfrak{X}_\lambda^{(\alpha)}) (\lambda V_2 + (1 - \lambda) \check{V}_2) + \frac{K'''(\mathfrak{X}_\lambda^{(\alpha)})}{2} (\lambda V_1 + (1 - \lambda) \check{V}_1)^2 \right] U_1 \check{U}_1. \end{aligned}$$

Step 2: We compute $\mathbb{E}_\alpha$ of W_2, W_3, W_4 above. Note that $\mathfrak{X}^{(\alpha)}, \check{\mathfrak{X}}^{(\alpha)}, G^{(\alpha)}, \check{G}^{(\alpha)}$ are independent of row α of X . Then for W_2 , we have

$$\begin{aligned} \mathbb{E}_\alpha[W_2] &= s_\alpha \check{s}_\alpha K'(\mathfrak{X}_\lambda^{(\alpha)}) \sum_{k,l} G_{ik}^{(\alpha)} \check{G}_{il}^{(\alpha)} \mathbb{E}_\alpha[X_{\alpha k} X_{\alpha l}] \\ (49) \quad &= s_\alpha \check{s}_\alpha K'(\mathfrak{X}_\lambda^{(\alpha)}) \frac{1}{N} \sum_k G_{ik}^{(\alpha)} \check{G}_{ik}^{(\alpha)}, \end{aligned}$$

where we have used $\mathbb{E}[X_{\alpha k} X_{\alpha l}] = 1/N$ if $k = l$ and 0 otherwise.

For W_3 , let us introduce

$$\begin{aligned} \mathfrak{Y}_{3,12'}^{(\alpha)} &= K'(\mathfrak{X}_\lambda^{(\alpha)}) (m_N^{(\alpha)} - m_*) \frac{1}{N} \sum_k G_{ik}^{(\alpha)} \check{G}_{ik}^{(\alpha)}, \\ \mathcal{Z}_{3,12'}^{(\alpha)} &= K'(\mathfrak{X}_\lambda^{(\alpha)}) (\check{m}_N^{(\alpha)} - \check{m}_*) \frac{1}{N} \sum_k G_{ik}^{(\alpha)} \check{G}_{ik}^{(\alpha)}, \\ \mathfrak{Y}_{3,3}^{(\alpha)} &= K'(\mathfrak{X}_\lambda^{(\alpha)}) \frac{1}{N^2} \sum_{k,l} G_{ik}^{(\alpha)} G_{kl}^{(\alpha)} \check{G}_{il}^{(\alpha)}, \\ \mathcal{Z}_{3,3}^{(\alpha)} &= K'(\mathfrak{X}_\lambda^{(\alpha)}) \frac{1}{N^2} \sum_{k,l} G_{ik}^{(\alpha)} \check{G}_{kl}^{(\alpha)} \check{G}_{il}^{(\alpha)}, \\ \mathfrak{Y}_{3,2\tilde{2}'}^{(\alpha)} &= K''(\mathfrak{X}_\lambda^{(\alpha)}) \frac{1}{N^2} \sum_{j,k,l} G_{ik}^{(\alpha)} \check{G}_{ik}^{(\alpha)} \operatorname{Im} \int (\tilde{G}_{jl}^{(\alpha)})^2, \\ \mathcal{Z}_{3,2\tilde{2}'}^{(\alpha)} &= K''(\mathfrak{X}_\lambda^{(\alpha)}) \frac{1}{N^2} \sum_{j,k,l} G_{ik}^{(\alpha)} \check{G}_{ik}^{(\alpha)} \operatorname{Im} \int (\tilde{\check{G}}_{jl}^{(\alpha)})^2, \\ \mathfrak{Y}_{3,2\tilde{2}}^{(\alpha)} &= K''(\mathfrak{X}_\lambda^{(\alpha)}) \frac{1}{N^2} \sum_{j,k,l} G_{ik}^{(\alpha)} \check{G}_{il}^{(\alpha)} \operatorname{Im} \int \tilde{G}_{jk}^{(\alpha)} \tilde{G}_{jl}^{(\alpha)}, \\ \mathcal{Z}_{3,2\tilde{2}}^{(\alpha)} &= K''(\mathfrak{X}_\lambda^{(\alpha)}) \frac{1}{N^2} \sum_{j,k,l} G_{ik}^{(\alpha)} \check{G}_{il}^{(\alpha)} \operatorname{Im} \int \tilde{\check{G}}_{jk}^{(\alpha)} \tilde{\check{G}}_{jl}^{(\alpha)}, \end{aligned}$$

which are versions of $\mathfrak{X}_{3,*}$ that don't depend on row α of X and with various instances of $m_N, m_*, G, \mathfrak{X}$ replaced by $\check{m}_N, \check{m}_*, \check{G}, \mathfrak{X}_\lambda$. Consider the first term of W_3 and write

$$\begin{aligned} &\mathbb{E}_\alpha[K'(\mathfrak{X}_\lambda^{(\alpha)}) U_2 \check{U}_1] \\ &= \mathbb{E}_\alpha \left[-t_\alpha s_\alpha^2 \check{s}_\alpha K'(\mathfrak{X}_\lambda^{(\alpha)}) \left(\sum_{p,q} G_{pq}^{(\alpha)} X_{\alpha p} X_{\alpha q} - m_* \right) \sum_{k,l} G_{ik}^{(\alpha)} X_{\alpha k} \check{G}_{il}^{(\alpha)} X_{\alpha l} \right] \\ &= -t_\alpha s_\alpha^2 \check{s}_\alpha K'(\mathfrak{X}_\lambda^{(\alpha)}) \sum_{k,l,p,q} \left(G_{pq}^{(\alpha)} \mathbb{E}_\alpha[X_{\alpha p} X_{\alpha q} X_{\alpha k} X_{\alpha l}] \right. \\ &\quad \left. - \frac{1}{N} m_* \mathbb{1}\{p=q\} \mathbb{E}_\alpha[X_{\alpha k} X_{\alpha l}] \right) G_{ik}^{(\alpha)} \check{G}_{il}^{(\alpha)}. \end{aligned}$$

The summand corresponding to (k, l, p, q) is 0 unless each distinct index appears at least twice in (k, l, p, q) . Furthermore, the case where all four indices are equal is negligible:

$$\sum_k \left(G_{kk}^{(\alpha)} \mathbb{E}_\alpha[X_{\alpha k}^4] - \frac{1}{N} m_* \mathbb{E}_\alpha[X_{\alpha k}^2] \right) G_{ik}^{(\alpha)} \check{G}_{ik}^{(\alpha)} < N \cdot N^{-2} \cdot \Psi^2 < \Psi^5.$$

(The $k = i$ case of the sum may be bounded separately as $O_{\prec}(N^{-2})$.) Thus up to $O_{\prec}(\Psi^5)$, we need only consider summands where each distinct index appears exactly twice. Considering the one case where $k = l$ and the two cases where $k = p$ and $k = q$,

$$\begin{aligned} \mathbb{E}_{\alpha}[K'(\mathfrak{X}_{\lambda}^{(\alpha)})U_2\check{U}_1] \\ = -t_{\alpha}s_{\alpha}^2\check{s}_{\alpha}K'(\mathfrak{X}_{\lambda}^{(\alpha)})\left(\frac{1}{N^2}\sum_k\sum_p^{(k)}(G_{pp}^{(\alpha)}-m_*)G_{ik}^{(\alpha)}\check{G}_{ik}^{(\alpha)}+\frac{2}{N^2}\sum_k\sum_l^{(k)}G_{ik}^{(\alpha)}\check{G}_{il}^{(\alpha)}G_{kl}^{(\alpha)}\right) \\ + O_{\prec}(\Psi^5). \end{aligned}$$

Re-including $p = k$ and $l = k$ into the double summations introduces an additional $O_{\prec}(\Psi^5)$ error; hence we obtain for the first term of W_3

$$(50) \quad \mathbb{E}_{\alpha}[K'(\mathfrak{X}_{\lambda}^{(\alpha)})U_2\check{U}_1] = -t_{\alpha}s_{\alpha}^2\check{s}_{\alpha}(\mathfrak{Y}_{3,12'}^{(\alpha)} + 2\mathfrak{Y}_{3,3}^{(\alpha)}) + O_{\prec}(\Psi^5).$$

Similar arguments apply for the remaining three terms of W_3 . For the terms involving an integral, we may apply Lemma 5.5 and also move $X_{\alpha k}$ outside of the integral and imaginary part because X is real and does not depend on the variable of integration $\tilde{y}$. We obtain

$$(51) \quad \mathbb{E}_{\alpha}[K'(\mathfrak{X}_{\lambda}^{(\alpha)})U_1\check{U}_2] = -\check{t}_{\alpha}\check{s}_{\alpha}^2s_{\alpha}(\mathcal{Z}_{3,12'}^{(\alpha)} + 2\mathcal{Z}_{3,3}^{(\alpha)}) + O_{\prec}(\Psi^5),$$

$$(52) \quad \mathbb{E}_{\alpha}[\lambda K''(\mathfrak{X}_{\lambda}^{(\alpha)})V_1U_1\check{U}_1] = -\lambda t_{\alpha}s_{\alpha}^2\check{s}_{\alpha}(\mathfrak{Y}_{3,2'\tilde{2}'}^{(\alpha)} + 2\mathfrak{Y}_{3,2\tilde{2}}^{(\alpha)}) + O_{\prec}(\Psi^5),$$

$$(53) \quad \mathbb{E}_{\alpha}[(1-\lambda)K''(\mathfrak{X}_{\lambda}^{(\alpha)})\check{V}_1U_1\check{U}_1] = -(1-\lambda)\check{t}_{\alpha}\check{s}_{\alpha}^2s_{\alpha}(\mathcal{Z}_{3,2'\tilde{2}'}^{(\alpha)} + 2\mathcal{Z}_{3,2\tilde{2}}^{(\alpha)}) + O_{\prec}(\Psi^5),$$

and $\mathbb{E}_{\alpha}[W_3]$ is the sum of (50–53).

For W_4 , consider the first term and write

$$\begin{aligned} \mathbb{E}_{\alpha}[K'(\mathfrak{X}_{\lambda}^{(\alpha)})U_3\check{U}_1] \\ = \mathbb{E}_{\alpha}\left[t_{\alpha}^2s_{\alpha}^3\check{s}_{\alpha}K'(\mathfrak{X}_{\lambda}^{(\alpha)})\left(\sum_{p,q}G_{pq}^{(\alpha)}X_{\alpha p}X_{\alpha q}-m_*\right)^2\sum_{k,l}G_{ik}^{(\alpha)}X_{\alpha k}\check{G}_{il}^{(\alpha)}X_{\alpha l}\right] \\ = t_{\alpha}^2s_{\alpha}^3\check{s}_{\alpha}K'(\mathfrak{X}_{\lambda}^{(\alpha)})\sum_{p,q,r,s,k,l}\left(G_{pq}^{(\alpha)}G_{rs}^{(\alpha)}\mathbb{E}_{\alpha}[X_{\alpha p}X_{\alpha q}X_{\alpha r}X_{\alpha s}X_{\alpha k}X_{\alpha l}] \right. \\ \left. - \frac{1}{N}m_*\mathbb{1}\{p=q\}G_{rs}^{(\alpha)}\mathbb{E}_{\alpha}[X_{\alpha r}X_{\alpha s}X_{\alpha k}X_{\alpha l}] - \frac{1}{N}m_*\mathbb{1}\{r=s\}G_{pq}^{(\alpha)}\mathbb{E}_{\alpha}[X_{\alpha p}X_{\alpha q}X_{\alpha k}X_{\alpha l}] \right. \\ \left. + \frac{1}{N^2}m_*^2\mathbb{1}\{p=q\}\mathbb{1}\{r=s\}\mathbb{E}[X_{\alpha k}X_{\alpha l}]\right)G_{ik}^{(\alpha)}\check{G}_{il}^{(\alpha)}. \end{aligned}$$

A summand corresponding to (k, l, p, q, r, s) is 0 unless each distinct index in (k, l, p, q, r, s) appears at least twice. Furthermore, as in the computations for W_3 above, all summands for which (k, l, p, q, r, s) do not form three distinct pairs may be omitted and reincluded after taking $\mathbb{E}_{\alpha}$, introducing an $O_{\prec}(\Psi^5)$ error. Considering all pairings of these indices,

$$\begin{aligned} \mathbb{E}_{\alpha}[K'(\mathfrak{X}_{\lambda}^{(\alpha)})U_3\check{U}_1] \\ = t_{\alpha}^2s_{\alpha}^3\check{s}_{\alpha}K'(\mathfrak{X}_{\lambda}^{(\alpha)})\left((m_N^{(\alpha)}-m_*)^2\frac{1}{N}\sum_kG_{ik}^{(\alpha)}\check{G}_{ik}^{(\alpha)}+4(m_N^{(\alpha)}-m_*)\frac{1}{N^2}\sum_{k,l}G_{ik}^{(\alpha)}G_{kl}^{(\alpha)}\check{G}_{il}^{(\alpha)} \right. \\ \left. + 8\frac{1}{N^3}\sum_{j,k,l}G_{ik}^{(\alpha)}G_{jk}^{(\alpha)}G_{jl}^{(\alpha)}\check{G}_{il}^{(\alpha)}+2\frac{1}{N^3}\sum_{j,k,l}G_{ik}^{(\alpha)}\check{G}_{ik}^{(\alpha)}(G_{jl}^{(\alpha)})^2\right)+O_{\prec}(\Psi^5). \end{aligned}$$

At this point, let us apply Lemmas 5.6 and 5.7 to remove each superscript (α) above and to convert each $\check{G}$ to G , introducing an $O_{<}(\Psi^5)$ error. We may also remove the superscript (α) and convert $\mathfrak{X}_\lambda$ to $\mathfrak{X}$ in $K'(\mathfrak{X}_\lambda^{(\alpha)})$, via the second-derivative bounds

$$\begin{aligned} K'(\mathfrak{X}_\lambda^{(\alpha)}) - K'(\mathfrak{X}_\lambda) &\leq \|K''\|_\infty |\mathfrak{X}_\lambda^{(\alpha)} - \mathfrak{X}_\lambda| < \Psi. \\ K'(\mathfrak{X}_\lambda) - K'(\mathfrak{X}) &\leq \|K''\|_\infty |\mathfrak{X}_\lambda - \mathfrak{X}| < \Psi. \end{aligned}$$

We thus obtain

$$\mathbb{E}_\alpha[K'(\mathfrak{X}_\lambda^{(\alpha)})U_3\check{U}_1] = t_\alpha^2 s_\alpha^3 \check{s}_\alpha (\mathfrak{X}_{4,22'} + 4\mathfrak{X}_{4,13} + 8\mathfrak{X}_{4,4} + 2\mathfrak{X}_{4,4'}) + O_{<}(\Psi^5).$$

Applying a similar computation to each term of W_4 , we obtain

$$\begin{aligned} (54) \quad \mathbb{E}_\alpha[K'(\mathfrak{X}_\lambda^{(\alpha)})(U_3\check{U}_1 + U_2\check{U}_2 + U_1\check{U}_3)] \\ = s_\alpha \check{s}_\alpha (t_\alpha^2 s_\alpha^2 + t_\alpha s_\alpha \check{t}_\alpha \check{s}_\alpha + \check{t}_\alpha^2 \check{s}_\alpha^2) (\mathfrak{X}_{4,22'} + 4\mathfrak{X}_{4,13} + 8\mathfrak{X}_{4,4} + 2\mathfrak{X}_{4,4'}) + O_{<}(\Psi^5), \end{aligned}$$

$$\begin{aligned} (55) \quad \mathbb{E}_\alpha[K''(\mathfrak{X}_\lambda^{(\alpha)})(\lambda V_1 + (1-\lambda)\check{V}_1)(U_2\check{U}_1 + U_1\check{U}_2)] \\ = s_\alpha \check{s}_\alpha (\lambda t_\alpha s_\alpha + (1-\lambda)\check{t}_\alpha \check{s}_\alpha) (t_\alpha s_\alpha + \check{t}_\alpha \check{s}_\alpha) \\ \cdot (\mathfrak{X}_{4,12'\tilde{2}'} + 2\mathfrak{X}_{4,12\tilde{2}} + 2\mathfrak{X}_{4,3\tilde{2}'} + 2\mathfrak{X}_{4,3'\tilde{2}} + 8\mathfrak{X}_{4,3\tilde{2}}) + O_{<}(\Psi^5), \end{aligned}$$

$$\begin{aligned} (56) \quad \mathbb{E}_\alpha[K''(\mathfrak{X}_\lambda^{(\alpha)})(\lambda V_2 + (1-\lambda)\check{V}_2)U_1\check{U}_1] \\ = s_\alpha \check{s}_\alpha (\lambda t_\alpha^2 s_\alpha^2 + (1-\lambda)\check{t}_\alpha^2 \check{s}_\alpha^2) \\ \cdot (\mathfrak{X}_{4,2'\tilde{1}2'} + 2\mathfrak{X}_{4,2\tilde{1}2} + 2\mathfrak{X}_{4,2'\tilde{3}} + 2\mathfrak{X}_{4,2\tilde{3}'} + 8\mathfrak{X}_{4,2\tilde{3}}) + O_{<}(\Psi^5), \end{aligned}$$

$$\begin{aligned} (57) \quad \mathbb{E}_\alpha\left[\frac{K'''(\mathfrak{X}_\lambda^{(\alpha)})}{2}(\lambda V_1 + (1-\lambda)\check{V}_1)^2 U_1\check{U}_1\right] \\ = \frac{s_\alpha \check{s}_\alpha}{2} (\lambda t_\alpha s_\alpha + (1-\lambda)\check{t}_\alpha \check{s}_\alpha)^2 \\ \cdot (\mathfrak{X}_{4,2'\tilde{2}\tilde{2}'} + 2\mathfrak{X}_{4,2'\tilde{2}2} + 4\mathfrak{X}_{4,2\tilde{2}\tilde{2}'} + 8\mathfrak{X}_{4,2\tilde{2}2}) + O_{<}(\Psi^5), \end{aligned}$$

and $\mathbb{E}_\alpha[W_4]$ is the sum of (54–57).

The $O_{<}(\Psi^5)$ remainder in (48) is given by the difference of the left side with W_2 , W_3 , W_4 . As this is an integral over a polynomial of entries of $G^{(\alpha)}$ and X , its partial expectation is still $O_{<}(\Psi^5)$ by Lemmas D.2 and 5.8.

Summarizing the results of Steps 1 and 2, we collect (48), (49), (50–53), and (54–57):

$$\begin{aligned} (58) \quad \mathbb{E}_\alpha\left[K'(\mathfrak{X}_\lambda) \frac{G_{i\alpha}}{t_\alpha} \frac{\check{G}_{i\alpha}}{\check{t}_\alpha}\right] \\ = s_\alpha \check{s}_\alpha K'(\mathfrak{X}_\lambda^{(\alpha)}) \frac{1}{N} \sum_k G_{ik}^{(\alpha)} \check{G}_{ik}^{(\alpha)} - t_\alpha s_\alpha^2 \check{s}_\alpha (\mathfrak{Y}_{3,12'}^{(\alpha)} + 2\mathfrak{Y}_{3,3}^{(\alpha)}) - \check{t}_\alpha \check{s}_\alpha^2 s_\alpha (\mathfrak{Z}_{3,12'}^{(\alpha)} + 2\mathfrak{Z}_{3,3}^{(\alpha)}) \\ - \lambda t_\alpha s_\alpha^2 \check{s}_\alpha (\mathfrak{Y}_{3,2'\tilde{2}'}^{(\alpha)} + 2\mathfrak{Y}_{3,2\tilde{2}}^{(\alpha)}) - (1-\lambda) \check{t}_\alpha \check{s}_\alpha^2 s_\alpha (\mathfrak{Z}_{3,2'\tilde{2}'}^{(\alpha)} + 2\mathfrak{Z}_{3,2\tilde{2}}^{(\alpha)}) \\ + s_\alpha \check{s}_\alpha (t_\alpha^2 s_\alpha^2 + t_\alpha s_\alpha \check{t}_\alpha \check{s}_\alpha + \check{t}_\alpha^2 \check{s}_\alpha^2) (\mathfrak{X}_{4,22'} + 4\mathfrak{X}_{4,13} + 8\mathfrak{X}_{4,4} + 2\mathfrak{X}_{4,4'}) \\ + s_\alpha \check{s}_\alpha (\lambda t_\alpha s_\alpha + (1-\lambda)\check{t}_\alpha \check{s}_\alpha) (t_\alpha s_\alpha + \check{t}_\alpha \check{s}_\alpha) \\ \times (\mathfrak{X}_{4,12'\tilde{2}'} + 2\mathfrak{X}_{4,12\tilde{2}} + 2\mathfrak{X}_{4,3\tilde{2}'} + 2\mathfrak{X}_{4,3'\tilde{2}} + 8\mathfrak{X}_{4,3\tilde{2}}) \\ + s_\alpha \check{s}_\alpha (\lambda t_\alpha^2 s_\alpha^2 + (1-\lambda)\check{t}_\alpha^2 \check{s}_\alpha^2) (\mathfrak{X}_{4,2'\tilde{1}2'} + 2\mathfrak{X}_{4,2\tilde{1}2} + 2\mathfrak{X}_{4,2'\tilde{3}} + 2\mathfrak{X}_{4,2\tilde{3}'} + 8\mathfrak{X}_{4,2\tilde{3}}) \end{aligned}$$

$$+ \frac{s_\alpha \check{s}_\alpha}{2} (\lambda t_\alpha s_\alpha + (1 - \lambda) \check{t}_\alpha \check{s}_\alpha)^2 (\mathfrak{X}_{4,2'\tilde{2}'} + 2\mathfrak{X}_{4,2'\tilde{2}\tilde{2}} + 4\mathfrak{X}_{4,2\tilde{2}\tilde{2}'} + 8\mathfrak{X}_{4,2\tilde{2}\tilde{2}\tilde{2}}) + O_{\prec}(\Psi^5).$$

Step 3: In (58), we consider the first term on the right (of size $O_{\prec}(\Psi^2)$) and remove the superscripts (α) , keeping track of the $O_{\prec}(\Psi^3)$ and $O_{\prec}(\Psi^4)$ terms that arise.

Applying the resolvent identities and a Taylor expansion for $G_{\alpha\alpha}$, we write

$$\begin{aligned} G_{ik}^{(\alpha)} &= G_{ik} - \frac{G_{i\alpha} G_{k\alpha}}{G_{\alpha\alpha}} \\ &= G_{ik} - G_{\alpha\alpha} \sum_{r,s} G_{ir}^{(\alpha)} X_{\alpha r} G_{ks}^{(\alpha)} X_{\alpha s} \\ (59) \quad &= G_{ik} + t_\alpha s_\alpha \sum_{r,s} G_{ir}^{(\alpha)} X_{\alpha r} G_{ks}^{(\alpha)} X_{\alpha s} \\ &\quad - t_\alpha^2 s_\alpha^2 \left(\sum_{p,q} G_{pq}^{(\alpha)} X_{\alpha p} X_{\alpha q} - m_* \right) \sum_{r,s} G_{ir}^{(\alpha)} X_{\alpha r} G_{ks}^{(\alpha)} X_{\alpha s} + O_{\prec}(\Psi^4) \\ &\equiv G_{ik} + R_{2k} + R_{3k} + O_{\prec}(\Psi^4), \end{aligned}$$

where we defined the two remainder terms of sizes $O_{\prec}(\Psi^2)$, $O_{\prec}(\Psi^3)$ as R_{2k} , R_{3k} . Similarly we write

$$(60) \quad \check{G}_{ik}^{(\alpha)} = \check{G}_{ik} + \check{R}_{2k} + \check{R}_{3k} + O_{\prec}(\Psi^4).$$

For $K'(\mathfrak{X}_\lambda^{(\alpha)})$, we apply the Taylor expansion (45) and recall V_1 , $\check{V}_1$, V_2 , $\check{V}_2$ from (46, 47) to obtain

$$\begin{aligned} K'(\mathfrak{X}_\lambda^{(\alpha)}) &= K'(\mathfrak{X}_\lambda) - K''(\mathfrak{X}_\lambda^{(\alpha)})(\mathfrak{X}_\lambda - \mathfrak{X}_\lambda^{(\alpha)}) - \frac{K'''(\mathfrak{X}_\lambda^{(\alpha)})}{2} (\mathfrak{X}_\lambda - \mathfrak{X}_\lambda^{(\alpha)})^2 + O_{\prec}(\Psi^3) \\ (61) \quad &= K'(\mathfrak{X}_\lambda) - K''(\mathfrak{X}_\lambda^{(\alpha)})(\lambda V_1 + (1 - \lambda)\check{V}_1) - K''(\mathfrak{X}_\lambda^{(\alpha)})(\lambda V_2 + (1 - \lambda)\check{V}_2) \\ &\quad - \frac{K'''(\mathfrak{X}_\lambda^{(\alpha)})}{2} (\lambda V_1 + (1 - \lambda)\check{V}_1)^2 + O_{\prec}(\Psi^3). \end{aligned}$$

Taking the product of (59), (60), and (61), applying the identity

$$xyz = (x - \delta_x)(y - \delta_y)(z - \delta_z) + xy\delta_z + x\delta_y z + \delta_x yz - x\delta_y \delta_z - \delta_x y\delta_z - \delta_x \delta_y z + \delta_x \delta_y \delta_z$$

(with $x = G_{ik}^{(\alpha)}$, $x - \delta_x = G_{ik}$, and $\delta_x = R_{2k} + R_{3k}$, etc.), and averaging over $k \in \mathcal{I}_N$, we obtain

$$(62) \quad K'(\mathfrak{X}_\lambda^{(\alpha)}) \frac{1}{N} \sum_k G_{ik}^{(\alpha)} \check{G}_{ik}^{(\alpha)} \equiv S_2 + S_{3,1} + S_{3,2} + \sum_{j=1}^5 S_{4,j} + O_{\prec}(\Psi^5),$$

where

$$\begin{aligned} S_2 &= K'(\mathfrak{X}_\lambda) \frac{1}{N} \sum_k G_{ik} \check{G}_{ik}, \\ S_{3,1} &= K'(\mathfrak{X}_\lambda^{(\alpha)}) \frac{1}{N} \sum_k G_{ik}^{(\alpha)} \check{R}_{2k} + K'(\mathfrak{X}_\lambda^{(\alpha)}) \frac{1}{N} \sum_k R_{2k} \check{G}_{ik}^{(\alpha)}, \\ S_{3,2} &= -K''(\mathfrak{X}_\lambda^{(\alpha)})(\lambda V_1 + (1 - \lambda)\check{V}_1) \frac{1}{N} \sum_k G_{ik}^{(\alpha)} \check{G}_{ik}^{(\alpha)}, \end{aligned}$$

$$\begin{aligned}
S_{4,1} &= K'(\mathfrak{X}_\lambda^{(\alpha)}) \frac{1}{N} \sum_k G_{ik}^{(\alpha)} \check{R}_{3k} + K'(\mathfrak{X}_\lambda^{(\alpha)}) \frac{1}{N} \sum_k R_{3k} \check{G}_{ik}^{(\alpha)}, \\
S_{4,2} &= -K''(\mathfrak{X}_\lambda^{(\alpha)}) (\lambda V_2 + (1-\lambda) \check{V}_2) \frac{1}{N} \sum_k G_{ik}^{(\alpha)} \check{G}_{ik}^{(\alpha)}, \\
S_{4,3} &= -\frac{K'''(\mathfrak{X}_\lambda^{(\alpha)})}{2} (\lambda V_1 + (1-\lambda) \check{V}_1)^2 \frac{1}{N} \sum_k G_{ik}^{(\alpha)} \check{G}_{ik}^{(\alpha)}, \\
S_{4,4} &= -K'(\mathfrak{X}_\lambda^{(\alpha)}) \frac{1}{N} \sum_k R_{2k} \check{R}_{2k}, \\
S_{4,5} &= K''(\mathfrak{X}_\lambda^{(\alpha)}) (\lambda V_1 + (1-\lambda) \check{V}_1) \frac{1}{N} \sum_k G_{ik}^{(\alpha)} \check{R}_{2k} \\
&\quad + K''(\mathfrak{X}_\lambda^{(\alpha)}) (\lambda V_1 + (1-\lambda) \check{V}_1) \frac{1}{N} \sum_k R_{2k} \check{G}_{ik}^{(\alpha)}.
\end{aligned}$$

Recalling the definition of R_{2k} and applying $\mathbb{E}_\alpha$ to the $O_{<}(\Psi^3)$ terms,

$$\begin{aligned}
\mathbb{E}_\alpha[S_{3,1}] &= t_\alpha s_\alpha \mathfrak{Y}_{3,3}^{(\alpha)} + \check{t}_\alpha \check{s}_\alpha \mathcal{Z}_{3,3}^{(\alpha)}, \\
\mathbb{E}_\alpha[S_{3,2}] &= \lambda t_\alpha s_\alpha \mathfrak{Y}_{3,2'\tilde{2}'}^{(\alpha)} + (1-\lambda) \check{t}_\alpha \check{s}_\alpha \mathcal{Z}_{3,2'\tilde{2}'}^{(\alpha)}.
\end{aligned}$$

Similarly, we apply $\mathbb{E}_\alpha$ to each of the $O_{<}(\Psi^4)$ terms, considering all pairings of the four summation indices as in Step 2. Then applying Lemmas 5.6 and 5.7 to remove superscripts and convert $\check{G}$ to G , we obtain

$$\begin{aligned}
\mathbb{E}_\alpha[S_{4,1}] &= -(t_\alpha^2 s_\alpha^2 + \check{t}_\alpha^2 \check{s}_\alpha^2) (\mathfrak{X}_{4,13} + 2\mathfrak{X}_{4,4}) + O_{<}(\Psi^5), \\
\mathbb{E}_\alpha[S_{4,2}] &= -(\lambda t_\alpha^2 s_\alpha^2 + (1-\lambda) \check{t}_\alpha^2 \check{s}_\alpha^2) (\mathfrak{X}_{4,2'\tilde{1}2'} + 2\mathfrak{X}_{4,2'\tilde{3}}) + O_{<}(\Psi^5), \\
\mathbb{E}_\alpha[S_{4,3}] &= -\frac{1}{2} (\lambda t_\alpha s_\alpha + (1-\lambda) \check{t}_\alpha \check{s}_\alpha)^2 (\mathfrak{X}_{4,2'\tilde{2}\tilde{2}'} + 2\mathfrak{X}_{4,2'\tilde{2}2}) + O_{<}(\Psi^5), \\
\mathbb{E}_\alpha[S_{4,4}] &= -t_\alpha s_\alpha \check{t}_\alpha \check{s}_\alpha (\mathfrak{X}_{4,4'} + 2\mathfrak{X}_{4,4}) + O_{<}(\Psi^5), \\
\mathbb{E}_\alpha[S_{4,5}] &= -(\lambda t_\alpha s_\alpha + (1-\lambda) \check{t}_\alpha \check{s}_\alpha) (t_\alpha s_\alpha + \check{t}_\alpha \check{s}_\alpha) (\mathfrak{X}_{4,3\tilde{2}'} + 2\mathfrak{X}_{4,3\tilde{2}}) + O_{<}(\Psi^5).
\end{aligned}$$

Then applying $\mathbb{E}_\alpha$ to (62), noting that the remainder is again $O_{<}(\Psi^5)$ by Lemmas D.2 and 5.8, and substituting into (58),

$$\begin{aligned}
&\mathbb{E}_\alpha \left[K'(\mathfrak{X}_\lambda) \frac{G_{i\alpha}}{t_\alpha} \frac{\check{G}_{i\alpha}}{\check{t}_\alpha} \right] \\
&= s_\alpha \check{s}_\alpha \mathbb{E}_\alpha \left[K'(\mathfrak{X}_\lambda) \frac{1}{N} \sum_k G_{ik} \check{G}_{ik} \right] - t_\alpha s_\alpha^2 \check{s}_\alpha (\mathfrak{Y}_{3,12'}^{(\alpha)} + \mathfrak{Y}_{3,3}^{(\alpha)}) \\
&\quad - \check{t}_\alpha \check{s}_\alpha^2 s_\alpha (\mathcal{Z}_{3,12'}^{(\alpha)} + \mathcal{Z}_{3,3}^{(\alpha)}) - 2\lambda t_\alpha s_\alpha^2 \check{s}_\alpha \mathfrak{Y}_{3,2\tilde{2}}^{(\alpha)} - 2(1-\lambda) \check{t}_\alpha \check{s}_\alpha^2 s_\alpha \mathcal{Z}_{3,2\tilde{2}}^{(\alpha)} \\
&\quad + s_\alpha \check{s}_\alpha (t_\alpha^2 s_\alpha^2 + \check{t}_\alpha^2 \check{s}_\alpha^2) (\mathfrak{X}_{4,22'} + 3\mathfrak{X}_{4,13} + 6\mathfrak{X}_{4,4} + 2\mathfrak{X}_{4,4'}) \\
&\quad + s_\alpha \check{s}_\alpha (t_\alpha s_\alpha \check{t}_\alpha \check{s}_\alpha) (\mathfrak{X}_{4,22'} + 4\mathfrak{X}_{4,13} + 6\mathfrak{X}_{4,4} + \mathfrak{X}_{4,4'}) \\
&\quad + s_\alpha \check{s}_\alpha (\lambda t_\alpha s_\alpha + (1-\lambda) \check{t}_\alpha \check{s}_\alpha) (t_\alpha s_\alpha + \check{t}_\alpha \check{s}_\alpha) \\
&\quad \times (\mathfrak{X}_{4,12'\tilde{2}'} + 2\mathfrak{X}_{4,12\tilde{2}} + \mathfrak{X}_{4,3\tilde{2}'} + 2\mathfrak{X}_{4,3'\tilde{2}} + 6\mathfrak{X}_{4,3\tilde{2}})
\end{aligned} \tag{63}$$

$$\begin{aligned}
& + s_\alpha \check{s}_\alpha (\lambda t_\alpha^2 s_\alpha^2 + (1 - \lambda) \check{t}_\alpha^2 \check{s}_\alpha^2) (2\mathfrak{X}_{4,2\tilde{1}2} + 2\mathfrak{X}_{4,2\tilde{3}'} + 8\mathfrak{X}_{4,2\tilde{3}}) \\
& + \frac{s_\alpha \check{s}_\alpha}{2} (\lambda t_\alpha s_\alpha + (1 - \lambda) \check{t}_\alpha \check{s}_\alpha)^2 (4\mathfrak{X}_{4,2\tilde{2}2'} + 8\mathfrak{X}_{4,2\tilde{2}2}) + O_{\prec}(\Psi^5).
\end{aligned}$$

Step 4: In (63), we remove the superscript (α) from $\mathfrak{Y}_{3,*}$ and $\mathcal{Z}_{3,*}$, keeping track of the $O_{\prec}(\Psi^4)$ errors that arise. For each quantity $\mathfrak{Y}_{3,*}^{(\alpha)}$ or $\mathcal{Z}_{3,*}^{(\alpha)}$, let $\mathfrak{Y}_{3,*}$ or $\mathcal{Z}_{3,*}$ be the analogous quantity with each instance of $m_N^{(\alpha)}$, $G^{(\alpha)}$, $\tilde{G}^{(\alpha)}$, $\mathfrak{X}_\lambda^{(\alpha)}$ replaced by m_N , G , $\tilde{G}$, $\mathfrak{X}_\lambda$.

For $\mathfrak{Y}_{3,12'}^{(\alpha)}$, recall from (59) and (61) that

$$\begin{aligned}
G_{ik}^{(\alpha)} &= G_{ik} + R_{2k} + O_{\prec}(\Psi^3), \\
K'(\mathfrak{X}_\lambda^{(\alpha)}) &= K'(\mathfrak{X}_\lambda) - K''(\mathfrak{X}_\lambda^{(\alpha)}) (\lambda V_1 + (1 - \lambda) \check{V}_1) + O_{\prec}(\Psi^2).
\end{aligned}$$

For $m_N^{(\alpha)} - m_*$, we apply the resolvent identities and write

$$\begin{aligned}
m_N^{(\alpha)} - m_* &= m_N - m_* - \frac{1}{N} \sum_j \frac{G_{j\alpha}^2}{G_{\alpha\alpha}} \\
&= m_N - m_* - G_{\alpha\alpha} \frac{1}{N} \sum_{j,k,l} G_{jk}^{(\alpha)} X_{\alpha k} G_{jl}^{(\alpha)} X_{\alpha l} \\
&= m_N - m_* + t_\alpha s_\alpha \frac{1}{N} \sum_{j,k,l} G_{jk}^{(\alpha)} X_{\alpha k} G_{jl}^{(\alpha)} X_{\alpha l} + O_{\prec}(\Psi^3) \\
&\equiv m_N - m_* + Q + O_{\prec}(\Psi^3),
\end{aligned}$$

where Q is the $O_{\prec}(\Psi^2)$ term. Multiplying the above and averaging over k ,

$$\begin{aligned}
\mathfrak{Y}_{3,12'}^{(\alpha)} &= \mathfrak{Y}_{3,12'} + K'(\mathfrak{X}_\lambda^{(\alpha)}) (m_N^{(\alpha)} - m_*) \frac{1}{N} \sum_k G_{ik}^{(\alpha)} \check{R}_{2k} \\
&+ K'(\mathfrak{X}_\lambda^{(\alpha)}) (m_N^{(\alpha)} - m_*) \frac{1}{N} \sum_k \check{G}_{ik}^{(\alpha)} R_{2k} + K'(\mathfrak{X}_\lambda^{(\alpha)}) Q \frac{1}{N} \sum_k G_{ik}^{(\alpha)} \check{G}_{ik}^{(\alpha)} \\
&- K''(\mathfrak{X}_\lambda^{(\alpha)}) (\lambda V_1 + (1 - \lambda) \check{V}_1) (m_N^{(\alpha)} - m_*) \frac{1}{N} \sum_k G_{ik}^{(\alpha)} \check{G}_{ik}^{(\alpha)} + O_{\prec}(\Psi^5),
\end{aligned}$$

where each term except $\mathfrak{Y}_{3,12'}$ on the right is of size $O_{\prec}(\Psi^4)$. Taking $\mathbb{E}_\alpha$ and applying Lemmas 5.6 and 5.7 to remove superscripts and checks,

$$\begin{aligned}
(64) \quad \mathfrak{Y}_{3,12'}^{(\alpha)} &= \mathbb{E}_\alpha[\mathfrak{Y}_{3,12'}] + (t_\alpha s_\alpha + \check{t}_\alpha \check{s}_\alpha) \mathfrak{X}_{4,13} + t_\alpha s_\alpha \mathfrak{X}_{4,4'} \\
&+ (\lambda t_\alpha s_\alpha + (1 - \lambda) \check{t}_\alpha \check{s}_\alpha) \mathfrak{X}_{4,12'\tilde{2}'} + O_{\prec}(\Psi^5).
\end{aligned}$$

Similar arguments yield

$$\begin{aligned}
\mathcal{Z}_{3,12'}^{(\alpha)} &= \mathbb{E}_\alpha[\mathcal{Z}_{3,12'}] + (t_\alpha s_\alpha + \check{t}_\alpha \check{s}_\alpha) \mathfrak{X}_{4,13} + \check{t}_\alpha \check{s}_\alpha \mathfrak{X}_{4,4'} \\
&+ (\lambda t_\alpha s_\alpha + (1 - \lambda) \check{t}_\alpha \check{s}_\alpha) \mathfrak{X}_{4,12'\tilde{2}'} + O_{\prec}(\Psi^5), \\
\mathfrak{Y}_{3,3}^{(\alpha)} &= \mathbb{E}_\alpha[\mathfrak{Y}_{3,3}] + (2t_\alpha s_\alpha + \check{t}_\alpha \check{s}_\alpha) \mathfrak{X}_{4,4} \\
&+ (\lambda t_\alpha s_\alpha + (1 - \lambda) \check{t}_\alpha \check{s}_\alpha) \mathfrak{X}_{4,3\tilde{2}'} + O_{\prec}(\Psi^5), \\
\mathcal{Z}_{3,3}^{(\alpha)} &= \mathbb{E}_\alpha[\mathcal{Z}_{3,3}] + (t_\alpha s_\alpha + 2\check{t}_\alpha \check{s}_\alpha) \mathfrak{X}_{4,4}
\end{aligned}$$

$$\begin{aligned}
& + (\lambda t_\alpha s_\alpha + (1 - \lambda) \check{t}_\alpha \check{s}_\alpha) \mathfrak{X}_{4,3\tilde{2}'} + O_{\prec}(\Psi^5), \\
\mathfrak{Y}_{3,2\tilde{2}}^{(\alpha)} &= \mathbb{E}_\alpha[\mathfrak{Y}_{3,2\tilde{2}}] + (t_\alpha s_\alpha + \check{t}_\alpha \check{s}_\alpha) \mathfrak{X}_{4,3\tilde{2}} + 2t_\alpha s_\alpha \mathfrak{X}_{4,2\tilde{3}} \\
& + (\lambda t_\alpha s_\alpha + (1 - \lambda) \check{t}_\alpha \check{s}_\alpha) \mathfrak{X}_{4,2\tilde{2}\tilde{2}'} + O_{\prec}(\Psi^5), \\
\mathfrak{Z}_{3,2\tilde{2}}^{(\alpha)} &= \mathbb{E}_\alpha[\mathfrak{Z}_{3,2\tilde{2}}] + (t_\alpha s_\alpha + \check{t}_\alpha \check{s}_\alpha) \mathfrak{X}_{4,3\tilde{2}} + 2\check{t}_\alpha \check{s}_\alpha \mathfrak{X}_{4,2\tilde{3}} \\
& + (\lambda t_\alpha s_\alpha + (1 - \lambda) \check{t}_\alpha \check{s}_\alpha) \mathfrak{X}_{4,2\tilde{2}\tilde{2}'} + O_{\prec}(\Psi^5).
\end{aligned}$$

Substituting into (63),

$$\begin{aligned}
& \mathbb{E}_\alpha \left[K'(\mathfrak{X}_\lambda) \frac{G_{i\alpha}}{t_\alpha} \frac{\check{G}_{i\alpha}}{\check{t}_\alpha} \right] \\
&= s_\alpha \check{s}_\alpha \mathbb{E}_\alpha \left[K'(\mathfrak{X}_\lambda) \frac{1}{N} \sum_k G_{ik} \check{G}_{ik} \right] - t_\alpha s_\alpha^2 \check{s}_\alpha \mathbb{E}_\alpha[\mathfrak{Y}_{3,12'} + \mathfrak{Y}_{3,3}] \\
(65) \quad & - \check{t}_\alpha \check{s}_\alpha^2 s_\alpha \mathbb{E}_\alpha[\mathfrak{Z}_{3,12'} + \mathfrak{Z}_{3,3}] - 2\lambda t_\alpha s_\alpha^2 \check{s}_\alpha \mathbb{E}_\alpha[\mathfrak{Y}_{3,2\tilde{2}}] - 2(1 - \lambda) \check{t}_\alpha \check{s}_\alpha^2 s_\alpha \mathbb{E}_\alpha[\mathfrak{Z}_{3,2\tilde{2}}] \\
& + s_\alpha \check{s}_\alpha (t_\alpha^2 s_\alpha^2 + t_\alpha s_\alpha \check{t}_\alpha \check{s}_\alpha + \check{t}_\alpha^2 \check{s}_\alpha^2) (\mathfrak{X}_{4,22'} + 2\mathfrak{X}_{4,13} + 4\mathfrak{X}_{4,4} + \mathfrak{X}_{4,4'}) \\
& + s_\alpha \check{s}_\alpha (\lambda t_\alpha s_\alpha + (1 - \lambda) \check{t}_\alpha \check{s}_\alpha) (t_\alpha s_\alpha + \check{t}_\alpha \check{s}_\alpha) (2\mathfrak{X}_{4,12\tilde{2}} + 2\mathfrak{X}_{4,3'\tilde{2}} + 4\mathfrak{X}_{4,3\tilde{2}}) \\
& + s_\alpha \check{s}_\alpha (\lambda \check{t}_\alpha^2 s_\alpha^2 + (1 - \lambda) \check{t}_\alpha^2 \check{s}_\alpha^2) (2\mathfrak{X}_{4,2\tilde{1}\tilde{2}} + 2\mathfrak{X}_{4,2\tilde{3}'} + 4\mathfrak{X}_{4,2\tilde{3}}) \\
& + 4s_\alpha \check{s}_\alpha (\lambda t_\alpha s_\alpha + (1 - \lambda) \check{t}_\alpha \check{s}_\alpha)^2 \mathfrak{X}_{4,2\tilde{2}\tilde{2}'} + O_{\prec}(\Psi^5).
\end{aligned}$$

Step 5: We take the full expectation of both sides of (65), applying Lemma 5.9 to convert $\mathfrak{Y}_{3,*}$ and $\mathfrak{Z}_{3,*}$ into $\mathfrak{X}_{3,*}$. We illustrate the argument for $\mathfrak{Z}_{3,12'}$: For $k \neq i$, denote

$$Y = K'(\mathfrak{X}_\lambda)(\check{m}_N - \check{m}_*)G_{ik}, \quad Y^{(\alpha)} = K'(\mathfrak{X}_\lambda^{(\alpha)})(\check{m}_N^{(\alpha)} - \check{m}_*)G_{ik}^{(\alpha)}.$$

Then $Y \prec \Psi^2$, and $Y - Y^{(\alpha)} \prec \Psi^3$ for all $\alpha \in \mathcal{I}_M$, the latter from Lemma 5.6 and the second-derivative bound for K . Then applying Lemma 5.9, $\mathbb{E}[Y\check{G}_{ik}] = \mathbb{E}[YG_{ik}] + O_{\prec}(\Psi^5)$. Hence

$$(66) \quad \mathbb{E} \left[K'(\mathfrak{X}_\lambda)(\check{m}_N - \check{m}_*) \frac{1}{N} \sum_k G_{ik} (\check{G}_{ik} - G_{ik}) \right] = O_{\prec}(\Psi^5),$$

where the $k = i$ term is controlled directly by Lemma 5.7. Applying this argument again with $Y = K'(\mathfrak{X}_\lambda)G_{ik}^2$, together with the bound $\check{m}_* - m_* \leq C/N \prec \Psi^3$, we may convert the term $\check{m}_N - \check{m}_*$:

$$(67) \quad \mathbb{E} \left[K'(\mathfrak{X}_\lambda)(\check{m}_N - \check{m}_* - m_N + m_*) \frac{1}{N} \sum_k G_{ik}^2 \right] = O_{\prec}(\Psi^5).$$

Finally, a Taylor expansion of $K'(x)$ around $\mathfrak{X}$ yields

$$(68) \quad K'(\mathfrak{X}_\lambda) = K'(\mathfrak{X}) + (1 - \lambda)K''(\mathfrak{X})(\check{\mathfrak{X}} - \mathfrak{X}) + O_{\prec}(\Psi^2),$$

where we have used $\check{\mathfrak{X}} - \mathfrak{X} \prec \Psi$ by Lemma 5.7. Applying the third implication of Lemma 5.9 with $Y = K''(\mathfrak{X})(m_N - m_*)G_{ik}^2 \prec \Psi^3$ for $k \neq i$, we obtain

$$(69) \quad \mathbb{E} \left[K''(\mathfrak{X})(\check{\mathfrak{X}} - \mathfrak{X})(m_N - m_*) \frac{1}{N} \sum_k G_{ik}^2 \right] = O_{\prec}(\Psi^5).$$

Then combining (66–69), we obtain $\mathbb{E}[\mathfrak{Z}_{3,12'}] = \mathbb{E}[\mathfrak{X}_{3,12'}] + O_{\prec}(\Psi^5)$.

The same argument holds for the other terms $\mathfrak{Y}_{3,*}$ and $\mathfrak{Z}_{3,*}$. Then taking the full expectation of (65),

$$\begin{aligned}
 & \mathbb{E} \left[K'(\mathfrak{X}_\lambda) \frac{G_{i\alpha}}{t_\alpha} \frac{\check{G}_{i\alpha}}{\check{t}_\alpha} \right] \\
 &= s_\alpha \check{s}_\alpha \mathbb{E} \left[K'(\mathfrak{X}_\lambda) \frac{1}{N} \sum_k G_{ik} \check{G}_{ik} \right] - (t_\alpha s_\alpha^2 \check{s}_\alpha + \check{t}_\alpha \check{s}_\alpha^2 s_\alpha) \mathbb{E}[\mathfrak{X}_{3,12'} + \mathfrak{X}_{3,3}] \\
 (70) \quad & - 2(\lambda t_\alpha s_\alpha^2 \check{s}_\alpha + (1-\lambda) \check{t}_\alpha \check{s}_\alpha^2 s_\alpha) \mathbb{E}[\mathfrak{X}_{3,2\tilde{2}}] \\
 & + s_\alpha \check{s}_\alpha (t_\alpha^2 s_\alpha^2 + t_\alpha s_\alpha \check{t}_\alpha \check{s}_\alpha + \check{t}_\alpha^2 \check{s}_\alpha^2) \mathbb{E}[\mathfrak{X}_{4,22'} + 2\mathfrak{X}_{4,13} + 4\mathfrak{X}_{4,4} + \mathfrak{X}_{4,4'}] \\
 & + s_\alpha \check{s}_\alpha (\lambda t_\alpha s_\alpha + (1-\lambda) \check{t}_\alpha \check{s}_\alpha) (t_\alpha s_\alpha + \check{t}_\alpha \check{s}_\alpha) \mathbb{E}[2\mathfrak{X}_{4,12\tilde{2}} + 2\mathfrak{X}_{4,3'\tilde{2}} + 4\mathfrak{X}_{4,3\tilde{2}}] \\
 & + s_\alpha \check{s}_\alpha (\lambda t_\alpha^2 s_\alpha^2 + (1-\lambda) \check{t}_\alpha^2 \check{s}_\alpha^2) \mathbb{E}[2\mathfrak{X}_{4,21\tilde{2}} + 2\mathfrak{X}_{4,2\tilde{3}} + 4\mathfrak{X}_{4,2\tilde{3}}] \\
 & + 4s_\alpha \check{s}_\alpha (\lambda t_\alpha s_\alpha + (1-\lambda) \check{t}_\alpha \check{s}_\alpha)^2 \mathbb{E}[\mathfrak{X}_{4,2\tilde{2}\tilde{2}}] + O_{\prec}(\Psi^5).
 \end{aligned}$$

Finally, we integrate (70) over $\lambda \in [0, 1]$, applying $\int \lambda = \int (1-\lambda) = 1/2$ and $\int \lambda^2 = \int 2\lambda(1-\lambda) = \int (1-\lambda)^2 = 1/3$. Simplifying and identifying the terms $\mathfrak{X}_3$, $\mathfrak{X}_4$, $\mathfrak{X}_4^-$, $\mathcal{P}_\alpha$, $\mathcal{Q}_\alpha$, and $\mathcal{R}_\alpha$ concludes the proof of the lemma.

5.5. Proof of optical theorems. We discuss briefly the proof of Lemma 5.11. In the setting $K' \equiv 1$, Lemma 5.11 corresponds to [19], Lemma B.1, upon taking the imaginary part.

The proof for general K is the same as that of [19], Lemma B.1, with additional terms arising from the Taylor expansion of K' as in the proof of Lemma 5.10. The computation may be broken down into the identities

$$\begin{aligned}
 & N^{-1} (\mathbb{E}[K'(\mathfrak{X})] + 2m_*^{-1} \mathbb{E}[K'(\mathfrak{X})(m_N - m_*)]) \\
 &= 2\mathbb{E}[\mathfrak{X}_3] - 2m_*^{-1} (z - E_*) \mathbb{E}[\mathfrak{X}_2] - (A_4 - 2m_*^{-1} - m_*^{-4}) \mathbb{E}[\mathfrak{X}_4] + O_{\prec}(\Psi^5), \\
 & N^{-1} \mathbb{E}[K'(\mathfrak{X})(m_N - m_*)] - 2\mathbb{E}[\mathfrak{X}_{4,22'} + \mathfrak{X}_{4,13} + \mathfrak{X}_{4,4} + \mathfrak{X}_{4,12\tilde{2}}] = O_{\prec}(\Psi^5), \\
 & \mathbb{E}[2\mathfrak{X}_{4,13} + 3\mathfrak{X}_{4,4} + \mathfrak{X}_{4,4'} + 2\mathfrak{X}_{4,3\tilde{2}}] = O_{\prec}(\Psi^5), \\
 & (z - E_*) \mathbb{E}[\mathfrak{X}_2] - \mathbb{E}[\mathfrak{X}_{4,22'} + 4\mathfrak{X}_{4,4} + \mathfrak{X}_{4,4'} + 2\mathfrak{X}_{4,3'\tilde{2}}] = O_{\prec}(\Psi^5), \\
 & \mathbb{E}[\mathfrak{X}_{4,12\tilde{2}} + 2\mathfrak{X}_{4,3\tilde{2}} + \mathfrak{X}_{4,3'\tilde{2}} + \mathfrak{X}_{4,21\tilde{2}} + 2\mathfrak{X}_{4,2\tilde{3}} + \mathfrak{X}_{4,2\tilde{3}} + 2\mathfrak{X}_{4,2\tilde{2}\tilde{2}}] = O_{\prec}(\Psi^5),
 \end{aligned}$$

where $\mathfrak{X}_2 = K'(\mathfrak{X}) N^{-1} \sum_k G_{ik}^2$. For $K' \equiv 1$, the first four identities above reduce to [19], eqs. (B.29), (B.33), (B.38), (B.51). The fifth identity is trivial for $K' \equiv 1$, as the left side is 0. It is analogous to [18], eq. (C.42), in the full computation for the deformed Wigner model, and may be derived as an “optical theorem” from $\mathfrak{X}_{3,2\tilde{2}}$.

Lemma 5.11 follows from substituting the second and fourth identities into the first, adding $4m_*^{-1}$ times the third and fifth, and taking the imaginary part (noting K' is real-valued). This concludes the proof of Theorem 5.1, and hence of Theorem 2.9.

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SUPPLEMENTARY MATERIAL

Supplementary appendices (DOI: [10.1214/21-AAP1754SUPP](https://doi.org/10.1214/21-AAP1754SUPP); .pdf). Appendices A–E may be found in the online supplement.

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