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Polynomial Control on Weighted Stability Bounds and Inversion Norms of Localized Matrices on Simple Graphs

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Abstract

The (un)weighted stability for some matrices on a graph is one of essential hypotheses in time-frequency analysis and applied harmonic analysis. In the first part of this paper, we show that for a localized matrix in a Beurling algebra, its weighted stabilities for different exponents and Muckenhoupt weights are equivalent to each other, and reciprocal of its optimal lower stability bound for one exponent and weight is controlled by a polynomial of reciprocal of its optimal lower stability bound for another exponent and weight. Banach algebras of matrices with certain off-diagonal decay is of great importance in many mathematical and engineering fields, and its inverse-closed property can be informally interpreted as localization preservation. Let $\mathcal{B}(\ell^p_w)$ be the Banach algebra of bounded linear operators on the weighted sequence space ℓ^p_w on a graph. In the second part of this paper, we prove that Beurling algebras of localized matrices on a connected simple graph are inverse-closed in $\mathcal{B}(\ell^p_w)$ for all $1 \le p < \infty$ and Muckenhoupt A_p-weights w, and the Beurling norm of the inversion of a matrix A is bounded by a bivariate polynomial of the Beurling norm of the matrix A and the operator norm of its inverse A^{-1} in $\mathcal{B}(\ell^p_w)$.

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1 Introduction

Let $\mathcal{G} := (V, E)$ be a connected simple graph with the vertex set V and edge set E. Our illustrative examples are (i) the d-dimensional lattice graph $\mathcal{Z}^d := (\mathbb{Z}^d, E^d)$ where there exists an edge between k and $l \in \mathbb{Z}^d$, i.e., $(k, l) \in E^d$, if the Euclidean distance between k and l is one; (ii) the (in)finite circulant graph $\mathcal{C}_G = (V_G, E_G)$ associated with an abelian group

$$V_G = \left\{ \prod_{i=1}^{k} g_i^{n_i},\ n_1, \ldots, n_k \in \mathbb{Z} \right\}$$

generated by $G = \{g_1, \ldots, g_k\}$, where $(\lambda, \lambda') \in E_G$ if and only if either $\lambda(\lambda')^{-1}$ or $\lambda'\lambda^{-1} \in G$ [5,6,27,32,38]; and (iii) the communication graph of a spatially distributed network (SDN) whose agents have limited sensing, data processing, and communication capacity for data transmission, where agents are used as elements in the vertex set and direct communication links between two agents as edges between two vertices [1,12,13,40].

For $1 \le p < \infty$ and a weight $w = (w(\lambda))_{\lambda \in V}$ on the graph $\mathcal{G}$, let $\ell_w^p := \ell_w^p(\mathcal{G})$ be the space of all weighted p-summable sequences/vectors $c = (c(\lambda))_{\lambda \in V}$ equipped with the standard norm

$$\|c\|_{p,w} = \left(\sum_{\lambda \in V} |c(\lambda)|^p w(\lambda) \right)^{1/p}.$$

For the trivial weight $w_0 = (w_0(\lambda))_{\lambda \in V}$, we will use the simplified notation ℓ^p and $\|\cdot\|_p$ instead of $\ell_{w_0}^p$ and $\|\cdot\|_{p,w_0}$, where $w_0(\lambda) = 1$ for all $\lambda \in V$. We say that a matrix

$$A := \big(a(\lambda, \lambda')\big)_{\lambda, \lambda' \in V} \tag{1.1}$$

on the graph $\mathcal{G}$ has ℓ_w^p-*stability* if there exist two positive constants B_1 and B_2 such that

$$B_1 \|c\|_{p,w} \le \|Ac\|_{p,w} \le B_2 \|c\|_{p,w}, \qquad c \in \ell_w^p \tag{1.2}$$

[2,40,42,49,50]. We call the maximal constant B_1 for the weighted stability inequality (1.2) to hold as the *optimal lower* ℓ_w^p-*stability bound* of the matrix A and denote by $\beta_{p,w}(A)$. The (un)weighted stability for matrices is an essential hypothesis in time-frequency analysis, applied harmonic analysis, and many other mathematical and engineering fields [3,15,21,33,47].

In practical sampling and reconstruction on an SDN of large size, signals and noises are usually contained in some range. For robust signal reconstruction and noise reduction, the sensing matrix on the SDN is required to have stability on ℓ^∞ [12], however there are some difficulties to numerically verify ℓ^p-stability of a matrix at the vertex level for $p \neq 2$ [12,34,45]. For a matrix A on a *finite* graph $\mathcal{G} = (V, E)$, its weighted ℓ^p_w-stability are equivalent to each other for different exponents $1 \le p \le \infty$ and weights w, since ℓ^p_w is isomorphic to ℓ^2 for any exponent $1 \le p \le \infty$ and weight w. In particular, for the unweighted case one may verify that the optimal lower stability bounds of a matrix A for different exponents are comparable,

$$\frac{\beta_{p,w_0}(A)}{\beta_{q,w_0}(A)} \le M^{|1/p-1/q|}, \quad 1 \le p, q \le \infty, \tag{1.3}$$

where $M = \#V$ is the number of vertices of the graph $\mathcal{G}$. The above estimation on optimal lower stability bounds for different exponents is unfavorable for matrices of large size, but it can be improved if the matrix A has some additional property, such as *off-diagonal decay*. For an *infinite* matrix $A = (a(i, j))_{i,j\in\mathbb{Z}^d}$ in the Baskakov-Gohberg-Sjöstrand algebra, it is proved in [2,27,42,50] that its unweighted stabilities are equivalent to each other for all exponents, i.e., for all $1 \le p, q < \infty$,

$$\beta_{q,w_0}(A) > 0 \text{ if and only if } \beta_{p,w_0}(A) > 0.$$

In [44], Beurling algebras of infinite matrices $A = (a(i, j))_{i,j\in\mathbb{Z}}$ are introduced. Comparing with the Baskakov-Gohberg-Sjöstrand algebras, matrices in the Baskakov-Gohberg-Sjöstrand algebra (resp. the Beurling algebra) are dominated by a bi-infinite Toeplitz matrix associated with a (resp. radially decreasing) sequence with certain decay, and they are bounded linear operators on unweighted sequence spaces $\ell^p_{w_0}$ (resp. on weighted spaces ℓ^p_w for all Muckenhoupt A_p-weights w). For an infinite matrix in a Beurling algebra on $\mathbb{Z}^d$, its weighted stabilities for different exponents and Muckenhoupt weights are established in [44],

$$\beta_{p,w}(A) > 0 \text{ if and only if } \beta_{q,w'}(A) > 0$$

where $1 \le p, q < \infty$ and w, w' are Muckenhoupt A_p- and A_q-weights respectively, however the optimal lower stability bound $\beta_{q,w'}$ on $\ell^q_{w'}$ is not explicitly expressed in terms of the optimal lower stability bound $\beta_{p,w}$ on ℓ^p_w. Obviously, the lattice $\mathbb{Z}^d$ is the vertex set of the lattice graph $\mathcal{Z}^d$. Inspired by the above observation, Beurling algebras $\mathcal{B}_{r,\alpha}(\mathcal{G})$ of matrices $A = (a(\lambda, \lambda'))_{\lambda,\lambda'\in V}$ on an arbitrary simple graph $\mathcal{G} = (V, E)$ are introduced in [40], where $1 \le r \le \infty$ and $\alpha \ge 0$. In [40], unweighted stabilities of a matrix $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ for different exponents are shown to be equivalent to each other, where $1 \le r \le \infty$, $\alpha > d_{\mathcal{G}}(1 - 1/r)$ and $d_{\mathcal{G}}$ is the Beurling dimension of the graph $\mathcal{G}$. Moreover, we have the following polynomial control on its optimal lower stability bounds for different exponents,

$$\frac{\beta_{p,w_0}(A)}{\beta_{q,w_0}(A)} \le D_1 \left(\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{\beta_{p,w_0}(A)}\right)^{D_0|1/p-1/q|}, \quad 1 \le p, q < \infty, \tag{1.4}$$

where D_0, D_1 are absolute constants independent of matrices A and the size M of the graph $\mathcal{G}$. The *first main contribution* of this paper is to establish the polynomial control property for matrices in Beurling algebras on a connected simple graph $\mathcal{G}$ on their optimal lower weighted stability bounds for different exponents and Muckenhoupt weights, see Theorem 3.1 and Remark 3.2 for the comparison with previous works.

Let $\mathcal{B}(\ell^p_w)$ be the Banach algebra of all matrices A which are bounded operators on the weighted vector space ℓ^p_w and denote the norm of $A \in \mathcal{B}(\ell^p_w)$ by $\|A\|_{\mathcal{B}(\ell^p_w)}$. The weighted ℓ^p_w-stability of a matrix A is usually considered as a weak notion of its invertibility, since

$$\beta_{p,w}(A) \ge \left(\|A^{-1}\|_{\mathcal{B}(\ell^p_w)}\right)^{-1}$$

when the matrix A is invertible in ℓ^p_w. However for a matrix A in a Beurling algebra, we discover that its weighted stability in ℓ^p_w implies the existence of its "inverse" $B = (b(\lambda, \lambda'))_{\lambda,\lambda' \in V}$ in the same Beurling algebra such that

$$|c(\lambda)| \le \sum_{\lambda' \in V} |b(\lambda, \lambda')||(Ac)(\lambda')|, \ \lambda \in V, \tag{1.5}$$

hold for all vectors $c = (c(\lambda))_{\lambda \in V} \in \ell^p_w$, see Lemma 3.4. The above *"weak invertibility"* of a matrix in Beurling algebras is crucial for us to discuss polynomial control on optimal lower weighted stability bounds for different exponents and Muckenhoupt weights, and also to establish norm-controlled inversion of Beurling algebras in $\mathcal{B}(\ell^p_w)$ in the second topic of this paper. We remark that the proof of the weak invertibility (1.5) depends on the concept of maximal disjoint set V_N in Sect. 2.1 and the crucial estimate (3.10) to the commutator $[\Psi^{2N}_\lambda, A]$ between a matrix A in Beurling algebra and smooth version Ψ^{2N}_λ of the truncation operator for $\lambda \in V_N$, cf. [8,40,43,44].

Given two Banach algebras $\mathcal{A}$ and $\mathcal{B}$ with common identity such that $\mathcal{A}$ is a Banach subalgebra of $\mathcal{B}$, we say that $\mathcal{A}$ is *inverse-closed* in $\mathcal{B}$ if $A \in \mathcal{A}$ and $A^{-1} \in \mathcal{B}$ implies $A^{-1} \in \mathcal{A}$ [7,8,26,28,43,44,46,48,51]. An equivalent condition for the inverse-closedness of $\mathcal{A}$ in $\mathcal{B}$ is that given an $A \in \mathcal{A}$, its spectral sets $\sigma_{\mathcal{A}}(A)$ and $\sigma_{\mathcal{B}}(A)$ in Banach algebras $\mathcal{A}$ and $\mathcal{B}$ are the same,

$$\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{B}}(A) \ \text{ for all } A \in \mathcal{A}.$$

In this paper, we also call the inverse-closed property for a Banach subalgebra as Wiener's lemma for that subalgebra [8,26,44,46,48,51]. For algebras of matrices with certain off-diagonal decay, Wiener's lemma can be informally interpreted as localization preservation under inversion. Such a localization preservation is of great importance in applied harmonic analysis, numerical analysis, and many mathematical and engineering fields, see the survey papers [22,31,41] and references therein for historical remarks. We remark that Wiener's lemma does not provide a norm estimate for the inversion, which is essential for some mathematical and engineering applications.

We say that a Banach subalgebra $\mathcal{A}$ of $\mathcal{B}$ admits *norm-controlled inversion* in $\mathcal{B}$ if there exists a continuous function h from $[0,\infty)\times[0,\infty)$ to $[0,\infty)$ such that

$$\|A^{-1}\|_{\mathcal{A}} \le h\left(\|A\|_{\mathcal{A}}, \|A^{-1}\|_{\mathcal{B}}\right) \tag{1.6}$$

for all $A\in\mathcal{A}$ being invertible in $\mathcal{B}$ [23,24,35,38,40]. By the norm-controlled inversion (1.6), we have the following estimate for the resolvent of $A\in\mathcal{A}$,

$$\|(\lambda I - A)^{-1}\|_{\mathcal{A}} \le h\left(\|\lambda I - A\|_{\mathcal{A}}, \|(\lambda I - A)^{-1}\|_{\mathcal{B}}\right), \quad \lambda \not\in \sigma_{\mathcal{B}}(A) = \sigma_{\mathcal{A}}(A), \tag{1.7}$$

where I is the common identity of Banach algebras $\mathcal{A}$ and $\mathcal{B}$. The norm-controlled inversion is a strong version of Wiener's lemma. The classical Wiener algebra of periodic functions with summable Fourier coefficients is an inverse-closed subalgebra of the Banach algebra of all periodic continuous functions [51], however it does not have norm-controlled inversion [9,35].

We say that $\mathcal{A}$ is a *differential subalgebra of order* $\theta\in(0,1]$ in $\mathcal{B}$ if there exists a positive constant $D := D(\mathcal{A},\mathcal{B},\theta)$ such that

$$\|AB\|_{\mathcal{A}} \le D\|A\|_{\mathcal{A}}\|B\|_{\mathcal{A}}\left(\left(\frac{\|A\|_{\mathcal{B}}}{\|A\|_{\mathcal{A}}}\right)^{\theta} + \left(\frac{\|B\|_{\mathcal{B}}}{\|B\|_{\mathcal{A}}}\right)^{\theta}\right) \quad \text{for all } A, B \in \mathcal{A}. \tag{1.8}$$

The concept of differential subalgebras of order θ was introduced in [11,30,36] for $\theta = 1$ and [14,24,40] for $\theta\in(0,1)$. It has been proved that a differential $*$-subalgebra $\mathcal{A}$ of a symmetric $*$-algebra $\mathcal{B}$ has norm-controlled inversion in $\mathcal{B}$ [23,24,38,39,48]. A crucial step in the proof is to introduce $B := I - \|A^*A\|_{\mathcal{B}}^{-1}A^*A$ for any $A\in\mathcal{A}$ being invertible in $\mathcal{B}$, whose spectrum is contained in an interval on the positive real axis. The above reduction depends on the requirements that $\mathcal{B}$ is symmetric and both $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras with common identity and involution $*$.

Several algebras of localized matrices with certain off-diagonal decay, including some subfamilies of Gröchenig–Schur algebra, Baskakov–Gohberg–Sjöstrand algebra, Beurling algebra and Jaffard algebra, have been shown to be differential $*$-subalgebras of the symmetric $*$-algebra $\mathcal{B}(\ell^2)$, and hence they admit norm-controlled inversion in $\mathcal{B}(\ell^2)$ [23–25,28,37,38,40,44,46,48]. In [23,24,40], the authors show that for the Baskakov-Gohberg-Sjöstrand algebra, Jaffard algebra, and Beurling algebra of matrices, a bivariate polynomial can be selected to be the norm-control function h in (1.6).

For applications in some mathematical and engineering fields, the widely-used algebras $\mathcal{B}$ of infinite matrices are the operator algebras $\mathcal{B}(\ell^p_w)$, $1\le p\le\infty$, which are symmetric only when $p=2$. Unlike norm-controlled inversion in symmetric algebras [23,24,35,38,40], to our knowledge, norm-controlled inversion in a *nonsymmetric* algebra is not well studied [17,44]. The *second main contribution* is to show that Beurling algebras of localized matrices admit norm-controlled inversion in $\mathcal{B}(\ell^p_w)$ for all exponents $1\le p<\infty$ and Muckenhoupt A_p-weights w, see Theorem 4.1. Moreover, we prove that the Beurling algebra norm of the inversion of a matrix A is bounded by a bivariate polynomial of its Beurling algebra norm of the matrix A and the operator norm of its inverse A^{-1} in $\mathcal{B}(\ell^p_w)$, see Remark 4.2 for the comparison of previous works.

The paper is organized as follows. In Sect. 2, we recall some preliminary results on a connected simple graph $\mathcal{G}$, Beurling algebras of matrices on the graph $\mathcal{G}$ and on its maximal disjoint sets, and weighted norm inequalities for matrices in a Beurling algebra. For matrices in a Beurling algebra, we consider the equivalence of their weighted stability for different exponents $1 \le p < \infty$ and Muckenhoupt A_p-weights w in Sect. 3, and their norm-controlled inversion in $\mathcal{B}(\ell^p_w)$ in Sect. 4. All proofs, except the proof of Theorem 3.1 in Sect. 3, are collected in Sect. 5.

1.1 Notation

For a real number t, we use the standard notation $\lfloor t \rfloor$ and $\lceil t \rceil$ to denote its floor and ceiling, respectively. For two terms A and B, we write $A \lesssim B$ if $A \le CB$ for some absolute constant C, and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2 Preliminaries

In Sect. 2.1, we recall the doubling property for the counting measure μ on a connected simple graph $\mathcal{G}$ [12,40,52], we show that the counting measure μ has the strong polynomial growth property (2.11), and then we define generalized Beurling dimension of the graph $\mathcal{G}$. In Sects. 2.2 and 2.3, we recall the definition of two closely-related Beurling algebras of matrices on the graph $\mathcal{G}$ and on its maximal disjoint sets [10,40,44], and we provide some algebraic and approximation properties of those two Banach algebras of matrices. In Sect. 2.4, we prove that any matrix in a Beurling algebra is a bounded linear operator on weighted vector spaces ℓ^p_w for all $1 \le p < \infty$ and Muckenhoupt A_p-weights w.

2.1 Generalized Beurling Dimension of a Connected Simple Graph

Let ρ be the *geodesic distance* on the connected simple graph $\mathcal{G}$, which is the nonnegative function on $V \times V$ such that $\rho(\lambda, \lambda) = 0, \lambda \in V$, and $\rho(\lambda, \lambda')$ is the number of edges in a shortest path connecting distinct vertices $\lambda, \lambda' \in V$ [16]. This geodesic distance ρ is a metric on V of a connected simple graph $\mathcal{G}$. For the lattice graph $\mathcal{Z}^d$, one may verify that its geodesic distance between two points $k = (k_1, \ldots, k_d)$ and $\ell = (\ell_1, \ldots, \ell_d)$ is given by $\rho(k, \ell) := \sum_{j=1}^d |k_j - \ell_j|$; for the circulant graph $\mathcal{C}_G$ generated by $G = \{g_1, \ldots, g_k\}$, we have

$$\rho(\lambda, \lambda') = \inf \left\{ \sum_{i=1}^{k} |n_i|,\ \lambda' \lambda^{-1} = \prod_{i=1}^{k} g_i^{n_i}, n_1, \ldots, n_k \in \mathbb{Z} \right\};$$

and for the communication graph of an SDN, $\rho(\lambda, \lambda')$ is the time delay of data transmission between two agents λ and λ'. Using the geodesic distance ρ, we define the closed ball with center $\lambda \in V$ and radius $r > 0$ by

$$B(\lambda, r) = \{\lambda' \in V,\ \rho(\lambda, \lambda') \le r\},$$

which contains all r-neighboring vertices of $\lambda \in V$.

Let μ be the counting measure on the vertex set V, i.e., $\mu(F)$ is the number of vertices in $F \subset V$. In this paper, we always assume that the counting measure μ has *doubling property*, i.e., there exists a positive constant D such that

$$\mu\left(B(\lambda, 2r)\right) \le D\mu\left(B(\lambda, r)\right) \quad \text{for all } \lambda \in V \text{ and } r > 0 \tag{2.1}$$

[12,40,52]. We denote the minimal constant D in the doubling property (2.1) by $D(\mu)$, which is also known as the *doubling constant* of the measure μ. Applying the doubling property (2.1) repeatedly, we have

$$\mu(B(\lambda, r)) \le \mu\big(B(\lambda, 2^{\lceil \log_2(r/r')\rceil} r')\big) \le D(\mu)(r/r')^{\log_2 D(\mu)}\mu(B(\lambda, r')) \tag{2.2}$$

for all $r \ge r' > 0$. Taking $r' = 1-\epsilon$ in (2.2) for sufficiently small $\epsilon > 0$, we conclude that the counting measure μ has *polynomial growth* in the sense that

$$\mu(B(\lambda, r)) \le D_1(r+1)^{d_1} \quad \text{for all } \lambda \in V \text{ and } r \ge 0, \tag{2.3}$$

where D_1 and d_1 are positive constants. The notion of polynomial growth for the counting measure μ is introduced in [12], where the minimal constants d_1 and D_1 in (2.3), to be denoted by $d_{\mathcal{G}}$ and $D_{\mathcal{G}}$, are known as the *Beurling dimension* and *density* of the graph $\mathcal{G}$ respectively.

Let $N \ge 0$. We say that a subset V_N of the vertex set V is *maximal N-disjoint* if

$$B(\lambda, N) \cap \left(\cup_{\lambda_m \in V_N} B(\lambda_m, N)\right) \ne \emptyset \ \text{ for all } \lambda \in V \tag{2.4}$$

and

$$B(\lambda_m, N) \cap B(\lambda_n, N) = \emptyset \ \text{ for all distinct } \lambda_m, \lambda_n \in V_N. \tag{2.5}$$

For $N = 0$, one may verify that the whole set V is the only maximal N-disjoint set V_N, i.e.,

$$V_N = V \text{ if } N = 0, \tag{2.6}$$

while for $N \ge 1$, one may construct many maximal N-disjoint sets V_N. For example, we can construct a maximal N-disjoint set $V_N = \{\lambda_m, m \ge 1\}$ by taking a vertex $\lambda_1 \in V$ and defining vertices $\lambda_m, m \ge 2$, recursively by

$$\lambda_m = \arg\min_{\lambda \in A_m} \rho(\lambda, \lambda_1),$$

where $A_m = \{\lambda \in V, B(\lambda, N) \cap \cup_{m'=1}^{m-1} B(\lambda_{m'}, N) = \emptyset\}$ [12]. For a maximal N-disjoint set V_N, it is observed in [12,40] that for any $N' \geq 2N$, $B(\lambda_m, N')$, $\lambda_m \in V_N$, form a finite covering of the whole set V, and

$$1 \leq \inf_{\lambda \in V} \sum_{\lambda_m \in V_N} \chi_{B(\lambda_m, N')}(\lambda) \leq \sup_{\lambda \in V} \sum_{\lambda_m \in V_N} \chi_{B(\lambda_m, N')}(\lambda) \leq \big(D(\mu)\big)^{\lceil \log_2(2N'/N+1) \rceil}. \tag{2.7}$$

For $\lambda \in V$ and $R \geq 0$, set

$$A_R(\lambda, N) := \big\{\lambda_m \in V_N : \ \rho(\lambda_m, \lambda) \leq (N+1)R\big\}, \tag{2.8}$$

and let $\lambda_{m_0} \in A_R(\lambda, N)$ be so chosen that

$$\mu(B(\lambda_{m_0}, N)) = \inf_{\lambda_m \in A_R(\lambda, N)} \mu(B(\lambda_m, N)). \tag{2.9}$$

Then we obtain from (2.2), (2.5) and (2.9) that

$$\mu(A_R(\lambda, N)) \leq \frac{\sum_{\lambda_m \in A_R(\lambda, N)} \mu(B(\lambda_m, N))}{\mu(B(\lambda_{m_0}, N))} = \frac{\mu\big(\cup_{\lambda_m \in A_R(\lambda, N)} B(\lambda_m, N)\big)}{\mu(B(\lambda_{m_0}, N))} \leq \frac{\mu(B(\lambda_{m_0}, N + 2(N+1)R)}{\mu(B(\lambda_{m_0}, N))} \leq (D(\mu))^3 (R+1)^{\log_2 D(\mu)}. \tag{2.10}$$

Therefore the counting measure μ on the graph $\mathcal{G}$ has *strong polynomial growth* since there exist two positive constants D and d such that

$$\sup_{\lambda \in V} \mu\big(\big\{\lambda_m \in V_N : \ \rho(\lambda_m, \lambda) \leq (N+1)R\big\}\big) \leq D(R+1)^d \tag{2.11}$$

hold for all $R, N \geq 0$ and maximal N-disjoint set V_N. Recall that the whole set V is the only maximal N-disjoint set V_N for $N = 0$. So in this paper the minimal constants d and D in (2.11), to be denoted by $\tilde{d}_{\mathcal{G}}$ and $\tilde{D}_{\mathcal{G}}$, are considered as *generalized Beurling dimension* and *density* respectively. Moreover it follows from (2.6) and (2.10) that

$$d_{\mathcal{G}} \leq \tilde{d}_{\mathcal{G}} \leq \log_2 D(\mu), \tag{2.12}$$

where $d_{\mathcal{G}}$ is the Beurling dimension of the graph $\mathcal{G}$.

We say that the counting measure μ on the graph $\mathcal{G}$ is *Ahlfors d_0-regular* if there exist positive constants B_3 and B_4 such that

$$B_3(r+1)^{d_0} \leq \mu\big(B(\lambda, r)\big) \leq B_4(r+1)^{d_0} \tag{2.13}$$

hold for all balls $B(\lambda, r)$ with center $\lambda \in V$ and radius $0 \leq r \leq \operatorname{diam} \mathcal{G}$, where $\operatorname{diam} \mathcal{G}$ denotes the diameter of the graph $\mathcal{G}$ [29,52]. Clearly for a graph $\mathcal{G}$ with its counting measure μ being Ahlfors d_0-regular, its Beurling dimension $d_{\mathcal{G}}$ is equal to d_0. In the following proposition, we show that the generalized Beurling dimension $\tilde{d}_{\mathcal{G}}$ is also equal to d_0, see Sect. 5.1 for the proof.

Proposition 2.1 *Let $\mathcal{G}$ be a connected simple graph. If the counting measure μ is Ahlfors d_0-regular, then $\tilde{d}_{\mathcal{G}} = d_0$.*

2.2 Beurling Algebras of Matrices on Graphs

Let $\mathcal{G} := (V, E)$ be a connected simple graph with its counting measure μ satisfying the doubling property (2.1). For $1 \le r \le \infty$ and $\alpha \ge 0$, we define the Beurling algebra $\mathcal{B}_{r,\alpha} := \mathcal{B}_{r,\alpha}(\mathcal{G})$ by

$$\mathcal{B}_{r,\alpha}(\mathcal{G}) := \Big\{A = \big(a(\lambda, \lambda')\big)_{\lambda,\lambda' \in V} : \ \|A\|_{\mathcal{B}_{r,\alpha}} < \infty\Big\}, \tag{2.14}$$

where $d_{\mathcal{G}}$ is the Beurling dimension of the graph $\mathcal{G}$,

$$\|A\|_{\mathcal{B}_{r,\alpha}} := \begin{cases} \big(\sum_{n=0}^{\infty} h_A(n)^r (n+1)^{\alpha r + d_{\mathcal{G}} - 1}\big)^{1/r} & \text{if } 1 \le r < \infty \\ \sup_{n \ge 0} h_A(n)(n+1)^{\alpha} & \text{if } r = \infty, \end{cases} \tag{2.15}$$

and

$$h_A(n) = \sup_{\rho(\lambda,\lambda') \ge n} |a(\lambda, \lambda')|, \quad n \ge 0.$$

The Beurling algebra $\mathcal{B}_{r,\alpha}(\mathcal{G})$ is introduced in [44] for the lattice graph $\mathcal{Z}^d$ and for an arbitrary simple graph $\mathcal{G}$ in [40], see also [8]. For a matrix $A = (a(\lambda, \lambda'))_{\lambda,\lambda' \in V}$ in the Beurling algebra $\mathcal{B}_{r,\alpha}(\mathcal{G})$, we define approximation matrices $A_K,\ K \ge 1$, with finite bandwidth by

$$A_K := \big(a(\lambda, \lambda')\chi_{[0,1]}(\rho(\lambda, \lambda')/K)\big)_{\lambda,\lambda' \in V}. \tag{2.16}$$

For the Beurling algebra $\mathcal{B}_{r,\alpha}(\mathcal{G})$, we recall some elementary properties where the first four conclusions have been established in [40], see Sect. 5.2 for the proof.

Proposition 2.2 *Let $\mathcal{G} := (V, E)$ be a connected simple graph such that its counting measure μ satisfies the doubling property (2.1) with the doubling constant $D(\mu)$. Then the following statements hold.*

(i) *$\mathcal{B}_{r,\alpha}(\mathcal{G})$ with $1 \le r \le \infty$ and $\alpha \ge 0$ are solid in the sense that*

$$\|A\|_{\mathcal{B}_{r,\alpha}} \le \|B\|_{\mathcal{B}_{r,\alpha}} \tag{2.17}$$

hold for all $A = (a(\lambda, \lambda'))_{\lambda,\lambda' \in V}$ and $B = (b(\lambda, \lambda'))_{\lambda,\lambda' \in V}$ satisfying $|a(\lambda, \lambda')| \le |b(\lambda, \lambda')|$ for all $\lambda, \lambda' \in V$.

(ii) $\mathcal{B}_{1,0}(\mathcal{G})$ *is a Banach algebra, and*

$$\|AB\|_{\mathcal{B}_{1,0}} \le d_{\mathcal{G}} D_{\mathcal{G}} 2^{d_{\mathcal{G}}+1} \|A\|_{\mathcal{B}_{1,0}} \|B\|_{\mathcal{B}_{1,0}} \text{ for all } A, B \in \mathcal{B}_{1,0}(\mathcal{G}). \quad (2.18)$$

(iii) $\mathcal{B}_{r,\alpha}(\mathcal{G})$ *with* $1 \le r \le \infty$ *and* $\alpha > d_{\mathcal{G}}(1-1/r)$ *are Banach algebras, and*

$$\|AB\|_{\mathcal{B}_{r,\alpha}} \le d_{\mathcal{G}} D_{\mathcal{G}} 2^{\alpha+1+d_{\mathcal{G}}/r} \Big(\frac{\alpha-(d_{\mathcal{G}}-1)(1-1/r)}{\alpha-d_{\mathcal{G}}(1-1/r)}\Big)^{1-1/r} \|A\|_{\mathcal{B}_{r,\alpha}} \|B\|_{\mathcal{B}_{r,\alpha}} \quad (2.19)$$

hold for all $A, B \in \mathcal{B}_{r,\alpha}(\mathcal{G})$.

(iv) $\mathcal{B}_{r,\alpha}(\mathcal{G})$ *with* $1 \le r \le \infty$ *and* $\alpha > d_{\mathcal{G}}(1-1/r)$ *are Banach subalgebras of* $\mathcal{B}_{1,0}(\mathcal{G})$*, and*

$$\|A\|_{\mathcal{B}_{1,0}} \le \Big(\frac{\alpha-(d_{\mathcal{G}}-1)(1-1/r)}{\alpha-d_{\mathcal{G}}(1-1/r)}\Big)^{1-1/r} \|A\|_{\mathcal{B}_{r,\alpha}} \text{ for all } A \in \mathcal{B}_{r,\alpha}(\mathcal{G}). \quad (2.20)$$

(v) *A matrix* A *in* $\mathcal{B}_{r,\alpha}(\mathcal{G})$ *with* $1 \le r \le \infty$ *and* $\alpha > d_{\mathcal{G}}(1-1/r)$ *is well approximated by its truncation* A_K, $K \ge 1$, *in the norm* $\|\cdot\|_{\mathcal{B}_{1,0}}$,

$$\|A - A_K\|_{\mathcal{B}_{1,0}} \le C_0 \|A\|_{\mathcal{B}_{r,\alpha}} K^{-\alpha+d_{\mathcal{G}}(1-1/r)}, \quad (2.21)$$

where

$$C_0 = \begin{cases} 2^{\alpha+1} & \text{if } r = 1 \\ \frac{2^{\alpha+1-d_{\mathcal{G}}(1-1/r)}}{(\alpha/(1-1/r)-d_{\mathcal{G}})^{1-1/r}} & \text{if } r > 1. \end{cases}$$

2.3 Beurling Algebras of Matrices on a Maximal Disjoint Set

Let $\mathcal{G} = (V, E)$ be a connected simple graph. Given $1 \le r \le \infty, \tilde{\alpha} \ge 0$ and a maximal N-disjoint subset V_N of the vertex set V, we define Beurling algebras of matrices $B := \big(b(\lambda_m, \lambda_k)\big)_{\lambda_m, \lambda_k \in V_N}$ on V_N by

$$\mathcal{B}_{r,\tilde{\alpha};N}(V_N) := \big\{B,\ \|B\|_{\mathcal{B}_{r,\tilde{\alpha};N}} < \infty\big\} \quad (2.22)$$

where

$$\|B\|_{\mathcal{B}_{r,\tilde{\alpha};N}} := \begin{cases} \Big(\sum_{n=0}^{\infty} (n+1)^{\tilde{\alpha}r+\tilde{d}_{\mathcal{G}}-1} \Big(\sup_{\rho(\lambda_m,\lambda_k)\ge n(N+1)} |b(\lambda_m,\lambda_k)|\Big)^r\Big)^{1/r} & \text{if } 1 \le r < \infty, \\ \sup_{n\ge 0} (n+1)^{\tilde{\alpha}} \big(\sup_{\rho(\lambda_m,\lambda_k)\ge n(N+1)} |b(\lambda_m,\lambda_k)|\big) & \text{if } r = \infty. \end{cases} \quad (2.23)$$

The Banach algebra $\mathcal{B}_{r,\tilde{\alpha};N}(V_N)$ is introduced in [40], where the counting measure μ is assumed to be Ahlfors regular in which the generalized Beurling dimension $\tilde{d}_{\mathcal{G}}$ and the Beurling dimension $d_{\mathcal{G}}$ coincides by Proposition 2.1. Following the argument used in the proof of Proposition 2.2 with the polynomial growth property (2.3) replaced by the strong polynomial growth property (2.11), we have the following properties for Banach algebras $\mathcal{B}_{r,\tilde{\alpha};N}(V_N)$ of matrices on V_N.

Proposition 2.3 *Let $\mathcal{G} := (V, E)$ be a connected simple graph such that its counting measure μ satisfies the doubling property (2.1), and V_N be a maximal N-disjoint set. Then the following statements hold.*

(i) *$\mathcal{B}_{1,0;N}(V_N)$ is a Banach algebra and*

$$\|AB\|_{\mathcal{B}_{1,0;N}} \le \tilde{d}_{\mathcal{G}} \tilde{D}_{\mathcal{G}} 2^{3\tilde{d}_{\mathcal{G}}+1} \|A\|_{\mathcal{B}_{1,0;N}} \|B\|_{\mathcal{B}_{1,0;N}}, \ \ A, B \in \mathcal{B}_{1,0;N}(V_N). \quad (2.24)$$

(ii) *$\mathcal{B}_{r,\tilde{\alpha};N}(V_N)$ with $1 \le r \le \infty$ and $\tilde{\alpha} > \tilde{d}_{\mathcal{G}}(1-1/r)$ are Banach subalgebras of $\mathcal{B}_{1,0;N}(V_N)$, and*

$$\|A\|_{\mathcal{B}_{1,0;N}} \le \left(\frac{\tilde{\alpha} - (\tilde{d}_{\mathcal{G}} - 1)(1-1/r)}{\tilde{\alpha} - \tilde{d}_{\mathcal{G}}(1-1/r)} \right)^{1-1/r} \|A\|_{\mathcal{B}_{r,\tilde{\alpha};N}}, \ A \in \mathcal{B}_{r,\tilde{\alpha};N}(V_N). \quad (2.25)$$

(iii) *$\mathcal{B}_{r,\tilde{\alpha};N}(V_N)$ with $1 \le r \le \infty$ and $\tilde{\alpha} > \tilde{d}_{\mathcal{G}}(1-1/r)$ are Banach algebras, and*

$$\|AB\|_{\mathcal{B}_{r,\tilde{\alpha};N}} \le \tilde{d}_{\mathcal{G}} \tilde{D}_{\mathcal{G}} 2^{\tilde{\alpha}+\tilde{d}_{\mathcal{G}}(2+1/r)+2} \left(\frac{\tilde{\alpha} - (\tilde{d}_{\mathcal{G}} - 1)(1-1/r)}{\tilde{\alpha} - \tilde{d}_{\mathcal{G}}(1-1/r)} \right)^{1-1/r} \times \|A\|_{\mathcal{B}_{r,\tilde{\alpha};N}} \|B\|_{\mathcal{B}_{r,\tilde{\alpha};N}}, \ \ A, B \in \mathcal{B}_{r,\tilde{\alpha};N}(V_N). \quad (2.26)$$

Beurling algebra on the graph $\mathcal{G}$ and on its maximal N-disjoint set V_N are closely related. For $N = 0$, we have

$$\mathcal{B}_{r,\tilde{\alpha};0}(V_0) = \mathcal{B}_{r,\tilde{\alpha}+(\tilde{d}_{\mathcal{G}}-d_{\mathcal{G}})/r}(\mathcal{G}) \quad (2.27)$$

as the only maximal 0-disjoint set V_0 is the whole vertex set V. For $N \ge 1$, we have the following results about Beurling algebras on a graph and its maximal disjoint sets, which will be used in our proofs to establish the equivalence of weighted stability for different exponents and weights and also the norm-controlled inversion. The detailed proof will be given in Sect. 5.3.

Proposition 2.4 *Let $1 \le r \le \infty$, $\mathcal{G} := (V, E)$ be a connected simple graph such that its counting measure μ satisfies the doubling property (2.1), and V_N, $N \ge 1$, be a maximal N-disjoint set. Then the following statements hold.*

(i) *If $A = (a(\lambda, \lambda'))_{\lambda,\lambda' \in V} \in \mathcal{B}_{r,\alpha}(\mathcal{G}), \alpha \ge 0$, then its submatrix $B = (a(\lambda_m, \lambda_k))_{\lambda_m,\lambda_k \in V_N}$ belongs to $\mathcal{B}_{r,\alpha-(\tilde{d}_{\mathcal{G}}-d_{\mathcal{G}})/r;N}$, and*

$$\|B\|_{\mathcal{B}_{r,\alpha-(\tilde{d}_{\mathcal{G}}-d_{\mathcal{G}})/r;N}} \le \|A\|_{\mathcal{B}_{r,\alpha}}. \quad (2.28)$$

(ii) *If* $B = (b(\lambda_m, \lambda_k))_{\lambda_m, \lambda_k \in V_N} \in \mathcal{B}_{r,\alpha;N}$, $\alpha \geq 0$, *the matrix*

$$A = \Big(\sum_{\lambda_m \in B(\lambda, 2N)} \sum_{\lambda_k \in B(\lambda', 4N)} b(\lambda_m, \lambda_k) \Big)_{\lambda, \lambda' \in V} \tag{2.29}$$

on the graph $\mathcal{G}$ *belongs to* $B_{r,\alpha+(\tilde{d}_\mathcal{G}-d_\mathcal{G})/r}(\mathcal{G})$, *and*

$$\|A\|_{\mathcal{B}_{r,\alpha+(\tilde{d}_\mathcal{G}-d_\mathcal{G})/r}} \leq 8^{\alpha+\tilde{d}_\mathcal{G}/r} (D(\mu))^7 N^{\alpha+\tilde{d}_\mathcal{G}/r} \|B\|_{\mathcal{B}_{r,\alpha;N}}. \tag{2.30}$$

(iii) *If* $A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in V} \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ *for some* $\alpha > d_\mathcal{G}(1 - 1/r)$, *then the matrix*

$$S_{A,N} = \big(S_{A,N}(\lambda_m, \lambda_k)\big)_{\lambda_m, \lambda_k \in V_N} \tag{2.31}$$

belongs to $\mathcal{B}_{r,\alpha-(\tilde{d}_\mathcal{G}-d_\mathcal{G})/r;N}$, *and*

$$\|S_{A,N}\|_{\mathcal{B}_{r,\alpha-(\tilde{d}_\mathcal{G}-d_\mathcal{G})/r;N}} \leq \tilde{C}_0 \|A\|_{\mathcal{B}_{r,\alpha}} \times \begin{cases} N^{-\min(1,\alpha-d_\mathcal{G}/r')} & \text{if } \alpha \neq d_\mathcal{G}/r' + 1 \\ N^{-1}(\ln(N+1))^{1/r'} & \text{if } \alpha = d_\mathcal{G}/r' + 1, \end{cases} \tag{2.32}$$

where $1/r' = 1 - 1/r$, $h_A(n) = \sup_{\rho(\lambda,\lambda') \geq n} |a(\lambda, \lambda')|$, $n \geq 0$,

$$\tilde{C}_0 = 2^{4\alpha+4d_\mathcal{G}/r+2} \times \begin{cases} \Big(\frac{1+|\alpha-1-d_\mathcal{G}/r'|}{|\alpha-1-d_\mathcal{G}/r'|}\Big)^{1/r'} & \text{if } \alpha \neq d_\mathcal{G}/r' + 1 \\ 1 & \text{if } \alpha = d_\mathcal{G}/r' + 1, \end{cases}$$

and for $\lambda_m, \lambda_k \in V_N$,

$$S_{A,N}(\lambda_m, \lambda_k) = \begin{cases} N^{d_\mathcal{G}} h_A\big(\rho(\lambda_m, \lambda_k)/2\big) & \text{if } \rho(\lambda_m, \lambda_k) > 12(N+1) \\ N^{-1} \sum_{n=0}^{2N} h_A(n)(n+1)^{d_\mathcal{G}} & \text{if } \rho(\lambda_m, \lambda_k) \leq 12(N+1). \end{cases} \tag{2.33}$$

We conclude this subsection with an example to illustrate the construction of the matrix $S_{A,N}$ in (2.33). This also demonstrates that the exponent $\min(1, \alpha - d_\mathcal{G}/r')$ in (2.32) is optimal in the sense that the estimate (2.32) could fail if it is replaced by a large exponent $\beta > \min(1, \alpha - d_\mathcal{G}/r')$.

Example 2.5 Let $1 \leq r \leq \infty$, $\alpha > 1 - 1/r$, and $A_\kappa := (a_\kappa(n - n'))_{n,n' \in \mathbb{Z}}$, where $\kappa \in (0, 1)$ is a constant sufficiently close to one, and

$$a_\kappa(n) = \begin{cases} 1 & \text{if } n = 0 \\ -\kappa & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.34}$$

Then its inverse $(A_\kappa)^{-1} = (\check{a}_\kappa(n-n'))_{n,n'\in\mathbb{Z}}$ is given by

$$\check{a}_\kappa(n) = \begin{cases} \kappa^n & \text{if } n \ge 0 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\|A_\kappa\|_{\mathcal{B}_{r,\alpha}} \approx 1 \ \text{ and } \ \|A_\kappa^{-1}\|_{\mathcal{B}_{r,\alpha}} \approx (1-\kappa)^{-\alpha-1/r}. \tag{2.35}$$

Observe that $V_N = \{\lambda_m = (2N+1)m, m \in \mathbb{Z}\}$ is a maximal N-disjoint subset of $\mathbb{Z}$. For the matrix A_κ and the above maximal disjoint subset V_N, one may verify that the matrix $S_{A,N}$ in (2.33) associated with the matrix A_κ and its inverse $(A_\kappa)^{-1}$ are given by

$$S_{A_\kappa,N}(\lambda_m, \lambda_k) = \begin{cases} 0 & \text{if } |\lambda_m - \lambda_k| > 12(N+1) \\ N^{-1}(1+2\kappa) & \text{if } |\lambda_m - \lambda_k| \le 12(N+1) \end{cases} \tag{2.36}$$

and

$$S_{(A_\kappa)^{-1},N}(\lambda_m, \lambda_k) = \begin{cases} N\kappa^{\lceil |\lambda_m - \lambda_k|/2 \rceil} & \text{if } |\lambda_m - \lambda_k| > 12(N+1) \\ \frac{1-(2N+2)\kappa^{2N}+(2N+1)\kappa^{2N+1}}{N(1-\kappa)^2} & \text{if } |\lambda_m - \lambda_k| \le 12(N+1) \end{cases}$$

respectively, where the last equality as $h_{(A_\kappa)^{-1}}(n) = \kappa^n, n \ge 0$, and

$$\sum_{n=0}^{2N}(n+1)t^n = \Big(\sum_{n=0}^{2N} t^{n+1}\Big)' = \frac{1-(2N+2)t^{2N+1}+(2N+1)t^{2N+1}}{(1-t)^2}, \quad 0 < t < 1.$$

By direct calculation, we obtain

$$\|S_{A_\kappa,N}\|_{\mathcal{B}_{r,\alpha-(\tilde{d}_\mathcal{G}-d_\mathcal{G})/r;N}} = \|S_{A_\kappa,N}\|_{\mathcal{B}_{r,\alpha;N}} \approx N^{-1}$$

and

$$\|S_{(A_\kappa)^{-1},N}\|_{\mathcal{B}_{r,\alpha;N}} \approx (1-\kappa)^{-1}(\max(N(1-\kappa), 1))^{-\min(1,\alpha-1/r')}.$$

This together with (2.35) demonstrates that the estimate (2.32) holds with the matrix A replaced by matrices A_κ and also by the inverse $(A_\kappa)^{-1}$ for $\kappa \in (0, 1)$ close to one, and that the exponent $\min(1, \alpha - d_\mathcal{G}/r')$ in (2.32) is optimal.

2.4 Weighted Norm Inequalities

Let $\mathcal{G} := (V, E)$ be a connected simple graph with its counting measure μ satisfying the doubling property (2.1). For $1 \le p < \infty$, a positive function $w = (w(\lambda))_{\lambda\in V}$ on the vertex set V is a *Muckenhoupt A_p-weight* if there exists a positive constant C such that

$$\left(\frac{1}{\mu(B)}\sum_{\lambda\in B} w(\lambda)\right)\left(\frac{1}{\mu(B)}\sum_{\lambda\in B}(w(\lambda))^{-1/(p-1)}\right)^{p-1} \le C \tag{2.37}$$

for $1 < p < \infty$, and a Muckenhoupt A_1-weight if

$$\frac{1}{\mu(B)}\sum_{\lambda\in B} w(\lambda) \le C \inf_{\lambda\in B} w(\lambda) \tag{2.38}$$

for any ball $B \subset V$ [20]. The smallest constant C for which (2.37) holds for $1 < p < \infty$, and (2.38) holds for $p = 1$, respectively is known as the A_p-bound of the weight w and is denoted by $A_p(w)$. An equivalent definition of a Muckenhoupt A_p-weight $w := (w(\lambda))_{\lambda\in V}$ is that

$$\left(\frac{1}{\mu(B)}\sum_{\lambda\in B}|c(\lambda)|\right)^p\left(\frac{1}{\mu(B)}\sum_{\lambda\in B} w(\lambda)\right) \le \frac{A_p(w)}{\mu(B)}\sum_{\lambda\in B}|c(\lambda)|^p w(\lambda) \tag{2.39}$$

holds for all balls $B \subset V$ and sequences $c := \big(c(\lambda)\big)_{\lambda\in V} \in \ell^p_w$, where $A_p(w)$ is A_p-bound of the weight w. For $\lambda \in V$ and $r \ge 0$, set

$$w(B(\lambda, r)) = \sum_{\lambda'\in B(\lambda,r)} w(\lambda').$$

It is well known that a Muckenhoupt A_p-weight w is a doubling measure. In fact, replacing the ball B and the sequence c by $B(\lambda, 2^j r)$, $1 \le j \in \mathbb{Z}$ and the index sequence on $B(\lambda, r)$ in (2.39) and using the doubling condition (2.1) for the counting measure μ, we obtain that

$$\begin{aligned} w(B(\lambda, 2^j r)) &\le A_p(w)\left(\frac{\mu(B(\lambda, 2^j r)}{\mu(B(\lambda, r)}\right)^p w(B(\lambda, r)) \\ &\le (D(\mu))^{jp} A_p(w) w(B(\lambda, r)) \end{aligned} \tag{2.40}$$

hold for all $\lambda \in V$, $r \ge 0$ and positive integers j.

Weighted norm inequalities of linear operators are an important topic in harmonic analysis, see [20] and references therein for historical remarks. In the following proposition, we show that the Banach algebra $\mathcal{B}_{1,0}(\mathcal{G})$ is a Banach subalgebra of $\mathcal{B}(\ell^p_w)$, see Sect. 5.4 for the proof.

Proposition 2.6 *Let $\mathcal{G} := (V, E)$ be a connected simple graph such that its counting measure μ satisfies the doubling property (2.1). Then $\mathcal{B}_{1,0}(\mathcal{G})$ is a subalgebra of $\mathcal{B}(\ell^p_w)$ for any $1 \le p < \infty$ and Muckenhoupt A_p-weight w, and*

$$\|Ac\|_{p,w} \le 2^{3d_\mathcal{G}} D_\mathcal{G}(A_p(w))^{1/p}\|A\|_{\mathcal{B}_{1,0}}\|c\|_{p,w} \text{ for all } A \in \mathcal{B}_{1,0}(\mathcal{G}) \text{ and } c \in \ell^p_w. \tag{2.41}$$

By Propositions 2.2 and 2.6, we conclude that $\mathcal{B}_{r,\alpha}(\mathcal{G})$ with $1 \le r \le \infty$ and $\alpha > d_{\mathcal{G}}(1-1/r)$ are Banach subalgebras of $\mathcal{B}(\ell^p_w)$ too. We remark that the subalgebra property in Proposition 2.6 was established in [12,40] for the unweighted case and in [44] for the weighted case on the lattice graph $\mathcal{Z}^d$.

3 Polynomial Control on Optimal Lower Stability Bounds

In this section, we show that weighted stabilities of matrices in a Beurling algebra for different exponents and Muckenhoupt weights are equivalent to each other, and reciprocal of the optimal lower stability bound for one exponent and weight is dominated by a polynomial of reciprocal of the optimal lower stability bound for another exponent and weight.

Theorem 3.1 *Let $1 \le r \le \infty$, $1 \le p, q < \infty$, $\mathcal{G} := (V, E)$ be a connected simple graph satisfying the doubling property (2.1), w, w' be Muckenhoupt A_p-weight and A_q-weight respectively, and let $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ for some $\alpha > \tilde d_{\mathcal{G}} - d_{\mathcal{G}}/r$, where $d_{\mathcal{G}}$ and $\tilde d_{\mathcal{G}}$ are the Beurling and generalized Beurling dimension of the graph $\mathcal{G}$ respectively. If A has ℓ^p_w-stability with the optimal lower stability bound $\beta_{p,w}(A)$,*

$$\beta_{p,w}(A)\|c\|_{p,w} \le \|Ac\|_{p,w} \quad \text{for all } c \in \ell^p_w, \tag{3.1}$$

then A has $\ell^q_{w'}$-stability with the optimal lower stability bound denoted by $\beta_{q,w'}(A)$,

$$\beta_{q,w'}(A)\|c\|_{q,w'} \le \|Ac\|_{q,w'} \quad \text{for all } c \in \ell^q_{w'}. \tag{3.2}$$

Moreover, there exists an absolute constant C, independent of matrices $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ and weights w and w', such that

$$\begin{aligned}\frac{\beta_{p,w}(A)}{\beta_{q,w'}(A)} \le\, & C\left(A_q(w')\right)^{1/q}\left(A_p(w)\right)^{1/p}\left(\frac{\left(A_p(w)\right)^{2/p}\|A\|_{\mathcal{B}_{r,\alpha}}}{\beta_{p,w}(A)}\right)^{E(\alpha,r)}\\ &\times\begin{cases}1 & \text{if } \alpha \ne 1+d_{\mathcal{G}}/r'\\ \left(\ln\left(\frac{(A_p(w))^{2/p}\|A\|_{\mathcal{B}_{r,\alpha}}}{\beta_{p,w}(A)}\right)\right)^{(2d_{\mathcal{G}}+1)/r'} & \text{if } \alpha = 1+d_{\mathcal{G}}/r',\end{cases}\end{aligned} \tag{3.3}$$

where $1/r' = 1 - 1/r$ and

$$E(\alpha, r) = \frac{\tilde d_{\mathcal{G}} + d_{\mathcal{G}} + 1}{\min\left(\alpha - d_{\mathcal{G}}/r', 1\right)}.$$

Remark 3.2 The equivalence of unweighted stabilities for different exponents is discussed for matrices in Baskakov–Gohberg–Sjöstrand algebras, Jaffard algebras and Beurling algebras [2,12,27,42,44,50], for convolution operators [4], and for localized integral operators of non-convolution type [18,19,37,42]. For a matrix A in the Jaffard algebra $\mathcal{J}_\alpha(\mathcal{G}) = \mathcal{B}_{\infty,\alpha}(\mathcal{G})$ with $\alpha > d_{\mathcal{G}}$, Cheng, Jiang and Sun use differential subalgebra approach to prove in [12, Theorem 5.2] that

$$\frac{\|A\|_{\mathcal{B}_{\infty,\alpha}}}{\beta_{q,w_0}} \le h\left(\frac{\|A\|_{\mathcal{B}_{\infty,\alpha}}}{\beta_{2,w_0}}\right) \tag{3.4}$$

for some positive function h with subexponential growth. The above conclusion is improved by Shin and Sun in [40, Theorem 4.1]. They used the boot-strap argument to show that, for a matrix A in the Beurling algebra $\mathcal{B}_{r,\alpha}(\mathcal{G})$ with $1 \le r \le \infty$ and $\alpha > d_{\mathcal{G}}(1-1/r)$, reciprocal of its optimal lower unweighted stability bound for one exponent is dominated by a polynomial of reciprocal of its optimal lower unweighted stability bound for another exponent,

$$\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{\beta_{q,w_0}} \le C \begin{cases} \left(\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{\beta_{p,w_0}}\right)^{(1+\theta(p,q))^{K_0}} & \text{if } \alpha \ne 1+d_{\mathcal{G}}/r' \\ \left(\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{\beta_{p,w_0}} \ln\left(1+\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{\beta_{p,w_0}}\right)\right)^{(1+\theta(p,q))^{K_0}} & \text{if } \alpha = 1+d_{\mathcal{G}}/r', \end{cases} \tag{3.5}$$

where C is an absolute constant, K_0 is a positive integer satisfying $K_0 > \frac{d_{\mathcal{G}}}{\min(\alpha-d_{\mathcal{G}}/r',1)}$, and

$$\theta(p,q) = \frac{d_{\mathcal{G}}|1/p-1/q|}{K_0 \min(\alpha - d_{\mathcal{G}}/r', 1) - d_{\mathcal{G}}|1/p-1/q|}.$$

Given $1 \le p < \infty$, we remark that for an exponent q close to p, the conclusion (3.5) provides a better estimate to the optimal lower unweighted stability bound $\beta_{q,w_0}(A)$ than the one in (3.3) with $w = w' = w_0$, while the conclusion (3.3) with $w = w' = w_0$ gives a tighter estimate to the optimal lower unweighted stability bound $\beta_{q,w_0}(A)$ than the one in (3.5) when q is close to one or infinity.

For $1 \le N \in \mathbb{Z}$ and $\lambda \in V$, we introduce a truncation operator χ_λ^N and its smooth version Ψ_λ^N by

$$\chi_\lambda^N : \big(c(\lambda)\big)_{\lambda\in V} \longmapsto \big(\chi_{[0,N]}\big(\rho(\lambda,\lambda')\big)c(\lambda')\big)_{\lambda'\in V} \tag{3.6}$$

and

$$\Psi_\lambda^N : \big(c(\lambda)\big)_{\lambda\in V} \longmapsto \big(\psi_0\big(\rho(\lambda,\lambda')/N\big)c(\lambda')\big)_{\lambda'\in V}, \tag{3.7}$$

where $\psi_0(t) = \max\{0, \min(1, 3-2|t|)\}$ is the trapezoid function satisfying

$$\chi_{[-1,1]}(t) \le \psi_0(t) \le \chi_{[-3/2,3/2]}(t), \ \ t \in \mathbb{R}.$$

The operators χ_λ^N and Ψ_λ^N localize a sequence to a neighborhood of λ and they can be considered as diagonal matrices with entries $\chi_{B(\lambda,N)}(\lambda')$ and $\psi_0(\rho(\lambda,\lambda')/N)$, $\lambda' \in V$ respectively.

Let V_N be a maximal N-disjoint set. To prove Theorem 3.1, we start from estimating the weighted terms $\big(w\big(B(\lambda_m, 4N)\big)\big)^{-1/p}\|\Psi_{\lambda_m}^{2N}c\|_{p,w}$, $\lambda_m \in V_N$, for sufficiently large N, which is established in [40] for the trivial weight w_0.

Lemma 3.3 *Let $1 \le p < \infty$, $1 \le r \le \infty$, $\alpha > d_{\mathcal{G}}(1-1/r)$, w be a Muckenhoupt A_p-weight, and $A = (a(\lambda, \lambda'))_{\lambda,\lambda' \in V} \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ have ℓ^p_w-stability. Assume that $N \ge 1$ is a positive integer such that*

$$2C_1(A_p(w))^{1/p}\|A\|_{\mathcal{B}_{r,\alpha}} N^{-\alpha+d_{\mathcal{G}}(1-1/r)} \le \beta_{p,w}(A), \tag{3.8}$$

where $\beta_{p,w}(A)$ is the optimal lower ℓ^p_w-stability bound, C_0 is the constant in (2.21) and $C_1 = 2^{3d_{\mathcal{G}}} C_0 D_{\mathcal{G}}$. Then for all maximal N-disjoint sets V_N and weighted sequences $c \in \ell^p_w$, we have

$$\begin{aligned} &\beta_{p,w}(A)\big(w\big(B(\lambda_m, 4N)\big)\big)^{-1/p}\|\Psi_{\lambda_m}^{2N}c\|_{p,w} \\ &\le 2\big(w\big(B(\lambda_m, 4N)\big)\big)^{-1/p}\|\Psi_{\lambda_m}^{2N}Ac\|_{p,w} + C_2\big(A_p(w)\big)^{2/p} \\ &\quad \times \sum_{\lambda_k \in V_N} S_{A,N}(\lambda_m, \lambda_k)\big(w\big(B(\lambda_k, 4N)\big)\big)^{-1/p}\|\Psi_{\lambda_k}^{2N}c\|_{p,w},\ \lambda_m \in V_N, \end{aligned} \tag{3.9}$$

where the smooth truncation operators $\Psi_{\lambda_m}^{2N}$, $\lambda_m \in V_N$, are defined in (3.7), the matrix $S_{A,N} = (S_{A,N}(\lambda_m, \lambda_k))_{\lambda_m,\lambda_k \in V_N}$ is given in (2.33), and $C_2 \ge 2$ is an absolute constant.

Let $[\Psi_{\lambda_m}^{2N}, A] := \Psi_{\lambda_m}^{2N}A - A\Psi_{\lambda_m}^{2N}$ be the commutator between the smooth truncation operator $\Psi_{\lambda_m}^{2N}$ and the matrix A [40,42,43], and the matrix $S_{A,N} = (S_{A,N}(\lambda_m, \lambda_k))_{\lambda_m,\lambda_k \in V_N}$ be given in (2.33). A crucial step in the proof of Lemma 3.3 is the following estimate to the commutator $[\Psi_{\lambda_m}^{2N}, A]$,

$$\|\chi_{\lambda_m}^{4N}[\Psi_{\lambda_m}^{2N}, A]\chi_{\lambda_k}^{3N}\|_{\mathcal{B}_{1,0}} \lesssim S_{A,N}(\lambda_m, \lambda_k),\ \lambda_m, \lambda_k \in V_N, \tag{3.10}$$

see Sect. 5.5 for the detailed argument. By Propositions 2.3 and 2.4, we have the following estimate for l-th power of the matrix $S_{A,N}$ for all $l \ge 1$,

$$\begin{aligned} \big\|\big(S_{A,N}\big)^{\ell}\big\|_{\mathcal{B}_{r,\alpha-(\tilde d_{\mathcal{G}}-d_{\mathcal{G}})/r;N}} &\le (C_3\|A\|_{\mathcal{B}_{r,\alpha}})^l \\ &\quad \times \begin{cases} N^{-\min(1,\alpha-d_{\mathcal{G}}/r')l} & \text{if } \alpha \ne d_{\mathcal{G}}/r'+1 \\ N^{-l}(\ln(N+1))^{l/r'} & \text{if } \alpha = d_{\mathcal{G}}/r'+1 \end{cases} \end{aligned} \tag{3.11}$$

where $1/r' = 1 - 1/r$ and C_3 is an absolute constant independent on $N \ge 1, l \ge 1$ and matrices $A \in \mathcal{B}_{r,\alpha}$. Applying (3.9) and (3.11) repeatedly, we have the following crucial estimates (3.13) and (3.14), see Sect. 5.6 for the proof.

Lemma 3.4 *Let $1 \le p < \infty$, w be a Muckenhoupt A_p-weight, and $A = (a(\lambda, \lambda'))_{\lambda,\lambda' \in V} \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ for some $1 \le r \le \infty$ and $\alpha > \tilde d_{\mathcal{G}} - d_{\mathcal{G}}/r$. Assume* *that A has ℓ^p_w-stability with the optimal lower stability bound $\beta_{p,w}(A)$ and that the positive integer N satisfies*

$$\beta_{p,w}(A) \geq 2\max(C_1, C_2C_3)\big(A_p(w)\big)^{2/p}\|A\|_{\mathcal{B}_{r,\alpha}} \times \begin{cases} N^{-\min(1,\alpha-d_{\mathcal{G}}/r')} & \text{if } \alpha \neq d_{\mathcal{G}}/r'+1 \\ N^{-1}(\ln(N+1))^{1/r'} & \text{if } \alpha = d_{\mathcal{G}}/r'+1, \end{cases} \tag{3.12}$$

where C_1, C_2, C_3 are absolute constants in (3.8), (3.9) and (3.11) respectively. Then there exist a matrix $H_{A,N} = (H_{A,N}(\lambda,\lambda'))_{\lambda,\lambda'\in V}$ and two absolute constants C_4 and C_5 such that

$$\|H_{A,N}\|_{\mathcal{B}_{r,\alpha}} \leq C_4 N^{\alpha+d_{\mathcal{G}}/r} \tag{3.13}$$

and

$$|c(\lambda)| \leq C_5(A_p(w))^{1/p}(\beta_{p,w}(A))^{-1}N^{d_{\mathcal{G}}}\sum_{\lambda'\in V} H_{A,N}(\lambda,\lambda')|Ac(\lambda')|,\ c\in\ell^p_w. \tag{3.14}$$

Next we provide an explicit construction of the matrix $H_{A_\kappa,N}$ in the above lemma for the matrix A_κ in Example 2.5 when w is the trivial weight w_0.

Example 3.5 (Continuation of Example 2.5) Let A_κ be as in Example 2.5 with $\kappa \in (0,1)$ sufficiently close to one. Take the trivial weight $w_0 \equiv 1$ and the maximal N-disjoint set $V_N = \{\lambda_m = (2N+1)m, m \in \mathbb{Z}\}$ with $N \geq 3$ satisfying (3.12). In the above setting, we obtain from (2.36) that the estimate in (3.9) becomes

$$(1-\kappa)\|\Psi^{2N}_{\lambda_m}c\|_p \leq 2\|\Psi^{2N}_{\lambda_m}A_\kappa c\|_p + C_2N^{-1}(1+\kappa)\sum_{|k-m|\leq 6}\|\Psi^{2N}_{\lambda_k}c\|_p,\ \lambda_m\in V_N.$$

Applying the above inequality repeatedly for $L \geq 1$ times, we obtain

$$\begin{aligned}
\|\Psi^{2N}_{\lambda_m}c\|_p &\leq \frac{2}{1-\kappa}\|\Psi^{2N}_{\lambda_m}A_\kappa c\|_p + \frac{C_2(1+\kappa)}{N(1-\kappa)}\sum_{k\in\mathbb{Z}} B_1(k-m)\|\Psi^{2N}_{\lambda_k}c\|_p \\
&\leq \frac{2}{1-\kappa}\|\Psi^{2N}_{\lambda_m}A_\kappa c\|_p + \frac{2C_2(1+\kappa)}{N(1-\kappa)^2}\sum_{k\in\mathbb{Z}} B_1(k-m)\|\Psi^{2N}_{\lambda_k}A_\kappa c\|_p \\
&\quad + \frac{2C_2(1+\kappa)}{N(1-\kappa)^2}\sum_{k\in\mathbb{Z}} B_2(k-m)\|\Psi^{2N}_{\lambda_k}c\|_p \\
&\leq \cdots \\
&\leq \frac{2}{1-\kappa}\sum_{k\in\mathbb{Z}}\Big(\sum_{l=0}^{L-1}\Big(\frac{C_2(1+\kappa)}{N(1-\kappa)}\Big)^l B_l(k-m)\Big)\|\Psi^{2N}_{\lambda_k}A_\kappa c\|_p \\
&\quad + \frac{2}{1-\kappa}\Big(\frac{C_2(1+\kappa)}{N(1-\kappa)}\Big)^{L-1}\sum_{k\in\mathbb{Z}} B_L(k-m)\|\Psi^{2N}_{\lambda_k}c\|_p,
\end{aligned} \tag{3.15}$$

where B_0 is the delta sequence, and $B_n := (B_n(k))_{k\in\mathbb{Z}}, n \geq 1$, are the n-th discrete convolution of $B_1 := (\chi_{[-6,6]}(k))_{k\in\mathbb{Z}}$. Set

$$h_{A_\kappa}(m) = \sum_{l=0}^{\infty} \Big(\frac{C_2(1+\kappa)}{N(1-\kappa)}\Big)^l B_l(m),\ m \in \mathbb{Z}. \tag{3.16}$$

For $n \in \mathbb{Z}$, let $\lambda_m = (2N+1)m$ be the unique vertex in V_N such that $0 \leq n-\lambda_m \leq 2N$. Then

$$\begin{aligned} |c(n)| \leq \|\Psi^{2N}_{\lambda_m} c\|_p &\leq \frac{2}{1-\kappa}\sum_{k\in\mathbb{Z}} h_{A_\kappa}(k-m)\|\Psi^{2N}_{\lambda_k} A_\kappa c\|_p \\ &\leq \frac{2}{1-\kappa}\sum_{k\in\mathbb{Z}} h_{A_\kappa}(k-m) \sum_{|n'-(2N+1)k|\leq 3N} |A_\kappa c(n')|, \end{aligned} \tag{3.17}$$

where the first and third inequalities follow as ψ_0 is bounded above and below by the characteristic function on the interval $[-1, 1]$ and $[-3/2, 3/2]$ respectively, and the second inequality is obtained from taking limit $L \to \infty$ in (3.15). By (3.17), the matrix $H_{A,N}$ in Lemma 3.4 with A replaced by A_κ can be defined by

$$H_{A_\kappa,N} = \left(\frac{2}{1-\kappa}\sum_{|n'-(2N+1)k|\leq 3N} h_{A_\kappa}(k - \lfloor n/(2N+1)\rfloor)\right)_{n,n'\in\mathbb{Z}}.$$

Now we are ready to finish the proof of Theorem 3.1.

Proof of Theorem 3.1 As for $\alpha' \geq \alpha$, $\mathcal{B}_{r,\alpha'}(\mathcal{G})$ is a Banach subalgebra of $\mathcal{B}_{r,\alpha}(\mathcal{G})$. Then it suffices to prove (3.3) for all α satisfying

$$d_\mathcal{G}/r' \leq \tilde{d}_\mathcal{G} - d_\mathcal{G}/r < \alpha \leq \tilde{d}_\mathcal{G} - d_\mathcal{G}/r + 1. \tag{3.18}$$

Define

$$N_0 = \begin{cases} \tilde{N}_0 & \text{if } \alpha \neq d_\mathcal{G}/r' + 1 \\ 2\tilde{N}_0(\ln(\tilde{N}_0+1))^{1-1/r} & \text{if } \alpha = d_\mathcal{G}/r' + 1, \end{cases} \tag{3.19}$$

where

$$\tilde{N}_0 = \left\lfloor \left(\frac{2\max(C_1, C_2C_3)\big(A_p(w)\big)^{2/p}\|A\|_{\mathcal{B}_{r,\alpha}}}{\beta_{p,w}(A)}\right)^{1/\min(1,\alpha-d_\mathcal{G}/r')} \right\rfloor + 2.$$

Then one may verify that (3.12) is satisfied for $N = N_0$. Applying Lemma 3.4 with N replaced by N_0 and also Proposition 2.6, we obtain

$$\|c\|_{q,w'} \lesssim (A_q(w'))^{1/q}(A_p(w))^{1/p}(\beta_{p,w}(A))^{-1}N_0^{d_\mathcal{G}}\|H_{A,N_0}\|_{\mathcal{B}_{1,0}}\|Ac\|_{q,w'},$$

$$c \in \ell^p_w \cap \ell^q_{w'}, \tag{3.20}$$

where w' is an A_q-weight with $1 \le q < \infty$ and the matrix $H_{A,N}$ is given in Lemma 3.4. This together with (3.13) and the density of $\ell^p_w \cap \ell^q_{w'}$ in $\ell^q_{w'}$ implies that

$$\|c\|_{q,w'} \lesssim (A_q(w'))^{1/q}(A_p(w))^{1/p}(\beta_{p,w}(A))^{-1} N_0^{\alpha+d_{\mathcal{G}}(1+1/r)} \|Ac\|_{q,w'}$$

for all $c \in \ell^q_{w'}$. Therefore

$$\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{\beta_{q,w'}(A)} \lesssim \frac{\big(A_q(w')\big)^{1/q}}{\big(A_p(w)\big)^{1/p}} \frac{\big(A_p(w)\big)^{2/p}\|A\|_{\mathcal{B}_{r,\alpha}}}{\beta_{p,w}(A)} N_0^{\alpha+d_{\mathcal{G}}(1+1/r)}, \tag{3.21}$$

where $\beta_{q,w'}(A)$ is the optimal lower $\ell^q_{w'}$-stability bound of the matrix A. This together with (3.18) and (3.19) completes the proof of Theorem 3.1. □

4 Norm-Controlled Inversion

In this section, we show that Banach algebras $\mathcal{B}_{r,\alpha}(\mathcal{G})$ with $1 \le r \le \infty$ and $\alpha > \tilde{d}_{\mathcal{G}} - d_{\mathcal{G}}/r$ admit a polynomial norm-controlled inversion in $\mathcal{B}(\ell^p_w)$ for all $1 \le p < \infty$ and Muckenhoupt A_p-weights, see Sect. 5.7 for the proof.

Theorem 4.1 *Let $1 \le r \le \infty$, $1 \le p < \infty$, $\mathcal{G} := (V, E)$ be a connected simple graph satisfying the doubling property (2.1), and w be a Muckenhoupt A_p-weight. If A belongs to $\mathcal{B}_{r,\alpha}(\mathcal{G})$ for some $\alpha > \tilde{d}_{\mathcal{G}} - d_{\mathcal{G}}/r$ and it is invertible in $\mathcal{B}(\ell^p_w)$, then $A^{-1} \in \mathcal{B}_{r,\alpha}$. Moreover, there exists an absolute constant C such that*

$$\begin{aligned} \|A^{-1}\|_{\mathcal{B}_{r,\alpha}} &\le C(A_p(w))^{1/p}\|A^{-1}\|_{\mathcal{B}(\ell^p_w)}\Big((A_p(w))^{2/p}\|A^{-1}\|_{\mathcal{B}(\ell^p_w)}\|A\|_{\mathcal{B}_{r,\alpha}}\Big)^{\frac{\alpha+d_{\mathcal{G}}(1+1/r)}{\min(\alpha-d_{\mathcal{G}}/r',1)}} \\ &\times \begin{cases} 1 & \text{if } \alpha \ne 1+d_{\mathcal{G}}/r', \\ \Big(\ln\big((A_p(w))^{2/p}\|A^{-1}\|_{\mathcal{B}(\ell^p_w)}\|A\|_{\mathcal{B}_{r,\alpha}}+1\big)\Big)^{(2d_{\mathcal{G}}+1)/r'} & \text{if } \alpha = 1+d_{\mathcal{G}}/r', \end{cases} \end{aligned} \tag{4.1}$$

where $1/r' = 1 - 1/r$.

Remark 4.2 In [12, Theorem 5.3], Jaffard algebras $\mathcal{J}_\alpha(\mathcal{G}) = \mathcal{B}_{\infty,\alpha}(\mathcal{G})$ with $\alpha > d_{\mathcal{G}}$ are shown to admit norm-controlled inversion in the symmetric $*$-algebra $\mathcal{B}(\ell^2) = \mathcal{B}(\ell^2_{w_0})$, and moreover

$$\|A^{-1}\|_{\mathcal{B}_{\infty,\alpha}}\|A\|_{\mathcal{B}_{\infty,\alpha}} \le h\big(\|A^{-1}\|_{\mathcal{B}(\ell^2)}\|A\|_{\mathcal{B}_{\infty,\alpha}}\big) \tag{4.2}$$

hold for all matrices $A \in \mathcal{J}_\alpha(\mathcal{G})$ being invertible in $\mathcal{B}(\ell^2)$, where h is a positive function with subexponential growth. Recall from (2.12) that $\tilde{d}_{\mathcal{G}} \ge d_{\mathcal{G}}$. Therefore for the case that $\alpha > \tilde{d}_{\mathcal{G}}$, the estimate in (4.1) with the parameter r, the exponent p and Muckenhoupt A_p-weight w replaced by ∞, 2 and the trivial weight w_0 respectively provides a better estimate than the one in (4.2), while the case that $\tilde{d}_{\mathcal{G}} \ge \alpha > d_{\mathcal{G}}$ and

$r = \infty$ is not covered in Theorem 4.1. The estimate (4.2) was improved by Shin and Sun under the additional assumption that the counting measure μ is Ahlfors regular in which $\tilde{d}_{\mathcal{G}} = d_{\mathcal{G}}$ by Proposition 2.1. They show in [40, Theorem 5.1] that for any matrix A in Beurling algebras $\mathcal{B}_{r,\alpha}(\mathcal{G})$ with $1 \le r \le \infty$ and $\alpha > d_{\mathcal{G}}(1 - /r)$,

$$\|A^{-1}\|_{\mathcal{B}_{r,\alpha}} \le C\|A^{-1}\|_{\mathcal{B}(\ell^2)}\big(\|A^{-1}\|_{\mathcal{B}(\ell^2)}\|A\|_{\mathcal{B}_{r,\alpha}}\big)^{(\alpha+d_{\mathcal{G}}/r)/\min(\alpha-d_{\mathcal{G}}/r',1)} \times \begin{cases} 1 & \text{if } \alpha \ne 1 + d_{\mathcal{G}}/r' \\ \big(\ln\big(\|A^{-1}\|_{\mathcal{B}(\ell^2)}\|A\|_{\mathcal{B}_{r,\alpha}} + 1\big)\big)^{(d_{\mathcal{G}}+1)/r'} & \text{if } \alpha = 1 + d_{\mathcal{G}}/r', \end{cases} \tag{4.3}$$

where $1/r' = 1 - 1/r$ and C is an absolute constant. Hence under the additional assumption that the counting measure μ is Ahlfors regular, the conclusion (4.3) provides a better upper bound estimate to $\|A^{-1}\|_{\mathcal{B}_{r,\alpha}}$ than the one in (4.1) with the exponent p and Muckenhoupt A_p-weight w replaced by 2 and the trivial weight w_0 respectively.

In the following example, we demonstrate the almost optimality of the norm estimate (4.1) for the inversion.

Example 4.3 (Continuation of Example 2.5) Let A_κ be as in Example 2.5, where $\kappa \in (0, 1)$ is a constant sufficiently close to one, and let $w_\theta = ((|n| + 1)^\theta)_{n\in\mathbb{Z}}$, $-1 < \theta < p - 1$. Then w_θ is a Muckenhoupt A_p-weight and

$$\|A_\kappa^{-1}\|_{\mathcal{B}(\ell^p_{w_\theta})} \lesssim \|A_\kappa^{-1}\|_{\mathcal{B}_{1,0}} \lesssim (1-\kappa)^{-1}. \tag{4.4}$$

Take $c_0 = (c_0(n))_{n\in\mathbb{Z}}$, where

$$c_0(n) := \begin{cases} \kappa^n & \text{if } n \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\|c_0\|_{p,w_\theta} = \left(\sum_{n=0}^{\infty} \kappa^{np}(n+1)^\theta\right)^{1/p} \approx (1-\kappa)^{-(\theta+1)/p} \tag{4.5}$$

and

$$\|A_\kappa^{-1}c_0\|_{p,w_\theta} = \left(\sum_{n=0}^{\infty} \kappa^{nq}(n+1)^p(n+1)^\theta\right)^{1/p} \approx (1-\kappa)^{-(\theta+p+1)/p}. \tag{4.6}$$

By (4.4), (4.5) and (4.6), we have

$$\|A_\kappa^{-1}\|_{\mathcal{B}(\ell^p_{w_\theta})} \approx (1-\kappa)^{-1}. \tag{4.7}$$

Combining (2.35) and (4.7) yields

$$\|A_\kappa^{-1}\|_{\mathcal{B}_{r,\alpha}} \approx \|A_\kappa^{-1}\|_{\mathcal{B}(\ell^p_{w_\theta})}\big(\|A_\kappa^{-1}\|_{\mathcal{B}(\ell^p_{w_\theta})}\|A_\kappa\|_{\mathcal{B}_{r,\alpha}}\big)^{\alpha+d_{\mathcal{G}}/r-d_{\mathcal{G}}}, \tag{4.8}$$

while the estimate (4.1) in Theorem 4.1 for $\alpha > 1 + d_{\mathcal{G}}(1 - 1/r)$ is

$$\|A_\kappa^{-1}\|_{\mathcal{B}_{r,\alpha}} \lesssim \|A_\kappa^{-1}\|_{\mathcal{B}(\ell^p_w)} \big(\|A_\kappa^{-1}\|_{\mathcal{B}(\ell^p_w)} \|A_\kappa\|_{\mathcal{B}_{r,\alpha}}\big)^{\alpha + d_{\mathcal{G}}/r + d_{\mathcal{G}}}. \tag{4.9}$$

We observe that the difference of exponents in (4.8) and (4.9) is $2d_{\mathcal{G}}$ (independent on α) when $\alpha > 1 + d_{\mathcal{G}}/r$.

5 Proofs

In this section, we collect the proofs of Propositions 2.1, 2.2, 2.4 and 2.6, Lemmas 3.3 and 3.4, and Theorem 4.1.

5.1 Proof of Proposition 2.1

By (2.6), it suffices to establish (2.11) for $N \geq 1$ and $R \geq 3$. Let V_N be a maximal N-disjoint set, and define $A_R(\lambda, N)$ as in (2.8). Then we obtain from (2.5) and (2.13) that

$$\begin{aligned}
\mu\left(A_R(\lambda, N)\right) &\leq \frac{\sum_{\lambda_m \in A_R(\lambda,N)} \mu(B(\lambda_m, N))}{\inf_{\lambda_{m'} \in A_R(\lambda,N)} \mu(B(\lambda_{m'}, N))} \\
&\leq B_3^{-1} N^{-d_0} \mu\left(\cup_{\lambda_m \in A_R(\lambda,N)} B(\lambda_m, N)\right) \\
&\leq B_3^{-1} N^{-d_0} \mu\left(B(\lambda, N + (N+1)R\right) \leq 2^{d_0} B_3^{-1} B_4 (R+1)^{d_0},
\end{aligned}$$

which implies that

$$\tilde{d}_{\mathcal{G}} \leq d_0. \tag{5.1}$$

Similarly by (2.7) and (2.13), we get

$$\begin{aligned}
\mu\big(A_R(\lambda, N)\big) &\geq \frac{\sum_{\lambda_m \in A_R(\lambda,N)} \mu(B(\lambda_m, 2N))}{\max_{\lambda_m \in A_R(\lambda,N)} \mu(B(\lambda_m, 2N))} \\
&\geq B_4^{-1} (2N+1)^{-d_0} \mu\big(\cup_{\lambda_m \in A_R(\lambda,N)} B(\lambda_m, 2N)\big) \\
&\geq 3^{-d_0} B_4^{-1} N^{-d_0} \mu(B(\lambda, N(R-2)) \\
&\geq 2^{-2d_0} 3^{-d_0} B_4^{-1} B_3 (R+1)^{d_0}, \ \ R \geq 3,
\end{aligned}$$

where the third inequality holds as $B(\lambda_m, 2N)) \cap B(\lambda, N(R-2)) = \emptyset$ for all $\lambda_m \notin A_R(\lambda, N)$. This show that

$$\tilde{d}_{\mathcal{G}} \geq d_0. \tag{5.2}$$

Combining (5.1) and (5.2) completes the proof.

5.2 Proof of Proposition 2.2

The conclusion (i) is obvious and the conclusions in (ii), (iii) and (iv) are presented in [40, Propositions 3.3 and 3.4]. Now we prove the conclusion (v). Write $A = (a(\lambda, \lambda'))_{\lambda,\lambda'\in V}$ and set $h_A(n) = \sup_{\rho(\lambda,\lambda')\ge n} |a(\lambda, \lambda')|$, $n \ge 0$. Then for $K \ge 1$ and $1 < r \le \infty$, we have

$$\begin{aligned}
\|A - A_K\|_{\mathcal{B}_{1,0}} &\le 2 \sum_{n=\lceil (K+1)/2\rceil}^{\infty} h_A(n)(n+1)^{d_{\mathcal{G}}-1} \\
&\le 2\|A\|_{\mathcal{B}_{r,\alpha}} \left(\sum_{n=\lceil (K+1)/2\rceil}^{\infty} (n+1)^{-\alpha r' + d_{\mathcal{G}}-1} \right)^{1/r'} \\
&\le 2\|A\|_{\mathcal{B}_{r,\alpha}} \left(\int_{\lceil (K+1)/2\rceil}^{\infty} x^{d_{\mathcal{G}}-1-\alpha r'} dx \right)^{1/r'} \\
&\le \frac{2^{\alpha - d_{\mathcal{G}}/r' + 1}}{(\alpha r' - d_{\mathcal{G}})^{1/r'}} \|A\|_{\mathcal{B}_{r,\alpha}} K^{-\alpha + d_{\mathcal{G}}/r'},
\end{aligned}$$

where $r' = r/(r-1)$. This proves (2.21) for $1 < r \le \infty$. Similarly we can prove (2.21) for $r = 1$.

5.3 Proof of Proposition 2.4

The conclusion (i) follows from the definition of Beurling algebras on the graph $\mathcal{G}$ and on its maximal disjoint set V_N.

Now we prove the conclusion (ii). Set $\check{\alpha} = \alpha + (\tilde{d}_{\mathcal{G}} - d_{\mathcal{G}})/r$. For $1 \le r < \infty$, we obtain

$$\begin{aligned}
\|A\|^r_{\mathcal{B}_{r,\check{\alpha}}} &\le \sum_{n'=0}^{\infty} \sum_{n=n'(N+1)}^{(n'+1)(N+1)-1} (n+1)^{\alpha r + \tilde{d}_{\mathcal{G}}-1} \left(\sup_{\rho(\lambda,\lambda')\ge n} \sum_{\substack{\lambda_m \in B(\lambda,2N) \\ \lambda_k \in B(\lambda',4N)}} |b(\lambda_m, \lambda_k)| \right)^r \\
&\le (N+1)^{\alpha r + \tilde{d}_{\mathcal{G}}} \sum_{n'=0}^{\infty} (n'+1)^{\alpha r + \tilde{d}_{\mathcal{G}}-1} \\
&\quad \times \left(\sup_{\rho(\lambda,\lambda')\ge n'(N+1)} \sum_{\substack{\lambda_m \in B(\lambda,2N) \\ \lambda_k \in B(\lambda',4N)}} |b(\lambda_m, \lambda_k)| \right)^r \\
&\le (D(\mu))^{7r} (N+1)^{\alpha r + \tilde{d}_{\mathcal{G}}} \sum_{n'=0}^{\infty} (n'+1)^{\alpha r + \tilde{d}_{\mathcal{G}}-1}
\end{aligned}$$

$$\times \left(\sup_{\rho(\lambda_m,\lambda_k)\geq \max(n'-6,0)(N+1)} |b(\lambda_m,\lambda_k)| \right)^r$$
$$\leq 8^{\alpha r+\tilde{d}_{\mathcal{G}}} (D(\mu))^{7r} N^{\alpha r+\tilde{d}_{\mathcal{G}}} \|B\|^r_{\mathcal{B}_{r,\alpha;N}}, \tag{5.3}$$

where the third inequality follows from (2.7). This proves (2.30) and the conclusion (ii) for $1 \leq r < \infty$.

Similarly for $r = \infty$, we have

$$\begin{aligned} \|A\|_{\mathcal{B}_{\infty,\check{\alpha}}} &\leq \sup_{\lambda,\lambda'\in V} \sum_{\lambda_m\in B(\lambda,2N),\lambda_k\in B(\lambda',4N)} |b(\lambda_m,\lambda_k)|(\rho(\lambda,\lambda')+1)^\alpha \\ &\leq \|B\|_{\infty,\alpha;N} \sup_{\lambda,\lambda'\in V} \sum_{\lambda_m\in B(\lambda,2N),\lambda_k\in B(\lambda',4N)} \left(\frac{\rho(\lambda,\lambda')+1}{\lfloor \rho(\lambda_m,\lambda_k)/N \rfloor + 1} \right)^\alpha \\ &\leq 8^\alpha (D(\mu))^7 N^\alpha \|B\|_{\infty,\alpha;N}. \end{aligned} \tag{5.4}$$

Combining (5.3) and (5.4) proves (2.30) and the conclusion (ii).

Finally we prove the conclusion (iii). Set $\tilde{\alpha} = \alpha - (\tilde{d}_{\mathcal{G}} - d_{\mathcal{G}})/r$ and $1/r' = 1 - 1/r$. Then for $1 < r < \infty$, we have

$$\begin{aligned} \|S_{A,N}\|_{\mathcal{B}_{r,\tilde{\alpha};N}} &\leq N^{d_{\mathcal{G}}} \left(\sum_{n=13}^{\infty} (h_A(n(N+1)/2))^r \, (n+1)^{\alpha r+d_{\mathcal{G}}-1} \right)^{1/r} \\ &\quad + N^{-1} \left(\sum_{n=0}^{2N} h_A(n)(n+1)^{d_{\mathcal{G}}} \right) \times \left(\sum_{n=0}^{12} (n+1)^{\alpha r+d_{\mathcal{G}}-1} \right)^{1/r} \\ &\leq 2^{2\alpha+(2d_{\mathcal{G}}-1)/r} N^{-\alpha+d_{\mathcal{G}}/r'} \left(\sum_{m=4N}^{\infty} (h_A(m))^r (m+1)^{\alpha r+d_{\mathcal{G}}-1} \right)^{1/r} \\ &\quad + 13^{\alpha+d_{\mathcal{G}}/r} N^{-1} \sum_{n=0}^{2N} h_A(n)(n+1)^{d_{\mathcal{G}}} \\ &\leq 2^{2\alpha+(2d_{\mathcal{G}}-1)/r} N^{-\alpha+d_{\mathcal{G}}/r'} \|A\|_{\mathcal{B}_{r,\alpha}} \\ &\quad + 13^{\alpha+d_{\mathcal{G}}/r} \|A\|_{\mathcal{B}_{r,\alpha}} N^{-1} \left(\sum_{n=0}^{2N} (n+1)^{-(\alpha-1)r'+d_{\mathcal{G}}-1} \right)^{1/r'} \\ &\leq 2^{2\alpha+(2d_{\mathcal{G}}-1)/r} N^{-\alpha+d_{\mathcal{G}}/r'} \|A\|_{\mathcal{B}_{r,\alpha}} + 13^{\alpha+d_{\mathcal{G}}/r} \|A\|_{\mathcal{B}_{r,\alpha}} \\ &\quad \times \begin{cases} 2\left(\frac{1+|\alpha-1-d_{\mathcal{G}}/r'|}{|\alpha-1-d_{\mathcal{G}}/r'|} \right)^{1/r'} N^{-\min(1,\alpha-d_{\mathcal{G}}/r')} & \text{if } \alpha \neq d_{\mathcal{G}}/r'+1, \\ 3^{1/r'} N^{-1} (\ln(N+1))^{1/r'} & \text{if } \alpha = d_{\mathcal{G}}/r'+1. \end{cases} \end{aligned} \tag{5.5}$$

This proves (2.32) for $1 < r < \infty$. Using similar argument, we can prove (2.32) for $r = 1, \infty$.

5.4 Proof of Proposition 2.6

For the completeness of this paper, we follow the argument in [20,44] and give a sketch of the proof. Write $A = (a(\lambda, \lambda'))_{\lambda,\lambda' \in V}$, and set $h_A(n) = \sup_{\rho(\lambda,\lambda')\geq n} |a(\lambda, \lambda')|$, $n \geq 0$. Then for any $c \in \ell^p_w$ with $1 < p < \infty$, we have

$$\begin{aligned}
\|Ac\|_{p,w} &\leq \Big(\sum_{\lambda\in V}\Big(\sum_{\lambda'\in V} h_A(\rho(\lambda,\lambda'))|c(\lambda')|\Big)^p w(\lambda)\Big)^{1/p}\\
&\leq h_A(0)\|c\|_{p,w} + \Big(\sum_{\lambda\in V}\Big(\sum_{l=1}^{\infty} h_A(2^{l-1}) \sum_{2^{l-1}\leq\rho(\lambda,\lambda')<2^l} |c(\lambda')|\Big)^p w(\lambda)\Big)^{1/p}\\
&\leq h_A(0)\|c\|_{p,w} + \Big(\sum_{l=1}^{\infty} h_A(2^{l-1})2^{ld_{\mathcal{G}}}\Big)^{1-1/p}\\
&\quad\times \Big(\sum_{l=1}^{\infty} h_A(2^{l-1})2^{-(p-1)ld_{\mathcal{G}}}\sum_{\lambda\in V}\Big(\sum_{2^{l-1}\leq\rho(\lambda,\lambda')<2^l} |c(\lambda')|\Big)^p w(\lambda)\Big)^{1/p}. \qquad (5.6)
\end{aligned}$$

By the equivalent definition (2.39) of the Muckenhoupt A_p-weight w and the polynomial property (2.3) of the counting measure μ, we obtain

$$\begin{aligned}
&\sum_{\lambda\in V}\Big(\sum_{\rho(\lambda,\lambda')<2^l} |c(\lambda')|\Big)^p w(\lambda)\\
&\leq \sum_{\lambda\in V}\Big(\sum_{\rho(\lambda,\lambda')<2^l} |c(\lambda')|^p w(\lambda')\Big)\Big(\sum_{\rho(\lambda,\lambda'')<2^l} (w(\lambda''))^{-1/(p-1)}\Big)^{p-1} w(\lambda)\\
&\leq A_p(w)\sum_{\lambda\in V} w(\lambda)\Big(\sum_{\rho(\lambda',\lambda)<2^l} |c(\lambda')|^p w(\lambda')\Big) \times \frac{\big(\mu(B(\lambda, 2^{l+1}-2))\big)^p}{\sum_{\rho(\lambda,\lambda'')\leq 2^{l+1}-2} w(\lambda'')}\\
&\leq A_p(w)(D_{\mathcal{G}})^p 2^{p(l+1)d_{\mathcal{G}}} \sum_{\lambda'\in V} |c(\lambda')|^p w(\lambda') \times \sum_{\rho(\lambda,\lambda')<2^l} \frac{w(\lambda)}{\sum_{\rho(\lambda',\lambda'')\leq 2^l-1} w(\lambda'')}\\
&= A_p(w)(D_{\mathcal{G}})^p 2^{p(l+1)d_{\mathcal{G}}} \|c\|^p_{p,w}.
\end{aligned}$$

This together with (5.6) and the following estimate

$$\begin{aligned}
&h_A(0) + \sum_{l=1}^{\infty} h(2^{l-1})2^{ld_{\mathcal{G}}}\\
&\leq h_A(0) + 2^{2d_{\mathcal{G}}}\sum_{l=1}^{\infty}\sum_{2^{l-2}<n\leq 2^{l-1}} h_A(n)(n+1)^{d_{\mathcal{G}}-1} \leq 2^{2d_{\mathcal{G}}}\|A\|_{\mathcal{B}_{1,0}}
\end{aligned}$$

proves (2.41) for $1 < p < \infty$.

Applying a similar argument as above, we can verify (2.41) for $p = 1$.

5.5 Proof of Lemma 3.3

We follow the procedure used in [40], where a similar result is established for the unweighted case. Take $\lambda_m \in V_N$. Denote the commutator between the smooth truncation operator $\Psi_{\lambda_m}^{2N}$ and the matrix A by $[\Psi_{\lambda_m}^{2N}, A] := \Psi_{\lambda_m}^{2N} A - A\Psi_{\lambda_m}^{2N}$, and set $\Phi^{2N} := \big(\sum_{\lambda_k \in V_N} \Psi_{\lambda_k}^{2N}\big)^{-1}$. Replacing c in (3.1) by $\Psi_{\lambda_m}^{2N} c$ and applying Proposition 2.6, we have

$$\begin{aligned}
&\beta_{p,w}(A)\|\Psi_{\lambda_m}^{2N} c\|_{p,w} \le \|A\Psi_{\lambda_m}^{2N} c\|_{p,w} \le \|\Psi_{\lambda_m}^{2N} Ac\|_{p,w} + \|[\Psi_{\lambda_m}^{2N}, A]c\|_{p,w}\\
&\le \|\Psi_{\lambda_m}^{2N} Ac\|_{p,w} + \|\chi_{\lambda_m}^{4N}[\Psi_{\lambda_m}^{2N}, A]c\|_{p,w} + \|(I-\chi_{\lambda_m}^{4N})A\chi_{\lambda_m}^{3N}\Psi_{\lambda_m}^{2N} c\|_{p,w}\\
&\le \|\Psi_{\lambda_m}^{2N} Ac\|_{p,w} + \sum_{\lambda_k \in V_N} \|\chi_{\lambda_m}^{4N}[\Psi_{\lambda_m}^{2N}, A]\chi_{\lambda_k}^{3N}\Phi^{2N}\Psi_{\lambda_k}^{2N} c\|_{p,w}\\
&\quad + 2^{3d_{\mathcal{G}}} D_{\mathcal{G}}\big(A_p(w)\big)^{1/p}\|(I-\chi_{\lambda_m}^{4N})A\chi_{\lambda_m}^{3N}\|_{\mathcal{B}_{1,0}}\|\Psi_{\lambda_m}^{2N} c\|_{p,w},
\end{aligned} \tag{5.7}$$

where the second inequality holds, as

$$(I-\chi_{\lambda_m}^{4N})[\Psi_{\lambda_m}^{2N}, A] = (I-\chi_{\lambda_m}^{4N})A\Psi_{\lambda_m}^{2N} = (I-\chi_{\lambda_m}^{4N})A\chi_{\lambda_m}^{3N}\Psi_{\lambda_m}^{2N}$$

by the supporting properties for $\chi_{\lambda_m}^{3N}$, $\chi_{\lambda_m}^{4N}$ and $\Psi_{\lambda_m}^{2N}$. From (2.17) and (2.21) in Proposition 2.2, we obtain

$$\|(I-\chi_{\lambda_m}^{4N})A\chi_{\lambda_m}^{3N}\|_{\mathcal{B}_{1,0}} \le \|A - A_N\|_{\mathcal{B}_{1,0}} \le C_0\|A\|_{\mathcal{B}_{r,\alpha}} N^{-\alpha+d_{\mathcal{G}}(1-1/r)}. \tag{5.8}$$

Combining (5.7) and (5.8) yields

$$\begin{aligned}
\beta_{p,w}(A)\|\Psi_{\lambda_m}^{2N} c\|_{p,w} &\le \|\Psi_{\lambda_m}^{2N} Ac\|_{p,w} + \sum_{\lambda_k \in V_N} \|\chi_{\lambda_m}^{4N}[\Psi_{\lambda_m}^{2N}, A]\chi_{\lambda_k}^{3N}\Phi^{2N}\Psi_{\lambda_k}^{2N} c\|_{p,w}\\
&\quad + 2^{3d_{\mathcal{G}}} C_0 D_{\mathcal{G}}(A_p(w))^{1/p}\|A\|_{\mathcal{B}_{r,\alpha}} N^{-\alpha+d_{\mathcal{G}}(1-1/r)}\|\Psi_{\lambda_m}^{2N} c\|_{p,w}.
\end{aligned}$$

This together with (3.8) proves that

$$\beta_{p,w}(A)\|\Psi_{\lambda_m}^{2N} c\|_{p,w} \le 2\|\Psi_{\lambda_m}^{2N} Ac\|_{p,w} + 2\sum_{\lambda_k \in V_N} \|\chi_{\lambda_m}^{4N}[\Psi_{\lambda_m}^{2N}, A]\chi_{\lambda_k}^{3N}\Phi^{2N}\Psi_{\lambda_k}^{2N} c\|_{p,w}. \tag{5.9}$$

For $\lambda_k \in V_N$ with $\rho(\lambda_m, \lambda_k) > 12(N+1)$, we obtain from the finite covering property (2.7) for the maximal N-disjoint set V_N, the equivalent definition (2.39) of the weight w, the polynomial growth property (2.3) of the counting measure μ, and the monotonicity of $h_A(n)$, $n \ge 0$, that

$$\|\chi_{\lambda_m}^{4N}[\Psi_{\lambda_m}^{2N}, A]\chi_{\lambda_k}^{3N}\Phi^{2N}\Psi_{\lambda_k}^{2N} c\|_{p,w} = \|\Psi_{\lambda_m}^{2N} A\chi_{\lambda_k}^{3N}\Phi^{2N}\Psi_{\lambda_k}^{2N} c\|_{p,w}$$

$$\le h_A\Big(\frac{\rho(\lambda_m,\lambda_k)}{2}\Big)\Big(\sum_{\lambda\in B(\lambda_m,4N)}\Big(\sum_{\lambda'\in B(\lambda_k,4N)}|\Psi_{\lambda_k}^{2N}c(\lambda')|\Big)^p w(\lambda)\Big)^{1/p}$$
$$\lesssim (A_p(w))^{1/p} S_{A,N}(\lambda_m,\lambda_k)\|\Psi_{\lambda_k}^{2N}c\|_{p,w}\left(\frac{w\big(B(\lambda_m,4N)\big)}{w\big(B(\lambda_k,4N)\big)}\right)^{1/p}. \tag{5.10}$$

Set $\tilde{A}_M = \big(|a(\lambda,\lambda')|\rho(\lambda,\lambda')\chi_{[0,M]}(\rho(\lambda,\lambda'))\big)_{\lambda,\lambda'\in V}$, $M\ge 0$. For $\lambda_k\in V_N$ with $\rho(\lambda_m,\lambda_k) < 12(N+1)$, we have

$$\begin{aligned}&\|\chi_{\lambda_m}^{4N}[\Psi_{\lambda_m}^{2N},A]\chi_{\lambda_k}^{3N}\Phi^{2N}\Psi_{\lambda_k}^{2N}c\|_{p,w}\\ &\lesssim \big(A_p(w)\big)^{1/p}\|\chi_{\lambda_m}^{4N}[\Psi_{\lambda_m}^{2N},A]\chi_{\lambda_k}^{3N}\|_{\mathcal{B}_{1,0}}\|\Phi^{2N}\Psi_{\lambda_k}^{2N}c\|_{p,w}\\ &\lesssim \big(A_p(w)\big)^{1/p}N^{-1}\|\tilde{A}_{19N+12}\|_{\mathcal{B}_{1,0}}\|\Psi_{\lambda_k}^{2N}c\|_{p,w},\end{aligned} \tag{5.11}$$

where the first inequality follows from the weighted norm inequality (2.41) in Proposition 2.6, and the second one holds by the solidness of the Banach algebra $\mathcal{B}_{1,0}(\mathcal{G})$ in Proposition 2.2 and the Lipschitz property for the trapezoid function ψ_0. Observe that

$$w\big(B(\lambda_k,4N)\big)\le w\big(B(\lambda_m,19N+12)\big)\le (D(\mu))^{3p}A_p(w)w\big(B(\lambda_m,4N)\big) \tag{5.12}$$

by the double property (2.40) for the A_p-weight w, and

$$\begin{aligned}\|\tilde{A}_{19N+12}\|_{\mathcal{B}_{1,0}} &= \sum_{n=0}^{19N+12}(n+1)^{d_{\mathcal{G}}-1}\sup_{n\le\rho(\lambda,\lambda')\le 19N+12}|a(\lambda,\lambda')|\rho(\lambda,\lambda')\\ &\le 2\sum_{n=0}^{19N+12}(n+1)^{d_{\mathcal{G}}-1}\sum_{n/2\le m\le 19N+12}h_A(m)\lesssim\sum_{n=0}^{2N}h_A(n)(n+1)^{d_{\mathcal{G}}}\end{aligned} \tag{5.13}$$

by the monotonicity of $h_A(n)$, $n\ge 0$. Combining (5.11), (5.12) and (5.13), we get

$$\begin{aligned}&\|\chi_{\lambda_m}^{4N}[\Psi_{\lambda_m}^{2N},A]\chi_{\lambda_k}^{3N}\Phi^{2N}\Psi_{\lambda_k}^{2N}c\|_{p,w}\\ &\lesssim \big(A_p(w)\big)^{2/p}S_{A,N}(\lambda_m,\lambda_k)\|\Psi_{\lambda_k}^{2N}c\|_{p,w}\left(\frac{w\big(B(\lambda_m,4N)\big)}{w\big(B(\lambda_k,4N)\big)}\right)^{1/p}\end{aligned} \tag{5.14}$$

if $\rho(\lambda_m,\lambda_k)\le 12(N+1)$.

Combining (5.9), (5.10) and (5.14) proves (3.9).

5.6 Proof of Lemma 3.4

Set $\alpha_{\lambda_m} := w\big(B(\lambda_m, 4N)\big)$, $\lambda_m \in V_N$, and write

$$\big(S_{A,N}\big)^l := \big(S_{A,N;l}(\lambda_m, \lambda_k)\big)_{\lambda_m,\lambda_k\in V_N}, \, l \geq 1.$$

By (3.12), the integer N satisfies (3.8) and hence (3.9) holds by Lemma 3.3. Applying (3.9) repeatedly, we get

$$\begin{aligned}
&\big(\alpha_{\lambda_m}\big)^{-1/p}\|\Psi^{2N}_{\lambda_m}c\|_{p,w}\\
&\leq 2(\beta_{p,w}(A))^{-1}\big(\alpha_{\lambda_m}\big)^{-1/p}\|\Psi^{2N}_{\lambda_m}Ac\|_{p,w} + C_2\big(A_p(w)\big)^{2/p}(\beta_{p,w}(A))^{-1}\\
&\quad\times\sum_{\lambda_k\in V_N} S_{A,N}(\lambda_m,\lambda_k)\big(\alpha_{\lambda_k}\big)^{-1/p}\|\Psi^{2N}_{\lambda_k}c\|_{p,w}\\
&\leq\cdots\\
&\leq 2(\beta_{p,w}(A))^{-1}\big(\alpha_{\lambda_m}\big)^{-1/p}\|\Psi^{2N}_{\lambda_m}Ac\|_{p,w}\\
&\quad+2(\beta_{p,w}(A))^{-1}\sum_{l=1}^{L-1}\big(C_2\big(A_p(w)\big)^{2/p}(\beta_{p,w}(A))^{-1}\big)^l\\
&\quad\times\sum_{\lambda_k\in V_N} S_{A,N;l}(\lambda_m,\lambda_k)\big(\alpha_{\lambda_k}\big)^{-1/p}\|\Psi^{2N}_{\lambda_k}Ac\|_{p,w}\\
&\quad+\big(C_2\big(A_p(w)\big)^{2/p}(\beta_{p,w}(A))^{-1}\big)^L\\
&\quad\times\sum_{\lambda_k\in V_N} S_{A,N;L}(\lambda_m,\lambda_k)\big(\alpha_{\lambda_k}\big)^{-1/p}\|\Psi^{2N}_{\lambda_k}c\|_{p,w},
\end{aligned}\tag{5.15}$$

where $L \geq 2$. Define

$$W_{A,N} = 2I + 2\sum_{l=1}^{\infty}\big(C_2\big(A_p(w)\big)^{2/p}(\beta_{p,w}(A))^{-1}\big)^l(S_{A,N})^l.\tag{5.16}$$

Then by (3.11) and (3.12), we have

$$\begin{aligned}
\|W_{A,N}\|_{\mathcal{B}_{r,\alpha-(\tilde d_{\mathcal{G}}-d_{\mathcal{G}})/r;N}} &\leq 2+2\sum_{l=1}^{\infty}\big(C_2C_3\big(A_p(w)\big)^{2/p}(\beta_{p,w}(A))^{-1}\|A\|_{\mathcal{B}_{r,\alpha}}\big)^l\\
&\qquad\times\begin{cases} N^{-\min(1,\alpha-d_{\mathcal{G}}/r')l} & \text{if } \alpha\neq d_{\mathcal{G}}/r'+1\\ N^{-l}(\ln(N+1))^{l/r'} & \text{if } \alpha = d_{\mathcal{G}}/r'+1,\end{cases}\\
&\leq 2+2\sum_{l=1}^{\infty}2^{-l} = 4.
\end{aligned}\tag{5.17}$$

Following the argument used in the proof of Proposition 2.6, we obtain

$$\begin{aligned}&\Big(\sum_{\lambda_m\in V_N}\Big|\sum_{\lambda_k\in V_N}S_{A,N;L}(\lambda_m,\lambda_k)\big(\alpha_{\lambda_k}\big)^{-1/p}\|\Psi^{2N}_{\lambda_k}c\|_{p,w}\Big|^p\alpha_{\lambda_m}\Big)^{1/p}\\&\lesssim\big\|(S_{A,N})^L\big\|_{\mathcal{B}_{1,0;N}}\Big(\sum_{\lambda_k\in V_N}\Big|\big(\alpha_{\lambda_k}\big)^{-1/p}\|\Psi^{2N}_{\lambda_k}c\|_{p,w}\Big|^p\alpha_{\lambda_k}\Big)^{1/p}\\&\le C_6\big\|(S_{A,N})^L\big\|_{\mathcal{B}_{r,\alpha-(\tilde d_{\mathcal{G}}-d_{\mathcal{G}})/r;N}}\|c\|_{p,w},\end{aligned}\tag{5.18}$$

where C_6 is an absolute constant. This together with (3.11) and (3.12) implies that

$$\begin{aligned}&\Big(\sum_{\lambda_m\in V_N}\Big|\sum_{\lambda_k\in V_N}S_{A,N;L}(\lambda_m,\lambda_k)\big(\alpha_{\lambda_k}\big)^{-1/p}\|\Psi^{2N}_{\lambda_k}c\|_{p,w}\Big|^p\alpha_{\lambda_m}\Big)^{1/p}\\&\times\Big(C_2\big(A_p(w)\big)^{2/p}(\beta_{p,w}(A))^{-1}\Big)^L\le C_6 2^{-L}\|c\|_{p,w}\to 0 \text{ as } L\to\infty.\end{aligned}\tag{5.19}$$

Taking limit $L\to\infty$ in (5.15) and applying (5.17) and (5.19), we obtain

$$\beta_{p,w}(A)\big(\alpha_{\lambda_m}\big)^{-1/p}\|\Psi^{2N}_{\lambda_m}c\|_{p,w}\le\sum_{\lambda_k\in V_N}W_{A,N}(\lambda_m,\lambda_k)\big(\alpha_{\lambda_k}\big)^{-1/p}\|\Psi^{2N}_{\lambda_k}Ac\|_{p,w}\tag{5.20}$$

where $\lambda_m\in V_N$ and $c\in\ell^p_w$.

Define $H_{A,N}:=\big(H_{A,N}(\lambda,\lambda')\big)_{\lambda,\lambda'\in V}$ by

$$H_{A,N}(\lambda,\lambda'):=\sum_{\lambda_m\in B(\lambda,2N)}\sum_{\lambda_k\in B(\lambda',4N)}W_{A,N}(\lambda_m,\lambda_k).\tag{5.21}$$

Then the desired norm estimate (3.13) follows from (2.17), (2.30) and (5.17).

Let $\lambda\in V$ and select $\lambda_m\in V_N$ such that $\lambda\in B(\lambda_m,2N)$. Such a vertex λ_m exists by the covering property (2.7). Replacing the vector $(c(\lambda'))_{\lambda'\in V}$ and the ball B by the delta vector $(\delta_0(\lambda,\lambda'))_{\lambda'\in V}$ and $B(\lambda_m,4N)$ in (2.39), respectively, we get

$$\alpha_{\lambda_m}\lesssim A_p(w)N^{pd_{\mathcal{G}}}w(\lambda).\tag{5.22}$$

Combining (3.9), (5.20) and (5.22), we obtain

$$\begin{aligned}|c(\lambda)|&\lesssim(A_p(w))^{1/p}N^{d_{\mathcal{G}}}\sum_{\lambda_m\in B(\lambda,2N)}\alpha^{-1/p}_{\lambda_m}\|\Psi^{2N}_{\lambda_m}c\|_{p,w}\\&\lesssim(A_p(w))^{1/p}(\beta_{p,w}(A))^{-1}N^{d_{\mathcal{G}}}\\&\qquad\sum_{\lambda_m\in B(\lambda,2N)}\sum_{\lambda_k\in V_N}W_{A,N}(\lambda_m,\lambda_k)\alpha^{-1/p}_{\lambda_k}\|\Psi^{2N}_{\lambda_k}Ac\|_{p,w}\end{aligned}$$

$$\lesssim (A_p(w))^{1/p}(\beta_{p,w}(A))^{-1}N^{d_\mathcal{G}}\sum_{\lambda'\in V} H_{A,N}(\lambda,\lambda')|Ac(\lambda')| \tag{5.23}$$

for all $c \in \ell^p_w$, where the last inequality holds as $\alpha_{\lambda_k}^{-1/p}\|\Psi^{2N}_{\lambda_k}Ac\|_{p,w} \le \|\Psi^{2N}_{\lambda_k}Ac\|_{p,w_0} \le \|\Psi^{2N}_{\lambda_k}Ac\|_{1,w_0}$. This proves (3.14).

5.7 Proof of Theorem 4.1

By the invertibility assumption of the matrix A in ℓ^p_w, it has the ℓ^p_w-stability (3.1) and its optimal lower stability bound $\beta_{p,w}(A)$ satisfies

$$\beta_{p,w}(A) \ge \left(\|A^{-1}\|_{\mathcal{B}(\ell^p_w)}\right)^{-1}. \tag{5.24}$$

Let r' be the conjugate exponent of r, i.e., $1/r + 1/r' = 1$, $N \ge 2$ be an integer satisfying

$$\left(\|A^{-1}\|_{\mathcal{B}(\ell^p_w)}\right)^{-1} \ge 2\max(C_2C_3, C_1)\big(A_p(w)\big)^{2/p}\|A\|_{\mathcal{B}_{r,\alpha}} \times \begin{cases} N^{-\min(1,\alpha-d_\mathcal{G}/r')} & \text{if } \alpha \ne d_\mathcal{G}/r'+1 \\ N^{-1}(\ln(N+1))^{1/r'} & \text{if } \alpha = d_\mathcal{G}/r'+1, \end{cases} \tag{5.25}$$

and $H_{A,N} = (H_{A,N}(\lambda,\lambda'))_{\lambda,\lambda'\in V}$ be as in (5.21) except replacing $\beta_{p,w}(A)$ by $(\|A^{-1}\|_{\mathcal{B}(\ell^p_w)})^{-1}$. Following the argument used in the proof of Lemma 3.4, we obtain

$$\|H_{A,N}\|_{\mathcal{B}_{r,\alpha}} \lesssim N^{\alpha+d_\mathcal{G}/r} \tag{5.26}$$

and

$$|c(\lambda)| \lesssim (A_p(w))^{1/p}\|A^{-1}\|_{\mathcal{B}(\ell^p_w)}N^{d_\mathcal{G}}\sum_{\lambda'\in V} H_{A,N}(\lambda,\lambda')|(Ac)(\lambda')| \tag{5.27}$$

for $c = (c(\lambda))_{\lambda\in V} \in \ell^p_w$.

Write $A^{-1} := (\check{a}(\lambda',\lambda))_{\lambda',\lambda\in V}$ and denote $\check{a}_\lambda := (\check{a}(\lambda',\lambda))_{\lambda'\in V}$, $\lambda \in V$. Then $\check{a}_\lambda \in \ell^p_w$ by (5.24) and the invertibility of the matrix A. Replacing c in (5.27) by $\check{a}_\lambda$, we get

$$\begin{aligned}|\check{a}(\lambda',\lambda)| &\lesssim (A_p(w))^{1/p}\|A^{-1}\|_{\mathcal{B}(\ell^p_w)}N^{d_\mathcal{G}}\sum_{\lambda''\in V} H_{A,N}(\lambda',\lambda'')|(A\check{a}_\lambda)(\lambda'')| \\ &= (A_p(w))^{1/p}\|A^{-1}\|_{\mathcal{B}(\ell^p_w)}N^{d_\mathcal{G}}H_{A,N}(\lambda',\lambda) \ \text{ for all } \lambda,\lambda'\in V.\end{aligned} \tag{5.28}$$

This together with (5.26) and the solidness of the Beurling algebra $\mathcal{B}_{r,\alpha}(\mathcal{G})$ in Proposition 2.2 implies that

$$\|A^{-1}\|_{\mathcal{B}_{r,\alpha}} \lesssim (A_p(w))^{1/p}\|A^{-1}\|_{\mathcal{B}(\ell^p_w)}N^{d_\mathcal{G}}\|H_{A,N}\|_{\mathcal{B}_{r,\alpha}}$$

$$\lesssim (A_p(w))^{1/p} \|A^{-1}\|_{\mathcal{B}(\ell^p_w)} N^{\alpha+d_{\mathcal{G}}(1+1/r)}. \tag{5.29}$$

Define

$$N_1 = \begin{cases} \tilde{N}_1 & \text{if } \alpha \neq d_{\mathcal{G}}/r' + 1 \\ 2\tilde{N}_1(\ln(\tilde{N}_1 + 1))^{1/r'} & \text{if } \alpha = d_{\mathcal{G}}/r' + 1, \end{cases} \tag{5.30}$$

where

$$\tilde{N}_1 = \left\lfloor \Big(2\max(C_1, C_2C_3)\big(A_p(w)\big)^{2/p} \|A^{-1}\|_{\mathcal{B}(\ell^p_w)} \|A\|_{\mathcal{B}_{r,\alpha}}\Big)^{1/\min(1,\alpha-d_{\mathcal{G}}/r')} \right\rfloor + 2$$

and C_1, C_2, C_3 are absolute constants in (3.8), (3.9) and (3.11) respectively. One may verify that N_1 satisfies (5.25). Then replacing N in (5.29) by the above integer N_1 completes the proof.

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