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Robert Bartnik Collection

It is a great pleasure to comment on three articles [2, 1, 3] that Robert Bartnik wrote [2, 1] and co-authored [3]. These results have had immense impact on various levels. The following describes a selection.

R. Bartnik. The mass of an asymptotically flat manifold. Comm. Pure Appl. Math. 39 (1986). 661-693.

In this article [2], R. Bartnik proved that the mass of an asymptotically flat n-manifold is a geometric invariant, and that the positive mass theorem holds for n-dimensional spin manifolds. The condition of admitting a spin structure is naturally satisfied for oriented 3-manifolds.

The positive mass theorem states that for an asymptotically flat manifold with non-negative scalar curvature the total mass is non-negative. For obvious physical reasons, this is one of the major theorems in general relativity.

Not only did Bartnik generalize the celebrated positive mass theorem, proven by R. Schoen and S.-T. Yau [9, 10] and independently by E. Witten [11], to allow for more general decay conditions and higher dimensions, but he also established the independence of the ADM mass from the coordinate system. Thereby, he generalized the ADM mass to manifolds of much slower decay. In particular, for 3-manifolds (M, g), the decay conditions roughly are $|g - \delta| = o(r^{-\frac{1}{2}}), |\partial g| = o(r^{-\frac{3}{2}})$. Independently, P.T. Chruściel gave a different proof of the invariance of the ADM mass in [8].

Clearly, one of the great achievements in Bartnik's article lies in extending the positive mass theorem. At the same time, the result on the invariance of the mass cannot be overestimated. In particular, from a physical point of view, it is crucial that the ADM mass is unambiguously defined and independent from the choice of coordinates.

The spaectimes that I constructed in my work [4, 5, 6], generalizing D. Christodoulou's and S. Klainerman's proof of the nonlinear stability of Minkowski spacetime, obey the above loose decay conditions. Therefore, Bartnik's [2] version of the positive mass theorem applies rather than the original ones.

The results in [2] marked important breakthroughs to include much more general isolated gravitating systems. Many crucial mathematical theorems with direct physical applications in general relativity and astrophysics build on these results.

R. Bartnik. Existence of maximal surfaces in asymptotically flat space-times. Comm. Math. Phys. 94. (1984). 155-175.

R. Bartnik, P.T. Chruściel, N. O'Murchadha. On Maximal Surfaces in Asymptotically Flat Spacetimes. Comm. Math. Phys. 130. (1990). 95-109.

R. Bartnik proved [1] the existence of maximal hypersurfaces in asymptotically flat spacetimes, then together with P.T. Chruściel and N. O'Murchadha generalized [3] these results.

Maximal hypersurfaces are spacelike submanifolds of a Lorentzian manifold which locally maximize the induced volume functional. They are characterized by the vanishing of the mean extrinsic curvature $k = 0$.

The original results [1] were established for asymptotically flat spacetimes obeying a certain barrier condition. The article [3] pushed the asymptotic decay conditions so far as to include situations where the metric decays to the flat one at a rate $r^{-\alpha}$ for $\alpha > 0$. In [3] the authors also proved the existence of almost maximal hypersurfaces, that is slices on which the trace of the extrinsic curvature has compact support. - The latter is especially useful in dealing with challenging topologies.

These results have had enormous impact in general relativity. Perhaps the most important feature of the constant mean curvature (CMC) hypersurfaces is their uniqueness property. In cosmological spacetimes, the CMC foliations $k = const.$ yield a canonical choice of a global time function. Similarly, maximal hypersurfaces, thus $k = 0$, parametrized by the time at infinity serve that purpose in asymptotically flat spacetimes.

As the constraint equations in general relativity simplify considerably when choosing $k = 0$ (respectively when $k = const.$) these results have enabled the construction of large classes of initial data for the Einstein equations. In particular, when $k = 0$, the momentum constraint becomes conformally invariant.

In [7] D. Christodoulou and S. Klainerman proved the global nonlinear stability of Minkowski spacetime for maximal initial data, $k = 0$, obeying certain smallness conditions, establishing a breakthrough. Thereby the structure equations of the time-foliation together with the condition that the surfaces are maximal give rise to an elliptic system of equations for the parameters of the foliation. The authors ensure that the time function is unique by stipulating that the lapse function of the foliation approaches 1 at infinity on each level set. Here, Bartnik's original theorem [1] applies. And most generalizations of [7] make use of [1].

The situation is different for my generalizations [4, 5, 6] of [7]. In particular, these spacetimes feature much slower decay properties. The latter are included in the work [3] by Bartnik, Chruściel and O'Murchadha. Again, the maximal time foliation puts the equations in a good form, simplifying the proof tremendously.

The spacetimes constructed in these stability proofs also yield insights into large data behavior. The smallness of the initial data was required to establish existence and uniqueness of solutions. However, the main behavior along null hypersurfaces towards future null infinity remains largely independent from the smallness assumptions. Therefore, they provide excellent playgrounds to study gravitational radiation. Many of the results that I have proved and co-authored about gravitational waves and their memory build on the afore-mentioned works using a maximal time foliation.

Further important applications extend to numerical relativity, and a multitude of geometric achievements.

The articles [1, 3] have opened up whole new alleys of research. They pushed forward hugely the knowledge in general relativity as well as geometric analysis.

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